Periodic solutions of wave equations for asymptotically full measure sets of frequencies

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1 Introduction

The aim of this Note is to prove existence and multiplicity of small amplitude periodic solutions of the completely resonant wave equation

\[
\begin{align*}
\Box u + f(x, u) &= 0 \\
u(t, 0) &= u(t, \pi) = 0
\end{align*}
\]

where \(\Box := \partial_{tt} - \partial_{xx}\) is the D’Alambertian operator and

\[
f(x, u) = a_2 u^2 + a_3(x) u^3 + O(u^4) \quad \text{or} \quad f(x, u) = a_4 u^4 + O(u^5)
\]

for a Cantor-like set of frequencies \(\omega\) of asymptotically full measure at \(\omega = 1\).

Equation (1) is called completely resonant because any solution \(v = \sum_{j \geq 1} a_j \cos(jt + \vartheta_j) \sin(jx)\) of the linearized equation at \(u = 0\)

\[
\begin{align*}
u_{tt} - \nu_{xx} &= 0 \\
u(t, 0) &= u(t, \pi) = 0
\end{align*}
\]

is \(2\pi\)-periodic in time.

Existence and multiplicity of periodic solutions of completely resonant wave equations had been proved for a zero measure, uncountable Cantor set of frequencies in [4] for \(f(u) = u^3 + O(u^5)\) and in [5]-[6] for any nonlinearity \(f(u) = a_p u^p + O(u^{p+1})\), \(p \geq 2\).

Existence of periodic solutions for a Cantor-like set of frequencies of asymptotically full measure has been recently proved in [7] where, due to the well known “small divisor difficulty”, the “0th order bifurcation equation” is required to possess non-degenerate periodic solutions. Such property was verified in [7] for nonlinearities like \(f = a_2 u^2 + O(u^4)\), \(f = a_3(x) u^3 + O(u^4)\). See also [11] for \(f = u^3 + O(u^5)\).

In this Note we shall prove that, for quadratic, cubic and quartic nonlinearities \(f(x, u)\) like in (2), the corresponding 0th order bifurcation equation possesses non-degenerate periodic solutions – Propositions [11] and [12] –, implying, by the results of [7], Theorem [11] and Corollary [11] below.

We remark that our proof is purely analytic (it does not use numerical calculations) being based on the analysis of the variational equation and exploiting properties of the Jacobi elliptic functions.

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1.1 Main results

Normalizing the period to $2\pi$, we look for solutions of

\[
\begin{align*}
\omega^2 u_{tt} - u_{xx} + f(x, u) &= 0 \\
u(t, 0) &= u(t, \pi) = 0
\end{align*}
\]

in the Hilbert algebra (for $s > 1/2$, $\sigma > 0$)

\[
X_{\sigma,s} := \left\{ u(t, x) = \sum_{l \geq 0} \cos(l t)\ u_l(x) \mid u_l \in H^1_0((0, \pi), \mathbb{R}) \ \forall l \in \mathbb{N} \text{ and } \right. \\
\left. ||u||^2_{\sigma,s} := \sum_{l \geq 0} \exp(2\sigma l)(l^2 + 1)||u_l||^2_{H^1_0} < +\infty \right\}.
\]

It is natural to look for solutions which are even in time because equation (1) is reversible. We look as well for solutions of (1) in the subalgebras

\[
X_{\sigma,s,n} := \left\{ u \in X_{\sigma,s} \mid u \text{ is } \frac{2\pi}{n}\text{-periodic} \right\} \subset X_{\sigma,s}, \ n \in \mathbb{N}
\]

(they are particular $2\pi$-periodic solutions).

The space of the solutions of the linear equation (2) that belong to $H^1_0(\mathbb{T} \times (0, \pi), \mathbb{R})$ and are even in time is

\[
V := \left\{ v(t, x) = \sum_{l \geq 1} \cos(l t)\sin(l x) \mid u_l \in \mathbb{R}, \ \sum_{l \geq 1} l^2|u_l|^2 < +\infty \right\} \\
= \left\{ v(t, x) = \eta(t + x) - \eta(t - x) \mid \eta \in H^1(\mathbb{T}, \mathbb{R}) \text{ with } \eta \text{ odd} \right\}.
\]

**Theorem 1.** Let

\[
f(x, u) = a_2 u^2 + a_3(x)u^3 + \sum_{k \geq 4} a_k(x)u^k
\]

where $(a_2, \langle a_3 \rangle) \neq (0, 0)$, $\langle a_3 \rangle := \pi^{-1}\int_0^\pi a_3(x)dx$, or

\[
f(x, u) = a_4 u^4 + \sum_{k \geq 5} a_k(x)u^k
\]

where $a_4 \neq 0$, $a_5(-\pi - x) = -a_5(x)$, $a_6(-\pi - x) = a_6(x)$, $a_7(-\pi - x) = -a_7(x)$. Assume moreover $a_k(x) \in H^1((0, \pi), \mathbb{R})$ with $\sum_k \|a_k\|_{H^1} \rho^k < +\infty$ for some $\rho > 0$.

Then there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ there is $\delta_0 > 0$, $\sigma > 0$ and a $C^\infty$-curve $\{0, \delta_0 \} \ni \delta \to u_\delta \in X_{\sigma,2,s,n}$ with the following properties:

- (i) $\|u_\delta - \delta \tilde{v}_n\|_{\sigma/2,s,n} = O(\delta^2)$ for some $\tilde{v}_n \in V \cap X_{\sigma,s,n} \setminus \{0\}$ with minimal period $2\pi/n$;
- (ii) there exists a Cantor set $C_n \subset (0, \delta_0)$ of asymptotically full measure, i.e. satisfying

\[
\lim_{\varepsilon \to 0^+} \frac{\text{meas}(C_n \cap (0, \varepsilon))}{\varepsilon} = 1,
\]

such that, $\forall \delta \in C_n$, $u_\delta(\omega(\delta)t, x)$ is a $2\pi/(\omega(\delta)n)$-periodic, classical solution of (1) with

\[
\omega(\delta) = \begin{cases} \\
\sqrt{1 - 2s^* \delta^2} & \text{if } f \text{ is like in } (\mathcal{A}) \\
\sqrt{1 - 2s^\star \delta^2} & \text{if } f \text{ is like in } (\mathcal{B})
\end{cases}
\]

and

\[
s^* = \begin{cases} \\
-1 & \text{if } \langle a_3 \rangle \geq \pi^2 a_2^2/12 \\
\pm 1 & \text{if } 0 < \langle a_3 \rangle < \pi^2 a_2^2/12 \\
1 & \text{if } \langle a_3 \rangle \leq 0.
\end{cases}
\]

\(^1\) Note how the interaction between the second and the third order terms $a_2 u^2$, $a_3(x)u^3$ changes the bifurcation diagram, i.e. existence of periodic solutions for frequencies $\omega$ less or/and greater of $\omega = 1$. 

By (6) also each Cantor-like set of frequencies \( W_n := \{ \omega(\delta) \mid \delta \in \mathcal{C}_n \} \) has asymptotically full measure at \( \omega = 1 \).

**Corollary 1. (Multiplicity)** There exists a Cantor-like set \( W \) of asymptotically full measure at \( \omega = 1 \), such that \( \forall \omega \in \mathcal{C} \), equation (1) possesses geometrically distinct periodic solutions

\[
u_{n_0}, \ldots, \nu_n, \ldots, \nu_{N_\omega}, \quad N_\omega \in \mathbb{N}
\]

with the same period \( 2\pi/\omega \). Their number increases arbitrarily as \( \omega \) tends to 1:

\[
\lim_{\omega \to 1} N_\omega = +\infty.
\]

**Proof.** The proof is like in [7] and we report it for completeness. If \( \delta \) belongs to the asymptotically full measure set (by (6))

\[D_n := \mathcal{C}_{n_0} \cap \ldots \cap \mathcal{C}_n, \quad n \geq n_0\]

there exist \((n - n_0 + 1)\) geometrically distinct periodic solutions of (1) with the same period \( 2\pi/\omega(\delta) \) (each \( \nu_n \) has minimal period \( 2\pi/\omega(\delta) \)).

There exists a decreasing sequence of positive \( \varepsilon_n \to 0 \) such that

\[\text{meas}(D_n \cap (0, \varepsilon_n)) \leq \varepsilon_n 2^{-n}.\]

Let define the set \( \mathcal{C} \equiv D_n \) on each \([\varepsilon_{n+1}, \varepsilon_n]\). \( \mathcal{C} \) has asymptotically full measure at \( \delta = 0 \) and for each \( \delta \in \mathcal{C} \) there exist \( N(\delta) := \max\{n \in \mathbb{N} : \delta < \varepsilon_n\} \) geometrically distinct periodic solutions of (1) with the same period \( 2\pi/\omega(\delta) \). \( N(\delta) \to +\infty \) as \( \delta \to 0 \).

**Remark 1.** Corollary 1 is an analogue for equation (1) of the well known multiplicity results of Weinstein-Moser [15]-[13] and Fadell-Rabinowitz [10] which hold in finite dimension. The solutions form a sequence of functions with increasing norms and decreasing minimal periods. Multiplicity of solutions was also obtained in [6] (with the "optimal" number \( N_\omega \approx C/\sqrt{|\omega - 1|} \)) but only for a zero measure set of frequencies.

The main point for proving Theorem 1 relies in showing the existence of non-degenerate solutions of the 0th order bifurcation equation for \( f \) like in (2). In these cases the 0th order bifurcation equation involves higher order terms of the nonlinearity, and, for \( n \) large, can be reduced to an integro-differential equation (which physically describes an averaged effect of the nonlinearity with Dirichlet boundary conditions).

**Case** \( f(x, u) = a_4 u^4 + O(u^5) \). Performing the rescaling

\[u \to \delta u, \quad \delta > 0\]

we look for \( 2\pi/n \)-periodic solutions in \( X_{\sigma,s,n} \) of

\[
\begin{align*}
\omega^2 u_{tt} - u_{xx} + \delta^3 g(\delta, x, u) &= 0 \\
u(t, 0) = u(t, \pi) &= 0
\end{align*}
\]

(7)

where

\[g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^4} = a_4 u^4 + \delta a_5(x) u^5 + \delta^2 a_6(x) u^6 + \ldots.\]

To find solutions of (7) we implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

\[X_{\sigma,s,n} = (V_n \cap X_{\sigma,s,n}) \oplus (W \cap X_{\sigma,s,n})\]

where

\[V_n := \left\{ v(t, x) = \eta(nt + nx) - \eta(nt - nx) \mid \eta \in H^1(T, \mathbb{R}) \text{ with } \eta \text{ odd} \right\}\]
and

\[ W := \left\{ w = \sum_{i \geq 0} \cos(lt) \, w_l(x) \in X_{0,s} \mid \int_0^\pi w_l(x) \sin(lx) \, dx = 0, \forall l \geq 0 \right\}. \]

Looking for solutions \( u = v + w \) with \( v \in V_n \cap X_{\sigma,s,n}, \) \( w \in W \cap X_{\sigma,s,n} \), and imposing the frequency-amplitude relation

\[ \frac{(\omega^2 - 1)}{2} = -\delta^6 \]

we are led to solve the bifurcation equation and the range equation

\[ \begin{cases} \Delta v = \delta^{-3} \Pi_{V_n} g(\delta, x, v + w) \\ L_\omega w = \delta^3 \Pi_{W_n} g(\delta, x, v + w) \end{cases} \tag{8} \]

where

\[ \Delta v := v_{xx} + v_{tt}, \quad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx} \]

and \( \Pi_{V_n} : X_{\sigma,s,n} \to V_n \cap X_{\sigma,s,n}, \) \( \Pi_{W_n} : X_{\sigma,s,n} \to W \cap X_{\sigma,s,n} \) denote the projectors.

With the further rescaling \( w \to \delta^3 w \)

and since \( v^4 \in W_n \) (Lemma 3.4 of [3]), \( a_5(x) v^5, a_6(x) v^6, a_7(x) v^7 \in W_n \) because \( a_5(\pi - x) = -a_5(x), \)
\( a_6(\pi - x) = a_6(x), \) \( a_7(\pi - x) = -a_7(x) \) (Lemma 7.1 of [7]), system (8) is equivalent to

\[ \begin{cases} \Delta v = \Pi_{V_n} \left( 4a_4 v^5 \delta + \tilde{r}(\delta, x, v, w) \right) \\ L_\omega w = a_4 v^4 + \delta \Pi_{W_n} \tilde{r}(\delta, x, v, w) \end{cases} \tag{9} \]

where \( \tilde{r}(\delta, x, v, w) = a_5(x) v^8 + 5a_5(x) v^4 w + O(\delta) \) and \( \tilde{r}(\delta, x, v, w) = a_5(x) v^5 + O(\delta). \)

For \( \delta = 0 \) system (9) reduces to \( w = -a_4 \Box^{-1} v^4 \) and to the 0th order bifurcation equation

\[ \Delta v + 4a_4^2 \Pi_{V_n} (v^3 \Box^{-1} v^4) = 0 \tag{10} \]

which is the Euler-Lagrange equation of the functional \( \Phi_0 : V_n \to \mathbb{R} \)

\[ \Phi_0(v) = \frac{||v||^2_{H_4}}{2} - \frac{a_4^2}{2} \int_\Omega v^4 \Box^{-1} v^4 \tag{11} \]

where \( \Omega := \mathbb{T} \times (0, \pi). \)

**Proposition 1.** Let \( a_4 \neq 0. \) \( \exists n_0 \in \mathbb{N} \) such that \( \forall n \geq n_0 \) the 0th order bifurcation equation (11) has a solution \( \tilde{v}_n \in V_n \) which is non-degenerate in \( V_n \) (i.e. \( \text{Ker} D^2 \Phi_0 = \{0\} \)), with minimal period \( 2\pi/n. \)

**Case** \( f(x, u) = a_2 u^2 + a_3(x) u^3 + O(u^4). \) Performing the rescaling \( u \to \delta u \) we look for \( 2\pi/n \)-periodic solutions of

\[ \begin{cases} \omega^2 u_{tt} - u_{xx} + \delta g(\delta, x, u) = 0 \\ u(t, 0) = u(t, \pi) = 0 \end{cases} \]

where

\[ g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^2} = a_2 u^2 + \delta a_3(x) u^3 + \delta^2 u_4(x) u^4 + \ldots. \]

With the frequency-amplitude relation

\[ \frac{\omega^2 - 1}{2} = -s^* \delta^2 \]

where \( s^* = \pm 1 \), we have to solve

\[ \begin{cases} -\Delta v = -s^* \delta^{-1} \Pi_{V_n} g(\delta, x, v + w) \\ L_\omega w = \delta \Pi_{W_n} g(\delta, x, v + w). \end{cases} \tag{12} \]
With the further rescaling \( w \to \delta w \) and since \( v^2 \in W_n \), system (12) is equivalent to

\[
\begin{aligned}
-\Delta v &= s^4 \Pi_v V_n \left(-2a_2 w - a_2 \delta w^2 - a_3(x)(v + \delta w)^3 - \delta r(\delta, x, v + \delta w)\right) \\
L_w w &= a_2 v^2 + \delta \Pi W_n \left(2a_2 w + \delta a_2 w^2 + a_3(x)(v + \delta w)^3 + a_8(x)v^5 r(\delta, x, v + \delta w)\right)
\end{aligned}
\]

where \( r(\delta, x, u) := \delta^{-4}[f(x, \delta u) - a_2 \delta^2 u^2 - \delta^3 a_3(x)u^3] = a_4(x)u^4 + \ldots \)

For \( \delta = 0 \) system (13) reduces to \( w = -a_2 \square^{-1} v^2 \) and the 0th order bifurcation equation

\[
-s^4 \Delta v = 2a_2^2 \Pi V_n (v \square^{-1} v^2) - \Pi V_n (a_3(x)v^3)
\]

which is the Euler-Lagrange equation of \( \Phi_0 : V_n \to \mathbb{R} \)

\[
\Phi_0(v) := s^4 \frac{\|v\|_{H^1}^2}{2} - \frac{a_2^2}{2} \int_\Omega v^2 \square^{-1} v^2 + \frac{1}{4} \int_\Omega a_3(x)v^4.
\]

**Proposition 2.** Let \((a_2, \langle a_3 \rangle) \neq 0\). \( \exists \ n_0 \in \mathbb{N} \) such that \( \forall n \geq n_0 \) the 0th order bifurcation equation (14) has a solution \( \bar{v}_n \in V_n \) which is non-degenerate in \( V_n \), with minimal period \( 2\pi/n \).

**2 Case** \( f(x, u) = a_4 u^4 + O(u^5) \)

We have to prove the existence of non-degenerate critical points of the functional

\[
\Phi_n : V \to \mathbb{R}, \quad \Phi_n(v) := \Phi_0(\mathcal{H}_n v)
\]

where \( \Phi_0 \) is defined in (11). Let \( \mathcal{H}_n : V \to V \) be the linear isomorphism defined, for \( v(t, x) = \eta(t + x) - \eta(t - x) \in V \), by

\[
a_8(x)v^5(\mathcal{H}_n v)(t, x) := \eta(n(t + x)) - \eta(n(t - x))
\]

so that \( V_n \equiv \mathcal{H}_n V \).

**Lemma 1.** See [6]. \( \Phi_n \) has the following development: for \( v(t, x) = \eta(t + x) - \eta(t - x) \in V \)

\[
\Phi_n(\beta n^{1/3} v) = 4\pi \beta^2 n^{2/3} \left[ \Psi(\eta) + \frac{\mathcal{R}(\eta)}{n^2} \right]
\]

where \( \beta := (3/(\pi^2 a_4^2))^{1/6}, \alpha := a_2^2/(8\pi), \)

\[
\Psi(\eta) := \frac{1}{2} \int_T \eta^2(t) \, dt - \frac{2\pi}{8} \left( \langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2 \right)^2,
\]

\( \langle \rangle \) denotes the average on \( T \), and

\[
\mathcal{R}(\eta) := -\int_\Omega v^{4} \square^{-1} v^4 \, dt \, dx + \frac{\pi^4}{6} \left( \langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2 \right)^2.
\]

**Proof.** Firstly the quadratic term writes

\[
\frac{1}{2} \left\| \mathcal{H}_n v \right\|_{H^1}^2 = \frac{n^2}{2} \left\| v \right\|_{H^1}^2 = n^2 2\pi \int_T \eta^2(t) \, dt.
\]

By Lemma 4.8 in [6] the non-quadratic term can be developed as

\[
\int_\Omega (\mathcal{H}_n v)^4 \square^{-1} (\mathcal{H}_n v) = \frac{\pi^4}{6} m^2 - \frac{\mathcal{R}(\eta)}{n^2}
\]

where \( m : T^2 \to \mathbb{R} \) is \( m(s_1, s_2) := (\eta(s_1) - \eta(s_2))^4 \), \( \langle m \rangle := (2\pi)^{-2} \int_{T^2} m(s_1, s_2) \, ds_1 \, ds_2 \) denotes its average, and

\[
\mathcal{R}(\eta) := \left( -\int_\Omega v^{4} \square^{-1} v^4 + \frac{\pi^4}{6} \langle m \rangle^2 \right)
\]
is homogeneous of degree 8. Since $\eta$ is odd we find
\[
\langle m \rangle = 2\left(\langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2 \right)
\]  
(22)
where $\langle \cdot \rangle$ denotes the average on $\mathbb{T}$.

Collecting (14), (20), (21) and (22) we find out
\[
\Phi_n(\eta) = 2\pi n^2 \int_{\mathbb{T}} \eta^2(t) \, dt - \frac{\pi^4}{3} a_4^2 \left(\langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2 \right)^2 + \frac{a_4^2}{2n^2} R(\eta).
\]
Via the rescaling $\eta \to \beta n^{1/3} \eta$ we get the expressions (17) and (18).

By (16), in order to find for $n$ large enough a non-degenerate critical point of $\Phi_n$, it is sufficient to find a non-degenerate critical point of $\Psi(\eta)$ defined on
\[
E := \left\{ \eta \in H^1(\mathbb{T}), \eta \text{ odd} \right\},
\]
namely non-degenerate solutions in $E$ of
\[
\ddot{\eta} + A(\eta) \left(3\langle \eta^2 \rangle \eta + \eta^3\right) = 0 \quad A(\eta) := \langle \eta^4 \rangle + 3\langle \eta^2 \rangle^2.
\]  
(23)

**Proposition 3.** There exists an odd, analytic, $2\pi$-periodic solution $g(t)$ of (23) which is non-degenerate in $E$. $g(t) = V sn(\Omega t, m)$ where $sn$ is the Jacobi elliptic sine and $V > 0$, $\Omega > 0$, $m \in (-1, 0)$ are suitable constants (therefore $g(t)$ has minimal period $2\pi$).

We will construct the solution $g$ of (23) by means of the Jacobi elliptic sine in Lemma 4. The existence of a solution $g$ follows also directly applying to $\Psi : E \to \mathbb{R}$ the Mountain-Pass Theorem [2]. Furthermore such solution is an analytic function arguing as in Lemma 2.1 of [11].

### 2.1 Non-degeneracy of $g$

We now want to prove that $g$ is non-degenerate. The linearized equation of (23) at $g$ is
\[
\ddot{h} + 3A(g) \left([g^2]h + g^2 h \right) + 6A(g)g(gh) + A'(g)[h] \left(3\langle g^2 \rangle + g^3 \right) = 0
\]
\[
\ddot{h} + 3A(g) \left([g^2] + g^2 \right) h + 6g(gh) \left([g^4] + 3\langle g^2 \rangle^2 \right) + 4g \left([g^2] + 3\langle g^2 \rangle \langle gh \rangle \right) \left(3\langle g^2 \rangle + g^2 \right) = 0
\]
that we write as
\[
\ddot{h} + 3A(g) \left([g^2] + g^2 \right) h = -\langle gh \rangle I_1 - \langle g^3 h \rangle I_2
\]  
(24)
where
\[
\begin{align*}
I_1 & := 6 \left(9\langle g^2 \rangle^2 + \langle g^4 \rangle \right) g + 12\langle g^2 \rangle g^3 \\
I_2 & := 12g\langle g^2 \rangle^2 + 4g^3.
\end{align*}
\]  
(25)
For $f \in E$, let $H := L(f)$ be the unique solution belonging to $E$ of the non-homogeneous linear system
\[
\ddot{h} + 3A(g) \left([g^2] + g^2 \right) H = f;
\]  
(26)
an integral representation of the Green operator $L$ is given in Lemma 4. Thus (24) becomes
\[
\ddot{h} = -\langle gh \rangle L(I_1) - \langle g^3 h \rangle L(I_2).
\]  
(27)
Multiplying (24) by $g$ and taking averages we get
\[
\langle gh \rangle \left[1 + \langle gL(I_1) \rangle \right] = -\langle g^3 h \rangle \langle gL(I_2) \rangle,
\]  
(28)
while multiplying (27) by $g^3$ and taking averages
\[
\langle g^3 h \rangle \left[1 + \langle g^3 L(I_2) \rangle \right] = -\langle gh \rangle \langle g^3 L(I_1) \rangle.
\]  
(29)
Since $g$ solves (23) we have the following identities.
Lemma 2. There holds
\[ 2 A(g) (g^2 L(g)) = \langle g^2 \rangle \]  \hfill (30)
\[ 2 A(g) (g^3 L(g^3)) = \langle g^4 \rangle. \]  \hfill (31)

Proof. \ref{30} is obtained by the identity for \( L(g) \)
\[ \frac{d^2}{dt^2} (L(g)) + 3 A(g) \left( \langle g^2 \rangle + \langle g^2 \rangle \right) L(g) = g \]
multiplying by \( g \), taking averages, integrating by parts,
\[ \langle g L(g) \rangle + 3 A(g) \left[ \langle g^2 \rangle L(g) + \langle g^3 L(g) \rangle \right] = \langle g^2 \rangle \]
and using that \( g \) solves \ref{23}.

Analogously, \ref{31} is obtained by the identity for \( L(g^3) \)
\[ \frac{d^2}{dt^2} (L(g^3)) + 3 A(g) \left( \langle g^2 \rangle + \langle g^3 \rangle \right) L(g^3) = g^3 \]
multiplying by \( g \), taking averages, integrating by parts, and using that \( g \) solves \ref{23}.

Since \( L \) is a symmetric operator we can compute the following averages using \ref{20}, \ref{30}, \ref{31}:
\[
\begin{align*}
\langle g L(I_1) \rangle &= 6 \langle \langle g^4 \rangle + 9 \langle g^2 \rangle^2 \rangle L(g) + 6 A(g)^{-1} \langle g^2 \rangle^2 \\
\langle g L(I_2) \rangle &= 12 \langle g^2 \rangle \langle g L(g) \rangle + 2 A(g)^{-1} \langle g^2 \rangle \\
\langle g^3 L(I_1) \rangle &= 9 \langle g^2 \rangle \\
\langle g^3 L(I_2) \rangle &= 2.
\end{align*}
\]  \hfill (32)

Thanks to the identities \ref{32}, equations \ref{28}, \ref{29} simplify to
\[
\begin{align*}
\langle gh \rangle \left[ A(g) + 6 \langle g^2 \rangle^2 \right] B(g) &= -2 \langle g^2 \rangle B(g) \langle g^3 h \rangle \\
\langle g^3 h \rangle &= -3 \langle g^2 \rangle \langle gh \rangle
\end{align*}
\]  \hfill (33)

where
\[ B(g) := 1 + 6 A(g) \langle g L(g) \rangle. \]  \hfill (34)

Solving \ref{33} we get
\[ B(g) \langle gh \rangle = 0. \]

We will prove in Lemma \ref{5} that \( B(g) \neq 0 \), so \( \langle gh \rangle = 0 \). Hence by \ref{33} also \( \langle g^3 h \rangle = 0 \) and therefore, by \ref{27}, \( h = 0 \). This concludes the proof of the non-degeneracy of the solution \( g \) of \ref{23}.

It remains to prove that \( B(g) \neq 0 \). The key is to express the function \( L(g) \) by means of the variation of constants formula.

We first look for a fundamental set of solutions of the homogeneous equation
\[ \ddot{h} + 3 A(g) \left( \langle g^2 \rangle + \langle g^2 \rangle \right) h = 0. \]  \hfill (HOM)

Lemma 3. There exist two linearly independent solutions of \( (HOM) \), \( \bar{u} := \dot{g}(t)/\dot{g}(0) \) and \( \bar{v} \), such that
\[
\begin{align*}
\bar{u} \text{ is even, } 2\pi \text{ periodic } & \quad \bar{v} \text{ is odd, not periodic } \\
\bar{u}(0) = 1, \bar{u}(0) = 0 & \quad \bar{v}(0) = 0, \bar{v}(0) = 1
\end{align*}
\]
and
\[ \bar{v}(t + 2\pi) - \bar{v}(t) = \rho \bar{u}(t) \]  \hfill (35)

for some \( \rho > 0 \).
PROOF. Since (23) is autonomous, \( \dot{g}(t) \) is a solution of the linearized equation (HOM). \( \dot{g}(t) \) is even and \( 2\pi \)-periodic.

We can construct another solution of (HOM) in the following way. The super-quadratic Hamiltonian system (with constant coefficients)

\[
\dot{y} + 3A(g)(g^2) y + A(g) y^3 = 0
\]

possesses a one-parameter family of odd, \( T(E) \)-periodic solutions \( y(E, t) \) close to \( g \), parametrized by the energy \( E \). Let \( \bar{E} \) denote the energy level of \( g \), i.e. \( g = \bar{g}(\bar{E}, t) \) and \( T(\bar{E}) = 2\pi \).

Therefore \( l(t) := (\partial_E y(E, t))_{|E = \bar{E}} \) is an odd solution of (HOM).

Deriving the identity \( y(E, t + T(E)) = y(E, t) \) with respect to \( E \) we obtain at \( E = \bar{E} \)

\[
l(t + 2\pi) - l(t) = - (\partial_E T(E))_{|E = \bar{E}} \dot{g}(t)
\]

and, normalizing \( \bar{v}(t) := l(t)/\bar{l}(0) \), we get (35) with

\[
\rho := - (\partial_E T(E))_{|E = \bar{E}} \left( \frac{\dot{g}(0)}{\bar{l}(0)} \right).
\]

Since \( y(E, 0) = 0 \) \( \forall E \), the energy identity gives \( E = \frac{1}{2} (\dot{g}(E, 0))^2 \). Deriving w.r.t \( E \) at \( E = \bar{E} \), yields \( 1 = \dot{g}(0) \bar{l}(0) \) which, inserted in (37), gives

\[
\rho = - (\partial_E T(E))_{|E = \bar{E}} (\dot{g}(0))^2.
\]

\( \rho > 0 \) because \( (\partial_E T(E))_{|E = \bar{E}} < 0 \) by the superquadraticity of the potential of (23). It can be checked also by a computation, see Remark after Lemma 3.

Now we write an integral formula for the Green operator \( L \).

**Lemma 4.** For every \( f \in E \) there exists a unique solution \( H = L(f) \) of (23) which can be written as

\[
L(f) = \left( \int_0^t f(s) \bar{u}(s) ds + \frac{1}{\rho} \int_0^{2\pi} f \bar{v}(s) ds \right) \bar{v}(t) - \left( \int_0^t f(s) \bar{v}(s) ds \right) \bar{u}(t) \in E.
\]

**Proof.** The non-homogeneous equation (24) possesses the particular solution

\[
\bar{H}(t) = \left( \int_0^t f(s) \bar{u}(s) ds \right) \bar{v}(t) - \left( \int_0^t f(s) \bar{v}(s) ds \right) \bar{u}(t)
\]

as can be verified noting that the Wronskian \( \bar{u}(t) \dot{v}(t) - \dot{u}(t) \bar{v}(t) \equiv 1, \forall t \). Notice that \( \bar{H} \) is odd.

Any solution \( \bar{H}(t) \) of (25) can be written as

\[
\bar{H}(t) = \bar{H}(t) + a\bar{u} + b\bar{v}, \quad a, b \in \mathbb{R}.
\]

Since \( \bar{H} \) is odd, \( \bar{u} \) is even and \( \bar{v} \) is odd, requiring \( \bar{H} \) to be odd, implies \( a = 0 \). Imposing now the \( 2\pi \)-periodicity yields

\[
0 = \left( \int_0^{t+2\pi} f \bar{u} \right) \bar{v}(t + 2\pi) - \left( \int_0^t f \bar{v} \right) \bar{u}(t + 2\pi) - \left( \int_0^t f \bar{u} \right) \bar{v}(t) + \left( \int_0^t f \bar{v} \right) \bar{u}(t) + b(\bar{v}(t + 2\pi) - \bar{v}(t))
\]

\[
= b + \left( \int_0^t f \bar{u} \right) (\bar{v}(t + 2\pi) - \bar{v}(t)) - \bar{u}(t) \left( \int_0^{t+2\pi} f \bar{v} \right)
\]

using that \( \bar{u} \) and \( f \bar{u} \) are \( 2\pi \)-periodic and \( \langle f \bar{u} \rangle = 0 \). By (10) and \( 35 \) we get

\[
\rho \left( b + \int_0^t f \bar{u} \right) - \int_0^{t+2\pi} f \bar{v} = 0.
\]
The left hand side in (41) is constant in time because, deriving w.r.t. \( t \),
\[
\rho f(t) \dddot{u}(t) - f(t) \left( \dddot{v}(t) + 2\pi \right) - \dddot{v}(t) = 0
\]
again by (39). Hence evaluating (41) for \( t = 0 \) yields \( b = \rho^{-1} \int_0^{2\pi} \dddot{v} \). So there exists a unique solution \( H = L(f) \) of (26) belonging to \( E \) and (39) follows.

Finally

**Lemma 5.** There holds

\[
\langle gL(g) \rangle = \frac{\rho}{4\pi A(g)} + \frac{1}{2\pi \rho} \left( \int_0^{2\pi} g\dddot{v} \right)^2 > 0
\]
because \( A(g), \rho > 0 \).

**Proof.** Using (39) we can compute

\[
\langle gL(g) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^t \dddot{u}(t)g(t) \right) dt + \frac{1}{2\pi \rho} \left( \int_0^{2\pi} g\dddot{v} \right)^2 - \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^t \dddot{u}(t)g(t) \right) dt
\]

because, by \( \int_0^{2\pi} g\dddot{u} = 0 \), we have

\[
0 = \int_0^{2\pi} \frac{d}{dt} \left[ \left( \int_0^t \dddot{u}(t)g(t) \right) \right] dt = \int_0^{2\pi} \left[ \left( \int_0^t \dddot{u}(t)g(t) \right) + \left( \int_0^t \dddot{u}(t)g(t) \right) \right] dt.
\]

Now, since \( \dddot{u}(t) = \dddot{g}(t)/\dddot{g}(0) \) and \( g(0) = 0 \),

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^t \dddot{u}(t)g(t) \right) dt = \frac{1}{2\pi \dddot{g}(0)} \int_0^{2\pi} \left( \int_0^t \frac{d}{d\tau} g^2(\tau) \right) d\tau = \frac{1}{4\pi \dddot{g}(0)} \int_0^{2\pi} g^3 \dddot{v}.
\]

We claim that

\[
\int_0^{2\pi} g^3 \dddot{v} = \frac{\rho \dddot{g}(0)}{2A(g)}.
\]

By (12), (33), (41), we have the thesyz.

Let us prove (41). Since \( g \) solves (26) multiplying by \( \dddot{v} \) and integrating

\[
\int_0^{2\pi} \dddot{v}(t)\dddot{g}(t) + 3A(g)(g^2)\dddot{v}(t)g(t) + A(g)g^3(t) \dddot{v}(t) dt = 0
\]

Next, since \( \dddot{v} \) solves (HOM), multiplying by \( g \) and integrating

\[
\int_0^{2\pi} g(t)\dddot{v}(t) + 3A(g)(g^2)\dddot{v}(t)g(t) + 3A(g)g^3(t) \dddot{v}(t) dt = 0.
\]

Subtracting (45) and (46), gives

\[
\int_0^{2\pi} \dddot{v}(t)\dddot{g}(t) - g(t)\dddot{v}(t) = 2A(g) \int_0^{2\pi} g^3 \dddot{v}.
\]

Integrating by parts the left hand side, since \( g(0) = g(2\pi) = 0, \dddot{u}(0) = 1 \) and (15), gives

\[
\int_0^{2\pi} \dddot{v}(t)\dddot{g}(t) - g(t)\dddot{v}(t) = \dddot{g}(0)[\dddot{v}(2\pi) - \dddot{v}(0)] = \rho \dddot{g}(0).
\]
2.2 Explicit computations

We now give the explicit construction of $g$ by means of the Jacobi elliptic sine defined as follows. Let $\text{am}(\cdot,m): \mathbb{R} \rightarrow \mathbb{R}$ be the inverse function of the Jacobi elliptic integral of the first kind

$$\varphi \mapsto F(\varphi,m) := \int_0^\varphi \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}}.$$  

The Jacobi elliptic sine is defined by

$$\text{sn}(t,m) := \sin(\text{am}(t,m)).$$

$\text{sn}(t,m)$ is $4K(m)$-periodic, where $K(m)$ is the complete elliptic integral of the first kind

$$K(m) := F(\frac{\pi}{2},m) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}},$$

and admits an analytic extension with a pole in $iK(1/(1-m))/\sqrt{1-m}$ for $m < 0$. Moreover, since

$$\partial_t \text{am}(t,m) = \sqrt{1 - m \text{sn}^2(t,m)},$$

the elliptic sine satisfies

$$(\text{sn})^2 = (1 - \text{sn}^2)(1 - m \text{sn}^2). \quad (49)$$

**Lemma 6.** There exist $V > 0$, $\Omega > 0$, $m \in (-1,0)$ such that $g(t) := V \text{sn}(\Omega t,m)$ is an odd, analytic, $2\pi$-periodic solution of (23) with pole in $iK(1/(1-m))/\sqrt{1-m}$.

**Proof.** Deriving (49) we have

$$\ddot{g} + (1 + m)g - 2m \Omega^2 \text{sn}^3 = 0. \quad (50)$$

The function $g(V,\Omega,m)$ will be a solution of (23) if $(V,\Omega,m)$ verify

$$\begin{cases}
\Omega^2(1 + m) = 3A(g(V,\Omega,m)) \langle g^2(V,\Omega,m) \\
-2m\Omega^2 = V^2A(g(V,\Omega,m)) \\
2K(m) = \Omega \pi.
\end{cases} \quad (51)$$

Dividing the first equation of (51) by the second one

$$-\frac{1 + m}{6m} = \langle \text{sn}^2(\cdot,m) \rangle. \quad (52)$$

The right hand side can be expressed as

$$\langle \text{sn}^2(\cdot,m) \rangle = \frac{K(m) - E(m)}{mK(m)} \quad (53)$$

where $E(m)$ is the complete elliptic integral of the second kind

$$E(m) := \int_0^{\pi/2} \sqrt{1 - m \sin^2 \vartheta} d\vartheta = \int_0^{K(m)} 1 - m \text{sn}^2(\xi,m) d\xi$$

(in the last passage we make the change of variable $\vartheta = \text{am}(\xi,m)$).

Now, we show that system (51) has a unique solution. By (52) and (53)

$$(7 + m)K(m) - 6E(m) = 0. \quad (54)$$
By the definitions of $E(m)$ and $K(m)$ we have

$$\psi(m) := (7 + m)K(m) - 6E(m) = \int_0^{\pi/2} \frac{1 + m(1 + 6 \sin^2 \vartheta)}{(1 - m \sin^2 \vartheta)^{1/2}} d\vartheta. \quad (55)$$

For $m = 0$ it holds $\psi(0) = \pi/2 > 0$ and, for $m = -1$, $\psi(-1) = -\int_0^{\pi/2} 6 \sin^2 \vartheta (1 + \sin^2 \vartheta)^{-1/2} d\vartheta < 0$. Since $\psi$ is continuous there exists a solution $\bar{m} \in (-1,0)$ of (54). Next the third equation in (51) fix $\bar{\Omega}$ and finally we find $\bar{V}$. Hence $g(t) = \bar{V} \text{ sn}(\bar{\Omega}, \bar{m})$ solves (58).

Analyticity and poles follow from [11], 16.2, 16.10.2, pp. 570, 573.

At last, $\bar{m}$ is unique because $\psi'(m) > 0$ for $m \in (-1,0)$ as can be verified by (55). One can also compute that $\bar{m} \in (-0.30, -0.28)$.

**Remark.** We can compute explicitly the sign of $dT/dE$ and $\rho$ of (58) in the following way.

The functions $g(v, \Omega, m)$ are solutions of the Hamiltonian system (55) imposing

$$\begin{cases}
\Omega^2(1 + m) = \alpha \\
-2m\Omega^2 = V^2 \beta
\end{cases} \quad (56)$$

where $\alpha := 3A(g) \langle g^2 \rangle$, $\beta := A(g)$ and $g$ is the solution constructed in Lemma [11].

We solve (56) w.r.t $m$ finding the one-parameter family $(y_m)$ of odd periodic solutions $y_m(t) := V(m) \text{ sn}(\Omega(m), t, m)$, close to $g$, with energy and period

$$E(m) = \frac{1}{2} V^2 m \Omega^2(m) = -\frac{1}{\beta} m \Omega^4(m), \quad T(m) = \frac{4K(m)}{\Omega(m)}.$$

It holds

$$\frac{dT(m)}{dm} = \frac{4K'(m)\Omega(m) - 4K(m)\Omega'(m)}{\Omega^2(m)} > 0$$

because $K'(m) > 0$ and from (56) $\Omega'(m) = -\Omega(m)(2(1 + m))^{-1} < 0$. Then

$$\frac{dE(m)}{dm} = -\frac{1}{\beta} \Omega^4(m) - \frac{1}{\beta} m 4\Omega^3(m)\Omega'(m) < 0,$$

so

$$\frac{dT}{dE} = \frac{dT(m)}{dm} \left( \frac{dE(m)}{dm} \right)^{-1} < 0$$

as stated by general arguments in the proof of Lemma [11].

We can also write an explicit formula for $\rho$,

$$\rho = \frac{m}{m - 1} \left[ 2\pi + (1 + m) \int_0^{2\pi} \frac{\text{ sn}^2(\Omega, t, m)}{\text{ dh}^2(\Omega, t, m)} dt \right]. \quad (57)$$

From (57) it follows that $\rho > 0$ because $-1 < m < 0$.

### 3. Case $f(x, u) = a_2 u^2 + a_3(x) u^3 + O(u^4)$

We have to prove the existence of non-degenerate critical points of the functional $\Phi_n(v) := \Phi_0(\mathcal{H}_n v)$ where $\Phi_0$ is defined in [15].

**Lemma 7.** See [12]. $\Phi_n$ has the following development: for $v(t, x) = \eta(t + x) - \eta(t - x) \in V$,

$$\Phi_n(\beta_n v) = 4\pi \beta^2 n^4 \left[ \Psi(\eta) + \frac{\beta^2}{4\pi} \left( R_2(\eta) + R_3(\eta) \right) \right] \quad (58)$$
where

\[ \Psi(\eta) := \frac{s^*}{2} \int_\Omega \dot{\eta}^2 + \frac{\beta^2}{4\pi} \left[ \alpha \left( \int_\Omega \eta^2 \right)^2 + \gamma \int_\Omega \eta^4 \right] \]

\[ R_2(\eta) := -\frac{\alpha^2}{2} \left[ \int_\Omega v^2 \partial^2 v^2 - \frac{\pi^2}{6} \left( \int_\Omega \eta^2 \right)^2 \right], \quad R_3(\eta) := \frac{1}{4} \int_\Omega (a_3(x) - \langle a_3 \rangle)(\mathcal{H}_n v)^4, \quad (59) \]

\[ \alpha := (9a_3 - \pi^2 a_3^2)/12, \quad \gamma := \pi(a_3)/2, \quad \text{and} \]

\[ \beta = \begin{cases} \frac{(2|\alpha|)^{-1/2}}{\alpha} & \text{if } \alpha \neq 0, \\ \left(\pi/\gamma\right)^{1/2} & \text{if } \alpha = 0. \end{cases} \]

PROOF. By Lemma 4.8 in [6] with \( m(s_1, s_2) = (\eta(s_1) - \eta(s_2))^2 \), for \( v(t, x) = \eta(t + x) - \eta(t - x) \) the operator \( \Phi_n \) admits the development

\[ \Phi_n(v) = 2\pi s^* n^2 \int_\Omega \dot{\eta}^2(t) dt - \frac{\pi^2 a_3^2}{12} \left( \int_\Omega \eta^2(t) dt \right)^2 - \frac{a_3^2}{2n^2} \left( \int_\Omega v^2 \partial^2 v^2 - \frac{\pi^2}{6} \left( \int_\Omega \eta^2(t) dt \right)^2 \right) + \int_\Omega v^4 + \frac{1}{4} \int_\Omega (a_3(x) - \langle a_3 \rangle)(\mathcal{H}_n v)^4. \]

Since

\[ \int_\Omega v^4 = 2\pi \int_\Omega \eta^4 + 3 \left( \int_\Omega \eta^2 \right)^2, \]

we write

\[ \Phi_n(v) = 2\pi s^* n^2 \int_\Omega \dot{\eta}^2 + \alpha \left( \int_\Omega \eta^2 \right)^2 + \gamma \int_\Omega \eta^4 + \frac{R_2(\eta)}{n^2} + R_3(\eta), \]

where \( R_2, R_3 \) defined in [69] are both homogenous of degree 4. So

\[ \Phi_n(v) = 2\pi s^* n^2 \int_\Omega \dot{\eta}^2 + \alpha \left( \int_\Omega \eta^2 \right)^2 + \gamma \int_\Omega \eta^4 + \frac{R_2(\eta)}{n^2} + R_3(\eta) \]

where \( \alpha, \gamma \) are defined above. With the rescaling \( \eta \rightarrow \eta/\eta_n \) we get decomposition [65].

In order to find for \( n \) large a non-degenerate critical point of \( \Phi_n \), by [58] it is sufficient to find critical points of \( \Psi \) on \( E = \{ \eta \in H^1(\Omega), \eta \text{ odd} \} \) (like in Lemma 6.2 of [71] also the term \( R_3(\eta) \) tends to 0 with its derivatives).

If \( \langle a_3 \rangle \in (-\infty, 0) \cup (\pi^2 a_3^2/9, +\infty) \), then \( \alpha \neq 0 \) and we must choose \( s^* = -\text{sign}(\alpha) \), so that the functional becomes

\[ \Psi(\eta) = \text{sign}(\alpha) \left( -\frac{1}{2} \int_\Omega \dot{\eta}^2 + \frac{1}{8\pi} \left( \int_\Omega \eta^2 \right)^2 + \frac{\gamma}{\alpha} \int_\Omega \eta^4 \right). \]

Since in this case \( \gamma/\alpha > 0 \), the functional \( \Psi \) clearly has a mountain pass critical point, solution of

\[ \ddot{\eta} + \langle \eta^2 \rangle \eta + \lambda \eta^3 = 0, \quad \lambda = \frac{\gamma}{2\pi \alpha} > 0. \quad (60) \]

The proof of the non-degeneracy of the solution of [60] is very simple using the analytical arguments of the previous section (since \( \lambda > 0 \) it is sufficient a positivity argument).

If \( \langle a_3 \rangle = 0 \), then the equation becomes \( \ddot{\eta} + \langle \eta^2 \rangle \eta = 0 \), so we find again what proved in [7] for \( a_3(x) \equiv 0 \).

If \( \langle a_3 \rangle = \pi^2 a_3^2/9 \), then \( \alpha = 0 \). We must choose \( s^* = -1 \), so that we obtain

\[ \Psi(\eta) = -\frac{1}{2} \int_\Omega \dot{\eta}^2 + \int_\Omega \eta^4, \quad \ddot{\eta} + \eta^3 = 0. \]

This equation has periodic solutions which are non-degenerate because of non-isocronicity, see Proposition 2 in [5].
Finally, if \((a_3) \in (0, \pi^2a_2^2/9)\), then \(\alpha < 0\) and there are both solutions for \(s^* = \pm 1\). The functional

\[
\Psi(\eta) = \frac{s^*}{2} \int_T \dot{\eta}^2 + \frac{1}{8\pi} \left[ - \left( \int_T \eta^2 \right)^2 + \frac{\gamma}{|\alpha|} \int_T \eta^4 \right]
\]

\[
= \frac{s^*}{2} \int_T \dot{\eta}^2 + \frac{1}{4} \int_T \eta^4 \left[ \lambda - Q(\eta) \right]
\]

where

\[
\lambda := \frac{\gamma}{2\pi|\alpha|} > 0, \quad Q(\eta) := \frac{\left( \int_T \eta^2 \right)^2}{2\pi \int_T \eta^4}
\]

possesses Mountain pass critical points for any \(\lambda > 0\) because (like in Lemma 3.14 of \(\text{[6]}\))

\[
\inf_{\eta \in E \setminus \{0\}} Q(\eta) = 0, \quad \sup_{\eta \in E \setminus \{0\}} Q(\eta) = 1
\]

(for \(\lambda \geq 1\) if \(s^* = -1\), and for \(0 < \lambda < 1\) for both \(s^* = \pm 1\)).

Such critical points satisfy the Euler Lagrange equation

\[
-s^* \ddot{\eta} - (\eta^2)\dot{\eta} + \lambda \eta^3 = 0 \tag{61}
\]

but their non-degeneracy is not obvious. For this, it is convenient to express this solutions in terms of the Jacobi elliptic sine.

**Proposition 4.** (i) Let \(s^* = -1\). Then for every \(\lambda \in (0, +\infty)\) there exists an odd, analytic, \(2\pi\)-periodic solution \(g(t)\) of \(\text{[51]}\) which is non-degenerate in \(E\). \(g(t) = V \text{sn}(\Omega t, m)\) for \(V > 0\), \(\Omega > 0\), \(m \in (-\infty, -1)\) suitable constants.

(ii) Let \(s^* = 1\). Then for every \(\lambda \in (0, 1)\) there exists an odd, analytic, \(2\pi\)-periodic solution \(g(t)\) of \(\text{[51]}\) which is non-degenerate in \(E\). \(g(t) = V \text{sn}(\Omega t, m)\) for \(V > 0\), \(\Omega > 0\), \(m \in (0, 1)\) suitable constants.

We prove Proposition 4 in several steps. First we construct the solution \(g\) like in Lemma 6.

**Lemma 8.** (i) Let \(s^* = -1\). Then for every \(\lambda \in (0, +\infty)\) there exist \(V > 0\), \(\Omega > 0\), \(m \in (-\infty, -1)\) such that \(g(t) = V \text{sn}(\Omega t, m)\) is an odd, analytic, \(2\pi\)-periodic solution of \(\text{[51]}\) with a pole in \(\frac{\Omega^2(1 + m)}{\pi \sqrt{1 - m} K \left( \frac{1}{1 - m} \right)}\).

(ii) Let \(s^* = 1\). Then for every \(\lambda \in (0, 1)\) there exist \(V > 0\), \(\Omega > 0\), \(m \in (0, 1)\) such that \(g(t) = V \text{sn}(\Omega t, m)\) is an odd, analytic, \(2\pi\)-periodic solution of \(\text{[51]}\) with a pole in \(iK(1 - m)/\Omega\).

**Proof.** We know that \(g(V, \Omega, m)(t) := V \text{sn}(\Omega t, m)\) is an odd, \((4K(m)/\Omega)\)-periodic solution of \(\text{[50]}\), see Lemma 6. So it is a solution of \(\text{[51]}\) if \(V, \Omega, m\) verify

\[
\begin{cases}
\Omega^2(1 + m) = s^*V^2 \langle \text{sn}^2(\cdot, m) \rangle \\
2m\Omega^2 = s^*V^2 \lambda \\
2K(m) = \Omega \pi.
\end{cases} \tag{62}
\]

Conditions \(\text{[62]}\) give the connection between \(\lambda\) and \(m\):

\[
\lambda = \frac{2m}{1 + m} \langle \text{sn}^2(\cdot, m) \rangle. \tag{63}
\]

Moreover system \(\text{[62]}\) imposes

\[
\begin{cases}
m \in (-\infty, -1) \quad \text{if } s^* = -1 \\
m \in (0, 1) \quad \text{if } s^* = 1.
\end{cases}
\]

We know that \(m \mapsto \langle \text{sn}^2(\cdot, m) \rangle\) is continuous, strictly increasing on \((-\infty, 1)\), it tends to 0 for \(m \to -\infty\) and to 1 for \(m \to 1\), see Lemma 12. So the right-hand side of \(\text{[63]}\) covers \((0, +\infty)\) for \(m \in (-\infty, 0)\), and
it covers $(0, 1)$ for $m \in (0, 1)$. For this reason for every $\lambda > 0$ there exists a unique $\bar{m} < -1$ satisfying (06), and for every $\lambda \in (0, 1)$ there exists a unique $\bar{m} \in (0, 1)$ satisfying (03).

The value $\bar{m}$ and system (02) determine uniquely the values $\bar{V}, \Omega$. Analyticity and poles follow from [1], 16.2, 16.10.2, pp.570,573. □

Now we have to prove the non-degeneracy of $g$. The linearized equation of (01) at $g$ is
\[ \frac{\bar{h}}{s}(g^2) + 3\lambda g^2)h = -2s(gh)g. \] (64)

Let $L$ be the Green operator, i.e. for $f \in E$, let $H := L(f)$ be the unique solution belonging to $E$ of the non-homogeneous linear system
\[ \bar{H} + s(g^2) - 3\lambda g^2)H = f. \]

We can write (64) as
\[ h = -2s(gh)L(g). \] (65)

Multiplying by $g$ and integrating we get
\[ \langle gh \rangle[1 + 2s(gL(g))] = 0. \]

If $A_0 := 1 + 2s(gL(g)) \neq 0$, then $\langle gh \rangle = 0$, so by (65) $h = 0$ and the non-degeneracy is proved.

It remains to show that $A_0 \neq 0$. As before, the key is to express $L(g)$ in a suitable way. We first look for a fundamental set of solutions of the homogeneous equation
\[ \bar{h} + s(g^2) - 3\lambda g^2)h = 0. \] (66)

Lemma 9. There exist two linearly independent solutions of (66), $\bar{u}$ even, $2\pi$-periodic and $\bar{v}$ odd, not periodic, such that $\bar{u}(0) = 1$, $\bar{u}(0) = 0$, $\bar{v}(0) = 0$, $\bar{v}(0) = 1$, and
\[ \bar{v}(t + 2\pi) - \bar{v}(t) = \rho \bar{u}(t) \quad \forall t \] (67)

for some $\rho \neq 0$. Moreover there hold the following expressions for $\bar{u}$, $\bar{v}$:
\[ \bar{u}(t) = \bar{g}(t)\bar{g}(0) = \sin(\Omega t, \bar{m}) \] (68)
\[ \bar{v}(t) = \frac{1}{\Omega(1 - \bar{m})}\sin(\Omega t) + \frac{\bar{m}}{\bar{m} - 1}\sin(\Omega t) \left[ t + \frac{1 + \bar{m}}{\Omega} \int_0^{\Omega t} \frac{1}{\sin^2(\xi, \bar{m})} d\xi \right]. \] (69)

Proof. $g$ solves (01) so $\bar{g}$ solves (63); normalizing we get (68).

By (60), the function $y(t) = V\sin(\Omega t, m)$ solves
\[ \bar{y} + s(g^2)y - s^*\lambda y^3 = 0 \] (70)

if $(V, \Omega, m)$ satisfy
\[ \begin{cases} \Omega^2(1 + m) = s^*(g^2) \\ 2m\Omega^2 + 2sV^2\lambda. \end{cases} \] (71)

We solve (64) w.r.t. $m$ finding the one-parameter family $(y_m)$ of odd periodic solutions of (64), $y_m(t) = V(m)\sin(\Omega(m)t, m)$. So $l(t) := (\partial_m y_m)|_{m = \bar{m}}$ solves (66). We normalize $\bar{v}(t) := l(t)/l(0)$ and we compute the coefficients differentiating (64) w.r.t. $m$. From the definitions of the Jacobi elliptic functions it holds
\[ \partial_m \sin(x, m) = -\sin(x, m) \frac{1}{2} \int_0^x \frac{\sin^2(\xi, m)}{\sin^2(\xi, m)} d\xi; \]

thanks to this formula we obtain (69).
Since $2\pi\tilde{\Omega} = 4K(\tilde{m})$ is the period of the Jacobi functions $\text{sn}$ and $\text{dn}$, by (63), (64) we obtain (72) with

$$\rho = \frac{\tilde{m}}{m - 1} 2\pi \left( 1 + (1 + \tilde{m})\left(\frac{\text{sn}^2}{\text{dn}^2}\right) \right).$$

If $s^* = 1$, then $\tilde{m} \in (0,1)$ and directly we can see that $\rho < 0$. If $s^* = -1$, then $\tilde{m} < -1$. From the equality $\langle \text{sn}^2/\text{dn}^2 \rangle = (1 - m)^{-1} (1 - \langle \text{sn}^2 \rangle)$ (see [3], Lemma 3, (L.2)), it results $\rho > 0$. 

We can note that the integral representation (39) of the Green operator $L$ holds again in the present case. The proof is just like in Lemma 1.

**Lemma 10.** We can write $A_0 := 1 + 2s^*(gL(g))$ as function of $\lambda, \tilde{m}$,

$$A_0 = \frac{\lambda(1 - m)^2 q - (1 - \lambda)^2(1 + m)^2 + \tilde{m} q^2}{\lambda(1 - m)^2 q}, \quad q = q(\lambda, \tilde{m}) := 2 - \frac{(1 + \tilde{m})^2}{2\tilde{m}} > 0. \quad (72)$$

**Proof.** First, we calculate $gL(g)$ with the integral formula (39) of $L$. The equalities (12), (13) still hold, while similar calculations give

$$\int_0^{2\pi} g^3\bar{v} = -s^* \frac{\rho(0)}{2\lambda}$$

instead of (14). So

$$\langle gL(g) \rangle = -s^* \frac{\rho}{4\pi\lambda} + \frac{1}{2\pi\rho} \left( \int_0^{2\pi} g\bar{v} \right)^2 \quad (73)$$

and the sign of $A_0$ is not obvious. We calculate $\int_0^{2\pi} g\bar{v}$ recalling that $g(t) = \bar{V}\text{sn}(\tilde{\Omega}t, \tilde{m})$, using formula (10) for $\bar{v}$ and integrating by parts

$$\int_0^{2\pi} \text{sn}(\bar{\Omega}t)\text{sn}(\bar{\Omega}t)\mu(t) dt = -\frac{1}{2\bar{\Omega}} \int_0^{2\pi} \text{sn}^2(\bar{\Omega}t)\bar{\mu}(t) dt$$

where $\mu(t) := t + (1 + \tilde{m})\bar{\Omega}^{-1} \int_0^{\tilde{\Omega}} \text{sn}^2(t)\text{dn}^2(t) dt$. From [3], (L.2), (L.3) in Lemma 3, we obtain the formula

$$\langle \text{sn}^4/\text{dn}^2 \rangle = \frac{1 + (m - 2)\langle \text{sn}^2 \rangle}{m(1 - m)}$$

and consequently

$$\int_0^{2\pi} g\bar{v} = \frac{\pi\bar{V}}{\bar{\Omega}(1 - \tilde{m})^2} (1 + \tilde{m} - 2\tilde{m}\langle \text{sn}^2 \rangle). \quad (74)$$

By the second equality of (12) and (13) we get

$$A_0 = 1 + \frac{2}{\lambda} \left[ -\frac{\rho}{4\pi} + \frac{\pi\tilde{m}}{\rho(1 - \tilde{m})^4} (1 + \tilde{m} - 2\tilde{m}\langle \text{sn}^2 \rangle)^2 \right] \quad (75)$$

both for $s^* = \pm 1$. From the proof of Lemma 3 we have $\rho = -2\pi\tilde{m}q(1 - \tilde{m})^{-2}$, where $q$ is defined in (72); inserting this expression of $\rho$ in (75) we obtain (72).

Finally, for $\tilde{m} < -1$ we have immediately $q > 0$, while for $\tilde{m} \in (0,1)$ we get $q = 2 - (1 + \tilde{m})\langle \text{sn}^2 \rangle$ by (13). Since $\langle \text{sn}^2 \rangle < 1$, it results $q > 0$. 

**Lemma 11.** $A_0 \neq 0$. More precisely, $\text{sign}(A_0) = -s^*$. 

Proof. From (52), $A_0 > 0$ iff $\lambda(1 - \tilde{m})^2 q - (1 - \lambda)^2 (1 + \tilde{m})^2 + \tilde{m} q^2 > 0$. This expression is equal to $-(1 - \tilde{m})^2 p$, where

$$p = p(\lambda, \tilde{m}) = \frac{(1 + \tilde{m})^2}{4\tilde{m}} \lambda^2 - 2\lambda + 1,$$

so $A_0 > 0$ iff $p < 0$. The polynomial $p(\lambda)$ has degree 2 and its determinant is $\Delta = -(1 - \tilde{m})^2/\tilde{m}$. So, if $s^* = 1$, then $\tilde{m} \in (0, 1)$, $\Delta < 0$ and $p > 0$, so that $A_0 < 0$.

It remains the case $s^* = -1$. For $\lambda > 0$, we have $p(\lambda) < 0$ iff $\lambda > x^*$, where $x^*$ is the positive root of $p$, $x^* := 2R(1 + R)^{-1}$, $R := |\tilde{m}|^{1/2}$. By (53), $\lambda > x^*$ iff

$$\langle \sin^2(\cdot, \tilde{m}) \rangle > \frac{R - 1}{(R + 1)R}.$$  

By formula (53) and by definition of complete elliptic integrals $K$ and $E$ we can write (76) as

$$\int_0^{\pi/2} \left( \frac{R - 1}{(R + 1)R} - \sin^2 \vartheta \right) \frac{d\vartheta}{\sqrt{1 + R^2 \sin^2 \vartheta}} < 0. \tag{77}$$

We put $\sigma := R - 1/(R + 1)R$ and note that $\sigma < 1/2$ for every $R > 0$.

$\sigma - \sin^2 \vartheta > 0$ iff $\vartheta \in (0, \vartheta^*)$, where $\vartheta^* := \arcsin(\sqrt{\sigma})$, i.e. $\sin^2 \vartheta^* = \sigma$. Moreover $1 < 1 + R^2 \sin^2 \vartheta < 1 + R^2$ for every $\vartheta \in (0, \pi/2)$. So

$$\int_0^{\vartheta^*} \frac{\sigma - \sin^2 \vartheta}{\sqrt{1 + R^2 \sin^2 \vartheta}} d\vartheta < \int_0^{\vartheta^*} (\sigma - \sin^2 \vartheta) d\vartheta + \int_0^{\pi/2} \frac{\sigma - \sin^2 \vartheta}{\sqrt{1 + R^2}} d\vartheta. \tag{78}$$

Thanks to the formula

$$\int_a^b \sin^2 \vartheta d\vartheta = \frac{b - a}{2} - \frac{\sin(2b) - \sin(2a)}{4}$$

the right-hand side term of (78) is equal to

$$\frac{\sin(2\vartheta^*)}{4} \left( (2\sigma - 1) \left( \frac{2\vartheta^*}{\sin(2\vartheta^*)} + \frac{1}{\sqrt{1 + R^2}} \right) + \left( 1 - \frac{1}{\sqrt{1 + R^2}} \right) \right).$$

Since $2\sigma - 1 < 0$ and $\alpha > \sin \alpha$ for every $\alpha > 0$, this quantity is less than

$$\frac{\sin(2\vartheta^*)}{4} \left( (2\sigma - 1) \left( 1 + \frac{1}{\sqrt{1 + R^2}} \right) + \left( 1 - \frac{1}{\sqrt{1 + R^2}} \right) \right).$$

By definition of $\sigma$, the last quantity is negative for every $R > 0$, so (78) is true. Consequently $\lambda > x^*$, $p < 0$ and $A_0 > 0$. $\blacksquare$

As Appendix, we show the properties of the function $m \mapsto \langle \sin^2(\cdot, m) \rangle$ used in the proof of Lemma S

**Lemma 12.** The function $\varphi : (-\infty, 1) \to \mathbb{R}$, $m \mapsto \langle \sin^2(\cdot, m) \rangle$ is continuous, differentiable, strictly increasing, and $\lim_{m \to -\infty} \varphi(m) = 0$, $\lim_{m \to 1} \varphi(m) = 1$.

**Proof.** By (53) and by definition of complete elliptic integrals $K$ and $E$,

$$\varphi(m) = \frac{K(m) - E(m)}{mK(m)} = \int_0^{\pi/2} \frac{\sin^2 \vartheta d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}} \left( \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m \sin^2 \vartheta}} \right)^{-1},$$

so the continuity of $\varphi$ is evident.

Using the equality $\sin^2 \vartheta + \cos^2 \vartheta = 1$ and the change of variable $\vartheta \to \pi/2 - \vartheta$ir in the integrals which define $K$ and $E$, we obtain the formulæ

$$K(m) = \frac{1}{\sqrt{1 - m}} K\left( \frac{m}{m - 1} \right), \quad E(m) = \sqrt{1 - m} E\left( \frac{m}{m - 1} \right) \quad \forall m < 1. \tag{79}$$
We put $\mu := m/(m - 1)$, so it results

$$\varphi(m) = 1 - \frac{1}{\mu} + \frac{E(\mu)}{\mu K(\mu)}. \tag{80}$$

Since $\mu$ tends to 1 as $m \to -\infty$, $E(1) = 1$ and $\lim_{\mu \to 1} K(\mu) = +\infty$, (79), (80) give $\lim_{m \to -\infty} \varphi(m) = 0$. Since $E(m)/K(m)$ tends to 0 as $m \to 1$, (53) gives $\lim_{m \to 1} \varphi(m) = 1$.

Differentiating the integrals which define $K$ and $E$ w.r.t. $m$ we obtain the formulae

$$E'(m) = \frac{E(m) - K(m)}{2m}, \quad K'(m) = \frac{1}{2m} \left( \int_0^{\pi/2} \frac{d\theta}{(1 - m \sin^2 \theta)^{3/2}} - K(m) \right),$$

so the derivative is

$$\varphi'(m) = \frac{1}{2m^2 K^2(m)} \left[ E(m) \int_0^{\pi/2} \frac{d\theta}{(1 - m \sin^2 \theta)^{3/2}} - K^2(m) \right].$$

The term in the square brackets is positive by strict Hölder inequality for $(1 - m \sin^2 \theta)^{-3/4}$ and $(1 - m \sin^2 \theta)^{1/4}$.

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**References**


