Structural stability and bifurcations

References

Very instructive with simple approach

Strogatz, S. H., Nonlinear dynamics and chaos: with application to physics, biology, chemistry and engineering, Addison Wesley, New York 1994.

A complete overview of the problems

Wiggins S., Introduction to applied nonlinear dynamical systems and chaos, Springer Verlag, New York 1990 (2nd Ed. 2003)

Kuznetsov Y. A., Elements of applied bifurcation theory, Springer Verlag, New York 2004 (3rd Rev. Ed.)

Very detailed on some aspects

Carr J., Applications of centre manifold theory, Springer Verlag, New York, 1981

Previous lectures - Nonlinear dynamical systems

- Stability of **hyperbolic** equilibrium points and periodic orbits
- Stability of **nonhyperbolic** situations
- Center manifold theorem
- Dynamics on the center manifold
  - **The linearized dynamics does not give information on stability**

Motivations - Theory

- Models contain parameters
- **What happens to the geometry of the phase space when a parameter changes?**
  - Quantitative changes
  - Qualitative changes
    - Implication on the safety
- Qualitative changes will be called **bifurcations**.
  - **When can we observe qualitative changes?**
- Can we a priori know the possible scenarios for different dynamical systems?
- Dynamical systems can be quite “large”: do we need to account for details of their “largeness” or we can limit to something simpler?

Motivations - Applications

Nonlinear models of engineering systems exhibit instabilities:

Multiplicity, Ignitions
Symmetry breakings
Phase transitions...

→

These phenomena must be understood for a correct design and optimization

↓

Software for the stability analysis

Automatic
Reliable
Large scale systems

Outline

- Introduction
- Topological equivalence
- Structural stability
- Bifurcations
  - Local
  - Global
- Simplifications on the center manifold
- Normal forms

General frame

- In this lecture we will only consider systems of the types:

Continuous

$$\frac{dx}{dt} = f(x,\mu),\ x \in \mathbb{R}^n,\ f \in C^r(R^n, R^n)\ with\ r \geqslant 1,\quad \mu \in \mathbb{R}^m\ with\ m \geqslant 1$$

Discrete

$$x_{k+1} = F(x_k,\mu),\ x \in \mathbb{R}^n,\ F \in C^r(R^n, R^n)\ with\ r \geqslant 1,\quad \mu \in \mathbb{R}^m\ with\ m \geqslant 1$$

- **The parameters μ are now explicitly considered to change**
  - **for a change in the operating conditions**
  - **to account for uncertainties**
  - **in time**

A very simple linear example

- We start from a very **simple linear example** $\dot{x} = \mu x$
- $x=0$ is an equilibrium point
  - It is **asymptotically stable** if $\mu<0$, the phase portrait does not change if the parameter is perturbed a little bit.
  - It is **unstable** if $\mu>0$, the phase portrait does not change if the parameter is perturbed a little bit.
  - It is **stable** if $\mu=0$.
- The latter condition merits some attention: it is a nonhyperbolic equilibrium point
  - Undecided
  - Any perturbation of the parameter value determines a qualitative change in the phase portrait of the system

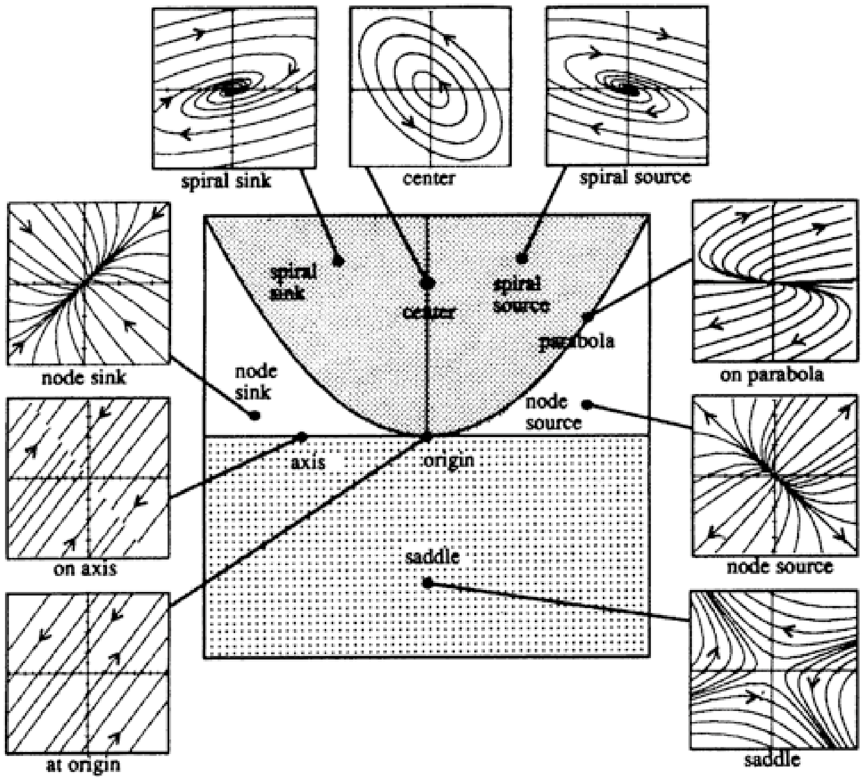
Motivation

- Two important considerations:
- **Hyperbolic points seems to be “indifferent” to small parameter changes, while the nonhyperbolic point is strongly affected from them**
  - Is this a kind of stability with respect to parameter changes?
    - NB: the stability we know was related to changes in the state of the system.
- **It seems that a qualitative change occurs when the system passes through a nonhyperbolic point: Bifurcations?**
- These two aspects will be addressed in detail in this lecture.

Topological equivalence

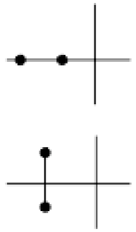
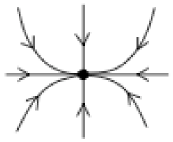
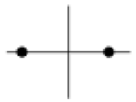
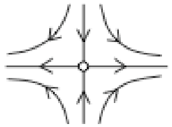
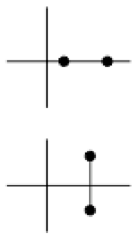
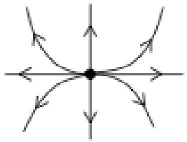
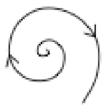
Topological equivalence of linear systems

- Let's reconsider the case of a **2D linear system**
- It is apparent that we can describe the hyperbolic systems with just 3 cases:
  - Sources, saddles and sinks
- In the case of nonhyperbolic systems we can identify 5 situations (not all shown in this figure)
- Similar conclusions could be drawn for larger dimension systems (the saddles could be of different types in such cases)



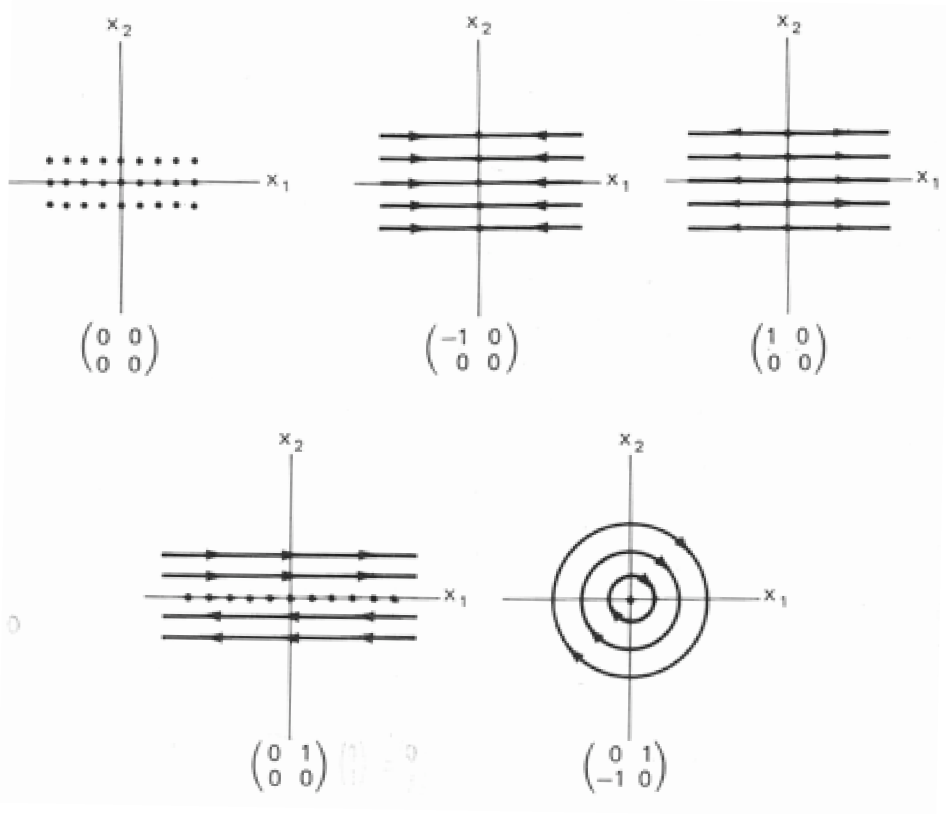
Topological equivalence of linear systems

- For linear systems one could define an equivalence relation and equivalence classes **on the basis of the eigenvalues**.
- For hyperbolic 2D systems for example

(n_+, n_-)	Eigenvalues	Phase portrait	Stability
$(0,2)$		 node  focus	stable
$(1,1)$		 saddle	unstable
$(2,0)$		 node  focus	unstable

Topological equivalence of linear systems

- In the case of nonhyperbolic conditions one has to account for algebraic and geometrical multiplicity of the eigenvalues
- In the 2D case the possible situations are:



Topological equivalence of systems

- We are strongly tempted to try to develop an equivalence criterion of the phase portrait for any kind of dynamical system (linear, nonlinear, continuous, discrete)

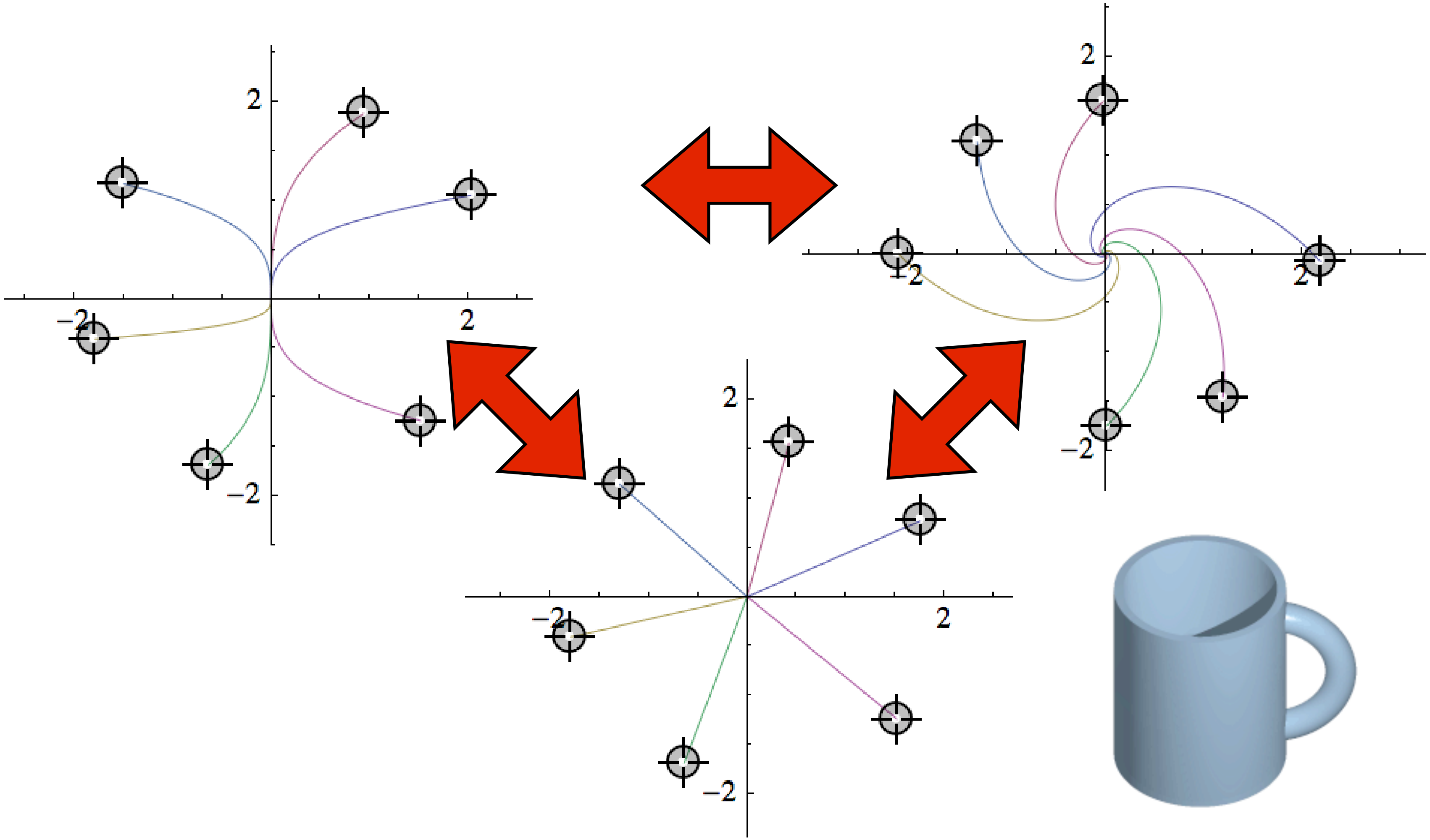
$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m$$
$$\dot{y} = g(y, \beta), \quad y \in \mathbb{R}^n, \ \beta \in \mathbb{R}^m$$

DEFINITION

Two systems SA1 (phase space X) and SA2 (phase space Y) of the same order n share the same qualitative behavior and are topologically equivalent if and only if there exists a homeomorphism that maps orbits of SA1 on those of SA2 preserving the direction of time.

- A homeomorphism is an invertible map such that both the map and its inverse are continuous $\psi \in C^0(X, Y) \quad \psi^{-1} \in C^0(Y, X)$.
- **It is important to remark that the two systems could be the same system for a different value of the parameters.**

Why homeomorphism?



Topological equivalence of systems

- The definition fulfills the following three properties

1. **Reflection**: SA1 is Topologically Equivalent to SA1
2. **Symmetry**: SA1 Topologically Equivalent to SA2 implies SA2 Topologically Equivalent to SA1
3. **Transitivity**: If SA1 is Topologically Equivalent to SA2 and SA2 is Topologically Equivalent to SA3 then SA1 is Topologically Equivalent to SA3

- With this definition: **two hyperbolic continuous linear systems with stable and unstable eigenspaces of the same dimensions are topologically equivalent.**
- The case of maps: two linear maps with stable and and unstable eigenspaces of the same dimensions and the matrices characterizing the dynamics on both eigenspaces have the same determinant are topologically equivalent

Topological equivalence of nonlinear systems

- In the case of nonlinear systems the phase portrait are topologically equivalent if:

1. They have the same number of equilibrium points with the same stability properties;
2. They have the same number of periodic orbits with the same stability properties
3. They have the same invariant in one to one correspondence

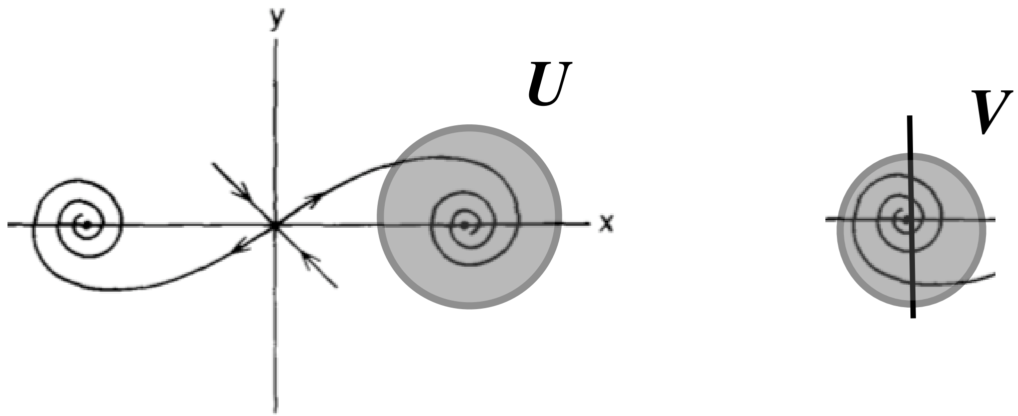
- **For nonlinear systems a local equivalence is of course useful**

DEFINITION

The system SA1 in the subset U of the phase space X is said topologically equivalent to the system SA2 in the subset V of the phase space Y if there exists a homeomorphism that transforms the orbits (or pieces of orbits) of the first system into orbits (or pieces of orbits) of the second system by preserving the time direction of points corresponding orbits.

Topological equivalence of systems

- An important case: if U includes an equilibrium point x_E of the first system and V includes an equilibrium y_E point of the second system
- In such a case the system SA1 is said to be close to x_E topologically equivalent to SA2 in y_E
- An example: **the nonlinear system in U is topologically equivalent to the associate linearized system in V if the latter is hyperbolic** (another way to state the Hartman-Grobmann theorem)



Bifurcation conditions

Structural stability

DEFINITION

A system $\dot{x} = f(x, \mu_0),\ or\ x_{k+1} = F(x_K, \mu_0)$ is **structurally stable** (with respect to the parameter) if and only if there exists an $\varepsilon>0$ such that its phase portrait is topologically equivalent to that of the system $\dot{x} = f(x, \mu),\ or\ x_{k+1} = F(x_K, \mu)$ for any μ such that $\|\mu-\mu_0\|<\varepsilon$.

- A hyperbolic equilibrium is structurally stable under smooth perturbations
- A sufficiently small perturbation of the vector field or of the map does not induce a qualitatively change of a structurally stable dynamical system
- A local version of the definition is of course available
- **The global and local definitions of the structural stability are the basis of the definitions of global and local bifurcations.**

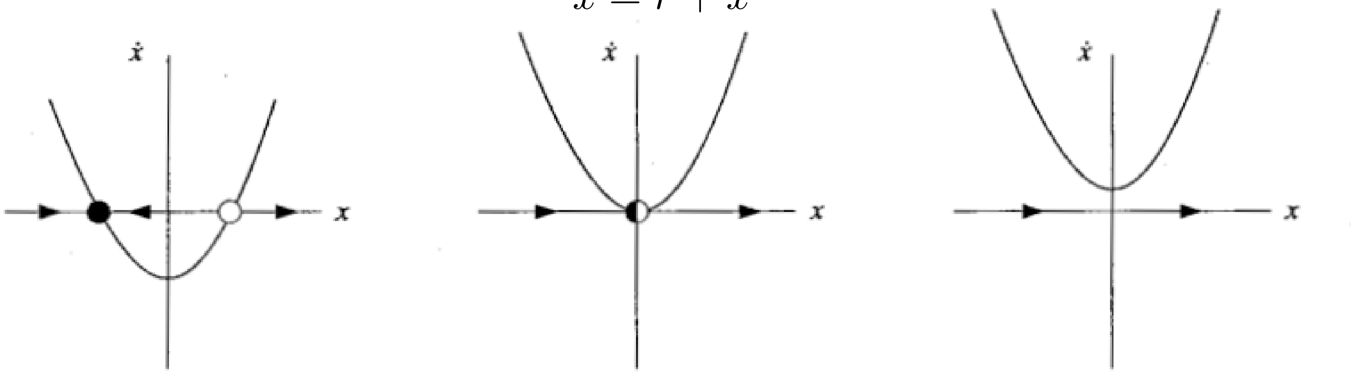
Bifurcation conditions

- The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a **bifurcation**.
- Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.

DEFINITION

The system SA $\dot{x} = f(x, \mu_C),\ or\ x_{k+1} = F(x_K, \mu_C)$ is in (critical) bifurcation condition for a value of the parameter $\mu = \mu_C$ (bifurcation value) if any neighborhood of μ_C contains at least a value μ such that the system is not topologically equivalent to SA

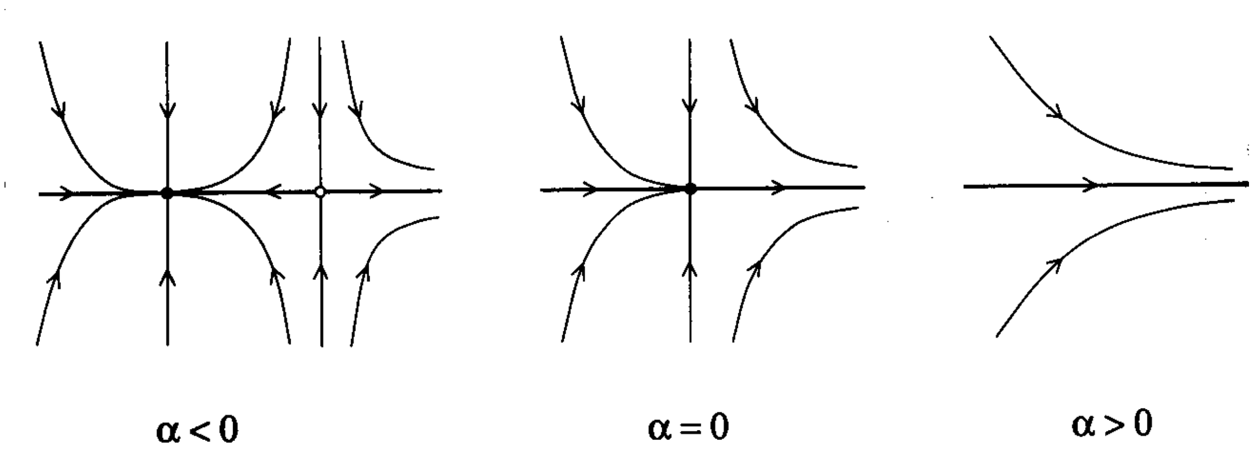
$$\dot{x} = r + x^2$$



(a) $r<0$ (b) $r=0$ (c) $r>0$

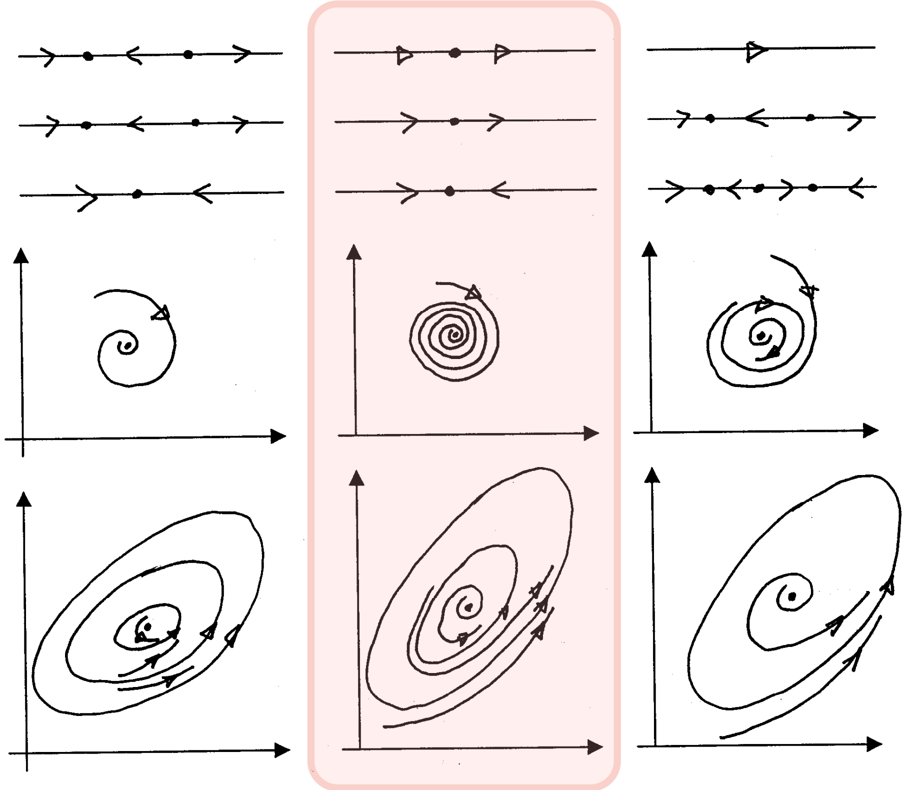
Bifurcations

- Generally, as the vector field f or the map F change (for example by varying a parameter value) the phase portrait of the dynamical system changes continuously and remain topologically equivalent.
- On the other hand, **it can happen that for a critical value of the parameter α a sudden change of the phase portrait of the dynamical system occurs**



Bifurcation conditions

- Some other examples



Local and global bifurcations

- **Bifurcations are classified as local or global**
- Continuous systems
  - We will examine bifurcations of both equilibrium points and limit cycles
    - The latter will be studied as bifurcation of fixed points of maps (Poincaré map)
- Discrete systems
  - We will examine bifurcations of fixed points and of m-periodic points (fixed points of iterated maps)
- More on bifurcations in NDSII

Bifurcation of equilibrium points

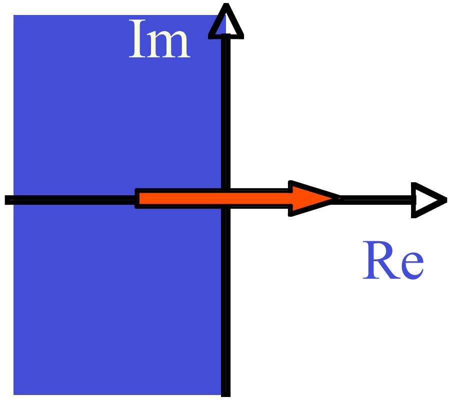
DEFINITION

The equilibrium point x_C of the dynamical system $\dot{x} = f(x, \mu)$ $(x_{k+1} = F(x_k, \mu))$ is in bifurcation conditions for the parameter value $\mu = \mu_C$ if a couple of neighborhoods $I(\mu_C)$ and $I(x_C)$ such that all systems with parameter contained in $I(\mu_C)$ are topological equivalent in $I(x_C)$ do not exist.

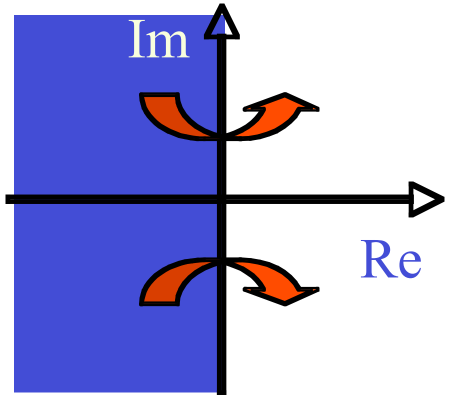
- With x_E an equilibrium point of a nonlinear dynamical system:
  - Hartman-Grobman theorems states that any nonlinear system in a neighborhood of x_E is locally topologically equivalent to the associate linearized system if such linearized system is hyperbolic. Bifurcations of hyperbolic equilibrium points are not possible.
  - Nonhyperbolicity of x_E is **necessary condition** for the occurrence of local bifurcations and the critical conditions of local bifurcations have to be searched for among nonhyperbolic equilibrium points
  - **Local bifurcation of equilibrium points are signaled by the nonhyperbolicity: eigenvalues with zero real parts (continuous systems) or unit magnitude Floquet multipliers (discrete systems)**

Bifurcation of equilibrium points

- Necessary conditions for bifurcation of equilibrium points of continuous systems



Saddle-node

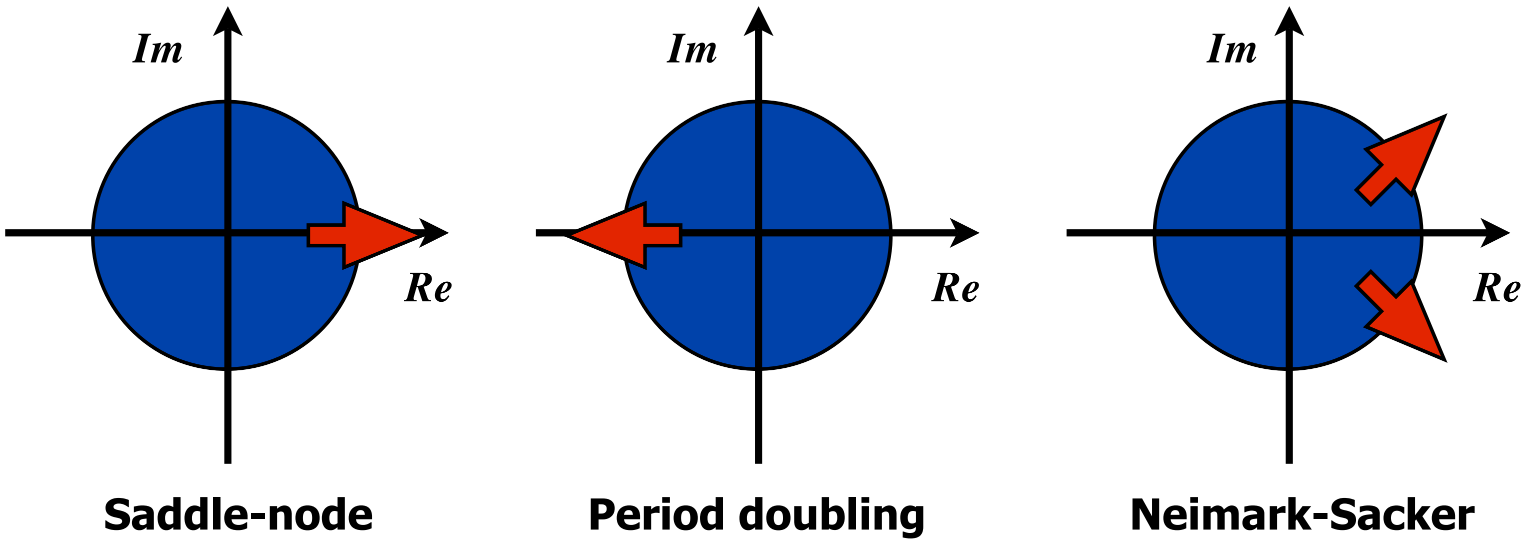


Hopf-Andronov

- Local bifurcations are linked to change in stability of the equilibrium point

Bifurcation of equilibrium points

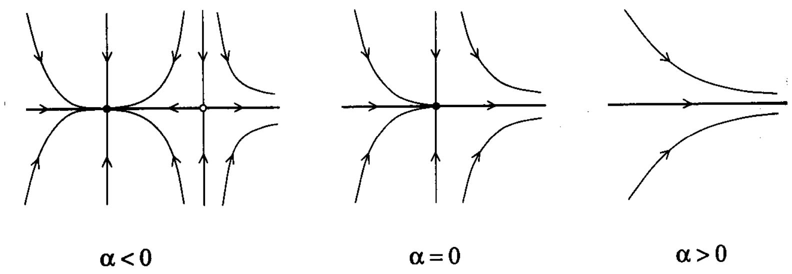
- Necessary conditions for bifurcation of equilibrium points of discrete systems



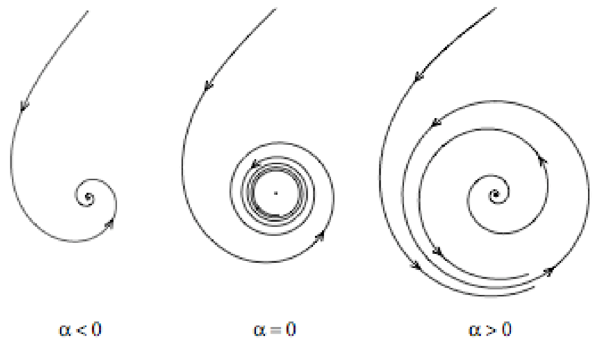
- Local bifurcations are linked to change in stability of the equilibrium point

Bifurcation of equilibrium points

- If we observe a bifurcation in the phase space we may recognize that quite often bifurcations imply the collisions of limit sets with different stability properties (equilibrium points, limit cycles, or equilibrium point with limit cycle



Collision between equilibrium points



Collision between an equilibrium point and a limit cycles

Global bifurcations

- Global bifurcations are difficult to detect. They cannot be pinpointed with a local analysis (in the neighborhood of an equilibrium point) rather one has to analyze the phase portrait (or a part of it).
- It proves useful to define some special orbits:

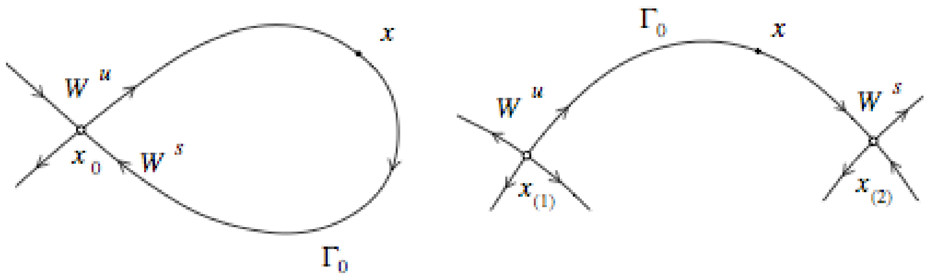
DEFINITIONS

An orbit Γ_0 starting at a point $x \in R^n$ is called **homoclinic** to the saddle point x_0 of dynamical system if $\phi_t\, x \to x_0$ as $t \to \pm\infty$.

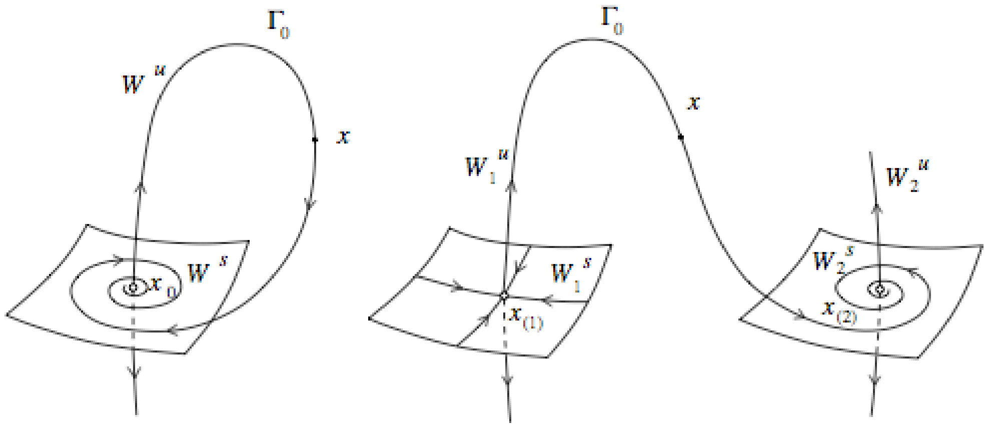
An orbit Γ_0 starting at a point $x \in R^n$ is called **heteroclinic** to the saddle points x_1 and x_2 of the dynamical system if $\phi_t\, x \to x_1$ as $t \to -\infty$ and $\phi_t\, x \to x_2$ as $t \to +\infty$.

Homoclinic and heteroclinic orbits

- In 2D



- In 3D



- **Examples of homoclinic and heteroclinic orbits of a dynamical system is structurally unstable.**

Transversality

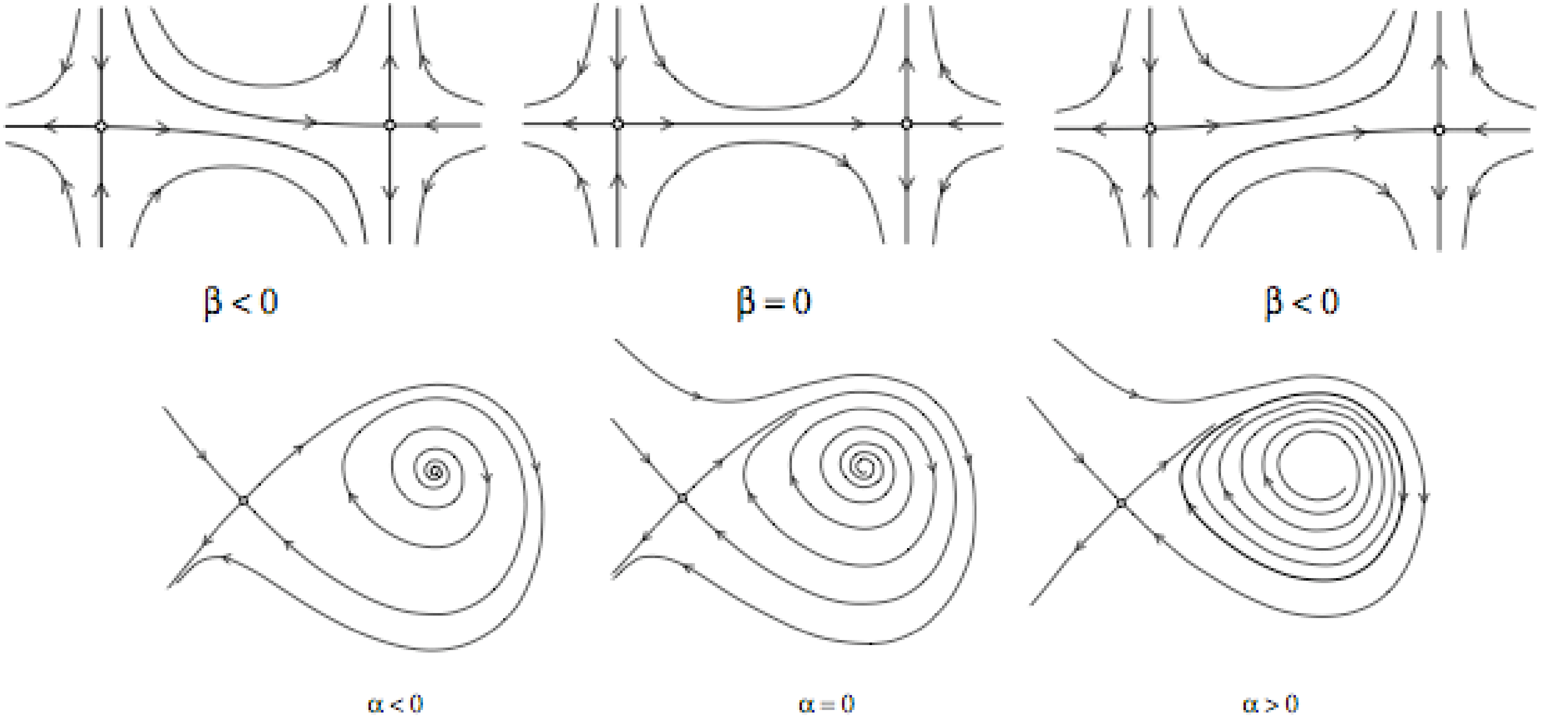
DEFINITION

Two smooth manifolds $M, N \subset R^n$ intersect transversally if there exist n linearly independent vectors that are tangent to at least one of these manifolds at any intersection point.

- For example, a surface and a curve intersecting with a nonzero angle at some point in R^3 are transversal. The main property of transversal intersection is that it persists under small C^1 perturbations of the manifolds.
- If the manifolds intersect nontransversally, generic perturbations make them either nonintersecting or transversally intersecting.

Global bifurcation

- **Global bifurcation can be interpreted as nontransversal collisions of stable and unstable manifolds** for both two bounded orbits (heteroclinic orbits) and one bounded orbit (homoclinic).
- The study of global bifurcation is easy in 2D systems as in the plane transversal collisions are not possible.



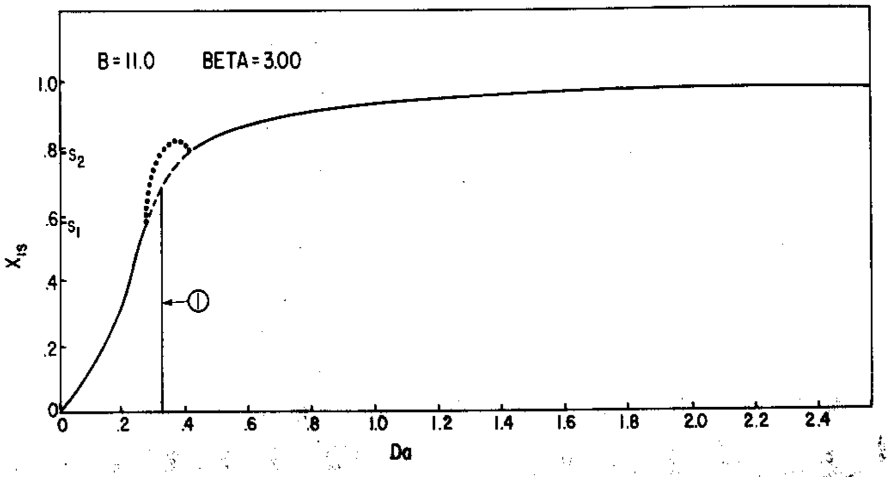
USEFUL TOOLS

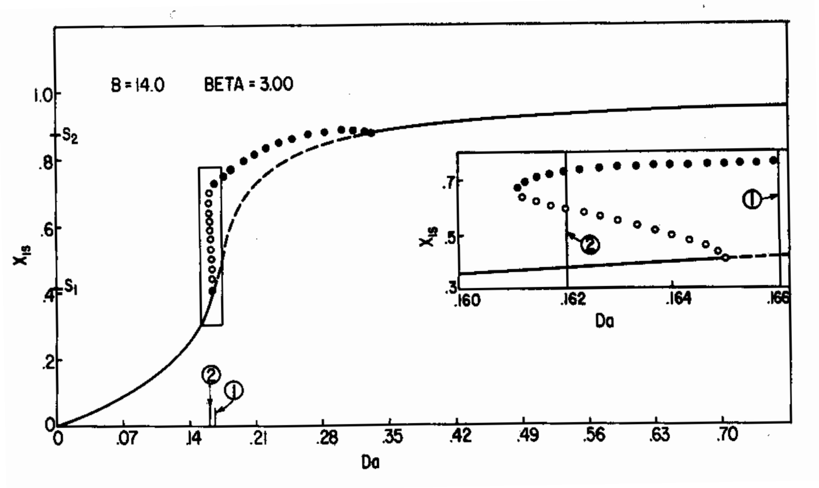
Solution diagrams

- It is very useful to have a graphic representation of the regime solutions (stationary and dynamics) and of their bifurcations as one parameter value is varied (**bifurcation parameter**).
- A simple graph is obtained by plotting a significant state variable (or its maximum for the case of dynamic regime solutions) as a function of the bifurcation parameter.
- As an alternative one can use a proper norm of the state vector in regime conditions.
- These graphs are called (regime) solution diagrams and can be efficiently built with parameter continuation software (e.g., Auto or MatCont)
- You will learn how to use such software in a subsequent lecture

Solution diagrams

- An example for a 2D system (CSTR)
  - Bifurcation parameter Da
  - Conventions
  - Bifurcation points





Bifurcation diagrams

- If the model contains more than one single parameter, i.e., if μ is a vector with more than one component, the critical values for a parameter depend on the other parameter values.
- One can than build a diagram in the parameter space where bifurcation values are reported. The plot of the manifolds corresponding to bifurcation conditions in the parameter space of interest.
- When a parameter value is varied one moves in this space
  - If no bifurcation line is crossed the systems are all topologically equivalent
  - If a bifurcation line is crossed the system show a qualitative change in its properties.

Bifurcation diagrams

- The CSTR: solution and bifurcation diagrams

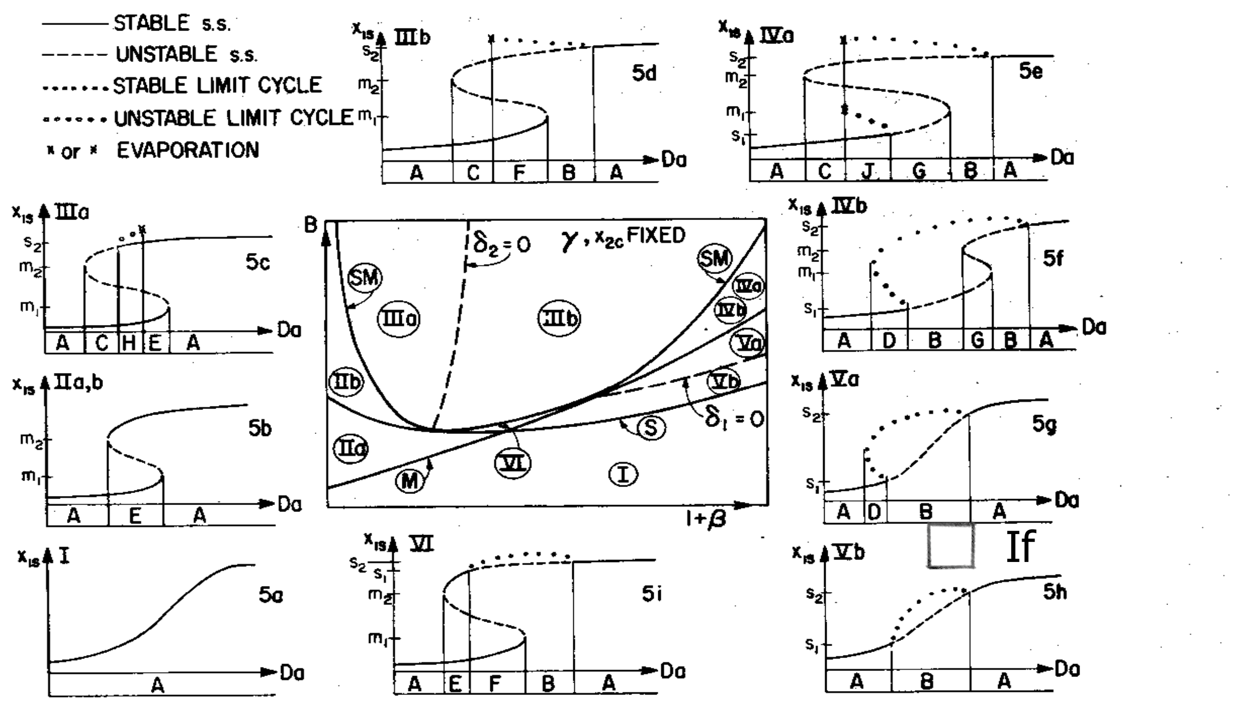


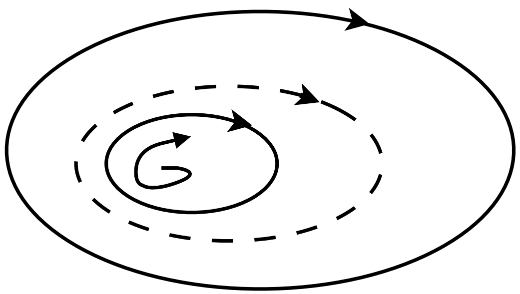
Fig. 5. Typical conversion vs *Da* plots for the different regions in parameter space.

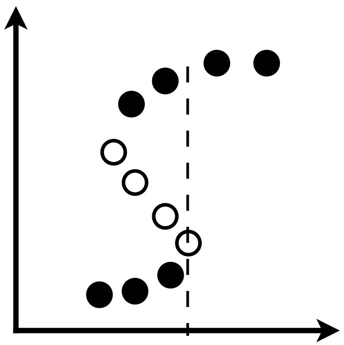
Catastrophic bifurcations

- Parameters may change in time. If such variations are slow (with respect to the characteristic times of the dynamical system) the system will remain in regime conditions by attaining the new (with respect to the parameter value) stable states.
- One would observe a qualitative change of the regime solution if the (slowly changing) parameter crosses a bifurcation value.
  - A change of regime would then be observed: for example from a steady state to a periodic solution.
- We are not guaranteed that when crossing a bifurcation value the state of the system would experience a “small” change when attaining the new stable solution (if any!)
- **When the change is not small we say that the corresponding bifurcation is catastrophic (in the sense that the change might have strong consequences)i: explosions, ignitions, extinctions, runaway, …., static failures (Tacoma Narrows bridge), ….**

Catastrophic bifurcations (Wikipedia)

- The first Tacoma Narrows Bridge opened to traffic on July 1, 1940. It collapsed four months later on November 7, 1940, at 11:00 AM (Pacific time) due to a physical phenomenon known as aeroelastic flutter caused by a 67 kilometres per hour (42 mph) wind.





- Flutter is a self-feeding and potentially destructive vibration where aerodynamic forces on an object couple with a structure's natural mode of vibration to produce rapid periodic motion.

An example from chemical engineering

- An industrial incident in an ammonia production plant. The incident was caused by a sudden loss of stability induced by a decrease of reactor pressure.
- Three beds in series with fresh feed between each bed and preheating of the feed with the effluents.

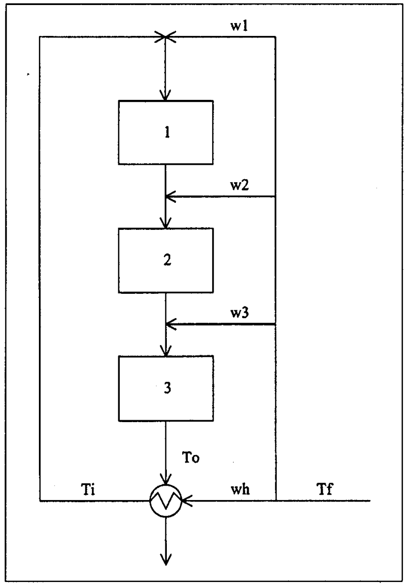


Figure 1. Ammonia reactor.

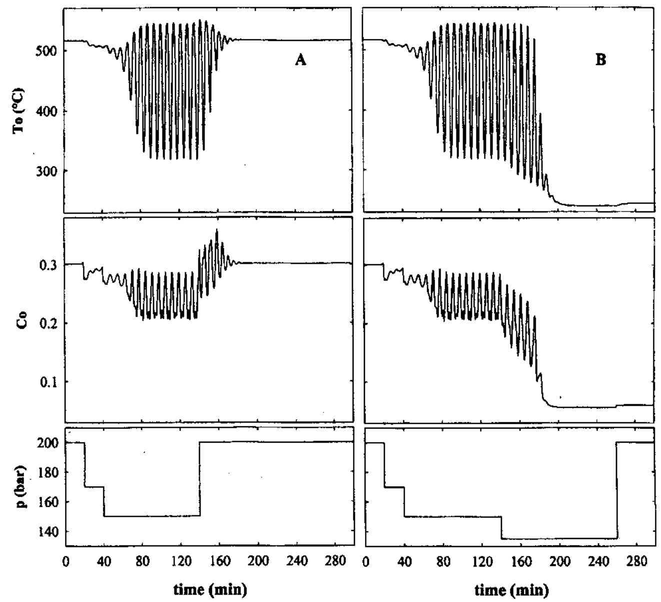


Figure 3. Possible loss of stability scenarios.
(A) Situation illustrated by Morud and Skogestad (1998); (B) Scenario revealed by the nonlinear analysis proposed in the present work.

An example from chemical engineering

- Catastrophic or not?

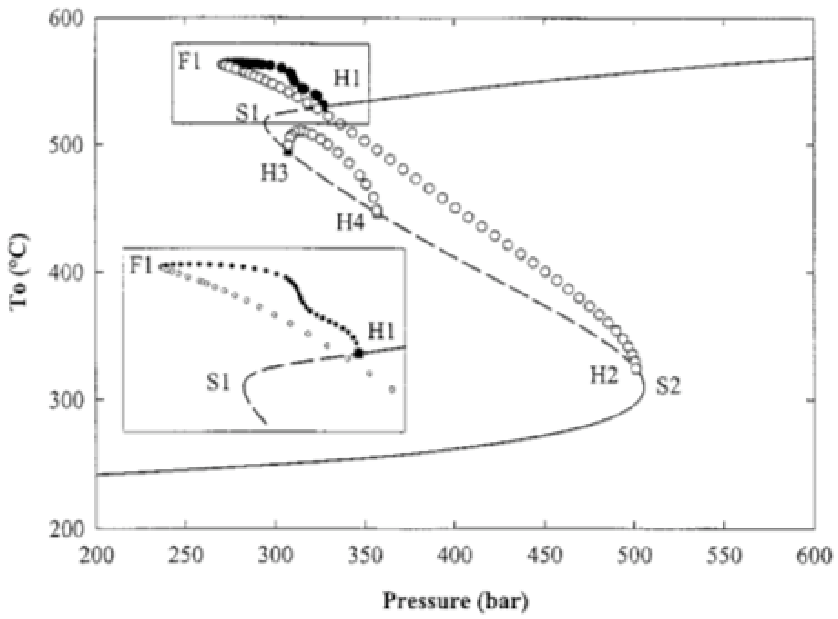


Figure 5. Solution for $\epsilon = 0.35$.

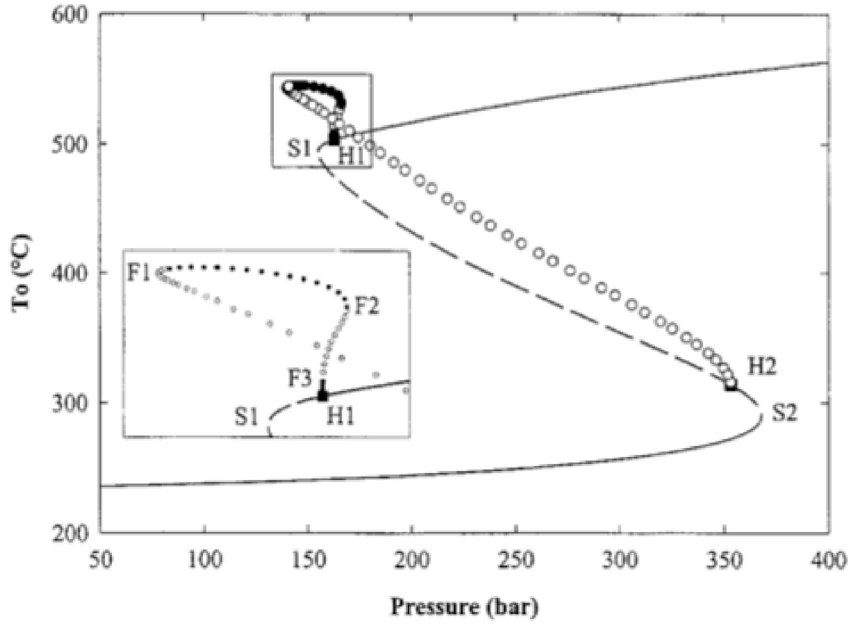


Figure 2. Solution for $\epsilon = 0.628$.

ANALYSIS OF LOCAL BIFURCATIONS
CENTER MANIFOLD THEOREM

Analysis of local bifurcations

- The analysis of local bifurcations of equilibrium points can be performed on a simplified version of the dynamical system (both continuous and discrete)
  - By reducing its dimensions → **Center manifold theory**
    - We anticipated this during Lecture 4
  - By simplifying the nonlinear terms → **Normal form theory**
- If the discrete system is a Poincaré map, the analysis is related to the local bifurcation of limit cycles

Center manifolds and local bifurcations of continuous systems

- In the case of continuous systems under investigation, if x_E is the equilibrium point one can translate the origin of the phase space in x_E and by separating the linear terms one ends up with:

$$\dot{x} = Df(x_E, \mu)x + R(x)$$

- Then, by with a Jordan transformation ($x=Ty$) one gets:

$$\dot{y} = \begin{pmatrix} \dot{y}_s \\ \dot{y}_u \\ \dot{y}_c \end{pmatrix} = \begin{pmatrix} J_s & 0 & 0 \\ 0 & J_u & 0 \\ 0 & 0 & J_c \end{pmatrix} \begin{pmatrix} y_s \\ y_u \\ y_c \end{pmatrix} + \begin{pmatrix} R_s\left(y_s, y_u y_c\right) \\ R_u\left(y_s, y_u y_c\right) \\ R_c\left(y_s, y_u y_c\right) \end{pmatrix}$$

Center manifolds and local bifurcations of continuous systems

- If x_E is a nonhyperbolic equilibrium point

1. There exists at least one center manifold $W^C(0)$ with the same dimensions of E^C of the associate linearized system
2. There exists one and only one stable manifold $W^S(0)$ with the same dimensions of E^S of the associate linearized system
3. There exists one and only one unstable manifold $W^U(0)$ with the same dimensions of E^U of the associate linearized system
4. The three manifolds crosses at the origin and are there tangent to the eigenspaces of the associate linearized system.

- **The (possible) bifurcation of the nonhyperbolic equilibrium point x_E can be studied on a system with lower dimensions (those of the center eigenspace).**

Center manifolds and local bifurcations of continuous systems

- Indeed, it can be demonstrated that local bifurcations take place on the center manifold (which is locally attracting).
- As we have already learned the center manifold close to the origin is described by the equation:

$$\boxed{W^c_{loc} \; : \; \begin{cases} y_s = h_s(y_c) \\ y_u = h_u(y_c) \end{cases}}$$

- Thus, the local bifurcations can be studied on a center manifold on the reduced system:

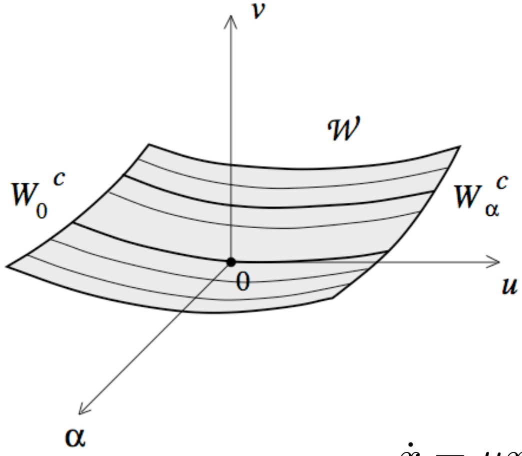
$$\dot{y}_c = J_c\, y_c + R_c(h_s(y_c), h_u(y_c), y_c)$$

- Where are the parameters? We want to study the local bifurcation on the center manifold, so it must exist in a neighborhood of the critical parameter value.

Center manifolds in parameter-dependent systems

- We consider the case: $\dot{x} = f(x, \mu), \; x \in \mathbb{R}^n, \; \mu \in \mathbb{R}^1.$
- Suppose that at μ=0 the system has a nonhyperbolic equilibrium $x = 0$ with n^0 eigenvalues on the imaginary axis and ($n - n^0$) eigenvalues with nonzero real parts. (n^- of them have negative real parts, n^+ of them have positive real parts).
- Consider the extended system: $$\begin{aligned} \dot{\mu} &= 0 \\ \dot{x} &= f(x, \mu) \end{aligned}$$
  - This system can be nonlinear even if the original system was linear (why?).
- The Jacobian is $J = \begin{pmatrix} 0 & 0 \\ f_\mu(0,0) & f_x(0,0) \end{pmatrix}$
- Nonhyperbolic equilibrium $x = 0$ with n^0+1 eigenvalues on the imaginary axis and ($n - n^0$) eigenvalues with nonzero real parts.
- **We can apply the Center Manifold Theorem: a center manifold exists in the neighborhood of the critical value of the parameter**

Center manifolds in parameter-dependent systems



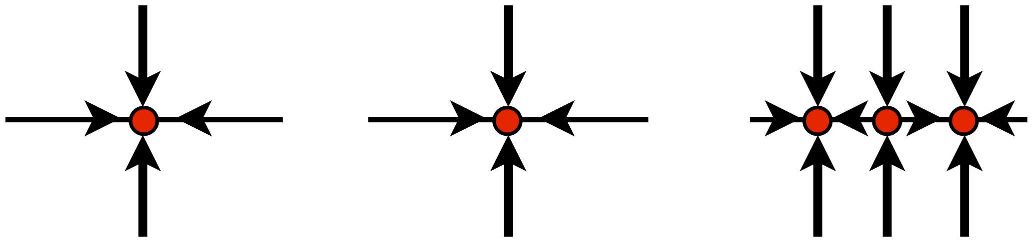
- An example

$$\dot{x} = \mu x - x^3$$
$$\dot{y} = -y$$

$$\dot{x} = \mu x - x^3$$
$$\dot{y} = -y$$
$$\dot{\mu} = 0$$

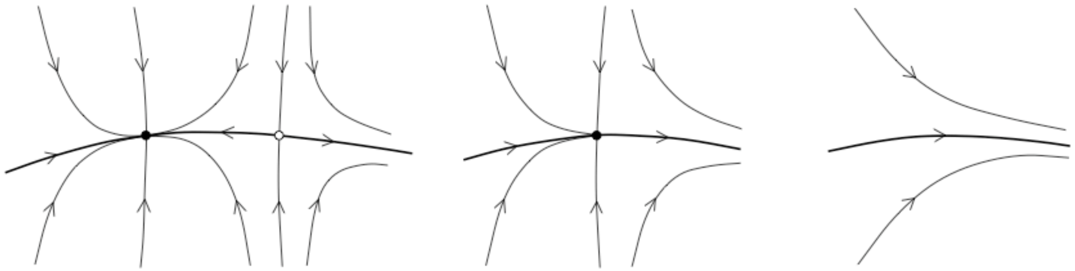
2D Center manifold

$\mu < 0$ $\mu = 0$ $\mu > 0$



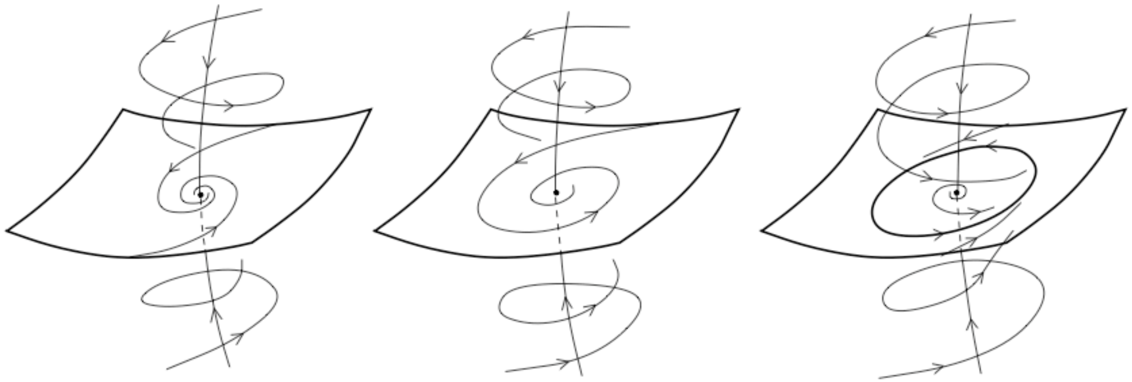
Center manifolds in parameter-dependent systems

- Saddle-node bifurcation (more in Lecture 6)



$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

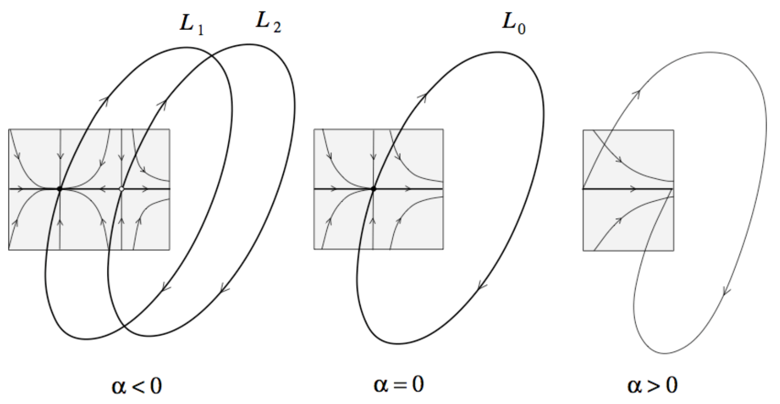
- Hopf bifurcation (more in Lecture 6)



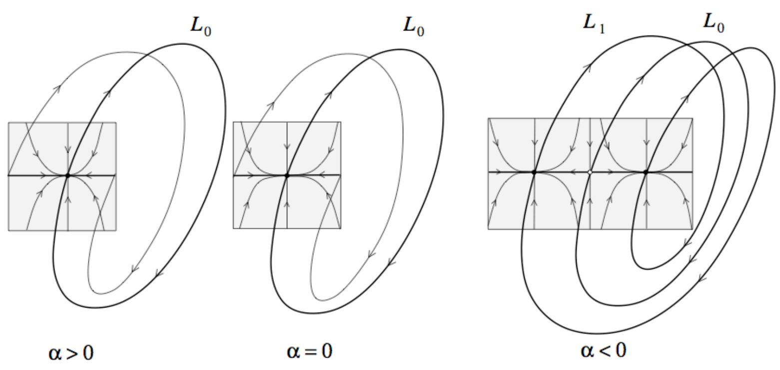
$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

Center manifolds in parameter-dependent systems

- Fold bifurcation of limit cycles (more in Lecture 7)



- Flip bifurcation of limit cycles (more in Lecture 7)



Center manifolds in parameter-dependent systems

- The analysis is the local and nonlinear terms can be reduced to polynomials (from second order up) and polynomial transformation can lead to very simple expressions (normal form theory).
- Generally, a translation of the parameter is also adopted
- **A similar procedure can be applied to the case of discrete systems as well:**

$$y_{c,k+1} = J_c \, y_{c,k} + R_c(h_s(y_{c,k}), h_u(y_{c,k}), y_{c,k})$$

$$\mu_{k+1} = \mu_k$$

ANALYSIS OF LOCAL BIFURCATIONS
NORMAL FORMS

Normal forms

- We have reduced the problem to the center manifold. For example for the continuous case:

$$\dot{y}_c = J_c\, y_c + R_c(h_s(y_c), h_u(y_c), y_c)$$

- We have to deal with the nonlinear part R_C. Depending on it different bifurcation scenario will be encountered.

- We now consider the problem of the classification of all possible local (i.e. near bifurcation boundaries in the parameter space and corresponding critical orbits in the phase space) bifurcation diagrams of generic systems

- For local bifurcations of equilibria and fixed points, universal bifurcation diagrams are provided by **normal forms**.

Normal forms

- **A normal form of a mathematical object, broadly speaking, is a simplified form of the object obtained by applying a transformation (often a change of coordinates) that is considered to preserve the essential features of the object.**
  - For instance, a matrix can be brought into Jordan normal form by applying a similarity transformation.
- Now we consider **normal forms for autonomous systems of differential equations (vector fields or flows) near an equilibrium point**.
  - Similar ideas can be used for discrete-time dynamical systems near a fixed point, or for flows near a periodic orbit.

Normal forms

- The idea of a normal form is to find a polynomial which would be topologically equivalent to a given system around a bifurcation point.
- Questions

1. Can an equivalent polynomial be found, i.e., does it exist?
2. Is the normal form unique?
3. Which properties of the bifurcation determine the minimal degree of such a polynomial?

DEFINITION

Given a bifurcation, a polynomial dynamical system $\dot{x} = f(x, \lambda)$ is called a normal form of the bifurcation at $(\lambda, x) = (\lambda_0, x_0)$ if it satisfies the generic bifurcation conditions, and is topologically equivalent to any system satisfying the same bifurcation conditions

Normal forms

- The method is local: the coordinate transformations are generated in the neighborhood of a known solution
- The coordinate transformation will in general be nonlinear functions of the dependent variables
  - Solution of a series of linear problems
- The structure of the normal form is determined entirely by the nature of the linear part of the vector field

Normal forms

- The first step in the reduction was to obtain the center manifold.
- Our goal is to find an equivalent polynomial system for R_C with as low degree as possible.
- We would like to find a coordinate change $y = z + p_m(z)$
  - where p_{m} is a homogeneous polynomial of degree m. Their coefficient are chosen to eliminate the largest number of nonlinear terms
  - The transformations leave the terms of degree less than m unaltered
  - The transformation is locally a diffeomorphism and thus the tranformed system is topologically equivalent to the original system
- The best one can hope is that the latter equation will be linear.
- **We are at a bifurcation point and the linear part of f has zero real part eigenvalues. At such equilibrium point the linearization problem cannot be solved and there are (nonlinear) resonant terms in f which cannot be removed by coordinate change.**

Normal forms - Technicalities

- On the center manifold $\dot{y}_c = J_c\, y_c + R_c(h_s(y_c), h_u(y_c), y_c)$

$$\dot{y}_c = J_c\, y_c + F_{2,c}(y_c) + F_{3,c}(y_c) + ... + F_{r-1,c}(y_c) + O\left(\|y_c\|^r\right)$$

- To eliminate second order terms one can use the transformation

$$y_c = z + p_2\,(z)$$

$$\dot{y}_c = \dot{z} + Dp_2(z)\dot{z}$$

$$(I + Dp_2\,(z))\,\dot{z} = J_c\, z + J_c\, p_2\,(z) + F_{2,c}(z) + \tilde{F}_{3,c}(y_c) + ... + \tilde{F}_{r-1,c}(y_c) + O\left(\|y_c^r\|\right)$$

$$\dot{z} = (I + Dp_2\,(z))^{-1}\left(J_c\, z + J_c\, p_2\,(z) + F_{2,c}(z) + \tilde{F}_{3,c}(y_c) + ... + \tilde{F}_{r-1,c}(y_c) + O\left(\|y_c^r\|\right)\right)$$

$$(I + Dp_2\,(z))^{-1} = I - Dp_2\,(z) + O\left(\|z\|^2\right)$$

$$\dot{z} = J_c\, z + J_c\, p_2\,(z) - Dp_2\,(z)\, J_c\, z + F_{2,c}(z) + \tilde{F}_{3,c}(z) + ... + \tilde{F}_{r-1,c}(z) + O\left(\|z\|^r\right)$$

$$L_{J_c}^2\,(p_2\,(z)) = -\left(Dp_2\,(z)\, J_c\, z - J_c\, p_2\,(z)\right)$$

- With $-L_{J_c}^2\,(p_2\,(z)) = F_{2,c}(z)$ one could eliminate all second order terms!

Normal forms - Technicalities

- In the end one would obtain a reduced system with all second order terms eliminated or simplified

$$\dot{z} = J_c \, z \ + F^r_{2,c} + \ \tilde{F}_{3,c}(z) + ... + \tilde{F}_{r-1,c}(z) + O\left(\|z\|^r\right)$$

- The same procedure can be applied to third order terms and so on...

Normal forms

- Thus, bifurcations of "common" equilibrium points could be studied by analyzing the behavior of local bifurcations of low dimensional (1D or 2D) systems with nonlinear terms of second or third (when second degree term are nil) degree.
- **Continuous systems**
  - One has to study a 1D system if only one eigenvalue has zero real part, or a 2D system if a conjugate pair of complex eigenvalues has zero real part.
- **Discrete systems**
  - One has to study a 1D system if only one Floquet multiplier has unit magnitude, or a 2D system if a pair of conjugate of complex Floquet multipliers has unit magnitude

Final remarks

- The concept of topological equivalence
- Structural stability: another kind of stability
- Effect of parameter changes
- Bifurcations as passage through structural instability
- Catastrophic bifurcations
- Center manifold theory to describe bifurcation in low dimensions
- Normal forms: a way to classify bifurcations