Steady periodic water waves under nonlinear elastic membranes

Pietro Baldi and John F. Toland

Abstract

This is a study of two-dimensional steady periodic travelling waves on the surface of an infinitely deep irrotational ocean, when the top streamline is in contact with a membrane which has a nonlinear response to stretching and bending, and the pressure in the air above is constant. It is not supposed that the waves have small amplitude. The problem of existence of such waves is addressed using methods from the calculus of variations. The analysis involves the Hilbert transform and a Riemann-Hilbert formulation.

Keywords: hydrodynamics, nonlinear elasticity, free boundary problems, travelling waves, variational methods, Hilbert transform, Riemann-Hilbert problems.

2000 Mathematics Subject Classification: 35R35, 74B20, 74F10, 35Q15, 76M30.

Contents

1 Introduction

Regarding water as an inviscid incompressible liquid, we study two-dimensional steady waves on the surface of an ocean of infinite depth, moving under the influence of gravity when the surface is in contact with a thin frictionless elastic membrane that responds nonlinearly to bending, compression and stretching, and above the membrane there is constant atmospheric pressure. We suppose that the steady fluid motion is irrotational and the top streamline is a space-periodic curve that travels with constant velocity, without changing its shape. We suppose also that the two-dimensional cross-section of the elastic surface behaves mechanically like a thin (unshearable) hyperelastic Cosserat rod, as described by Antman in [3], Ch. 4. The physical significance of such a problem is evident; for example in the theory of very large floating structures or platforms (see [2] and the references therein), or, possibly, flow under ice. We refer to this as a hydroelastic travelling wave problem.

The mathematical study of these waves began with the linear theory of Greenhill in the nineteenth century [9], but an analysis of nonlinear models has only recently been attempted. In [14], the existence question was formulated as a variational problem, and existence was proved, for the case of a class of membranes that have an infinite elastic energy when the stretching or bending exceed certain fixed values, by maximising a Lagrangian over a set of admissible functions. Other recent work extend the theory of [14] in different ways. For example, in [15] membranes with positive densities are included in the theory provided the resulting variational problem is convex. This is a restriction on the membrane density and on one of the wave-speed parameters in the problem. In [11], that restriction was removed and the general problem of membranes with positive mass was studied, using Young's measures to deal with the problem of non-convexity.

In the present paper we generalize and simplify the theory of surface membranes with zero density [14], by proving the existence of steady periodic hydroelastic waves for membranes when the stored elastic energy remains finite but has power-law growth as the bending or stretching/compression increases, as is more or less standard in the mathematical theory of nonlinear elasticity. A further novelty is the use of a Riemann-Hilbert formulation in the context of hydroelastic waves. This approach simplifies and clarifies the reduction of the problem to one for a single function of a single real variable.

In the rest of this Introduction, we describe the physical problem, summarize the main results and the methods, and discuss the hypotheses on the elastic properties of the membranes under which they are obtained.

1.1 The physical problem

The physical system under investigation was studied in [14], Sec. 1.1. We seek waves that are two-dimensional and steady with prescribed period. More precisely, we consider waves such that

(i) in three-dimensional space (X, Y, Z), with gravity g acting in the negative Y direction, the flow beneath the free surface is irrotational;

(ii) the Z-component of the fluid velocity is everywhere zero and all components tend to zero as $Y \to -\infty$;

(iii) the Y-coordinates of points on the surface are independent of Z and the surface moves without change of form and with constant speed c in the X-direction;

(iv) the flow is 2π-periodic and stationary with respect to axes moving with the wave speed.

Because the membrane has zero density, it is equivalent to study, in a frame moving with the wave, steady 2π-periodic waves for which the speed of the flow at infinite depth is $-c$ horizontally. In this frame, the intersection of the surface membrane with the plane $Z = 0$, called the membrane section, is supposed to behave like a nonlinear, unshearable, hyperelastic rod for which the stored energy depends on stretch and curvature. By the reference membrane is meant the line $Y = 0$ and one period of it refers to a line segment of length 2π. We study waves for which

(v) one period of the reference membrane is deformed to become one period of the hydroelastic wave surface.

The unknown region occupied by the liquid is characterized by the kinematic requirement that the surface is a streamline and the dynamic condition that the pressure P in the fluid and internal forces are those required to deform the membrane. Therefore a steady hydroelastic wave with speed c satisfying (i-iv) corresponds to a non-self-intersecting smooth curve $\mathcal{S}$ in the plane (X, Y) which is 2π-periodic in the horizontal direction X and for which there exists a solution of the following system:

(1.1a) $$\Delta \psi = 0 \quad \text{below } \mathcal{S},$$

(1.1b) $$\psi = 0 \quad \text{on } \mathcal{S} \text{ (the kinematic boundary condition)},$$

(1.1c) $$\nabla \psi(X, Y) \to (0, c) \quad \text{as } Y \to -\infty,$$

with the dynamic boundary condition

(1.1d) $$\frac{1}{2} |\nabla \psi|^2 + gY = \frac{c^2}{2} - P \quad \text{on } \mathcal{S}.$$

Moreover, suppose that $\boldsymbol{r}$ is the physical deformation that carries a material point x of the reference membrane into its new position $\boldsymbol{r}(x)$. Then, by assumption, the profile $\{\boldsymbol{r}(x) : x \in \mathbb{R}\}$ of the deformed membrane must coincide with the free surface $\mathcal{S}$ of the fluid, and the constraint (v) reads

(1.1e) $$\mathcal{S} \cap \{0 \le X \le 2\pi\} = \{\boldsymbol{r}(x) : x \in [x_0, x_0 + 2\pi]\}$$

3

for some $x_0 \in \mathbb{R}$. In [14], Antman's treatment [3] of unshearable Cosserat rods is used to derive a formula for the pressure P in (1.1d). Here we simply recall that formula after introducing some notation.

Consider an interval of membrane in its rest position. Its material points are labelled $x \in [x_1, x_2]$. We consider a deformation $\boldsymbol{r}$ that move any point x in its new position $\boldsymbol{r}(x)$. The stretch of the deformed membrane at the point $\boldsymbol{r}(x)$ is then

$$(1.2a) \qquad\qquad \nu(x) := |\boldsymbol{r}'(x)|,$$

where $'$ denotes the derivative with respect to x. Let $\vartheta(x)$ denote the angle formed by the membrane and the positive horizontal semiaxis at the point $\boldsymbol{r}(x)$, and let

$$(1.2b) \qquad\qquad \mu(x) := \vartheta'(x).$$

Then the curvature of the membrane at $\boldsymbol{r}(x)$ is

$$\hat{\sigma}(\boldsymbol{r}(x)) := \frac{\mu(x)}{\nu(x)}.$$

We assume that the material is hyperelastic (see, for example, [3], Ch. 4) with stored elastic energy function,

$$E(\nu, \mu) \geq 0, \quad \nu > 0, \ \mu \in \mathbb{R},$$

of class C^2. Denote by E_1 and E_2 the partial derivatives of E with respect to its variables, ν and μ, respectively. From the balance law for forces and moments acting on the membrane, it follows (see [14], eqs. (1.1)) that

$$(1.3a) \qquad \nu(x)\, E_1(\nu(x), \mu(x))' + \mu(x)\, E_2(\nu(x), \mu(x))' = 0,$$

and

$$(1.3b) \qquad P(\boldsymbol{r}(x)) = \frac{1}{\nu(x)} \left(\frac{E_2(\nu(x), \mu(x))'}{\nu(x)} \right)' - \hat{\sigma}(\boldsymbol{r}(x))\, E_1(\nu(x), \mu(x)),$$

where $P(\boldsymbol{r})$ is the pressure that is needed to produce the deformation $\boldsymbol{r}$. Hence the physical deformation $\boldsymbol{r}$ of the material points of the reference membrane enters in the hydroelastic wave problem (1.1d), through the term P.

Remark 1. At this point in [14], eqn. (1.6), $\nu(x)$ was calculated in terms of $\hat{\sigma}(\boldsymbol{r}(x))$, using equation (1.3a) and the constraint (v). In the present paper we avoid that calculation at this stage. The formula for ν will emerge later, in the final part of the regularity proof, see (4.16). $\qquad\qquad\square$

Now we introduce the hypotheses on E and explain briefly their roles in the theory.

1.2 Hypotheses

The first seven hypotheses are used in the theory of maximization of the Lagrangian. The next five are needed to ensure that maximizers satisfy the Euler-Lagrange equation and are sufficiently regular to give a solution of the physical problem (1.1). The first hypothesis is needed to define the potential energy stored by elasticity in the deformed membrane. The absence of a shear variable (see [3], Ch. 4) in the argument of E reflects our assumption that the membrane is thin and unshearable.

Hypothesis (H1). (*Unshearable hyperelasticity*) There exists a stored elastic energy function,

$$E(\nu, \mu) \geq 0, \quad \nu > 0, \ \mu \in \mathbb{R},$$

of class C^2, such that the elastic energy in a segment $[x_1, x_2]$ of material, when deformed by $x \mapsto \boldsymbol{r}(x)$, is

$$(1.4) \qquad \mathcal{E}(\boldsymbol{r}) = \int_{x_1}^{x_2} E(\nu(x), \mu(x)) \, dx,$$

where $\nu(x), \mu(x)$ are defined in (1.2). □

The following four hypotheses are used to obtain the existence of a maximizer of the Lagrangian J_0, using the direct method of the calculus of variations. First we assume that the material response is even with respect to curvature and then that the elastic energy is minimized when the material is neither stretched nor bent (we normalize the elastic energy of this rest state to be zero).

Hypothesis (H2). (*Evenness with respect to curvature*) $E(\nu, \mu) = E(\nu, -\mu)$ for all $\nu > 0$, $\mu \in \mathbb{R}$. □

Hypothesis (H3). (*Rest state*) $E(\nu, \mu) \geq 0 = E(1, 0)$ for all $\nu > 0$, $\mu \in \mathbb{R}$. □

The next condition ensures the upper semi-continuity of the functional to be maximized. However, it has a wider significance. For example, in the dynamic theory of nonlinear rods it ensures that the equations of motion are strictly hyperbolic and accordingly have rich wave-like behavior. It is also an exact analog of the 3-dimensional Strong Ellipticity Condition [10]. By Lemma 4 below, (H4) coincides with the convexity assumption in [15].

Hypothesis (H4). (*Strict joint convexity*)

$$E_{22} > 0, \quad E_{11} > 0, \quad E_{11} \, E_{22} - (E_{12})^2 > 0$$

at all points (ν, μ) of the semi-plane $\nu > 0$, $\mu \in \mathbb{R}$. □

It is natural to assume that for infinite stretch, compression, or curvature, an infinite amount of energy is required, that is, $E(\nu, \mu)$ tends to infinity as ν goes to 0 or $+\infty$, or $|\mu|$ goes to $+\infty$. The next hypothesis quantifies that assumption.

Hypothesis (H5). (*Growth condition*)

$$E(\nu, \mu) \geq K_0 \left(\nu^r + \frac{1}{\nu^s} + |\mu|^p \right) - K_0' \quad \forall \nu > 0, \ \mu \in \mathbb{R},$$

for some positive constants K_0, K_0', and some exponents $r > 2$, $s > 0$, $p > 1$. □

With the existence question settled under the above hypotheses we need two further hypotheses to guarantee basic properties of the maximizer. The first ensures that the maximizer is non-trivial (does not correspond to the laminar flow of a wave with zero elevation), see Lemma 2.

Hypothesis (H6). (*Non-trivial maximizers*) $c^2 > g + E_{22}(1, 0)$. □

The second is needed to show that the curve that emerges is non-self-intersecting (see Lemma 3). Obviously, this is essential if it is to be the surface of a travelling wave. However (H7) has a further role. It guarantees sufficient compactness of a maximizing sequence to yield the existence of a maximizer. For $\ell > 1$, let $A(\ell)$ be the area in a circle enclosed between an arc of length $2\pi\ell$ and a chord of length 2π (Figure 1). The asymptotics for $A(\ell)$ is then

$$\lim_{\ell \to 1} \frac{A(\ell)}{\sqrt{\ell - 1}} \to 2\sqrt{2/3}\,\pi^2, \qquad \lim_{\ell \to +\infty} \frac{A(\ell)}{\ell^2} = \pi,$$
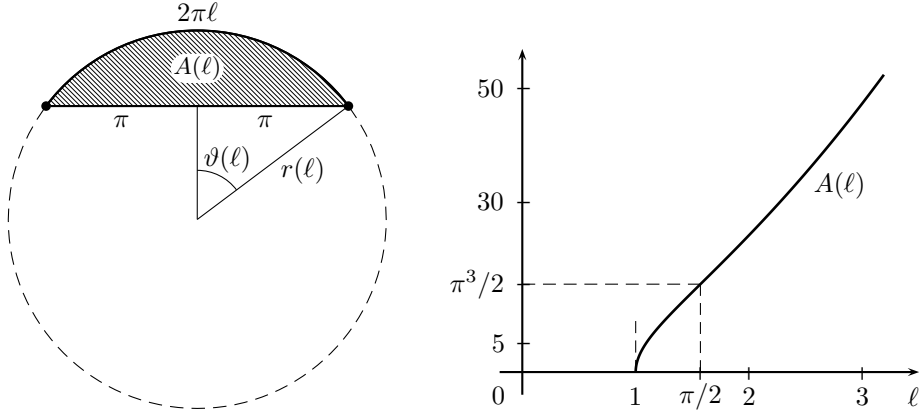
see Lemma 1.

Figure 1: Geometric definition and plot of $A(\ell)$

Hypothesis (H7). (*Non-self-intersecting maximizers*) There exists $\mu^* \in (0, 1)$ such that

$$E(\nu, \mu^*) \geq \frac{g}{8\pi^2} A(\nu)^2 + \frac{c^2}{4\pi} A(\nu) + \frac{g}{2} A(\nu) \sqrt{\nu^2 - 1} \quad \forall \nu \geq 1. \qquad \square$$

6

Remark 2. Note that, by (3.6) below, (H7) is implied by the stronger condition

$$E(\nu, \mu^*) \geq \frac{g}{2}\nu^4 + g\pi\,\nu^3 + \frac{c^2}{2}\nu^2 \quad \forall \nu \geq 1,$$

and the exponent r in (H5) must be not less than 4. □

Remark 3. (H6) and (H7) are simultaneously satisfied if

$$(1.5) \qquad g + E_{22}(1,0) < c^2 \leq \inf_{\nu > 1}\left\{ \frac{4\pi E(\nu, \mu^*)}{A(\nu)} - \frac{gA(\nu)}{2\pi} - 2\pi g\sqrt{\nu^2 - 1} \right\}.$$

As an example, we consider the case when E splits, $E(\nu, \mu) = S(\nu) + B(\mu)$, and show that the interval (1.5) for c^2 is nonempty provided the growth of $S(\nu)$ for large ν and that of $B(\mu)$ on the interval $\mu \in [0, \mu^*]$ are sufficiently rapid.

We denote $b := B''(0)$. Let us fix $M > 0$. If the growth of $S(\nu)$ as $\nu \to +\infty$ is rapid enough, then there exists $\bar{\nu} > 1$ (depending on M and b) such that

$$\frac{4\pi S(\nu)}{A(\nu)} - \frac{gA(\nu)}{2\pi} - 2\pi g\sqrt{\nu^2 - 1} \geq g + b + M \quad \forall \nu \geq \bar{\nu}.$$

Next, since $A(\nu)$ is increasing,

$$\frac{4\pi B(\mu^*)}{A(\nu)} - \frac{gA(\nu)}{2\pi} - 2\pi g\sqrt{\nu^2 - 1} \geq \frac{4\pi B(\mu^*)}{A(\bar{\nu})} - \frac{gA(\bar{\nu})}{2\pi} - 2\pi g\sqrt{\bar{\nu}^2 - 1}$$

for all $\nu \in (1, \bar{\nu}]$. Then

$$\frac{4\pi E(\nu, \mu^*)}{A(\nu)} - \frac{gA(\nu)}{2\pi} - 2\pi g\sqrt{\nu^2 - 1} \geq g + b + M \quad \forall \nu > 1$$

provided

$$(1.6) \qquad B(\mu^*) \geq \frac{g}{8\pi^2}A(\bar{\nu})^2 + \frac{g}{2}A(\bar{\nu})\sqrt{\bar{\nu}^2 - 1} + \frac{(g + b + M)}{4\pi}A(\bar{\nu}).$$

(1.6) holds if $B(\mu^*)$ is sufficiently large, depending on b and M. For example, if $B(\mu) = (b/2)\mu^2 + b_1\mu^4$, then (1.6) holds if b_1 is sufficiently large. In that case, (1.5) holds for all c^2 in the interval

$$g + b \leq c^2 \leq g + b + M. \qquad \square$$

The remaining five hypotheses are needed to show that maximizers of the Lagrangian yield steady hydroelastic travelling waves. The first is an assumption on ∇E, which will lead to the conclusion that the stretch ν of the membrane is bounded above (see Lemma 6).

Hypothesis (H8). (*Bounded stretch*)

$$\lim_{\nu \to +\infty}\left\{ \inf_{\mu \in \mathbb{R}}\left(\nabla E(\nu, \mu) \cdot (\nu, \mu) - E(\nu, \mu) \right) \right\} = +\infty. \qquad \square$$

7

Remark 4. In the "splitting" case when $E(\nu, \mu) = S(\nu) + B(\mu)$, (H8) is automatically satisfied when (H4,5) hold. Indeed,

$$\big(B'(\mu)\,\mu - B(\mu) \big)' = B''(\mu)\,\mu$$

has the same sign as μ, therefore

$$\inf_{\mu \in \mathbb{R}} \big(\nabla E(\nu, \mu) \cdot (\nu, \mu) - E(\nu, \mu) \big) = S'(\nu)\,\nu - S(\nu) - B(0).$$

Now, $S'(\nu)\,\nu - S(\nu)$ is strictly increasing in ν, thus its limit as $\nu \to +\infty$ exists. Suppose that such a limit is a real number. Then

$$\Big(\frac{S(\nu)}{\nu} \Big)' = \frac{S'(\nu)\,\nu - S(\nu)}{\nu^2} \to 0 \quad \text{as } \nu \to +\infty.$$

Hence there exists $\bar{\nu}$ such that

$$\Big| \Big(\frac{S(\nu)}{\nu} \Big)' \Big| \leq 1 \quad \forall \nu \geq \bar{\nu}.$$

It follows that

$$\frac{S(\nu)}{\nu} = \frac{S(\bar{\nu})}{\bar{\nu}} + \int_{\bar{\nu}}^{\nu} \Big(\frac{S(\xi)}{\xi} \Big)' d\xi \leq C + \nu \quad \forall \nu \geq \bar{\nu},$$

for some constant C. But this violates (H5), because $r > 2$. Hence $S'(\nu)\,\nu - S(\nu)$ goes to $+\infty$ as $\nu \to +\infty$, and (H8) follows. $\square$

The next two growth conditions ensure the differentiability of J_0 at a maximizer.

Hypothesis (H9). There exist positive constants $K_1, \bar{\nu}_1$ and $\bar{\mu}_1$ such that

$$|E_1(\nu, \mu)| \leq K_1 \Big(\frac{1}{\nu^{s+1}} + |\mu|^p \Big) \quad \forall \nu \leq \bar{\nu}_1,\ |\mu| \geq \bar{\mu}_1. \qquad \square$$

Hypothesis (H10). There exist positive constants $K_2, \bar{\nu}_2$ and $\bar{\mu}_2$ such that

$$|E_2(\nu, \mu)| \leq K_2 \Big(\frac{1}{\nu^{\frac{s(p-1)}{p}}} + |\mu|^{p-1} \Big) \quad \forall \nu \leq \bar{\nu}_2,\ |\mu| \geq \bar{\mu}_2. \qquad \square$$

Remark 5. Recalling Young's inequality

(1.7) $$xy \leq \varepsilon\, x^q + C_\varepsilon\, y^{q'} \quad \forall\, x, y, \varepsilon > 0,$$

with $q > 1$, $1/q + 1/q' = 1$ and $C_\varepsilon := \varepsilon^{-1/(q-1)}$, we note that (H9,10) are compatible with the presence of "mixed term" of the type $|\mu|^\alpha / \nu^\delta$ in $E(\nu, \mu)$, provided these couplings are not too strong with respect to the leading "pure terms" of the form ν^r, $1/\nu^s$ and $|\mu|^p$; see the example in Subsection 1.4. $\square$

Hypothesis (H11). For every $\gamma \in \mathbb{R}$ there exist positive constants K_γ, K'_γ, $\bar{\nu}_\gamma$, $\bar{\mu}_\gamma$ with the following property. If (ν, μ), with $\nu \leq \bar{\nu}_\gamma$ or $|\mu| \geq \bar{\mu}_\gamma$, satisfy

$$E(\nu, \mu) - \nabla E(\nu, \mu) \cdot (\nu, \mu) = \gamma,$$

then

$$\frac{K_\gamma}{\nu^s} \leq |\mu|^p \leq \frac{K'_\gamma}{\nu^s} . \qquad \square$$

Remark 6. In the case when $E(\nu, \mu) = S(\nu) + B(\mu)$, one can show that, if (H4,5,9,10) hold, then

$$-\frac{C}{\nu^s} \leq \nu S'(\nu) - S(\nu) \leq -\frac{C'}{\nu^s} , \quad \mu B'(\mu) - B(\mu) \leq C''|\mu|^p$$

for all small ν, all large $|\mu|$, for some positive constants C, C', C''. If, in addition,

$$(1.8) \qquad \mu B'(\mu) - B(\mu) \geq C|\mu|^p$$

for all large $|\mu|$, for some C, then (H11) is satisfied. We note that (1.8) holds if and only if the ratio $B(\mu)/\mu^\alpha$ is non-decreasing for all large μ, for some $\alpha > 1$. Also, if B satisfies

$$C_0|\mu|^{p-1} \leq |B'(\mu)| \leq C_1|\mu|^{p-1}$$

for all μ large, for some $C_0, C_1 > 0$ such that $C_1 < p\,C_0$, then $B(\mu) \leq C + (C_1/p)|\mu|^p$ for all μ large, therefore (1.8) holds.

Finally, we note that the simplest case $B(\mu) = |\mu|^p +$ (lower order terms) satisfies (1.8) trivially. $\qquad \square$

The final assumption leads to regularity properties of solutions (see Lemma 10).

Hypothesis (H12). There exist positive constants $K_3, \bar{\nu}_3, \bar{\mu}_3$ and an exponent $\alpha := s(p-1)/p - \varepsilon$, with $\varepsilon > 0$, such that

$$|E_2(\nu, \mu)| \geq K_3 \nu^\alpha |\mu|^{p-1} \quad \forall \nu \leq \bar{\nu}_3, \ |\mu| \geq \bar{\mu}_3. \qquad \square$$

Remark 7. In the splitting case $E(\nu, \mu) = S(\nu) + B(\mu)$, (H12) is automatically satisfied when (H4,5) hold, because, by the convexity of $B(\mu)$ and its growth condition,

$$|E_2(\nu, \mu)| = |B'(\mu)| \geq \frac{B(\mu)}{|\mu|} \geq C|\mu|^{p-1}$$

for all $|\mu|$ sufficiently large, uniformly in ν. $\qquad \square$

1.3 Main result and methods

Under the above hypotheses on the elastic properties of the membrane, our main result on the existence of $\mathcal{S}, \psi$ and $\boldsymbol{r}$ satisfying the hydroelastic wave problem (1.1) is the following:

9

Theorem 1. (Existence). *Suppose that the stored elastic energy function $E(\nu, \mu)$ satisfies (H1-12). Then, for admissible velocities c^2 in a certain interval, see (1.5), there exist a free surface curve $\mathcal{S}$ of class $W^{3,\infty}$ and a membrane deformation $\boldsymbol{r}$ of class $W^{2,\infty}$, satisfying the constraint (1.1e), such that the stream function ψ that solves (1.1a,b,c) is also a solution of the dynamic boundary equation (1.1d).*

(Regularity). If $E \in C^k$, with $k \geq 2$, then $\mathcal{S}$ is of class $W^{k+1,\infty}$, and $\boldsymbol{r} \in W^{k,\infty}$.[1]

Our strategy to prove Theorem 1 is the following. We approach the free boundary problem by defining a Lagrangian in terms of the kinetic and potential energies, including the elastic energy in the membrane, in one period of a steady wave (Section 2). We use conformal mappings in a variational setting [6] to overcome the difficulty that the flow domain is the unknown (Subsection 2.1). We then use the direct method of the calculus of variations to maximize the Lagrangian (Section 3). Key ingredients in the existence theory are Hurwitz's analytical version of the classical isoperimetrical inequality [4], which we use to control kinetic and gravitational potential energies in terms of the length of a deformed period of the mebrane, and Zygmund's theorem [17] for exponential of holomorphic functions on the unit disc, to recover both the non-self-intersection property for the wave profile and compactness. After that we use the growth hypotheses on the stored energy function to deduce an a priori bounds for the maximizer sufficient to infer that they satisfy the corresponding Euler-Lagrange system (Section 4). This is a coupled system of equations, for a pair of periodic function of a real variable which involves the Hilbert transform, that can be reformulated as a Riemann-Hilbert problem [12, 13]. Using this observation it is shown that hydroelastic waves arise from maximizers of the Lagrangian (Section 5). The results in the variational formulation are listed in Theorem 2 (Subsection 2.2), from which Theorem 1 follows (see Lemma 12 and Proposition 2).

We remark that this is a global variational theory; in particular, it is not a theory of small-amplitude solutions. Its successful application to existence questions is restricted to membranes that are resistant to both bending and stretching: this maximization argument cannot be used for Stokes waves or simple surface tension waves.

1.4 Illustrative Example

The following is a simple illustration of our hypotheses. It shows that our result is valid even for stored elastic energy functions $E(\nu, \mu)$ that include a nontrivial "mixed term" of the form $|\mu|^\alpha / \nu^\delta$.

Suppose that E is given by

$$E(\nu, \mu) = \frac{a}{s} \frac{1}{\nu^s} + \frac{a}{r} \nu^r + b|\mu|^p + \beta\mu^2 + d \frac{|\mu|^\alpha}{\nu^\delta} - \frac{a(s+r)}{s\, r},$$

---

[1]The regularity of ψ follows by that of $\mathcal{S}$ by classical theory.

with $a, b, \beta, d, s, \delta > 0$, $r > 1$, $\alpha \geq 2$, and $p > 2$.

The coefficients of ν^r and $1/\nu^s$ are such that the minimum of $E(\nu, 0)$ occur at $\nu = 1$. The constant term $-a(s+r)/sr$ guarantees that $E(1, 0) = 0$. Since α and p are not less than 2, $E(\nu, \mu)$ is of class C^2. As a consequence, (H1,3) are satisfied.

(H2,5,12) and the fact that E_{11} and E_{22} are positive everywhere can be immediately verified. If

$$(1.9a) \qquad\qquad \alpha > \delta + 1,$$

then the mixed term $|\mu|^\alpha / \nu^\delta$ is strictly jointly convex, and (H4) follows. We also assume (1.9a) to prove (H8).

(H6) holds provided

$$(1.9b) \qquad\qquad c^2 > g + 2\beta_0,$$

where $\beta_0 := \beta$ if $\alpha > 2$, and $\beta_0 := \beta + d$ if $\alpha = 2$.

To prove (H7), it is sufficient to assume that

$$(1.9c) \qquad r \geq 4, \qquad \frac{a}{r} \geq \frac{g}{2} + g\pi + \frac{c^2}{2}, \qquad b > \frac{a(s+r)}{sr},$$

by Remark 2 and the continuity of $\mu \mapsto b\mu^p$ near $\mu = 1$.

Using Young's inequality (1.7) to control the mixed terms, one can see that (H9,10,11) hold if

$$\alpha(s+1) + \delta p \leq sp.$$

Note that this inequality, when (1.9a) holds, is implyed by the stronger condition

$$(1.9d) \qquad\qquad \alpha \leq \frac{p(s+1)}{p+s+1}.$$

Thus, (1.9a,b,c,d) imply (H1-12), with

$$g + 2\beta_0 < c^2 \leq \frac{2a}{r} - g(1 + 2\pi)$$

as an interval of admissible velocities. A necessary condition for this interval to be nonempty is then

$$\beta_0 < \frac{bs}{s+r} - g(1 + \pi).$$

2 The Lagrangian

The strategy for proving this result is to maximize the natural Lagrangian of the physical problem and to observe that such a maximizer yields a non trivial solution of (1.1) in which P is given by (1.3b). The Lagrangian involves the fluid's kinetic and potential energies, and the elastic energy of the membrane. As in [6, 11, 14, 15], to deal with the unknown flow domain, it is convenient to

formulate the Lagrangian using conformal mappings. We begin by considering it in its physical context.

The Lagrangian for travelling waves is the difference between kinetic and potential energies in one period, relative to a frame in which the fluid velocity is stationary. Formally suppose that one period of the wave profile $\mathcal{S}$ in the moving frame is given by

$$\mathcal{S}_{2\pi} = \{(U(\tau), V(\tau)) \,:\, \tau \in [0, 2\pi]\},$$

where

$$U(\tau + 2\pi) = 2\pi + U(\tau), \quad V(\tau + 2\pi) = V(\tau).$$

Let $\boldsymbol{U}_{2\pi}$ denote one period of the steady flow below $\mathcal{S}_{2\pi}$. Then, in terms of the stream function ψ, which satisfies (1.1a,b,c), where c is given, the kinetic energy in one period is

$$K := \frac{1}{2} \int_{\boldsymbol{U}_{2\pi}} |\nabla(\psi(X, Y) - cY)|^2 \, dY \, dX,$$

the gravitational potential energy is

$$V_g := \frac{g}{2} \int_0^{2\pi} V(\tau)^2 \, U'(\tau) \, d\tau,$$

and, by (1.4), the elastic potential energy is

$$(2.1) \qquad V_e := \int_0^{2\pi} E\big(|\boldsymbol{r}'(x)|, \, |\boldsymbol{r}'(x)| \, \hat{\sigma}(\boldsymbol{r}(x))\big) \, dx.$$

Note that V_e does not depend on the number x_0 that appears in (1.1e), because $\boldsymbol{r}'(x)$ and $\hat{\sigma}(\boldsymbol{r}(x))$ are 2π-periodic functions. For this reason we fix $x_0 = 0$ in (1.1e).

Remark 8. K and V_g are determined by any parametrization of the surface, namely by the *shape* of $\mathcal{S}$ alone, and not by the displacement of the material points $x \mapsto \boldsymbol{r}(x)$ of the undeformed membrane $\mathcal{S}$. By contrast, V_e also depends on both the physical deformation $\boldsymbol{r}$ and on the shape of $\mathcal{S}$. $\qquad\square$

Thus, the Lagrangian of the travelling waves problem is

$$\mathcal{L} = K - V_g - V_e.$$

Remark 9. K involves the solution ψ of a Dirichlet problem (1.1a,b,c) on a domain which is itself the main unknown in the problem, and V_g and V_e are integrals on its unknown boundary $\mathcal{S}$. As a consequence, this is not a Lagrangian in the usual sense, since variations in the domain are involved when discussing critical points. In this context we mention a paper of Alt & Caffarelli [1] in which a class of variational free-boundary problems that includes the variational principle for $K - V_g$ is discussed. $\qquad\square$

2.1 The mathematical formulation

In [14], following the work of [6] on Stokes waves, the difficulty explained in Remark 9 was overcome by regarding one period of the flow domain as a conformal image of the unit disc, the wave surface being the image of the unit circle. Here we use the same technology, and we refer to [12, 14] for the details.

Let $L^p_{2\pi}$ denote the usual Lebesgue space of 2π-periodic functions on $\mathbb{R}$, which are p-power locally integrable, and $W^{k,p}_{2\pi}$ the Sobolev space of 2π-periodic functions whose kth weak derivative lies in $L^p_{2\pi}$, $p \in [1, +\infty]$, $k \in \mathbb{N}$. Let $[v]$ denote the mean on $[0, 2\pi]$ of $v \in L^1_{2\pi}$.

For any $v \in L^1_{2\pi}$, its conjugate function (or Hilbert transform) $\mathcal{C}v$ from harmonic analysis is defined almost everywhere by

$$\mathcal{C}v(\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{v(s)}{\tan \frac{1}{2}(\xi - s)} \, ds.$$

For $p \in (1, +\infty)$ and $k \in \mathbb{N}$, $\mathcal{C}$ is a bounded linear operator on $L^p_{2\pi}$ and $W^{k,p}_{2\pi}$ with $\mathcal{C}1 = 0$ and $\mathcal{C}(e^{int}) = -i \operatorname{sign}\{n\} e^{int}$, $n \neq 0$.

Now, we consider deformations $\boldsymbol{r}$ such that the shape of the deformed membrane, that is the curve $\{\boldsymbol{r}(x) : x \in \mathbb{R}\}$, is 2π-periodic in the horizontal direction. According to constraint (v) in Section 1.1, we assume that $\boldsymbol{r}$ deforms the material points x of any interval of length 2π into one period of the deformed membrane. Following [12, 14], introduce a special parametrization

$$\rho(w)(\tau) := (-\tau - \mathcal{C}w(\tau), \, w(\tau)), \quad \tau \in \mathbb{R},$$

for the curve $\{\boldsymbol{r}(x) : x \in \mathbb{R}\}$, where $w(\tau)$ is a 2π-periodic real function representing the elevation of the wave, and $\mathcal{C}w$ is its Hilbert transform.

Remark 10. It is shown in [14] that when a curve $\mathcal{S}(w)$ is defined in terms of w as

(2.2) $$\mathcal{S}(w) := \{\rho(w)(\tau) : \tau \in \mathbb{R}\},$$

its slope $\Theta(w)(\tau)$ and curvature $\sigma(w)(\tau)$ at $\rho(w)(\tau)$ are given by

$$\Theta(w)(\tau) := -\mathcal{C} \log \Omega(w)(\tau), \quad \sigma(w)(\tau) := \frac{\Theta(w)'(\tau)}{\Omega(w)(\tau)},$$

where
$$\Omega(w)(\tau) := \sqrt{(1 + \mathcal{C}w'(\tau))^2 + w'(\tau)^2}\,.$$

If $\mathcal{S}(w)$ is non-self intersecting, it is also shown that

$$K = \frac{c^2}{2} \int_0^{2\pi} w\mathcal{C}w' \, d\tau \quad \text{and} \quad V_g = \frac{g}{2} \int_0^{2\pi} w^2(1 + \mathcal{C}w') \, d\tau.$$

By [12], Thm. 2.7, any rectifiable 2π-periodic curve $\mathcal{S}$ in the plane can be represented as $\mathcal{S}(w)$ for some w with w' and $\mathcal{C}w'$ in $L^1_{2\pi}$. We do not make an a

13

priori assumption that w is such that the curve $\mathcal{S}(w)$ is the graph of a function. We will prove that this is so *a posteriori*, for maximizers of J_0 below. The non-self-intersection property of a curve $\mathcal{S}$, for given w, is a key aspect of this problem. $\qquad\square$

To find a formula for V_e, we consider diffeomorphisms $\chi(\tau)$ of the interval $[0, 2\pi]$ such that

$$\chi(0) = 0, \quad \chi(2\pi) = 2\pi, \quad \chi'(\tau) > 0 \ \text{ for a.e. } \tau,$$

and

(2.3) $$x = \chi(\tau) \quad \forall \tau \in \mathbb{R}.$$

In this way, when the surface $\mathcal{S}$ is defined by w, as described above, the position $\boldsymbol{r}(x)$ of the material point x after the deformation is

$$\boldsymbol{r}(x) = \rho(w)(\tau),$$

and the stretch of the membrane is

(2.4) $$\nu(x) = \frac{|\rho(w)'(\tau)|}{\chi'(\tau)}.$$

Note that the curvature $\hat{\sigma}(\boldsymbol{r}(x))$ at $\boldsymbol{r}(x)$ depends only on the shape, and not on any particular parametrization, of the curve. Then, since $\boldsymbol{r}(x) = \rho(w)(\tau)$,

(2.5) $$\hat{\sigma}(\boldsymbol{r}(x)) = \sigma(w)(\tau).$$

With the change of variable (2.3), the elastic energy (2.1) of the deformation $\boldsymbol{r}(x)$ has the form

$$\mathcal{E}(w, \chi) = \int_0^{2\pi} \chi'(\tau) \, E\Big(\frac{\Omega(w)(\tau)}{\chi'(\tau)}, \, \frac{\Omega(w)(\tau)}{\chi'(\tau)} \, \sigma(w)(\tau)\Big) \, d\tau.$$

Thus formally the hydroelastic wave problem is one of finding critical points for the Lagrangian functional

$$J(w, \chi) := I(w) - \mathcal{E}(w, \chi),$$

where

$$I(w) := \frac{c^2}{2} \int_0^{2\pi} w' \mathcal{C} w \, d\tau - \frac{g}{2} \int_0^{2\pi} w^2 (1 + \mathcal{C} w') \, d\tau,$$

w is a real 2π-periodic function belonging to the admissible set $\mathcal{A}_0$ below, and χ belongs to

(2.6) $$\mathcal{D} := \big\{\chi \in W^{1,1}(\mathbb{R}) : \ \chi' \in L^1_{2\pi}, \ \chi' \geq 0 \text{ a.e.}, \ \chi(0) = 0, \ \chi(2\pi) = 2\pi\big\}.$$

If we consider $w = a + \widetilde{w}$, with $a \in \mathbb{R}$ and $[\widetilde{w}] = 0$, we see immediately that

$$\max_{a \in \mathbb{R}} J(a + \widetilde{w}, \chi)$$

14

is attained at

$$(2.7) \qquad a = -\frac{1}{2\pi} \int_0^{2\pi} \widetilde{w} \mathcal{C} \widetilde{w}' \, d\tau,$$

and that this value of a is the one for which the area of the region delimited by the profile $\mathcal{S}(w)$ and the horizontal axis is 0. In other words, (2.7) corresponds to a law of conservation of the mass. Hence, maximizing J is equivalent to seeking a maximum of

$$J_0(w, \chi) := I_0(w) - \mathcal{E}(w, \chi)$$

(we have dropped the tilde over w), where

$$I_0(w) := I(w) + \frac{g}{4\pi} \left(\int_0^{2\pi} w' \mathcal{C} w \, d\tau \right)^2,$$

with the restriction that $[w] = 0$.

Remark on $I_0(w)$

For the existence of the integral in the definition of I_0, we need at least that $w \in \mathcal{H}_{\mathbb{R}}^{1,1}$, that is w is a 2π-periodic, real, absolutely continuous function with derivative $w' \in L_{2\pi}^1$ and $\mathcal{C}w' \in L_{2\pi}^1$ also. For such functions, $\Omega(w) \in L_{2\pi}^1$. $\qquad \square$

Remark on $\mathcal{E}(w, \chi)$

The integrand of the integral $\mathcal{E}(w, \chi)$ is defined when the curvature $\sigma(w)$ of the curve $\mathcal{S}(w)$ is defined, at least for almost every τ. The formula for the curvature is

$$\sigma(w) = \frac{\Theta(w)'}{\Omega(w)} = -\frac{1}{\Omega(w)} \mathcal{C}\left(\frac{\Omega(w)'}{\Omega(w)} \right)$$

where $\Theta(w) := -\mathcal{C} \log \Omega(w)$. The Hilbert transform $\mathcal{C}$ can be applied to the quotient Ω'/Ω provided it is integrable, that is, when $\log \Omega(w) \in W_{2\pi}^{1,1}$. Moreover, $\mathcal{C}(\Omega'/\Omega)$ is integrable when $\log \Omega(w) \in \mathcal{H}_{\mathbb{R}}^{1,1}$ (see above). In that case, $\log \Omega(w)$ is absolutely continuous and periodic, and hence there are two positive constants a, b such that

$$0 < a \leq \Omega(w) \leq b \quad \text{for a.e. } \tau.$$

As a consequence, $\sigma(w) \in L_{2\pi}^1$, since it is the product of a bounded and an integrable function.

In conclusion, the functional J_0 is well-defined for w in the set

$$\mathcal{A}_0 := \left\{ w \in \mathcal{H}_{\mathbb{R}}^{1,1} : [w] = 0, \ \log \Omega(w) \in \mathcal{H}_{\mathbb{R}}^{1,1}, \ [\log \Omega(w)] = 0 \right\},$$

which is a subset of the Hardy space $\mathcal{H}_{\mathbb{R}}^{1,1}$. The condition $[\log \Omega(w)] = 0$ is related to the complex formulation of the original water waves problem (see [14] and the references therein).

Note that $\mathcal{E}(w, \chi)$ may be infinite for some $(w, \chi) \in \mathcal{A}_0 \times \mathcal{D}$. $\qquad \square$

2.2 The theorem in the variational setting

The variational problem is the one of finding a maximizer of

$$(2.8) \qquad \max_{(w,\chi)\in\mathcal{A}_0\times\mathcal{D}} J_0(w,\chi),$$

with $J_0 : \mathcal{A}_0 \times \mathcal{D} \to \mathbb{R} \cup \{-\infty\}$, from which all else follows. In this variational setting, the complete result is the following.

Theorem 2. *(a) (Existence of a maximiser). If (H1-7) hold, there exists a nontrivial maximizer (w_0, χ_0) of problem (2.8). Moreover,*

$$w_0 \in \mathcal{A}_0 \cap W^{2,\rho}_{2\pi}, \quad \chi_0 \in \mathcal{D} \cap W^{1,s+1}(0,2\pi),$$

where and p and s are the exponents in (H5) and $\rho := \frac{p+ps}{p+s} > 1$.

(b) If (H1-8) hold, then $\chi'_0 \geq C > 0$ a.e., for some constant C.

(c) (Euler equation for the deformation variable χ_0). If (H1-10) hold, χ_0 is a solution of the Euler equation (4.5).

(d) (Euler equation for the wave elevation w_0). If (H1-11) hold, then w_0 is a solution of the Euler equation (4.12).

(e) (Regularity). If (H1-12) hold, and $E(\nu,\mu)$ is of class C^k, with $k \geq 2$, then

$$w_0 \in W^{k+1,\beta}_{2\pi}, \quad \chi_0 \in W^{k,\infty}(0,2\pi),$$

$$\frac{\Omega(w_0)}{\chi'_0} \in W^{k-1,\infty}_{2\pi}, \quad \sigma(w_0) \in W^{k-1,\infty}_{2\pi}.$$

(f) (Dynamic boundary condition). If (H1-12) hold, then (w_0, χ_0) satisfies the dynamic boundary equation (5.8).

We divide the proof of Theorem 2 into distinct parts, introducing Hypotheses only when needed.

3 Existence theory

In this section we prove part (a) of Theorem 2. Before anything else, we make some key technical observations.

3.1 Three technical observations

Observation 1. Among all the curves of length $2\pi\ell$, $\ell > 1$, which intersect the horizontal axis at 0 and 2π, the one that achieves the largest possible distance from the horizontal axis is an isosceles triangle. Therefore

$$(3.1) \qquad \|w\|_\infty \leq \pi\sqrt{\ell(w)^2 - 1}$$

where

$$\ell(w) = \frac{L(w)}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} \Omega(w)(\tau)\, d\tau.$$

Observation 2. This is based on Hurwitz's analytic version ([4], page 29) of the classical isoperimetric inequality: when $U, V : [a, b] \to \mathbb{R}$ are absolutely continuous with $U(a) = U(b)$ and $V(a) = V(b)$, then

$$\left| \int_a^b U'(x)V(x)dx \right| \le \pi R^2, \quad \text{where} \quad R := \frac{1}{2\pi} \int_a^b \sqrt{U'(x)^2 + V'(x)^2}\, dx.$$

In other words, $|\int_a^b U'(x)V(x)dx|$ is bounded above by the area of the circle of radius R, and equality holds if and only if $\{(U(x), V(x)) : x \in [a, b]\}$ is such a circle.

At this point we refer the reader to Figure 1.2. For $\ell > 1$, a circle of radius $r(\ell)$, where

$$r(\ell) \sin \vartheta(\ell) = \pi$$

and

$$(3.2) \qquad \vartheta(\ell) \in (0, \pi), \quad \ell = \frac{\vartheta(\ell)}{\sin \vartheta(\ell)},$$

is uniquely determined (up to congruence) by the requirement that the end-points of a chord of length 2π and the end-points of a circular arc of length $2\pi\ell$ coincide. Let $A(\ell)$ be the area enclosed between the circular arc of length $2\pi\ell$ and the chord of length 2π. Then it is easily seen that

$$(3.3) \qquad A(\ell) = \pi^2 \frac{2\vartheta(\ell) - \sin(2\vartheta(\ell))}{1 - \cos(2\vartheta(\ell))}$$

where $\vartheta(\ell)$ is defined in (3.2). For future convenience, we prove some properties of the function $A(\ell)$.

Lemma 1. $A(\ell)$ *is strictly increasing, concave on* $(1, \pi/2)$ *and convex on* $(\pi/2, +\infty)$, *and*

$$(3.4) \qquad A'(\ell) = \frac{2\pi^2}{\sin \vartheta(\ell)}.$$

Therefore $A'(\ell) > 2\pi^2$ *for all* $\ell \ne \pi/2$. *Moreover,*

$$(3.5) \qquad \lim_{\ell \to 1} \frac{A(\ell)}{\sqrt{\ell - 1}} \to 2\sqrt{2/3}\, \pi^2, \qquad \lim_{\ell \to +\infty} \frac{A(\ell)}{\ell^2} = \pi,$$

$A(\ell)/\sqrt{\ell - 1}$ *is an increasing function of* ℓ, *and*

$$(3.6) \qquad A(\ell) \le 2\pi\, \ell^2 \quad \text{for all } \ell > 1.$$

Proof. First of all, we note that the map $(1, +\infty) \ni \ell \mapsto \vartheta(\ell) \in (0, \pi)$ is strictly increasing. Indeed,

$$(3.7) \qquad \vartheta'(\ell) = \frac{\sin^2 \vartheta(\ell)}{\sin \vartheta(\ell) - \vartheta(\ell) \cos \vartheta(\ell)} > 0$$

17

because
$$\sin\vartheta - \vartheta\cos\vartheta > 0 \quad \forall\vartheta \in (0,\pi).$$

Now,
$$\frac{d}{d\vartheta}\left(\frac{2\vartheta - \sin(2\vartheta)}{1 - \cos(2\vartheta)}\right) = \frac{2(\sin\vartheta - \vartheta\cos\vartheta)}{\sin^3\vartheta}$$

so that (3.4) follows by (3.7). Hence $A'(\ell)$ is positive for all ℓ because $\vartheta(\ell) \in (0,\pi)$. More, since $\vartheta(\ell)$ is strictly increasing in ℓ and $\vartheta(\pi/2) = \pi/2$, formula (3.4) shows that $A'(\ell) > 2\pi^2$ for all $\ell \neq \pi/2$, and it is decreasing on $(1, \pi/2)$ and increasing on $(\pi/2, \infty)$.

Now we note that
$$\frac{d}{d\ell}\frac{A(\ell)}{\sqrt{\ell-1}} = \frac{2A'(\ell)(\ell-1) - A(\ell)}{2(\ell-1)^{3/2}}.$$

By (3.2), (3.3) and (3.4)
$$2A'(\ell)(\ell-1) - A(\ell) = \frac{\pi^2}{\sin^2\vartheta(\ell)}\left\{3\vartheta(\ell) - 4\sin\vartheta(\ell) + \sin\vartheta(\ell)\cos\vartheta(\ell)\right\}$$

and $3\vartheta - 4\sin\vartheta + \sin\vartheta\cos\vartheta > 0$ for all $\vartheta > 0$ because its value at $\vartheta = 0$ is zero and its derivative is
$$\frac{d}{d\vartheta}\left(3\vartheta - 4\sin\vartheta + \sin\vartheta\cos\vartheta\right) = 2(1 - \cos\vartheta)^2 \geq 0.$$

Thus, $A(\ell)/\sqrt{\ell-1}$ is an increasing function of ℓ.

The first limit in (3.5) can be proved by Taylor series, because $\vartheta(\ell) \to 0$ as $\ell \to 1$. By (3.3) and (3.2),
$$\frac{A(\ell)}{\ell^2} = \pi^2\frac{2\vartheta(\ell) - \sin(2\vartheta(\ell))}{2\vartheta(\ell)^2},$$

and the second limit in (3.5) follows because $\vartheta(\ell) \to \pi$ as $\ell \to \infty$.

To prove (3.6), we differentiate
$$\frac{d}{d\ell}\frac{A(\ell)}{\ell^2} = \frac{A'(\ell)\ell - 2A(\ell)}{\ell^3}.$$

By (3.2), (3.3) and (3.4)
$$A'(\ell)\ell - 2A(\ell) = \frac{2\pi^2\cos\vartheta(\ell)}{\sin\vartheta(\ell)}.$$

Then $A(\ell)/\ell^2$ has one global maximum at $\ell = \pi/2$, and (3.6) follows because $A(\pi/2) = \pi^3/2$. $\qquad\square$

Proposition 1. *Suppose that $\{(u(\tau), v(\tau)) : \tau \in [0, 2\pi]\}$ is a parametrization of a rectifiable curve of length $2\pi\ell$, $\ell > 1$, with $v(0) = v(2\pi)$ and $u(2\pi) - u(0) = 2\pi$. Then*
$$\int_0^{2\pi} u'(\tau)v(\tau)d\tau \leq A(\ell).$$

18

Proof. Suppose that this is false for (u, v) and define continuous functions U, V on the interval $[0, 3\pi]$ as follows. Let (U, V) coincide with (u, v) on $[0, 2\pi]$, let $(U(3\pi), V(3\pi)) = (u(0), v(0))$, and let $\{(U(x), V(x)) : x \in [2\pi, 3\pi]\}$ be an injective parametrization of the arc of the circle with radius $r(\ell)$ which is complementary to $c(\ell)$. Therefore, by the divergence theorem,

$$\int_{2\pi}^{3\pi} U'(x)V(x)\, dx = \pi r(\ell)^2 - A(\ell).$$

Since the proposition is supposed to be false, we find from the definition of (U, V) that

$$\int_0^{3\pi} U'(x)V(x)\, dx > \pi r(\ell)^2,$$

where, by construction,

$$\int_0^{3\pi} \sqrt{U'(x)^2 + V'(x)^2}\, dx = 2\pi r(\ell).$$

This contradicts the isoperimetric inequality and proves the result. $\qquad\square$

Observation 3. The third observation is the vector version of Jensen's inequality: if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, $U \subset \mathbb{R}^m$ has unit measure and $u : U \to \mathbb{R}^n$, then

$$(3.8) \qquad \int_U f(u(x))\, dx \geq f\left(\int_U u(x)\, dx \right).$$

This is immediate from the fact ([7], Ch. I.3) that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

$$f(x) = \sup \big\{ a \cdot x + b \ : \ a \in \mathbb{R}^n, \ b \in \mathbb{R}, \ a \cdot y + b \leq f(y) \ \forall y \in \mathbb{R}^n \big\}.$$

3.2   Estimates of $\Theta(w)$ when $J_0(w, \chi) \geq 0$

We seek a positive maximum of J_0 on $\mathcal{A}_0 \times \mathcal{D}$.

Lemma 2. *Suppose that* (H1,2,3,6) *hold. Then*

$$\Sigma := \sup_{(w, \chi) \in \mathcal{A}_0 \times \mathcal{D}} J_0(w, \chi) > 0.$$

Proof. Let $w_\varepsilon(\tau) := \varepsilon \cos \tau$ and $\chi'_\varepsilon(\tau) := 1 + \varepsilon \cos \tau$. Then $\mathcal{C}w'_\varepsilon = \varepsilon \cos \tau$, and calculations with Taylor series for $\varepsilon \to 0$ give

$$I_0(w_\varepsilon) = \frac{\pi}{2}\, (c^2 - g)\, \varepsilon^2 + O(\varepsilon^4),$$

$$\frac{\Omega(w_\varepsilon)}{\chi'_\varepsilon} = 1 + O(\varepsilon^2), \qquad \frac{\Omega(w_\varepsilon)\, \sigma(w_\varepsilon)}{\chi'_\varepsilon} = -\varepsilon \cos \tau + O(\varepsilon^2).$$

By (H1,2,3), the Taylor series of $E(\nu, \mu)$ near $(1, 0)$ is

$$E(\nu, \mu) = \frac{E_{11}(1,0)}{2} (\nu - 1)^2 + \frac{E_{22}(1,0)}{2} \mu^2 + o((\nu - 1)^2 + \mu^2).$$

Hence

$$\chi_\varepsilon' \, E\Big(\frac{\Omega(w_\varepsilon)}{\chi_\varepsilon'}, \, \frac{\Omega(w_\varepsilon) \, \sigma(w_\varepsilon)}{\chi_\varepsilon'}\Big) = \frac{E_{22}(1,0)}{2} \, \varepsilon^2 \cos^2 \tau + o(\varepsilon^2),$$

and

$$\mathcal{E}(w_\varepsilon, \chi_\varepsilon) = \frac{\pi}{2} \, E_{22}(1,0) \, \varepsilon^2 + o(\varepsilon^2).$$

Thus (H6) implies that $J_0(w_\varepsilon, \chi_\varepsilon) > 0$ for all $\varepsilon \neq 0$ sufficiently small. $\qquad \square$

Lemma 3. *Suppose that* (H1,2,4,7) *hold and* $\mu^* < 1$ *is that defined in* (H7). *Then, for all* $(w, \chi) \in \mathcal{A}_0 \times \mathcal{D}$ *such that* $J_0(w, \chi) > 0$,

$$|\Theta(w)(\tau_1) - \Theta(w)(\tau_2)| < \mu^* \pi \quad \forall \tau_1, \tau_2 \in \mathbb{R}.$$

Proof. First, we consider the elastic energy $\mathcal{E}(w, \chi)$, which is

$$\mathcal{E}(w, \chi) = \int_0^{2\pi} \chi' \, E\Big(\frac{\Omega(w)}{\chi'}, \, \frac{\Omega(w) \, |\sigma(w)|}{\chi'}\Big) \, d\tau$$

by (H2). From (H4), and Jensen's inequality (3.8) it follows that

(3.9) $$\mathcal{E}(w, \chi) \geq 2\pi \, E\big(\ell(w), m(w)\big),$$

where

$$\ell(w) := \frac{1}{2\pi} \int_0^{2\pi} \Omega(w)(\tau) \, d\tau, \quad m(w) := \frac{1}{2\pi} \int_0^{2\pi} |\Theta(w)'(\tau)| \, d\tau.$$

Now we consider

$$I_0(w) = \frac{g}{4\pi} \Big(\int_0^{2\pi} w \mathcal{C} w' d\tau \Big)^2 + \frac{c^2}{2} \int_0^{2\pi} w \mathcal{C} w' d\tau - \frac{g}{2} \int_0^{2\pi} w^2 (1 + \mathcal{C} w') \, d\tau.$$

Since w is 2π-periodic and has zero mean, by Proposition 1,

$$0 \leq \int_0^{2\pi} w \mathcal{C} w' d\tau = \int_0^{2\pi} w(1 + \mathcal{C} w') d\tau \leq A(\ell(w)),$$

because the length of a period of $\mathcal{S}(w)$ is $2\pi \ell(w)$. Integrating by parts shows that

$$\int_0^{2\pi} w^2 \mathcal{C} w' d\tau = \int_0^{2\pi} w(w \mathcal{C} w' - \mathcal{C}(ww')) \, d\tau + \int_0^{2\pi} w \mathcal{C}(ww') \, d\tau$$

$$= \int_0^{2\pi} w(w \mathcal{C} w' - \mathcal{C}(ww')) \, d\tau + \frac{1}{2} \int_0^{2\pi} w^2 \mathcal{C} w' d\tau.$$

20

Since $w\mathcal{C}w' - \mathcal{C}(ww') \geq 0$ almost everywhere (see [13], Prop. 3.1),

$$\left| \int_0^{2\pi} w^2 \mathcal{C}w' d\tau \right| \leq 2\|w\|_\infty \int_0^{2\pi} \left(w\mathcal{C}w' - \mathcal{C}(ww') \right) d\tau = 2\|w\|_\infty \int_0^{2\pi} w\mathcal{C}w' d\tau,$$

because $\mathcal{C}(ww')$ has zero mean. From these inequalities and (3.1),

$$(3.10) \qquad I_0(w) \leq \frac{g}{4\pi} A(\ell(w))^2 + \frac{c^2}{2} A(\ell(w)) + g\pi A(\ell(w))\sqrt{\ell(w)^2 - 1}\,.$$

Let $J_0(w, \chi) > 0$. Then $\mathcal{E}(w, \chi) < I_0(w)$, whence, by (3.9) and (3.10),

$$E(\ell, m) < \frac{g}{8\pi^2} A(\ell)^2 + \frac{c^2}{4\pi} A(\ell) + \frac{g}{2} A(\ell)\sqrt{\ell^2 - 1},$$

where $\ell = \ell(w)$ and $m = m(w)$. Now, suppose, for contradiction, that $m \geq \mu^*$. Since, by (H2) and (H4), $\mu \mapsto E(\nu, \mu)$ is increasing on $[0, +\infty)$ for every fixed ν,

$$E(\ell, \mu^*) \leq E(\ell, m) < \frac{g}{8\pi^2} A(\ell)^2 + \frac{c^2}{4\pi} A(\ell) + \frac{g}{2} A(\ell)\sqrt{\ell^2 - 1},$$

violating (H7). Thus, $m < \mu^*$. Therefore, for every $\tau_1 < \tau_2 < \tau_1 + 2\pi$,

$$|\Theta(w)(\tau_1) - \Theta(w)(\tau_2)| \leq \frac{1}{2} \left| \int_{\tau_1}^{\tau_2} \Theta(w)'(\tau)\, d\tau - \int_{\tau_2}^{\tau_1 + 2\pi} \Theta(w)'(\tau)\, d\tau \right|$$

$$\leq \frac{1}{2} \int_0^{2\pi} |\Theta(w)'(\tau)|\, d\tau = \pi m < \mu^* \pi < \pi. \qquad \square$$

3.3 Maximising sequences

Assume (H6), so that $\Sigma > 0$ by Lemma 2, and let $\{(w_k, \chi_k)\} \subset \mathcal{A}_0 \times \mathcal{D}$ be a maximising sequence, with

$$0 < J_0(w_k, \chi_k) \to \Sigma \quad (k \to \infty).$$

For convenience, let $\Omega_k := \Omega(w_k)$, $\sigma_k := \sigma(w_k)$, $\Theta_k := \Theta(w_k)$. By Lemma 3 we have that

$$|\Theta_k(\tau_1) - \Theta_k(\tau_2)| < \mu^* \pi \quad \forall \tau_1, \tau_2 \in \mathbb{R},$$

and the bound is uniform for $k \in \mathbb{N}$. Let

$$M_k := \frac{1}{2} \left(\max_{\tau \in [0, 2\pi]} \Theta_k(\tau) + \min_{\tau \in [0, 2\pi]} \Theta_k(\tau) \right)$$

and set

$$\widetilde{\Theta}_k(\tau) := \Theta_k(\tau) - M_k.$$

Then $\|\widetilde{\Theta}_k\|_\infty < \mu^* \pi / 2$. Moreover, since $\Theta_k = -\mathcal{C} \log \Omega_k$ and $[\log \Omega_k] = 0$,

$$\mathcal{C}\widetilde{\Theta}_k = \mathcal{C}\Theta_k = \log \Omega_k.$$

21

We recall Zygmund's Theorem [17], Vol. I, page 257: If $f \in L_{2\pi}^{\infty}$, then

$$\int_0^{2\pi} \exp\{q|\mathcal{C}f(\tau)|\}\, d\tau \leq \frac{4\pi}{\cos(q\|f\|_\infty)} \quad \text{for all } 0 \leq q < \frac{\pi}{2\|f\|_\infty}\,.$$

We apply this result to $f = \widetilde{\Theta}_k$ and we get

$$\int_0^{2\pi} \Omega_k^q\, d\tau \leq \int_0^{2\pi} e^{q|\log \Omega_k|}\, d\tau < \frac{4\pi}{\cos(q\mu^*\pi/2)} \quad \text{for all } 0 \leq q < \frac{1}{\mu^*}\,.$$

Since $\mu^* < 1$, in what follows we can fix an exponent $q \in (1, 1/\mu^*)$ and obtain a uniform bound

$$(3.11) \qquad \qquad \|\Omega_k\|_{L_{2\pi}^q} \leq C \quad \forall k$$

for the maximising sequence.

By (3.11) and Hölder's inequality we get a uniform bound for Ω_k in $L_{2\pi}^1$,

$$L_k := L(w_k) = \int_0^{2\pi} \Omega_k\, d\tau \leq C \quad \forall k,$$

where here, and more generally in this section, C denotes positive, possibly different constants. By (3.10) $I_0(w)$ is dominated by a function of $L(w)$. Then, since $J_0(w_k, \chi_k) > 0$, we have a uniform bound for the elastic energy

$$(3.12) \qquad \qquad \mathcal{E}(w_k, \chi_k) < I_0(w_k) \leq C \quad \forall k.$$

Since $[\chi_k'] = 1$ for all k, from (H5) it follows that

$$(3.13) \qquad \int_0^{2\pi} \left\{ \left(\frac{\Omega_k}{\chi_k'}\right)^r + \left(\frac{\chi_k'}{\Omega_k}\right)^s + \left(\frac{\Omega_k|\sigma_k|}{\chi_k'}\right)^p \right\} \chi_k'\, d\tau \leq C \quad \forall k.$$

Now, let

$$\alpha := 1 + \frac{s}{q}, \quad \beta := \frac{s}{\alpha}, \quad \gamma := \frac{s+1}{\alpha}\,.$$

By Hölder's inequality,

$$\int_0^{2\pi} (\chi_k')^\gamma\, d\tau \leq \left(\int_0^{2\pi} \left(\frac{(\chi_k')^\gamma}{\Omega_k^\beta}\right)^\alpha d\tau\right)^{\frac{1}{\alpha}} \left(\int_0^{2\pi} (\Omega_k^\beta)^{\alpha'}\, d\tau\right)^{\frac{1}{\alpha'}}.$$

Then, by (3.13) and (3.11),

$$(3.14) \qquad \qquad \|\chi_k'\|_{L_{2\pi}^\gamma} \leq C \quad \forall k,$$

because $\gamma\alpha = s + 1$, $\beta\alpha = s$ and $\alpha'\beta = q$. Note that $\gamma > 1$ because $q > 1$.

Now let

$$\lambda := 1 + \frac{p-1}{\gamma}, \quad \eta := \frac{p}{\lambda}, \quad \xi := \frac{p-1}{\lambda}\,.$$

22

By Hölder's inequality,

$$\int_0^{2\pi} |\Theta_k'|^\eta \, d\tau \le \Big(\int_0^{2\pi} \Big(\frac{|\Theta_k'|^\eta}{(\chi_k')^\xi} \Big)^\lambda \, d\tau \Big)^{\frac{1}{\lambda}} \Big(\int_0^{2\pi} (\chi_k')^{\xi\lambda'} \, d\tau \Big)^{\frac{1}{\lambda'}}.$$

Then, by (3.13) and (3.14), and recalling that $\Omega_k \, \sigma_k = \Theta_k'$,

$$\|\Theta_k'\|_{L_{2\pi}^\eta} \le C \quad \forall k,$$

because $\eta\lambda = p$, $\xi\lambda = p - 1$ and $\xi\lambda' = \gamma$. Note that $\eta > 1$ because $\gamma > 1$. Since $[\Theta_k] = [-\mathcal{C} \log \Omega_k] = 0$,

$$\|\Theta_k\|_{W_{2\pi}^{1,\eta}} \le C \quad \forall k.$$

It follows that

$$\|\mathcal{C}\Theta_k\|_{W_{2\pi}^{1,\eta}} = \|\log \Omega_k\|_{W_{2\pi}^{1,\eta}} \le C \quad \forall k$$

and then $\log \Omega_k$ are absolutely continuous functions with a uniform bound $\|\log \Omega_k\|_\infty \le C$. This means that there are two constants C, C' such that

(3.15) $$0 < C \le \Omega_k(\tau) \le C' \quad \forall \tau, \ \forall k.$$

Thanks to (3.15), the bound (3.13) becomes

(3.16) $$\int_0^{2\pi} \Big(\frac{1}{(\chi_k')^{r-1}} + (\chi_k')^{s+1} + \frac{|\sigma_k|^p}{(\chi_k')^{p-1}} \Big) \, d\tau \le C \quad \forall k.$$

Thus (3.14) can be improved to

$$\|\chi_k'\|_{L_{2\pi}^{s+1}} \le C \quad \forall k.$$

Then, since $\chi_k(0) = 0$ for all k,

$$\|\chi_k\|_{W^{1,s+1}(0,2\pi)} \le C \quad \forall k.^2$$

Also, let

$$a := 1 + \frac{p-1}{s+1}, \quad b := \frac{p-1}{a}, \quad \rho := \frac{p}{a} = \frac{p(s+1)}{s+p}.$$

By Hölder's inequality,

$$\int_0^{2\pi} |\sigma_k|^\rho \, d\tau \le \Big(\int_0^{2\pi} \Big(\frac{|\sigma_k|^\rho}{(\chi_k')^b} \Big)^a \, d\tau \Big)^{\frac{1}{a}} \Big(\int_0^{2\pi} (\chi_k')^{ba'} \, d\tau \Big)^{\frac{1}{a'}}.$$

Then, by (3.16),

$$\|\sigma_k\|_{L_{2\pi}^\rho} \le C \quad \forall k,$$

because $\rho a = p$, $ba = p-1$ and $ba' = s+1$. Note that $\rho > \eta > 1$ by construction. By the last inequality and (3.15),

$$\|\Theta_k'\|_{L_{2\pi}^\rho} \le C \quad \forall k$$

---

[2] We write $W^{1,s+1}(0, 2\pi)$, and not $W_{2\pi}^{1,s+1}$, because the diffeomorphism χ_k, unlike its derivative χ_k', is not a periodic function; see (2.6).

and then $\|\mathcal{C}\Theta_k'\|_{L^\rho} \leq C$,

$$\|\Omega_k'\|_{L^\rho_{2\pi}} \leq C \quad \forall k.$$

Since $w_k' = \Omega_k \sin \Theta_k$ and $\|\Omega_k\|_\infty \leq C$, by the previous bound we get

$$(3.17) \qquad \|w_k''\|_{L^\rho_{2\pi}} \leq C \quad \forall k, \quad \rho = \frac{p + sp}{p + s},$$

and so w_k is a bounded sequence in $W^{2,\rho}_{2\pi}$, because $[w_k] = 0$ for all k.

3.4 Weak limit (w_0, χ_0) of the maximising sequence

By (3.17), and since $[w_k] = 0$, there exists a subsequence, which we denote (w_k) as well, and a function $w_0 \in W^{2,\rho}_{2\pi}$ such that $w_k \rightharpoonup w_0$ in $W^{2,\rho}_{2\pi}$ weakly; more precisely:

$$\begin{aligned}
w_k \to w_0, \quad &\mathcal{C}w_k \to \mathcal{C}w_0 \quad \text{in } L^\infty_{2\pi} \text{ strongly,} \\
w_k' \to w_0', \quad &\mathcal{C}w_k' \to \mathcal{C}w_0' \quad \text{in } L^\infty_{2\pi} \text{ strongly,} \\
w_k'' \rightharpoonup w_0'', \quad &\mathcal{C}w_k'' \rightharpoonup \mathcal{C}w_0'' \quad \text{in } L^\rho_{2\pi} \text{ weakly.}
\end{aligned}$$

In particular, $\Omega_k \to \Omega_0 := \Omega(w_0)$ uniformly and, as a consequence of (3.15), Ω_0 is bounded below, so that

$$(3.18) \qquad \Omega_k \to \Omega_0 \quad \text{and} \quad \log \Omega_k \to \log \Omega_0 \quad \text{in } L^\infty_{2\pi}.$$

It follows that $[\log \Omega_0] = [w_0] = 0$ because $[\log \Omega_k] = [w_k] = 0$ for all k. Thus $w_0 \in \mathcal{A}_0$. Since $\mathcal{C}w_k' \to \mathcal{C}w_0'$ and $w_k' \to w_0'$ in $L^\infty_{2\pi}$, it follows that

$$\Omega_k' \rightharpoonup \Omega_0' \quad \text{and} \quad \frac{\Omega_k'}{\Omega_k} \rightharpoonup \frac{\Omega_0'}{\Omega_0} \quad \text{in } L^\rho_{2\pi} \text{ weakly}$$

and so

$$\Theta_k' = -\mathcal{C}\Big(\frac{\Omega_k'}{\Omega_k}\Big) \rightharpoonup -\mathcal{C}\Big(\frac{\Omega_0'}{\Omega_0}\Big) = \Theta_0' \quad \text{in } L^\rho_{2\pi} \text{ weakly}.$$

Thus $\Theta_k'/\Omega_k \rightharpoonup \Theta_0'/\Omega_0$ weakly, that is

$$(3.19) \qquad \sigma_k \rightharpoonup \sigma_0 \quad \text{in } L^\rho_{2\pi} \text{ weakly}$$

where $\sigma_0 := \sigma(w_0)$.

Moreover, a subsequence (χ_k) converges to some $\chi_0 \in W^{1,s+1}(0, 2\pi) \cap \mathcal{D}$,

$$(3.20) \qquad \begin{aligned}
\chi_k \to \chi_0 \quad &\text{in } L^\infty(\mathbb{R}) \text{ strongly,} \\
\chi_k' \rightharpoonup \chi_0' \quad &\text{in } L^{s+1}_{2\pi} \text{ weakly.}
\end{aligned}$$

Obviously

$$\int_0^{2\pi} w_k \mathcal{C}w_k' \, d\tau \to \int_0^{2\pi} w_0 \mathcal{C}w_0' \, d\tau$$

and

$$\int_0^{2\pi} w_k^2 (1 + \mathcal{C}w_k') \, d\tau \to \int_0^{2\pi} w_0^2 (1 + \mathcal{C}w_0') \, d\tau,$$

so that $I_0(w_k) \to I_0(w_0)$.

3.5 The existence of a maximum

To prove that (w_0, χ_0) is a maximiser for J_0, it is more convenient to write the elastic energy as

$$\mathcal{E}(w, \chi) = \int_0^{2\pi} \Omega(w) \, E^\star \Big(\frac{\chi'}{\Omega(w)}, \sigma(w) \Big) \, d\tau,$$

where

$$E^\star(t, \sigma) := t E \Big(\frac{1}{t}, \frac{\sigma}{t} \Big), \quad \forall t > 0, \ \sigma \in \mathbb{R}.$$

Note that $(E^\star)^\star = E$, that is

$$E(\nu, \mu) = \nu E^\star \Big(\frac{1}{\nu}, \frac{\mu}{\nu} \Big), \quad \forall \nu > 0, \ \mu \in \mathbb{R}.$$

$E^\star$ is jointly convex in both its argument by (H4) and the following lemma. Note that $E^\star$ coincides with $\widetilde{\mathbf{e}}$ in [14], Remark 3.1.

Lemma 4. $E^\star(t, \sigma)$ is jointly convex in (t, σ) if and only if $E(\nu, \mu)$ is jointly convex in (ν, μ).

Proof. Differentiating gives

$$E_{22}(\nu, \mu) = \frac{1}{\nu} E_{22}^\star \Big(\frac{1}{\nu}, \frac{\nu}{\mu} \Big)$$

and

$$E_{11}(\nu, \mu) \, E_{22}(\nu, \mu) - E_{12}(\nu, \mu)^2 = \frac{1}{\nu^4} \Big\{ E_{11}^\star \Big(\frac{1}{\nu}, \frac{\nu}{\mu} \Big) E_{22}^\star \Big(\frac{1}{\nu}, \frac{\nu}{\mu} \Big) - E_{12}^\star \Big(\frac{1}{\nu}, \frac{\nu}{\mu} \Big)^2 \Big\}.$$

Then the jointly convexity of E, that is (H4), implies that $E_{11}^\star$, $E_{22}^\star$ and $E_{11}^\star E_{22}^\star - (E_{12}^\star)^2$ are positive at every (t, σ) with $t > 0$ and $\sigma \in \mathbb{R}$.

The opposite is true because $E = (E^\star)^\star$. $\qquad \square$

Now, by (3.20), $\chi_k' \rightharpoonup \chi_0'$ in $L_{2\pi}^\rho$ weakly because $\rho < s + 1$. Therefore, since $1/\Omega_k \to 1/\Omega_0$ uniformly,

$$\frac{\chi_k'}{\Omega_k} \rightharpoonup \frac{\chi_0'}{\Omega_0} \quad \text{in } L_{2\pi}^\rho \text{ weakly},$$

and so, by (3.19), the pairs $(\chi_k'/\Omega_k, \sigma_k)$ converge to $(\chi_0'/\Omega_0, \sigma_0)$ weakly in the product space $L_{2\pi}^\rho \times L_{2\pi}^\rho$. We define

$$F : L_{2\pi}^\rho \times L_{2\pi}^\rho \to \mathbb{R}, \quad F(u, v) := \int_0^{2\pi} \Omega_0 \, E^\star(u, v) \, d\tau.$$

Since $E^\star$ is continuous and non-negative (recall that $E \geq 0$ by (H3)), by Fatou's Lemma F is strongly lower semicontinuous on $L_{2\pi}^\rho \times L_{2\pi}^\rho$. Moreover, by (H4)

25

and Lemma 4, F is also convex. Hence F is weakly lower semicontinuous on $L_{2\pi}^{\rho} \times L_{2\pi}^{\rho}$. It follows that

$$\mathcal{E}(w_0, \chi_0) = F\Big(\frac{\chi_0'}{\Omega_0}, \sigma_0\Big) \leq \liminf_k F\Big(\frac{\chi_k'}{\Omega_k}, \sigma_k\Big) = \liminf_k \int_0^{2\pi} \Omega_0 \, E^\star\Big(\frac{\chi_k'}{\Omega_k}, \sigma_k\Big) d\tau.$$

We note that

$$\Big|F\Big(\frac{\chi_k'}{\Omega_k}, \sigma_k\Big) - \mathcal{E}(w_k, \chi_k)\Big| = \Big|\int_0^{2\pi} \frac{(\Omega_0 - \Omega_k)}{\Omega_k} \Omega_k \, E^\star\Big(\frac{\chi_k'}{\Omega_k}, \sigma_k\Big) d\tau\Big|$$
$$\leq C \, \|\Omega_0 - \Omega_k\|_{L^\infty} \to 0$$

as $k \to \infty$ by (3.15), (3.12) and (3.18). Hence

$$\liminf_k F\Big(\frac{\chi_k'}{\Omega_k}, \sigma_k\Big) = \liminf_k \mathcal{E}(w_k, \chi_k).$$

On the other hand, by the definition of the maximising sequence (w_k, χ_k),

$$I_0(w_k) - \mathcal{E}(w_k, \chi_k) = J_0(w_k, \chi_k) \to \Sigma,$$

and, since $I_0(w_k) \to I_0(w_0)$,

$$\mathcal{E}(w_k, \chi_k) \to I_0(w_0) - \Sigma.$$

Then

(3.21) $$\mathcal{E}(w_0, \chi_0) \leq \liminf_k \mathcal{E}(w_k, \chi_k) = I_0(w_0) - \Sigma,$$

that is, $J_0(w_0, \chi_0) \geq \Sigma$. Therefore

$$J_0(w_0, \chi_0) = \Sigma$$

and (w_0, χ_0) is a maximum for J_0 on $\mathcal{A}_0 \times \mathcal{D}$.

We have proved that, when (H1-7) hold, there exists a maximizer (w_0, χ_0) for problem (2.8). Since $J_0(0, 1) = 0$ and $J_0(w_0, \chi_0) = \Sigma > 0$, the maximum is nontrivial. Moreover, we have also proved that

$$w_0 \in \mathcal{A}_0 \cap W_{2\pi}^{2,\rho}, \quad \chi_0 \in \mathcal{D} \cap W^{1,s+1}(0, 2\pi),$$

and the proof of part (a) of Theorem 2 is concluded.

4   Euler equation and regularity of the solution

We next prove parts (b), (c) and (d) of Theorem 2. First, to see that χ_0' is bounded below, we note the following facts about $E^\star$.

Lemma 5. *Suppose that* (H2,4,5) *hold. Then, for all* $\sigma \in \mathbb{R}$,

$$\lim_{t \to 0^+} E_1^\star(t, \sigma) = -\infty, \quad \lim_{t \to +\infty} E_1^\star(t, \sigma) = +\infty.$$

Proof. By (H2,4) and Lemma 4, $E^\star(t,\sigma)$ is even in σ and jointly convex, therefore

$$(4.1) \qquad E^\star(t,\sigma) \geq E^\star(t,0) \quad \forall t > 0, \ \sigma \in \mathbb{R}.$$

Since $E^\star(t,0) = tE(1/t,0)$, by (H5) we have that

$$\lim_{t \to 0^+} E^\star(t,0) = +\infty, \quad \lim_{t \to +\infty} \frac{E^\star(t,0)}{t} = +\infty.$$

Then, by (4.1),

$$\lim_{t \to 0^+} E^\star(t,\sigma) = +\infty, \quad \lim_{t \to +\infty} \frac{E^\star(t,\sigma)}{t} = +\infty$$

for all $\sigma \in \mathbb{R}$. The lemma easily follows by the convexity of the map $t \mapsto E^\star(t,\sigma)$, for every fixed σ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We have proved that $E_1^\star(t,\sigma) \to -\infty$ as $t \to 0^+$. (H8) is equivalent to assuming that such a limit is uniform in σ. Indeed, we observe that, for every ν,

$$\inf_{\mu \in \mathbb{R}} \big(\nabla E(\nu,\mu) \cdot (\nu,\mu) - E(\nu,\mu) \big) = \inf_{\sigma \in \mathbb{R}} \big(\nabla E(\nu,\sigma\nu) \cdot (\nu,\sigma\nu) - E(\nu,\sigma\nu) \big).$$

Hence (H8) holds if and only if

$$\lim_{t \to 0^+} \{ \sup_{\sigma \in \mathbb{R}} E_1^\star(t,\sigma) \} = -\infty,$$

that is

$$(4.2) \qquad E_1^\star(t,\sigma) \leq f(t) \quad \forall t > 0, \ \sigma \in \mathbb{R}, \quad \text{and} \quad \lim_{t \to 0^+} f(t) = -\infty,$$

where $f(t) := \sup_\sigma E_1^\star(t,\sigma)$.

As a consequence of (4.2), there exists a positive constant α^* such that $f(t) < 0$ for all $t \in (0,\alpha^*)$, therefore

$$(4.3) \qquad E_1^\star(t,\sigma) < 0 \quad \forall t \in (0,\alpha^*), \ \sigma \in \mathbb{R}.$$

Note that the map $t \mapsto E_1^\star(t,\sigma)$ is strictly increasing. Then, by Lemma 5, for every σ there exists a unique $t^\star(\sigma) > 0$ such that $E_1^\star(t^\star(\sigma),\sigma) = 0$, and $t^\star(0) = 1$. By (4.3), $t^\star$ is bounded below, namely $t^\star(\sigma) \geq \alpha^*$ for all σ.

Lemma 6. *Let (w_0, χ_0) be the maximum for J_0 described in part (a) of Theorem 2. Suppose that (H1,4,8) hold. Then there exists a constant C such that*

$$\chi_0'(\tau) \geq C > 0 \quad \text{for a.e. } \tau.$$

27

Proof. Suppose that $1/\chi_0'$ does not belong to L^∞. Then all the sets

$$A_n := \left\{ \tau \in (0, 2\pi) : \chi_0'(\tau) \le \frac{1}{n} \right\}, \quad n \in \mathbb{N},$$

have positive Lebesgue measure denoted by $|A_n| > 0$. Note also that

$$\{ \tau \in (0, 2\pi) : \chi_0'(\tau) \ge 1 \}$$

has positive measure — if not, then $\chi_0' < 1$ almost everywhere, whence

$$\chi_0(2\pi) - \chi_0(0) = \int_0^{2\pi} \chi_0'(\tau) \, d\tau < 2\pi,$$

violating the fact that $\chi_0 \in \mathcal{D}$. Since σ_0 and χ_0' are integrable we can therefore choose N large enough that

$$B := \{ 1 \le \chi_0' < N, \ |\sigma_0| < N \}$$

has positive measure. Then, for every n we define φ_n by

$$\varphi_n(\tau) := \begin{cases} 2/n & \text{if } \tau \in A_n, \\ \chi_0'(\tau) - \lambda_n & \text{if } \tau \in B, \\ \chi_0'(\tau) & \text{everywhere else}, \end{cases}$$

where

$$\lambda_n := \frac{1}{|B|} \int_{A_n} \left(\frac{2}{n} - \chi_0' \right) d\tau,$$

so that

$$\int_0^{2\pi} \varphi_n(\tau) \, d\tau = 2\pi$$

for all n. Also, we note that

(4.4) $$\frac{1}{|B|} \frac{|A_n|}{n} \le \lambda_n \le \frac{2}{|B|} \frac{|A_n|}{n}$$

for all n. We define

$$\widetilde{\chi}_n(\tau) := \int_0^\tau \varphi_n(\tilde{\tau}) \, d\tilde{\tau}$$

and observe that $\widetilde{\chi}_n \in \mathcal{D}$ for all n sufficiently large. We calculate the difference

$$\mathcal{E}(w_0, \chi_0) - \mathcal{E}(w_0, \widetilde{\chi}_n) = \int_0^{2\pi} \Omega_0 \left\{ E^\star \left(\frac{\chi_0'}{\Omega_0}, \sigma_0 \right) - E^\star \left(\frac{\varphi_n}{\Omega_0}, \sigma_0 \right) \right\} d\tau = a_n + b_n,$$

where

$$a_n := \int_{A_n} \Omega_0 \left\{ E^\star \left(\frac{\chi_0'}{\Omega_0}, \sigma_0 \right) - E^\star \left(\frac{2}{\Omega_0 \, n}, \sigma_0 \right) \right\} d\tau$$

28

and
$$b_n := \int_B \Omega_0 \left\{ E^\star\!\left(\frac{\chi_0'}{\Omega_0}, \sigma_0\right) - E^\star\!\left(\frac{\chi_0' - \lambda_n}{\Omega_0}, \sigma_0\right) \right\} d\tau.$$

Since $\Omega_0(\tau) \geq C > 0$ for all τ,

$$\frac{\chi_0'(\tau)}{\Omega_0(\tau)} \leq \frac{1}{\Omega_0(\tau)\, n} < \frac{2}{\Omega_0(\tau)\, n} \leq \frac{C'}{n} < \alpha^* \quad \forall \tau \in A_n,$$

for all n sufficiently large, where α^* is defined in (4.3), using (H8). Then, by (4.3) and the fact that $E_{11}^\star > 0$,

$$E^\star\!\left(\frac{\chi_0'}{\Omega_0}, \sigma_0\right) - E^\star\!\left(\frac{2}{\Omega_0\, n}, \sigma_0\right) > -E_1^\star\!\left(\frac{C'}{n}, \sigma_0\right)\left(\frac{2}{\Omega_0\, n} - \frac{\chi_0'}{\Omega_0}\right)$$

$$\geq -f\!\left(\frac{C'}{n}\right)\frac{1}{\Omega_0\, n} \quad \forall \tau \in A_n,$$

where f is defined in (4.2). Hence

$$a_n > -f\!\left(\frac{C'}{n}\right)\frac{|A_n|}{n}.$$

To estimate b_n, we observe that χ_0'/Ω_0 and $(\chi_0' - \lambda_n)/\Omega_0$ are confined in a compact interval K which does not contain zero, for all $\tau \in B$, for all n sufficiently large. Hence we define

$$M := \max_{t \in K,\; |\sigma| \leq N} |E_1^\star(t, \sigma)|,$$

and so

$$|b_n| \leq \int_B M \lambda_n \, d\tau = |B|\, M \lambda_n.$$

Then, by (4.4),

$$\frac{a_n}{|b_n|} > -\frac{f(C'/n)}{2\,M} \to +\infty$$

as $n \to \infty$ by (4.2). This implies that

$$a_n + b_n > 0$$

for n sufficiently large, so that $\mathcal{E}(w_0, \widetilde{\chi}_n) < \mathcal{E}(w_0, \chi_0)$, violating the maximality of (w_0, χ_0) for J_0. $\qquad\square$

4.1 The Euler equation for χ_0

Now we prove part (c) of Theorem 2. By (3.21), $\mathcal{E}(w_0, \chi_0) < \infty$, therefore, by (H5) and the fact that $0 < C \leq \Omega_0 \leq C'$, we know that

$$\int_0^{2\pi} \frac{|\sigma_0|^p}{(\chi_0')^{p-1}} \, d\tau < \infty.$$

Since χ_0' is bounded below (Lemma 6),

$$\int_0^{2\pi} \Big(\frac{|\sigma_0|}{\chi_0'}\Big)^p \, d\tau \le \Big\|\frac{1}{\chi_0'}\Big\|_\infty \int_0^{2\pi} \frac{|\sigma_0|^p}{(\chi_0')^{p-1}} \, d\tau < \infty,$$

that is σ_0/χ_0' belongs to $L_{2\pi}^p$. Moreover, recall that

$$\chi_0' \in L_{2\pi}^{s+1}, \quad E\Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \sigma_0}{\chi_0'}\Big) \in L_{2\pi}^1.$$

To study the differentiability of the functional $J_0(w, \chi)$ with respect to χ, we assume (H9,10). (H9) implies that

$$E_{1,0} := E_1\Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) \in L_{2\pi}^1,$$

and (H10) implies that

$$E_{2,0} := E_2\Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) \in L_{2\pi}^{p'},$$

where $1/p + 1/p' = 1$, because $\Omega_0 \, \sigma_0/\chi_0' \in L_{2\pi}^p$, $\chi_0'/\Omega_0 \in L_{2\pi}^{s+1}$ and $\Omega_0/\chi_0' \in L_{2\pi}^\infty$. As a consequence, the functional $\mathcal{E}(w, \chi)$ is Gateaux-differentiable with respect to χ, and its partial derivative in any direction $\psi \in W_{2\pi}^{1,\infty}$, with $\chi_0 + \psi \in \mathcal{D}$, at (w_0, χ_0) is

$$d_\chi \mathcal{E}(w_0, \chi_0) \, \psi$$
$$= \int_0^{2\pi} \Big\{ E\Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) - \nabla E\Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) \cdot \Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) \Big\} \psi' \, d\tau,$$

where

$$\nabla E(\nu, \mu) \cdot (\nu, \mu) = \nu \, E_1(\nu, \mu) + \mu \, E_2(\nu, \mu).$$

Then the maximiser (w_0, χ_0) satisfies the Euler-Lagrange equation for the functional $J_0(w, \chi)$ with respect to χ, that is

$$(4.5) \qquad E\Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) - \nabla E\Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) \cdot \Big(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \, \sigma_0}{\chi_0'}\Big) \equiv \gamma_0,$$

for some constant $\gamma_0 \in \mathbb{R}$. Note that (4.5) is equation (1.2) of [14].

We have proved that, if (w_0, χ_0) is the maximum of J_0 described in part (a) of Theorem 2, with $\chi_0'(\tau) \ge C > 0$, and (H5,9,10) hold, then (w_0, χ_0) solves the Euler-Lagrange equation (4.5). Thus part (c) of Theorem 2 is proved.

4.2   The Euler equation for w_0

We prove part (d) of Theorem 2. The maps

$$W_{2\pi}^{1,\rho} \to \mathbb{R}, \quad w \mapsto \int_0^{2\pi} w \mathcal{C} w' \, d\tau \quad \text{and} \quad w \mapsto \int_0^{2\pi} w^2 \mathcal{C} w' \, d\tau$$

30

are Fréchet differentiable at w_0. Hence I_0 is differentiable on $W_{2\pi}^{1,\rho}$, and its differential at w_0 in the direction $h \in W_{2\pi}^{1,\rho}$ is

$$dI_0(w_0)h = \int_0^{2\pi} \nabla I_0 \, h \, d\tau,$$

where

(4.6) $$\nabla I_0 := \nabla I_0(w_0) = (c^2 - 2ga_0)\mathcal{C}w_0' - g\{w_0(1 + \mathcal{C}w_0') + \mathcal{C}(w_0 w_0')\}$$

and

(4.7) $$a_0 := -[w_0 \mathcal{C} w_0'].$$

In the following, we will denote

$$H_0^{k,\rho} := \{w \in W_{2\pi}^{k,\rho} \; : \; [w] = 0\}, \quad k = 1, 2.$$

To investigate the differentiability (at least in the Gateaux sense) of the elastic energy term $\mathcal{E}(w, \chi)$ with respect to w, we recall the following fact.

Lemma 7. w_0 is an interior point of $\mathcal{A}_0$ in the topology of $W_{2\pi}^{2,\rho}$, that is there exists $\varepsilon_0 = \varepsilon_0(w_0) > 0$ such that

$$\mathcal{W}_0 := \{w \in H_0^{2,\rho} \; : \; \|w - w_0\|_{W_{2\pi}^{2,\rho}} < \varepsilon_0\} \subset \mathcal{A}_0,$$

and there exist constants C, C' such that

$$0 < C \leq \Omega(w) \leq C' \quad \forall w \in \mathcal{W}_0.$$

Proof. See [14], Lemma 4.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The map

$$\Omega : \mathcal{W}_0 \to W_{2\pi}^{1,\rho}, \quad w \mapsto \Omega(w)$$

is of class C^1, and its differential at w in the direction h is

$$d\Omega(w)h = \frac{w'h' + (1 + \mathcal{C}w')\mathcal{C}h'}{\Omega(w)} \quad \forall h \in H_0^{2,\rho}.$$

Also the map

$$\Omega\sigma : \mathcal{W}_0 \to L_{2\pi}^\rho, \quad w \mapsto \Omega(w)\sigma(w)$$

is of class C^1, and

$$d(\Omega(w)\sigma(w))h = -\mathcal{C}\Big(\frac{d\Omega(w)h}{\Omega(w)}\Big)' = -\mathcal{C}\Big(\frac{w'h' + (1 + \mathcal{C}w')\mathcal{C}h'}{\Omega(w)^2}\Big)',$$

because, by definition, $\Omega(w)\sigma(w) = \Theta(w)'$ and $\Theta(w) = -\mathcal{C}\log\Omega(w)$. We define

$$\mathcal{L}(u) := \frac{w_0'u + (1 + \mathcal{C}w_0')\,\mathcal{C}u}{\Omega_0^2},$$

31

so that
$$d\Omega(w_0)h = \Omega_0\,\mathcal{L}(h'), \quad d(\Omega(w_0)\sigma(w_0))h = -\,\mathcal{C}(\mathcal{L}(h'))'.$$

Note that, in general, $\mathcal{L}(h')$ belongs to $W^{1,\rho}_{2\pi}$ only, even for $h \in C^\infty$.

The functional $\mathcal{E}(w,\chi)$ is Gateaux-differentiable with respect to w at (w_0,χ_0) in the direction h if
$$E_{1,0}\,\mathcal{L}(h') - E_{2,0}\,\mathcal{C}(\mathcal{L}(h'))' \in L^1_{2\pi}.$$

Since $\mathcal{L}(h') \in W^{1,\rho}_{2\pi}$ and $E_{1,0} \in L^1_{2\pi}$, this holds when

(4.8)
$$E_{2,0} \in L^{\rho'}_{2\pi},$$

where
$$\rho' = \frac{p(s+1)}{s(p-1)}$$

is the conjugate exponent of $\rho = (p+sp)/(p+s)$. Now (4.8) holds if we assume (H11). Indeed, recalling that Ω_0 is bounded both below and above, (H11) and the Euler equation (4.5) imply that, where $|\sigma_0|/\chi'_0$ is larger than some constant (depending on the constant γ_0 of (4.5)), then

$$\left(\frac{|\sigma_0|}{\chi'_0}\right)^p \le C(\chi'_0)^s,$$

for some $C > 0$. Hence
$$\frac{\sigma_0}{\chi'_0} \in L^{\frac{p(s+1)}{s}}$$

because $\chi'_0 \in L^{s+1}$. Then (4.8) follows by (H10), and, as a consequence, the functional $\mathcal{E}(w,\chi)$ is Gateaux-differentiable at (w_0,χ_0) with respect to w in all directions $h \in W^{2,\rho}_{2\pi}$. Hence the maximiser (w_0,χ_0) solves the Euler equation in weak form

(4.9) $$\int_0^{2\pi} \left\{\nabla I_0\, h \ - \ E_{1,0}\Omega_0\,\mathcal{L}(h') \ + \ E_{2,0}\,\mathcal{CL}(h')'\right\} d\tau \ = \ 0 \quad \forall h \ \in \ H^{2,\rho}_0.$$

Lemma 8. *The linear operator $\mathcal{L}$ is an isomorphism of $H^{1,\rho}_0$ into itself, and*
$$\mathcal{L}(u) = v \quad iff \quad u = w'_0 v - (1 + \mathcal{C}w'_0)\,\mathcal{C}v =: \mathcal{L}^{-1}(v),$$

for all $u, v \in H^{1,\rho}_0$.

Proof. The proof is elementary once the following facts from complex function theory are taken into account. For $p > 0$, write $U \in \mathcal{H}^p_\mathbb{C}$ if U is holomorphic in the unit disc D and
$$\sup_{r\in(0,1)} \int_0^{2\pi} |U(re^{i\tau})|^p \, d\tau < \infty.$$

It is well known [5, 8] that when $U \in \mathcal{H}^p_\mathbb{C}$ for any $p > 0$, then
$$U^*(t) := \lim_{r\nearrow 1} U(re^{it}) \quad \text{exists for a.e. } t$$

and if, for some $q \in (0, \infty)$, $|U^*|^q \in L_{2\pi}^1$, then $U \in \mathcal{H}_\mathbb{C}^q$; if $|U^*| \in L_{2\pi}^\infty$, then U is bounded on D. Moreover, if $U \in \mathcal{H}_\mathbb{C}^p$, $p > 0$, then $U \in \mathcal{H}_\mathbb{C}^q$ for some $q > 1$ if and only if $U^*(t) = u(t) + i\mathcal{C}u(t) + i\alpha$ for some $u \in L_{2\pi}^q$ where

$$\frac{1}{2\pi} \int_0^{2\pi} u(\tau) \, d\tau + i\alpha = U(0).$$

Conversely, if $u \in L_{2\pi}^q$, $q > 1$, there exists $U \in \mathcal{H}_\mathbb{C}^q$ with $U^* = u + i\mathcal{C}u$.

Finally ([5], Theorem 3.11), if U is holomorphic on D and continuous on $\overline{D}$, then U^* is absolutely continuous if and only if $U' \in \mathcal{H}_\mathbb{C}^1$, in which case

$$\frac{d}{d\tau} U^*(\tau) = i e^{i\tau} (U')^*(\tau).$$

Moreover, if $U' \in \mathcal{H}_\mathbb{C}^1$ and $[\log U'^*] = \log |U'(0)|$, then U' has no zeros in D.

Now we turn to our proof. Since $w_0' \in H_0^{1,\rho}$, there exists a function W which is holomorphic in the unit disc D, continuous on the closed unit disc $\overline{D}$, $W(0) = i$, $W' \in \mathcal{H}_\mathbb{C}^\rho$ and

(4.10) $$W^* = w_0' + i(1 + \mathcal{C}w_0'), \quad |W^*| = \Omega(w_0).$$

Since $w_0 \in \mathcal{A}$, it follows that W and $1/W$ are bounded on $\overline{D}$.

Now for $u \in H_0^{1,\rho}$, let U be holomorphic on D and continuous on $\overline{D}$ such that $U' \in \mathcal{H}_\mathbb{C}^\rho$ and $U^* = u + i\mathcal{C}u$. Note that $\mathcal{L}u = \operatorname{Re}(U/W)^*$. It follows from the above remarks that $(U/W)' \in \mathcal{H}_\mathbb{C}^\rho$. Hence $\mathcal{L}u$ is absolutely continuous and $(\mathcal{L}u)' \in L_{2\pi}^\rho$. Also, since $W(0) \neq 0$, it follows that $[\mathcal{L}u] = 0$ if and only if $U(0) = 0$, i.e. if and only if $[u] = 0$. If, on the other hand, $0 = \mathcal{L}u = \operatorname{Re}(U/W)^*$, then $U \equiv 0$ and hence $\mathcal{L}$ is injective from $H_0^{1,\rho}$ to itself.

Finally for any $v \in H_0^{1,\rho}$ let V be holomorphic on D and continuous on $\overline{D}$ such that $V' \in \mathcal{H}_\mathbb{C}^\rho$ and $V^* = v + i\mathcal{C}v$. Then $V(0) = 0$ and $\mathcal{L}u = v$ if and only if $U = WV$ on D. Thus $U' \in \mathcal{H}_\mathbb{C}^\rho$ and $U(0) = 0$. In other words, $u \in H_0^{1,\rho}$ and $u = w_0'v - (1 + \mathcal{C}w_0')\mathcal{C}v$. This completes the proof. $\qquad\square$

Remark 11. In Lemma 8 we have proved that $\mathcal{L}$ is an isomorphism of $H_0^{1,\rho}$. However, $h \in H_0^{2,\rho}$ does not imply that $\mathcal{L}(h) \in H_0^{2,\rho}$, because w_0' has only regularity $W^{1,\rho}$. This means that, in the present problem, $\mathcal{L}(h)$ cannot be taken as test function, as was done in [14]. So instead, here we will take $\mathcal{L}(h') \in H_0^{1,\rho}$ as test function "of lower order". $\qquad\square$

Now we seek an expression for (4.9) that involves only $\mathcal{L}(h')$ as test function. First, we note that, for every $h \in H_0^{2,\rho}$,

$$\int_0^{2\pi} \nabla I_0 \, h \, d\tau = \int_0^{2\pi} (\nabla I_0 - \lambda_0) \, h \, d\tau,$$

where $\lambda_0 := [\nabla I_0]$. Integrating by parts yields

$$\int_0^{2\pi} (\nabla I_0 - \lambda_0) \, h \, d\tau = -\int_0^{2\pi} \left(\int_0^\tau (\nabla I_0 - \lambda_0) \right) h'(\tau) \, d\tau$$

$$= -\int_0^{2\pi} m_0 \, \mathcal{L}(h') \, d\tau,$$

33

where
$$m_0 := (\mathcal{L}^{-1})^* \left(\int_0^\tau (\nabla I_0 - \lambda_0) \right)$$

and $(\mathcal{L}^{-1})^*$ is the adjoint operator of $\mathcal{L}^{-1}$ in the usual $L_{2\pi}^2$ sense,

$$(\mathcal{L}^{-1})^*(f) = w_0' f + \mathcal{C}\big((1 + \mathcal{C}w_0')f\big) \quad \forall f.$$

Now, for any $h \in H_0^{2,\rho}$, let
$$\varphi := \mathcal{L}(h').$$

Since $[h'] = 0$, by Lemma 8, $\varphi \in H_0^{1,\rho}$. We observe that

$$\{\varphi = \mathcal{L}(h') : \ h \in H_0^{2,\rho}\} = H_0^{1,\rho}.$$

Indeed, given any $\varphi \in H_0^{1,\rho}$, there exists a unique primitive h of $\mathcal{L}^{-1}(\varphi)$ having zero mean. As a consequence, (4.9) can be written as

$$(4.11) \qquad \int_0^{2\pi} (m_0 + \Omega_0 E_{1,0}) \, \varphi \, d\tau + \int_0^{2\pi} \mathcal{C} E_{2,0} \, \varphi' \, d\tau = 0 \qquad \forall \varphi \in H_0^{1,\rho},$$

where $\mathcal{C}(E_{2,0})$ is well defined, by (4.8).

Lemma 9. *Suppose that $a(\tau) \in L_{2\pi}^{\rho'}$, $b(\tau) \in L_{2\pi}^1$ satisfy*

$$\int_0^{2\pi} b\varphi \, d\tau + \int_0^{2\pi} a\varphi' \, d\tau = 0 \quad \forall \varphi \in H_0^{1,\rho}.$$

Then $a(\tau) \in W_{2\pi}^{1,1}$, and

$$a(\tau) = const. + \int_0^\tau \big(b(t) - [b]\big) \, dt.$$

Proof. The proof is elementary. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By Lemma 9 and (4.11) we deduce the Euler equation

$$(4.12) \qquad \mathcal{C} E_{2,0}(\tau) = const. + \int_0^\tau (m_0 + \Omega_0 E_{1,0} - b_0) \, dt,$$

where $b_0 := [m_0 + \Omega_0 E_{1,0}]$.

We have proved that, if the maximizer (w_0, χ_0) of J_0 described in part (a) of Theorem 2 satisfies the Euler equation (4.5), with $\chi_0'(\tau) \geq C > 0$, and (H10,11) hold, then (w_0, χ_0) also solves the Euler equation (4.12). In Section 5, this will be shown to imply the dynamic boundary condition (1.1d). But first we examine the smoothness of solutions.

4.3 Regularity of the solution

We prove part (e) of Theorem 2. From (4.12) it follows that $\mathcal{C}E_{2,0} \in W_{2\pi}^{1,1} \subset L^\infty \subset L_{2\pi}^\beta$ for all $\beta \in (1,\infty)$. Hence

$$(4.13) \qquad\qquad E_{2,0} \in L^\beta \quad \forall \beta \in (1,\infty).$$

In (H9-12), let $\bar{\nu}_4 := \min\{\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3, \bar{\nu}_{\gamma_0}\}$ and $\bar{\mu}_4 := \max\{\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_{\gamma_0}\}$, where γ_0 is the constant in the Euler equation (4.5), and let

$$A^* := \Big\{\tau \in (0, 2\pi): \ \frac{\Omega_0}{\chi_0'} \leq \bar{\nu}_4, \quad \frac{\Omega_0|\sigma_0|}{\chi_0'} \geq \bar{\mu}_4\Big\}.$$

By (H11,12),

$$\int_{A^*} |E_{2,0}|^\beta \, d\tau \geq C \int_{A^*} \Big(\frac{\Omega_0}{\chi_0'}\Big)^{\alpha\beta} \Big(\frac{\Omega_0|\sigma_0|}{\chi_0'}\Big)^{(p-1)\beta} d\tau$$

$$\geq C' \int_{A^*} (\chi_0')^{\varepsilon\beta} \, d\tau.$$

Note that χ_0' is bounded above on $(0, 2\pi) \setminus A^*$ by (H11) and Euler equation (4.5). Hence, by (4.13), $\chi_0' \in L_{2\pi}^\beta$ for all $\beta \in (1,\infty)$, and therefore

$$\frac{\sigma_0}{\chi_0'} \in L_{2\pi}^\beta \quad \forall \beta \in (1,\infty),$$

by (H11). Thus, by (H9),

$$|E_{1,0}| \leq C\Big\{(\chi_0')^{s+1} + \Big(\frac{|\sigma_0|}{\chi_0'}\Big)^p\Big\} \in L_{2\pi}^\beta \quad \forall \beta \in (1,\infty),$$

for some C. Since $m_0 \in W_{2\pi}^{1,\rho} \subset L_{2\pi}^\infty$, by (4.12) it follows that $\mathcal{C}E_{2,0} \in W_{2\pi}^{1,\beta}$ for all β, therefore

$$(4.14) \qquad\qquad E_{2,0} \in W_{2\pi}^{1,\beta} \quad \forall \beta \in (1,\infty).$$

In particular, $E_{2,0} \in L_{2\pi}^\infty$. Hence, by (H11,12),

$$\|E_{2,0}\|_\infty \geq C\frac{1}{(\chi_0')^\alpha} \Big(\frac{|\sigma_0|}{\chi_0'}\Big)^{p-1} \geq C'(\chi_0')^\varepsilon, \quad \text{on } A^*.$$

Thus

$$\chi_0' \in L_{2\pi}^\infty.$$

By (H11), also $\sigma_0/\chi_0' \in L_{2\pi}^\infty$, therefore

$$\sigma_0 \in L_{2\pi}^\infty.$$

By the definition of $E^\star(t, \sigma)$, the Euler equation (4.5) for χ can be written as

$$(4.15) \qquad\qquad E_1^\star\Big(\frac{\chi_0'}{\Omega_0}, \sigma_0\Big) \equiv \gamma_0.$$

By (H4), the map $t \mapsto E_1^\star(t, \sigma)$ is strictly increasing. Hence, by Lemma 5, it follows that for every $\sigma \in \mathbb{R}$ there exists a unique $t > 0$ such that $E_1^\star(t, \sigma) = \gamma_0$. In other words, there is defined a function $\varpi : \mathbb{R} \to (0, +\infty)$ such that

$$(4.16) \qquad E_1^\star(\varpi(\sigma), \sigma) = \gamma_0 \quad \forall \sigma \in \mathbb{R}.$$

Since $E(\nu, \mu)$ is of class C^2 (H1), ϖ is of class C^1 by the implicit function theorem, and

$$(4.17) \qquad \varpi'(\sigma) = -\frac{E_{12}^\star(\varpi(\sigma), \sigma)}{E_{11}^\star(\varpi(\sigma), \sigma)}.$$

We rewrite (4.15) as

$$(4.18) \qquad \frac{\chi_0'}{\Omega_0} = \varpi(\sigma_0),$$

therefore

$$(4.19) \qquad E_{2,0} = E_2\left(\frac{\Omega_0}{\chi_0'}, \frac{\Omega_0 \sigma_0}{\chi_0'}\right) = E_2\left(\frac{1}{\varpi(\sigma_0)}, \frac{\sigma_0}{\varpi(\sigma_0)}\right) = E_2^\star(\varpi(\sigma_0), \sigma_0)$$

by the definition of $E_{2,0}$ and $E^\star$. Now we consider the map

$$\psi : \mathbb{R} \to \mathbb{R}, \quad y \mapsto \psi(y) := E_2^\star(\varpi(y), y).$$

ψ is differentiable and

$$\psi'(y) = \frac{E_{11}^\star(\varpi(y), y)\, E_{22}^\star(\varpi(y), y) - (E_{12}^\star(\varpi(y), y))^2}{E_{11}^\star(\varpi(y), y)}$$

by (4.17). Thus, by (H4) and Lemma 4, ψ is strictly increasing, and therefore invertible. Hence (4.19) can be written as

$$(4.20) \qquad \sigma_0 = \psi^{-1}(E_{2,0}).$$

By (4.14), $\psi^{-1}(E_{2,0})$ is differentiable, and

$$\partial_\tau \{\psi^{-1}(E_{2,0}(\tau))\} = \frac{1}{\psi'(\sigma_0(\tau))}\, (E_{2,0})'(\tau),$$

for almost all τ. Since ψ' is positive and continuous, and $|\sigma_0(\tau)| \leq \|\sigma_0\|_{L^\infty}$ for almost all τ,

$$\psi'(\sigma_0) \geq C > 0 \quad \text{a.e.},$$

for some constant C. Therefore, by (4.14), $\psi^{-1}(E_{2,0})$ belongs to $W_{2\pi}^{1,\beta}$ for all β, that is,

$$(4.21) \qquad \sigma_0 \in W_{2\pi}^{1,\beta} \quad \forall \beta \in (1, \infty).$$

By the usual sequence of (non-trivial) implications, one can then prove that

$$\Omega_0 \in W_{2\pi}^{2,\beta}, \quad w_0 \in W_{2\pi}^{3,\beta} \quad \forall \beta \in (1, \infty).$$

By the fact that

$$(4.22) \qquad\qquad 0 < C \leq \varpi(\sigma_0) = \frac{\chi_0'}{\Omega_0} \leq C'$$

and by (4.21) and (4.17), it follows that $\varpi(\sigma_0) \in W_{2\pi}^{1,\beta}$ for all β. Then by (4.18)

$$\chi_0 \in W_{2\pi}^{2,\beta} \quad \forall \beta \in (1, \infty).$$

Hence $E_{1,0}$ is differentiable, and, by (4.22), (4.21) and the continuity of the second derivatives E_{11} and E_{12}, we have proved that

$$E_{1,0} \in W_{2\pi}^{1,\beta} \quad \forall \beta \in (1, \infty).$$

Next, the fact that $w_0 \in W_{2\pi}^{3,\beta}$ implies that

$$m_0 \in W_{2\pi}^{2,\beta} \quad \forall \beta \in (1, \infty)$$

by the definition of m_0 and ∇I_0. Then, by (4.12),

$$(4.23) \qquad\qquad E_{2,0} \in W_{2\pi}^{2,\beta} \quad \forall \beta \in (1, \infty).$$

In particular, $(E_{2,0})'$ is bounded. Since $\sigma_0' = (E_{2,0})'/\psi'(\sigma_0)$, we conclude that

$$\sigma_0 \in W_{2\pi}^{1,\infty}.$$

Hence, by (4.22),

$$\frac{\Omega_0}{\chi_0'} \in W_{2\pi}^{1,\infty}.$$

Moreover, since Ω_0 and $1/\Omega_0$ belong to $W_{2\pi}^{1,\infty}$, also

$$\frac{\chi_0'}{\Omega_0}, \ \chi_0', \ \frac{1}{\chi_0'} \in W_{2\pi}^{1,\infty}.$$

Differentiating (4.5) with respect to τ (which is possible now because Ω_0/χ_0' and σ_0 are differentiable) yields

$$(4.24) \qquad\qquad (E_{1,0})' + \sigma_0(E_{2,0})' = 0,$$

which is (1.3a). By (4.23) and (4.21), (4.24) implies that

$$E_{1,0} \in W_{2\pi}^{2,\beta}.$$

We have proved that, if $E(\nu, \mu)$ is of class C^2, then the curvature σ_0 and the stretch Ω_0/χ_0' of the membrane belong to $W_{2\pi}^{1,\infty}$. By bootstrap, when $E(\nu, \mu)$ enjoys more regularity, σ_0 and Ω_0/χ_0' are also more regular, as the following result shows.

Lemma 10. *Suppose that* (H1,2,4,5,9,11,12) *hold, and let* (w_0, χ_0) *satisfy Euler equations* (4.5) *and* (4.12). *Suppose that* $E(\nu, \sigma)$ *is of class* C^k, $k \geq 2$. *Then*

$$w_0 \in W_{2\pi}^{k+1,\beta}, \quad \chi_0 \in W_{2\pi}^{k,\infty}, \quad \sigma_0 \in W_{2\pi}^{k-1,\infty}, \quad \frac{\Omega_0}{\chi_0'} \in W_{2\pi}^{k-1,\infty},$$

$$E_{1,0} \in W_{2\pi}^{k,\beta}, \quad E_{2,0} \in W_{2\pi}^{k,\beta},$$

for all $\beta \in (1, \infty)$.

Proof. We have already proved the case $k = 2$. By induction, suppose that the statement holds for all $j = 2, \ldots, k$, and let $E(\nu, \mu) \in C^{k+1}$. Hence Ω_0 and m_0 belong to $W_{2\pi}^{k,\beta}$, because $w_0 \in W_{2\pi}^{k+1,\beta}$. Then, by the Euler equation (4.12) for w_0,

(4.25)
$$E_{2,0} \in W_{2\pi}^{k+1,\beta}$$

for all β. Recalling the definition of ϖ and ψ, we note that, since $E \in C^{k+1}$, both ϖ and ψ are of class C^k. Then, with a direct calculation, one can see that the kth derivative of $\psi^{-1}(E_{2,0})$ with respect to τ is a finite sum of terms, each of which is a quotient where the numerator is a polynomial involving the partial derivatives of E at $(\Omega_0/\chi_0', \Omega_0\sigma_0/\chi_0')$ of order $\leq (k+1)$, and the denominator is an integer power of $\psi'(\sigma_0)$. Since $E \in C^{k+1}$, by (4.22) and the fact that σ_0 is bounded it follows that the kth derivative of $\psi^{-1}(E_{2,0})$ is bounded. Thus, by (4.20),

(4.26)
$$\sigma_0 \in W_{2\pi}^{k,\infty}.$$

Using (4.24), (4.25) and (4.26) imply that

$$E_{1,0} \in W_{2\pi}^{k+1,\beta}$$

for all β.

Next, with a direct calculation, one can see that the kth derivative of $\varpi(\sigma_0)$ with respect to τ is a polynomial involving the derivatives of ϖ at σ_0 of order $\leq k$, and the derivatives of σ_0 at τ of order $\leq k$. Since $\varpi \in C^k$, and by (4.26), the kth derivative of $\varpi(\sigma_0)$ is bounded. By the Euler equation (4.18) for χ_0,

$$\frac{\Omega_0}{\chi_0'} \in W_{2\pi}^{k,\infty}.$$

Moreover, since $\Omega_0/\chi_0' \geq C > 0$ for some constant C, also

(4.27)
$$\frac{\chi_0'}{\Omega_0} \in W_{2\pi}^{k,\infty}.$$

By (4.26), with the usual sequence of implications, it follows that

$$\Omega_0 \in W_{2\pi}^{k+1,\beta}, \quad w_0 \in W_{2\pi}^{k+2,\beta}$$

38

for all β. Hence $\Omega_0 \in W_{2\pi}^{k,\infty}$, therefore, by (4.27),

$$\chi_0' = \frac{\chi_0'}{\Omega_0} \, \Omega_0 \in W_{2\pi}^{k,\infty},$$

and the proof is complete. $\qquad\qquad\square$

Remark 12. *As an obvious consequence of Lemma 10, if $E(\nu, \sigma) \in C^\infty$, then w_0, χ_0, σ_0, Ω_0/χ_0', $E_{1,0}$ and $E_{2,0}$ are also C^∞.* $\qquad\square$

By using the inverse diffeomorphism χ_0^{-1}, and recalling (2.3), (2.4) and (2.5), the same regularity result holds for the stretch

$$\nu_0(x) := \frac{\Omega_0(\chi_0^{-1}(x))}{\chi_0'(\chi_0^{-1}(x))} = \frac{\Omega_0(\tau)}{\chi_0'(\tau)}$$

and the curvature

$$\hat{\sigma}_0(\boldsymbol{r}(x)) := \sigma_0(\chi_0^{-1}(x)) = \sigma_0(\tau)$$

as functions of the Lagrangian coordinate x of material points.

Proposition 2. *Under the same assumptions as in Lemma 10,*

$$\chi_0^{-1}(x) \in W_{2\pi}^{k,\infty}, \quad \nu(x) \in W_{2\pi}^{k-1,\infty}, \quad \hat{\sigma}(\boldsymbol{r}(x)) \in W_{2\pi}^{k-1,\infty}.$$

Proof. We know that $0 < C \le \chi_0'(\tau) \le C'$ for all τ, for some constants C, C'. The kth derivatives of the inverse diffeomorphism $\chi_0^{-1}(x)$ is a finite sum of terms, each of which is a quotient where the numerator is a polynomial in the derivatives of $\chi_0(\tau)$ of order $\le k$, the denominator is an integer power of $\chi_0'(\tau)$, and $\tau = \chi_0^{-1}(x)$. Hence also

$$\chi_0^{-1} \in W_{2\pi}^{k,\infty}.$$

Then the proposition follows by (2.3), (2.4), (2.5) and Lemma 10. $\qquad\square$

5 The dynamic boundary condition: Riemann-Hilbert theory

We now derive the dynamic boundary condition for the physical boundary-value problem and prove part (f) of Theorem 2. We recall that the pressure at the free surface in terms of the Lagrangian coordinate x of material points in the reference configuration of the surface membrane is given by (1.3b). When this is rewritten as a function of τ, we find the formula

$$(5.1) \qquad\qquad P(\tau) = \frac{1}{\Omega_0} \Big(\frac{(E_{2,0})'}{\Omega_0} \Big)' - \sigma_0 E_{1,0},$$

where $\Omega_0 = \Omega_0(\tau)$ etc., and $'$ is, as usual, the derivative with respect to τ. However it is not obvious how to deduce the dynamic boundary condition for hydroelastic waves directly from the existence of a maximizer of J_0. In this section we derive it by interpreting the Euler-Lagrange equation (4.9) as a Riemann–Hilbert problem in the manner of [12]. We begin with a special case of a result in [13], and include a short proof for the sake of completeness.

Lemma 11. *Suppose that* $f(\tau) \in L^{\beta}_{2\pi}$, $\beta > 1$, *and* $a \in \mathbb{R}$. *Then*

$$(i) \ \mathcal{C}(fw'_0) + f(1 + \mathcal{C}w'_0) \equiv a \qquad \text{iff} \qquad (ii) \ \Omega_0^2 f \equiv a.$$

Proof. Recall the notation from the proof of Lemma 8.

$(i) \Rightarrow (ii)$. Consider the holomorphic function $U \in \mathcal{H}^{\beta}_{\mathbb{C}}$ such that

$$U^* = fw'_0 + i(\mathcal{C}(fw'_0) - a), \quad \operatorname{Im} U(0) = -ia.$$

By (i),

$$U^* = f\,\overline{W^*},$$

therefore, multiplying by W^*,

$$U^*W^* = f|W^*|^2 = \Omega_0^2 f.$$

So the holomorphic function $UW \in \mathcal{H}^{\beta}_{\mathbb{C}}$ is real on the unit circle. Then on the whole unit disc $UW \equiv b$ for some real constant b. Since at the origin

$$U(0)W(0) = ([fw'_0] - ia)i = a + i[fw'_0],$$

it follows that $b = a$ and $[fw'_0] = 0$. In particular, on the unit circle $a \equiv U^*W^* \equiv \Omega_0^2 f$.

$(ii) \Rightarrow (i)$. We consider the holomorphic function

$$V := \frac{a}{W} \in \mathcal{H}^{\infty}_{\mathbb{C}},$$

and denote $v := \operatorname{Re} V^*$. Since $V(0) = a/W(0) = -ia$, on the unit circle

$$V^* = v + i(-a + \mathcal{C}v).$$

On the other hand, (ii) implies that

$$V^* = \frac{a}{W^*} = \frac{a\overline{W^*}}{|W^*|^2} = f\overline{W^*} = fw'_0 - if(1 + \mathcal{C}w'_0).$$

Then

$$fw'_0 = v, \quad -f(1 + \mathcal{C}w'_0) = -a + \mathcal{C}v,$$

and (i) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Let

$$f := \frac{c^2}{2} - g(a_0 + w_0) - P,$$

where P is defined in (5.1), and note that $f \in L^{\beta}_{2\pi}$ for all $\beta \in (1, \infty)$, by the regularity results we have proved in the previous section.

By a simple calculation using (4.6), we see that

$$(5.2) \qquad\qquad\qquad \mathcal{C}(fw'_0) + f(1 + \mathcal{C}w'_0) \equiv \frac{c^2}{2}$$

if and only if

$$(5.3) \qquad \nabla I_0 - ga_0 = \mathcal{C}(w_0'P) + P(1 + \mathcal{C}w_0').$$

We now prove that (5.3) follows from the two Euler-Lagrange equations and the regularity results which lead to (4.24). After integrating the Euler equation (4.9) by parts, we get

$$
\begin{aligned}
0 &= \int_0^{2\pi} \nabla I_0 \, h \, d\tau + \int_0^{2\pi} \{\mathcal{C}(E_{2,0})' - E_{1,0}\Omega_0\} \, \mathcal{L}(h') \, d\tau \\
&= \int_0^{2\pi} \nabla I_0 \, h \, d\tau - \int_0^{2\pi} \left\{\mathcal{L}^*\big(\mathcal{C}(E_{2,0})' - E_{1,0}\Omega_0\big)\right\}' h \, d\tau \quad \forall h \in H_0^{2,\rho},
\end{aligned}
$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. Hence

$$\nabla I_0 - \left\{\mathcal{L}^*\big(\mathcal{C}(E_{2,0})' - E_{1,0}\Omega_0\big)\right\}' \equiv \text{const.},$$

therefore

$$\nabla I_0 - ga_0 = \left\{\mathcal{L}^*\big(\mathcal{C}(E_{2,0})' - E_{1,0}\Omega_0\big)\right\}'$$

because

$$[\nabla I_0] = -g[w_0 \mathcal{C}w_0'] = ga_0.$$

Thus, (5.3) can be written as

$$(5.4) \qquad \{\mathcal{L}^*(\mathcal{C}T - Q)\}' - \mathcal{C}(w_0'P) - P(1 + \mathcal{C}w_0') = 0,$$

where, for convenience, we let

$$T := (E_{2,0})' \in W_{2\pi}^{1,\beta}, \quad Q := E_{1,0}\Omega_0 \in W_{2\pi}^{2,\beta}.$$

To calculate the left-hand term in (5.4), we use the formula

$$\mathcal{L}^* u = \frac{w_0' u}{\Omega_0^2} - \mathcal{C}\Big(\frac{(1 + \mathcal{C}w_0')u}{\Omega_0^2}\Big)$$

and the equalities

$$\Big(\frac{w_0'}{\Omega_0^2}\Big)' = \frac{1 + \mathcal{C}w_0'}{\Omega_0^2} \, \Theta_0' - \frac{w_0'}{\Omega_0^2} \mathcal{C}\Theta_0',$$

$$\Big(\frac{1 + \mathcal{C}w_0'}{\Omega_0^2}\Big)' = -\frac{w_0'}{\Omega_0^2} \, \Theta_0' - \frac{1 + \mathcal{C}w_0'}{\Omega_0^2} \mathcal{C}\Theta_0'.$$

These follow from the fact that

$$w_0' = \Omega_0 \sin \Theta_0, \quad 1 + \mathcal{C}w_0' = \Omega_0 \cos \Theta_0, \quad \Omega_0'/\Omega_0 = \mathcal{C}\Theta_0',$$

the identity

$$T' = \Omega_0 \Big(\frac{T}{\Omega_0}\Big)' + T\mathcal{C}\Theta_0'$$

41

and the formula
$$Q' = Q\,\mathcal{C}\Theta_0' - T\,\Theta_0'.$$

This formula follows from (4.24), which was obtained by differentiating the Euler equation (4.5), using the regularity already proved. In this way (5.4) can be written explicitly, in terms of Θ_0', Q, T and their Hilbert transform, as

$$\left(\frac{1+\mathcal{C}w_0'}{\Omega_0^2}\,\Theta_0' - \frac{w_0'}{\Omega_0^2}\,\mathcal{C}\Theta_0'\right)(\mathcal{C}T - Q) + \mathcal{C}\left\{\left(\frac{w_0'}{\Omega_0^2}\,\Theta_0' + \frac{1+\mathcal{C}w_0'}{\Omega_0^2}\,\mathcal{C}\Theta_0'\right)(\mathcal{C}T - Q)\right\}$$

(5.5)

$$+ \frac{w_0'}{\Omega_0^2}\,\mathcal{C}\left\{\Omega_0\left(\frac{T}{\Omega_0}\right)' + T\mathcal{C}\Theta_0'\right\} - \frac{w_0'}{\Omega_0^2}\,(Q\,\mathcal{C}\Theta_0' - T\Theta_0')$$

$$- \mathcal{C}\left\{\frac{1+\mathcal{C}w_0'}{\Omega_0^2}\left(\mathcal{C}\left\{\Omega_0\left(\frac{T}{\Omega_0}\right)' + T\,\mathcal{C}\Theta_0'\right\} - Q\,\mathcal{C}\Theta_0' + T\Theta_0'\right)\right\}$$

$$- \mathcal{C}\left\{\frac{w_0'}{\Omega_0^2}\left(\Omega_0\left(\frac{T}{\Omega_0}\right)' - Q\Theta_0'\right)\right\} - \frac{1+\mathcal{C}w_0'}{\Omega_0^2}\left\{\Omega_0\left(\frac{T}{\Omega_0}\right)' - Q\Theta_0'\right\} = 0.$$

The four terms in (5.5) involving $u := \Omega_0(T/\Omega_0)' \in L_{2\pi}^\beta$ cancel because

(5.6) $\quad \dfrac{w_0'\,\mathcal{C}u - (1+\mathcal{C}w_0')\,u}{\Omega_0^2} - \mathcal{C}\left(\dfrac{w_0'u + (1+\mathcal{C}w_0')\,\mathcal{C}u}{\Omega_0^2}\right)$

$$= \mathrm{Im}\left(\frac{U^*}{W^*}\right) - \mathcal{C}\,\mathrm{Re}\left(\frac{U^*}{W^*}\right) = 0,$$

where $U(z)$ is the holomorphic function of the unit disc such that $U^* = u + i\mathcal{C}u$, and W has been defined in (4.10). The eight terms involving Q simply cancel by pairs. Now we note that

(5.7) $\qquad\qquad \Theta_0'T - (\mathcal{C}\Theta_0')(\mathcal{C}T) + \mathcal{C}(T\,\mathcal{C}\Theta_0') = -\mathcal{C}(\Theta_0'\,\mathcal{C}T),$

because

$$-\mathrm{Re}\left\{(\Theta_0' + i\mathcal{C}\Theta_0')(T + i\mathcal{C}T)\right\} = \mathcal{C}\,\mathrm{Im}\left\{(\Theta_0' + i\mathcal{C}\Theta_0')(T + i\mathcal{C}T)\right\}.$$

Using (5.7), the eight terms involving T cancel because

$$-\frac{w_0'\,\mathcal{C}\xi - (1+\mathcal{C}w_0')\,\xi}{\Omega_0^2} + \mathcal{C}\left(\frac{w_0'\xi + (1+\mathcal{C}w_0')\,\mathcal{C}\xi}{\Omega_0^2}\right) = 0,$$

with $\xi := \Theta_0'\,\mathcal{C}T$, for the same reason as in (5.6). Hence (5.5) holds. This implies (5.2) and so, by Lemma 11,

(5.8) $\qquad\qquad\qquad 1 - \dfrac{2g}{c^2}\,(a_0 + w_0) - \dfrac{2}{c^2}\,P = \dfrac{1}{\Omega_0^2}.$

We have showed that, if (w_0, χ_0) satisfies the Euler equations (4.5) and (4.12), then it solves (5.8), proving part (f) of Theorem 2.

Now, the maximum (w_0, χ_0) for J_0 corresponds to the maximum (w_0^*, χ_0) for J, where

$$w_0^* := a_0 + w_0$$

(recall (2.7) and definition (4.7)). Then (5.8) writes

(5.9)
$$1 - \frac{2}{c^2}(gw_0^* + P) = \frac{1}{\Omega_0^2}.$$

Note that (5.9) is the same equation as in [14], par. 7.4 (in [14] the pressure was denoted by $-\mathcal{F}$). By Lemma 10, $\Omega_0 \in W_{2\pi}^{k,\beta}$ for all $\beta \in (1, \infty)$. So

$$\frac{1}{\Omega_0^2} \in W_{2\pi}^{k,\beta} \quad \forall \beta \in (1, \infty),$$

because $\Omega_0 \geq C > 0$ for some C. Hence, by (5.9),

$$P = \frac{c^2}{2} - gw_0^* - \frac{c^2}{2}\frac{1}{\Omega_0^2} \in W_{2\pi}^{k,\beta}$$

for all $\beta \in (1, \infty)$. In particular, in the case when E is of class C^2, the pressure belongs to $W_{2\pi}^{2,\beta} \subset W_{2\pi}^{1,\infty}$. The same holds for the pressure $P(\boldsymbol{r}(x))$ (see (1.3b)) as a function of the Lagrangian coordinate x.

Lemma 12. *The solution (w_0, χ_0) of the variational problem described in Theorem 2 gives a solution of the physical problem (1.1), where the free boundary $\mathcal{S}$ and the material deformation of the membrane $\boldsymbol{r}(x)$ are*

$$\mathcal{S} = \mathcal{S}(w_0) := \{\rho(w_0)(\tau) : \tau \in \mathbb{R}\}, \quad \boldsymbol{r}(x) = \rho(w_0)(\chi_0^{-1}(x)).$$

Proof. It was proved in [12] that, given the curve $\mathcal{S}(w)$ parametrized by (2.2), the solution $\widetilde{\psi}$ of the problem

$$\Delta\widetilde{\psi} = 0 \quad \text{below } \mathcal{S}(w),$$
$$\widetilde{\psi} = 0 \quad \text{on } \mathcal{S}(w),$$
$$\nabla\widetilde{\psi} \to (0, 1) \quad \text{as } Y \to -\infty$$

satisfies

$$|\nabla\widetilde{\psi}|^2 = \frac{1}{\Omega(w)^2} \quad \text{on } \mathcal{S}(w).$$

Now, we consider the curve $\mathcal{S} = \mathcal{S}(w_0)$, where (w_0, χ_0) is the solution of the variational problem in Theorem 2. The solution ψ of problem (1.1a,b,c) is $\psi = c\widetilde{\psi}$, therefore

$$|\nabla\psi|^2 = \frac{c^2}{\Omega(w)^2} \quad \text{on } \mathcal{S}(w).$$

By construction, $w_0^*(\tau)$ is the elevation of the point $\rho(w)(\tau) = \boldsymbol{r}(x)$ of the membrane, that is the vertical coordinate of the deformed point $\boldsymbol{r}(x)$ with respect to the rest frame, and $P(\tau)$ is the pressure at the point $\rho(w)(\tau) = \boldsymbol{r}(x)$. Then (5.9) is exactly (1.1d).

Note that the constraint (1.1e) is satisfied by (w_0, χ_0) by construction. $\square$

References

[1] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary.* J. reine angew. Math. **325** (1981), 105–144.

[2] A. Andrianov, *Hydroelastic Analysis of Floating Structures.* PhD Thesis, Technische Universiteit Delft, September 2005.

[3] S. S. Antman, *Nonlinear Problems of Elasticity.* Springer-Verlag, New York, 1995.

[4] S. S. Chern, *Curves and surfaces in Euclidian space.* In: Studies in Global Geometry and Analysis, Studies in Mathematics, Volume 4 (editor S. S. Chern). Mathematical Association of America, 1967.

[5] P. L. Duren, *Theory of H^p-Spaces.* Dover, Mineola, 2000.

[6] A. I. Dyachenko, E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov, *Analytic description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping).* Physics Letters A **221** (1996), 73–79.

[7] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems.* Amsterdam, North-Holland, 1976.

[8] J. B. Garnett, *Bounded Analytic Functions*, revised 1st edition. Springer, New York, 2007.

[9] A. G. Greenhill, *Wave motion in hydrodynamics.* Amer. J. Math. **9** (1) (1886), 62–96. `http://jstor.org`

[10] C. B. Morrey, Jr., *Multiple Integrals in the Calculus of Variations.* Springer-Verlag, Berlin 1966.

[11] P. Plotnikov and J. F. Toland, *Strain-gradient theory of hydroelastic travelling waves and tehir singular limits.* In preparation.

[12] E. Shargorodsky and J. F. Toland, *Bernoulli free-boundary problems.* Memoirs of Amer. Math. Soc., to appear. `www.maths.bath.ac.uk/~jft/`

[13] J. F. Toland, *Stokes waves in Hardy spaces and as distributions.* J. Math. Pure et Appl. **79** (9) (2000), 901–917.

[14] J. F. Toland, *Steady periodic hydroelastic waves.* Arch. Rational Mech. Analysis (2008). (DOI 10.1007/s00205-007-0104-2)

[15] J. F. Toland, *Heavy hydroelastic travelling waves.* Proc. R. Soc. A **463** (2007), 2371–2397. (DOI : 10.1098/rspa.2007.1883)

[16] V. E. Zakharov, *Stability of periodic waves of finite amplitude on the surface of deep fluid.* J. Appl. Mech. Tech. Phys. **2** (1968), 190–194.

[17] A. Zygmund, *Trigonometric Series I & II.* Corrected reprint (1968) of 2nd edition, Cambridge University Press, Cambridge, 1959.

Department of Mathematical Sciences,
University of Bath,
Bath BA2 7AY, U.K.

E-mail: `P.Baldi@bath.ac.uk, jft@maths.bath.ac.uk`