Periodic solutions of wave equations for asymptotically full measure sets of frequencies

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1 Introduction

The aim of this Note is to prove existence and multiplicity of small amplitude periodic solutions of the completely resonant wave equation

$$\begin{cases} \Box u + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1}$$

where $\Box := \partial_{tt} - \partial_{xx}$ is the D'Alambertian operator and

$$f(x,u) = a_2 u^2 + a_3(x) u^3 + O(u^4) \qquad \text{or} \qquad f(x,u) = a_4 u^4 + O(u^5) \tag{2}$$

for a Cantor-like set of frequencies ω of asymptotically full measure at $\omega = 1$.

Equation (1) is called completely resonant because any solution $v = \sum_{j\geq 1} a_j \cos(jt + \vartheta_j)\sin(jx)$ of the linearized equation at $u = 0$

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{3}$$

is 2π-periodic in time.

Existence and multiplicity of periodic solutions of completely resonant wave equations had been proved for a zero measure, uncontable Cantor set of frequencies in [4] for $f(u) = u^3 + O(u^5)$ and in [5]-[6] for any nonlinearity $f(u) = a_p u^p + O(u^{p+1})$, $p \geq 2$.

Existence of periodic solutions for a Cantor-like set of frequencies of asymptotically full measure has been recently proved in [7] where, due to the well known "small divisor difficulty", the "0th order bifurcation equation" is required to possess non-degenerate periodic solutions. Such property was verified in [7] for nonlinearities like $f = a_2 u^2 + O(u^4)$, $f = a_3(x) u^3 + O(u^4)$. See also [11] for $f = u^3 + O(u^5)$.

In this Note we shall prove that, for quadratic, cubic and quartic nonlinearities $f(x,u)$ like in (2), the corresponding 0th order bifurcation equation possesses non-degenerate periodic solutions – Propositions 1 and 2 –, implying, by the results of [7], Theorem 1 and Corollary 1 below.

We remark that our proof is purely analytic (it does not use numerical calculations) being based on the analysis of the variational equation and exploiting properties of the Jacobi elliptic functions.

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1.1 Main results

Normalizing the period to 2π, we look for solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases}$$

in the Hilbert algebra (for $s > 1/2$, $\sigma > 0$)

$$X_{\sigma,s} := \Big\{ u(t,x) = \sum_{l\geq 0} \cos(lt)\ u_l(x) \ \Big| \ u_l \in H_0^1((0,\pi),\mathbb{R}) \ \ \forall l \in \mathbb{N} \ \text{ and } \|u\|^2_{\sigma,s} := \sum_{l \geq 0} \exp(2\sigma l)(l^{2s}+1)\|u_l\|^2_{H^1} < +\infty \Big\}.$$

It is natural to look for solutions which are even in time because equation (1) is reversible.

We look as well for solutions of (1) in the subalgebras

$$X_{\sigma,s,n} := \Big\{ u \in X_{\sigma,s} \mid u \text{ is } \frac{2\pi}{n}\text{-periodic} \Big\} \subset X_{\sigma,s}\,, \quad n \in \mathbb{N}$$

(they are particular 2π-periodic solutions).

The space of the solutions of the linear equation (3) that belong to $H_0^1(\mathbb{T}\times(0,\pi),\mathbb{R})$ and are even in time is

$$\begin{aligned} V \ &:= \ \Big\{ v(t,x) = \sum_{l\geq 1} \cos(lt) u_l \sin(lx) \ \Big| \ u_l \in \mathbb{R}\,, \ \sum_{l \geq 1} l^2 |u_l|^2 < +\infty \Big\} \\ &= \ \Big\{ v(t,x) = \eta(t+x) - \eta(t-x) \ \Big| \ \eta \in H^1(\mathbb{T},\mathbb{R}) \text{ with } \eta \text{ odd} \Big\}. \end{aligned}$$

Theorem 1. *Let*

$$f(x,u) = a_2 u^2 + a_3(x) u^3 + \sum_{k\geq 4} a_k(x) u^k \tag{4}$$

where $(a_2, \langle a_3\rangle) \neq (0,0)$, $\langle a_3 \rangle := \pi^{-1}\int_0^\pi a_3(x)dx$, *or*

$$f(x,u) = a_4 u^4 + \sum_{k\geq 5} a_k(x) u^k \tag{5}$$

where $a_4 \neq 0$, $a_5(\pi - x) = -a_5(x)$, $a_6(\pi-x) = a_6(x)$, $a_7(\pi-x) = -a_7(x)$. *Assume moreover* $a_k(x) \in H^1((0,\pi),\mathbb{R})$ *with* $\sum_k \|a_k\|_{H^1}\rho^k < +\infty$ *for some* $\rho > 0$.

Then there exists $n_0 \in \mathbb{N}$ *such that* $\forall n \geq n_0$ *there is* $\delta_0 > 0$, $\bar\sigma > 0$ *and a* C^∞*-curve* $[0,\delta_0) \ni \delta \to u_\delta \in X_{\bar\sigma/2,s,n}$ *with the following properties:*

- *(i)* $\|u_\delta - \delta \bar v_n\|_{\bar\sigma/2,s,n} = O(\delta^2)$ *for some* $\bar v_n \in V \cap X_{\bar\sigma,s,n} \setminus \{0\}$ *with minimal period* $2\pi/n$;
- *(ii) there exists a Cantor set* $\mathcal{C}_n \subset [0,\delta_0)$ *of asymptotically full measure, i.e. satisfying*

$$\lim_{\varepsilon\to 0^+} \frac{\text{meas}(\mathcal{C}_n \cap (0,\varepsilon))}{\varepsilon} = 1\,, \tag{6}$$

such that, $\forall\ \delta \in \mathcal{C}_n$, $u_\delta(\omega(\delta)t,x)$ *is a* $2\pi/(\omega(\delta)n)$*-periodic, classical solution of (1) with*

$$\omega(\delta) = \begin{cases} \sqrt{1-2s^*\delta^2} & \text{if } f \text{ is like in (4)} \\ \sqrt{1-2\delta^6} & \text{if } f \text{ is like in (5)} \end{cases}$$

and[1]

$$s^* = \begin{cases} -1 & \text{if} \quad \langle a_3\rangle \geq \pi^2 a_2^2/12 \\ \pm 1 & \text{if} \quad 0 < \langle a_3 \rangle < \pi^2 a_2^2/12 \\ 1 & \text{if} \quad \langle a_3 \rangle \leq 0\,. \end{cases}$$

[1] Note how the interaction between the second and the third order terms a_2u^2, $a_3(x)u^3$ changes the bifurcation diagram, i.e. existence of periodic solutions for frequencies ω less or/and greater of $\omega = 1$.

By (6) also each Cantor-like set of frequencies $\mathcal{W}_n := \{\omega(\delta) \mid \delta \in \mathcal{C}_n\}$ has asymptotically full measure at $\omega = 1$.

Corollary 1. ***(Multiplicity)*** *There exists a Cantor-like set $\mathcal{W}$ of asymptotically full measure at $\omega = 1$, such that $\forall \omega \in \mathcal{C}$, equation (1) possesses geometrically distinct periodic solutions*

$$u_{n_0}, \ldots, u_n, \ldots u_{N_\omega}, \qquad N_\omega \in \mathbb{N}$$

with the same period $2\pi/\omega$. Their number increases arbitrarily as ω tends to 1:

$$\lim_{\omega \to 1} N_\omega = +\infty .$$

Proof. The proof is like in [7] and we report it for completeness. If δ belongs to the asymptotically full measure set (by (6))

$$D_n := \mathcal{C}_{n_0} \cap \ldots \cap \mathcal{C}_n , \qquad n \geq n_0$$

there exist $(n - n_0 + 1)$ geometrically distinct periodic solutions of (1) with the same period $2\pi/\omega(\delta)$ (each u_n has minimal period $2\pi/(n\omega(\delta))$).

There exists a decreasing sequence of positive $\varepsilon_n \to 0$ such that

$$\text{meas}(D_n^c \cap (0, \varepsilon_n)) \leq \varepsilon_n 2^{-n} .$$

Let define the set $\mathcal{C} \equiv D_n$ on each $[\varepsilon_{n+1}, \varepsilon_n)$. $\mathcal{C}$ has asymptotically full measure at $\delta = 0$ and for each $\delta \in \mathcal{C}$ there exist $N(\delta) := \max\{n \in \mathbb{N} : \delta < \varepsilon_n\}$ geometrically distinct periodic solutions of (1) with the same period $2\pi/\omega(\delta)$. $N(\delta) \to +\infty$ as $\delta \to 0$. ■

Remark 1. *Corollary 1 is an analogue for equation (1) of the well known multiplicity results of Weinstein-Moser [15]-[13] and Fadell-Rabinowitz [10] which hold in finite dimension. The solutions form a sequence of functions with increasing norms and decreasing minimal periods. Multiplicity of solutions was also obtained in [6] (with the "optimal" number $N_\omega \approx C/\sqrt{|\omega - 1|}$) but only for a zero measure set of frequencies.*

The main point for proving Theorem 1 relies in showing the existence of non-degenerate solutions of the 0th order bifurcation equation for f like in (2). In these cases the 0th order bifurcation equation involves higher order terms of the nonlinearity, and, for n large, can be reduced to an integro-differential equation (which physically describes an averaged effect of the nonlinearity with Dirichlet boundary conditions).

Case $f(x, u) = a_4 u^4 + O(u^5)$. Performing the rescaling

$$u \to \delta u , \qquad \delta > 0$$

we look for $2\pi/n$-periodic solutions in $X_{\sigma,s,n}$ of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^3 g(\delta, x, u) = 0 \\ u(t, 0) = u(t, \pi) = 0 \end{cases} \tag{7}$$

where

$$g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^4} = a_4 u^4 + \delta a_5(x) u^5 + \delta^2 a_6(x) u^6 + \ldots .$$

To find solutions of (7) we implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

$$X_{\sigma,s,n} = (V_n \cap X_{\sigma,s,n}) \oplus (W \cap X_{\sigma,s,n})$$

where

$$V_n \quad := \Big\{ v(t, x) = \eta(nt + nx) - \eta(nt - nx) \mid \eta \in H^1(\mathbb{T}, \mathbb{R}) \ \text{ with } \ \eta \text{ odd} \Big\}$$

and

$$W := \Big\{ w = \sum_{l \geq 0} \cos(lt)\ w_l(x) \in X_{0,s} \ \Big| \quad \int_0^\pi w_l(x) \sin(lx)\, dx = 0,\ \forall l \geq 0 \ \Big\}.$$

Looking for solutions $u = v + w$ with $v \in V_n \cap X_{\sigma,s,n}$, $w \in W \cap X_{\sigma,s,n}$, and imposing the frequency-amplitude relation

$$\frac{(\omega^2 - 1)}{2} = -\delta^6$$

we are led to solve the bifurcation equation and the range equation

$$\begin{cases} \Delta v = \delta^{-3} \Pi_{V_n} g(\delta, x, v + w) \\ L_\omega w = \delta^3 \Pi_{W_n} g(\delta, x, v + w) \end{cases} \tag{8}$$

where

$$\Delta v := v_{xx} + v_{tt}, \qquad\qquad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}$$

and $\Pi_{V_n} : X_{\sigma,s,n} \to V_n \cap X_{\sigma,s,n}$, $\Pi_{W_n} : X_{\sigma,s,n} \to W \cap X_{\sigma,s,n}$ denote the projectors.

With the further rescaling

$$w \to \delta^3 w$$

and since $v^4 \in W_n$ (Lemma 3.4 of [5]), $a_5(x)v^5$, $a_6(x)v^6$, $a_7(x)v^7 \in W_n$ because $a_5(\pi - x) = -a_5(x)$, $a_6(\pi - x) = a_6(x)$, $a_7(\pi - x) = -a_7(x)$ (Lemma 7.1 of [7]), system (8) is equivalent to

$$\begin{cases} \Delta v = \Pi_{V_n} \Big(4 a_4 v^3 w + \delta r(\delta, x, v, w) \Big) \\ L_\omega w = a_4 v^4 + \delta \Pi_{W_n} \widetilde{r}(\delta, x, v, w) \end{cases} \tag{9}$$

where $r(\delta, x, v, w) = a_8(x)v^8 + 5a_5(x)v^4 w + O(\delta)$ and $\widetilde{r}(\delta, x, v, w) = a_5(x)v^5 + O(\delta)$.

For $\delta = 0$ system (9) reduces to $w = -a_4 \Box^{-1} v^4$ and to the 0th order bifurcation equation

$$\Delta v + 4 a_4^2 \Pi_{V_n} \Big(v^3 \Box^{-1} v^4 \Big) = 0 \tag{10}$$

which is the Euler-Lagrange equation of the functional $\Phi_0 : V_n \to \mathbb{R}$

$$\Phi_0(v) = \frac{\|v\|_{H_1}^2}{2} - \frac{a_4^2}{2} \int_\Omega v^4 \Box^{-1} v^4 \tag{11}$$

where $\Omega := \mathbb{T} \times (0, \pi)$.

Proposition 1. *Let $a_4 \neq 0$. $\exists\ n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the 0th order bifurcation equation (10) has a solution $\bar{v}_n \in V_n$ which is non-degenerate in V_n (i.e.* $\mathrm{Ker} D^2 \Phi_0 = \{0\}$*), with minimal period $2\pi/n$.*

Case $f(x, u) = a_2 u^2 + a_3(x) u^3 + O(u^4)$. Performing the rescaling $u \to \delta u$ we look for $2\pi/n$-periodic solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta g(\delta, x, u) = 0 \\ u(t, 0) = u(t, \pi) = 0 \end{cases}$$

where

$$g(\delta, x, u) := \frac{f(x, \delta u)}{\delta^2} = a_2 u^2 + \delta a_3(x) u^3 + \delta^2 u_4(x) u^4 + \ldots .$$

With the frequency-amplitude relation

$$\frac{\omega^2 - 1}{2} = -s^* \delta^2$$

where $s^* = \pm 1$, we have to solve

$$\begin{cases} -\Delta v = -s^* \delta^{-1} \Pi_{V_n} g(\delta, x, v + w) \\ L_\omega w = \delta \Pi_{W_n} g(\delta, x, v + w)\,. \end{cases} \tag{12}$$

With the further rescaling $w \to \delta w$ and since $v^2 \in W_n$, system (12) is equivalent to

$$\begin{cases} -\Delta v = s^* \Pi_{V_n} \Big(-2a_2 vw - a_2 \delta w^2 - a_3(x)(v+\delta w)^3 - \delta r(\delta, x, v+\delta w) \Big) \\ L_\omega w = a_2 v^2 + \delta \Pi_{W_n} \Big(2a_2 vw + \delta a_2 w^2 + a_3(x)(v+\delta w)^3 + \alpha_8(x) v^8 dr(\delta, x, v+\delta w) \Big) \end{cases} \tag{13}$$

where $r(\delta, x, u) := \delta^{-4}[f(x, \delta u) - a_2 \delta^2 u^2 - \delta^3 a_3(x) u^3] = a_4(x) u^4 + \ldots$

For $\delta = 0$ system (13) reduces to $w = -a_2 \Box^{-1} v^2$ and the 0th order bifurcation equation

$$-s^* \Delta v = 2a_2^2 \Pi_{V_n}(v \Box^{-1} v^2) - \Pi_{V_n}(a_3(x) v^3) \tag{14}$$

which is the Euler-Lagrange equation of $\Phi_0 : V_n \to \mathbb{R}$

$$\Phi_0(v) := s^* \frac{\|v\|_{H^1}^2}{2} - \frac{a_2^2}{2} \int_\Omega v^2 \Box^{-1} v^2 + \frac{1}{4} \int_\Omega a_3(x) v^4 \, . \tag{15}$$

Proposition 2. *Let $(a_2, \langle a_3 \rangle) \neq 0$. $\exists \; n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the 0th order bifurcation equation (14) has a solution $\bar{v}_n \in V_n$ which is non-degenerate in V_n, with minimal period $2\pi/n$.*

2 Case $f(x, u) = a_4 u^4 + O(u^5)$

We have to prove the existence of *non-degenerate* critical points of the functional

$$\Phi_n : V \to \mathbb{R} \, , \qquad \Phi_n(v) := \Phi_0(\mathcal{H}_n v)$$

where Φ_0 is defined in (11). Let $\mathcal{H}_n : V \to V$ be the linear isomorphism defined, for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$, by

$$a_8(x) v^8 (\mathcal{H}_n v)(t,x) := \eta(n(t+x)) - \eta(n(t-x))$$

so that $V_n \equiv \mathcal{H}_n V$.

Lemma 1. ***See [6].*** *Φ_n has the following development: for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$*

$$\Phi_n(\beta n^{1/3} v) = 4\pi \beta^2 n^{8/3} \Big[\Psi(\eta) + \alpha \frac{\mathcal{R}(\eta)}{n^2} \Big] \tag{16}$$

where $\beta := (3/(\pi^2 a_4^2))^{1/6}$, $\alpha := a_4^2/(8\pi)$,

$$\Psi(\eta) := \frac{1}{2} \int_{\mathbb{T}} \eta'^2(t) \, dt - \frac{2\pi}{8} \Big(\langle \eta^4 \rangle + 3 \langle \eta^2 \rangle^2 \Big)^2 \, , \tag{17}$$

$\langle \; \rangle$ denotes the average on $\mathbb{T}$, and

$$\mathcal{R}(\eta) := - \int_\Omega v^4 \Box^{-1} v^4 \, dt dx + \frac{\pi^4}{6} 4 \Big(\langle \eta^4 \rangle + 3 \langle \eta^2 \rangle^2 \Big)^2 . \tag{18}$$

Proof. Firstly the quadratic term writes

$$\frac{1}{2} \|\mathcal{H}_n v\|_{H^1}^2 = \frac{n^2}{2} \|v\|_{H^1}^2 = n^2 2\pi \int_{\mathbb{T}} \eta'^2(t) \, dt \, . \tag{19}$$

By Lemma 4.8 in [6] the non-quadratic term can be developed as

$$\int_\Omega (\mathcal{H}_n v)^4 \Box^{-1} (\mathcal{H}_n v)^4 = \frac{\pi^4}{6} \langle m \rangle^2 - \frac{\mathcal{R}(\eta)}{n^2} \tag{20}$$

where $m : \mathbb{T}^2 \to \mathbb{R}$ is $m(s_1, s_2) := (\eta(s_1) - \eta(s_2))^4$, $\langle m \rangle := (2\pi)^{-2} \int_{\mathbb{T}^2} m(s_1, s_2) \, ds_1 ds_2$ denotes its average, and

$$\mathcal{R}(\eta) := \Big(- \int_\Omega v^4 \Box^{-1} v^4 + \frac{\pi^4}{6} \langle m \rangle^2 \Big) \tag{21}$$

is homogeneous of degree 8. Since η is odd we find

$$\langle m\rangle = 2\Big(\langle\eta^4\rangle + 3\langle\eta^2\rangle^2\Big) \tag{22}$$

where $\langle\ \rangle$ denotes the average on $\mathbb{T}$.

Collecting (19), (20), (21) and (22) we find out

$$\Phi_n(\eta) = 2\pi n^2\int_{\mathbb{T}}\eta'^2(t)\,dt - \frac{\pi^4}{3}a_4^2\Big(\langle\eta^4\rangle + 3\langle\eta^2\rangle^2\Big)^2 + \frac{a_4^2}{2n^2}\mathcal{R}(\eta)\,.$$

Via the rescaling $\eta\to\beta n^{1/3}\eta$ we get the expressions (17) and (18). ■

By (16), in order to find for n large enough a non-degenerate critical point of Φ_n, it is sufficient to find a non-degenerate critical point of $\Psi(\eta)$ defined on

$$E := \Big\{\eta\in H^1(\mathbb{T}),\ \eta \text{ odd}\Big\}\,,$$

namely non-degenerate solutions in E of

$$\ddot\eta + A(\eta)\Big(3\langle\eta^2\rangle\eta + \eta^3\Big) = 0 \qquad A(\eta) := \langle\eta^4\rangle + 3\langle\eta^2\rangle^2\,. \tag{23}$$

Proposition 3. *There exists an odd, analytic, 2π-periodic solution $g(t)$ of (23) which is non-degenerate in E. $g(t) = V\mathrm{sn}(\Omega t, m)$ where sn is the Jacobi elliptic sine and $V>0$, $\Omega>0$, $m\in(-1,0)$ are suitable constants (therefore $g(t)$ has minimal period 2π).*

We will construct the solution g of (23) by means of the Jacobi elliptic sine in Lemma 6. The existence of a solution g follows also directly applying to $\Psi : E\to\mathbb{R}$ the Mountain-Pass Theorem [2]. Furthermore such solution is an analytic function arguing as in Lemma 2.1 of [7].

2.1 Non-degeneracy of g

We now want to prove that g is non-degenerate. The linearized equation of (23) at g is

$$\begin{aligned}&\ddot h + 3A(g)\Big[\langle g^2\rangle h + g^2 h\Big] + 6A(g)g\langle gh\rangle + A'(g)[h]\Big(3\langle g^2\rangle g + g^3\Big) = \\ &\ddot h + 3A(g)\left[\langle g^2\rangle + g^2\right]h + 6g\langle gh\rangle\Big(\langle g^4\rangle + 3\langle g^2\rangle^2\Big) + 4g\Big(\langle g^3h\rangle + 3\langle g^2\rangle\langle gh\rangle\Big)\Big(3\langle g^2\rangle + g^2\Big) = 0\end{aligned}$$

that we write as

$$\ddot h + 3A(g)\left(\langle g^2\rangle + g^2\right)h = -\langle gh\rangle I_1 - \langle g^3h\rangle I_2 \tag{24}$$

where

$$\begin{cases} I_1 := 6\Big(9\langle g^2\rangle^2 + \langle g^4\rangle\Big)g + 12\langle g^2\rangle g^3 \\ I_2 := 12g\langle g^2\rangle + 4g^3\,. \end{cases} \tag{25}$$

For $f\in E$, let $H := L(f)$ be the unique solution belonging to E of the non-homogeneous linear system

$$\ddot H + 3A(g)\left(\langle g^2\rangle + g^2\right)H = f\,; \tag{26}$$

an integral representation of the Green operator L is given in Lemma 4. Thus (24) becomes

$$h = -\langle gh\rangle L(I_1) - \langle g^3h\rangle L(I_2)\,. \tag{27}$$

Multiplying (27) by g and taking averages we get

$$\langle gh\rangle\left[1 + \langle gL(I_1)\rangle\right] = -\langle g^3h\rangle\,\langle gL(I_2)\rangle, \tag{28}$$

while multiplying (27) by g^3 and taking averages

$$\langle g^3h\rangle\left[1 + \langle g^3L(I_2)\rangle\right] = -\langle gh\rangle\,\langle g^3L(I_1)\rangle. \tag{29}$$

Since g solves (23) we have the following identities.

Lemma 2. *There holds*

$$2A(g)\langle g^3 L(g)\rangle = \langle g^2\rangle \tag{30}$$

$$2A(g)\langle g^3 L(g^3)\rangle = \langle g^4\rangle\,. \tag{31}$$

Proof. (30) is obtained by the identity for $L(g)$

$$\frac{d^2}{dt^2}(L(g)) + 3A(g)\left(\langle g^2\rangle + g^2\right)L(g) = g$$

multiplying by g, taking averages, integrating by parts,

$$\langle \ddot g L(g)\rangle + 3A(g)\left[\langle g^2\rangle\langle L(g)g\rangle + \langle g^3 L(g)\rangle\right] = \langle g^2\rangle$$

and using that g solves (23).

Analogously, (31) is obtained by the identity for $L(g^3)$

$$\frac{d^2}{dt^2}(L(g^3)) + 3A(g)\left(\langle g^2\rangle + g^2\right)L(g^3) = g^3$$

multiplying by g, taking averages, integrating by parts, and using that g solves (23). ∎

Since L is a symmetric operator we can compute the following averages using (25), (30), (31):

$$\begin{cases} \langle gL(I_1)\rangle = 6\left(\langle g^4\rangle + 9\langle g^2\rangle^2\right)\langle gL(g)\rangle + 6\,A(g)^{-1}\,\langle g^2\rangle^2 \\ \langle gL(I_2)\rangle = 12\langle g^2\rangle\,\langle gL(g)\rangle + 2\,A(g)^{-1}\,\langle g^2\rangle \\ \langle g^3 L(I_1)\rangle = 9\langle g^2\rangle \\ \langle g^3 L(I_2)\rangle = 2\,. \end{cases} \tag{32}$$

Thanks to the identities (32), equations (28), (29) simplify to

$$\begin{cases} \langle gh\rangle\left[A(g) + 6\langle g^2\rangle^2\right]B(g) = -2\,\langle g^2\rangle\,B(g)\,\langle g^3 h\rangle \\ \langle g^3 h\rangle = -3\langle g^2\rangle\,\langle gh\rangle \end{cases} \tag{33}$$

where

$$B(g) := 1 + 6A(g)\langle gL(g)\rangle\,. \tag{34}$$

Solving (33) we get

$$B(g)\langle gh\rangle = 0\,.$$

We will prove in Lemma 5 that $B(g) \neq 0$, so $\langle gh\rangle = 0$. Hence by (33) also $\langle g^3 h\rangle = 0$ and therefore, by (27), $h = 0$. This concludes the proof of the non-degeneracy of the solution g of (23).

It remains to prove that $B(g) \neq 0$. The key is to express the function $L(g)$ by means of the variation of constants formula.

We first look for a fundamental set of solutions of the homogeneous equation

$$\ddot h + 3A(g)\left(\langle g^2\rangle + g^2\right)h = 0\,. \tag{HOM}$$

Lemma 3. *There exist two linearly independent solutions of (HOM), $\bar u := \dot g(t)/\dot g(0)$ and $\bar v$, such that*

$$\begin{cases} \bar u \ \text{ is even, } 2\pi\,\text{periodic} \\ \bar u(0) = 1\,,\ \dot{\bar u}(0) = 0 \end{cases} \qquad \begin{cases} \bar v \ \text{ is odd, not periodic} \\ \bar v(0) = 0\,,\ \dot{\bar v}(0) = 1 \end{cases}$$

and

$$\bar v(t + 2\pi) - \bar v(t) = \rho\bar u(t) \qquad \text{for some } \rho > 0\,. \tag{35}$$

PROOF. Since (23) is autonomous, $\dot g(t)$ is a solution of the linearized equation (HOM). $\dot g(t)$ is even and 2π-periodic.

We can construct another solution of (HOM) in the following way. The super-quadratic Hamiltonian system (with constant coefficients)

$$\ddot y + 3A(g)\langle g^2\rangle\, y + A(g)\, y^3 = 0 \tag{36}$$

possesses a one-parameter family of odd, $T(E)$-periodic solutions $y(E,t)$, close to g, parametrized by the energy E. Let $\bar E$ denote the energy level of g, i.e. $g = y(\bar E, t)$ and $T(\bar E) = 2\pi$.

Therefore $l(t) := (\partial_E y(E,t))_{|E=\bar E}$ is an odd solution of (HOM).

Deriving the identity $y(E, t+T(E)) = y(E,t)$ with respect to E we obtain at $E = \bar E$

$$l(t+2\pi) - l(t) = -(\partial_E T(E))_{|E=\bar E}\, \dot g(t)$$

and, normalizing $\bar v(t) := l(t)/\dot l(0)$, we get (35) with

$$\rho := -(\partial_E T(E))_{|E=\bar E} \Big(\frac{\dot g(0)}{\dot l(0)} \Big) . \tag{37}$$

Since $y(E,0) = 0\ \forall E$, the energy identity gives $E = \frac{1}{2}(\dot y(E,0))^2$. Deriving w.r.t E at $E = \bar E$, yields $1 = \dot g(0)\dot l(0)$ which, inserted in (37), gives

$$\rho = -(\partial_E T(E))_{|E=\bar E}\, (\dot g(0))^2 . \tag{38}$$

$\rho > 0$ because $(\partial_E T(E))_{|E=\bar E} < 0$ by the superquadraticity of the potential of (36). It can be checked also by a computation, see Remark after Lemma 6. ∎

Now we write an integral formula for the Green operator L.

Lemma 4. *For every $f \in E$ there exists a unique solution $H = L(f)$ of (26) which can be written as*

$$L(f) = \Big(\int_0^t f(s)\bar u(s)\, ds + \frac{1}{\rho}\int_0^{2\pi} f\bar v \Big) \bar v(t) - \Big(\int_0^t f(s)\bar v(s)\, ds \Big) \bar u(t) \ \in E . \tag{39}$$

PROOF. The non-homogeneous equation (26) possesses the particular solution

$$\bar H(t) = \Big(\int_0^t f(s)\bar u(s)\, ds \Big) \bar v(t) - \Big(\int_0^t f(s)\bar v(s)\, ds \Big) \bar u(t)$$

as can be verified noting that the Wronskian $\bar u(t)\dot{\bar v}(t) - \dot{\bar u}(t)\bar v(t) \equiv 1$, $\forall t$. Notice that $\bar H$ is odd.

Any solution $H(t)$ of (26) can be written as

$$H(t) = \bar H(t) + a\bar u + b\bar v , \qquad a, b \in \mathbb{R} .$$

Since $\bar H$ is odd, $\bar u$ is even and $\bar v$ is odd, requiring H to be odd, implies $a = 0$. Imposing now the 2π-periodicity yields

$$\begin{aligned} 0 &= \Big(\int_0^{t+2\pi} f\bar u \Big) \bar v(t+2\pi) - \Big(\int_0^{t+2\pi} f\bar v \Big) \bar u(t+2\pi) - \Big(\int_0^t f\bar u \Big) \bar v(t) + \Big(\int_0^t f\bar v \Big) \bar u(t) + b\Big(\bar v(t+2\pi) - \bar v(t) \Big) \\ &= \Big(b + \int_0^t f\bar u \Big) \Big(\bar v(t+2\pi) - \bar v(t) \Big) - \bar u(t) \Big(\int_t^{t+2\pi} f\bar v \Big) \end{aligned} \tag{40}$$

using that $\bar u$ and $f\bar u$ are 2π-periodic and $\langle f\bar u\rangle = 0$. By (40) and (35) we get

$$\rho\Big(b + \int_0^t f\bar u \Big) - \int_t^{t+2\pi} f\bar v = 0 . \tag{41}$$

The left hand side in (41) is constant in time because, deriving w.r.t. t,

$$\rho f(t)\bar{u}(t) - f(t)\Big(\bar{v}(t+2\pi) - \bar{v}(t)\Big) = 0$$

again by (35). Hence evaluating (41) for $t = 0$ yields $b = \rho^{-1}\int_0^{2\pi} f\bar{v}$. So there exists a unique solution $H = L(f)$ of (26) belonging to E and (39) follows. ■

Finally

Lemma 5. *There holds*

$$\langle gL(g)\rangle = \frac{\rho}{4\pi A(g)} + \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2 > 0$$

because $A(g)$, $\rho > 0$.

Proof. Using (39) we can compute

$$\begin{aligned}
\langle gL(g)\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t)\,dt + \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2 - \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^t g\bar{v}\Big)\bar{u}(t)g(t)\,dt \\
&= 2\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t)\,dt + \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2
\end{aligned} \tag{42}$$

because, by $\int_0^{2\pi} g\bar{u} = 0$, we have

$$0 = \int_0^{2\pi}\frac{d}{dt}\Big[\Big(\int_0^t g\bar{v}\Big)\Big(\int_0^t g\bar{u}\Big)\Big]dt = \int_0^{2\pi}\Big[\Big(\int_0^t g\bar{v}\Big)\bar{u}(t)g(t) + \Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t)\Big]\,dt\,.$$

Now, since $\bar{u}(t) = \dot{g}(t)/\dot{g}(0)$ and $g(0) = 0$,

$$\frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^t g\bar{u}\Big)\bar{v}(t)g(t) = \frac{1}{2\pi\dot{g}(0)}\int_0^{2\pi}\Big(\int_0^t \frac{d}{d\tau}\frac{g^2(\tau)}{2}\,d\tau\Big)\bar{v}(t)g(t) = \frac{1}{4\pi\dot{g}(0)}\int_0^{2\pi} g^3\bar{v}\,. \tag{43}$$

We claim that

$$\int_0^{2\pi} g^3\bar{v} = \frac{\rho\dot{g}(0)}{2A(g)}\,. \tag{44}$$

By (42), (43), (44) we have the thesys.

Let us prove (44). Since g solves (23) multiplying by $\bar{v}$ and integrating

$$\int_0^{2\pi}\bar{v}(t)\ddot{g}(t) + 3A(g)\langle g^2\rangle g(t)\bar{v}(t) + A(g)g^3(t)\bar{v}(t)\,dt = 0 \tag{45}$$

Next, since $\bar{v}$ solves (HOM), multiplying by g and integrating

$$\int_0^{2\pi} g(t)\ddot{\bar{v}}(t) + 3A(g)\langle g^2\rangle\bar{v}(t)g(t) + 3A(g)g^3(t)\bar{v}(t)\,dt = 0\,. \tag{46}$$

Subtracting (45) and (46), gives

$$\int_0^{2\pi}\bar{v}(t)\ddot{g}(t) - g(t)\ddot{\bar{v}}(t) = 2A(g)\int_0^{2\pi} g^3\bar{v}\,. \tag{47}$$

Integrating by parts the left hand side, since $g(0) = g(2\pi) = 0$, $\bar{u}(0) = 1$ and (35), gives

$$\int_0^{2\pi}\bar{v}(t)\ddot{g}(t) - g(t)\ddot{\bar{v}}(t) = \dot{g}(0)[v(2\pi) - v(0)] = \rho\dot{g}(0)\,. \tag{48}$$

(47) and (48) give (44). ■

2.2 Explicit computations

We now give the explicit construction of g by means of the Jacobi elliptic sine defined as follows. Let $\mathrm{am}(\cdot, m) : \mathbb{R} \to \mathbb{R}$ be the inverse function of the Jacobi elliptic integral of the first kind

$$\varphi \mapsto F(\varphi, m) := \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}\,.$$

The Jacobi elliptic sine is defined by

$$\mathrm{sn}(t, m) := \sin(\mathrm{am}(t, m))\,.$$

$\mathrm{sn}(t, m)$ is $4K(m)$-periodic, where $K(m)$ is the complete elliptic integral of the first kind

$$K(m) := F\Big(\frac{\pi}{2}, m\Big) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}$$

and admits an analytic extension with a pole in $iK(1-m)$ for $m \in (0, 1)$ and in $iK\big(1/(1-m)\big)/\sqrt{1-m}$ for $m < 0$. Moreover, since

$$\partial_t \mathrm{am}(t, m) = \sqrt{1 - m\,\mathrm{sn}^2(t, m)}\,,$$

the elliptic sine satisfies

$$(\dot{\mathrm{sn}})^2 = (1 - \mathrm{sn}^2)(1 - m\,\mathrm{sn}^2)\,. \tag{49}$$

Lemma 6. *There exist $V > 0$, $\Omega > 0$, $m \in (-1, 0)$ such that $g(t) := V\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of (23) with pole in $iK\big(1/(1-m)\big)/\big(\Omega\sqrt{1-m}\big)$.*

Proof. Deriving (49) we have $\ddot{\mathrm{sn}} + (1+m)\,\mathrm{sn} - 2m\,\mathrm{sn}^3 = 0$. Therefore $g_{(V,\Omega,m)}(t) := V\mathrm{sn}(\Omega t, m)$ is an odd, $(4K(m)/\Omega)$-periodic solution of

$$\ddot{g} + \Omega^2(1+m)g - 2m\frac{\Omega^2}{V^2}g^3 = 0\,. \tag{50}$$

The function $g_{(V,\Omega,m)}$ will be a solution of (23) if (V, Ω, m) verify

$$\begin{cases} \Omega^2(1+m) = 3A(g_{(V,\Omega,m)})\,\langle g^2_{(V,\Omega,m)}\rangle \\ -2m\Omega^2 = V^2 A(g_{(V,\Omega,m)}) \\ 2K(m) = \Omega\pi\,. \end{cases} \tag{51}$$

Dividing the first equation of (51) by the second one

$$-\frac{1+m}{6m} = \langle \mathrm{sn}^2(\cdot, m)\rangle\,. \tag{52}$$

The right hand side can be expressed as

$$\langle \mathrm{sn}^2(\cdot, m)\rangle = \frac{K(m) - E(m)}{mK(m)} \tag{53}$$

where $E(m)$ is the complete elliptic integral of the second kind

$$E(m) := \int_0^{\pi/2} \sqrt{1 - m\sin^2\vartheta}\,d\vartheta = \int_0^{K(m)} 1 - m\,\mathrm{sn}^2(\xi, m)\,d\xi$$

(in the last passage we make the change of variable $\vartheta = \mathrm{am}(\xi, m)$).

Now, we show that system (51) has a unique solution. By (52) and (53)

$$(7+m)K(m) - 6E(m) = 0\,. \tag{54}$$

By the definitions of $E(m)$ and $K(m)$ we have

$$\psi(m) := (7+m)K(m) - 6E(m) = \int_0^{\pi/2} \frac{1 + m(1+6\sin^2\vartheta)}{\big(1 - m\sin^2\vartheta\big)^{1/2}}\, d\vartheta\,. \tag{55}$$

For $m=0$ it holds $\psi(0) = \pi/2 > 0$ and, for $m=-1$, $\psi(-1) = -\int_0^{\pi/2} 6\sin^2\vartheta\,(1+\sin^2\vartheta)^{-1/2}\, d\vartheta < 0$. Since ψ is continuous there exists a solution $\bar m \in (-1,0)$ of (54). Next the third equation in (51) fix $\bar\Omega$ and finally we find $\bar V$. Hence $g(t) = \bar V\,\mathrm{sn}(\bar\Omega t, \bar m)$ solves (23).

Analyticity and poles follow from [1], 16.2, 16.10.2, pp.570,573.

At last, $\bar m$ is unique because $\psi'(m) > 0$ for $m \in (-1,0)$ as can be verified by (55). One can also compute that $\bar m \in (-0.30, -0.28)$. ■

Remark. We can compute explicitly the sign of dT/dE and ρ of (38) in the following way.

The functions $g_{(V,\Omega,m)}$ are solutions of the Hamiltonian system (36) imposing

$$\begin{cases} \Omega^2(1+m) = \alpha \\ -2m\Omega^2 = V^2\beta \end{cases} \tag{56}$$

where $\alpha := 3A(g)\,\langle g^2\rangle$, $\beta := A(g)$ and g is the solution constructed in Lemma 6.

We solve (56) w.r.t m finding the one-parameter family (y_m) of odd periodic solutions $y_m(t) := V(m)\,\mathrm{sn}(\Omega(m)t,\, m)$, close to g, with energy and period

$$E(m) = \frac{1}{2}V^2(m)\Omega^2(m) \;=\; -\frac{1}{\beta}\, m\,\Omega^4(m)\,, \qquad T(m) = \frac{4K(m)}{\Omega(m)}\,.$$

It holds

$$\frac{dT(m)}{dm} = \frac{4K'(m)\Omega(m) - 4K(m)\Omega'(m)}{\Omega^2(m)} > 0$$

because $K'(m) > 0$ and from (56) $\Omega'(m) = -\Omega(m)\big(2(1+m)\big)^{-1} < 0$. Then

$$\frac{dE(m)}{dm} = -\frac{1}{\beta}\,\Omega^4(m) - \frac{1}{\beta}\, m\, 4\Omega^3(m)\Omega'(m) < 0\,,$$

so

$$\frac{dT}{dE} = \frac{dT(m)}{dm}\Big(\frac{dE(m)}{dm}\Big)^{-1} < 0$$

as stated by general arguments in the proof of Lemma 3.

We can also write an explicit formula for ρ,

$$\rho = \frac{m}{m-1}\left[2\pi + (1+m)\int_0^{2\pi} \frac{\mathrm{sn}^2(\Omega t, m)}{\mathrm{dn}^2(\Omega t, m)}\, dt\right]. \tag{57}$$

From (57) it follows that $\rho > 0$ because $-1 < m < 0$.

3 Case $f(x,u) = a_2 u^2 + a_3(x)u^3 + O(u^4)$

We have to prove the existence of *non-degenerate* critical points of the functional $\Phi_n(v) := \Phi_0(\mathcal{H}_n v)$ where Φ_0 is defined in (15).

Lemma 7. *See [6]. Φ_n has the following development: for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$,*

$$\Phi_n(\beta n v) = 4\pi\beta^2 n^4\Big[\Psi(\eta) + \frac{\beta^2}{4\pi}\Big(\frac{R_2(\eta)}{n^2} + R_3(\eta)\Big)\Big] \tag{58}$$

where

$$\Psi(\eta) := \frac{s^*}{2}\int_{\mathbb{T}} \dot\eta^2 + \frac{\beta^2}{4\pi}\Big[\alpha\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \gamma\int_{\mathbb{T}}\eta^4\Big]$$

$$R_2(\eta) := -\frac{a_2^2}{2}\Big[\int_\Omega v^2\square^{-1}v^2 - \frac{\pi^2}{6}\Big(\int_{\mathbb{T}}\eta^2\Big)^2\Big], \qquad R_3(\eta) := \frac{1}{4}\int_\Omega \big(a_3(x)-\langle a_3\rangle\big)(\mathcal{H}_n v)^4\,, \tag{59}$$

$\alpha := \big(9\langle a_3\rangle - \pi^2 a_2^2\big)/12,\ \gamma := \pi\langle a_3\rangle/2,$ *and*

$$\beta = \begin{cases} (2|\alpha|)^{-1/2} & \text{if } \alpha\neq 0,\\ (\pi/\gamma)^{1/2} & \text{if } \alpha = 0.\end{cases}$$

Proof. By Lemma 4.8 in [6] with $m(s_1,s_2) = \big(\eta(s_1)-\eta(s_2)\big)^2$, for $v(t,x) = \eta(t+x)-\eta(t-x)$ the operator Φ_n admits the development

$$\begin{aligned}\Phi_n(v) \;&=\; 2\pi s^* n^2\int_{\mathbb{T}}\dot\eta^2(t)dt - \frac{\pi^2 a_2^2}{12}\Big(\int_{\mathbb{T}}\eta^2(t)\,dt\Big)^2 - \frac{a_2^2}{2n^2}\Big(\int_\Omega v^2\square^{-1}v^2 - \frac{\pi^2}{6}\Big(\int_{\mathbb{T}}\eta^2(t)\,dt\Big)^2\Big)\\ &\quad+\frac{1}{4}\langle a_3\rangle\int_\Omega v^4 + \frac{1}{4}\int_\Omega\big(a_3(x)-\langle a_3\rangle\big)(\mathcal{H}_n v)^4\,.\end{aligned}$$

Since

$$\int_\Omega v^4 = 2\pi\int_{\mathbb{T}}\eta^4 + 3\Big(\int_{\mathbb{T}}\eta^2\Big)^2,$$

we write

$$\Phi_n(v) \;=\; 2\pi s^* n^2\int_{\mathbb{T}}\dot\eta^2(t)dt - \frac{\pi^2a_2^2}{12}\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \frac{1}{4}\langle a_3\rangle\Big[2\pi\int_{\mathbb{T}}\eta^4 + 3\Big(\int_{\mathbb{T}}\eta^2\Big)^2\Big] + \frac{R_2(\eta)}{n^2} + R_3(\eta)\,,$$

where R_2, R_3 defined in (59) are both homogenous of degree 4. So

$$\Phi_n(v) = 2\pi s^* n^2\int_{\mathbb{T}}\dot\eta^2 + \alpha\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \gamma\int_{\mathbb{T}}\eta^4 + \frac{R_2(\eta)}{n^2} + R_3(\eta)$$

where α, γ are defined above. With the rescaling $\eta\to\eta\beta n$ we get decomposition (58). ■

In order to find for n large a non-degenerate critical point of Φ_n, by (58) it is sufficient to find critical points of Ψ on $E = \{\eta\in H^1(\mathbb{T}),\ \eta \text{ odd}\}$ (like in Lemma 6.2 of [7] also the term $R_3(\eta)$ tends to 0 with its derivatives).

If $\langle a_3\rangle\in(-\infty,0)\cup(\pi^2a_2^2/9,+\infty)$, then $\alpha\neq 0$ and we must choose $s^* = -\text{sign}(\alpha)$, so that the functional becomes

$$\Psi(\eta) = \text{sign}(\alpha)\Big(-\frac{1}{2}\int_{\mathbb{T}}\dot\eta^2 + \frac{1}{8\pi}\Big[\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \frac{\gamma}{\alpha}\int_{\mathbb{T}}\eta^4\Big]\Big)\,.$$

Since in this case $\gamma/\alpha > 0$, the functional Ψ clearly has a mountain pass critical point, solution of

$$\ddot\eta + \langle\eta^2\rangle\eta + \lambda\eta^3 = 0\,, \qquad \lambda = \frac{\gamma}{2\pi\alpha} > 0\,. \tag{60}$$

The proof of the non-degeneracy of the solution of (60) is very simple using the analytical arguments of the previous section (since $\lambda > 0$ it is sufficient a positivity argument).

If $\langle a_3\rangle = 0$, then the equation becomes $\ddot\eta + \langle\eta^2\rangle\eta = 0$, so we find again what proved in [7] for $a_3(x)\equiv 0$.

If $\langle a_3\rangle = \pi^2a_2^2/9$, then $\alpha = 0$. We must choose $s^* = -1$, so that we obtain

$$\Psi(\eta) = -\frac{1}{2}\int_{\mathbb{T}}\dot\eta^2 + \frac{1}{4}\int_{\mathbb{T}}\eta^4, \qquad \ddot\eta + \eta^3 = 0.$$

This equation has periodic solutions which are non-degenerate because of non-isocronicity, see Proposition 2 in [8].

Finally, if $\langle a_3 \rangle \in (0,\, \pi^2 a_2^2/9)$, then $\alpha < 0$ and there are both solutions for $s^* = \pm 1$. The functional

$$\begin{aligned}\Psi(\eta) &= \frac{s^*}{2}\int_{\mathbb{T}} \dot\eta^2 + \frac{1}{8\pi}\Big[-\Big(\int_{\mathbb{T}}\eta^2\Big)^2 + \frac{\gamma}{|\alpha|}\int_{\mathbb{T}}\eta^4\Big] \\ &= \frac{s^*}{2}\int_{\mathbb{T}} \dot\eta^2 + \frac{1}{4}\int_{\mathbb{T}}\eta^4\Big[\lambda - Q(\eta)\Big]\end{aligned}$$

where

$$\lambda := \frac{\gamma}{2\pi|\alpha|} > 0\,, \qquad Q(\eta) := \frac{\Big(\int_{\mathbb{T}}\eta^2\Big)^2}{2\pi\int_{\mathbb{T}}\eta^4}$$

possesses Mountain pass critical points for any $\lambda > 0$ because (like in Lemma 3.14 of [6])

$$\inf_{\eta\in E\setminus\{0\}} Q(\eta) = 0\,, \qquad \sup_{\eta\in E\setminus\{0\}} Q(\eta) = 1$$

(for $\lambda \geq 1$ if $s^* = -1$, and for $0 < \lambda < 1$ for both $s^* = \pm 1$).

Such critical points satisfy the Euler Lagrange equation

$$-s^*\ddot\eta - \langle \eta^2\rangle\eta + \lambda\eta^3 = 0 \tag{61}$$

but their non-degeneracy is not obvious. For this, it is convenient to express this solutions in terms of the Jacobi elliptic sine.

Proposition 4. *(i) Let $s^* = -1$. Then for every $\lambda \in (0,+\infty)$ there exists an odd, analytic, 2π-periodic solution $g(t)$ of (61) which is non-degenerate in E. $g(t) = V\mathrm{sn}(\Omega t, m)$ for $V > 0$, $\Omega > 0$, $m \in (-\infty,-1)$ suitable constants.*
(ii) Let $s^* = 1$. Then for every $\lambda \in (0,1)$ there exists an odd, analytic, 2π-periodic solution $g(t)$ of (61) which is non-degenerate in E. $g(t) = V\mathrm{sn}(\Omega t, m)$ for $V > 0$, $\Omega > 0$, $m \in (0,1)$ suitable constants.

We prove Proposition 4 in several steps. First we construct the solution g like in Lemma 6.

Lemma 8. *(i) Let $s^* = -1$. Then for every $\lambda \in (0,+\infty)$ there exist $V > 0$, $\Omega > 0$, $m \in (-\infty,-1)$ such that $g(t) = V\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of (61) with a pole in $\frac{i}{\Omega\sqrt{1-m}}K\big(\frac{1}{1-m}\big)$.*
(ii) Let $s^* = 1$. Then for every $\lambda \in (0,1)$ there exist $V > 0$, $\Omega > 0$, $m \in (0,1)$ such that $g(t) = V\mathrm{sn}(\Omega t, m)$ is an odd, analytic, 2π-periodic solution of (61) with a pole in $iK(1-m)/\Omega$.

PROOF. We know that $g_{(V,\Omega,m)}(t) := V\mathrm{sn}(\Omega t, m)$ is an odd, $(4K(m)/\Omega)$-periodic solution of (50), see Lemma 6. So it is a solution of (61) if (V,Ω,m) verify

$$\begin{cases}\Omega^2(1+m) = s^* V^2\langle \mathrm{sn}^2(\cdot,m)\rangle \\ 2m\Omega^2 = s^* V^2\lambda \\ 2K(m) = \Omega\pi\,.\end{cases} \tag{62}$$

Conditions (62) give the connection between λ and m:

$$\lambda = \frac{2m}{1+m}\langle \mathrm{sn}^2(\cdot,m)\rangle\,. \tag{63}$$

Moreover system (62) imposes

$$\begin{cases} m \in (-\infty,-1) & \text{if } s^* = -1 \\ m \in (0,1) & \text{if } s^* = 1\,.\end{cases}$$

We know that $m \mapsto \langle \mathrm{sn}^2(\cdot,m)\rangle$ is continuous, strictly increasing on $(-\infty,1)$, it tends to 0 for $m \to -\infty$ and to 1 for $m \to 1$, see Lemma 12. So the right-hand side of (63) covers $(0,+\infty)$ for $m \in (-\infty,0)$, and it covers $(0,1)$ for $m \in (0,1)$. For this reason for every $\lambda > 0$ there exists a unique $\bar{m} < -1$ satisfying (63), and for every $\lambda \in (0,1)$ there exists a unique $\bar{m} \in (0,1)$ satisfying (63).

The value $\bar{m}$ and system (62) determine uniquely the values $\bar{V}$, $\bar{\Omega}$.

Analyticity and poles follow from [1], 16.2, 16.10.2, pp.570,573. ■

Now we have to prove the non-degeneracy of g. The linearized equation of (61) at g is

$$\ddot{h} + s^*\big(\langle g^2\rangle - 3\lambda g^2\big)h = -2s^*\langle gh\rangle g. \tag{64}$$

Let L be the Green operator, i.e. for $f \in E$, let $H := L(f)$ be the unique solution belonging to E of the non-homogeneous linear system

$$\ddot{H} + s^*\left(\langle g^2\rangle - 3\lambda g^2\right) H = f\,.$$

We can write (64) as

$$h = -2s^*\langle gh\rangle L(g)\,. \tag{65}$$

Multiplying by g and integrating we get

$$\langle gh\rangle\big[1 + 2s^*\langle gL(g)\rangle\big] = 0\,.$$

If $A_0 := 1 + 2s^*\langle gL(g)\rangle \neq 0$, then $\langle gh\rangle = 0$, so by (65) $h = 0$ and the non-degeneracy is proved.

It remains to show that $A_0 \neq 0$. As before, the key is to express $L(g)$ in a suitable way. We first look for a fundamental set of solutions of the homogeneous equation

$$\ddot{h} + s^*\big(\langle g^2\rangle - 3\lambda g^2\big)h = 0\,. \tag{66}$$

Lemma 9. *There exist two linearly independent solutions of (66), $\bar{u}$ even, 2π-periodic and $\bar{v}$ odd, not periodic, such that $\bar{u}(0) = 1$, $\dot{\bar{u}}(0) = 0$, $\bar{v}(0) = 0$, $\dot{\bar{v}}(0) = 1$, and*

$$\bar{v}(t + 2\pi) - \bar{v}(t) = \rho\,\bar{u}(t) \quad \forall\, t \tag{67}$$

for some $\rho \neq 0$. Moreover there hold the following expressions for $\bar{u}$, $\bar{v}$:

$$\bar{u}(t) = \dot{g}(t)/\dot{g}(0) = \dot{\mathrm{sn}}(\bar{\Omega}t, \bar{m}) \tag{68}$$

$$\bar{v}(t) = \frac{1}{\bar{\Omega}(1-\bar{m})}\,\mathrm{sn}(\bar{\Omega}t) + \frac{\bar{m}}{\bar{m}-1}\,\dot{\mathrm{sn}}(\bar{\Omega}t)\Big[t + \frac{1+\bar{m}}{\bar{\Omega}}\int_0^{\bar{\Omega}t}\frac{\mathrm{sn}^2(\xi,\bar{m})}{\mathrm{dn}^2(\xi,\bar{m})}\,d\xi\Big]\,. \tag{69}$$

Proof. g solves (61) so $\dot{g}$ solves (66); normalizing we get (68).

By (50), the function $y(t) = V\mathrm{sn}(\Omega t, m)$ solves

$$\ddot{y} + s^*\langle g^2\rangle y - s^*\lambda y^3 = 0 \tag{70}$$

if (V, Ω, m) satisfy

$$\begin{cases} \Omega^2(1+m) = s^*\langle g^2\rangle \\ 2m\Omega^2 = s^*V^2\lambda\,. \end{cases} \tag{71}$$

We solve (71) w.r.t. m finding the one-parameter family (y_m) of odd periodic solutions of (70), $y_m(t) = V(m)\mathrm{sn}(\Omega(m)t, m)$. So $l(t) := (\partial_m y_m)_{|m=\bar{m}}$ solves (66). We normalize $\bar{v}(t) := l(t)/\dot{l}(0)$ and we compute the coefficients differentiating (71) w.r.t. m. From the definitions of the Jacobi elliptic functions it holds

$$\partial_m \mathrm{sn}(x,m) = -\dot{\mathrm{sn}}(x,m)\frac{1}{2}\int_0^x \frac{\mathrm{sn}^2(\xi,m)}{\mathrm{dn}^2(\xi,m)}\,d\xi\,;$$

thanks to this formula we obtain (69).

Since $2\pi\bar{\Omega} = 4K(\bar{m})$ is the period of the Jacobi functions sn and dn, by (68),(69) we obtain (67) with

$$\rho = \frac{\bar{m}}{\bar{m}-1}\, 2\pi\Big(1 + (1+\bar{m})\langle\frac{\mathrm{sn}^2}{\mathrm{dn}^2}\rangle\Big)\,.$$

If $s^* = 1$, then $\bar{m} \in (0,1)$ and directly we can see that $\rho < 0$. If $s^* = -1$, then $\bar{m} < -1$. From the equality $\langle \mathrm{sn}^2/\mathrm{dn}^2\rangle = (1-m)^{-1}\left(1 - \langle \mathrm{sn}^2\rangle\right)$ (see [3], Lemma 3, (L.2)), it results $\rho > 0$. ■

We can note that the integral representation (39) of the Green operator L holds again in the present case. The proof is just like in Lemma 4.

Lemma 10. *We can write $A_0 := 1 + 2s^*\langle gL(g)\rangle$ as function of λ, $\bar{m}$,*

$$A_0 = \frac{\lambda(1-\bar{m})^2 q - (1-\lambda)^2(1+\bar{m})^2 + \bar{m}q^2}{\lambda(1-\bar{m})^2 q}\,, \qquad q = q(\lambda,\bar{m}) := 2 - \lambda\frac{(1+\bar{m})^2}{2\bar{m}} > 0\,. \tag{72}$$

Proof. First, we calculate $\langle gL(g)\rangle$ with the integral formula (39) of L. The equalities (42),(43) still hold, while similar calculations give

$$\int_0^{2\pi} g^3\bar{v} = -s^*\frac{\dot{g}(0)\rho}{2\lambda}$$

instead of (44). So

$$\langle gL(g)\rangle = -s^*\,\frac{\rho}{4\pi\lambda} + \frac{1}{2\pi\rho}\Big(\int_0^{2\pi} g\bar{v}\Big)^2 \tag{73}$$

and the sign of A_0 is not obvious. We calculate $\int_0^{2\pi} g\bar{v}$ recalling that $g(t) = \bar{V}\mathrm{sn}(\bar{\Omega}t,\bar{m})$, using formula (69) for $\bar{v}$ and integrating by parts

$$\int_0^{2\pi} \mathrm{sn}(\bar{\Omega}t)\dot{\mathrm{sn}}(\bar{\Omega}t)\mu(t)\,dt = -\frac{1}{2\bar{\Omega}}\int_0^{2\pi} \mathrm{sn}^2(\bar{\Omega}t)\dot{\mu}(t)\,dt$$

where $\mu(t) := t + (1+\bar{m})\bar{\Omega}^{-1}\int_0^{\bar{\Omega}t} \mathrm{sn}^2(\xi)/\mathrm{dn}^2(\xi)\,d\xi$. From [3], (L.2),(L.3) in Lemma 3, we obtain the formula

$$\langle\frac{\mathrm{sn}^4}{\mathrm{dn}^2}\rangle = \frac{1 + (m-2)\langle \mathrm{sn}^2\rangle}{m(1-m)}$$

and consequently

$$\int_0^{2\pi} g\bar{v} = \frac{\pi\bar{V}}{\bar{\Omega}(1-\bar{m})^2}\left(1 + \bar{m} - 2\bar{m}\langle\mathrm{sn}^2\rangle\right)\,. \tag{74}$$

By the second equality of (62) and (73) we get

$$A_0 = 1 + \frac{2}{\lambda}\Big[-\frac{\rho}{4\pi} + \frac{\pi\bar{m}}{\rho(1-\bar{m})^4}\left(1 + \bar{m} - 2\bar{m}\langle\mathrm{sn}^2\rangle\right)^2\Big] \tag{75}$$

both for $s^* = \pm 1$. From the proof of Lemma 9 we have $\rho = -2\pi\bar{m}q\,(1-\bar{m})^{-2}$, where q is defined in (72); inserting this expression of ρ in (75) we obtain (72).

Finally, for $\bar{m} < -1$ we have immediately $q > 0$, while for $\bar{m} \in (0,1)$ we get $q = 2 - (1+\bar{m})\langle\mathrm{sn}^2\rangle$ by (63). Since $\langle\mathrm{sn}^2\rangle < 1$, it results $q > 0$. ■

Lemma 11. $A_0 \neq 0$. *More precisely, sign$(A_0) = -s^*$.*

Proof. From (72), $A_0 > 0$ iff $\lambda(1-\bar{m})^2 q - (1-\lambda)^2(1+\bar{m})^2 + \bar{m}q^2 > 0$. This expression is equal to $-(1-\bar{m})^2 p$, where

$$p = p(\lambda, \bar{m}) = \frac{(1+\bar{m})^2}{4\bar{m}}\,\lambda^2 - 2\lambda + 1\,,$$

so $A_0 > 0$ iff $p < 0$. The polynomial $p(\lambda)$ has degree 2 and its determinant is $\Delta = -(1-\bar{m})^2/\bar{m}$. So, if $s^* = 1$, then $\bar{m} \in (0,1)$, $\Delta < 0$ and $p > 0$, so that $A_0 < 0$.

It remains the case $s^* = -1$. For $\lambda > 0$, we have $p(\lambda) < 0$ iff $\lambda > x^*$, where x^* is the positive root of p, $x^* := 2R(1+R)^{-2}$, $R := |\bar{m}|^{1/2}$. By (63), $\lambda > x^*$ iff

$$\langle \mathrm{sn}^2(\cdot,\bar{m})\rangle > \frac{R-1}{(R+1)R}\,. \tag{76}$$

By formula (53) and by definition of complete elliptic integrals K and E we can write (76) as

$$\int_0^{\pi/2} \Big(\frac{R-1}{(R+1)R} - \sin^2\vartheta\Big)\, \frac{d\vartheta}{\sqrt{1+R^2\sin^2\vartheta}} < 0\,. \tag{77}$$

We put $\sigma := R - 1/(R+1)R$ and note that $\sigma < 1/2$ for every $R > 0$.

$\sigma - \sin^2\vartheta > 0$ iff $\vartheta \in (0,\vartheta^*)$, where $\vartheta^* := \arcsin(\sqrt{\sigma})$, i.e. $\sin^2\vartheta^* = \sigma$. Moreover $1 < 1 + R^2\sin^2\vartheta < 1 + R^2$ for every $\vartheta \in (0,\pi/2)$. So

$$\int_0^{\pi/2} \frac{\sigma - \sin^2\vartheta}{\sqrt{1+R^2\sin^2\vartheta}}\, d\vartheta \;<\; \int_0^{\vartheta^*} \big(\sigma - \sin^2\vartheta\big)\, d\vartheta + \int_{\vartheta^*}^{\pi/2} \frac{\sigma - \sin^2\vartheta}{\sqrt{1+R^2}}\, d\vartheta\,. \tag{78}$$

Thanks to the formula

$$\int_a^b \sin^2\vartheta\, d\vartheta \;= \frac{b-a}{2} - \frac{\sin(2b) - \sin(2a)}{4}$$

the right-hand side term of (78) is equal to

$$\frac{\sin(2\vartheta^*)}{4}\left[(2\sigma-1)\Big(\frac{2\vartheta^*}{\sin(2\vartheta^*)} + \frac{1}{\sqrt{1+R^2}}\,\frac{\pi - 2\vartheta^*}{\sin(2\vartheta^*)}\Big) + \Big(1 - \frac{1}{\sqrt{1+R^2}}\Big)\right].$$

Since $2\sigma - 1 < 0$ and $\alpha > \sin\alpha$ for every $\alpha > 0$, this quantity is less than

$$\frac{\sin(2\vartheta^*)}{4}\left[(2\sigma-1)\Big(1 + \frac{1}{\sqrt{1+R^2}}\Big) + \Big(1 - \frac{1}{\sqrt{1+R^2}}\Big)\right].$$

By definition of σ, the last quantity is negative for every $R > 0$, so (77) is true. Consequently $\lambda > x^*$, $p < 0$ and $A_0 > 0$. ■

As Appendix, we show the properties of the function $m \mapsto \langle \mathrm{sn}^2(\cdot,m)\rangle$ used in the proof of Lemma 8.

Lemma 12. *The function* $\varphi : (-\infty, 1) \to \mathbb{R}$, $m \mapsto \langle \mathrm{sn}^2(\cdot,m)\rangle$ *is continuous, differentiable, strictly increasing, and* $\lim_{m\to-\infty}\varphi(m) = 0$, $\lim_{m\to 1}\varphi(m) = 1$.

Proof. By (53) and by definition of complete elliptic integrals K and E,

$$\varphi(m) = \frac{K(m) - E(m)}{mK(m)} = \int_0^{\pi/2} \frac{\sin^2\vartheta\, d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}\,\Big(\int_0^{\pi/2}\frac{d\vartheta}{\sqrt{1-m\sin^2\vartheta}}\Big)^{-1},$$

so the continuity of φ is evident.

Using the equality $\sin^2 + \cos^2 = 1$ and the change of variable $\vartheta \to \pi/2 - \vartheta$ in the integrals which define K and E, we obtain the formulae

$$K(m) = \frac{1}{\sqrt{1-m}}\, K\Big(\frac{m}{m-1}\Big)\,, \qquad E(m) = \sqrt{1-m}\, E\Big(\frac{m}{m-1}\Big) \qquad \forall\, m < 1\,. \tag{79}$$

We put $\mu := m/(m-1)$, so it results

$$\varphi(m) = 1 - \frac{1}{\mu} + \frac{E(\mu)}{\mu K(\mu)}\,. \tag{80}$$

Since μ tends to 1 as $m \to -\infty$, $E(1) = 1$ and $\lim_{\mu\to 1} K(\mu) = +\infty$, (79),(80) give $\lim_{m\to-\infty} \varphi(m) = 0$. Since $E(m)/K(m)$ tends to 0 as $m \to 1$, (53) gives $\lim_{m\to 1} \varphi(m) = 1$.

Differentiating the integrals which define K and E w.r.t. m we obtain the formulae

$$E'(m) = \frac{E(m) - K(m)}{2m}\,, \qquad K'(m) = \frac{1}{2m}\Big(\int_0^{\pi/2} \frac{d\vartheta}{(1 - m\sin^2\vartheta)^{3/2}} - K(m)\Big)\,,$$

so the derivative is

$$\varphi'(m) = \frac{1}{2m^2 K^2(m)}\Big[E(m)\int_0^{\pi/2} \frac{d\vartheta}{(1 - m\sin^2\vartheta)^{3/2}} - K^2(m)\Big]\,.$$

The term in the square brackets is positive by strict Hölder inequality for $(1 - m\sin^2\vartheta)^{-3/4}$ and $(1 - m\sin^2\vartheta)^{1/4}$. ■

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