Higher Integrability of the Gradient of Minimizers of Functionals with Nonstandard Growth Conditions

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Introduction

Let us consider the functional

\[ I(v) = \int_{\Omega} F(|Dv|) \, dx, \]

where \( F: [0, \infty] \to [0, \infty] \) is an increasing function. These functionals have been extensively studied when \( F \) satisfies control assumptions of the type

\[ t^p \leq F(t) \leq c(1 + t^q), \quad t \geq 0, \]

with \( 1 < p = q \).

In this paper we give a contribution to the study of the case in which \( p < q \) (see Theorem 3.2). Namely, assuming that \( F \) is a convex function satisfying

\[ \frac{F(t)}{t^p} \text{ is increasing,} \]

\[ \frac{F(t)}{t^q} \text{ is decreasing,} \]

and \( q_* < p \leq q \), where \( q_* = \frac{nq}{n + q} \), we prove that if \( u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N) \) is a local minimizer of \( I(v) \), then there exists \( r > 1 \) such that \( F(|Du|) \in L^r_{\text{loc}}(\Omega) \). As an example of a function to which our result applies, one can take

\[ F(t) = t^p \log(1 + t), \quad p > 1, \]

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for any dimension $n$, or

$$F(t) = \begin{cases} et^3, & 0 \leq t \leq e, \\
t^{4 + \sin(\log \log t)}, & e \leq t, \end{cases}$$

if $n = 2, 3, 4$ (see Section 3).

The higher integrability result is obtained by proving a suitable extension of the so-called Gehring's lemma. The proof given here is different from the ones given in [1], [7], [3], and follows the lines of the one suggested by [2] in the case $F(t) = t^p$.

We would like to point out that the results obtained here rely heavily on the fact that our integrand depends only on the modulus of the gradient. Moreover, the restrictions on $p, q$ seem related to some counterexamples contained in [5], [10].

1. Preliminaries and Notation

In the following we shall denote by $\Omega$ a bounded open set of $\mathbb{R}^n$, by $Q$ a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, and by $l(Q)$ its side; moreover, by $aQ$ we shall denote the cube with the same center as $Q$ such that $l(aQ) = al(Q)$. If $Q_o$ is a fixed cube in $\mathbb{R}^n$, $f$ is a $L_{\text{loc}}(Q_o)$ non-negative function, we shall denote by

$$Mf(x) = \sup_{x \in Q \subset Q_o} \int_Q f(y) \, dy$$

(where $f_Q$ stands for $(1/|Q|)$) the local maximal function of $f$ in $Q_o$. Among the properties of $Mf$, we recall the following "weak type" inequality (see [1]).

**Proposition 1.1.** *If $Q_o$ is a cube in $\mathbb{R}^n$, $f \in L^1(Q_o)$, $f \geq 0$, then, for any $t > 0$,

$$\left| \left\{ x \in Q_o : Mf(x) > t \right\} \right| \leq \left[ c(n)/t \right] \int_{\{f > t/2\}} f \, dx.$$*

In the following, we shall denote by $A$ a continuous function $A : [0, \infty[ \rightarrow [0, \infty[,$ such that

(i) $A(2t) \leq kA(t)$ for all $t > 0,$

(ii) there exists $p > 1$ such that $A(t)/t^p$ is increasing.

We remark that if $A$ is an increasing convex function, condition (i), which is known as the $\Delta_2$ condition, is equivalent to the assumption that there exists $q > 1$ such that $A(t)/t^q$ is decreasing. On the other hand, condition (ii) is equivalent to the $\Delta_2$ condition for the conjugate function of $A$ (see [9], §4).
We now state the following extension of the maximal theorem of Hardy-Littlewood given in [1].

**Proposition 1.2.** If $A$ satisfies (i) and (ii), $f$ is a non-negative function such that $A(f) \in L^1(Q_0)$, then there exists a constant $c = c(n, p, k)$ such that

\begin{equation}
\int_{Q_n} A(Mf) \, dx \leq c \int_{Q_n} A(f) \, dx.
\end{equation}

**Proof:** If, for any $t > 0$, we set

$$\lambda(t) = |\{x \in Q_0 : Mf(x) > t\}|,$$

then, using Proposition 1.1, we have by Fubini's theorem and integration by parts

\begin{align*}
\int_{Q_n} A(Mf) \, dx &= \int_0^{+\infty} A'(t) \lambda(t) \, dt \\
&\leq c(n) \int_0^{+\infty} \left[ A'(t) / t \right] dt \int_{\{f > t/2\}} f \, dx \\
&= c(n) \int_{Q_n} f(x) \, dx \int_0^{2f(x)} \left[ A'(t) / t \right] dt \\
&= c(n) \int_{Q_n} A(2f(x)) / 2 \, dx \\
&\quad + c(n) \int_{Q_n} f(x) \int_0^{2f(x)} A(t) / t^2 \, dx \, dt \\
&\leq \frac{1}{2} kc(n) \int_{Q_n} A(f) \, dx \\
&\quad + c(n) \int_{Q_n} f(x) \int_0^2 \left[ A(sf(x)) / (s^2f(x)) \right] \, dx \, dt.
\end{align*}
Using again the assumption on \( A \) it follows that

\[
\int_0^2 [A(s f(x))/s^2] \, ds = \int_0^1 [A(s f(x))/s^2] \, ds + \int_1^2 [A(s f(x))/s^2] \, ds
\]

(1.3)

\[
\leq A(f(x)) \int_0^1 s^{p-2} \, ds + kA(f(x)) \int_1^2 s^{-2} \, ds.
\]

Then, combining (1.2) and (1.3) we obtain (1.1).

Remark 1.3. From the proof it is clear that Proposition 1.2 still holds if, instead of (ii), we assume that there exist \( p > 1, H > 0 \) such that

\[
A(\lambda t) \leq H\lambda^p(t) \quad \text{for all} \quad t > 0, 0 \leq \lambda \leq 1.
\]

Let us now recall the following useful result of [12].

**Theorem 1.4.** If \( f(x) \) is a non-negative \( L^1(Q_o) \) function such that, for any cube \( Q \subset Q_o \),

\[
\frac{1}{Q} \int_Q f(x) \, dx \leq k \text{ess sup}_Q f(x),
\]

then there exists \( r > 1, L > 0 \) depending only on \( k \) and \( n \) such that

\[
\left( \frac{1}{Q_r} \int_{Q_r} f'(x) \, dx \right)^{1/r} \leq L \frac{1}{Q} \int_Q f(x) \, dx.
\]

We conclude this section, by proving the following lemma about cubes.

**Lemma 1.5.** If \( Q, Q' \) are two cubes contained in a fixed cube \( Q_o \) and \( Q \subsetneq 2Q', Q \cap Q' \neq \emptyset \), then there exists a cube \( \tilde{Q} \supset Q \cup Q' \), \( \tilde{Q} \subset Q_o \) such that \( \ell(\tilde{Q}) \leq 3\ell(Q) \).

**Proof:** Let us suppose that \( Q_o = [0, a]^n \). If \( Q = \prod_{i=1}^n [a_i, b_i] \) and \( Q' = \prod_{i=1}^n [a'_i, b'_i] \), then since \( Q \cap Q' \neq \emptyset, [a_i, b_i] \cap [a'_i, b'_i] \neq \emptyset \) for all \( i = 1, \ldots, n \). If we set

\[
l = \max_{1 \leq i \leq n} \{ \max\{b_i, b'_i\} - \min\{a_i, a'_i\} \},
\]
then \( l < a \) and there exists a cube \( \tilde{Q} = \prod_{i=1}^n [c_i, c_i + 1] \) containing \( Q \cup Q' \).
Now \( l \leq l(Q) + l(Q') \) and, since \( Q \not\subset 2Q' \), \( Q \cap Q' \not= \emptyset \), we have \( l(Q') \leq 2l(Q) \).

2. An Extension of Gehring's Lemma

Now we give a suitable extension of Gehring's lemma for functions \( A \) satisfying (i) and (ii), following an idea of [2] relative to the case \( A(t) = t^p \).

**Proposition 2.1.** If \( A \) satisfies (i) and (ii) and \( f \) is an \( L^1_{\text{loc}}(\Omega) \) function, \( f \geq 0 \), such that, for any cube \( Q \subset \Omega \), for which \( 2Q \subset \subset \Omega \),

\[
\left(2.1\right) \quad \int_Q A(f) \leq b_1 A\left(\int_{2Q} f\right) + b_2,
\]

then there exist \( c_1, c_2 > 0, r > 1 \), depending only on \( b_1, b_2, n, k, p \), such that, for any \( 2Q \subset \subset \Omega \),

\[
\left(2.2\right) \quad \int_Q A'(f) \leq c_1 A'\left(\int_{2Q} f\right) + c_2.
\]

Proof: Let us fix \( 2Q_o \subset \subset \Omega \) and denote by \( Mf(x) \) the local maximal function of \( f \) in \( Q_o \). Let us fix a cube \( Q' \subset Q_o \) such that \( 4Q' \subset Q_o \) and a point \( z \in Q' \). If \( \tilde{Q} \subset Q_o \) is any cube containing \( z \) we have: if \( Q \subset 2Q' \), then

\[
\left(2.3\right) \quad \int_Q f(y) \, dy = \int_Q f(y) x_{2Q}(y) \, dy \leq M(f x_{2Q})(z).
\]

If \( Q \not\subset 2Q' \), then, using Lemma 1.5, there exists a cube \( \tilde{Q} \subset Q_o \) such that \( Q \cup Q' \subset Q', l(\tilde{Q}) \leq 3l(Q) \), and so

\[
\int_Q f(y) \, dy \leq 3^n \int_{Q'} f(y) \, dy \leq 3^n \inf_{Q'} Mf.
\]

From this inequality and (2.3), taking the supremum over all cubes \( Q \subset Q_o \), \( z \in Q \), and then using assumption (i) we have

\[
A(Mf)(z) \leq c(n, k) \left[ A(M(f x_{2Q}))(z) + \inf_{Q'} A(Mf) \right].
\]
Thus, integrating over $Q'$ and using (2.1) and Proposition 1.2, we obtain

$$\int_{Q'} A(Mf)(z) \, dz \leq c \left[ \int_{Q'} A(M(fx_{2Q'})) + \inf_{Q'} A(Mf) \right]$$

$$\leq c \left[ \frac{1}{|Q'|} \int_{Q_3} A(fx_{2Q'}) + \inf_{Q'} A(Mf) \right]$$

$$\leq c \left[ \int_{2Q'} A(f) + \inf_{Q'} A(Mf) \right]$$

$$\leq c \left[ b_1 A \left( \int_{Q'} f \right) + b_2 + \inf_{Q'} A(Mf) \right]$$

$$\leq c \left[ \inf_{Q'} A(Mf) + 1 \right],$$

which implies that, for any cube $Q' \subset \frac{1}{2}Q_0$,

$$\int_{Q'} (A(Mf) + 1) \, dz \leq c \inf_{Q'} (A(Mf) + 1).$$

Applying now Theorem 1.4, we deduce that there exist $r > 1$, $c > 0$ such that, (using again Proposition 1.2):

$$\int_{Q_r/4} A'(f) \, dx \leq \int_{Q_r/4} (A(Mf) + 1) \, dx \leq c \left( \int_{Q_r/4} A(Mf) + 1 \right)$$

$$\leq c \left( \int_{Q_r} A(Mf) + 1 \right) \leq c \left( \int_{Q_r} A(f) + 1 \right)$$

$$\leq c \left[ b_1 A \left( \int_{2Q_r} f \right) + b_2 + 1 \right].$$

So we have shown that, for any cube $2Q \subset \subset \Omega$,

$$\int_{Q_r/4} A'(f) \, dx \leq c \left[ A'(\int_{2Q} f) + 1 \right],$$

which implies (2.2) by a standard argument.
Remark 2.2. We point out that in Proposition 2.1, in inequalities (2.1) and (2.2), cubes may easily be replaced by balls. Moreover, it is clear from the proof that if \( b_2 = 0 \), then also \( c_2 = 0 \).

3. The Main Result

In this section we shall suppose that the function \( A \) introduced in Section 1 satisfies:

(i) \[ A(2t) \leq kA(t) \quad \text{for any} \quad t > 0, \]

(ii) there exist \( 1 < p \leq q \) such that \( A(t)/t^p \) is increasing, \( A(t)/t^q \) is decreasing and \( p > q_* = nq/(n + q) \).

Remark 3.1. Assumption (ii) implies that \( A(t) \) satisfies the growth condition

(iii) \[ c_1 t^p - c_2 \leq A(t) \leq c_3(t^q + 1), \]

but it may happen that the exponents \( p, q \) appearing in (ii) are not necessarily the best ones in order for (iii) to hold. For example, the convex function

\[ A(t) = \begin{cases} et^3 & \text{if} \ 0 \leq t \leq \varepsilon, \\ t^{4 + \sin \log \log t} & \text{if} \ t \geq \varepsilon, \end{cases} \]

satisfies (ii) with \( p = 4 - \sqrt{2} \) and \( q = 4 + \sqrt{2} \) and (iii) with \( p = 3 \) and \( q = 5 \). Moreover, if \( 4 - \sqrt{2} < r < 4 + \sqrt{2} \), \( A(t)/t^r \) is neither strictly increasing nor decreasing.

In the following we shall say that a function \( u \in W^{1,1}_\text{loc}(\Omega; \mathbb{R}^N) \) is a minimizer for the functional

\[ I(\Omega; v) = \int_{\Omega} A(|Dv|) \, dx \quad (3.1) \]

if and only if, for every \( \psi \in W^{1,1}(\Omega; \mathbb{R}) \) with compact support,

\[ I(\text{supp} \, \psi; u) \leq I(\text{supp} \, \psi; u + \psi). \]

The first step in deriving the higher integrability result is the following version of the well-known Caccioppoli inequality (see [7], [13]).

Theorem 3.2. If \( A \) satisfies (i), and there exists \( q > 1 \) such that \( A(t)/t^q \) is decreasing, and \( u \in W^{1,1}(\Omega; \mathbb{R}^N) \) is a minimizer of the functional (3.1), then, for

\[ 1 < q < n/(n - 1), \]

we can replace \( q \) by \( n/(n - 1) \). With such a choice we still have \( A(t)/t^q \) decreasing and \( q_* = 1 \).
any ball $B_R \subset \subset \Omega$,

\[(3.2) \quad \int_{B_{R/2}} A(|Du|) \, dx \leq c(k, q) \int_{B_{R/2}} A\left(\frac{|u - u_R|}{R}\right) \, dx,
\]

where $u_R = \int_{B_R} u$, and $c$ depends only on $A$.

We shall also need the following

**Lemma 3.3.** If $A$ satisfies (ii), then, setting, for any $t > 0$,

\[
K(t) = \int_0^t \left[ A\left(s^{1/q}\right)/s\right]^{(n+q)/q} \, ds, \quad H(t) = \frac{\left(A(t^{1/q})\right)^{(n+q)/q}}{t^{n/q}} \frac{1}{t^{(n+q)/q}},
\]

we see that $K(t)$ is a concave function such that there exists a $c$ for which

\[(3.3) \quad H(t) \leq K(t) \leq cH(t) \quad \text{for all} \quad t > 0.
\]

The concavity of $K(t)$ follows observing that $K'(t) = [A(t^{1/q})/t]^{(n+q)/q}$ which is decreasing by (ii). Note also that

\[
H'(t) = \frac{n + q}{q^2} \frac{A^{n/q}(t^{1/q})}{t^{1+n/q}} \left[A'(t^{1/q})t^{1/q} - \frac{nq}{n+q}A(t^{1/q})\right].
\]

Thus, by (ii), one easily gets

\[
H'(t) \leq K'(t) \leq \frac{q^2}{(n+q)(p-q_s)} H'(t),
\]

which implies (3.3), since $H(0) = K(0) = 0$.

We can now prove the following higher integrability result.

**Theorem 3.4.** If $A$ satisfies (i), (ii) and $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ is a minimizer for the functional (3.1), then there exist $r > 1, c > 0$ such that, for any $B_R \subset \subset \Omega$,

\[(3.4) \quad \frac{1}{r} \int_{B_{R/2}} A^r(|Du|) \, dx \leq c \left(\int_{B_R} A(|Du|) \, dx\right)^r.
\]

**Proof:** From Theorem 3.2, we deduce that

\[
\int_{B_{R/2}} A(|Du|) \, dx \leq c \int_{B_{R/2}} \frac{A(|u - u_R|/R)}{\left|(u - u_R)/R\right|^{nq/(n+q)}} \left|\frac{u - u_R}{R}\right|^{mq/(n+q)} \, dx
\]
and thus, by Hölder’s inequality

\[
\int_{B_{R/2}} A(|Du|) \, dx \leq c \left[ \int_{B_R} A^{(n+q)/q} \left( \frac{|u-u_R|}{R} \right)^{q/(n+q)} \, dx \right]^q \left[ \int_{B_R} \frac{\left( |(u-u_R)/R\right)^q \, dx}{\left( |u-u_R|/R\right)^n} \right]^{n/(n+q)}.
\]

If \( H(t), K(t) \) are defined as in Lemma 3.3, using that lemma and noting that \( K \) is concave, we have

\[
\int_{B_{R/2}} A(|Du|) \, dx \leq c \left[ \int_{B_R} K \left( \frac{|u-u_R|}{R} \right)^q \, dx \right]^{q/(n+q)} \int_{B_R} |Du|^{q*} \, dx
\]

\[
\leq c K^{q/(n+q)} \left( \int_{B_R} \left( |u-u_R|/R \right)^q \, dx \right) \int_{B_R} |Du|^{q*} \, dx
\]

\[
\leq c H^{q/(n+q)} \left( \int_{B_R} |Du|^{q*} \, dx \right)^{q/q*} \int_{B_R} |Du|^{q*} \, dx
\]

\[
= c A \left( \left[ \int_{B_R} |Du|^{q*} \, dx \right]^{1/q*} \right) \int_{B_R} |Du|^{q*} \, dx
\]

\[
= c A \left( \left[ \int_{B_R} |Du|^{q*} \, dx \right]^{1/q*} \right).
\]

If we set \( B(t) = A(t^{1/q*}) \), we have

\[
B(2t) \leq kB(t),
\]

\[
B'(t) \geq \frac{p}{q*} \frac{B(t)}{t}, \quad p/q* > 1,
\]

and so with \( f(x) = |Du|^{q*} \), we deduce by (3.5) that

\[
\int_{B_{R/2}} B(f) \, dx \leq cB \left( \int_{B_R} f \, dx \right),
\]

from which it follows that \( B \) and \( f \) satisfy the assumptions of Proposition 2.1.
Thus there exists $r > 1$ such that
\[
\int_{B_{r/2}} B'(f) \, dx \leq B'\left(\int_{B_r} f \, dx\right),
\]
i.e., for any $B_R \subset \Omega$,
\[
\int_{B_{r/2}} A'(|Du|) \, dx \leq c A'\left(\left[\int_{B_R} |Du|^{q_*} \, dx\right]^{1/q_*}\right).
\]
(3.6)

Setting
\[
\tilde{A}(t) = \int_0^t \frac{A(s)}{s} \, ds,
\]
$t > 0$,

it is easy to check that
\[
\frac{1}{q} A(t) \leq \tilde{A}(t) \leq A(t),
\]
(3.7)

that $\tilde{A}(t)$ is a convex function and also that $\tilde{A}(t^{1/p})$ is convex, since its derivative is $(1/p)A(t^{1/p})/t$ which is increasing. From a result of [9], page 169, it follows that
\[
\left[\int_{B_R} |Du|^p \, dx\right]^{1/p} \leq \tilde{A}^{-1}\left(\int_{B_R} \tilde{A}(|Du|) \, dx\right).
\]
(3.8)

Taking into account that $p > q_*$, we see from (3.7), (3.8) that
\[
\frac{1}{q} A\left(\left[\int_{B_R} |Du|^{q_*} \, dx\right]^{1/q_*}\right) \leq \frac{1}{q} A\left(\left[\int_{B_R} |Du|^p \, dx\right]^{1/p}\right) \leq c \int_{B_R} A(|Du|) \, dx.
\]
(3.9)

From (3.6), (3.9) we deduce (3.4).

Remark 3.5. Let $F(x, s, z): \Omega \times \mathbb{R}^N \times \mathbb{R}^{n \times N} \to \mathbb{R}$ be a function satisfying:

\[
A(|z|) \leq F(x, s, z) \leq c [A(|z|) + 1]
\]

with $A$ as in Theorem 3.4. Then if $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ is a minimizer of $\int F(x, u(x), Du(x)) \, dx$, using the general form of Proposition 2.1 (with $b_2 > 0$)
one easily obtains that, for any $B_R \subset \subset \Omega$,

$$
\int_{B_{R/2}} A'(|Du|) \, dx \leq C \left[ A' \left( \left( \int_{B_R} |Du|^{q/(n+q)} \, dx \right)^{(n+q)/nq} \right) + 1 \right].
$$

Remark 3.6. If $p \geq n$, then $q$ in Theorem 3.4 can be any number greater than $p$ and (3.4) implies, by the Sobolev-Poincaré imbedding theorem, that the minimizer $u$ is locally Hölder continuous.

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