Lezione 4

Geometry and stability of nonlinear dynamical systems

References

Very instructive with simple approach

Strogatz, S. H., Nonlinear dynamics and chaos: with application to physics, biology, chemistry and engineering, Addison Wesley, New York 1994.

A complete overview of the problems

Wiggins S., Introduction to applied nonlinear dynamical systems and chaos, Springer Verlag, New York 1990 (2nd Ed. 2003)

Kuznetsov Y. A., Elements of applied bifurcation theory, Springer Verlag, New York 2004 (3rd Rev. Ed.)

Very detailed on some aspects

Carr J., Applications of centre manifold theory, Springer Verlag, New York, 1981

Previous lectures - Linear autonomous systems

- Dynamics completely determined by the eigenvalues
- For linear systems each eigenspace, generalized eigenspace and their composition is a subspace U of the phase space invariant under the evolution operator:

$$\frac{dx}{dt} = Ax,\ x \in \mathbb{R}^n \ \Rightarrow\ x(t;x_0) = \phi_t x_0 = e^{At}x_0$$

$$x_{k+1} = Ax_k,\ x \in \mathbb{R}^n \ \Rightarrow\ x_k = \phi_k x_0 = A^k x_0$$

$$\Downarrow$$

$$x \in U \ \Rightarrow\ e^{At}x,\ A^k x \in U$$

Previous lectures - Linear autonomous systems

- Through a Jordan transformation $x=Ty$ the state vector can be partitioned in three components:
  - Stable: y^s
  - Unstable: y^u
  - Central: y^c
- Their evolution is uncoupled from the others

$$\dot{y}^s = A^S y^s, \quad y^s \in \mathbb{R}^S, \quad s = \sum_{\lambda_i \in \mathbb{C}^-} m_a(\lambda_i)$$

$$\dot{y}^u = A^U y^u, \quad y^u \in \mathbb{R}^U, \quad u = \sum_{\lambda_i \in \mathbb{C}^+} m_a(\lambda_i)$$

$$\dot{y}^c = A^C y^c, \quad y^c \in \mathbb{R}^C, \quad c = \sum_{\lambda_i \in \mathbb{C}^0} m_a(\lambda_i)$$

$$\sigma(A^S) \in \mathbb{C}^-, \ \sigma(A^U) \in \mathbb{C}^+, \ \sigma(A^C) \in \mathbb{C}^0$$

$$A^S \in M_s(\mathbb{R}), A^U \in M_u(\mathbb{R}), A^C \in M_c(\mathbb{R})$$

$$s + u + c = n$$

Previous lectures - Linear autonomous systems

- Stable (Unstable) Eigenspace E^S (E^U):
  - Invariant
  - The orbits tend to (move from) the origin as time increases
- Central Eigenspace E^C:
  - Invariant
  - The dynamics may be simply stable or unstable
    - Continuous systems: depends on the algebraic and geometrical multiplicity of the eigenvalues with zero real part
    - Discrete time systems: depends on the algebraic and geometrical multiplicity of the eigenvalues with unit magnitude.

Previous lectures - Linear autonomous systems

- In the case of linear systems the stability is a **global** property, e.g., **the system is stable**.
- Hyperbolic systems:
  - Sinks (attractors)
  - Sources (repellors)
  - Saddles

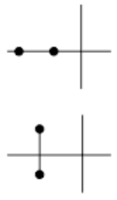
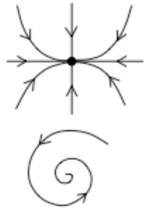
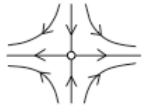
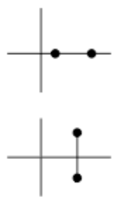
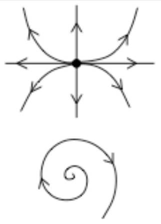
(n_+, n_-)	Eigenvalues	Phase portrait	Stability
$(0,2)$		 node focus	stable
$(1,1)$		 saddle	unstable
$(2,0)$		 node focus	unstable

FIGURE 2.5. Topological classification of hyperbolic equilibria on the plane.
From Kuznetsov Elements of Applied Bifurcation Theory, Springer 1998

- **WHAT HAPPENS WHEN THE DYNAMICAL SYSTEM IS NONLINEAR?**

Motivation of this lecture

- Stability in nonlinear dynamical systems
  - Definition of the (**local**) orbital stability
- Relationships between the **local** dynamics of a nonlinear system and the dynamics of associated linearized systems
- Invariant manifolds for nonlinear systems
  - Stable, unstable and central manifolds of bounded orbits (equilibrium points and limit cycles)
- Definitions of the **local** dynamical characteristics (**geometry of the orbits in the phase space**) of nonlinear systems through the dynamics on the invariant manifolds of bounded orbits
- Definitions of attractors, repulsors and saddles for nonlinear systems

Outline

- Introduction
- Stability
  - Hyperbolic points
  - Nonhyperbolic points
  - Limit Cycles
- Convention

Mathematical statements

Important statements

General frame

- In this lecture we will only consider systems of the types:

Continuous

$$\frac{dx}{dt} = f(x), \; x \in \mathbb{R}^n, \; f \in C^r(\mathbb{R}^n, \mathbb{R}^n) \; with \; r \geqslant 1$$

Discrete

$$x_{k+1} = F(x_k), \; x \in \mathbb{R}^n, \; F \in C^r(\mathbb{R}^n, \mathbb{R}^n) \; with \; r \geqslant 1$$

INTRODUCTION

Orbits of continuous systems

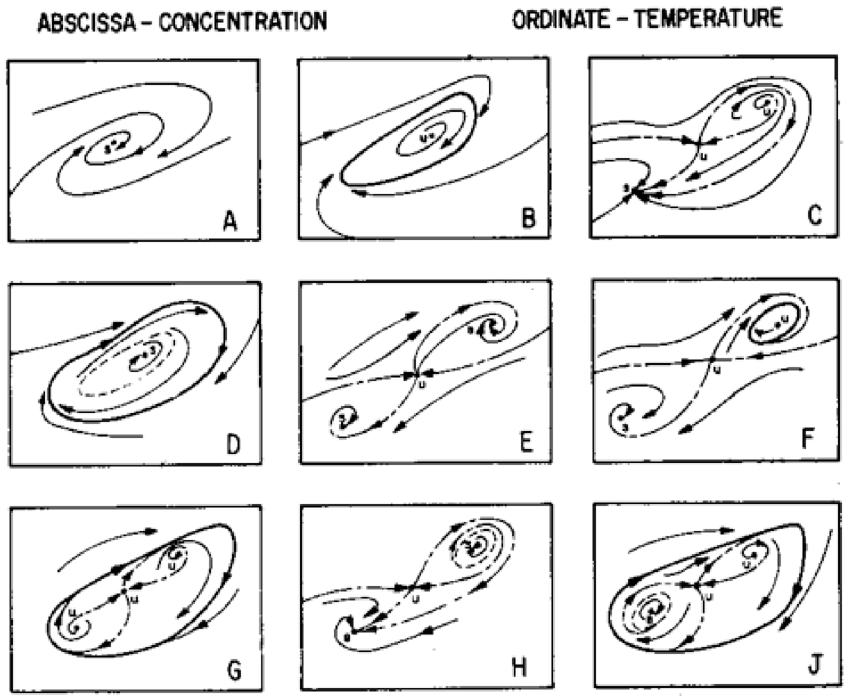
- In the case of continuous systems bounded orbits may represents the asymptotic behavior of an orbit for $t \to +\infty$ (regime) or $t \to -\infty$.

Bounded orbits can be:

Stationary (points in the phase space, for any n)
Periodic (closed curve in the phase space, only for n>1)
Quasiperiodic or k-periodic (k>1, k-tori in the phase space, n>k)
Aperiodic (chaotic, often fractal objects in the phase space, only for n>2)

Bounded orbits - Examples for Continuous Systems

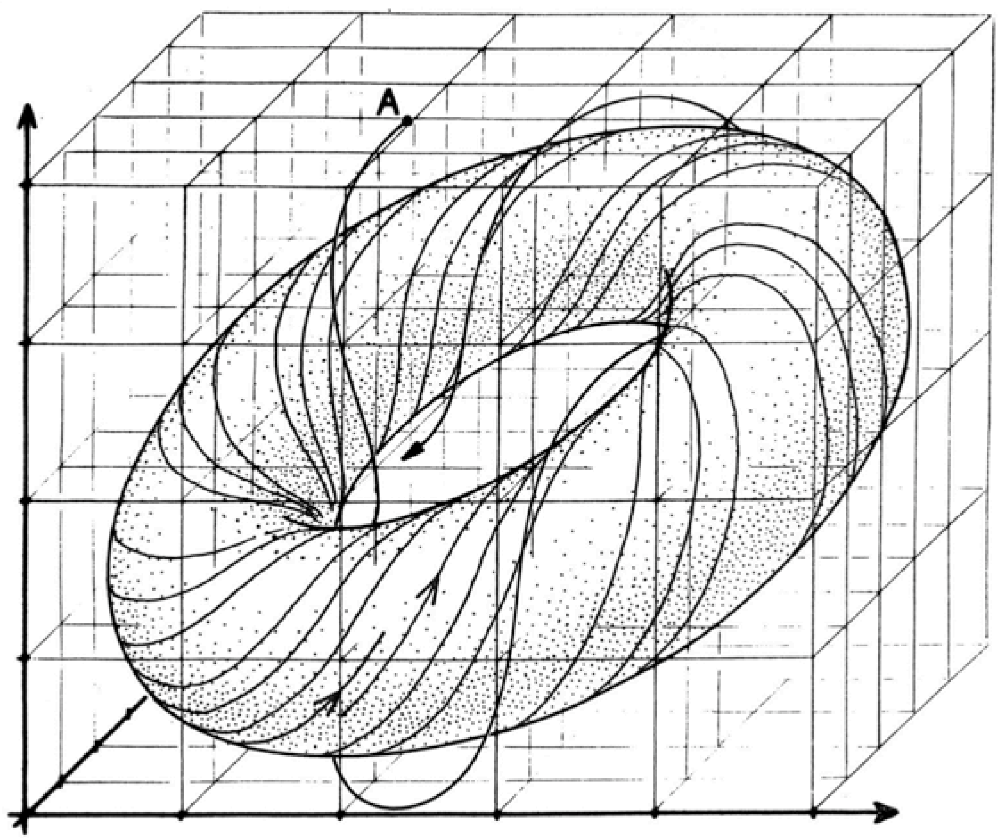
- Stationary and periodic orbits in n=2



- From a model for a Continuous Stirred Tank Reactor

Bounded orbits - Examples for Continuous Systems

- Quasiperiodic orbit in n=3



Orbits of discrete systems

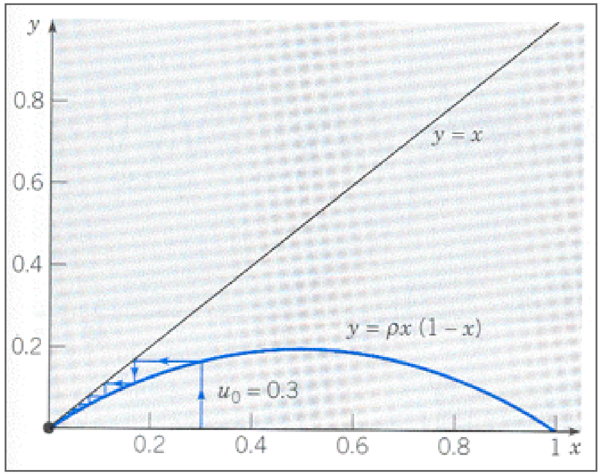
- Bounded (discrete!) orbits can represent the asymptotic behavior of orbits for $k \to \infty$ (regime) or $k \to -\infty$ also in discrete systems.

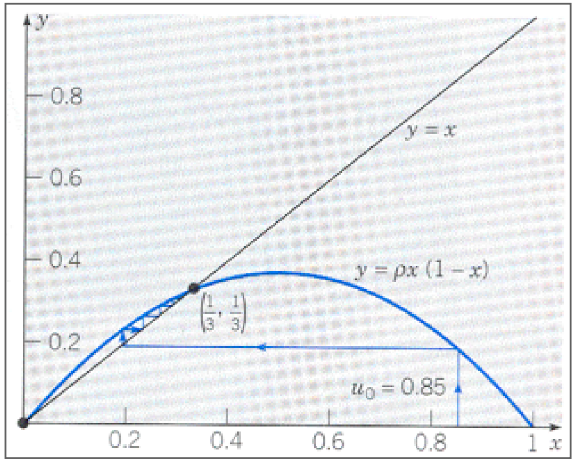
Bounded orbits can be:

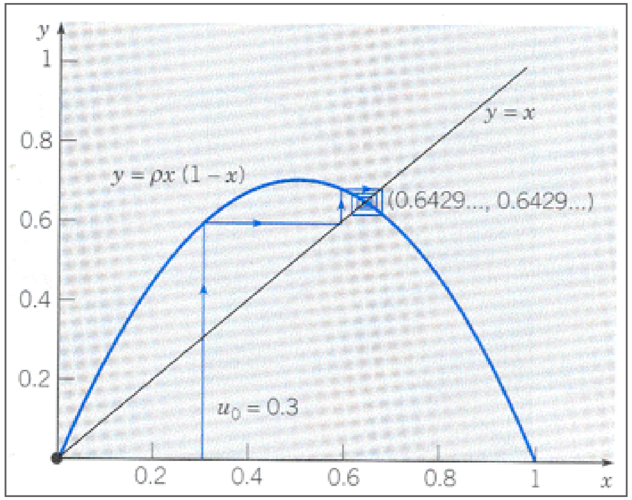
Stationary (points in the phase space, for any n)
m-Periodic (m points in the phase space, for any n)
Quasiperiodic (k-tori in the phase space, n>k)
Aperiodic (chaotic, often fractal objects in the phase space, only for any n)

Bounded orbits - Examples for discrete systems

- Stationary orbits (fixed points) n=1

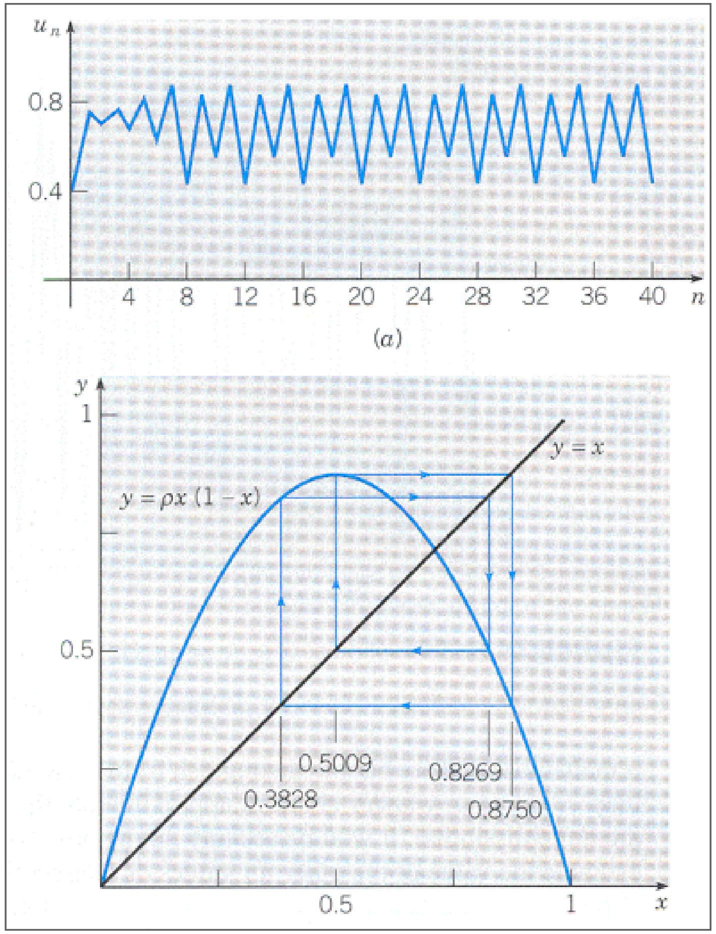






Bounded orbits - Examples for discrete systems

- m-periodic orbits (fixed points) n=1



Nonlinear Dynamical Systems - Examples

- Earliest important examples: The Newton Equations to derive and unify the three laws of Kepler, i.e., the basis of Classical Mechanics

   1. The orbit of every planet is an ellipse with the sun at a focus.
   2. A line joining a planet and the sun sweeps out equal areas during equal intervals of time.
   3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

- The Newton equations are conservation laws and are **nonlinear** in general
- A particle moving in a Force Field: the gravitational field of the sun
   - Similar examples: charged particles in electromagnetic fields
- Newton's second law $\mathbf{F} = m\mathbf{a}$
   - The force vector at the location of the particle at any instant equals the acceleration vector of the particle times the mass

Nonlinear Dynamical Systems - Examples

- Newton's second law $\mathbf{F} = m\mathbf{a}$

- $\boldsymbol{x}(t)$ the position of the particle at time t

  - Mathematically: $x : \mathbb{R} \to \mathbb{R}^3$ is a sufficiently differentiable curve

- The acceleration vector is **the second derivative** of $\boldsymbol{x}(t)$ with respect to time

$$\mathbf{F}(\boldsymbol{x}(t)) = m\, \ddot{\boldsymbol{x}}(t)$$

- Second order differential equation

$$\boxed{\ddot{\boldsymbol{x}} = \frac{1}{m}\mathbf{F}(\boldsymbol{x})}$$

Nonlinear Dynamical Systems - Examples

- Which is the **state** of the dynamical system?
  - From previous lectures:
    - A state of a dynamical system is information characterizing it at a given time

- Recast the problem as a set of first order differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{v}$$

$$\dot{\boldsymbol{v}} = \frac{1}{m}\mathbf{F}(\boldsymbol{x})$$

- The state variables are the position and the velocity
- A solution $(\boldsymbol{x}(t),\ \boldsymbol{v}(t))$ gives the passage of the state of the system in time

Nonlinear Dynamical Systems - Examples

- Depending on the nature of the force field one ends up with different dynamical systems describing different physical systems

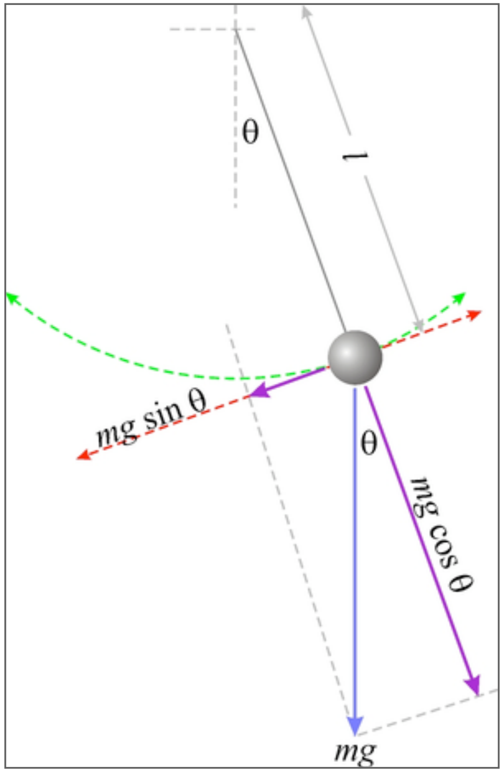
- **Frictionless Pendulum**

$$F = -mg\ sin\theta \qquad a = \ell \frac{d^2\theta}{dt^2}$$

$$\ell \frac{d^2\theta}{dt^2} = -g sin\theta$$

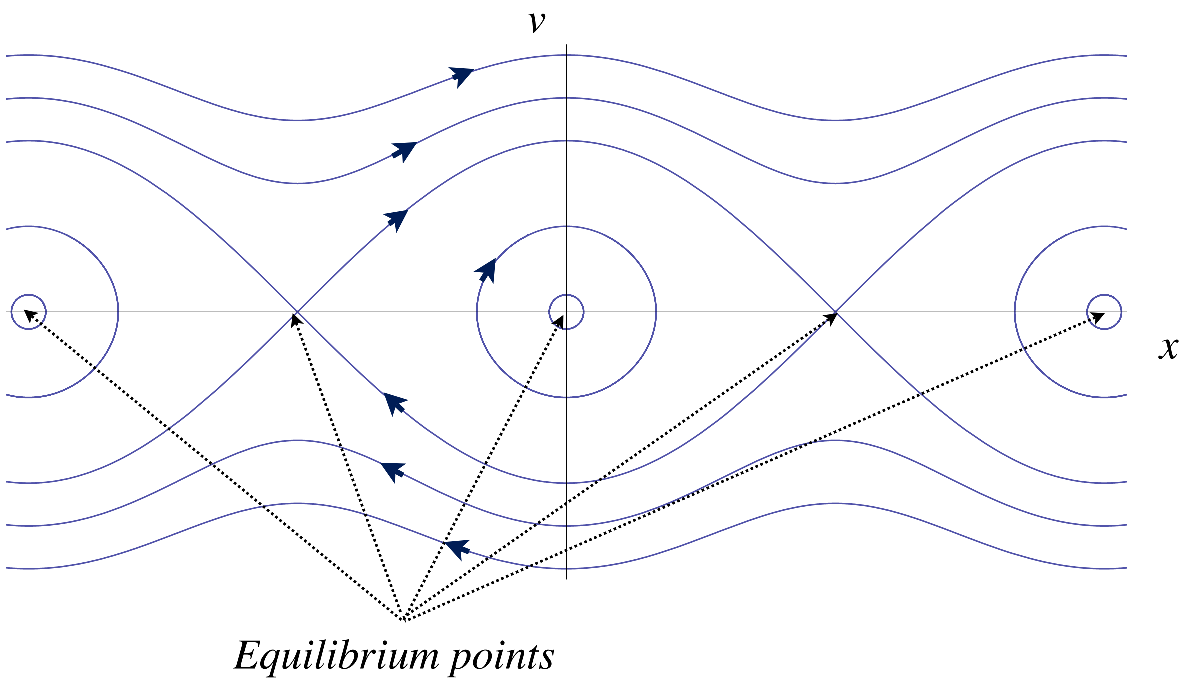
- **State variables:** $x = \ell\theta \qquad v = \ell \frac{d\theta}{dt}$

- **Nonlinear Dynamical system** $\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -g\ sinx \end{aligned}}$



Nonlinear Dynamical Systems - Examples

- **Pendulum - Phase Portrait**

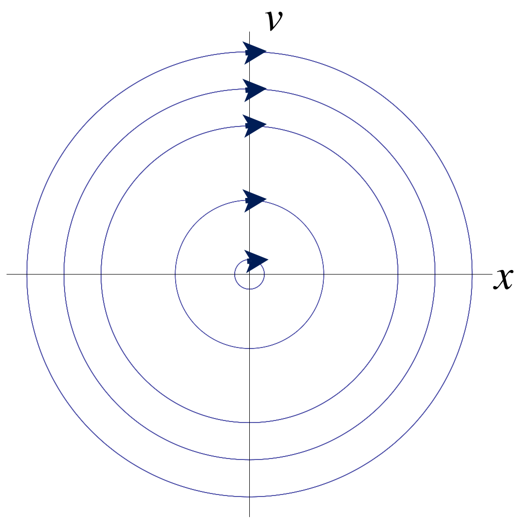


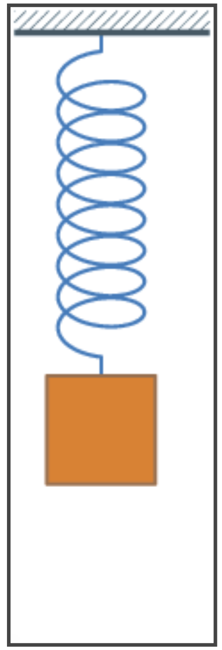
Stability?
Multiplicity

Nonlinear Dynamical Systems - Examples

- Close to the (0,0) equilibrium point the system can be linearized $sin x = x + O(x^2)$

- **Harmonic oscillator** $\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -gx\end{aligned}}$
  - A **linear and autonomous** dynamical system





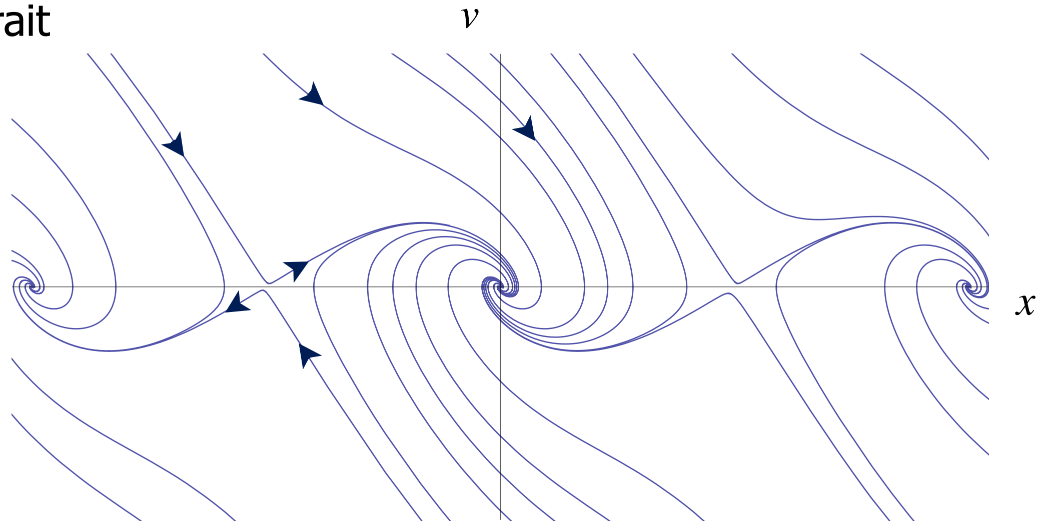
- What happens linearizing close to $(\pi,0)$?

Nonlinear Dynamical Systems - Examples

- **A damped harmonic oscillator**
  - In real oscillators friction, or damping, slows the motion of the system.

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -g\ sinx - \zeta v \end{aligned}}$$

  - Another **nonlinear and autonomous** dynamical system
- Phase portrait



Problems at hand

- **Multiplicity** of regime solutions
- We have to decide on the **stability** of the regime solutions
  - Which might be **stationary** or **dynamic**
- We think that we might use **linearization** and information derived from linear systems to take position on stability
  - Is this always possible?
- What happens if we vary the parameter values?

STABILITY

Orbital Stability - Poincaré

- The stability of an orbit of a dynamical system characterizes whether nearby orbits will remain in **a neighborhood of that orbit** or be repelled away from it.
- **Asymptotic stability** additionally characterizes attraction of nearby orbits to this orbit in the long-time limit.
- Let's consider a generic **bounded** orbit of a continuous system passing through x_0

$$\boxed{\gamma(x_0) = \{x \in \mathbb{R}^n | \exists t \in \mathbb{R} : x = x(t, x_0) = \Phi_t x_0\}}$$

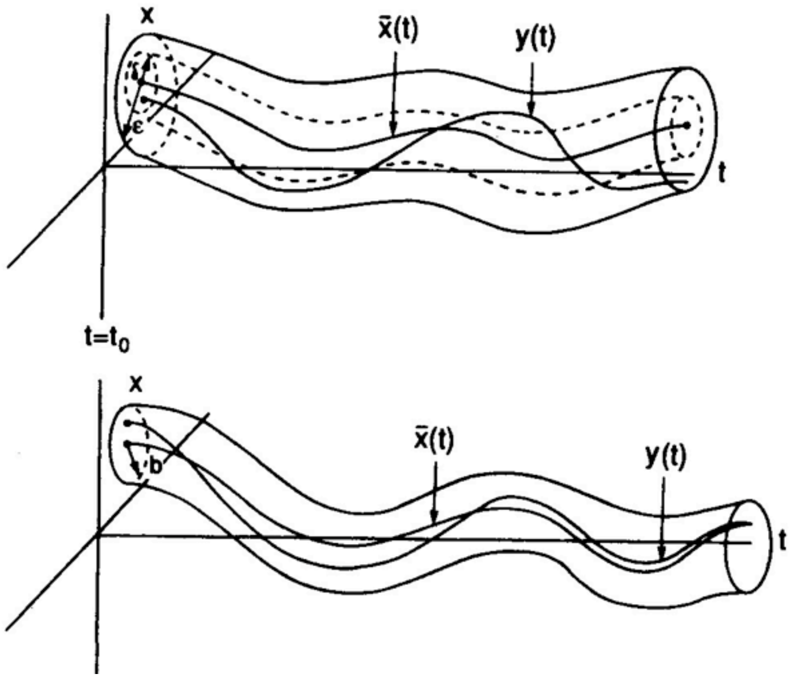
- An orbit $\gamma(x_0)$ is called **stable** if for any given neighborhood $U(\gamma(x_0))$ there exists another neighborhood $V(\gamma(x_0)) \subseteq U(\gamma(x_0))$ such that any solution starting in $V(\gamma(x_0))$ remains in $U(\gamma(x_0))$ for all $t \geq 0$.
- Similarly, an orbit $\gamma(x_0)$ is called **asymptotically stable** if it is stable and if there is a neighborhood $U(\gamma(x_0))$ such that

$$\lim_{t\to\infty} d(\phi(t,x), \gamma(x_0)) = 0 \ for \ all \ x \in U(x_0)$$

with $d(x, \gamma(x_0)) = sup_{y\in\gamma(x_0)} |x - y|$

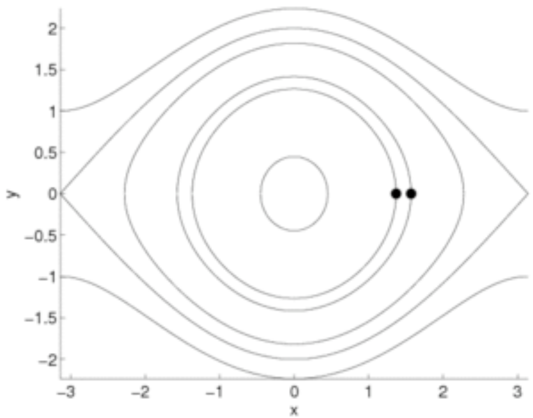
Orbital Stability - Poincaré

- An orbit is **stable** if a orbit sufficiently close to it remains close to it, and is **asymptotically stable** if they asymptotically tend to it.



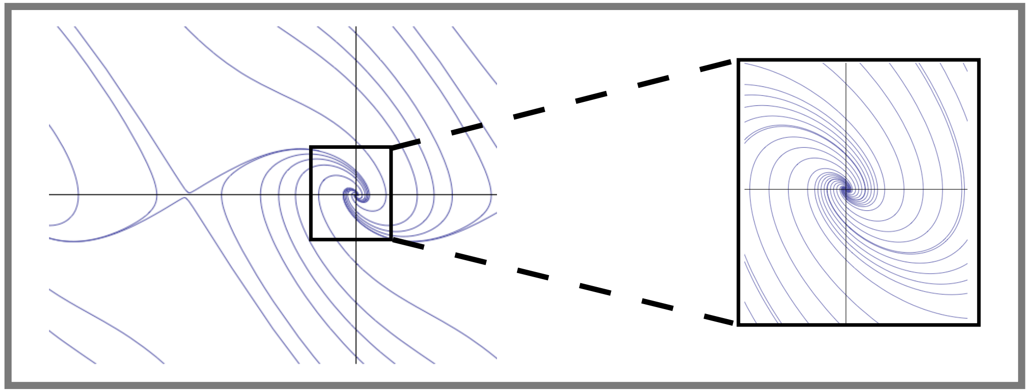
Orbital Stability - Poincaré

- Note that this definition ignores the time parametrization of the orbit. In particular, if x is close to $x_1 \in \gamma(x_0)$, **we do not require that $\phi(t, x)$ stays close to $\phi(t, x_1)$ (we only require that it stays close to $\gamma(x_0)$)**.
- To see that this definition is the right one, consider the pendulum.
- There all orbits are periodic, but the period is not the same. Hence, if we fix a point x_0, any point $x = x_0$ starting close will have a slightly larger respectively smaller period and thus $\phi(t, x)$ does not stay close to $\phi(t, x_0)$.
- Nevertheless, it will still stay close to the orbit of x.



Linearization

- The definition of stability of an orbit is **LOCAL**
- This suggests the idea of analyzing a linearization around it to obtain information from the characteristic local dynamics (local phase portrait)



- **In most cases the local geometry (and the stability properties) of a nonlinear dynamical system can be identified through the knowledge of the associate linearized system**
- This is not always possible. Even when this is not possible the knowledge of the geometry of the associate linearized system is useful.

Linearization at an equilibrium point - Machinery

- To characterize the local dynamics and the stability of an equilibrium point of a nonlinear dynamical system

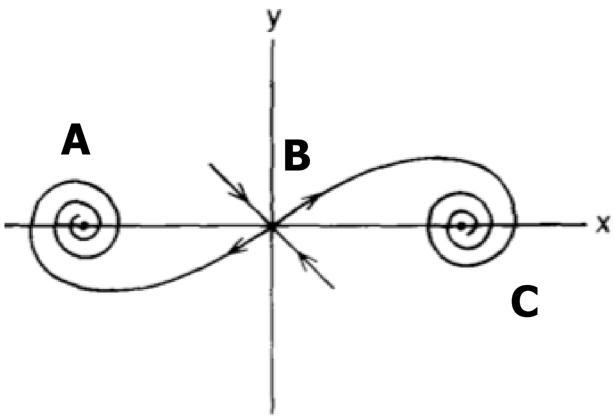
$$f(x_E) = 0$$

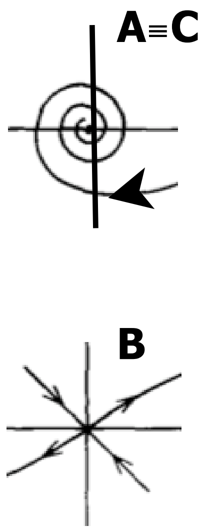
Jacobian

$$Df(x_E) = \left\| \begin{array}{ccc} \frac{\partial f_1}{\partial x} & \cdots & \frac{\partial f_1}{\partial x} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x} & \cdots & \frac{\partial f_n}{\partial x} \end{array} \right\|_{x=x_E}$$

- x_E identifies one equilibrium point
- A nearby point is $u := x - x_E, \quad with\ |u| \ll 1$
- The derivative $\dot{u} = f(u + x_E) = f(x_E) + Df(x_E) + R(u) = Au + R(u)$
  - with $\dfrac{R(u)}{|u|} \to 0\ as\ |u| \downarrow 0$
- Thus it is reasonable to believe that solutions of $\dot{u} = Au + R(u)$ behave similarly to solutions of $\dot{u} = Au$ for u near 0.
- Equivalently: $\dot{x} = A(x - x_E)$ for x close to x_E **Translation**

Linearization - Translation + Derivation





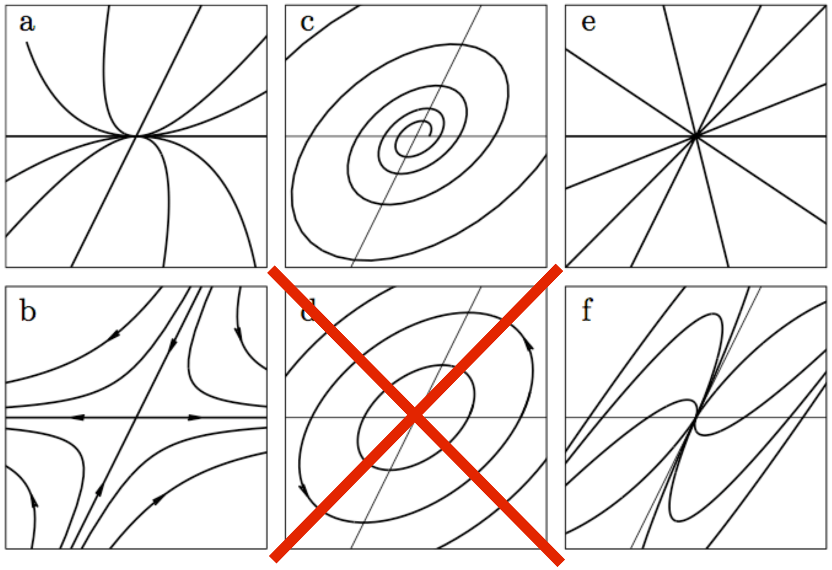
- The linearized system
  - around **A** and **C** are **stable foci** (eigenvalues?)
  - around **B** is a **saddle** (eigenvalues?)

Linearized stability

- An equilibrium point x_E is **hyperbolic** if none of the eigenvalues of $Df(x_E)$ have zero real part.
- If x_E is hyperbolic, then either all the eigenvalues of $Df(x_E)$ have negative real part or at least one has positive real part.
  - In the former case, we know that 0 is an asymptotically stable equilibrium solution;
  - In the latter case, we know that 0 is an unstable solution.
- **Theorem of Hartman-Grobman**
  - We can extend this conclusion to the case of nonlinear dynamical system
  - **Asymptotic Stability**: If x_E is an equilibrium point and all the eigenvalues of $Df(x_E)$ have negative real part, then x_E is asymptotically stable.
  - **Instability**: If x_E is an equilibrium point and some eigenvalue of $Df(x_E)$ has positive real part, then x_E unstable.

Linearized stability

- Consequence of the Hartman-Grobman Theorem:
  - if the associate linearized system is hyperbolic the local dynamics (geometry in the phase space) of the nonlinear system has the same features of that of the linearized system
  - In such conditions the stability of the equilibrium point can be determined by studying the stability of the associate linearized system
    - Sinks, sources, saddles



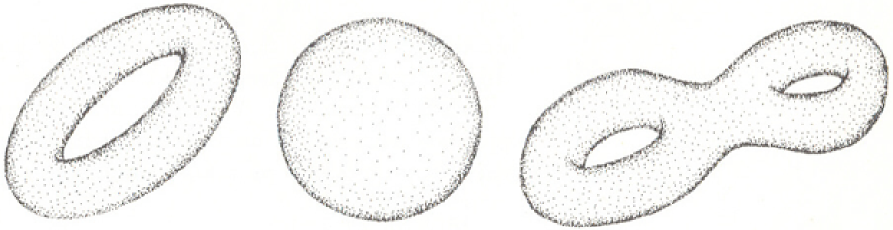
Nonlinear geometry of the phase space

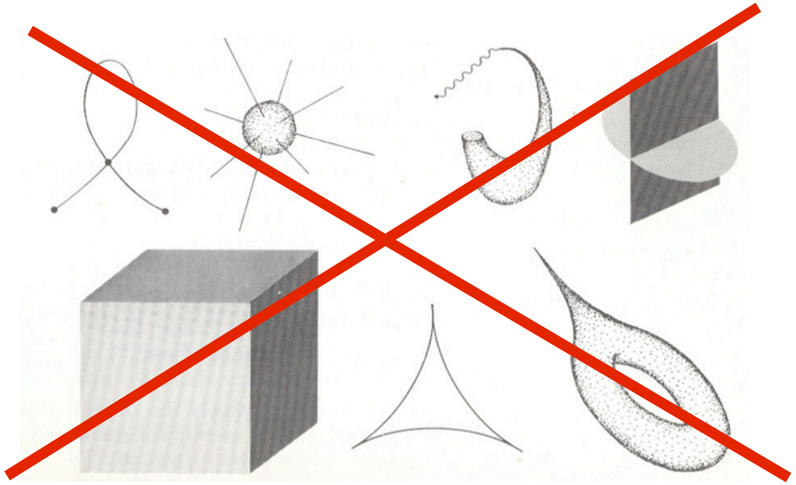
- Any equilibrium point (more in general any bounded orbit) of a nonlinear dynamical system possesses **manifolds** corresponding to the eigenspaces of the associated linearized system.
- **Such manifolds are invariant** (as the eigenspaces)
- If the dynamics on these manifolds is known the geometries of the orbits close to the equilibrium point (more in general the bounded orbit) is determined.
- Though superposition is no longer applicable, one can call for orbit continuity.

Nonlinear geometry of the phase space

- Manifolds of dimension m and class C^s locally posses the structure of m-dimensional Euclidean space (generalization of lines, surfaces ...) and are "sufficiently smooth".

Manifolds in R^3





Nonlinear geometry of the phase space - Manifold invariance

- A manifold S is said to be invariant under the vector field $\dot{x} = f(x)$ if for any $x_0 \in S$ we have $x_0(t, 0, x_0) \in S$ for all $t \in R$
- Similarly we can invoke and negative positive invariance
- Complete orbits are invariant manifolds

HYPERBOLIC EQUILIBRIUM POINTS

Linearized stability

- Hyperbolic equilibrium points: the analogies between nonlinear and linear systems can be pushed further.
- Invariant manifolds generalize the invariant subspaces of the linear case.
- We assume here that f is of class C^r with $r \geq 2$.

DEFINITION

Let x^* be a singular point of the system $\dot{x} = f(x)$, and let U be a neighborhood of x^*. The **local stable and unstable manifolds** of x^* in U are defined as

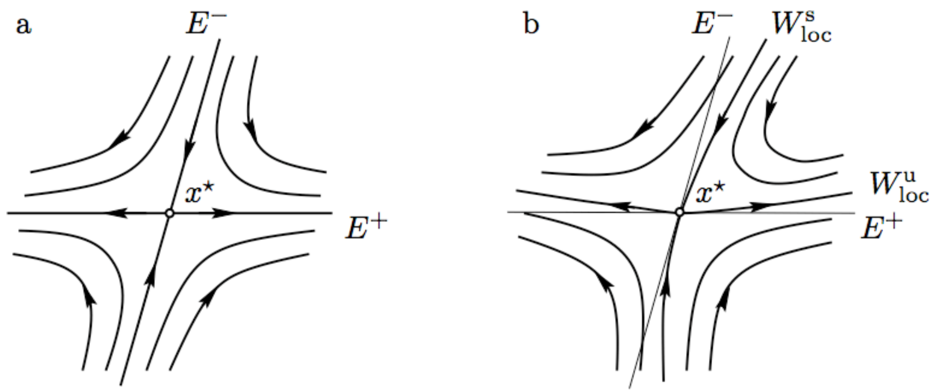
$$W^S_{loc}(x*) := \{x \in U : \lim_{t \to \infty} \phi_t(x) = x* \;\; and \;\; \phi_t(x) \in U \;\; \forall t \geq 0\}$$

$$W^U_{loc}(x*) := \{x \in U : \lim_{t \to -\infty} \phi_t(x) = x* \;\; and \;\; \phi_t(x) \in U \;\; \forall t \leq 0\}$$

Linearized stability

STABLE MANIFOLD THEOREM

- Let x^* be a hyperbolic equilibrium point of the system $\dot{x} = f(x)$ such that the matrix $\partial f/\partial x(x^*)$ has n^+ eigenvalues with positive real parts and n^- eigenvalues with negative real parts, with n^+, $n^- \geq 1$. Then x^* admits, in a neighborhood U
  - a local stable manifold $W^S_{loc}(x^*)$, which is a differentiable manifold of class C^r and dimension n^-, tangent to the stable subspace E^- at x^*, and which can be represented as a graph;
  - a local unstable manifold $W^U_{loc}(x^*)$, which is a differentiable manifold of class C^r and dimension n^+, tangent to the stable subspace E^+ at x^*, and which can be represented as a graph;



Remarks

1. Whenever we speak of manifolds (either stable, or unstable or center) **we speak of manifolds of equilibrium points** (or of other invariant sets)
2. The nonlinear manifolds are tangent to the associated linear manifolds at the equilibrium point
3. The theorem applies only when the center eigenspace of the associated linearized system is absent
    1. The nature of the solutions in the center manifold if present cannot be inferred by the nature of solutions in the center eigenspace. **MORE REFINED TECHNIQUES ARE NEEDED** (see below)

Linearized stability

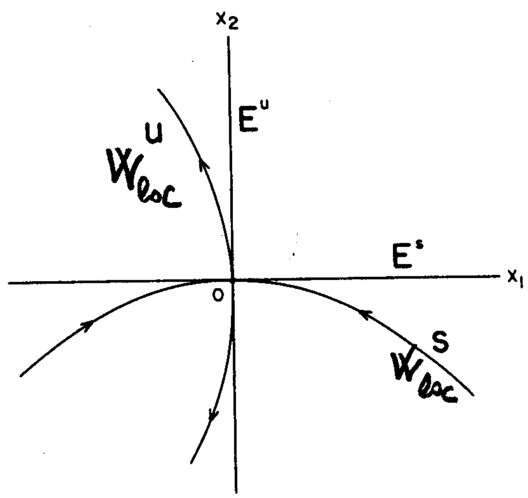
WHAT DOES THE STABLE MANIFOLD THEOREM MEAN?

- For the linear system, there is an n^-dimensional space (x^- arbitrary and $x^+=0$) in which solutions approach zero as $t\rightarrow\infty$, and an n^+dimensional space ($x^-=0$ and x^+ arbitrary) in which solutions approach zero as $t\rightarrow-\infty$.
  - An example of is a simple saddle, where there is a line along which solutions move towards the origin, and another line along which they move away from the origin.
- The stable manifold theorem says that the nonlinear system behaves in a qualitatively similar fashion; **the only difference is that the linear subspaces must be replaced by the nonlinear manifolds**.
- The line $x^+=0$ becomes the function $x^+=g(x^-)$, and the line $x^-=0$ becomes the function $x^-=h(x^+)$ with the following properties:
  - The manifolds $x^+=g(x^-)$ (UNSTABLE) and $x^-=h(x^+)$ (STABLE) are invariant, i.e. if a solution starts on one of these manifolds, then it remains there.
  - The manifolds are tangent to the spaces $x^+=0$ and $x^-=0$, respectively.
  - A solution approaches the origin as $t\rightarrow\infty$ precisely if it lies on the stable manifold and it approaches the origin as $t\rightarrow-\infty$ precisely if it lies on the unstable manifold.

Stable and Unstable Manifolds of an equilibrium point

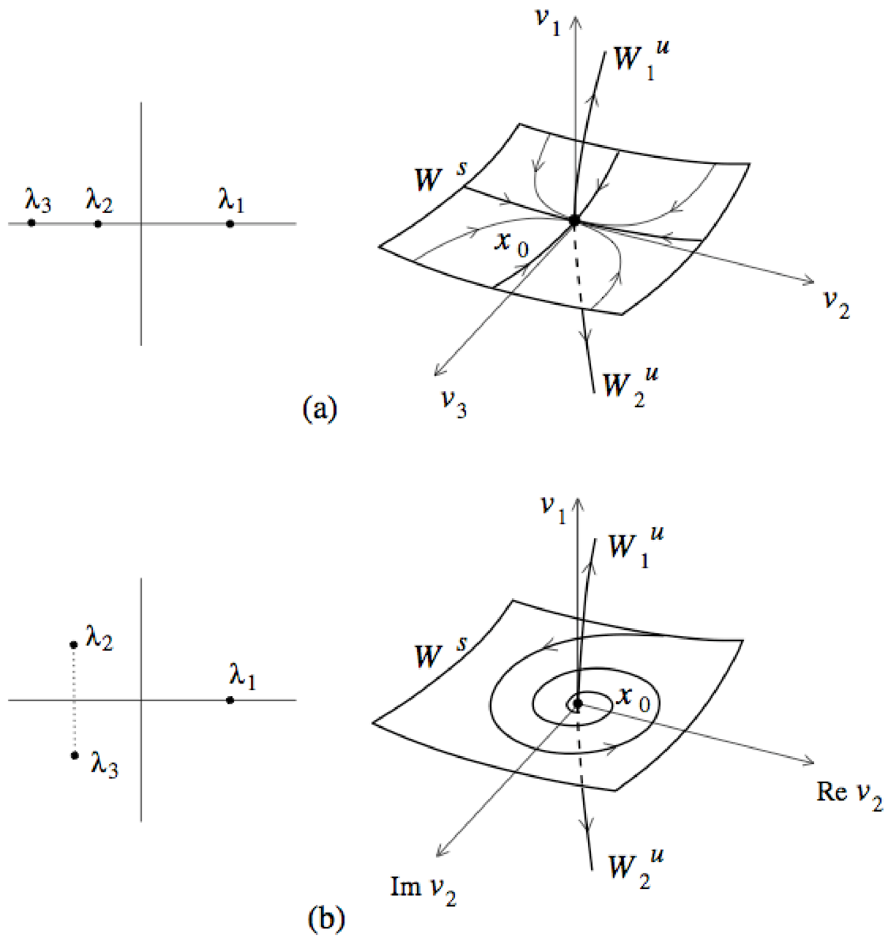
- Thus the local manifolds W^S and W^U close to an equilibrium point are graphs of functions

$$W^S_{loc}(x*) : y_u = h^s(y_s)$$
$$W^U_{loc}(x*) : y_s = h^u(y_u)$$



- Global manifolds can be reconstructed via simulation by starting close to the equilibrium point on the local manifolds...

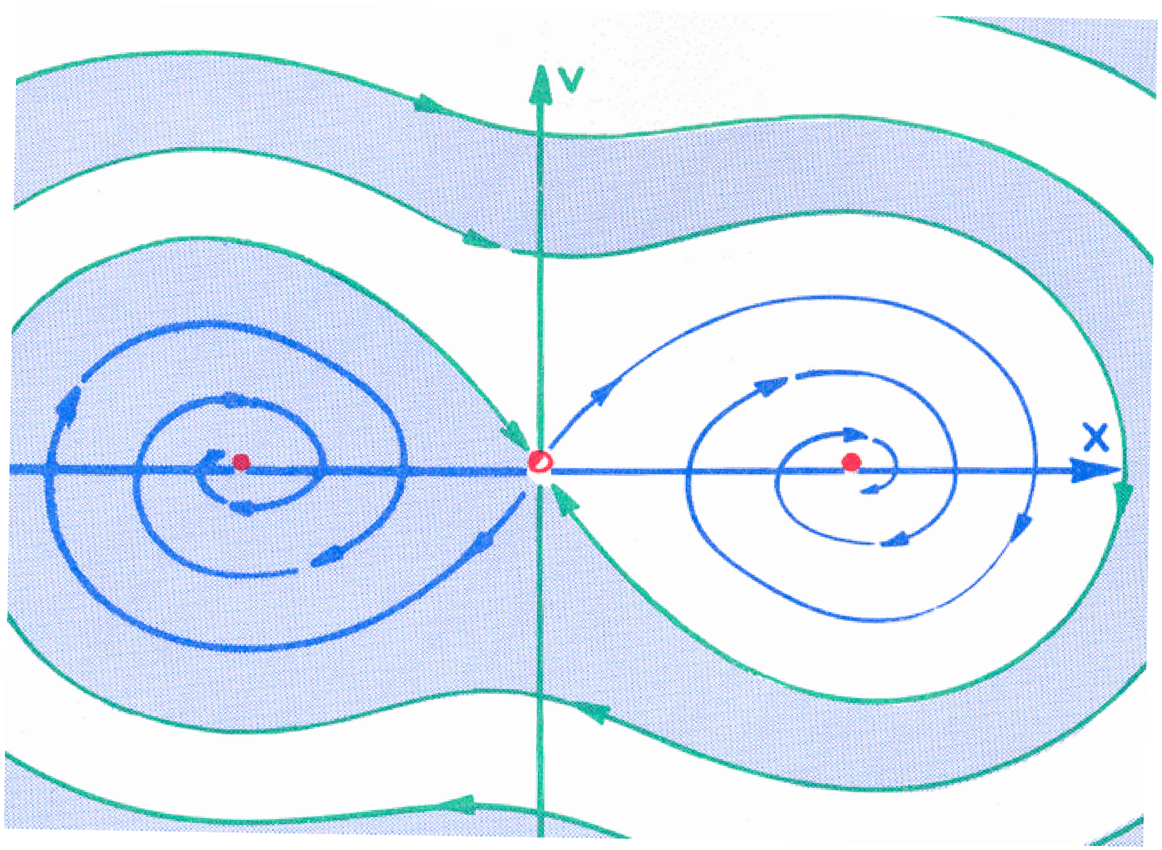
Manifolds and eigenspaces



From: Kuznetsov Y. A., Elements of applied bifurcation theory, Springer Verlag, New York 2004

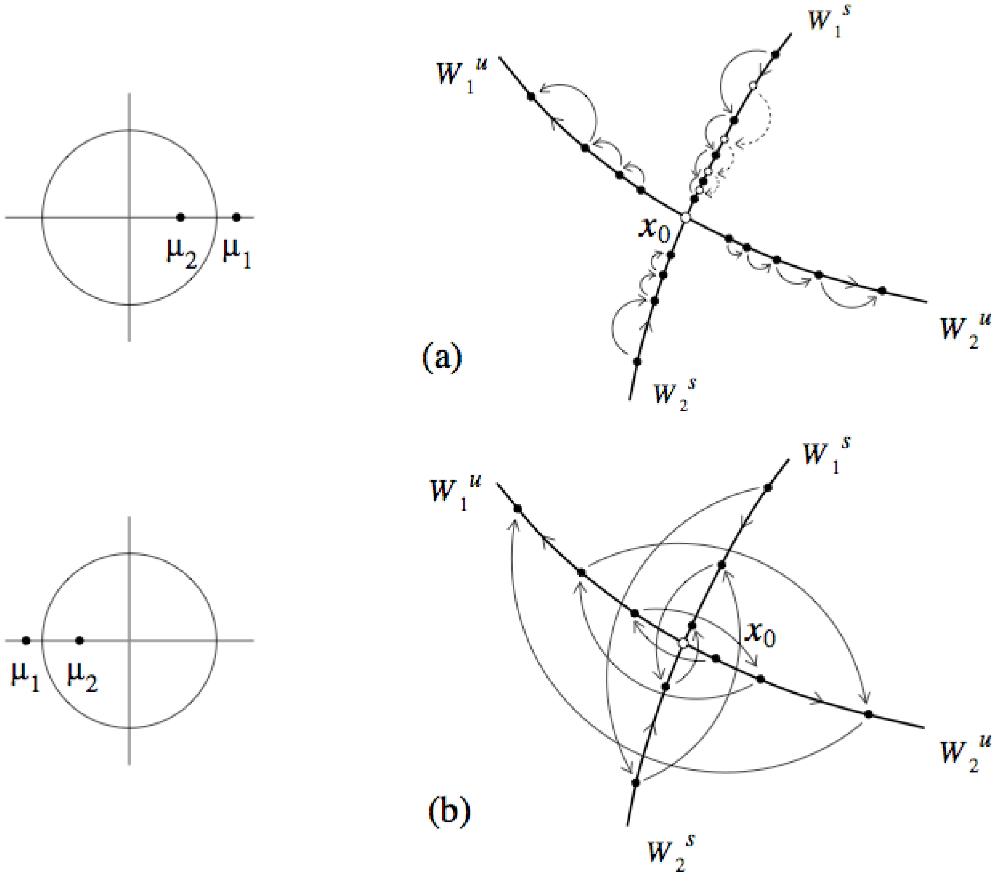
Global Manifolds

- **Stable Global manifold** and **Unstable Global Manifolds** for the saddle in the origin.



Invariant Manifolds for maps

- Similarly for the maps



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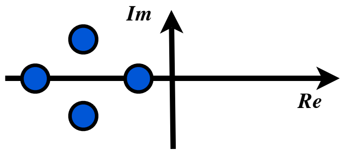
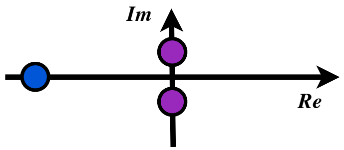
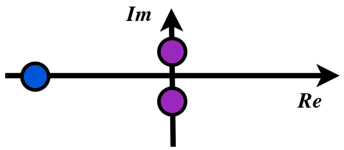
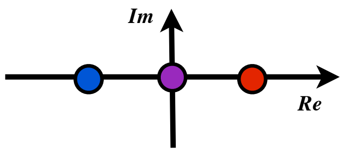
Linearized stability

- What happens if the system is nonhyperbolic?
  - Is this an important issue?

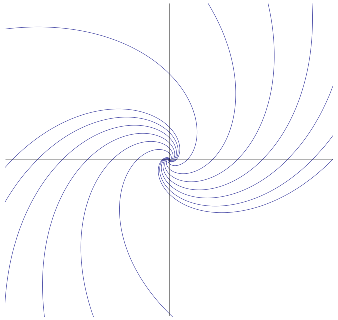
Eigenvalues	Linear system	Nonlinear system
	Asymptotically stable	Asymptotically stable
	Stable (simple imaginary roots)	?
	Weakly unstable (multiple imaginary roots)	?
	Unstable	Unstable

NONHYPERBOLIC EQUILIBRIUM POINTS

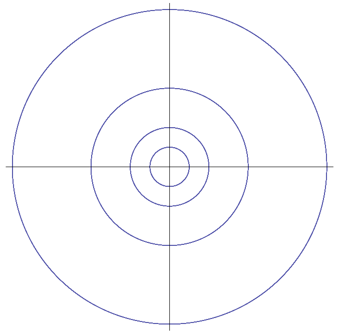
Linearized stability - Nonhyperbolic points

- The following nonlinear dynamical system has an equilibrium point in the origin (0,0)

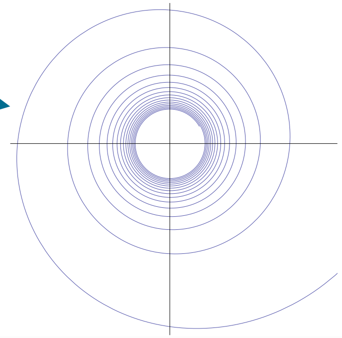
$$\dot{x_1} = \alpha x_1 + x_2 + \epsilon x_1(x_1^2 + x_2^2)$$
$$\dot{x_2} = -x_1 + \alpha x_2 + \epsilon x_1(x_1^2 + x_2^2)$$



- Hyperbolic conditions: The equilibrium point is stable (unstable) if $\alpha<0$ ($\alpha>0$).



- If $\alpha=0$ the linearized system is stable.



- What happens to the nonlinear system? The stability in this case depends on the nonlinear terms: the origin is asymptotically stable if $\varepsilon<0$.

Linearized stability - Nonhyperbolic points

- Another example

$$\dot{x} = xy$$
$$\dot{y} = -y + \alpha x^2$$

- The linearized system is nonhyperbolic with eigenvalues 0 and -1.
- The linearized system has monodimensional central and stable eigenspaces
- **Which is the stability of the origin?**
- **What about the local dynamics?**

Stable manifold theorem again

- There is also a version of the stable manifold theorem which covers the case where some of the eigenvalues are on the imaginary axis.
- Consider the system

$$\dot{x_1} = Ax_1 + f_1(x_1, x_2, x_3)$$
$$\dot{x_2} = Bx_2 + f_2(x_1, x_2, x_3)$$
$$\dot{x_3} = Cx_3 + f_3(x_1, x_2, x_3)$$

  - f_1, f_2, and f_3 are of quadratic order at the origin, ***A*** has eigenvalues with negative real parts, ***B*** has eigenvalues with positive real parts, and ***C*** has eigenvalues with zero real part.
  - The stable manifold in this case has the form $x_2=\phi_1(x_1)$, $x_3=\phi_2(x_1)$, and the unstable manifold has the form $x_1=\psi_1(x_2)$, $x_3=\psi_2(x_2)$, where, as before, the the functions are of quadratic order at the origin.
  - The difference is that we can no longer characterize the stable manifold as the locus of solutions which approach the origin for $t \to +\infty$. This is because of the presence of neutral eigenvalues. **Therefore, the nonlinear terms determine which orbits approach the origin.**

Center manifold theorem

- The center manifold is invariant and has the same dimensions of E^C, it is tangent to E^C in x_E, **BUT** the dynamics on it is not "similar" to that on E^C.
  - For a given dynamics on E^C the dynamics on W^C can be stable, a-stable or unstable and depends on the nonlinear terms.
- The knowledge of the center manifold and of the dynamics on it allows the determination of the stability of the equilibrium point **(or any bounded orbits)**
- When the equilibrium point does not posses an unstable manifold but only a stable and a center manifold the stability issues depend on the dynamics on the central manifold for the continuity of the vector field.
- These aspects are strictly related with the problem of bifurcation as will become clearer during next lecture.

Center manifold theorem

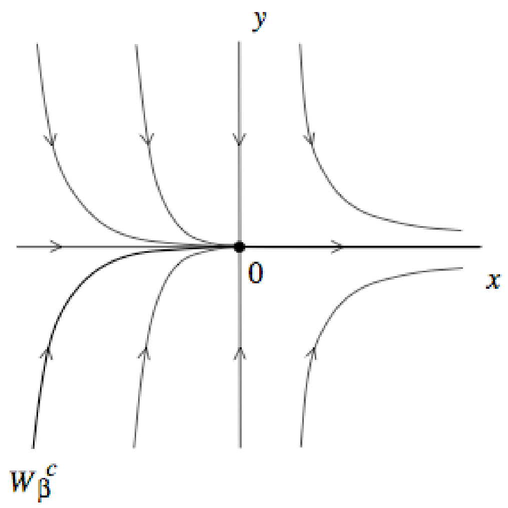
Center manifold theorem

- Let x^* be a singular point of f, where f is of class C^r, $r \geq 2$, in a neighbourhood of x^*. Let $A = \partial f / \partial x\,(x^*)$ have, respectively, n^+, n^0 and n^- eigenvalues with positive, zero and negative real parts, where $n^0 > 0$. Then there exist, in a neighbourhood of x^*, a local invariant C^r manifolds $W^U_{loc},\ W^C_{loc},\ W^S_{loc}$, of respective dimension n^+, n^0 and n^-, and such that
  - W^U_{loc} is the unique local invariant manifold tangent to E^+ at x^*, and $\phi_t(x) \to x^*$ as $t \to -\infty$ for all $x \in W^U_{loc}$.
  - W^S_{loc} is the unique local invariant manifold tangent to E^- at x^*, and $\phi_t(x) \to x^*$ as $t \to +\infty$ for all $x \in W^S_{loc}$.
  - W^C_{loc} is tangent to E^0, but **not necessarily unique**.
- As we have already done, a center manifold can be treated as a graph

$$W^C_{loc} = \{y \in R^n : y^S = h^S(y^C), y^U = h^U(y^C), h^J(0) = Dh^J(0) = 0\}$$

Center manifold theorem

- Orbits staying near the equilibrium for $t \geq 0$ or $t \leq 0$ tend to W^C_{loc} in the corresponding time direction.
- If we know a priori that all orbits starting in U remain in this region forever (a necessary condition for this is $n^+ = 0$), then the theorem implies that these orbits approach W^C_{loc}(0) as $t \rightarrow \infty$. In this case **the manifold is locally "attracting".**
- **Nonuniqueness**



From: Kuznetsov Y. A., Elements of applied bifurcation theory, Springer Verlag, New York 2004

Dynamics on the center manifold

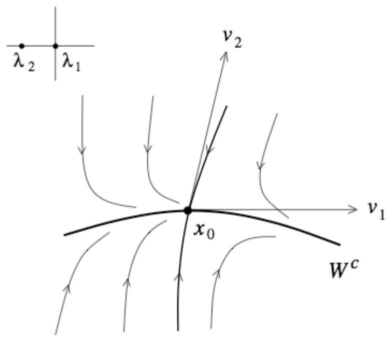
- The local analysis of the nonlinear dynamical system can be inferred from what happens on the center manifold

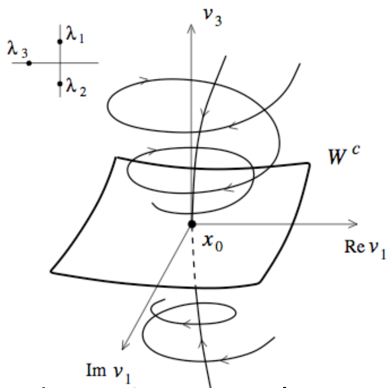
$$\dot{y}^C = A^C y^C + R^C(h^S(y^C), h^U(y^C), y^C)$$

- Of course, if unstable eigenvalues are present the equilibrium point is unstable. But if they are absent the dynamics on the center manifolds becomes:

$$\dot{y}^C = A^C y^C + R^C(h^S(y^C), y^C)$$

  - **The central idea is that the dynamics of the system is governed by the evolution of these center modes, while the stable modes follow in a passive fashion, they are "enslaved".**





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How to calculate the center manifold

- We want to determine the equation that represents the center manifold.
- Let's assume that no unstable component is present.
  - All orbits starting near the equilibrium point approach the invariant center manifold. The qualitative behavior of the local flow on the plane can then be determined from the flow of an appropriate scalar differential equation on the center manifold.
- The coordinate of a point on the center manifold must obey the following equation:

$$y^s = h^s(y^c)$$

Enslaving

- By deriving respect to time

$$\dot{y}^s = Dh^s(y^c)\dot{y}^c$$

- Recalling the relationships

$$\dot{y}^s = A^S y^s + R^S(y^s, y^c)$$
$$\dot{y}^c = A^C y^c + R^C(y^s, y^c)$$

How to calculate the center manifold

- Thus (as the center manifold is invariant)

$$A^S y^s + R^S(y^s, y^c) = Dh^S(y^c)\left[A^C y^c + R^C(y^s, y^c)\right]$$

$$\Downarrow$$

$$A^S h^S(y^c) + R^S(h^S(y^c), y^c) = Dh^S(y^c)\left[A^C y^c + R^C(h^S(y^c), y^c)\right]$$

- This is a partial differential equation for $h^S(y^c)$ with boundary conditions

$$h^S(0) = 0, \quad Dh^S(0) = 0$$

- The problem is complex but it can be solved by using polynomials at any degree of accuracy

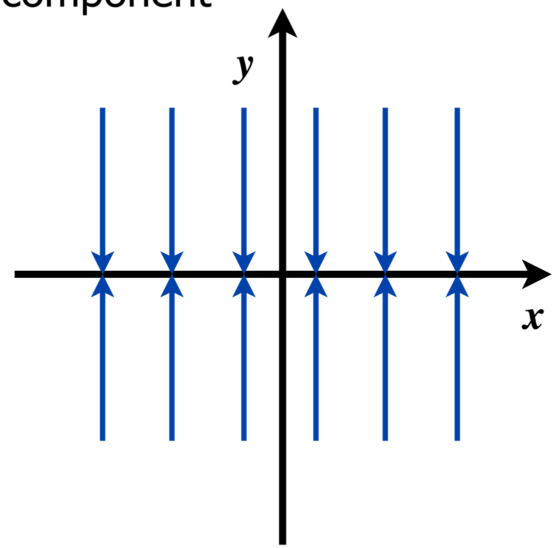
How to calculate the center manifold - An example

- The system:

$$\dot{x} = xy$$
$$\dot{y} = -y + \alpha x^2$$

- The linear part is already in the Jordan form, and the eigenvalues are 0 and -1
- We have one stable component and one central component
- The axis $x=0$ is the E^S while $y=0$ is E^C.
- The stable manifold coincides with E^S.
- For the center manifold:

1. $y = h(x)$
2. $\dot{y} = Dh(x)\dot{x}$
3. $-h(x) + \alpha x^2 = Dh(x)\ x\ h(x)$



How to calculate the center manifold - An example

- An approximate solution is:

$$h(x) = \ ax^2 \ + \ bx^3 \ + \ldots$$

- which is inserted in Eq. 3 gives

$$-ax^2 - bx^3 + \alpha x^2 = (2ax + 3bx^2)(ax^3 + bx^4)$$

- By equating the polynomials in RHS and LHS one obtains:

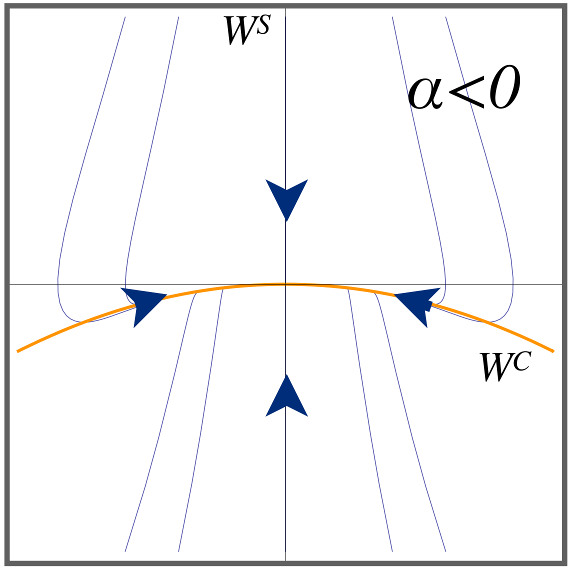
$$h(x) = \alpha x^2 + O(x^4)$$

- The center manifold is: $y = \alpha x^2$

- And the dynamics on W^C is: $\dot{x} = \alpha x^3 + O(x^5)$

How to calculate the center manifold - An example

- If $\alpha<0$ ($\alpha>0$) the origin is stable (unstable)
  - A nonhyperbolic attractor (saddle)



- It is wrong to approximate the center manifold with the center eigenspace. In such a case the dynamics would have been

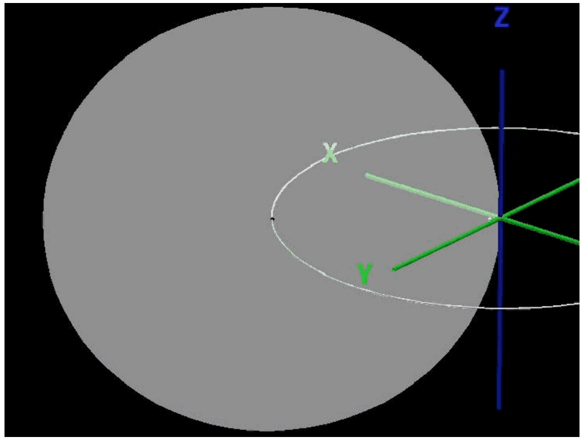
$$\dot{x} = 0$$

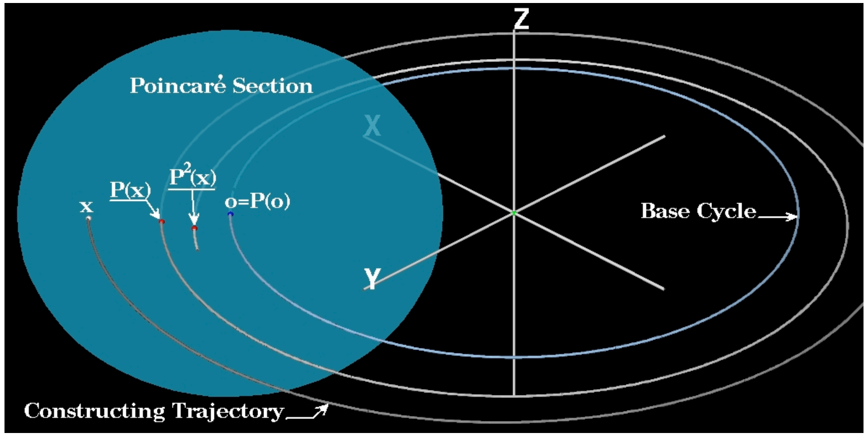
LIMIT CYCLES

Poincaré maps to study continuous time systems

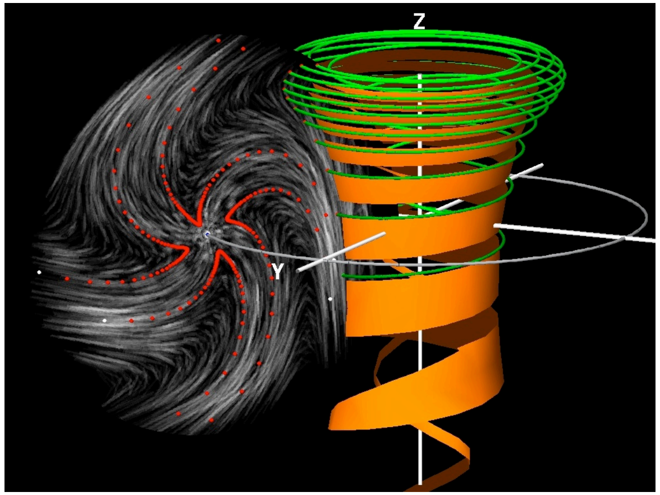
- Continuous nonlinear dynamical system can be associated to a discrete time system the so-called Poincarè map.
  - Among other things, the Poincaré map is a clear and powerful tool to understand and describe concepts related to nonlinear dynamical systems
    - An example: Orbital stability is easily linked to eigenvalues of the Poincarè map.
  - Unfortunately: There exists no general methods to construct the map. (Ingenuity is required)
  - Poincarè map is very useful
    - To study the orbit structure near a periodic orbit
    - When the phase space is periodic (nonautonomous systems)
    - To study the orbit structure near peculiar orbits (homoclinic and heteroclinic)

Poincaré maps - What is it?





http://www.cg.tuwien.ac.at/research/vis/dynsys/Poincare97/
by H. Löffelmann, T. Kucera, and E. Gröller



Poincaré map

- A Poincaré section is used to construct a $(n-1)$-dimensional discrete dynamical system, i.e., a Poincaré map, of a continuous flow given in n dimensions.
- **This reduced system of $n-1$ dimensions inherits many properties, e.g., periodicity or quasi-periodicity, of the original system.**
- We will concentrate in the following on the case of n being equal to three.
- A Poincaré section S is assumed to be a part of a plane, which is placed within the 3D phase space of the continuous dynamical system such that either the periodic orbit (or else) the Poincaré section.
- The Poincaré map is defined as a discrete function $P:S \rightarrow S$, which associates consecutive intersections of a trajectory of the 3D flow with S
- A cycle of the 3D system which intersects the Poincaré section in q points ($q \geq 1$) is related to a periodic point of Poincaré map P, i.e., it is a critical point of the map P^q.

Poincaré map

- **Furthermore stability characteristics of the cycle are inherited by the critical point: stable, unstable, or saddle cycles result in stable, unstable, or saddle nodes, respectively.**
- Many characteristics of periodic or quasi-periodic dynamical systems can be derived from the corresponding Poincaré map.

DEFINITION

A set $\Sigma\subset R^n$ is called a submanifold of codimension one (i.e., its dimension is $n-1$), if it can be written as $\Sigma = \{x \in U | S(x) = 0\}$ where $U\subset R^n$ is open, $S\in C^k(U)$, and $\partial S/\partial x\neq 0$ for all $x\in\Sigma$.

The submanifold Σ is said to be transversal to the vector field f if $\partial S/\partial x\, f(x)\neq 0$ for all $x\in\Sigma$.

Poincaré map and stability of periodic solutions

- Let Σ be a submanifold of codimension one transversal to f such that $\phi(\mathrm{T}, x) \in \Sigma$. Then there exists a neighborhood U of x and $\tau \in C^n(U)$ such that $\tau(x)=\mathrm{T}$ and $\phi(\tau(y), y) \in \Sigma$ for all $y \in U$.

- If x is periodic and $\mathrm{T}=\mathrm{T}(x)$ is its period, then $P_{\Sigma(y)} = \phi(\tau(y), y)$ is called Poincaré map.

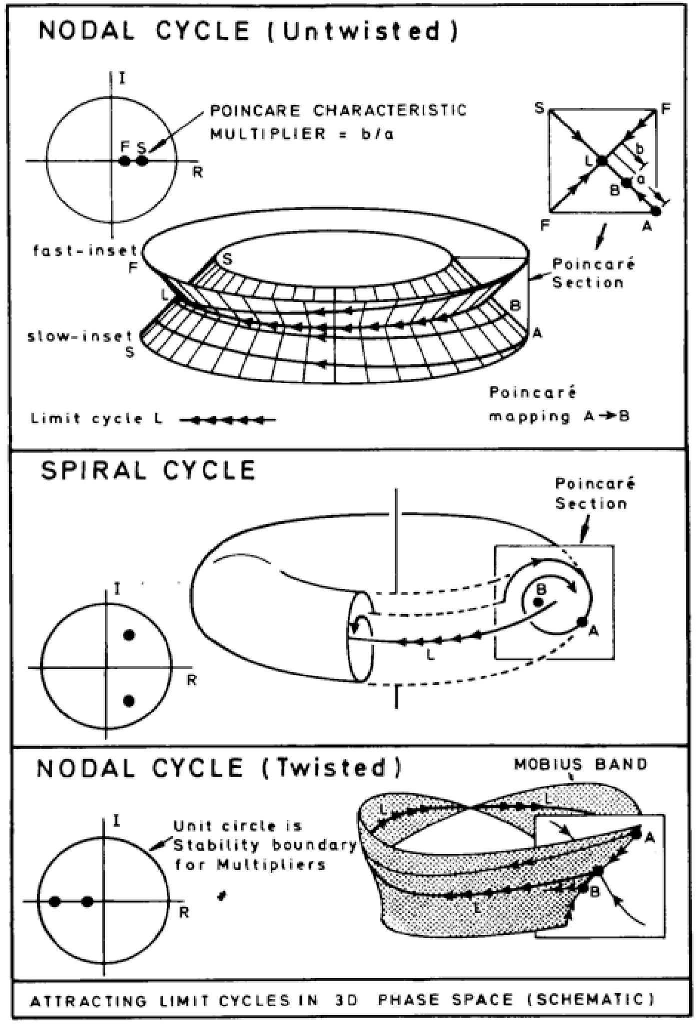
  - **It maps Σ into itself and every fixed point corresponds to a periodic orbit of the continuous system.**

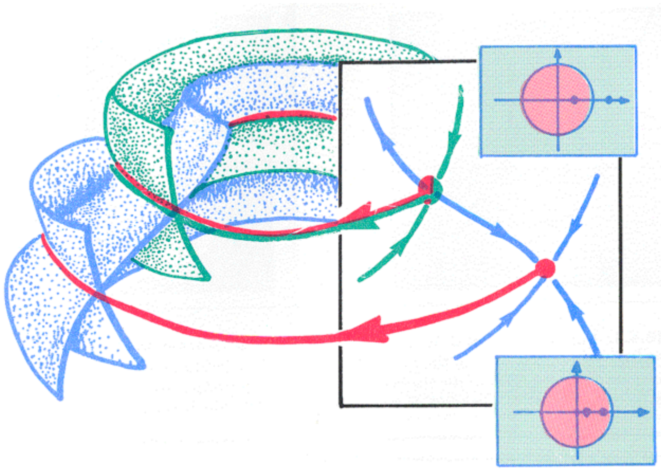
- The periodic orbit $\gamma(x_0)$ is an (asymptotically) stable orbit of f if and only if x_0 is an (asymptotically) stable fixed point of P_Σ.

- Moreover, if $y_n \rightarrow x_0$ then $\phi(t, y) \rightarrow \gamma(x_0)$ by continuity of ϕ and compactness of $[0, \mathrm{T}]$. Hence $\gamma(x_0)$ is asymptotically stable if x_0 is.

- Suppose $f \in C^n$ has a periodic orbit $\gamma(x_0)$. **If all eigenvalues of the Poincaré map lie inside the unit circle then the periodic orbit is asymptotically stable**.

Stability of periodic solutions - Examples





Basin of attraction

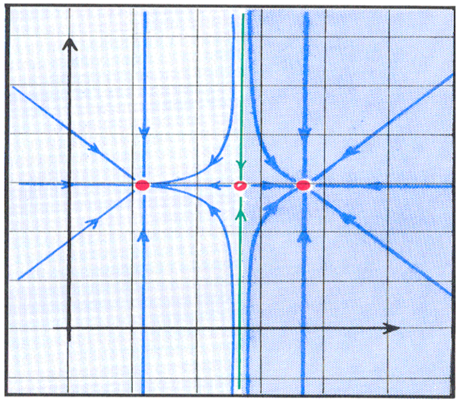
- It is interesting to known for a given attractor (equilibrium point, periodic orbit, ...) the subset of the phase space from which the orbits tend to it.
- This subset is invariant and is called basin of attraction, B_A, of the attractor

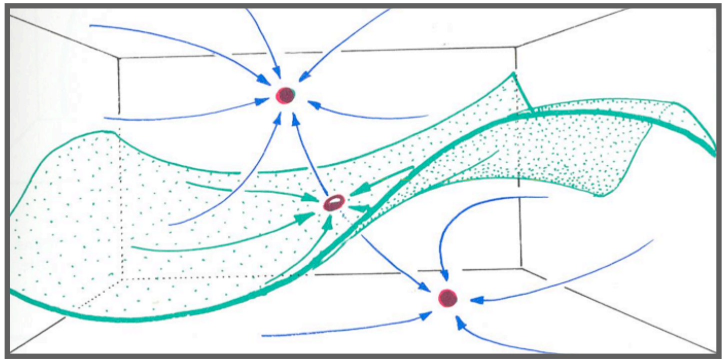
$$B_A = \left\{ x \in R^n : \lim_{t \to +\infty} d(\phi_t x, A) = 0 \right\}$$

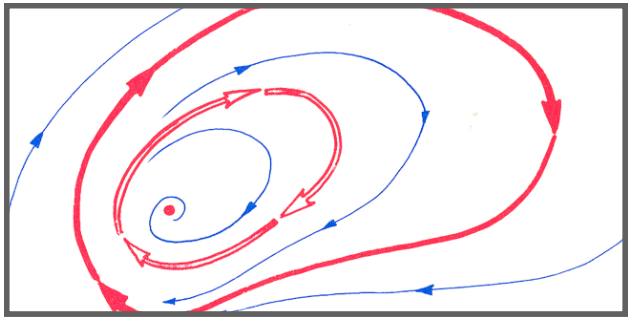
- A nonlinear dynamical system can have multiplicity of attractors, each of them with its own basin of attraction.
- These basins of attraction are open sets and their boundaries do not belong to any of them.

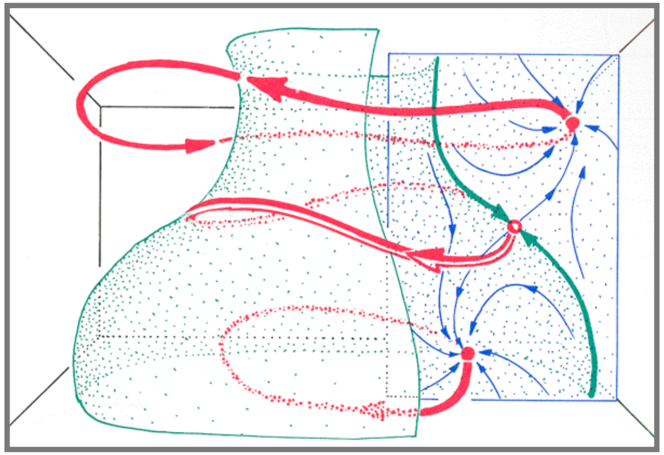
Basin of attraction

- Often the boundaries of the basin of attraction are the stable manifolds of other solutions (saddles, saddle-cycles).









Final remarks

- Stability and linearization of nonlinear dynamical systems
- Locality
- Hyperbolicity vs. nonhyperbolicity
  - Center manifold theorem (very useful for bifurcation analysis)
- Limit cycles and Poincaré map