Periodic solutions of forced Kirchhoff equations

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Abstract

We consider Kirchhoff equations for vibrating bodies in any dimension in presence of a time-periodic external forcing with period $2\pi/\omega$ and amplitude ε, both for Dirichlet and for space-periodic boundary conditions.

We prove existence, regularity and local uniqueness of time-periodic solutions of period $2\pi/\omega$ and order ε, by means of a Nash-Moser iteration scheme. The results hold for parameters (ω, ε) in Cantor sets having measure asymptotically full for $\varepsilon \to 0$.

1 Introduction

We consider the Kirchhoff equation

$$
(1) \qquad u_{tt} - \Delta u \Big(1 + \int_\Omega |\nabla u|^2 \, dx \Big) = \varepsilon g(x,t) \qquad x \in \Omega, \ t \in \mathbb{R}
$$

where g is a time-periodic external forcing with period $2\pi/\omega$ and amplitude ε, and the displacement $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is the unknown. We consider both Dirichlet boundary conditions

$$
(2) \qquad u(x,t) = 0 \quad \forall x \in \partial\Omega, \ t \in \mathbb{R}
$$

where $\Omega \subset \mathbb{R}^d$ is a bounded, connected open set with smooth boundary, $d \geq 1$, and periodic boundary conditions on $\mathbb{R}^d$

$$
(3) \qquad u(x,t) = u(x + 2\pi m, t) \quad \forall m \in \mathbb{Z}^d, \ x \in \mathbb{R}^d, \ t \in \mathbb{R}
$$

where $\Omega = (0, 2\pi)^d$.

Equation (1) is a quasi-linear integro-partial differential equation having the structure of an infinite-dimensional Hamiltonian system, with time-depending Hamiltonian

$$
H(u,v) = \int_\Omega \frac{v^2}{2} \, dx + \int_\Omega \frac{|\nabla u|^2}{2} \, dx + \Big(\int_\Omega \frac{|\nabla u|^2}{2} \, dx \Big)^2 - \int_\Omega \varepsilon g u \, dx.
$$

It describes nonlinear forced vibrations of a d-dimensional body (in particular, a string for $d = 1$ and a membrane for $d = 2$).

This model has been proposed first in 1876 by Kirchhoff [23] in dimension one, without forcing terms, with Dirichlet boundary conditions, namely

$$(4) \qquad u_{tt} - u_{xx}\Big(1 + \int_0^\pi u_x^2\, dx\Big) = 0\,, \quad u(0,t) = u(\pi,t) = 0$$

to describe transversal free vibrations of a clamped string in which the dependence of the tension on the deformation cannot be neglected. Independently, Carrier [14] and Narasimha [31] rediscovered the same equation as a nonlinear approximation of the exact model for the stretched string.

Kirchhoff equations have been studied by many authors from the point of view of the Cauchy problem

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)$$

starting from the pioneering paper of Bernstein [6]. Both local and global existence has been investigated, for initial data having Sobolev or analytic regularity. See for example [18, 33, 26, 3, 17, 4, 28] and the rich surveys [2, 37].

In spite of the wide study for the Cauchy problem, to the best of our knowledge nothing is known about the existence of periodic solutions of Kirchhoff equations, except for the normal modes.

Kirchhoff himself observed that equation (4), thanks to its special symmetry, possesses a sequence of normal modes, that is solutions of the form $u(x,t) = u_j(t)\sin jx$, $j = 1, 2, \ldots$ where $u_j(t)$ is periodic. In general, normal modes are solutions of the form $u_j(t)\varphi_j(x)$ where $\varphi_j(x)$ is an eigenfunction of the Laplacian on Ω.

In presence of a forcing term $g(x,t)$ this symmetry is broken and normal modes do not survive (except in the one-mode case $g(x,t) = g_j(t)\varphi_j(x)$). Indeed, decomposing $u(x,t) = \sum_j u_j(t)\varphi_j(x)$ shows that all components $u_j(t)$ are coupled in the integral term $\int_\Omega |\nabla u|^2 dx$, and problem (1) is equivalent to a system of infinitely many nonlinear coupled ODEs, namely

$$u_j''(t) + \lambda_j^2 u_j(t)\Big(1 + \sum_k \lambda_k^2 u_k^2(t)\Big) = \varepsilon g_j(t)\,, \quad j = 1, 2, \ldots$$

where $g(x,t) = \sum_j g_j(t)\varphi_j(x)$ and λ_j^2 are the eigenvalues of the Laplacian on Ω.

In this paper we prove the existence of periodic solutions of (1). We consider the amplitude ε and the frequency ω of the forcing term g as parameters of the problem. We prove that there exist periodic solutions of order ε and period $2\pi/\omega$ when ε is small and (ε,ω) belong to a Cantor set which has positive measure, asymptotically full for $\varepsilon \to 0$. We prove regularity estimates for the solutions, both in Sobolev and in analytic classes, and local uniqueness (see Theorem 1, Remark 1 and Theorem 2 in Section 2).

There are two main difficulties in looking for periodic solutions of (1). The first one is the so-called "small divisors problem", caused by resonances between the forcing frequency ω with its overtones and the eigenvalues λ_j^2 of the Laplacian on Ω. Such a problem arises in the inversion of the d'Alembert operator $\partial_{tt} - \Delta$, whose spectrum

$$\{-\omega^2 l^2 + \lambda_j^2 \ : \ l \in \mathbb{N},\ j = 1, 2, \ldots\}$$

accumulates to zero for almost every ω. For this reason, $(\partial_{tt} - \Delta)^{-1}$ cannot map, in general, a functional space in itself, but only in a larger space of less regular functions. This makes impossible the application of the standard implicit function theorem.

The other difficulty is the presence of derivatives in the nonlinearity. In general, little is known about periodic solutions of equations of the form

$$u_{tt} - u_{xx} = \varepsilon f(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}).$$

This problem has been studied by Rabinowitz [35] in presence of a dissipative term αu_t, $\alpha \neq 0$, and frequency $\omega = 1$; by Craig [15] for pseudodifferential operators

$$u_{tt} - u_{xx} = a(x)u + b(x, |\partial_x|^\beta u) = 0, \quad \beta < 1;$$

by Bourgain [13] in cases like $u_{tt} - u_{xx} + \rho u + u_t^2 = 0$ and, for quasiperiodic solutions, [10] $u_{tt} - u_{xx} = a(x)u + \varepsilon \partial_x^{1/2}(h(x, u))$. We remark that, in general, the presence of derivatives in the nonlinearity makes uncertain the existence of global (even not periodic) solutions, see for example the non-existence results in [21, 24] for the equation $u_{tt} - a(u_x)u_{xx} = 0$ when $a > 0$, $a(v) = O(v^p)$ near 0, $p \geq 1$.

Our proof overcomes these two difficulties by means of a modified Newton's method in scales of Banach spaces, that is a Nash-Moser method. At each step of the Newton's iteration we impose some "non-resonance conditions" on the parameter ω to control small divisors. For these non-resonance frequencies we can invert the linearised operator, which is a perturbation of the d'Alembertian, losing some amount of regularity. In this way we construct inductively a sequence of approximate solutions. The loss of regularity, which occurs at each step of the iteration, is overcome thanks to smoothing operators and to the high speed of convergence of the quadratic scheme.

The application of Nash-Moser methods to infinite-dimensional dynamical systems having small divisors problems has been introduced in the Nineties by Craig, Wayne and Bourgain, in analytic or Gevray classes [16, 12, 13, 15]. Further developments are for example in [9, 5, 7, 32].

This technique, combined with Lyapunov-Schmidt reductions, is a flexible alternative with respect to KAM procedures [38, 25, 34]. In particular, currently available KAM methods seem not to apply to the quasi-linear problem (1).

Since we deal with not only analytic, but also finite order regularity, the scheme we use here differs from that in [9, 5] and it does not rely on analyticity assumptions. Such a procedure goes back directly to ideas of the original methods of [29, 30, 39] and it is developed in [7]. Recently [8] this technique has made possible to prove the existence of periodic solutions of nonlinear wave equations for nonlinearities having only C^k differentiability. We point out that some of the difficulties of [8] are not present here, thanks to the special symmetry of the Kirchhoff nonlinearity. Moreover, the roles played here by space and time are inverted with respect to [8, 9, 5].

We remark that small divisors problems become more difficult in higher dimension. For this reason, not many works deal with such problems when the dimension is larger than one (e.g. [11, 12]). In that case, indeed, λ_j have a sublinear growth, see (6). In general this causes further difficulties in the inversion

of the linearised operators. In the present case, however, the structure of the Kirchhoff nonlinear integral term makes possible the inversion in any dimension (Section 6).

Finally, we note that in case of periodic boundary conditions (3) zero is an eigenvalue of the Laplacian. As a consequence, we have to solve a space-average equation which is not present in the Dirichlet case (2), see (57) in Section 7.

2 Functional setting and main results

Let $2\pi/\omega$ be the period of g. We look for solutions u with the same period. Normalising the time $t \to \omega t$ and rescaling $u \to \varepsilon^{1/3} u$, (1) becomes

$$(5) \qquad \omega^2 u_{tt} - \Delta u = \mu \Big(\Delta u \int_\Omega |\nabla u|^2 \, dx + g(x,t) \Big)$$

where $\mu := \varepsilon^{2/3}$ and g, u are 2π-periodic.

2.1 Case of Dirichlet boundary conditions

Assume that $\partial\Omega$ is C^∞. Let $\lambda_j^2, \varphi_j(x)$, $j = 1, 2, \ldots$ be the eigenvalues and eigenfunctions of the boundary-value problem

$$\begin{cases} -\Delta\varphi_j = \lambda_j^2 \, \varphi_j & \text{in } \Omega \\ \varphi_j = 0 & \text{on } \partial\Omega \end{cases}$$

with $\int_\Omega \varphi_j^2 \, dx = 1$ and $\lambda_1 < \lambda_2 \leq \ldots$. Weyl's formula for the asymptotic distribution of the eigenvalues gives $\lambda_j = O(j^{1/d})$ as $j \to \infty$, thus

$$(6) \qquad Cj^{1/d} \leq \lambda_j \leq C'j^{1/d} \quad \forall j = 1, 2, \ldots$$

for some positive C, C' depending on the dimension d and on the domain Ω (see e.g. [36, Vol. IV, XIII.15]).

By expansion in the basis $\{\varphi_j(x)\}$, we define the spaces

$$V_{\sigma,s}(\Omega) := \Big\{ v(x) = \sum_j v_j \varphi_j(x) : \sum_j |v_j|^2 \lambda_j^{2s} \, e^{2\sigma\lambda_j} < \infty \Big\}$$

for $s \geq 0$, $\sigma \geq 0$. Spaces $V_{0,s}$ with $\sigma = 0$ are used in [4]. They are the domains of the fractional powers $\Delta^{s/2}$ of the Laplace operator. See [4, 20] for a characterisation. For instance, $V_{0,2} = H^2(\Omega) \cap H_0^1(\Omega)$. We note that if $u \in V_{0,s}(\Omega)$ then $\Delta^k u \in H_0^1(\Omega)$ for all $0 \leq k \leq (s-1)/2$.

Spaces $V_{\sigma,0}$ with $s = 0$ are used in [3], where it is proved that $\cup_{\sigma>0} V_{\sigma,0}$ is the class of the $(-\Delta)$-analytic functions, that is, by definition, the set of functions $v(x) \in H_0^1(\Omega)$ such that

$$\Delta^k v \in H_0^1(\Omega) \quad \text{and} \quad \Big| \int_\Omega v\Delta^k v \, dx \Big|^{1/2} \leq CA^k k! \qquad \forall k = 0, 1, \ldots$$

for some constants C, A. In [3] it is observed that an important subset of $\cup_{\sigma>0} V_{\sigma,0}$ consists of the functions $v(x)$, analytic on some neighbourhood of $\overline{\Omega}$, such that

$$\Delta^k v = 0 \quad \text{on } \partial\Omega \qquad \forall k = 0, 1, \ldots$$

4

This subset coincides with the whole class of $(-\Delta)$-analytic functions when $\partial\Omega$ is a real analytic manifold of dimension $(d-1)$, leaving Ω on one side [27], or when Ω is a parallelepiped [1].

Clearly $V_{\sigma,s} = \{u \in V_{\sigma,0} : \Delta^{s/2}u \in V_{\sigma,0}\}$ and $V_{\sigma,0} \subset V_{\sigma',s} \subset V_{\sigma',0}$ for all $s > 0$, $\sigma > \sigma' > 0$. Moreover, all finite sums $\sum_{j \leq N} v_j \varphi_j(x)$ belong to $V_{\sigma,s}$ for all σ, s.

We set the problem in the spaces $X_{\sigma,s} = H^1(\mathbb{T}, V_{\sigma,s})$ of 2π-periodic functions $u : \mathbb{T} \to V_{\sigma,s}$, $t \mapsto u(\cdot, t)$ with H^1 regularity, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, namely

$$X_{\sigma,s} := \Big\{ u(x,t) = \sum_{j \geq 1} u_j(t)\varphi_j(x) : u_j \in H^1(\mathbb{T}, \mathbb{R}),$$

$$\|u\|_{\sigma,s}^2 := \sum_{j \geq 1} \|u_j\|_{H^1}^2 \lambda_j^{2s} e^{2\sigma\lambda_j} < \infty \Big\}.$$

Theorem 1. (Case of Dirichlet boundary conditions). *Suppose that $g \in X_{\sigma,s_0}$ for some $\sigma \geq 0$, $s_0 > 2d$. Let $s_1 \in (1 + d, 1 + s_0/2)$. There exist positive constants δ, C with the following properties.*

For every $\gamma \in (0, \lambda_1)$ there exists a Cantor set $\mathcal{A}_\gamma \subset (0, +\infty) \times (0, \delta\gamma)$ of parameters such that for every $(\omega, \mu) \in \mathcal{A}_\gamma$ there exists a classical solution $u(\omega, \mu) \in X_{\sigma,s_1}$ of $(5)(2)$. Such a solution satisfies

$$\|u(\omega, \mu)\|_{\sigma,s_1} \leq \frac{\mu}{\gamma} C, \qquad \|u(\omega, \mu)_{tt}\|_{\sigma,s_1-2} \leq \frac{\mu}{\gamma\omega^2} C$$

and it is unique in the ball $\{\|u\|_{\sigma,s_1} < 1\}$.

The set $\mathcal{A}_\gamma$ satisfies the following Lebesgue measure property: for every $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$ there exists a constant $\bar{C}$ independent on γ such that in the rectangular region $\mathcal{R}_\gamma := (\bar{\omega}_1, \bar{\omega}_2) \times (0, \delta\gamma)$ there holds

$$\frac{|\mathcal{R}_\gamma \cap \mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1 - \bar{C}\gamma.$$

We recall that (5) is obtained from (1) by the normalisation $t \to \omega t$ and the rescaling $u \to \varepsilon^{1/3}u$. Hence, going back, the solution $u(\omega, \mu)$ of (5) found in Theorem 1 gives a solution of (1) of order ε and period $2\pi/\omega$.

Remark 1. Theorem 1 covers both Sobolev and analytic cases:

- *(Sobolev regularity).* If g belongs to the Sobolev space X_{0,s_0}, then the solution u found in the theorem belongs to the Sobolev space X_{0,s_1}.

- *(Analytic regularity).* If g belongs to the analytic space $X_{\sigma_0,0}$, then $g \in X_{\sigma_1,s_0}$ for all $\sigma_1 \in (0, \sigma_0)$. Indeed,

$$\frac{\xi^{s_0}}{\exp[(\sigma_0 - \sigma_1)\xi]} \leq \Big(\frac{s_0}{(\sigma_0 - \sigma_1)e} \Big)^{s_0} =: C \quad \forall \xi \geq 0,$$

therefore

$$\|g\|_{\sigma_1,s_0}^2 = \sum_j \|g_j\|_{H^1}^2 \lambda_j^{2s_0} e^{2\sigma_1\lambda_j} \frac{e^{2\sigma_0\lambda_j}}{e^{2\sigma_0\lambda_j}} \leq C^2 \|g\|_{\sigma_0,0}^2.$$

Since $g \in X_{\sigma_1,s_0}$, the solution u found in the theorem belongs to the analytic space $X_{\sigma_1,s_1} \subset X_{\sigma_1,0}$.

Remark 2. If $g(x, \cdot) \in H^r(\mathbb{T})$, $r \geq 1$, then the solution u of (1) found in the theorem satisfies $u(x, \cdot) \in H^{r+2}(\mathbb{T})$ by bootstrap.

Remark 3. *(Nonplanar vibrations).* We can consider the Kirchhoff equation for a string in the 3-dimensional space

$$(7) \qquad u_{tt} - u_{xx}\left(1 + \int_0^\pi |u_x|^2\, dx\right) = \varepsilon g(x, t), \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where the forcing g and the displacement u are $\mathbb{R}^2$-vectors belonging to the plane orthogonal to the rest position of the string, see [14, 31]. In this case nonplanar vibrations of the string are permitted.

Setting $\|u_j\|_{H^1}^2 := \|u_{1,j}\|_{H^1}^2 + \|u_{2,j}\|_{H^1}^2$ in the definition of the spaces $X_{\sigma, s}$, Theorem 1 holds true for problem (7) as well.

2.2 Case of periodic boundary conditions

The eigenvalues and eigenfunctions of the Laplacian on $\mathbb{T}^d$ are $|m|^2$, $e^{im \cdot x}$ for $m \in \mathbb{Z}^d$. We consider a bijective numbering $\{m_j : j \in \mathbb{N}\}$ of $\mathbb{Z}^d$ such that $|m_j| \leq |m_{j+1}|$ for all $j \in \mathbb{N} = \{0, 1, \ldots\}$, and we denote

$$\tilde{\lambda}_j^2 := |m_j|^2, \quad \tilde{\varphi}_j(x) := e^{im_j \cdot x} \quad \forall j \in \mathbb{N}.$$

We note that $\tilde{\lambda}_0 = 0$, $\tilde{\varphi}_0(x) \equiv 1$ and $\tilde{\lambda}_j \geq 1$ for all $j \geq 1$. Weyl's estimate (6) holds true for $\tilde{\lambda}_j$ as well, because the number of integer vectors $m \in \mathbb{Z}^d$ such that $|m| \leq \lambda$ is $O(\lambda^d)$ for $\lambda \to +\infty$ (see [36, Vol. IV, XIII.15]).

We define

$$\tilde{X}_{\sigma, s} := \Big\{ u(x, t) = \sum_{j \geq 0} u_j(t)\tilde{\varphi}_j(x) : \ u_j \in H^1(\mathbb{T}, \mathbb{R}),$$

$$\|u\|_{\sigma, s}^2 := \|u_0\|_{H^1}^2 + \sum_{j \geq 1} \|u_j\|_{H^1}^2 \tilde{\lambda}_j^{2s} e^{2\sigma \tilde{\lambda}_j} < \infty \Big\}.$$

Theorem 2. (Case of periodic boundary conditions). *Suppose that $g \in \tilde{X}_{\sigma, s_0}$ for some $\sigma \geq 0$, $s_0 > 2d$, and*

$$(8) \qquad \int_{(0, 2\pi)^{d+1}} g(x, t)\, dxdt = 0\,.$$

Let $s_1 \in (1 + d, 1 + s_0/2)$. There exist positive constants δ, C with the following properties.

For every $\gamma \in (0, 1)$ there exists a Cantor set $\mathcal{A}_\gamma \subset (0, +\infty) \times (0, \delta\gamma)$ of parameters such that for every $(\omega, \mu) \in \mathcal{A}_\gamma$ there exists a classical solution $u(\omega, \mu) \in \tilde{X}_{\sigma, s_1}$ of (5)(3) satisfying

$$\int_{(0, 2\pi)^{d+1}} u(\omega, \mu)(x, t)\, dxdt = 0\,.$$

Such a solution satisfies

$$(9) \qquad \|u(\omega, \mu)\|_{\sigma, s_1} \leq \frac{\mu}{\gamma}\left(1 + \frac{1}{\omega^2}\right) C\,, \qquad \|u(\omega, \mu)_{tt}\|_{\sigma, s_1 - 2} \leq \frac{\mu}{\gamma \omega^2} C$$

and it is unique in the ball $\{\int_{(0, 2\pi)^{d+1}} u(x, t)\, dxdt = 0, \ \|u\|_{\sigma, s_1} < 1\}$.

The set $\mathcal{A}_\gamma$ satisfies the same measure property of Theorem 1.

Remark 4. If $u(\omega, \mu)$ is a solution of (5)(3), then also $u(\omega, \mu) + c$, $c \in \mathbb{R}$, solves (5)(3).

2.3 Outline of the proof

The rest of the paper is devoted to the proof of the theorems. In Sections 3,4,5,6 we develop the details for the proof of Theorem 1, then the same calculations are used to prove Theorem 2 in Section 7.

In Section 3 we perform the Nash-Moser iteration to construct the approximating sequence (u_n), for μ small and (ω, μ) belonging to smaller and smaller "non-resonant" sets A_n. Avoiding resonances allows to invert the linearised operator at each step of the iteration.

In Section 4 we prove that u_n converges to a solution of the Kirchhoff equation if $(\omega, \mu) \in A_n$ for all n. Local uniqueness of the solution is also proved.

In Section 5 we prove that the intersection of all A_n is a nonempty set, which is very large in a Lebesgue measure sense.

In Section 6 we prove the invertibility of the linearised operator for $(\omega, \mu) \in A_n$ and we give an estimate on the inverse operator.

In Section 7 we complete the proof of Theorem 1 and we prove Theorem 2.

3 The iteration scheme

We fix $\sigma \geq 0$ once for all. In the following, we write in short $X_s := X_{\sigma,s}$, $\|u\|_s := \|u\|_{\sigma,s}$. We remark that all the following calculations holds true both in the Sobolev case $\sigma = 0$ and in the analytic case $\sigma > 0$. Indeed, the only index used in the present Nash-Moser method is s.

We set the iterative scheme in the Banach spaces X_s endowed with the smoothing operators P_n, defined in the following way. We consider a constant $\chi \in (1, 2)$ and denote

$$(10) \qquad\qquad N_n := \exp(\chi^n)$$

for all $n \in \mathbb{N}$. We define the finite-dimensional space

$$X^{(n)} := \Big\{ u(x, t) = \sum_{\lambda_j \leq N_n} u_j(t) \varphi_j(x) \Big\}$$

and indicate P_n the projector onto $X^{(n)}$ (truncation operator). For all $s, \alpha \geq 0$ there holds the smoothing properties

$$(11) \qquad\qquad \|P_n u\|_{s+\alpha} \ \leq \ N_n^\alpha \|u\|_s \quad \forall u \in X_s$$
$$(12) \qquad\qquad \|(I - P_n)u\|_s \ \leq \ N_n^{-\alpha} \|u\|_{s+\alpha} \quad \forall u \in X_{s+\alpha}$$

where I is the identity map. We denote

$$L_\omega := \omega^2 \partial_{tt} - \Delta, \quad f(u) := \Delta u \int_\Omega |\nabla u|^2 \, dx,$$
$$F(u) := L_\omega u - \mu f(u) - \mu g$$

so that (5) can be written as

$$(13) \qquad\qquad F(u) = 0.$$

Note that f is not a composition operator, because of the presence of the integral. The map f is cubic: indeed $f(u) = A[u, u, u]$ where A is the three-linear map $A[u, v, w] = \Delta u \int_\Omega \nabla v \circ \nabla w \, dx$. Moreover, since the integral term $\int_\Omega |\nabla u|^2 dx$ depends only on time, there holds

$$f(u) \in X^{(n)} \quad \forall u \in X^{(n)}.$$

The quadratic remainder of f at u is

(14) $\quad Q(u, h) := f(u + h) - f(u) - f'(u)[h]$

$$= \Delta u \int_\Omega |\nabla h|^2 dx + \Delta h \int_\Omega \left(2\nabla u \circ \nabla h + |\nabla h|^2\right) dx.$$

We observe that, if $a(t)$ depends only on time, then

$$\|a(t)u(x, t)\|_s \leq \|a\|_{H^1} \|u\|_s$$

(we omit a factor given by the algebra constant of $H^1(\mathbb{T})$). As a consequence, by Hölder inequality it is easy to estimate $\|f(u)\|_s$, $\|f'(u)[h]\|_s$ and $\|Q(u, h)\|_s$.

We adapt the Newton's scheme with smoothing operators P_n to the special structure of problem (5). We will construct a sequence (u_n) defining

(15) $\quad u_0 := 0, \quad u_{n+1} := u_n - F'(u_n)^{-1}[L_\omega u_n - \mu f(u_n) - \mu P_{n+1} g]$

provided the linearised operator

$$F'(u_n) : h \mapsto L_\omega h - \mu f'(u_n)[h]$$

admits a bounded inverse $F'(u_n)^{-1}$ on $X^{(n+1)}$. In this inversion problem a small divisors difficulty arises. We will prove (Lemma 1) that $F'(u_n)$ can be inverted if the parameters (ω, μ) belong to some "nonresonant" set A_{n+1}, defined as follows. First,

$$A_0 := (0, +\infty) \times (0, 1).$$

By induction, suppose we have constructed A_n and u_n. We denote

(16) $$a_n(t) := \int_\Omega |\nabla u_n|^2 \, dx,$$

we consider the Hill's eigenvalue problem

$$\begin{cases} y'' + p^2 \left(1 + \mu a_n(t)\right) y = 0 \\ y(t) = y(t + 2\pi) \end{cases}$$

and indicate $(p_l^{(n)})^2$ its eigenvalues, $l \in \mathbb{N}$. For

$$\tau > d, \quad \gamma \in (0, \lambda_1)$$

we define

(17) $\quad A_{n+1} := \left\{ (\omega, \mu) \in A_n \ : \ |\omega p_l^{(n)} - \lambda_j| > \dfrac{\gamma}{\lambda_j^\tau} \quad \forall \lambda_j \leq N_{n+1}, \ l \in \mathbb{N} \right\}.$$

Remark 5. Note that for all μ, n the set $A_n(\mu) := \{\omega : (\omega, \mu) \in A_n\}$ is open. Indeed, for every $0 < \bar\omega_1 < \bar\omega_2 < \infty$ the intersection $(\bar\omega_1, \bar\omega_2) \cap A_n(\mu)$ is defined by means of finitely many strict inequalities (see (6) and (51)).

We fix a positive constant R such that, if $u \in X_1$ and $\|u\|_1 < R$, then $a(t) := \int_\Omega |\nabla u|^2 \, dx$ satisfies $\|a\|_{H^1} < 1$ and $\|a\|_\infty < 1/2$.

Lemma 1. (Inversion of the linearised operator). *There exist two universal constants K_1, K_1' with the following property. Let $u \in X^{(n)}$ with $\|u\|_1 < R$. Let $(\omega, \mu) \in A_{n+1}$. If*

$$(18) \qquad \frac{\mu}{\gamma} \|u\|_{\tau+1}^2 < K_1',$$

then $F'(u)$ is invertible, $F'(u)^{-1} : X^{(n+1)} \to X^{(n+1)}$ and

$$(19) \qquad \|F'(u)^{-1} h\|_0 \leq \frac{K_1}{\gamma} \|h\|_{\tau-1} \quad \forall h \in X^{(n+1)}.$$

Proof. In Section 6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Lemma 2. (Construction of the approximating sequence). *Let $g \in Y_{s_0}$, $s_0 > 2d$. Let $\tau \in (d, s_0/2)$. There exist a choice for χ in the definition (10) and positive constants K, b, δ_0, with $b(2 - \chi) > \tau + 1$, satisfying the following properties.*
(First step). If $(\omega, \mu) \in A_1$ and $\mu/\gamma < \delta_0$, then there exists $u_1 \in X^{(0)}$ defined by (15), and there holds

$$(20) \qquad \|u_1\|_0 < K \frac{\mu}{\gamma} \exp(-b\chi).$$

(Induction step). Suppose we have constructed $u_1, \ldots, u_n$ by (15) for $(\omega, \mu) \in A_n$, $n \geq 1$, where each A_{k+1} is defined by means of u_k by (17), and $u_k \in X^{(k)}$. Suppose that $\mu/\gamma < \delta_0$. Let

$$h_{k+1} := u_{k+1} - u_k.$$

Suppose that for all $k = 1, \ldots, n$ there holds

$$(21) \qquad \|h_k\|_0 < K \frac{\mu}{\gamma} \exp(-b\chi^k).$$

If $(\omega, \mu) \in A_{n+1}$ then there exists $h_{n+1} \in X^{(n+1)}$ defined by (15) and there holds

$$(22) \qquad \|h_{n+1}\|_0 < K \frac{\mu}{\gamma} \exp(-b\chi^{n+1}).$$

Proof. *(First step).* Since $u_0 = 0$ and $(\omega, \mu) \in A_1$, by Lemma 1 $F'(0)$ is invertible and (15) defines

$$(23) \qquad u_1 = -F'(0)^{-1}[F(0) + \mu(I - P_1)g] = \mu L_\omega^{-1} P_1 g.$$

By (19), the inequality (20) holds true provided

$$(24) \qquad K_1 \|g\|_{s_0} < K \exp(-b\chi), \quad \tau - 1 \leq s_0.$$

9

(Induction step). To define h_{n+1} by (15), we have to verify the hypotheses of Lemma 1. By (11) and (21)

$$(25) \qquad \|h_k\|_{\tau+1} \leq N_k^{\tau+1} \|h_k\|_0 \leq K \frac{\mu}{\gamma} \exp[(-b+\tau+1)\chi^k]$$

because $h_k \in X^{(k)}$. Then

$$(26) \qquad \|u_n\|_{\tau+1} \leq \sum_{k=1}^{n} \|h_k\|_{\tau+1} < K \frac{\mu}{\gamma} \sum_{k=1}^{+\infty} \exp[(-b+\tau+1)\chi^k]$$

which is finite for $b > \tau + 1$. Thus condition (18) is verified provided

$$(27) \qquad b > \tau + 1, \quad K^2 \left(\frac{\mu}{\gamma}\right)^3 C_0^2 < K_1', \quad C_0 := \sum_{k \geq 1} \exp[(-b+\tau+1)\chi^k].$$

Since $\|u_n\|_1 \leq \|u_n\|_{\tau+1}$, by (26) we have $\|u_n\|_1 < R$ provided

$$(28) \qquad K \frac{\mu}{\gamma} C_0 < R.$$

Since $(\omega, \mu) \in A_{n+1}$, we can apply Lemma 1 and we define h_{n+1} according to the scheme, namely

$$(29) \qquad h_{n+1} := -F'(u_n)^{-1}[F(u_n) + \mu(I - P_{n+1})g].$$

By (19) we have

$$(30) \qquad \|h_{n+1}\|_0 \leq \frac{K_1}{\gamma} \|F(u_n) + \mu(I - P_{n+1})g\|_{\tau-1}.$$

By construction (15), u_n satisfies

$$F'(u_{n-1})h_n = -L_\omega u_{n-1} + \mu f(u_{n-1}) + \mu P_n g.$$

By Taylor expansion $F(u_n) = F(u_{n-1}) + F'(u_{n-1})h_n - \mu Q(u_{n-1}, h_n)$, where Q is defined in (14). Thus

$$(31) \qquad F(u_n) = -\mu[(I - P_n)g + Q(u_{n-1}, h_n)]$$

and (30) gives

$$(32) \qquad \|h_{n+1}\|_0 \leq \frac{K_1 \mu}{\gamma} \|(P_{n+1} - P_n)g + Q(u_{n-1}, h_n)\|_{\tau-1}.$$

Now $(P_{n+1} - P_n)g = (I - P_n)P_{n+1}g$, then by (12)

$$\|(P_{n+1} - P_n)g\|_{\tau-1} \leq \frac{1}{N_n^\beta} \|P_{n+1}g\|_{\tau-1+\beta} \leq \frac{1}{N_n^\beta} \|g\|_{s_0}$$

for

$$\tau - 1 + \beta \leq s_0, \quad \beta > 0.$$

To estimate $\|Q(u_{n-1}, h_n)\|_{\tau-1}$, we note that

$$\left\|\Delta u_{n-1} \int_\Omega |\nabla h_n|^2\right\|_{\tau-1} \leq \|u_{n-1}\|_{\tau+1} \|h_n\|_1^2 < K \frac{\mu}{\gamma} C_0 \|h_n\|_1^2$$

10

by (25), and $\|h_n\|_1 \le N_n \|h_n\|_0$ by (11). For the second term, recalling that $2 < \tau + 1$,

$$
\left\| \Delta h_n \int_\Omega \nabla(2u_{n-1} + h_n) \circ \nabla h_n \right\|_{\tau-1} \le \|h_n\|_{\tau+1} \|h_n\|_0 \|2u_{n-1} + h_n\|_2
$$
$$
< 2K \frac{\mu}{\gamma} C_0 \|h_n\|_{\tau+1} \|h_n\|_0
$$

by (25), and $\|h_n\|_{\tau+1} \le N_n^{\tau+1} \|h_n\|_0$ by (11). Then

$$
\|Q(u_{n-1}, h_n)\|_{\tau-1} < 3K \frac{\mu}{\gamma} C_0 N_n^{\tau+1} \|h_n\|_0^2 .
$$

As a consequence, (22) holds true provided

$$
(33) \qquad\qquad K_1 \frac{1}{N_n^\beta} \|g\|_{s_0} < \frac{1}{2} K \exp(-b\chi^{n+1})
$$

and

$$
(34) \qquad\qquad 3K_1 \frac{\mu}{\gamma} C_0 N_n^{\tau+1} \|h_n\|_0^2 < \frac{1}{2} \exp(-b\chi^{n+1}) .
$$

Condition (33) is satisfied for

$$
(35) \qquad\qquad \beta > b\chi, \qquad K > \frac{2K_1 \|g\|_{s_0}}{\exp[(\beta - b\chi)\chi]}
$$

and, by (21), condition (34) is satisfied for

$$
(36) \qquad b(2-\chi) > \tau + 1, \qquad \frac{\mu}{\gamma} < \left\{ \frac{\exp[(b(2-\chi) - \tau - 1)\chi]}{6K_1 C_0 K^2} \right\}^{1/3} .
$$

Since $2\tau < s_0$, we can fix $\chi \in (1, 2)$ so close to 1 that

$$
\tau - 1 + (\tau + 1)\frac{\chi}{2 - \chi} < s_0 .
$$

Now we fix b such that

$$
b(2-\chi) > \tau + 1 , \qquad \tau - 1 + b\chi < s_0
$$

and then we fix β as

$$
\beta = s_0 - \tau + 1 .
$$

So (35) and (24) are satisfied for K big enough, and we fix K in such a way. Then (27),(28) and (36) are satisfied for μ/γ small enough. $\qquad\square$

4   The solution

Lemma 3. (Existence of a solution). *Assume the hypotheses of Lemma 2 and suppose that $(\omega, \varepsilon) \in A_n$ for all $n \in \mathbb{N}$. Then the sequence (u_n) constructed in Lemma 2 converges in $X_{\tau+1}$ to $u_\infty := \sum_{k \ge 1} h_k$. u_∞ is a solution of (13) and*

$$
(37) \qquad\qquad \|u_\infty\|_{\tau+1} \le \frac{\mu}{\gamma} C
$$

for some C. Moreover, $(u_n)_{tt}$ *converges to* $(u_\infty)_{tt}$ *in* $X_{\tau-1}$,

$$(38) \qquad \|(u_\infty)_{tt}\|_{\tau-1} \le \frac{\mu}{\gamma\omega^2}\, C$$

so u_∞ *is a classical solution of* (5).

Proof. By (25), the series $\sum_k \|h_k\|_{\tau+1}$ converges, u_n converges to u_∞ in $X_{\tau+1}$ and (37) holds true.

By (23) there holds $\omega^2(h_1)_{tt} = \Delta h_1 + \mu P_0 g$. By (11) and (21)

$$\|\Delta h_1\|_{\tau-1} + \mu\|P_0 g\|_{\tau-1} \le \frac{K\mu}{\gamma}\, \exp[(-b+\tau+1)\chi] + \mu\|g\|_{s_0},$$

so that

$$(39) \qquad \|(h_1)_{tt}\|_{\tau-1} \le C\,\frac{\mu}{\gamma\omega^2}$$

for some C (recall that $\gamma < \lambda_1$). For $n \ge 1$, by (29) and (31)

$$(40) \qquad F'(u_n)h_{n+1} = \mu[(P_{n+1} - P_n)g + Q(u_{n-1}, h_n)],$$

thus

$$\omega^2(h_{n+1})_{tt} = \Delta h_{n+1} + \mu\big(f'(u_n)[h_{n+1}] + (P_n - P_{n-1})g + Q(u_{n-1}, h_n)\big).$$

By (25),(32),(33) and (34) we get

$$(41) \qquad \|(h_{n+1})_{tt}\|_{\tau-1} \le C\,\frac{\mu}{\gamma\omega^2}\, \exp[(-b+\tau+1)\chi^{n+1}].$$

It follows that $(u_n)_{tt}$ converges in $X_{\tau-1}$, $(u_\infty)_{tt} \in X_{\tau-1}$, so that u_∞ has regularity $H^3 \subset C^2$ in time and (38) holds true. As a consequence $F(u_n)$ converges to $F(u_\infty)$ in $X_{\tau-1}$.

On the other hand, by (31),(33) and (34)

$$\|F(u_n)\|_{\tau-1} < \frac{K\mu}{K_1}\, \exp(-b\chi^{n+1}),$$

and $F(u_n) \to 0$ in $X_{\tau-1}$. Then $F(u_\infty) = 0$. $\qquad\square$

Remark 6. We will prove in Lemma 6 that the set $\{(\omega, \mu) \in A_n \ \forall n \in \mathbb{N}\}$ is nonempty and has positive, large measure. As a consequence, the sequence (u_n) of Lemma 3 is defined for all (ω, μ) in that large set.

Lemma 4. (Uniqueness of the solution). *Assume the hypotheses of Lemma 3. There exists* $\delta_1 \in (0, \delta_0]$ *such that, for* $\mu/\gamma < \delta_1$, u_∞ *is the unique solution of* (13) *in the ball* $\{v \in X_{\tau+1} : \|v\|_{\tau+1} < 1\}$.

Proof. Suppose v is another solution of (13), with $\|v\|_{\tau+1} < 1$. Let $v_n := P_n v$. Projecting the equation $F(v) = 0$ on $X^{(n)}$ gives

$$L_\omega v_n = \mu(f(v_n) + R_n(v) + P_n g), \qquad R_n(v) := \Delta v_n \int_\Omega |\nabla(v - v_n)|^2\, dx.$$

Since u_n solves (15), that is

$$F'(u_n)h_{n+1} = -L_\omega u_n + \mu(f(u_n) + P_{n+1}g),$$

the difference $w_n := v_n - u_n$ satisfies

$$L_\omega w_n - \mu(f(v_n) - f(u_n)) - \mu R_n(v) + \mu(P_{n+1} - P_n)g = F'(u_n)h_{n+1}.$$

Since $f(v_n) - f(u_n) = f'(u_n)[w_n] + Q(u_n, w_n)$, applying $F'(u_n)^{-1}$

(42) $\qquad w_n = h_{n+1} + \mu F'(u_n)^{-1}[Q(u_n, w_n) + R_n(v) - (P_{n+1} - P_n)g].$

Now, by (19)

$$\|\mu F'(u_n)^{-1}Q(u_n, w_n)\|_0 \le K_1 \frac{\mu}{\gamma} \|Q(u_n, w_n)\|_{\tau-1}.$$

By assumption and (37)

$$\|w_n\|_2 \le \|w_n\|_{\tau+1} \le \|v\|_{\tau+1} + \|u_\infty\|_{\tau+1} < C$$

and by (25) $\|u_n\|_2 \le \|u_n\|_{\tau+1} < C$, so that $\|Q(u_n, w_n)\|_{\tau-1} \le C\|w_n\|_0$ and

$$\|\mu F'(u_n)^{-1}Q(u_n, w_n)\|_0 \le K_1 \frac{\mu}{\gamma} C \|w_n\|_0 \le \frac{1}{2} \|w_n\|_0$$

provided μ/γ is small enough. Thus (42) gives

$$\frac{1}{2} \|w_n\|_0 \le \|h_{n+1}\|_0 + K_1 \frac{\mu}{\gamma} \|R_n(v) - (P_{n+1} - P_n)g\|_{\tau-1}.$$

By (22) and (12) the right-hand side tends to 0 as $n \to \infty$, so that $\|v_n - u_n\|_0 \to 0$. Since v_n converges to v and u_n to u_∞ in X_0, it follows that $v = u_\infty$. $\qquad \square$

5 The Cantor set of parameters

Lemma 5. (Regular dependence on the parameter ω). *Assume the hypotheses of Lemma 2. There exist $\delta_2 \in (0, \delta_1]$ such that all the maps*

$$h_n \; : \; A_n \cap \{(\omega, \mu) : \mu/\gamma < \delta_2\} \to X^{(n)}, \quad (\omega, \mu) \mapsto h_n(\omega, \mu)$$

are differentiable w.r.t. ω and

(43) $$\|\partial_\omega u_n\|_0 \le C \frac{\mu}{\gamma^2 \omega}$$

for some C.

Proof. $u_1 = h_1 \in X^{(1)}$ is defined for $(\omega, \mu) \in A_1$ and it solves $\omega^2(h_1)_{tt} = \Delta h_1 + \mu P_0 g$. Recalling Remark 5 and Lemma 1, by the classical implicit function theorem it follows that h_1 is differentiable w.r.t. ω. Differentiating w.r.t. ω gives

$$2\omega(h_1)_{tt} + L_\omega[\partial_\omega h_1] = 0.$$

We apply L_ω^{-1} and by (19) and (39)

$$\|\partial_\omega h_1\|_0 \le C \frac{\mu}{\gamma^2 \omega}$$

13

for some C.

Assume that for $n \geq 1$

(44) $\qquad \|\partial_\omega h_k\|_0 \leq \bar{C} \dfrac{\mu}{\gamma^2 \omega} \exp[(-b + \tau + 1)\chi^k] \quad \forall k = 1, \ldots, n.$

h_{n+1} solves (40), then it is differentiable w.r.t. ω. Differentiating (40) gives

$$F'(u_n)[\partial_\omega h_{n+1}] = -2\omega(h_{n+1})_{tt} + \mu f''(u_n)[\partial_\omega u_n, h_{n+1}] + \mu \partial_\omega(Q(u_{n-1}, h_n)).$$

We apply $F'(u_n)^{-1}$ and observe that

$$\|F'(u_n)^{-1}[\omega(h_{n+1})_{tt}]\|_0 \leq \dfrac{K_1 \omega}{\gamma} \|(h_{n+1})_{tt}\|_{\tau-1} \leq \dfrac{C\mu}{\gamma^2 \omega} \exp[(-b + \tau + 1)\chi^{n+1}]$$

by (41). To estimate

(45) $\qquad K_1 \dfrac{\mu}{\gamma} \|f''(u_n)[\partial_\omega u_n, h_{n+1}] + \partial_\omega(Q(u_{n-1}, h_n))\|_{\tau-1}$

we write all the integral terms and apply (11) and (44) to each of them. We write here the calculations for two terms, the other ones are analogous. First,

(46) $\qquad \|\Delta(\partial_\omega u_n) \displaystyle\int_\Omega \nabla u_n \circ \nabla h_{n+1}\|_{\tau-1} \leq \|\partial_\omega u_n\|_{\tau+1} \|u_n\|_2 \|h_{n+1}\|_0$

and $\|\partial_\omega u_n\|_{\tau+1} \leq \|\partial_\omega u_n\|_0 N_n^{\tau+1}$,

$$\|\partial_\omega u_n\|_0 \leq \sum_{k=1}^n \|\partial_\omega h_k\|_0 \leq C \dfrac{\mu}{\gamma^2 \omega} \sum_{k \geq 1} \exp[(-b + \tau + 1)\chi^k] = \dfrac{C'\mu}{\gamma^2 \omega},$$

$$\|u_n\|_2 \leq \|u_n\|_{\tau+1} < C\mu/\gamma \quad \text{by (25)},$$

$$N_n^{\tau+1} \|h_{n+1}\|_0 \leq \dfrac{K\mu}{\gamma} \exp[(-b + \tau + 1)\chi^{n+1}] \quad \text{by (22)},$$

so that $(46) \leq C(\mu/\gamma)^2 (\mu/\gamma^2 \omega) \exp[(-b + \tau + 1)\chi^{n+1}]$. As second example,

$$\|\Delta h_{n+1} \displaystyle\int_\Omega \nabla u_n \circ \nabla(\partial_\omega u_n)\|_{\tau-1} \leq \|h_{n+1}\|_0 N_{n+1}^{\tau+1} \|u_n\|_2 \|\partial_\omega u_n\|_0.$$

At the end we have

$$(45) \leq C \left(\dfrac{\mu}{\gamma}\right)^3 \dfrac{\mu}{\gamma^2 \omega} \exp[(-b + \tau + 1)\chi^{n+1}]$$

for some C, so that (44) holds true for $k = n + 1$ provided μ/γ is small enough, independently on n.

Finally, $\|\partial_\omega u_n\|_0 \leq \sum_{k=1}^n \|\partial_\omega h_k\|_0$ and (44) implies (43). $\qquad \square$

Lemma 6. (The Cantor set). *There exist $\delta_3 \leq \delta_2$ such that the Cantor set $\mathcal{A}_\gamma := A_\infty \cap \{(\omega, \mu) : \mu < \delta_3 \gamma\}$, $\gamma \in (0, \lambda_1)$, has the following measure property.*

For every interval $I = (\bar{\omega}_1, \bar{\omega}_2)$ with $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$ there is a constant $\bar{C}$ depending on I such that, denoted by $\mathcal{R}_\gamma$ the rectangular region $\mathcal{R}_\gamma = I \times (0, \delta_3 \gamma)$,

$$\dfrac{|\mathcal{R}_\gamma \cap \mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1 - \bar{C}\gamma.$$

14

Proof. We fix μ, we recall that $A_n(\mu) := \{\omega : (\omega, \mu) \in A_n\}$ and define

$$E_n := A_n(\mu) \setminus A_{n+1}(\mu), \quad n \in \mathbb{N}.$$

We have to prove that $\cup_{n \in \mathbb{N}} E_n$ has small measure. As a consequence, its complementary set $A_\infty(\mu) := \cap_{n \in \mathbb{N}} A_n(\mu)$ will be a large set. Let

$$\Omega_{j,l}^n := \left\{ \omega : |\omega p_l^{(n)}(\omega, \mu) - \lambda_j| \leq \frac{\gamma}{\lambda_j^\tau} \right\}.$$

We note that $\Omega_{j,0}^n = \emptyset$ for all j, n because $\gamma < \lambda_1$ and $p_l^{(n)} = 0$ for $l = 0$. Suppose that $\omega \in \Omega_{j,l}^n$. Then $C\lambda_j < \omega l < C'\lambda_j$ for some C, C' by (51). Moreover

$$|\partial_\omega p_l^{(n)}| \leq 2l\mu\|\partial_\omega a_n\|_\infty$$

by (52). Recalling the definition (16) of a_n, by (43) and (25)

$$\|\partial_\omega a_n\|_{H^1} = \|2 \int_\Omega \nabla u_n \circ \nabla(\partial_\omega u_n)\, dx\|_{H^1} \leq 2\|u_n\|_2\|\partial_\omega u_n\|_0 \leq C \frac{\mu^2}{\gamma^3 \omega}$$

for some C, so that

$$|\partial_\omega p_l^{(n)}| \leq C\, l\, \frac{\mu^3}{\gamma^3 \omega}.$$

By (51) it follows that

$$(47) \qquad \partial_\omega(\omega\, p_l^{(n)}(\omega, \mu)) \geq p_l^{(n)} - \omega C\, l\, \frac{\mu^3}{\gamma^3 \omega} > \frac{l}{4}$$

provided μ/γ is small enough, say $\mu/\gamma < \delta_3$. Fix $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$. If $\Omega_{j,l}^n \cap (\bar{\omega}_1, \bar{\omega}_2)$ is nonempty, then

$$(48) \qquad |\Omega_{j,l}^n| < \frac{8\gamma}{l\lambda_j^\tau} < C\bar{\omega}_2\, \frac{\gamma}{\lambda_j^{\tau+1}}, \quad l \in \left(\frac{C'}{\bar{\omega}_2}\, \lambda_j\, , \, \frac{C''}{\bar{\omega}_1}\, \lambda_j \right) =: \Lambda(j)$$

for some C, C', C''. Since $E_0 = \cup_{\lambda_j \leq N_1,\, l \geq 1} \Omega_{j,l}^0$, we have the estimate

$$(49) \qquad |E_0 \cap (\bar{\omega}_1, \bar{\omega}_2)| \leq \sum_{\lambda_j \leq N_1} \sum_{l \in \Lambda(j)} |\Omega_{j,l}^0| < \gamma\tilde{C} \sum_{\lambda_j \leq N_1} \frac{1}{\lambda_j^\tau}$$

for some $\tilde{C}$ depending on $(\bar{\omega}_1, \bar{\omega}_2)$.

To estimate $|E_n \cap (\bar{\omega}_1, \bar{\omega}_2)|$, $n \geq 1$, we notice that

$$E_n = \bigcup_{\lambda_j \leq N_{n+1},\, l \geq 1} \Omega_{j,l}^n \cap A_n(\mu).$$

For the sets $\Omega_{j,l}^n$ with $N_n < \lambda_j \leq N_{n+1}$ we use (48) and we get

$$\left| \bigcup_{\substack{N_n < \lambda_j \leq N_{n+1} \\ l \in \Lambda(j)}} \Omega_{j,l}^n \cap (\bar{\omega}_1, \bar{\omega}_2) \right| < \gamma\tilde{C} \sum_{N_n < \lambda_j \leq N_{n+1}} \frac{1}{\lambda_j^\tau}$$

15

where $\tilde{C}$ is the constant of (49). To estimate the remaining sets, suppose that $\omega \in \Omega_{j,l}^n$ for some $\lambda_j \le N_n$, $l \ge 1$. Then by (52)

$$\begin{aligned} |\lambda_j - \omega p_l^{(n-1)}| &\le |\lambda_j - \omega p_l^{(n)}| + \omega |p_l^{(n)} - p_l^{(n-1)}| \\ &\le \frac{\gamma}{\lambda_j^\tau} + 2\omega l \mu \|a_n - a_{n-1}\|_\infty . \end{aligned}$$

Since $\omega l \le C\lambda_j$ and $\|a_n - a_{n-1}\|_{H^1} \le \|h_n\|_0 \|2u_{n-1} + h_n\|_2$, by (21),(25) we have

$$|\lambda_j - \omega p_l^{(n-1)}| \le \frac{\gamma}{\lambda_j^\tau} + C\lambda_j \frac{\mu^3}{\gamma^2} \exp(-b\chi^n).$$

Thus

$$\Omega_{j,l}^n \cap A_n(\mu) \subseteq \left\{ \omega : \frac{\gamma}{\lambda_j^\tau} < |\lambda_j - \omega p_l^{(n-1)}| \le \frac{\gamma}{\lambda_j^\tau} + C\lambda_j \frac{\mu^3}{\gamma^2} \exp(-b\chi^n) \right\}$$

and by (47)

$$|\Omega_{j,l}^n \cap A_n(\mu)| \le C\bar{\omega}_2 \frac{\mu^3}{\gamma^2} \exp(-b\chi^n) .$$

It follows that

$$\left| \bigcup_{\substack{\lambda_j \le N_n \\ l \in \Lambda(j)}} \Omega_{j,l}^n \cap A_n(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2) \right| \le \gamma C \frac{\mu^3}{\gamma^3} \sum_{\lambda_j \le N_n} \lambda_j \exp(-b\chi^n)$$

for some C depending on $(\bar{\omega}_1, \bar{\omega}_2)$. Now, by (6) $\lambda_j \le N_n$ implies $j \le (N_n/C)^d$, then

$$\sum_{\lambda_j \le N_n} \lambda_j \le \sum_{j \le (N_n/C)^d} C' j^{1/d} \le C'' \int_0^{(N_n/C)^d} \xi^{1/d} d\xi \le C''' N_n^{d+1}$$

for some C'''. As a consequence

$$\sum_{\lambda_j \le N_n} \lambda_j \exp(-b\chi^n) \le C \exp[(-b + d + 1)\chi^n].$$

Then

$$|E_n \cap (\bar{\omega}_1, \bar{\omega}_2)| \le \gamma C \left(\sum_{N_n < \lambda_j \le N_{n+1}} \frac{1}{\lambda_j^\tau} + \exp[(-b + d + 1)\chi^n] \right)$$

for some C, and

$$\left| \bigcup_{n \in \mathbb{N}} E_n \cap (\bar{\omega}_1, \bar{\omega}_2) \right| \le \gamma C \left(\sum_{j \ge 1} \frac{1}{\lambda_j^\tau} + \sum_{n \in \mathbb{N}} \exp[(-b + d + 1)\chi^n] \right) .$$

The first series converges because by (6)

$$\sum_{j \ge 1} \frac{1}{\lambda_j^\tau} \le C \sum_{j \ge 1} \frac{1}{j^{\tau/d}} < \infty$$

16

being $\tau > d$. The second series converges because $b > d + 1$. Thus

$$\Big| \bigcup_{n \in \mathbb{N}} E_n \cap (\bar{\omega}_1, \bar{\omega}_2) \Big| \le C\gamma$$

and the relative measure of $A_\infty(\mu)$ in $(\bar{\omega}_1, \bar{\omega}_2)$ satisfies

$$\frac{|A_\infty(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2)|}{\bar{\omega}_2 - \bar{\omega}_1} \ge 1 - C\gamma$$

for some C.

Finally, we integrate in μ in the interval where $\mu/\gamma < \delta_3$,

$$|A_\infty \cap R_\gamma| = \int_0^{\delta_3 \gamma} |A_\infty(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2)| \, d\mu \,.$$

$\square$

6 Inversion of the linearised operator

In this section we prove Lemma 1. Let $u \in X^{(n)}, h \in X^{(n+1)}$. The linearised operator is

$$F'(u)h = L_\omega h - \mu f'(u)[h] = Dh + Sh$$

where we split $F'(u)$ in a diagonal part

$$Dh := \omega^2 h_{tt} - \Delta h \Big(1 + \mu \int_\Omega |\nabla u|^2 \, dx \Big)$$

and a "projection" part

$$Sh := -\mu \Delta u \int_\Omega 2 \nabla u \circ \nabla h \, dx \,.$$

We recall here some results on Hill's problems. The proof is in the Appendix.

Lemma 7. (Hill's problems). *Let $\alpha(t)$ be 2π-periodic and $\|\alpha\|_\infty < 1/2$. The eigenvalues p^2 of the periodic problem*

$$(50) \qquad \begin{cases} y'' + p^2(1 + \alpha(t)) \, y = 0 \\ y(t) = y(t + 2\pi) \end{cases}$$

form a sequence $\{p_l^2\}_{l \in \mathbb{N}}$ such that

$$(51) \qquad \frac{1}{3} l \le p_l \le 2l \quad \forall l \in \mathbb{N}.$$

For α, β 2π-periodic, $\|\alpha\|_\infty, \|\beta\|_\infty < 1/2$,

$$(52) \qquad \big| p_l(\alpha) - p_l(\beta) \big| \le 2l \, \|\alpha - \beta\|_\infty \quad \forall l \in \mathbb{N}.$$

The eigenfunctions $\psi_l(t)$ of (50) form an orthonormal basis of $L^2(\mathbb{T})$ w.r.t. the scalar product

$$(u, v)_{L_\alpha^2} = \int_0^{2\pi} uv(1 + \alpha) \, dt$$

17

and also an orthogonal basis of $H^1(\mathbb{T})$ w.r.t. the scalar product

$$(u,v)_{H^1_\alpha} = \int_0^{2\pi} u'v'\, dt + (u,v)_{L^2_\alpha}.$$

The corresponding norms are equivalent to the standard Sobolev norms,

(53) $\qquad \dfrac{1}{2}\,\|y\|_{L^2} \le \|y\|_{L^2_\alpha} \le 2\,\|y\|_{L^2}, \quad \dfrac{1}{2}\,\|y\|_{H^1} \le \|y\|_{H^1_\alpha} \le 2\,\|y\|_{H^1}.$

Lemma 8. (Inversion of D). *Let $u \in X^{(n)}$, $a(t) := \int_\Omega |\nabla u|^2\, dx \in H^1(\mathbb{T})$, $\|a\|_{H^1} < 1$, $\|a\|_\infty < 1/2$. Let p_l^2 be the eigenvalues of the Hill's problem*

(54)
$$\begin{cases} y'' + p^2(1 + \mu a(t))\, y = 0 \\ y(t) = y(t + 2\pi). \end{cases}$$

If (ω, μ) satisfy the non-resonant condition

$$|\omega p_l - \lambda_j| > \frac{\gamma}{\lambda_j^\tau} \quad \forall \lambda_j \le N_{n+1}, \;\; l \in \mathbb{N},$$

then D is invertible, $D^{-1} : X^{(n+1)} \to X^{(n+1)}$ and

(55) $\qquad\qquad \|D^{-1}h\|_0 \le \dfrac{C}{\gamma}\,\|h\|_{\tau-1} \quad \forall h \in X^{(n+1)}$

for some constant C.

Proof. If $h = \sum h_j(t)\,\varphi_j(x)$, then $Dh = \sum D_j h_j(t)\,\varphi_j(x)$, where

$$D_j z(t) = \omega^2 z''(t) + \lambda_j^2 z(t)\rho(t), \quad \rho(t) := 1 + \mu a(t).$$

Using the eigenfunctions $\psi_l(t)$ of (54) as a basis of $H^1(\mathbb{T})$,

$$D_j z(t) = \sum_{l \in \mathbb{N}} (\lambda_j^2 - \omega^2 p_l^2)\hat{z}_l\,\psi_l(t)\,\rho(t), \qquad z = \sum_{l \in \mathbb{N}} \hat{z}_l\,\psi_l(t),$$

and $K_j := (1/\rho)D_j$ is the diagonal operator $\{\lambda_j^2 - \omega^2 p_l^2\}_{l \in \mathbb{N}}$. Since $|\lambda_j^2 - \omega^2 p_l^2| > \gamma/\lambda_j^{\tau-1}$ for all $\lambda_j \le N_{n+1}$, we have that K_j is invertible and

$$\|K_j^{-1}z\|_{H^1_{\mu a}}^2 = \sum_{l \in \mathbb{N}} \left(\frac{\hat{z}_l}{\lambda_j^2 - \omega^2 p_l^2}\right)^2 \|\psi_l\|_{H^1_{\mu a}}^2 \le \frac{\lambda_j^{2(\tau-1)}}{\gamma^2}\,\|z\|_{H^1_{\mu a}}^2.$$

By (53) $\|K_j^{-1}z\|_{H^1} \le 4(\lambda_j^{\tau-1}/\gamma)\,\|z\|_{H^1}$. Since $D_j^{-1}z = K_j^{-1}(z/\rho)$ and $\|1/\rho\|_{H^1}$ is smaller than a universal constant,

$$\|D_j^{-1}z\|_{H^1} \le \frac{C\lambda_j^{\tau-1}}{\gamma}\,\|z\|_{H^1}.$$

Since $D^{-1}h = \sum_j D_j^{-1}h_j(t)\varphi_j(x)$ we obtain (55). $\qquad\square$

Lemma 9. (Control of S). *For all $s \ge 0$, if $u \in X_{s+2}$ then $S : X_0 \to X_s$ is bounded and*
$$\|Sh\|_s \le \mu\|u\|_{s+2}\|u\|_2\|h\|_0 \qquad \forall h \in X_0.$$

18

Proof. Since $\int_\Omega \nabla u \circ \nabla h \, dx$ does not depend on x,

$$\left\| \Delta u \int_\Omega \nabla u \circ \nabla h \, dx \right\|_s \leq \|\Delta u\|_s \left\| \int_\Omega \nabla u \circ \nabla h \, dx \right\|_{H^1(\mathbb{T})}.$$

$\int_\Omega \nabla u \circ \nabla h \, dx = \sum_j \lambda_j^2 u_j(t) h_j(t)$, so $\| \int_\Omega \nabla u \circ \nabla h \, dx \|_{H^1(\mathbb{T})} \leq \|u\|_2 \|h\|_0$ by Hölder inequality. $\qquad \square$

Proof of Lemma 1. $F'(u) = D + S = (I + SD^{-1})D$ where I is the identity map. Since D^{-1} satisfies (55), we have to prove the invertibility of $I + SD^{-1}$ in norm $\| \ \|_{\tau-1}$. By Neumann series it is sufficient to show that

(56) $$\|SD^{-1}h\|_{\tau-1} \leq \frac{1}{2}\|h\|_{\tau-1} \quad \forall h \in X^{(n+1)}.$$

By Lemmas 8 and 9

$$\|SD^{-1}h\|_{\tau-1} \leq \mu \|u\|_{\tau-1+2} \|u\|_2 \|D^{-1}h\|_0 \leq \frac{C\mu}{\gamma} \|u\|_{\tau+1}^2 \|h\|_{\tau-1}$$

because $\|u\|_2 \leq \|u\|_{\tau+1}$. Thus the condition

$$\frac{\mu}{\gamma} \|u\|_{\tau+1}^2 \leq \frac{1}{2C} =: K_1'$$

implies (56) and by Neumann series $\|(I + SD^{-1})^{-1}h\|_{\tau-1} \leq 2\|h\|_{\tau-1}$. $\qquad \square$

7 Proof of the theorems

Proof of Theorem 1. Let $g \in X_{\sigma,s_0}$ and $2d < 2(s_1 - 1) < s_0$ as assumed in the theorem. We apply Lemma 2 with

$$\tau := s_1 - 1 \,.$$

The construction of the sequence (u_n) is possible provided the parameters (ω, μ) belong to A_n for all $n \in \mathbb{N}$. Lemma 6 assures that, for μ/γ sufficiently small, the set $\mathcal{A}_\gamma$ of parameters satisfying this property is a nonempty set, which is very large in a Lebesgue measure sense. Lemmas 3 and 4 complete the proof. $\qquad \square$

Proof of Theorem 2. Let $g \in X_{\sigma,s_0}$ with $2d < 2(s_1 - 1) < s_0$ as assumed in the theorem. We consider a Lyapunov-Schmidt reduction splitting the space $\tilde{X}_{\sigma,s}$ in two subspaces $\tilde{X}_{\sigma,s} = Y \oplus (W \cap \tilde{X}_{\sigma,s})$,

$$Y := \{y(t) \in H^1(\mathbb{T}, \mathbb{R})\}, \quad W := \left\{ w \in \tilde{X}_{0,0} : w(x,t) = \sum_{j \geq 1} w_j(t) \tilde{\varphi}_j(x) \right\}.$$

We denote Π_Y, Π_W the projectors on Y, W, and observe that Π_Y is the map

$$u \mapsto \int_{(0,2\pi)^d} u(x,t) \, dx.$$

We define

$$g_0(t) := \Pi_Y g, \quad \bar{g}(x,t) := \Pi_W g.$$

We decompose $u(x,t) = y(t) + w(x,t)$, $y \in Y$, $w \in W$, and note that

$$f(u) = f(y+w) = f(w) \in W.$$

Then projecting equation (5) on Y gives

(57) $$\omega^2 y''(t) = \mu g_0(t) \quad (Y \ equation)$$

while projecting it on W

(58) $$L_\omega w = \mu(f(w) + \bar{g}) \quad (W \ equation).$$

Equation (57) is an ODE. With direct calculations (or by Fourier series) we see that (57) admits 2π-periodic solutions if and only if

(59) $$\int_0^{2\pi} g_0(t)\, dt = 0$$

and (59) is just assumption (8). We note that, if $y(t)$ solves (57), then also $y(t) + c$ solves (57), for all $c \in \mathbb{R}$. Moreover, the unique solution $y(t)$ of (57) such that $\int_0^{2\pi} y(t)\, dt = 0$ satisfies

$$\|y\|_{H^1} \le \|y''\|_{H^1} \le \frac{\mu}{\omega^2} \|g_0\|_{H^1}.$$

To solve (58), we consider all the calculations in Sections 3,4,5,6 replacing $X_{\sigma,s}$ with $\tilde{X}_{\sigma,s} \cap W$ and $\lambda_j, \varphi_j(x)$ with $\tilde{\lambda}_j, \tilde{\varphi}_j(x)$, $j \ge 1$. It follows the existence of a unique solution $w \in \tilde{X}_{\sigma,s_1} \cap W$ of (58) satisfying

$$\|w\|_{\sigma,s_1} \le \frac{\mu}{\gamma} C, \quad \|w_{tt}\|_{\sigma,s_1-2} \le \frac{\mu}{\gamma\omega^2} C.$$

Then $u = y + w$ solves (5)(3). Since

$$\|u\|_{\sigma,s_1}^2 = \|y\|_{H^1}^2 + \|w\|_{\sigma,s_1}^2, \quad \|u_{tt}\|_{\sigma,s_1-2}^2 = \|y''\|_{H^1}^2 + \|w_{tt}\|_{\sigma,s_1-2}^2,$$

we obtain estimates (9). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

8 Appendix

Proof of Lemma 7. The proof follows from classical results in [19, 22]. First, if $y'' + p^2(1+\alpha)y = 0$, then

$$\int_0^{2\pi} y'^2\, dt = p^2 \int_0^{2\pi} (1+\alpha)y^2\, dt,$$

so that $p^2 \ge 0$ because $1 + \alpha$ is positive. $p_0^2 = 0$ is an eigenvalue, the corresponding eigenfunctions are the constants, and all the other eigenvalues are positive.

By [19, Theorem 2.2.2, p.23], for every $k \in \mathbb{N}$ both p_{2k+1} and p_{2k+2} satisfy

$$\frac{2}{3}(k+1)^2 \le p^2 \le 2(k+1)^2$$

and (51) follows.

The equivalence (53) and the orthogonality of (ψ_l) w.r.t. $(,)_{H_\alpha^1}$ can be verified by direct calculations.

To prove (52), we define

$$q(\vartheta)(t) = q(t) := 1 + \alpha(t) + \vartheta(\beta(t) - \alpha(t)), \quad \vartheta \in [0, 1]$$

and the "Liouville's change of variable" $t \to \xi$

$$t = f(\xi) \quad \Leftrightarrow \quad \xi = g(t) := \frac{1}{c} \int_0^t \sqrt{q(s)}\, ds, \qquad c := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{q(t)}\, dt.$$

We note that $p^2, y(t)$ satisfy

(60)
$$\begin{cases} y''(t) + p^2 q(t) y(t) = 0 \\ y(t) = y(t + 2\pi) \end{cases}$$

if and only if $p^2, z(\xi)$ satisfy

(61)
$$\begin{cases} z''(\xi) + c^2[p^2 - Q(f(\xi))]z(\xi) = 0 \\ z(\xi) = z(\xi + 2\pi) \end{cases}$$

where

$$z(\xi) := y(f(\xi)) \sqrt[4]{q(f(\xi))}, \quad Q(t) := -\frac{5}{16}\frac{q'(t)^2}{q(t)^3} + \frac{1}{4}\frac{q''(t)}{q(t)^2}.$$

The operators $T(\vartheta) : z \mapsto -z'' + c^2 Q(f)z$ are selfadjoint in $L^2(0, 2\pi)$. We apply [22, Theorem 3.9, VII.3.5, p. 392] to the holomorphic family $\{T(\vartheta) : \vartheta \in [0, 1]\}$ (see [22, Definition VII.2.1, p. 375, Example 2.12, VII.2.3, p. 380 and Example 6.13, III.6.8, p. 187] to verify the hypotheses of the Theorem in the present case) to prove that the eigenvalues and eigenfunctions of (61) are analytic in ϑ. As a consequence, the eigenvalues and eigenfunctions of (60) are analytic in ϑ as well. This allows us to differentiate the equation

$$\psi_l(\vartheta)'' + p_l(\vartheta)^2(1 + \alpha + \vartheta(\beta - \alpha))\psi_l(\vartheta) = 0$$

w.r.t. ϑ. Recalling that $\int_0^{2\pi} q\psi_l^2\, dt = 1$, multiplying by $\psi_l(\vartheta)$ and integrating

$$\partial_\vartheta p_l(\vartheta) = -\frac{1}{2} p_l(\vartheta) \int_0^{2\pi} (\beta - \alpha)\psi_l(\vartheta)^2\, dt.$$

Since $p_l(\vartheta) \leq 2l$ and $q(\vartheta) \geq 1/2$,

$$|\partial_\vartheta p_l(\vartheta)| \leq l \int_0^{2\pi} \frac{|\beta - \alpha|}{q(\vartheta)} q(\vartheta)\psi_l(\vartheta)^2\, dt \leq 2l\|\beta - \alpha\|_\infty$$

and

$$|p_l(\beta) - p_l(\alpha)| \leq \int_0^1 |\partial_\vartheta p_l(\vartheta)|\, d\vartheta \leq 2l\|\beta - \alpha\|_\infty.$$

$\square$

21

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