Networks Self–Similarly Moving by Curvature with Two Triple Junctions

Pietro Baldi, Emanuele Haus, Carlo Mantegazza

August 30, 2016

Abstract

We prove that there are no networks homeomorphic to the Greek "Theta" letter (a double cell) embedded in the plane with two triple junctions with angles of 120 degrees, such that under the motion by curvature they are self–similarly shrinking. This fact completes the classification of the self–similarly shrinking networks in the plane with at most two triple junctions, see [5, 7, 18].

1 Introduction

Recently, the problem of the evolution by curvature of a network of curves in the plane got the interest of several authors [3, 7, 11, 12, 15–19]. It is well known that, after the work of Huisken [8] in the smooth case of the hypersurfaces in the Euclidean space and of Ilmanen [9, 10] in the more general weak settings of varifolds, that a suitable sequence of rescalings of the *subsets* of $\mathbb{R}^n$ which are evolving by mean curvature, approaching a singular time of the flow, converges to a so called "blow–up limit" set which, letting it flow again by mean curvature, simply moves by homothety, precisely, it shrinks down self–similarly toward the origin of the Euclidean space.

This procedure and the classification of these special sets (possibly under some hypotheses), called *shrinkers*, is a key point in understanding the asymptotic behavior of the flow at a singular time.

Dealing with the evolution of a single curve in the plane, it is easy to see that any C^2 curve $\gamma : I \to \mathbb{R}^2$ which moves by curvature, self–similarly shrinking, must satisfy the following "structural" equation (which is actually an ODE for γ)

$$\overline{k} + \gamma^\perp = 0, \tag{1.1}$$

where $\overline{k}$ is the vector curvature of the curve at the point γ and $\gamma^\perp$ denotes the normal component of the position vector γ. Introducing an arclength parameter s on the curve γ, we have a unit tangent vector field $\tau = \frac{d}{ds}\gamma$, a unit normal vector field ν which is the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector τ and the curvature vector given by $\overline{k} = k\nu = \frac{d^2}{ds^2}\gamma$, where k is then simply the curvature of γ. With these notations, the above equation can be rewritten as

$$k + \langle \gamma \,|\, \nu \rangle = 0. \tag{1.2}$$

It is then known, by the work of Abresch–Langer [1] and independently of Epstein–Weinstein [6], that the only complete, embedded, self–similarly shrinking curves in $\mathbb{R}^2$ without end–points, are the lines through the origin and the unit circle (they actually classify *all* the closed, embedded or not, self–similarly shrinking curves in the plane).

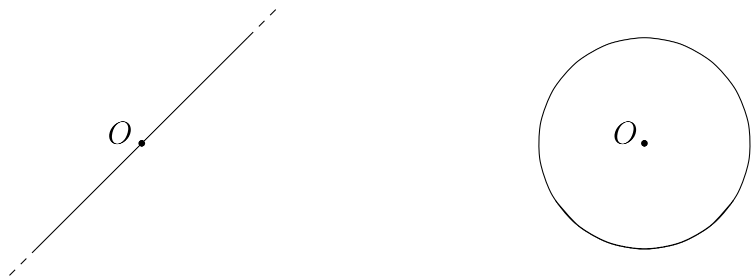

Figure 1: The only complete, embedded, self–similarly shrinking curves in $\mathbb{R}^2$: lines through the origin and the unit circle.

The same equation $\overline{k} + \gamma^\perp = 0$ (that is, $k + \langle \gamma \,|\, \nu \rangle = 0$) must be satisfied by every curve of a network in the plane which self–similarly shrinks to the origin moving by curvature (see [14, 15], for instance). Moreover, for "energetic" reasons, it is natural to consider networks with only triple junctions and such that the three concurring curves (which are C^∞) form three angles of 120 degrees between each other – "Herring" condition – such networks are called *regular*. In such class, the *embedded* shrinking regular networks (without self–intersections) play a crucial role, indeed, they "reasonably" arise as blow–up limits of the motion of networks without self–intersections (this is still a conjecture for a general network, but there holds for networks with at most two triple junctions - see the end of the section).

Our goal in this paper is to complete and describe the classification of the complete, embedded, self–similarly shrinking regular networks in the plane with at most two triple junctions, after the contributions in [5,7,18].

If one consider networks with only one triple junction, the only complete, embedded, regular shrinkers are given (up to rotations) by the "standard triod" and the "Brakke spoon" (first described in [4]), as in the following figure. Actually, the loop of the Brakke spoon is the only possible shape for a region of every regular shrinker (with any number of triple junctions) bounded by a single curve (and the curve "exiting" by such region is straight).

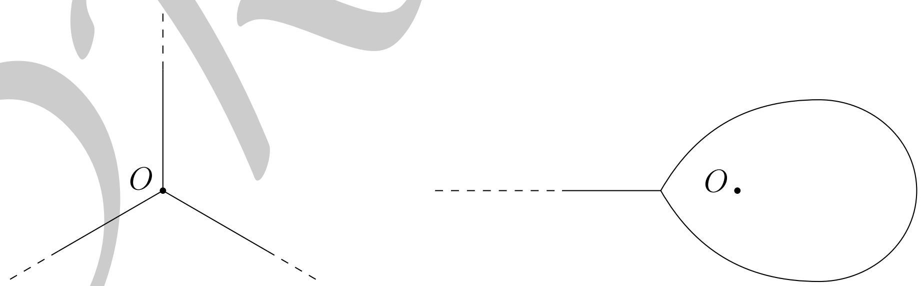

Figure 2: A *standard triod* and a *Brakke spoon*.

About networks with two triple junction, it is not difficult to show that the possible topological shapes for a connected, complete, embedded, regular network without end–points, are the ones depicted in the following figure.

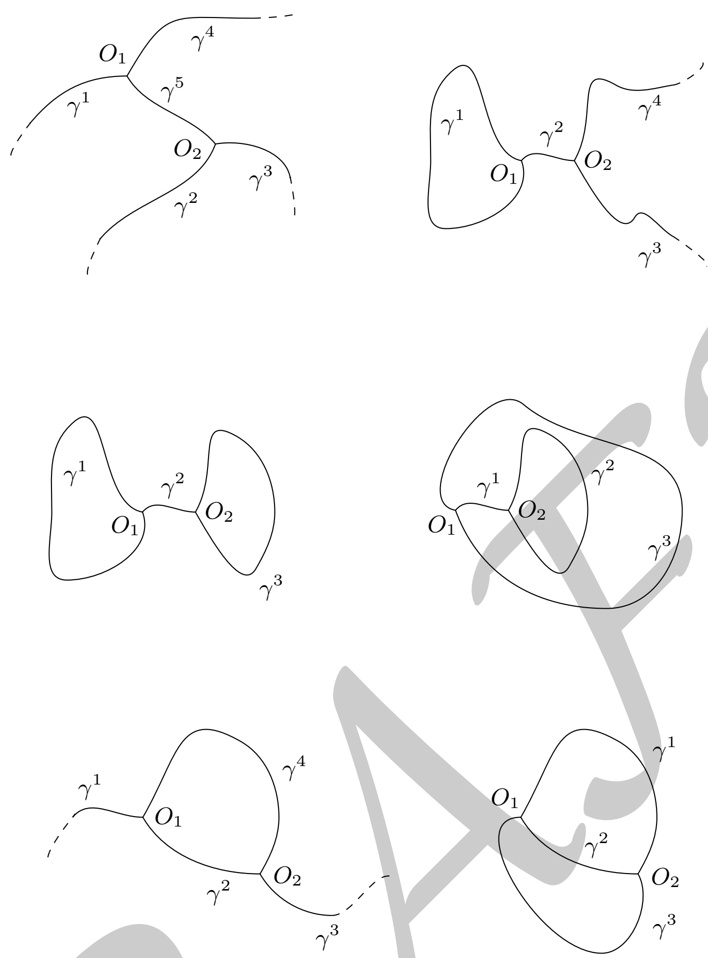

Figure 3: The possible topological shapes of a complete, connected, embedded network with two triple junctions.

Then, looking for shrinkers with one of these structure, by the cited work of Abresch and Langer [1], it follows that any unbounded curve of such shrinkers must be a piece of a halfline from the origin, going to infinity. Then, differentiating in arclength s the equation $k = -\langle \gamma \,|\, \nu \rangle$, we get the ODE for the curvature $k_s = k\langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, then it must also hold $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODEs we conclude that k is identically zero and we are dealing with a piece of a straight line, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in \gamma$. Notice that, if a curve γ contains the origin at such point its curvature is zero, by the equation $k + \langle \gamma \,|\, \nu \rangle = 0$, hence, it must be straight.

Now, if a regular shrinker would have the topological shape of the first drawing on the top of Figure 3, the four unbounded curves should be halflines, which implies that the two triple junctions should coincide with the origin, which is a contradiction (the curve γ^5 should be a non trivial segment between the triple junctions), thus, such a shape is excluded.
Then, by an argument of Hättenschweiler [7, Lemma 3.20], if a regular shrinker contains a region bounded by a single curve, the shrinker must be a Brakke spoon, that is, no other triple junctions can be present. This excludes the possibility for a regular shrinker also to have a shape like the second one in the first row of Figure 3 or the two in the second row.

It remains to discuss the last two cases: one is the "lens/fish" shape and the other is the shape of the Greek "theta" letter (or "double cell"). It is well known that there exist unique (up to a rotation) lens–

shaped or fish–shaped, complete, embedded, regular shrinkers which are symmetric with respect to a line through the origin of $\mathbb{R}^2$ (see [5,18]).

Figure 4: A lens–shaped and a fish–shaped shrinker.

It was instead unknown whether regular Θ–shaped shrinkers (or simply Θ–shrinkers) exist, with numerical evidence in favor of the conjecture of non–existence (see [7]). We are going to show that this is actually the case.

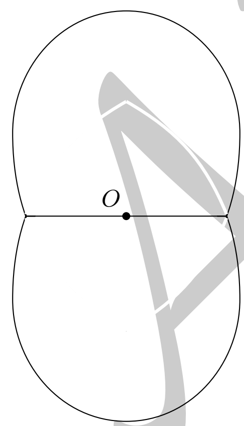

Figure 5: A hypothetical Θ–shrinker.

Theorem 1.1. *There are no regular Θ–shrinkers.*

As a consequence, we have the following classification result.

Theorem 1.2. *The shrinkers of Figure 4 ("lens " and "fish") are the only (up to rotations) complete, embedded, self–similarly shrinking regular networks in the plane with two triple junctions.*

We conclude this discussion mentioning that the main motivation for this problem is given by the fact that for an evolving network with at most *two* triple junctions, the so called *multiplicity–one conjecture* holds (see [14]), saying that any limit shrinker of a sequence of rescalings of the network at different times is again a "genuine" embedded network without "double" or "multiple" curves (curves that in such convergence go to coincide in the limit). This is a key point in the singularity analysis (actually, in general, for mean curvature flow), together with the classification of these limit shrinkers, which is complete after our result Theorem 1.1, for such "low complexity" networks, thus leading to a detailed description of their motion in [13].

To show Theorem 1.1 we first analyze the geometric properties that an hypothetical Θ–shrinker must satisfy, reducing the proof of non–existence to show that a certain parametric integral is always smaller than

$\pi/2$, for every value of the parameter. The proof of such estimate, mixing some approximation techniques and numerical computations based on *interval arithmetic* is shown in full detail in [2].

Acknowledgments. *We wish to thank Matteo Novaga and Alessandra Pluda for several discussions on the subject of this paper. This research was financially supported by UniNA and Compagnia di San Paolo, in the frame of Programme STAR, by the European Research Council under FP7 (ERC Project 306414) and by PRIN 2012 "Variational and perturbative aspects of nonlinear differential problems".*

2 Basic properties of shrinking curves

Consider a shrinking curve $\gamma : I \to \mathbb{R}^2$ parametrized in arclength s, where $I \subset \mathbb{R}$ is an interval. We denote with $R : \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation of 90 degrees. Then, the relation

$$\gamma_{ss} = \frac{d^2\gamma}{ds^2} = k = -\langle \gamma \,|\, \nu \rangle = -\Big\langle \gamma \,\Big|\, R\Big(\frac{d\gamma}{ds}\Big)\Big\rangle$$

gives an ODE satisfied by γ. It follows that the curve is smooth and it is not difficult to see that for every point $x_0 \in \mathbb{R}^2$ and unit velocity vector τ_0, there exists a unique shrinking curve (solution of such ODE) parametrized in arclength, passing at $s = 0$ through the point x_0 with velocity τ_0, defined for all $s \in \mathbb{R}$.

Differentiating in arclength the equation $k = -\langle \gamma \,|\, \nu \rangle$, we get the ODE for the curvature $k_s = k\langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, then it must also hold $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODEs we conclude that k is identically zero and we are dealing with a line L which, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in L$, must contain the origin of $\mathbb{R}^2$.

So we suppose that k is always nonzero and, by looking at the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$, we can see that the curve is then strictly convex with respect to the origin of $\mathbb{R}^2$. Another consequence (by the uniqueness theorem for ODE) is that the curve must be symmetric with respect to any critical point (maximum or minimum) of its curvature function: Notice that if the curve is not a piece of a circle, they are all nondegenerate and isolated (if the curve has bounded length, their number is finite).

Computing the derivative of $|\gamma|^2$,

$$\frac{d|\gamma|^2}{ds} = 2\langle \gamma \,|\, \tau \rangle = 2k_s/k = 2\frac{d\log k}{ds}$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C \in \mathbb{R}$, that is, the quantity

$$\mathcal{E} = \mathcal{E}(\gamma) := ke^{-|\gamma|^2/2}, \tag{2.1}$$

that we call *Energy*, is constant along the curve. Equivalently, $\langle \gamma \,|\, \nu \rangle e^{-|\gamma|^2/2}$ is constant. A solution γ has positive energy if $k > 0$, so that γ runs counterclockwise around the origin, γ has negative energy if $k < 0$, so that γ runs clockwise around the origin, γ has energy zero if $k = 0$, so that γ is a piece of a straight line through the origin.

We consider now a new coordinate $\theta = \arccos \langle e_1 \,|\, \nu \rangle$; this can be done for the whole curve as we know that it is convex (obviously, θ is only locally continuous, since it "jumps" after a complete round).

Differentiating with respect to the arclength parameter we have $\frac{d\theta}{ds} = k$ and

$$k_\theta = k_s/k = \langle \gamma \,|\, \tau \rangle \qquad k_{\theta\theta} = \frac{1}{k}\frac{dk_\theta}{ds} = \frac{1 + k\langle \gamma \,|\, \nu \rangle}{k} = \frac{1}{k} - k. \tag{2.2}$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\frac{d}{d\theta}[k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity

$$E := k_\theta^2 + k^2 - \log k^2$$

is constant along all the curve. Notice that such quantity E cannot be less than 1 (if $k \neq 0$), moreover, if $E = 1$ we have that k^2 must be constant and equal to one along the curve, which consequently must be a piece of the unit circle centered at the origin of $\mathbb{R}^2$.

As $E \geq 1$, it follows that k^2 is uniformly bounded from above and away from zero, hence, recalling that $k = \mathcal{E} e^{|\gamma|^2/2}$, the curve γ is contained in a ball of $\mathbb{R}^2$ (and it is outside some small ball around the origin).

Since we are interested in the curves of a nontrivial connected, compact (Θ–shaped), regular network, there will be no unbounded lines or complete circles and all the curves of the network will be images of a closed bounded interval, once parametrized in arclength.

Resuming, either γ is a segment or $k^2 > 0$, the equations (2.2) hold, the Energy $\mathcal{E} = ke^{-|\gamma|^2/2}$ and the quantity $E = k_\theta^2 + k^2 - \log k^2 > 1$ are constant along the curve, where $\theta = \arccos \langle e_1 \,|\, \nu \rangle$. Moreover, the curve is locally symmetric with respect to the critical points of the curvature, hence the curvature $k(\theta)$ is oscillating between its maximum and its minimum.

Suppose now that $k_{\min} < k_{\max}$ are these two consecutive critical values of k. It follows that they are two distinct positive zeroes of the function $k_\theta^2 = E + \log k^2 - k^2$, when $E > 1$, with $0 < k_{\min} < 1 < k_{\max}$.
We have then that the change $\Delta\theta$ in the angle θ along the piece of curve delimited by two consecutive points where the curvature assumes the values $k_{\min}$ and $k_{\max}$, is given by the integral

$$\Delta\theta = I(E) = \int_{k_{\min}}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}}. \tag{2.3}$$

Proposition 2.1 (Abresch and Langer [1]). *The function $I : (1, +\infty) \to \mathbb{R}$ satisfies*

1. $\lim_{E \to 1^+} I(E) = \pi/\sqrt{2}$,
2. $\lim_{E \to +\infty} I(E) = \pi/2$,
3. *$I(E)$ is monotone nonincreasing.*

As a consequence $I(E) > \pi/2$.

We write now the curve γ in polar coordinates, that is, $\gamma(s) = (\rho(s) \cos\phi(s), \rho(s) \sin\phi(s))$, then, the arclength constraint and the shrinker equation (1.2) become

$$\rho_s^2 + \rho^2 \phi_s^2 = 1, \tag{2.4}$$

$$\rho^2 \phi_s + \rho\rho_{ss}\phi_s - 2\rho_s^2\phi_s - \rho^2\phi_s^3 - \rho\rho_s\phi_{ss} = 0,$$

moreover,

$$\cos\big(\text{angle between } \gamma \text{ and } \gamma_s\big) = \frac{\gamma \cdot \gamma_s}{|\gamma||\gamma_s|} = \rho_s. \tag{2.5}$$

Notice that shrinking curves with positive energy have $\phi_s > 0$ everywhere, indeed, either ϕ_s is always different by zero or the curve is a segment of a straight line for the origin of $\mathbb{R}^2$.

The curvature and the Energy $\mathcal{E} = ke^{-|\gamma|^2/2}$ are given by

$$k = \rho^2 \phi_s, \qquad \mathcal{E} = \rho^2 \phi_s e^{-\frac{1}{2}\rho^2} \tag{2.6}$$

and, when the energy is positive, it will be useful to consider also the quantity $\mathcal{F} := -\log(\mathcal{E})$, that is,

$$\mathcal{F} = -\log(\mathcal{E}) = \frac{1}{2}\rho^2 - \log(\rho^2\phi_s). \tag{2.7}$$

Since $0 < \rho\phi_s \leq 1$, by equation (2.4), one has

$$\mathcal{F} \geq \frac{1}{2}\rho^2 - \log(\rho) \geq \frac{1}{2}.$$

Let us assume that γ is a shrinking curve with $k > 0$ (the assumption on the sign of k is not restrictive, up to a change of orientation of the curve). Then, by the definition of the Energy (2.1), it is immediate to see that the points where k attains its maximum (resp. minimum) coincide with the points where ρ attains its maximum (resp. minimum). Thus, at any extremal point of k there hold $k_\theta = 0$, $\rho_s = 0$ and also $\rho\phi_s = 1$, by equation (2.4), hence, by equation (2.6), we have $k = \rho$. Then, computing E and $\mathcal{F}$ at such point (clearly, $k_\theta = 0$), we get

$$E = k^2 - 2\log k \qquad\qquad \text{and} \qquad\qquad \mathcal{F} = k^2/2 - \log k,$$

that is, $E = 2\mathcal{F} = \log\left(\frac{1}{\mathcal{E}^2}\right)$.

Since the Energy and the quantity $\mathcal{F}$ are constant, this relation must hold along all the curve γ and $\mathcal{F} = \rho_{\min}^2/2 - \log\rho_{\min} = \rho_{\max}^2/2 - \log\rho_{\max}$.

Since the function $\mu(t) = t^2/2 - \log t$ is strictly convex with a minimum value $1/2$ at $t = 1$, to each value of $\mathcal{F} \geq \frac{1}{2}$, there correspond two values $\rho_{\min}(\mathcal{F})$ and $\rho_{\max}(\mathcal{F})$ which are the admissible (interior) minimum and maximum of ρ on γ, with $\rho_{\min}(\mathcal{F}) < 1 < \rho_{\max}(\mathcal{F})$ if $\mathcal{F} > \frac{1}{2}$. It follows easily that $\rho_{\max} : (1/2, +\infty) \to (1, +\infty)$ is an increasing function and $\rho_{\min} : (1/2, +\infty) \to (0, 1)$ is a decreasing function. Viceversa, the quantity $\mathcal{F}$ can be seen as a decreasing function of $\rho_{\min} \in (0, 1]$ and an increasing function of $\rho_{\max} \in [1, +\infty)$.

Let $s_{\min}, s_{\max} \in \mathbb{R}$ with $s_{\min} < s_{\max}$ be two consecutive (interior) extremal points of ρ (hence, also of k) such that $\rho(s_{\min}) = \rho_{\min}(\mathcal{F})$, $\rho(s_{\max}) = \rho_{\max}(\mathcal{F})$. Since at the interior extremal points of ρ the vectors γ, γ_s must be orthogonal, it follows that the quantity considered in formula (2.3) satisfies

$$\Delta\theta = \int_{s_{\min}}^{s_{\max}} \phi_s(s)\, ds \,:= \mathcal{I}(\mathcal{F}), \tag{2.8}$$

that is, the integral $\mathcal{I}(\mathcal{F})$ is the variation of the angle ϕ on the shortest arc such that ρ passes from $\rho_{\min}$ to $\rho_{\max}$.

Then, by the above discussion, $\mathcal{I}(\mathcal{F}) = I(E) = I(2\mathcal{F})$ and we can rephrase Proposition 2.1 in terms of the integral $\mathcal{I}(\mathcal{F})$ as follows.

Proposition 2.2. *The function $\mathcal{I} : (1/2, +\infty) \to \mathbb{R}$ satisfies*

1. $\lim_{\mathcal{F}\to(1/2)^+} \mathcal{I}(\mathcal{F}) = \frac{\pi}{\sqrt{2}}$*,*
2. $\lim_{\mathcal{F}\to+\infty} \mathcal{I}(\mathcal{F}) = \frac{\pi}{2}$*,*
3. *$\mathcal{I}(\mathcal{F})$ is monotone nonincreasing.*

As a consequence $\mathcal{I}(\mathcal{F}) > \frac{\pi}{2}$ for all $\mathcal{F} > \frac{1}{2}$.

3 The proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following lemma whose proof can be found in [2].

Lemma 3.1. *Let γ be a shrinking curve, parametrized counterclockwise by arclength, with positive curvature and let (s_0, s_1) be an interval where $s \mapsto \rho(s)$ is increasing. If $\rho_s(s_0) \geq \frac{1}{2}$, namely, if the angle formed by the vectors $\gamma(s_0)$ and $\gamma_s(s_0)$ is $\leq \frac{\pi}{3}$, then*

$$\int_{s_0}^{s_1} \phi_s(s)\, ds < \frac{\pi}{2}. \tag{3.1}$$

Similarly, if $s \mapsto \rho(s)$ is decreasing on (s_0, s_1) and $\rho_s(s_1) \leq -\frac{1}{2}$, namely the angle formed by the vectors $\gamma(s_1)$ and $\gamma_s(s_1)$ is $\geq \frac{2\pi}{3}$, then the same conclusion holds.

Remark 3.2. Proving estimate (3.1) is equivalent to show that

$$\int_{s_0}^{s_1} \theta_s(s)\, ds < \frac{2\pi}{3}, \tag{3.2}$$

where $\theta(s)$ is the angle formed by $e_1 = (1,0)$ and the normal vector $\nu(s)$. Indeed, clearly

$$\int_{s_0}^{s_1} \theta_s(s)\,ds \le \int_{\sigma_0}^{\sigma_1} \theta_s(s)\,ds, \quad \int_{s_0}^{s_1} \phi_s(s)\,ds \le \int_{\sigma_0}^{\sigma_1} \phi_s(s)\,ds,$$

where $k(\sigma_1) = k_{\text{max}}$, and σ_0 is the maximum $\sigma \le s_0$, assuming it exists, such that the angle formed by the vectors $\gamma(\sigma)$ and $\gamma_s(\sigma)$ equals $\frac{\pi}{3}$ and the map $s \mapsto \rho(s)$ is increasing on (σ, σ_1). Then one observes (by elementary angle geometry) that

$$\int_{\sigma_0}^{\sigma_1} \theta_s(s)\,ds = \int_{\sigma_0}^{\sigma_1} \phi_s(s)\,ds + \frac{\pi}{6}.$$

The integral in (3.2) can be expressed as before

$$\int_{s_0}^{s_1} \theta_s(s)\,ds = \int_{k(s_0)}^{k(s_1)} \frac{dk}{\sqrt{E - k^2 + \log k^2}},$$

hence it is bounded by $I(E)$, defined in formula (2.3) (because, in general, $k_{\text{min}} \le k(s_0) \le k(s_1) \le k_{\text{max}}$). We know that $I(E) < \frac{\pi}{\sqrt{2}}$, but being $\frac{2\pi}{3} < \frac{\pi}{\sqrt{2}}$, estimate (3.2) is not a direct consequence of Proposition 2.1.

Even if such integral is well studied, we found it easier to prove estimate (3.1) than to show that

$$\int_{s_0}^{s_1} \theta_s(s)\,ds = \int_{k(s_0)}^{k(s_1)} \frac{dk}{\sqrt{E - k^2 + \log k^2}} < \frac{2\pi}{3}$$

and this is the reason for our introduction and computation in polar coordinates (ρ, ϕ).

We assume now that a Θ–shrinker exists, described by three embedded shrinking curves $\gamma_i : [\underline{s}_i, \overline{s}_i] \to \mathbb{R}^2$, parametrized by arclength, expressed in polar coordinates by $\gamma_i = (\rho_i \cos(\phi_i), \rho_i \sin(\phi_i))$, for $i \in \{1,2,3\}$. The two triple junctions will be denoted with A, B and the three curves intersect each other only at A and B (which are their endpoints) forming angles of 120 degrees. Since the shrinker equation (1.1) is invariant by rotation, we can assume that the segment $\overline{AB}$ is contained in the straight line $\{(x,q) : x \in \mathbb{R}\}$ with $q \ge 0$ and we let $A = (x_A, q)$, $B = (x_B, q)$ with $x_A < x_B$.

We begin with some preliminary elementary lemmas. To simplify the notation, in all this section we will denote the arclength derivative $\frac{d}{ds}$ with $'$.

Lemma 3.3. *For all $i \in \{1,2,3\}$, the curve γ_i is either a straight line or such that*

$$\left| \int_{\underline{s}_i}^{\overline{s}_i} \phi_i'(s)\,ds \right| < 2\pi.$$

Proof. Without loss of generality, assume that all $\gamma_1, \gamma_2, \gamma_3$ start at B and end at A, namely $\gamma_i(\underline{s}_i) = B$, $\gamma_i(\overline{s}_i) = A$, for $i \in \{1,2,3\}$.

Assume, by contradiction, that γ_1 is a curve with positive energy and curvature such that

$$\int_{\underline{s}_1}^{\overline{s}_1} \phi_1'(s)\,ds \ge 2\pi.$$

Then there exist $\sigma_1, \tau_1 \in S_1$ such that

$$\int_{\underline{s}_1}^{\sigma_1} \phi_1'(s)\,ds = 2\pi, \quad \int_{\tau_1}^{\overline{s}_1} \phi_1'(s)\,ds = 2\pi.$$

Since γ_1 does not intersect itself, one has $(\rho_1(\sigma_1) - \rho_1(\underline{s}_1))(\rho_1(\overline{s}_1) - \rho_1(\tau_1)) > 0$. Assume, without loss of generality, that

$$\rho_1(\sigma_1) < \rho_1(\underline{s}_1), \quad \rho_1(\tau_1) > \rho_1(\overline{s}_1).$$

Now consider the triple junction at the point B, the straight line r passing through B and the origin, and let H_1 and H_2 be the open half-planes in which r divides $\mathbb{R}^2$, where H_1 is the one containing $\gamma_1'(\underline{s}_1)$. Since the three curves γ_1, γ_2, γ_3 form angles of $\frac{2\pi}{3}$ at B, at least one among $\gamma_2'(\underline{s}_2)$ and $\gamma_3'(\underline{s}_3)$ belongs to H_2. Without loss of generality, let $\gamma_2'(\underline{s}_2) \in H_2$. Since ϕ_2' never vanishes, the curve γ_2 cannot reach the endpoint A without crossing the curve γ_1 at some interior point, which is a contradiction. □

Lemma 3.4. *Let $S = [\underline{s}, \overline{s}]$ and $\gamma : S \to \mathbb{R}^2$ be a shrinking curve parametrized by arclength, expressed in polar coordinates by $\gamma = (\rho\cos(\phi), \rho\sin(\phi))$. Assume that $\phi'(s) > 0$ in S and*

$$0 < \Delta \leq \pi, \quad \textit{where} \;\; \Delta := \int_{\underline{s}}^{\overline{s}} \phi'(s)\, ds.$$

Let L be the straight line passing through the two points $\gamma(\underline{s})$, $\gamma(\overline{s})$ and H_1 and H_2 be the two closed half–planes in which L divides the plane $\mathbb{R}^2$. Then the arc $\gamma(S)$ is entirely contained in H_1 or H_2.

Moreover, if $\Delta < \pi$ and $\gamma(S) \subset H_1$, then the origin of $\mathbb{R}^2$ belongs to the interior of H_2.

Proof. By the assumption $\phi' > 0$, we have $k > 0$ and the arc $\gamma(S)$ is contained in the cone $\mathcal{C} := \{\phi(\underline{s}) \leq \phi \leq \phi(\overline{s})\}$, which is convex by the assumption $0 < \Delta \leq \pi$. Since the curvature is positive, the closed set $\mathcal{T}$ delimited by the arc $\gamma(S)$ and by the two line segments joining the origin with $\gamma(\underline{s})$ and $\gamma(\overline{s})$ is a convex subset of $\mathbb{R}^2$, hence, the line segment joining $\gamma(\underline{s})$ with $\gamma(\overline{s})$ is contained in $\mathcal{T}$, which implies the thesis. □

Coming back to our Θ–shrinker, because of its topological structure, one of the curves is contained in the region delimited by the other two, moreover the curvature of both these two "external" curves is always non zero, otherwise any such curve is a segment of a straight line passing for the origin, then the 120 degrees condition at its endpoints would imply that it must be contained in the region bounded by the other two curves, hence it could not be "external". Notice that, on the contrary, the "inner" curve could actually be a segment for the origin.

We call γ_2 the "inner" curve and, recalling that the origin of $\mathbb{R}^2$ is not over the straight line through the two triple junctions A and B, parametrizing counterclockwise the three curves, that is $\phi_i' > 0$ (in the case that the "inner" curve γ_2 is not a segment), we call γ_1 the "external" curve which starts at B. By Lemma 3.3, γ_1 reaches the point A after ϕ_1 changes of an angle $\Delta = \int_{\underline{s}_1}^{\overline{s}_1} \phi_1'(s)\, ds < 2\pi$ equal to the angle $\widehat{BOA}$, which is smaller or equal than π. Hence, by Lemma 3.4, all such curve γ_1 stays over the straight line passing for the two triple junctions A and B.

We call γ_3 the other extremal curve, hence since $\phi_1, \phi_3 > 0$, we have

$$\gamma_1(\underline{s}_1) = \gamma_3(\overline{s}_3) = B, \quad \gamma_1(\overline{s}_1) = \gamma_3(\underline{s}_3) = A.$$

Because of the shrinker equation (1.2), all the three curves are convex with respect to the origin. This implies that the origin is contained in the interior of the bounded area A_{13} enclosed by γ_1 and γ_3 (if the origin belongs to γ_1 or γ_3 such curve is a segment and cannot be "external", as we said before), which also contains $\gamma_2 \subset A_{13}$. We let A_{12} be the region enclosed by the curves γ_1 and γ_2 and we split the analysis into two cases.

Case 1. The origin does not belong to the interior of A_{12}.

Since the curve γ_2 is convex with respect to the origin, by the same argument used above for γ_1, it is contained in the upper half–plane determined by the straight line for the points A and B.

By the 120 degrees condition it follows that the angle β at B formed by the vector $(1, 0)$ and γ_1' is at most $\frac{\pi}{3}$. Similarly, also the angle α at A formed by the vector $(1, 0)$ and γ_1' is at most $\frac{\pi}{3}$. By the convexity of the region delimited by γ_2 and γ_3 containing the origin and again the 120 degrees condition at B, it is then easy to see that the angle at B formed by the vectors γ_1 and γ_1' is less or equal than $\frac{\pi}{3}$ and analogously, the angle at A formed by γ_1 and γ_1' is greater or equal than $\frac{2\pi}{3}$.

Hence, by equality (2.5), it follows

$$\rho_1'(\underline{s}_1) \geq \frac{1}{2} > 0, \quad \rho_1'(\overline{s}_1) \leq -\frac{1}{2} < 0.$$

As a consequence, there is a point of maximum radius $s_1^* \in (\underline{s}_1, \overline{s}_1)$ such that $\rho_1(s_1^*) \geq \rho_1(s)$ for all $s \in (\underline{s}_1, \overline{s}_1)$.

The vector $\gamma_1(s_1^*)$ forms an angle $\sigma \geq \frac{\pi}{2}$ with $(1,0)$ or $(-1,0)$. Assume that the angle between $\gamma_1(s_1^*)$ and $(1,0)$ is greater or equal than $\frac{\pi}{2}$ (the other case is analogous, switching A and B). We extend the curve γ_1 (still parametrized by arclength) "before" the point B till it intersects the x–axis at some $\widetilde{s}_1 \leq \underline{s}_1$ (this must happen because $\phi_1(s) > 0$ everywhere also on the extended curve) and we consider the (non relabeled) curve γ_1 defined in the interval $L_1 = [\widetilde{s}_1, s_1^*]$. Calling β_0 the angle formed by the vectors $\gamma_1'(\widetilde{s}_1)$ and $(1,0)$, by convexity and the fact that the angle β at B formed by the vector $(1,0)$ and γ_1' is at most $\frac{\pi}{3}$, we have that $\beta_0 \leq \beta \leq \frac{\pi}{3}$. Hence, by equality (2.5), we have $\rho_1'(\widetilde{s}_1) \geq \frac{1}{2} > 0$.

Considering now the function $s \mapsto \rho_1(s)$ on the interval $L_1 = [\widetilde{s}_1, s_1^*]$, since $\rho_1'(\widetilde{s}_1) > 0$ and s_1^* is a maximum point for ρ_1, either ρ_1 is increasing on L_1, or ρ_1 has another maximum and then a minimum in the interior of L_1 (notice that the map ρ_1 cannot be constant on an interval, otherwise γ_1 would be an arc of a circle centered at the origin, which is impossible since ρ_1 is not constant). But we know from formula (2.3) and Proposition 2.1 that the angle ϕ_1 must increase more than $\frac{\pi}{2}$ to go from a minimum to a maximum or viceversa (we can apply such proposition since γ_1 is not an arc of a circle). Since

$$\int_{\widetilde{s}_1}^{s_1^*} \phi_1'(s)\, ds \leq \pi,$$

there cannot be a maximum, then a minimum, then a second maximum in L_1. It follows that ρ_1 is increasing in such interval.

This, combined with the fact that $\beta_0 \leq \frac{\pi}{3}$ and that the angle σ is at least $\frac{\pi}{2}$, that is, $\int_{\widetilde{s}_1}^{s_1^*} \phi_1'(s)\, ds \geq \frac{\pi}{2}$, is in contradiction with Lemma 3.1. Therefore, this case cannot happen.

Case 2. The origin belongs to the interior of A_{12}.

Being the region A_{12} convex (by the shrinker equation (1.2), since it contains the origin), the curve γ_2 (which is oriented counterclockwise) goes from A to B. The fact that γ_2' and γ_3' form angles of $\frac{2\pi}{3}$ at the points A and B implies that:

(i) the angle in A formed by the vectors $\gamma_3(\underline{s}_3)$ and $\gamma_3'(\underline{s}_3)$ and the angle in B formed by the vectors $\gamma_2(\overline{s}_2)$ and $\gamma_2'(\overline{s}_2)$ are both less or equal than $\frac{\pi}{3}$;

(ii) the angle in B formed by the vectors $\gamma_3(\overline{s}_3)$ and $\gamma_3'(\overline{s}_3)$ and the angle in A formed by the vectors $\gamma_2(\underline{s}_2)$ and $\gamma_2'(\underline{s}_2)$ are both greater or equal than $\frac{2\pi}{3}$.

In particular, by equality (2.5), it follows

$$\rho_2'(\underline{s}_2) \leq -\frac{1}{2} < 0, \quad \rho_2'(\overline{s}_2) \geq \frac{1}{2} > 0, \quad \rho_3'(\underline{s}_3) \geq \frac{1}{2} > 0, \quad \rho_3'(\overline{s}_3) \leq -\frac{1}{2} < 0. \tag{3.3}$$

Hence, the function $s \mapsto \rho_3(s)$ has a maximum at some point $s_3^* \in (\underline{s}_3, \overline{s}_3)$, while the function $s \mapsto \rho_2(s)$ has a minimum at some point $s_2^\circ \in (\underline{s}_2, \overline{s}_2)$.

If s_3^* is the only point of maximum of ρ_3 in the interval $[\underline{s}_3, \overline{s}_3]$, then the function ρ_3 is strictly monotone on each of the two subintervals $[\underline{s}_3, s_3^*]$ and $[s_3^*, \overline{s}_3]$, moreover,

$$\int_{\underline{s}_3}^{s_3^*} \phi_3'(s)\, ds + \int_{s_3^*}^{\overline{s}_3} \phi_3'(s)\, ds = \int_{\underline{s}_3}^{\overline{s}_3} \phi_3'(s)\, ds \geq \pi,$$

since the origin is "below" the segment $\overline{AB}$. Thus, at least one of the two integrals on the left–hand side is greater or equal than $\frac{\pi}{2}$ and, by Lemma 3.1, this is not possible. As a consequence, there must be another point of maximum radius $s_3^{**} \in (\underline{s}_3, \overline{s}_3)$ (notice that the maximum points cannot be an interval, otherwise

γ_3 would be an arc of a circle centered at the origin, hence with $\rho_3' = 0$, against relations (3.3)). Hence, between these two points of maximum radius there is a minimum point s_3°. Without loss of generality, we assume that $\underline{s}_3 < s_3^* < s_3^\circ < s_3^{**} < \overline{s}_3$.

We observe that there cannot be a third maximum point for ρ_3 (hence also another minimum point) in the interval $[\underline{s}_3, \overline{s}_3]$ because, by Lemma 2.2, each of the four angles at the origin formed by the segment connecting the origin with two consecutive of the five extremal points for ρ_3 on γ_3 is greater than $\frac{\pi}{2}$ and, by Lemma 3.3, there holds $\int_{\underline{s}_3}^{\overline{s}_3} \phi_3'(s)\, ds < 2\pi$. Moreover, also the case of two minimum points and two maximum points for ρ_3 in the interval $[\underline{s}_3, \overline{s}_3]$ is not possible, because of the sign of the derivative ρ_3' at the endpoints in relations (3.3). Hence, we conclude that $s_3^*, s_3^\circ, s_3^{**}$ are the only extremal points for ρ_3 in the interval $[\underline{s}_3, \overline{s}_3]$.

Now consider the quantities $\mathcal{F}_2, \mathcal{F}_3$ of the curves γ_2, γ_3, respectively, given by formula (2.7). By relations (3.3), the curves γ_2 and γ_3 are not the unit circle (they would have ρ_2' or ρ_3' equal to zero everywhere), therefore $\mathcal{F}_2, \mathcal{F}_3 > \frac{1}{2}$. If we draw the line from the origin to $\gamma_3(s_3^\circ)$, this must intersect γ_2 in an intermediate point, implying that the minimal radius of the curve γ_2 is smaller than the minimal radius of the curve γ_3. By the discussion about the value of the quantity $\mathcal{F}$ in relation with the extremal values of ρ at the end of Section 2, we have $\mathcal{F}_2 > \mathcal{F}_3$. Then, if a maximum of ρ_2 is taken in the interior of γ_2, it must be larger than the maximal radius of γ_3 (which is taken in the interior of γ_3), which is not possible as γ_2 is contained in the region bounded by γ_3 and the segment $\overline{AB}$. From this argument we conclude that there are no points of maximal radius in the interior of γ_2, thus, the only extremal point for ρ_2 in the interval $[\underline{s}_2, \overline{s}_2]$ is the minimum point s_2°.

Defining the angle

$$\alpha := \int_{\underline{s}_2}^{\overline{s}_2} \phi_2'(s)\, ds = \int_{\underline{s}_3}^{\overline{s}_3} \phi_3'(s)\, ds,$$

by formula (2.8) and the symmetry of the curve γ_3 with respect to the straight line through the origin and the point $\gamma_3(s_3^\circ)$ of minimum distance, we have

$$\mathcal{I}(\mathcal{F}_3) = \int_{s_3^*}^{s_3^\circ} \phi_3'(s)\, ds = \int_{s_3^\circ}^{s_3^{**}} \phi_3'(s)\, ds < \frac{\alpha}{2}$$

while, since γ_2 does not contain any interior point of maximum radius,

$$\mathcal{I}(\mathcal{F}_2) > \max\left\{ \int_{\underline{s}_2}^{s_2^\circ} \phi_2'(s)\, ds,\ \int_{s_2^\circ}^{\overline{s}_2} \phi_2'(s)\, ds \right\} \geq \frac{\alpha}{2}.$$

Thus, $\mathcal{I}(\mathcal{F}_2) > \mathcal{I}(\mathcal{F}_3)$ and $\mathcal{F}_2 > \mathcal{F}_3$, which is in contradiction with the monotonicity of the function $\mathcal{I}$ given by Proposition 2.2. Hence, also this case can be excluded.

Since we excluded both cases, our hypothetical Θ–shrinker cannot exist and we are done with the proof of Theorem 1.1.

References

[1] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geom. **23** (1986), no. 2, 175–196.

[2] P. Baldi, E. Haus, and C. Mantegazza, *Non–existence of Theta–shaped self–similarly shrinking networks moving by curvature*, ArXiv Preprint Server – http://arxiv.org, 2016.

[3] G. Bellettini and M. Novaga, *Curvature evolution of nonconvex lens–shaped domains*, J. Reine Angew. Math. **656** (2011), 17–46.

[4] K. A. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, NJ, 1978.

[5] X. Chen and J.-S. Guo, *Self–similar solutions of a 2–D multiple–phase curvature flow*, Phys. D **229** (2007), no. 1, 22–34.

[6] C. L. Epstein and M. I. Weinstein, *A stable manifold theorem for the curve shortening equation*, Comm. Pure Appl. Math. **40** (1987), no. 1, 119–139.

[7] J. Hättenschweiler, *Mean curvature flow of networks with triple junctions in the plane*, Master's thesis, ETH Zürich, 2007.

[8] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom. **31** (1990), 285–299.

[9] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc., vol. 108(520), Amer. Math. Soc., 1994.

[10] ______, *Singularities of mean curvature flow of surfaces*, http://www.math.ethz.ch/~ilmanen/papers/sing.ps, 1995.

[11] T. Ilmanen, A. Neves, and F. Schulze, *On short time existence for the planar network flow*, ArXiv Preprint Server – http://arxiv.org, 2014.

[12] A. Magni, C. Mantegazza, and M. Novaga, *Motion by curvature of planar networks II*, Ann. Sc. Norm. Sup. Pisa **15** (2016), 117–144.

[13] C. Mantegazza, M. Novaga, and A. Pluda, *Motion by curvature of networks with two triple junctions*, in preparation.

[14] C. Mantegazza, M. Novaga, A. Pluda, and F. Schulze, *Evolution of networks with multiple junctions*, in preparation.

[15] C. Mantegazza, M. Novaga, and V. M. Tortorelli, *Motion by curvature of planar networks*, Ann. Sc. Norm. Sup. Pisa **3 (5)** (2004), 235–324.

[16] R. Mazzeo and M. Sáez, *Self–similar expanding solutions for the planar network flow*, Analytic aspects of problems in Riemannian geometry: elliptic PDEs, solitons and computer imaging, Sémin. Congr., vol. 22, Soc. Math. France, Paris, 2011, pp. 159–173.

[17] A. Pluda, *Evolution of spoon–shaped networks*, ArXiv Preprint Server – http://arxiv.org, to appear on Netw. Heterog. Media, 2015.

[18] O. C. Schnürer, A. Azouani, M. Georgi, J. Hell, J. Nihar, A. Koeller, T. Marxen, S. Ritthaler, M. Sáez, F. Schulze, and B. Smith, *Evolution of convex lens–shaped networks under the curve shortening flow*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2265–2294.

[19] O. C. Schnürer and F. Schulze, *Self–similarly expanding networks to curve shortening flow*, Ann. Sc. Norm. Super. Pisa **6** (2007), no. 4, 511–528.