A note on KAM theory for quasi-linear and fully nonlinear forced KdV

Pietro Baldi, Massimiliano Berti, Riccardo Montalto

Abstract

We present the recent results in [3] concerning quasi-periodic solutions for quasi-linear and fully nonlinear forced perturbations of KdV equations. For Hamiltonian or reversible nonlinearities the solutions are linearly stable. The proofs are based on a combination of different ideas and techniques: (*i*) a Nash-Moser iterative scheme in Sobolev scales. (*ii*) A regularization procedure, which conjugates the linearized operator to a differential operator with constant coefficients plus a bounded remainder. These transformations are obtained by changes of variables induced by diffeomorphisms of the torus and pseudo-differential operators. (*iii*) A reducibility KAM scheme, which completes the reduction to constant coefficients of the linearized operator, providing a sharp asymptotic expansion of the perturbed eigenvalues.

Partial Differential Equations.

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1 Introduction

One of the most challenging and open questions in KAM theory concerns its possible extension to *quasi-linear* and *fully nonlinear* PDEs, namely partial differential equations whose nonlinearities contain derivatives of the same order as the linear operator. Besides its mathematical interest, this question is also relevant in view of applications to physical real world nonlinear models, for example in fluid dynamics and elasticity.

The aim of this Note is to present the recent results in [3] about KAM theory for quasi-periodically forced KdV equations of the form

$$u_t + u_{xxx} + \varepsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx}) = 0\,, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}\,. \qquad (1)$$

To the best of our knowledge, these are the first KAM results for quasi-linear or fully nonlinear PDEs.

KAM and Nash-Moser theory for PDEs, which counts nowadays on a wide literature, started with the pioneering works of Kuksin [19] and Wayne [26], and was developed in the 1990s by Craig-Wayne [12], Bourgain [9], Pöschel [23] (see also [21], [11] for more references). These papers concern wave and Schrödinger equations with bounded Hamiltonian nonlinearities.

The first KAM results for unbounded perturbations have been obtained by Kuksin [20], [21], and, then, Kappeler-Pöschel [17], for Hamiltonian, analytic perturbations of KdV. Here the highest constant coefficients linear operator is ∂_{xxx}

and the nonlinearity contains one space derivative ∂_x. Their approach has been recently improved by Liu-Yuan [22] and Zhang-Gao-Yuan [27] for 1-dimensional derivative NLS (DNLS) and Benjamin-Ono equations, where the highest order constant coefficients linear operator is ∂_{xx} and the nonlinearity contains one derivative ∂_x. These methods apply to dispersive PDEs with derivatives like KdV, DNLS, but not to derivative wave equations (DNLW) which contain first order derivatives ∂_x, ∂_t in the nonlinearity.

For DNLW, KAM theorems have been recently proved by Berti-Biasco-Procesi for both Hamiltonian [7] and reversible [8] equations. The key ingredient is an asymptotic expansion of the perturbed eigenvalues that is sufficiently accurate to impose the second order Melnikov non-resonance conditions. In this way, the scheme produces a constant coefficients normal form around the invariant torus (*reducibility*), implying the linear stability of the solution. This is achieved introducing the notion of "quasi-Töplitz" vector field, which is inspired to "quasi-Töplitz" and "Töplitz-Lipschitz" Hamiltonians, developed, respectively, in Procesi-Xu [24] and Eliasson-Kuksin [13], [14].

Existence of quasi-periodic solutions can also be proved by imposing only the first order Melnikov conditions. This approach has been developed by Bourgain [9], [10] extending the work of Craig-Wayne [12] for periodic solutions. It is especially convenient for PDEs in higher space dimension, because of the high multiplicity of the eigenvalues, see also Berti-Bolle [6]. This method does not provide informations about the stability of the quasi-periodic solutions, because the linearized equations have variable coefficients.

All the aforementioned results concern "semilinear" PDEs, namely equations in which the nonlinearity depends on the unknown and its derivatives up to an order *strictly less* than that one of the linear differential operator. For quasi-linear or fully nonlinear PDEs the perturbative effect is much stronger and the possibility of extending KAM theory in this context is doubtful, see [17], [11], [22], because of the possible phenomenon of formation of singularities outlined in Klainerman and Majda [18]. For example Kappeler-Pöschel [17] (remark 3, page 19) wrote: "*...it would be interesting to obtain perturbation results which also include terms of higher order, at least in the region where the KdV approximation is valid. However, results of this type are still out of reach, if true at all*".

For quasi-linear and fully nonlinear PDEs, the literature concerns, so far, only *periodic* solutions. We quote the classical bifurcation results of Rabinowitz [25] for fully nonlinear forced wave equations with a small dissipation term. More recently, Baldi [1] proved existence of periodic forced vibrations for quasi-linear Kirchhoff equations. Here the quasi-linear perturbation term depends explicitly only on time. Both these results are proved via Nash-Moser methods.

For the water waves equations, which are a fully nonlinear PDE, we mention the pioneering work of Iooss-Plotnikov-Toland [15] about the existence of time periodic standing waves, and of Iooss-Plotinikov [16] for 3-dimensional traveling water waves. The key idea is to use diffeomorphisms of the torus $\mathbb{T}^2$ and pseudo-differential operators, in order to conjugate the linearized operator to a constant coefficients operator plus a sufficiently regularizing remainder. This is enough to invert the whole linearized operator by Neumann series, see remark 2.

Very recently Baldi [2] has further developed the techniques of [15], proving the existence of periodic solutions for fully nonlinear autonomous, reversible Benjamin-Ono equations.

These approaches do not imply the linear stability of the solutions (see comment 2 below) and, unfortunately, they do not work for quasi-periodic solutions, because stronger small divisors difficulties arise (see remark 2).

In [3] we combine different ideas and techniques. The key analysis concerns the linearized KdV operator (15) obtained at any step of the Nash-Moser iteration. First, we use changes of variables, like quasi-periodic time-dependent diffeomorphisms of the space variable x, a quasi-periodic reparametrization of time, multiplication operators and Fourier multipliers, in order to reduce the linearized operator to constant coefficients up to a bounded remainder (see (21)). These transformations, which are inspired to [2], [15], are very different from the usual KAM transformations. Then we perform a quadratic KAM reducibility scheme *à la* Eliasson-Kuksin, which completely diagonalizes the linearized operator. For reversible or Hamiltonian KdV perturbations we get that the eigenvalues of this diagonal operator are purely imaginary, i.e. we prove the linear stability. In section 3 we present the main ideas of the proof in more details.

We remark that the present approach could be also applied to quasi-linear and fully nonlinear perturbations of dispersive PDEs like 1-dimensional NLS and Benjamin-Ono equations (but not to the wave equation, which is not dispersive).

In order to highlight the main ideas, we have considered in [3] the simplest setting of nonlinear perturbations of the Airy-KdV operator $\partial_t + \partial_{xxx}$ and we look for small amplitude solutions.

2 Main results

We consider equation (1) where $\varepsilon > 0$ is a small parameter, the nonlinearity is quasi-periodic in time with Diophantine frequency vector

$$\omega = \lambda \bar{\omega} \in \mathbb{R}^\nu \,, \quad \lambda \in \Lambda := \Big[\frac{1}{2}, \frac{3}{2}\Big], \quad |\bar{\omega} \cdot l| \geq \frac{3\gamma_0}{|l|^{\tau_0}} \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\}, \tag{2}$$

and $f(\varphi, x, z)$, $\varphi \in \mathbb{T}^\nu$, $z := (z_0, z_1, z_2, z_3) \in \mathbb{R}^4$, is a finitely many times differentiable function, namely

$$f \in C^q(\mathbb{T}^\nu \times \mathbb{T} \times \mathbb{R}^4; \mathbb{R}) \tag{3}$$

for some $q \in \mathbb{N}$ large enough. For simplicity we fix in (2) the diophantine exponent $\tau_0 := \nu$. The only "external" parameter in (1) is λ, which is the length of the frequency vector (this corresponds to a time scaling).

We consider the following questions:

- *For ε small enough, do there exist quasi-periodic solutions of* (1) *for positive measure sets of $\lambda \in \Lambda$?*
- *Are these solutions linearly stable?*

Clearly, if $f(\varphi, x, 0)$ is not identically zero, then $u = 0$ is not a solution of (1) for $\varepsilon \neq 0$. Thus we look for non-trivial $(2\pi)^{\nu+1}$-periodic solutions $u(\varphi, x)$ of

$$\omega \cdot \partial_\varphi u + u_{xxx} + \varepsilon f(\varphi, x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{4}$$

in the Sobolev space

$$H^s := H^s(\mathbb{T}^\nu \times \mathbb{T}; \mathbb{R}) := \Big\{ u(\varphi, x) = \sum_{(l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}} u_{l,j}\, e^{\mathrm{i}(l\cdot\varphi + jx)} \in \mathbb{R}, \quad \bar{u}_{l,j} = u_{-l,-j}\,,$$
$$\|u\|_s^2 := \sum_{(l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}} (\max\{1, |l|, |j|\})^{2s} |u_{l,j}|^2 < \infty \Big\}\,.$$

From now on, we fix $\mathfrak{s}_0 := (\nu+2)/2 > (\nu+1)/2$, so that for all $s \geq \mathfrak{s}_0$ the Sobolev space H^s is a Banach algebra, and it is continuously embedded $H^s(\mathbb{T}^{\nu+1}) \hookrightarrow C(\mathbb{T}^{\nu+1})$.

We need some assumptions on the nonlinearity. We first consider *quasi-linear* perturbations satisfying

- Type (Q)

$$\partial^2_{z_3 z_3} f = 0, \quad \partial_{z_2} f = \alpha(\varphi)\Big(\partial^2_{z_3 x} f + z_1 \partial^2_{z_3 z_0} f + z_2 \partial^2_{z_3 z_1} f + z_3 \partial^2_{z_3 z_2} f\Big) \qquad (5)$$

  for some function $\alpha(\varphi)$ (independent on x).

We note that every Hamiltonian nonlinearity, see (9), satisfies (Q) with $\alpha(\varphi) = 2$. In step 3 in section 3 we explain the reason for assuming condition (Q).

Theorem 1. (Existence) *There exist $s := s(\nu) > 0$, $q := q(\nu) \in \mathbb{N}$, such that: For every quasi-linear nonlinearity $f \in C^q$ of the form*

$$f = \partial_x\big(g(\omega t, x, u, u_x, u_{xx})\big) \qquad (6)$$

satisfying the (Q)-condition (5), for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \varepsilon_0(f, \nu)$ is small enough, there exists a Cantor set $\mathcal{C}_\varepsilon \subset \Lambda$ of asymptotically full Lebesgue measure, i.e.

$$|\mathcal{C}_\varepsilon| \to 1 \quad \text{as} \quad \varepsilon \to 0, \qquad (7)$$

such that, $\forall \lambda \in \mathcal{C}_\varepsilon$ the perturbed KdV equation (4) has a solution $u(\varepsilon, \lambda) \in H^s$ with $\|u(\varepsilon, \lambda)\|_s \to 0$ as $\varepsilon \to 0$.

We may ensure the *linear stability* of the solutions requiring further conditions on the nonlinearity, see Theorem 5 for the precise statement. The first case concerns *Hamiltonian* KdV equations

$$u_t = \partial_x \nabla_{L^2} H(t, x, u, u_x)\,, \quad H(t, x, u, u_x) := \int_{\mathbb{T}} \frac{u_x^2}{2} + \varepsilon F(\omega t, x, u, u_x)\, dx\,, \qquad (8)$$

which have the form (1), (6) with

$$f(\varphi, x, u, u_x, u_{xx}, u_{xxx}) = \partial_x\Big(\partial_x\big\{(\partial_{z_1} F)(\varphi, x, u, u_x)\big\} - (\partial_{z_0} F)(\varphi, x, u, u_x)\Big). \qquad (9)$$

The phase space of (8) is

$$H^1_0(\mathbb{T}) := \Big\{ u(x) \in H^1(\mathbb{T}, \mathbb{R}) \,:\, \int_{\mathbb{T}} u(x)\, dx = 0 \Big\}$$

endowed with the non-degenerate symplectic form

$$\Omega(u, v) := \int_{\mathbb{T}} (\partial_x^{-1} u) v\, dx\,, \quad u, v \in H^1_0(\mathbb{T})\,, \qquad (10)$$

where $\partial_x^{-1} u$ is the periodic primitive of u with zero average, namely

$$\partial_x^{-1} e^{\mathrm{i}jx} := \frac{e^{\mathrm{i}jx}}{\mathrm{i}j} \quad \forall j \in \mathbb{Z} \setminus \{0\}, \qquad \partial_x^{-1} 1 = 0.$$

The Hamiltonian nonlinearity f in (9) satisfies both (6) and (5). As a consequence, Theorem 1 implies the existence of quasi-periodic solutions of (8). In addition, exploiting the symplectic structure, we also prove their linear stability.

Theorem 2. (Hamiltonian KdV) *For all Hamiltonian quasi-linear KdV equations* (8) *the quasi-periodic solution $u(\varepsilon, \lambda)$ found in Theorem 1 is* LINEARLY STABLE *(see Theorem 5).*

The stability of the quasi-periodic solutions also follows by the *reversibility* condition

$$f(-\varphi, -x, z_0, -z_1, z_2, -z_3) = -f(\varphi, x, z_0, z_1, z_2, z_3). \tag{11}$$

Condition (11) implies that the infinite-dimensional non-autonomous dynamical system

$$u_t = V(t, u), \quad V(t, u) := -u_{xxx} - \varepsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx})$$

is reversible with respect to the involution

$$S : u(x) \to u(-x), \quad S^2 = I,$$

namely

$$-SV(-t, u) = V(t, Su)\,.$$

In this case it is natural to look for "reversible" solutions of (4), namely

$$u(\varphi, x) = u(-\varphi, -x)\,. \tag{12}$$

In this case we also consider *fully nonlinear* perturbations f which may depend on u_{xxx} in a nonlinear way. We assume that

- TYPE (F)
$$\partial_{z_2} f = 0, \tag{13}$$

namely f is independent of u_{xx}, see step 3 in section 3.

Theorem 3. (Reversible KdV) *There exist $s := s(\nu) > 0$, $q := q(\nu) \in \mathbb{N}$, such that: for every nonlinearity $f \in C^q$ that satisfies*

(i) *the reversibility condition* (11)*,*

and

(ii) *either the (F)-condition* (13) *or the (Q)-condition* (5)*,*

for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \varepsilon_0(f, \nu)$ is small enough, there exists a Cantor set $\mathcal{C}_\varepsilon \subset \Lambda$ with Lebesgue measure satisfying (7)*, such that for all $\lambda \in \mathcal{C}_\varepsilon$ the perturbed KdV equation* (4) *has a solution $u(\varepsilon, \lambda) \in H^s$ that satisfies* (12)*, with $\|u(\varepsilon, \lambda)\|_s \to 0$ as $\varepsilon \to 0$. In addition, $u(\varepsilon, \lambda)$ is* LINEARLY STABLE.

Let us make some comments on the results.

1. — The previous theorems (in particular the Hamiltonian Theorem 2) give a positive answer to the question posed by Kappeler-Pöschel [17], page 19, Remark 3, about the possibility of KAM type results for quasi-linear perturbations of KdV.

2. — In Theorem 1 we do not have informations about the linear stability of the solutions because the nonlinearity f has no special structure and it may happen that some eigenvalues of the linearized operator have non zero real part (partially hyperbolic tori). We remark that, in any case, the approach of [3] allows to compute the eigenvalues (i.e. Lyapunov-exponents) of the linearized operator with any order of accuracy. With further conditions on the nonlinearity—like reversibility or in the Hamiltonian case—the eigenvalues are purely imaginary, and the torus is linearly stable. The present situation is very different with respect to [12], [10], [6] and also [15]-[16], [2], where the lack of stability informations is due to the fact that the linearized equation has variable coefficients, and it is not reduced as in Theorem 4 below.

3. — One cannot expect the existence of quasi-periodic solutions of (4) for *any* perturbation f. Actually, if $f = m \neq 0$ is a constant, then, integrating (4) in (φ, x) we find the contradiction $\varepsilon m = 0$. This is a consequence of the fact that

$$\mathrm{Ker}(\omega \cdot \partial_\varphi + \partial_{xxx}) = \mathbb{R} \tag{14}$$

is non trivial. Both the condition (6) (which is satisfied by the Hamiltonian nonlinearities) and the reversibility condition (11) allow to overcome this obstruction, working in a space of functions with zero average. The degeneracy (14) also reflects in the fact that the solutions of (4) appear as a 1-dimensional family $c + u_c(\varepsilon, \lambda)$ parametrized by the "average" $c \in \mathbb{R}$. We could also avoid this degeneracy by adding a "mass" term $+mu$ in (1), but it does not seem to have physical meaning.

4. — In Theorem 1 we have not considered the case in which f is fully nonlinear and satisfies condition (F) in (13), because any nonlinearity of the form (6) is automatically quasi-linear (and so the first condition in (5) holds) and (13) trivially implies the second condition in (5) with $\alpha(\varphi) = 0$.

5. — The solutions $u \in H^s$ have the same regularity in both variables (φ, x). The main reason is that the compositions operators that we use in the first (and fourth) step of the reduction procedure (see section 3) mix the time and space variables.

6. — In the Hamiltonian case (8), the nonlinearity f in (9) satisfies the reversibility condition (11) if and only if $F(-\varphi, -x, z_0, -z_1) = F(\varphi, x, z_0, z_1)$.

Theorems 1-3 are based on a Nash-Moser iterative scheme, as developed in [5]. An essential ingredient in the proof—which also implies the linear stability of the quasi-periodic solutions—is the *reducibility* of the linear operator

$$\mathcal{L} := \mathcal{L}(u) = \omega \cdot \partial_\varphi + (1 + a_3(\varphi, x))\partial_{xxx} + a_2(\varphi, x)\partial_{xx} + a_1(\varphi, x)\partial_x + a_0(\varphi, x) \tag{15}$$

obtained linearizing (4) at any approximate (or exact) solution u. The coefficients $a_i = a_i(\varphi, x) = a_i(u, \varepsilon)(\varphi, x)$ are periodic functions of (φ, x), depending on u, ε, obtained from the partial derivatives of $\varepsilon f(\varphi, x, z_0, z_1, z_2, z_3)$ as

$$a_i(\varphi, x) = \varepsilon(\partial_{z_i} f)\big(\varphi, x, u(\varphi, x), u_x(\varphi, x), u_{xx}(\varphi, x), u_{xxx}(\varphi, x)\big)\,. \tag{16}$$

Let $H^s_x := H^s(\mathbb{T})$ denote the usual Sobolev spaces of functions of $x \in \mathbb{T}$ only (phase space).

Theorem 4. (Reducibility) *There exist $\bar\sigma > 0$, $q \in \mathbb{N}$, depending on ν, such that:*

For every nonlinearity $f \in C^q$ that satisfies the hypotheses of Theorems 1 or 3, for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \varepsilon_0(f, \nu)$ is small enough, for all u in the ball $\|u\|_{\mathfrak{s}_0+\bar\sigma} \le 1$, there exists a Cantor like set $\Lambda_\infty(u) \subset \Lambda$ such that, for all $\lambda \in \Lambda_\infty(u)$:

i) for all $s \in (\mathfrak{s}_0, q - \bar\sigma)$, if $\|u\|_{s+\bar\sigma} < +\infty$ then there exist linear invertible bounded operators W_1, $W_2 : H^s(\mathbb{T}^{\nu+1}) \to H^s(\mathbb{T}^{\nu+1})$ with bounded inverse, that semi-conjugate the linear operator $\mathcal{L}(u)$ in (15) to the diagonal operator $\mathcal{L}_\infty$, namely

$$\mathcal{L}(u) = W_1 \mathcal{L}_\infty W_2^{-1}\,, \quad \mathcal{L}_\infty := \omega \cdot \partial_\varphi + \mathcal{D}_\infty \tag{17}$$

where

$$\mathcal{D}_\infty := \mathrm{diag}_{j\in\mathbb{Z}}\{\mu_j\}$$

and

$$\mu_j := \mathrm{i}(-m_3 j^3 + m_1 j) + r_j\,, \quad m_3, m_1 \in \mathbb{R}\,, \quad \sup_j |r_j| \le C\varepsilon\,. \tag{18}$$

ii) For each fixed $\varphi \in \mathbb{T}^\nu$, the operators $W_i(\varphi)$, defined by setting

$$\big(W_i(\varphi)h\big)(x) := (W_i h)(\varphi, x) \quad \forall h = h(x) \in H^s_x,$$

are also bounded linear bijections of the phase space H^s_x,

$$W_i(\varphi)\,, W_i^{-1}(\varphi) : H^s_x \to H^s_x\,, \quad i = 1, 2\,.$$

A curve $h(t) = h(t, \cdot) \in H^s_x$ is a solution of the quasi-periodically forced linear KdV equation

$$\partial_t h + (1 + a_3(\omega t, x))\partial_{xxx} h + a_2(\omega t, x)\partial_{xx} h + a_1(\omega t, x)\partial_x h + a_0(\omega t, x) h = 0 \tag{19}$$

if and only if the transformed curve

$$v(t) := v(t, \cdot) := W_2^{-1}(\omega t)[h(t)] \in H^s_x$$

is a solution of the constant coefficients dynamical system

$$\partial_t v + \mathcal{D}_\infty v = 0\,, \quad \dot v_j = -\mu_j v_j\,, \quad \forall j \in \mathbb{Z}\,. \tag{20}$$

In the reversible or Hamiltonian case all the μ_j are purely imaginary.

The exponents μ_j can be effectively computed. All the solutions of (20) are

$$v(t) = \sum_{j\in\mathbb{Z}} v_j(t) e^{\mathrm{i}jx}\,, \quad v_j(t) = e^{-\mu_j t} v_j(0)\,.$$

If the μ_j are purely imaginary—as in the reversible or the Hamiltonian cases—all the solutions of (20) are almost periodic in time (in general) and the Sobolev norm

$$\|v(t)\|_{H^s_x} = \Big(\sum_{j\in\mathbb{Z}} |v_j(t)|^2 \langle j\rangle^{2s}\Big)^{1/2} = \Big(\sum_{j\in\mathbb{Z}} |v_j(0)|^2 \langle j\rangle^{2s}\Big)^{1/2} = \|v(0)\|_{H^s_x}$$

is constant in time. As a consequence we have:

Theorem 5. (Linear stability) *Assume the hypothesis of Theorem 4 and, in addition, that f is Hamiltonian (see* (9)*) or it satisfies the reversibility condition* (11)*. Then, $\forall s \in (\mathfrak{s}_0, q - \bar{\sigma} - \mathfrak{s}_0)$, $\|u\|_{s+\mathfrak{s}_0+\bar{\sigma}} < +\infty$, there exists $K_0 > 0$ such that for all $\lambda \in \Lambda_\infty(u)$, $\varepsilon \in (0, \varepsilon_0)$, all the solutions of* (19) *satisfy*

$$\|h(t)\|_{H^s_x} \le K_0 \|h(0)\|_{H^s_x}$$

and, for some $\alpha \in (0,1)$,

$$\|h(0)\|_{H^s_x} - \varepsilon^\alpha K_0 \|h(0)\|_{H^{s+1}_x} \le \|h(t)\|_{H^s_x} \le \|h(0)\|_{H^s_x} + \varepsilon^\alpha K_0 \|h(0)\|_{H^{s+1}_x} .$$

3 Ideas of the proof

The proofs are based on a Nash-Moser iterative scheme in the Sobolev spaces H^s. The main issue concerns the invertibility of the linearized KdV operator $\mathcal{L}$ in (15), at each step of the iteration, and the proof of tame estimates for its right inverse $\mathcal{L}^{-1}$. These informations are obtained by conjugating $\mathcal{L}$ to constant coefficients.

We now explain the main ideas of the reducibility scheme. The term of $\mathcal{L}$ that produces the strongest perturbative effects to the spectrum (and eigenfunctions) is $a_3(\varphi, x)\partial_{xxx}$, and, then, $a_2(\varphi, x)\partial_{xx}$. The usual KAM transformations are not able to deal with these terms because they are "too close" to the identity. Our strategy is the following. First, we conjugate the operator $\mathcal{L}$ in (15) to a constant coefficients third order differential operator plus a zero order remainder

$$\mathcal{L}_5 = \omega \cdot \partial_\varphi + m_3 \partial_{xxx} + m_1 \partial_x + \mathcal{R}_0, \tag{21}$$

where $m_1, m_3 \in \mathbb{R}$, $m_3 = 1 + O(\varepsilon)$, $m_1 = O(\varepsilon)$. We use changes of variables induced by diffeomorphisms of the torus, reparametrization of time, and pseudo-differential operators, that we now shortly present.

1. — The first step is to eliminate the space variable dependence of the highest order perturbation $a_3(\varphi, x)\partial_{xxx}$. We use a φ-dependent change of variable of the form

$$(\mathcal{A}h)(\varphi, x) := h(\varphi, x + \beta(\varphi, x)) .$$

Note that $\mathcal{A}$ converges pointwise to the identity if $\beta \to 0$, but it does not converge in operatorial norm. Choosing β such that

$$\big(1 + a_3(\varphi, x)\big)\big(1 + \beta_x(\varphi, x)\big)^3 = b_3(\varphi) = \text{independent on } x, \tag{22}$$

the transformation $\mathcal{A}$ conjugates $\mathcal{L}$ to

$$\mathcal{L}_1 := \mathcal{A}^{-1}\mathcal{L}\mathcal{A} = \omega \cdot \partial_\varphi + b_3(\varphi)\partial_{yyy} + b_2(\varphi, y)\partial_{yy} + b_1(\varphi, y)\partial_y + b_0(\varphi, y) .$$

For β odd, $\mathcal{A}$ preserves the reversible structure.

For the Hamiltonian KdV (8) we use instead the modified transformation

$$(\mathcal{A}h)(\varphi, x) := (1 + \beta_x(\varphi, x))\, h(\varphi, x + \beta(\varphi, x)) \tag{23}$$

which is symplectic, namely, for each $\varphi \in \mathbb{T}^\nu$,

$$\Omega\big(\mathcal{A}(\varphi)h,\, \mathcal{A}(\varphi)v\big) = \Omega(h, v) \quad \forall h, v \in H^1_0,$$

where

$$(\mathcal{A}(\varphi)h)(x) := (1+\beta_x(\varphi,x))\, h(x+\beta(\varphi,x))\,, \quad \forall h \in H_0^1(\mathbb{T})\,.$$

Hence (23) preserves the Hamiltonian structure, namely the corresponding conjugated operator $\mathcal{L}_1$ is still Hamiltonian. Choosing β as in (22), the coefficient $b_3(\varphi)$ is the same as above, and, moreover, $b_2(\varphi,y) = 2\partial_y b_3(\varphi) = 0$.

2. — In the second step we eliminate the time dependence of the coefficient of ∂_{yyy} by a quasi-periodic time re-parametrization

$$(Bh)(\varphi,y) := h\big(\varphi+\omega\alpha(\varphi),\, y\big)\,, \quad \varphi \in \mathbb{T}^\nu, \quad \alpha(\varphi) \in \mathbb{R}\,.$$

Calling the new angle $\vartheta := \varphi + \omega\alpha(\varphi)$, we choose α so that

$$B^{-1}\mathcal{L}_1 B = \rho\,\mathcal{L}_2, \quad \mathcal{L}_2 := \omega\cdot\partial_\vartheta + m_3\,\partial_{yyy} + c_2(\vartheta,y)\,\partial_{yy} + c_1(\vartheta,y)\,\partial_y + c_0(\vartheta,y)$$

where $m_3 \in \mathbb{R}$ and $\rho(\varphi)$ is close to 1. This transformation preserves the reversible and the Hamiltonian structure.

3. — The next goal is to eliminate the term $c_2(\vartheta,y)\partial_{yy}$ obtaining an operator of the form

$$\mathcal{L}_3 := \mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \omega\cdot\partial_\vartheta + m_3\partial_{yyy} + d_1(\vartheta,y)\partial_y + d_0(\vartheta,y)\,.$$

This is achieved by a conjugation with a multiplication operators $\mathcal{M}$, assuming condition (Q) (see (5)) or (F) (see (13)). Indeed, after a computation, it turns out that the second order term is zero if

$$\int_{\mathbb{T}} \frac{a_2(\varphi,x)}{1+a_3(\varphi,x)}\, dx = 0\,. \tag{24}$$

If (F) holds, then the coefficient $a_2(\varphi,x) = 0$, and (24) is satisfied. If (Q) holds, then $a_2(\varphi,x) = \alpha(\varphi)\,\partial_x a_3(\varphi,x)$, and so

$$\int_{\mathbb{T}} \frac{a_2(\varphi,x)}{1+a_3(\varphi,x)}\, dx = \int_{\mathbb{T}} \alpha(\varphi)\,\partial_x\big(\log[1+a_3(\varphi,x)]\big)\, dx = 0\,.$$

In both cases (Q) and (F), condition (24) is satisfied.

We remark that, in the Hamiltonian case, this step is not needed because the term $c_2\partial_{yy}$ has already been eliminated (namely $b_2 \equiv 0$, see comment 7).

Remark 1. *Without assumptions (Q) or (F), we can always reduce $\mathcal{L}$ to a time dependent operator*

$$\mathcal{L}_3 = \omega\cdot\partial_\vartheta + m_3\partial_{yyy} + d_2(\vartheta)\partial_{yy} + d_1(\vartheta,y)\partial_y + d_0(\vartheta,y)\,.$$

If $d_2(\vartheta)$ were a constant, then this term would even simplify the analysis, killing the small divisors. The pathological situation that we want to eliminate assuming (Q) or (F) is when $d_2(\vartheta)$ changes sign. In such a case this term acts as a friction when $d_2(\vartheta) < 0$ and as an amplifier when $d_2(\vartheta) > 0$.

4. — Finally, in order obtain (21), we conjugate $\mathcal{L}_3$ via a translation of the space variable $\mathcal{T}h(\varphi,x) := h(\varphi,x+p(\varphi))$ (renaming the variables $\varphi := \vartheta$, $x := y$), and a transformation of the form

$$\mathcal{S} = I + w(\varphi,x)\partial_x^{-1}\,.$$

In the Hamiltonian case, we use the symplectic map

$$\mathcal{S} = \exp\{\pi_0 w(\varphi,x)\partial_x^{-1}\} = I + \pi_0 w(\varphi,x)\partial_x^{-1} + O(w^2\partial_x^{-2})$$

where π_0 is the projection $\pi_0 := \partial_x\partial_x^{-1}$ on $H^1_0(\mathbb{T})$, namely $\pi_0 e^{\mathrm{i}jx} = e^{\mathrm{i}jx}$ for $j \neq 0$, and $\pi_0 1 = 0$.

Remark 2. *We could iterate the regularization procedure at any finite order $k = 0, 1, \ldots$, conjugating $\mathcal{L}$ to an operator of the form $\mathfrak{D} + \mathcal{R}$, where*

$$\mathfrak{D} = \omega\cdot\partial_\varphi + \mathcal{D}, \quad \mathcal{D} = m_3\partial_x^3 + m_1\partial_x + \ldots + m_{-k}\partial_x^{-k}\,, \quad m_i \in \mathbb{R}\,,$$

has constant coefficients, and the rest $\mathcal{R}$ is arbitrarily regularizing in space, namely

$$\partial_x^k \circ \mathcal{R} = \textit{bounded}. \tag{25}$$

One cannot iterate this regularization infinitely many times, because it is not a quadratic scheme, and therefore, because of the small divisors, it does not converge. This regularization procedure is sufficient to prove the invertibility of $\mathcal{L}$, giving tame estimates for the inverse, in the periodic case, but it does not work for quasi-periodic solutions. In order to use Neumann series, one needs that $\mathfrak{D}^{-1}\mathcal{R} = (\mathfrak{D}^{-1}\partial_x^{-k})(\partial_x^k\mathcal{R})$ is bounded, namely, in view of (25), that $\mathfrak{D}^{-1}\partial_x^{-k}$ is bounded. In the region where the eigenvalues $(\mathrm{i}\omega\cdot l + \mathcal{D}_j)$ of $\mathfrak{D}$ are small, space and time derivatives are related, $|\omega\cdot l| \sim |j|^3$, where l is the Fourier index of time, j is that of space, and $\mathcal{D}_j = -\mathrm{i}m_3 j^3 + \mathrm{i}m_1 j + \ldots$ are the eigenvalues of $\mathcal{D}$. Imposing the first order Melnikov conditions $|\mathrm{i}\omega\cdot l + \mathcal{D}_j| > \gamma|l|^{-\tau}$, in that region $(\mathfrak{D}^{-1}\partial_x^{-k})$ has eigenvalues

$$\Big|\frac{1}{(\mathrm{i}\omega\cdot l + \mathcal{D}_j)j^k}\Big| < \frac{|l|^\tau}{\gamma|j|^k} < \frac{C|l|^\tau}{|\omega\cdot l|^{k/3}}\,.$$

In the periodic case, $\omega\in\mathbb{R}$, $l\in\mathbb{Z}$, $|\omega\cdot l| = |\omega||l|$, and this determines the order of regularization that is required by the procedure: $k \geq 3\tau$. In the quasi-periodic case, instead, $|l|$ is not controlled by $|\omega\cdot l|$, and the argument fails.

5. — Once (21) has been obtained, we implement a quadratic reducibility KAM scheme to diagonalize $\mathcal{L}_5$, namely to conjugate $\mathcal{L}_5$ to the diagonal operator $\mathcal{L}_\infty$ in (17). Since we work with finite regularity we perform a Nash-Moser smoothing regularization (in time). In order to decrease the size of the perturbation $\mathcal{R}$ at each step, we use standard KAM transformations of the form

$$\Phi = I + \Psi\,, \quad \Phi = e^\Psi \text{ in the Hamiltonian case}\,.$$

If Ψ is a solution of the *homological* equation

$$\omega\cdot\partial_\varphi\Psi + [\mathcal{D},\Psi] + \Pi_N\mathcal{R} = [\mathcal{R}] \qquad \text{where} \qquad [\mathcal{R}] := \mathrm{diag}_{j\in\mathbb{Z}}\mathcal{R}_j^j(0) \tag{26}$$

and Π_N is the time-Fourier truncation operator, then

$$\mathcal{L}_+ := \Phi^{-1}\mathcal{L}\Phi = \omega\cdot\partial_\varphi + \mathcal{D}_+ + \mathcal{R}_+\,,$$

where

$$\mathcal{D}_+ := \mathcal{D} + [\mathcal{R}]\,, \quad \mathcal{R}_+ := \Phi^{-1}\Big(\Pi_N^\perp\mathcal{R} + \mathcal{R}\Psi - \Psi[\mathcal{R}]\Big).$$

Note that $\mathcal{L}_+$ has the same form of $\mathcal{L}$, but the remainder $\mathcal{R}_+$ is the sum of a quadratic function of $\Psi, \mathcal{R}$ and a remainder supported on high modes.

This iterative scheme converges because the initial remainder $\mathcal{R}_0$ in (21) is a bounded operator (of the space variable x) and this property is preserved, along the iteration, passing from $\mathcal{R}$ to $\mathcal{R}_+$. This is the reason why we have performed the regularization procedure in steps 1-4 above, before starting with the KAM reducibility scheme. The homological equation (26) may be solved imposing the second order Melnikov non-resonance conditions

$$|\mathrm{i}\omega \cdot l + \mu_j(\lambda) - \mu_k(\lambda)| \geq \frac{\gamma |j^3 - k^3|}{\langle l \rangle^\tau} \,, \quad \forall l \in \mathbb{Z}^\nu \,, |l| \leq N \,, \, j, k \in \mathbb{Z} \,,$$

where $\mu_j(\lambda)$ are the eigenvalues of the diagonal operator $\mathcal{D}$. We may verify that for most parameters $\lambda \in [1/2, 3/2]$ these conditions are verified thanks to the sharp control of the eigenvalues $\mu_j(\lambda) := -\mathrm{i} m_3(\varepsilon,\lambda) j^3 + \mathrm{i} m_1(\varepsilon,\lambda) j + r_j(\varepsilon,\lambda)$ where $\sup_j |r_j(\varepsilon,\lambda)| = O(\varepsilon)$.

Note that the eigenvalues μ_j could be not purely imaginary, i.e. r_j could have a non-zero real part which depends on the nonlinearity (unlike the reversible or Hamiltonian case, where $r_j \in \mathrm{i}\mathbb{R}$). In such a case, the invariant torus could be (partially) hyperbolic. Since we do not control the real part of r_j (i.e. the hyperbolicity may vanish), we perform the measure estimates proving the diophantine lower bounds of the imaginary part of the small divisors.

All the above transformations, both those of the regularization procedure and those of the KAM reducibility scheme, are also quasi-periodically time-dependent families of transformations of the phase space (of functions of x only), namely they are "Töplitz in time". For this reason we deduce the dynamical consequence of Theorem 4-*ii*) concerning *all* the solutions of (19) and, therefore, Theorem 5.

We note that the transformations used in [15] (as well as those of [10], [6]) have not the Töplitz-in-time structure. This is another reason (in addition to comment 2) for which stability informations are not obtained in [15], [10], [6].

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Pietro Baldi, Massimiliano Berti, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi Napoli Federico II, Via Cintia, Monte S. Angelo, I-80126, Napoli, Italy. `pietro.baldi@unina.it`, `m.berti@unina.it`.

Riccardo Montalto, SISSA, Via Bonomea 265, 34136, Trieste, Italy.
`riccardo.montalto@sissa.it`.

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