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On the Classification of Networks Self–Similarly Moving by Curvature

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Abstract: We give an overview of the classification of networks in the plane with at most two triple junctions with the property that under the motion by curvature they are self–similarly shrinking. After the contributions in [7, 9, 20], such a classification was completed in the recent work in [4] (see also [3]), proving that there are no self–shrinking networks homeomorphic to the Greek "theta" letter (a double cell) embedded in the plane with two triple junctions with angles of 120 degrees. We present the main geometric ideas behind the work [4]. We also briefly introduce our work in progress in the higher–dimensional case of networks of surfaces in $\mathbb{R}^3$.

1 Introduction

Recently, the problem of the evolution by curvature of a network of curves in the plane got the interest of several authors [5, 9, 13, 14, 17–21]. It is well known, after the work of Huisken [10] in the smooth case of the hypersurfaces in the Euclidean space and of Ilmanen [11, 12] in the more general weak settings of varifolds, that a suitable sequence of rescalings of the *subsets* of $\mathbb{R}^n$ which are evolving by mean curvature, approaching a singular time of the flow, converges to a so called "blow–up limit" set. Such a limit set has the property that, letting it flow again by mean curvature, it simply moves by homothety, more precisely it shrinks down self–similarly toward the origin of the Euclidean space.

This procedure and the classification of these special sets (possibly under some hypotheses), called *shrinkers*, is a key point in understanding the asymptotic behavior of the flow at a singular time.

Dealing with the evolution of a single curve in the plane, it is easy to see that any C^2 curve $\gamma : I \to \mathbb{R}^2$ that moves by curvature, self–similarly shrinking, must satisfy the following "structural" equation (which is actually an ODE for γ)

$$\overline{k} + \gamma^\perp = 0, \tag{1.1}$$

where $\overline{k}$ is the vector curvature of the curve at the point γ and $\gamma^\perp$ denotes the normal component of the position vector γ. Introducing an arclength parameter s on the curve γ, we have a unit tangent vector field $\tau = \frac{d}{ds}\gamma$, a unit normal vector field ν which is the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2$ of the vector τ and the curvature vector given by $\overline{k} = k\nu = \frac{d^2}{ds^2}\gamma$, where k is then simply the curvature of γ. With this notation, the above equation can be rewritten as

$$k + \langle \gamma \,|\, \nu \rangle = 0. \tag{1.2}$$

It is then known, by the work of Abresch–Langer [1] and independently of Epstein–Weinstein [8], that the only complete, embedded, self–similarly shrinking curves in $\mathbb{R}^2$ without end–points, are the lines through the origin and the unit circle (they actually classify *all* the closed, embedded or not, self–similarly shrinking

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curves in the plane). The same equation $\overline{k} + \gamma^{\perp} = 0$ (that is, $k + \langle \gamma \,|\, \nu \rangle = 0$) must be satisfied by every curve

Figure 1: The only complete, embedded, self–similarly shrinking curves in $\mathbb{R}^2$: lines through the origin and the unit circle.

of a network in the plane which self–similarly shrinks to the origin moving by curvature (see [16, 17], for instance). Moreover, for "energetic" reasons, it is natural to consider networks with only triple junctions and such that the three concurring curves (which are C^∞) form three angles of 120 degrees between each other – "Herring" condition – such networks are called *regular*. In such a class, the *embedded* shrinking regular networks (without self–intersections) play a crucial role, indeed, they "reasonably" arise as blow–up limits of the motion of networks without self–intersections (this is still a conjecture for a general network, but there holds for networks with at most two triple junctions – see the end of the section).

In [3, 4] we were able to complete the classification of the complete, embedded, self–similarly shrinking regular networks in the plane with at most two triple junctions, after the contributions in [7, 9, 20]. We describe such a classification here below.

If one consider networks with only one triple junction, the only complete, embedded, regular shrinkers are given (up to rotations) by the "standard triod" and the "Brakke spoon" (first described in [6]), as in Figure 2.

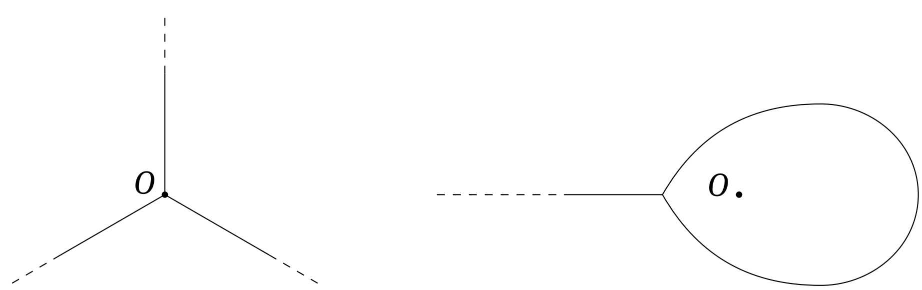

Figure 2: A *standard triod* and a *Brakke spoon*.

About networks with two triple junctions, it is not difficult to show that the possible topological shapes for a connected, complete, embedded, regular network without end–points, are the ones depicted in Figure 3. Then, looking for shrinkers with one of these topological shapes, by the cited work of Abresch and Langer [1] it follows that any unbounded curve of such shrinkers must be a piece of a halfline from the origin, going to infinity. Then, differentiating in arclength s the equation $k = -\langle \gamma \,|\, \nu \rangle$, we get the ODE for the curvature $k_s = k\langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, then it must also hold $k_s = 0$ at the same point, hence by the uniqueness theorem for ODEs we conclude that k is identically zero and we are dealing with a piece of a straight line through the origin of $\mathbb{R}^2$, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in \gamma$. Notice that, if a curve γ contains the origin, at such a point its curvature is zero by the equation $k + \langle \gamma \,|\, \nu \rangle = 0$, hence it must be straight.

Now, if a regular shrinker had the topological shape of the first drawing on the top of Figure 3, the four unbounded curves should be halflines, which implies that the two triple junctions should coincide with the

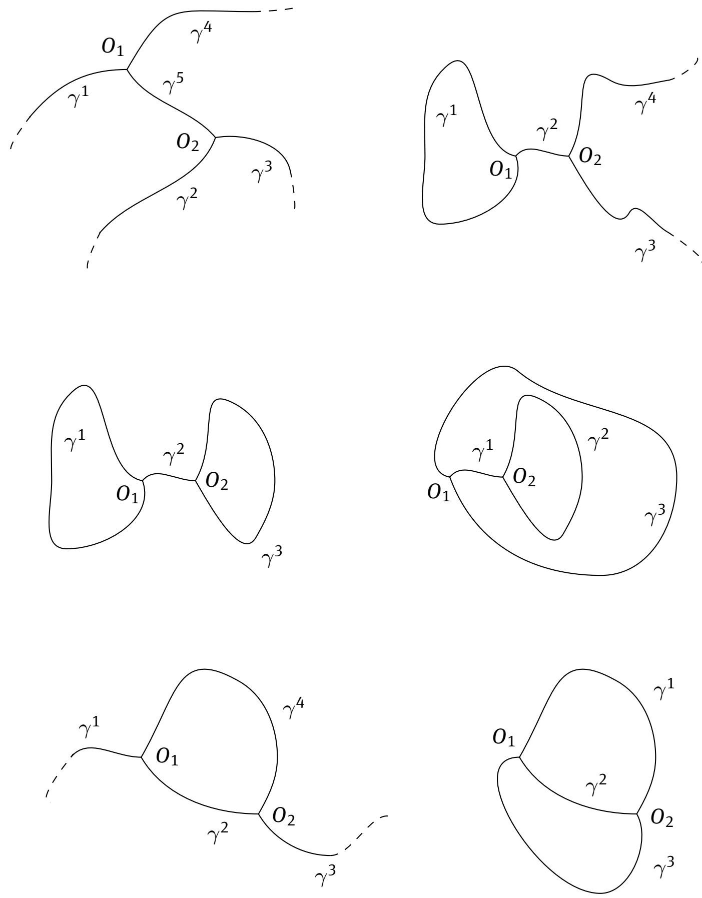

Figure 3: The possible topological shapes of a complete, connected, embedded network with two triple junctions.

origin, which is a contradiction (the curve γ^5 has to be a non trivial segment between the triple junctions), thus such a shape is excluded.
Then, thanks to an argument by Hättenschweiler [9, Lemma 3.20], if a regular shrinker contains a region bounded by a single curve, the shrinker must be a Brakke spoon, that is, no other triple junctions can be present. This excludes the possibility for a regular shrinker also to have a shape like the second one in the first row of Figure 3 or like the two in the second row.

It remains to discuss the last two cases: one is the "lens/fish" shape and the other is the shape of the Greek "theta" letter (or "double cell"). It is well known that there exist unique (up to a rotation) lens–shaped and fish–shaped, complete, embedded, regular shrinkers, which are symmetric with respect to a line through the origin of $\mathbb{R}^2$ (see [7, 20]). It was instead unknown whether regular Θ–shaped shrinkers (or simply Θ–

Figure 4: A lens–shaped and a fish–shaped shrinker.

shrinkers) exist, with numerical evidence in favor of the conjecture of non–existence (see [9]). We have proved that this is actually the case.

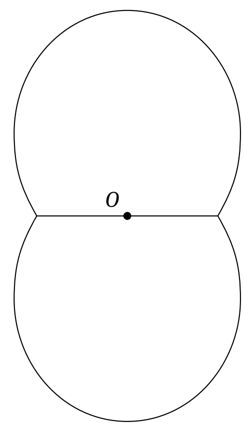

Figure 5: A hypothetical Θ–shrinker.

Theorem 1.1 ([4], Theorem 1.1)**.** *There are no regular Θ–shrinkers.*

As a consequence, one gets the following classification result.

Theorem 1.2 ([3], Theorem 1.2)**.** *The shrinkers of Figure 4 ("lens" and "fish") are the only (up to rotations) complete, embedded, self–similarly shrinking regular networks in the plane with two triple junctions.*

To prove Theorem 1.1 we first analyze the geometric properties that any hypothetical Θ–shrinker must satisfy, reducing the proof of non–existence to show that a certain parametric integral is always smaller than $\pi/2$, for every value of the parameter. The proof of such an estimate, mixing some approximation techniques and numerical computations based on *interval arithmetic*, is contained in the paper [4].

We conclude this discussion mentioning that the main motivation for this problem is given by the fact that for an evolving network with at most *two* triple junctions, the so called *multiplicity–one conjecture* holds (see [16]), saying that any limit shrinker of a sequence of rescalings of the network at different times is again a "genuine" embedded network without "double" or "multiple" curves (curves that in such a convergence go to coincide in the limit). This is a key point in the singularity analysis (actually, in general, for mean curvature flow), together with the classification of these limit shrinkers, which is complete after our result Theorem 1.1, for such "low complexity" networks, thus leading to a detailed description of their motion in [15].

For instance, from the non–existence of regular Θ–shrinkers one can deduce some information (which would otherwise be unknown) on the evolution by curvature of a symmetric "double bubble", one of the simplest examples of networks one may think of.

The symmetry of the picture and the non–existence of regular Θ–shrinkers lead to the conclusion that the evolution of such a double bubble will reach a singularity by shrinking the line segment to a single point and creating two symmetric "teardrops" joined by a quadruple point, forming two angles of 120 degrees and two angles of 60 degrees.

It is of course interesting (yet, in general, much harder) to study the possible blow–up limits of the mean curvature flow of networks also in higher dimension. For this reason, it is natural to try to classify – even if, at the moment, in this higher dimensional setting much less is known – those networks of hypersurfaces in $\mathbb{R}^n$ whose motion by mean curvature is homothetic. We have recently begun the study of a problem in this

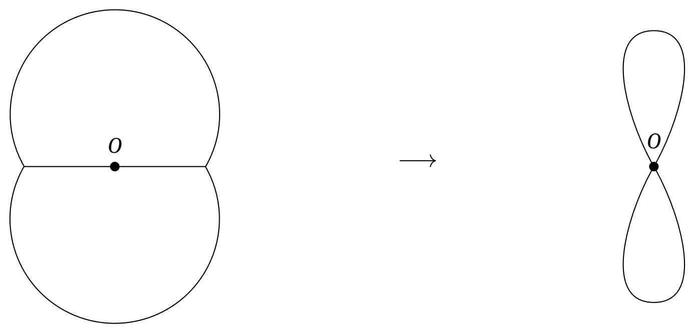

Figure 6: On the left, a symmetric double bubble: a Θ–shaped network with two symmetric arc circles and a horizontal segment, forming angles of 120 degrees. On the right, the type of singularity reached by the evolution by curvature of a double bubble.

context, namely we are working in order to prove or disprove the existence of self–shrinking networks of surfaces in $\mathbb{R}^3$ with some given simple topologies (see Section 4).

Plan of the paper. In Section 2 we recall some basic properties of shrinkers, while in Section 3 we summarize the geometric part of the proof of Theorem 1.1. Finally, in Section 4 we spend a few words about the ongoing work on the classification of self–shrinking surfaces in $\mathbb{R}^3$.

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2 Basic properties of shrinking curves

Consider a shrinking curve $\gamma : I \to \mathbb{R}^2$ parametrized in arclength s, where $I \subset \mathbb{R}$ is an interval. We denote with $R : \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation of 90 degrees. Then, the relation

$$\gamma_{ss} = \frac{d^2\gamma}{ds^2} = k\nu = -\langle \gamma \,|\, \nu \rangle \nu = -\Big\langle \gamma \,\Big|\, R\Big(\frac{d\gamma}{ds}\Big)\Big\rangle \nu$$

gives an ODE satisfied by γ. It follows that the curve is smooth and it is not difficult to see that for every point $x_0 \in \mathbb{R}^2$ and unit velocity vector τ_0, there exists a unique shrinking curve (solution of such an ODE) parametrized in arclength, passing at $s = 0$ through the point x_0 with velocity τ_0, defined for all $s \in \mathbb{R}$.

Differentiating in arclength the equation $k = -\langle \gamma \,|\, \nu \rangle$, we get the ODE for the curvature $k_s = k\langle \gamma \,|\, \tau \rangle$. Suppose that at some point $k = 0$, then it must also hold $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODEs, we conclude that k is identically zero and we are dealing with a piece of a straight line which, as $\langle x \,|\, \nu \rangle = 0$ for every $x \in \gamma$, must contain the origin of $\mathbb{R}^2$.

So we suppose that k is always nonzero and, by looking at the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$, we can see that the curve is then strictly convex with respect to the origin of $\mathbb{R}^2$. Another consequence (by the uniqueness theorem for ODE) is that the curve must be symmetric with respect to any critical point (maximum or minimum) of its curvature function. Notice that if the curve is not a piece of a circle, the critical points are all nondegenerate and isolated (if the curve has bounded length, their number is finite).

Computing the derivative of $|\gamma|^2$,

$$\frac{d|\gamma|^2}{ds} = 2\langle \gamma \,|\, \tau \rangle = 2k_s/k = 2\frac{d\log k}{ds}$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C \in \mathbb{R}$, that is, the quantity

$$\mathcal{E} = \mathcal{E}(\gamma) := ke^{-|\gamma|^2/2}, \tag{2.1}$$

that we call *Energy*, is constant along the curve. Equivalently, $\langle \gamma \mid \nu \rangle e^{-|\gamma|^2/2}$ is constant. A solution γ has positive energy if $k > 0$, so that γ runs counterclockwise around the origin, γ has negative energy if $k < 0$, so that γ runs clockwise around the origin, γ has energy zero if $k = 0$, so that γ is a piece of a straight line through the origin.

We consider now a new coordinate $\theta = \arccos \langle e_1 \mid \nu \rangle$; this can be done for the whole curve as we know that it is convex (obviously, θ is only locally continuous, since it "jumps" after a complete round).

Differentiating with respect to the arclength parameter we have $\frac{d\theta}{ds} = k$ and

$$k_\theta = k_s/k = \langle \gamma \mid \tau \rangle\,, \qquad k_{\theta\theta} = \frac{1}{k}\frac{dk_\theta}{ds} = \frac{1 + k\langle \gamma \mid \nu \rangle}{k} = \frac{1}{k} - k. \tag{2.2}$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\frac{d}{d\theta}[k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity

$$E := k_\theta^2 + k^2 - \log k^2$$

is constant along all the curve. Notice that the quantity E cannot be less than 1 (if $k \neq 0$), moreover, if $E = 1$ we have that k^2 must be constant and equal to one along the curve, which consequently must be a piece of the unit circle centered at the origin of $\mathbb{R}^2$.

As $E \geq 1$, it follows that k^2 is uniformly bounded from above and away from zero, hence, recalling that $k = \mathcal{E}e^{|\gamma|^2/2}$, the curve γ is contained in a ball of $\mathbb{R}^2$ (and it is outside some small ball around the origin).

Since we are interested in the curves of a nontrivial connected, compact (Θ–shaped), regular network, there will be no unbounded lines or complete circles and all the curves of the network will be images of a closed bounded interval, once parametrized in arclength.

Summarizing, either γ is a segment or $k^2 > 0$, the equations (2.2) hold, the Energy $\mathcal{E} = ke^{-|\gamma|^2/2}$ and the quantity $E = k_\theta^2 + k^2 - \log k^2 \geq 1$ are constant along the curve, where $\theta = \arccos \langle e_1 \mid \nu \rangle$. Moreover, the curve is locally symmetric with respect to the critical points of the curvature, hence the curvature $k(\theta)$ is oscillating between its maximum and its minimum.

Suppose now that $k_{\min} < k_{\max}$ are these two consecutive critical values of k. It follows that they are two distinct positive zeroes of the function $k_\theta^2 = E + \log k^2 - k^2$, when $E > 1$, with $0 < k_{\min} < 1 < k_{\max}$.
We have then that the change $\Delta\theta$ in the angle θ along the piece of curve delimited by two consecutive points where the curvature assumes the values $k_{\min}$ and $k_{\max}$ is given by the integral

$$\Delta\theta = I(E) = \int_{k_{\min}}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}}. \tag{2.3}$$

Proposition 2.1 (Abresch and Langer [1]). *The function $I : (1, +\infty) \to \mathbb{R}$ satisfies*

1. $\lim_{E\to 1^+} I(E) = \pi/\sqrt{2}$,
2. $\lim_{E\to +\infty} I(E) = \pi/2$,
3. *$I(E)$ is monotone decreasing.*

As a consequence $I(E) > \pi/2$.

We write now the curve γ in polar coordinates, that is, $\gamma(s) = (\rho(s)\cos\phi(s), \rho(s)\sin\phi(s))$, then, the arclength constraint and the shrinker equation (1.2) become

$$\rho_s^2 + \rho^2\phi_s^2 = 1, \tag{2.4}$$

$$\rho^2\phi_s + \rho\rho_{ss}\phi_s - 2\rho_s^2\phi_s - \rho^2\phi_s^3 - \rho\rho_s\phi_{ss} = 0,$$

moreover,

$$\cos\big(\text{angle between } \gamma \text{ and } \gamma_s\big) = \frac{\gamma \cdot \gamma_s}{|\gamma||\gamma_s|} = \rho_s. \tag{2.5}$$

Notice that shrinking curves with positive energy have $\phi_s > 0$ everywhere, indeed, either ϕ_s is always different by zero or the curve is a segment of a straight line through the origin of $\mathbb{R}^2$.

The curvature and the Energy $\mathcal{E} = ke^{-|\gamma|^2/2}$ are given by

$$k = \rho^2 \phi_s, \qquad \mathcal{E} = \rho^2 \phi_s e^{-\frac{1}{2}\rho^2} \tag{2.6}$$

and, when the energy is positive, it will be useful to consider also the quantity $\mathcal{F} := -\log(\mathcal{E})$, that is,

$$\mathcal{F} = -\log(\mathcal{E}) = \frac{1}{2}\rho^2 - \log(\rho^2 \phi_s). \tag{2.7}$$

Since $0 < \rho\phi_s \leq 1$, by equation (2.4), one has

$$\mathcal{F} \geq \frac{1}{2}\rho^2 - \log(\rho) \geq \frac{1}{2}.$$

Let us assume that γ is a shrinking curve with $k > 0$ (the assumption on the sign of k is not restrictive, up to a change of orientation of the curve). Then, by the definition of the Energy (2.1), it is immediate to see that the points where k attains its maximum (resp. minimum) coincide with the points where ρ attains its maximum (resp. minimum). Thus, at any extremal point of k there hold $k_\theta = 0$, $\rho_s = 0$ and also $\rho\phi_s = 1$, by equation (2.4), hence, by equation (2.6), we have $k = \rho$. Then, computing E and $\mathcal{F}$ at such a point (clearly, $k_\theta = 0$), we get

$$E = k^2 - 2\log k \qquad \text{and} \qquad \mathcal{F} = k^2/2 - \log k,$$

that is, $E = 2\mathcal{F} = \log\left(\frac{1}{\mathcal{E}^2}\right)$.

Since the Energy and the quantity $\mathcal{F}$ are constant, this relation must hold along all the curve γ and $\mathcal{F} = \rho_{\min}^2/2 - \log\rho_{\min} = \rho_{\max}^2/2 - \log\rho_{\max}$.

Since the function $\mu(t) = t^2/2 - \log t$ is strictly convex with a minimum value $1/2$ at $t = 1$, to each value of $\mathcal{F} \geq \frac{1}{2}$, there correspond two values $\rho_{\min}(\mathcal{F})$ and $\rho_{\max}(\mathcal{F})$ which are the admissible (interior) minimum and maximum of ρ on γ, with $\rho_{\min}(\mathcal{F}) < 1 < \rho_{\max}(\mathcal{F})$ if $\mathcal{F} > \frac{1}{2}$. It follows easily that $\rho_{\max} : (1/2, +\infty) \to (1, +\infty)$ is an increasing function and $\rho_{\min} : (1/2, +\infty) \to (0, 1)$ is a decreasing function. Viceversa, the quantity $\mathcal{F}$ can be seen as a decreasing function of $\rho_{\min} \in (0, 1]$ and an increasing function of $\rho_{\max} \in [1, +\infty)$.

Let $s_{\min}, s_{\max} \in \mathbb{R}$ with $s_{\min} < s_{\max}$ be two consecutive (interior) extremal points of ρ (hence, also of k) such that $\rho(s_{\min}) = \rho_{\min}(\mathcal{F})$, $\rho(s_{\max}) = \rho_{\max}(\mathcal{F})$. Since at the interior extremal points of ρ the vectors γ, γ_s must be orthogonal, it follows that the quantity considered in formula (2.3) satisfies

$$\Delta\theta = \int_{s_{\min}}^{s_{\max}} \phi_s(s)\, ds := \mathcal{I}(\mathcal{F}), \tag{2.8}$$

that is, the integral $\mathcal{I}(\mathcal{F})$ is the variation of the angle ϕ on the shortest arc such that ρ passes from $\rho_{\min}$ to $\rho_{\max}$.
Then, by the above discussion, $\mathcal{I}(\mathcal{F}) = I(E) = I(2\mathcal{F})$ and we can rephrase Proposition 2.1 in terms of the integral $\mathcal{I}(\mathcal{F})$ as follows.

Proposition 2.2. *The function $\mathcal{I} : (1/2, +\infty) \to \mathbb{R}$ satisfies*

1. $\lim_{\mathcal{F}\to(1/2)^+} \mathcal{I}(\mathcal{F}) = \frac{\pi}{\sqrt{2}}$,
2. $\lim_{\mathcal{F}\to+\infty} \mathcal{I}(\mathcal{F}) = \frac{\pi}{2}$,
3. *$\mathcal{I}(\mathcal{F})$ is monotone decreasing.*

As a consequence $\mathcal{I}(\mathcal{F}) > \frac{\pi}{2}$ for all $\mathcal{F} > \frac{1}{2}$.

3 The proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following lemma, whose proof – which is nontrivial and based also on a computer–assisted estimate – can be found in [4].

Lemma 3.1 ([4], Lemma 3.1). *Let γ be a shrinking curve, parametrized counterclockwise by arclength, with positive curvature and let (s_0, s_1) be an interval where $s \mapsto \rho(s)$ is increasing. If $\rho_s(s_0) \geq \frac{1}{2}$, namely, if the angle formed by the vectors $\gamma(s_0)$ and $\gamma_s(s_0)$ is $\leq \frac{\pi}{3}$, then*

$$\int_{s_0}^{s_1} \phi_s(s)\, ds < \frac{\pi}{2}.$$

Similarly, if $s \mapsto \rho(s)$ is decreasing on (s_0, s_1) and $\rho_s(s_1) \leq -\frac{1}{2}$, namely the angle formed by the vectors $\gamma(s_1)$ and $\gamma_s(s_1)$ is $\geq \frac{2\pi}{3}$, then the same conclusion holds.

We assume now that a Θ–shrinker exists, described by three embedded shrinking curves $\gamma_i : [\underline{s}_i, \overline{s}_i] \to \mathbb{R}^2$, parametrized by arclength, expressed in polar coordinates by $\gamma_i = (\rho_i \cos(\phi_i), \rho_i \sin(\phi_i))$, for $i \in \{1, 2, 3\}$. The two triple junctions will be denoted with A, B and the three curves intersect each other only at A and B (which are their endpoints) forming angles of 120 degrees. Since the shrinker equation (1.1) is invariant by rotation, we can assume that the segment $\overline{AB}$ is contained in the straight line $\{(x, q) : x \in \mathbb{R}\}$ with $q \geq 0$ and we let $A = (x_A, q)$, $B = (x_B, q)$ with $x_A < x_B$.

To simplify the notation, in all this section we will denote the arclength derivative $\frac{d}{ds}$ with $'$. We start by stating the following two preliminary lemmas, which are proved in [4].

Lemma 3.2 ([4], Lemma 3.3). *For all $i \in \{1, 2, 3\}$, the curve γ_i is either a straight line or such that*

$$\left| \int_{\underline{s}_i}^{\overline{s}_i} \phi_i'(s)\, ds \right| < 2\pi.$$

Lemma 3.3 ([4], Lemma 3.4). *Let $S = [\underline{s}, \overline{s}]$ and $\gamma : S \to \mathbb{R}^2$ be a shrinking curve parametrized by arclength, expressed in polar coordinates by $\gamma = (\rho \cos(\phi), \rho \sin(\phi))$. Assume that $\phi'(s) > 0$ in S and*

$$0 < \Delta \leq \pi, \quad \text{where } \Delta := \int_{\underline{s}}^{\overline{s}} \phi'(s)\, ds.$$

Let L be the straight line passing through the two points $\gamma(\underline{s})$, $\gamma(\overline{s})$ and H_1 and H_2 be the two closed half–planes in which L divides the plane $\mathbb{R}^2$. Then the arc $\gamma(S)$ is entirely contained in H_1 or H_2.

Moreover, if $\Delta < \pi$ and $\gamma(S) \subset H_1$, then the origin of $\mathbb{R}^2$ belongs to the interior of H_2.

Coming back to our Θ–shrinker, because of its topological structure, one of the curves is contained in the region delimited by the other two, moreover the curvature of both these two "external" curves is always non zero, otherwise any such curve is a segment of a straight line passing through the origin, then the 120 degrees condition at its endpoints would imply that it must be contained in the region bounded by the other two curves, hence it could not be "external". Notice that, on the contrary, the "inner" curve could actually be a segment for the origin.

We call γ_2 the "inner" curve and, recalling that the origin of $\mathbb{R}^2$ is not over the straight line through the two triple junctions A and B, parametrizing counterclockwise the three curves, that is $\phi_i' > 0$ (in the case that the "inner" curve γ_2 is not a segment), we call γ_1 the "external" curve which starts at B. By Lemma 3.2, γ_1 reaches the point A after ϕ_1 changes of an angle $\Delta = \int_{\underline{s}_1}^{\overline{s}_1} \phi_1'(s)\, ds < 2\pi$ equal to the angle $\widehat{BOA}$, which is smaller or equal than π. Hence, by Lemma 3.3, all of the curve γ_1 stays over the straight line passing for the two triple junctions A and B.

We call γ_3 the other extremal curve, hence since $\phi_1, \phi_3 > 0$, we have

$$\gamma_1(\underline{s}_1) = \gamma_3(\overline{s}_3) = B, \quad \gamma_1(\overline{s}_1) = \gamma_3(\underline{s}_3) = A.$$

Because of the shrinker equation (1.2), all the three curves are convex with respect to the origin. This implies that the origin is contained in the interior of the bounded area A_{13} enclosed by γ_1 and γ_3 (if the origin belongs to γ_1 or γ_3 such a curve is a segment and cannot be "external", as we said before), which also contains $\gamma_2 \subset A_{13}$. We let A_{12} be the region enclosed by the curves γ_1 and γ_2 and we split the analysis into two cases.

Case 1. The origin does not belong to the interior of A_{12}.

Since the curve γ_2 is convex with respect to the origin, by the same argument used above for γ_1, it is contained in the upper half–plane determined by the straight line for the points A and B.

By the 120 degrees condition it follows that the angle β at B formed by the vector $(1,0)$ and γ_1' is at most $\frac{\pi}{3}$. Similarly, also the angle α at A formed by the vector $(1,0)$ and γ_1' is at most $\frac{\pi}{3}$. By the convexity of the region delimited by γ_2 and γ_3 containing the origin and again the 120 degrees condition at B, it is then easy to see that the angle at B formed by the vectors γ_1 and γ_1' is less or equal than $\frac{\pi}{3}$ and analogously, the angle at A formed by γ_1 and γ_1' is greater or equal than $\frac{2\pi}{3}$.

Hence, by equality (2.5), it follows

$$\rho_1'(\underline{s}_1) \geq \frac{1}{2} > 0, \quad \rho_1'(\overline{s}_1) \leq -\frac{1}{2} < 0.$$

As a consequence, there is a point of maximum radius $s_1^\star \in (\underline{s}_1, \overline{s}_1)$ such that $\rho_1(s_1^\star) \geq \rho_1(s)$ for all $s \in (\underline{s}_1, \overline{s}_1)$.

The vector $\gamma_1(s_1^\star)$ forms an angle $\sigma \geq \frac{\pi}{2}$ with $(1,0)$ or $(-1,0)$. Assume that the angle between $\gamma_1(s_1^\star)$ and $(1,0)$ is greater or equal than $\frac{\pi}{2}$ (the other case is analogous, switching A and B). We extend the curve γ_1 (still parametrized by arclength) "before" the point B till it intersects the x–axis at some $\widetilde{s}_1 \leq \underline{s}_1$ (this must happen because $\phi_1(s) > 0$ everywhere also on the extended curve) and we consider the (non relabeled) curve γ_1 defined in the interval $L_1 = [\widetilde{s}_1, s_1^\star]$. Calling β_0 the angle formed by the vectors $\gamma_1'(\widetilde{s}_1)$ and $(1,0)$, by convexity and the fact that the angle β at B formed by the vector $(1,0)$ and γ_1' is at most $\frac{\pi}{3}$, we have that $\beta_0 \leq \beta \leq \frac{\pi}{3}$. Hence, by equality (2.5), we have $\rho_1'(\widetilde{s}_1) \geq \frac{1}{2} > 0$.

Considering now the function $s \mapsto \rho_1(s)$ on the interval $L_1 = [\widetilde{s}_1, s_1^\star]$, since $\rho_1'(\widetilde{s}_1) > 0$ and $s_1^\star$ is a maximum point for ρ_1, either ρ_1 is increasing on L_1, or ρ_1 has another maximum and then a minimum in the interior of L_1 (notice that the map ρ_1 cannot be constant on an interval, otherwise γ_1 would be an arc of a circle centered at the origin, which is impossible since ρ_1 is not constant). But we know from formula (2.3) and Proposition 2.1 that the angle ϕ_1 must increase more than $\frac{\pi}{2}$ to go from a minimum to a maximum or viceversa (we can apply such a proposition since γ_1 is not an arc of a circle). Since

$$\int_{\widetilde{s}_1}^{s_1^\star} \phi_1'(s)\,ds \leq \pi,$$

there cannot be a maximum, then a minimum, then a second maximum in L_1. It follows that ρ_1 is increasing in such an interval.

This, combined with the fact that $\beta_0 \leq \frac{\pi}{3}$ and that the angle σ is at least $\frac{\pi}{2}$, that is, $\int_{\widetilde{s}_1}^{s_1^\star} \phi_1'(s)\,ds \geq \frac{\pi}{2}$, is in contradiction with Lemma 3.1. Therefore, this case cannot happen.

Case 2. The origin belongs to the interior of A_{12}.

Being the region A_{12} convex (by the shrinker equation (1.2), since it contains the origin), the curve γ_2 (which is oriented counterclockwise) goes from A to B. The fact that γ_2' and γ_3' form angles of $\frac{2\pi}{3}$ at the points A and B implies that:

(*i*) the angle in A formed by the vectors $\gamma_3(\underline{s}_3)$ and $\gamma_3'(\underline{s}_3)$ and the angle in B formed by the vectors $\gamma_2(\overline{s}_2)$ and $\gamma_2'(\overline{s}_2)$ are both less or equal than $\frac{\pi}{3}$;

(*ii*) the angle in B formed by the vectors $\gamma_3(\overline{s}_3)$ and $\gamma_3'(\overline{s}_3)$ and the angle in A formed by the vectors $\gamma_2(\underline{s}_2)$ and $\gamma_2'(\underline{s}_2)$ are both greater or equal than $\frac{2\pi}{3}$.

In particular, by equality (2.5), it follows

$$\rho_2'(\underline{s}_2) \leq -\frac{1}{2} < 0, \quad \rho_2'(\overline{s}_2) \geq \frac{1}{2} > 0, \quad \rho_3'(\underline{s}_3) \geq \frac{1}{2} > 0, \quad \rho_3'(\overline{s}_3) \leq -\frac{1}{2} < 0. \tag{3.1}$$

Hence, the function $s \mapsto \rho_3(s)$ has a maximum at some point $s_3^\star \in (\underline{s}_3, \overline{s}_3)$, while the function $s \mapsto \rho_2(s)$ has a minimum at some point $s_2^\circ \in (\underline{s}_2, \overline{s}_2)$.

If $s_3^\star$ is the only point of maximum of ρ_3 in the interval $[\underline{s}_3, \overline{s}_3]$, then the function ρ_3 is strictly monotone on each of the two subintervals $[\underline{s}_3, s_3^\star]$ and $[s_3^\star, \overline{s}_3]$, moreover,

$$\int_{\underline{s}_3}^{s_3^\star} \phi_3'(s)\,ds + \int_{s_3^\star}^{\overline{s}_3} \phi_3'(s)\,ds = \int_{\underline{s}_3}^{\overline{s}_3} \phi_3'(s)\,ds \geq \pi,$$

since the origin is "below" the segment $\overline{AB}$. Thus, at least one of the two integrals on the left–hand side is greater or equal than $\frac{\pi}{2}$ and, by Lemma 3.1, this is not possible. As a consequence, there must be another point of maximum radius $s_3^{**} \in (\underline{s}_3, \overline{s}_3)$ (notice that the maximum points cannot be an interval, otherwise γ_3 would be an arc of a circle centered at the origin, hence with $\rho_3' = 0$, against relations (3.1)). Hence, between these two points of maximum radius there is a minimum point s_3°. Without loss of generality, we assume that $\underline{s}_3 < s_3^* < s_3^\circ < s_3^{**} < \overline{s}_3$.

We observe that there cannot be a third maximum point for ρ_3 (hence also another minimum point) in the interval $[\underline{s}_3, \overline{s}_3]$ because, by Proposition 2.2, each of the four angles at the origin formed by the segment connecting the origin with two consecutive of the five extremal points for ρ_3 on γ_3 is greater than $\frac{\pi}{2}$ and, by Lemma 3.2, there holds $\int_{\underline{s}_3}^{\overline{s}_3} \phi_3'(s)\,ds < 2\pi$. Moreover, also the case of two minimum points and two maximum points for ρ_3 in the interval $[\underline{s}_3, \overline{s}_3]$ is not possible, because of the sign of the derivative ρ_3' at the endpoints in relations (3.1). Hence, we conclude that $s_3^*, s_3^\circ, s_3^{**}$ are the only extremal points for ρ_3 in the interval $[\underline{s}_3, \overline{s}_3]$.

Now consider the quantities $\mathcal{F}_2, \mathcal{F}_3$ of the curves γ_2, γ_3, respectively, given by formula (2.7). By relations (3.1), the curves γ_2 and γ_3 are not the unit circle (they would have ρ_2' or ρ_3' equal to zero everywhere), therefore $\mathcal{F}_2, \mathcal{F}_3 > \frac{1}{2}$. If we draw the line from the origin to $\gamma_3(s_3^\circ)$, this must intersect γ_2 in an intermediate point, implying that the minimal radius of the curve γ_2 is smaller than the minimal radius of the curve γ_3. By the discussion about the value of the quantity $\mathcal{F}$ in relation with the extremal values of ρ at the end of Section 2, we have $\mathcal{F}_2 > \mathcal{F}_3$. Then, if a maximum of ρ_2 is taken in the interior of γ_2, it must be larger than the maximal radius of γ_3 (which is taken in the interior of γ_3), which is not possible as γ_2 is contained in the region bounded by γ_3 and the segment $\overline{AB}$. From this argument we conclude that there are no points of maximal radius in the interior of γ_2, thus, the only extremal point for ρ_2 in the interval $[\underline{s}_2, \overline{s}_2]$ is the minimum point s_2°.

Defining the angle

$$\alpha := \int_{\underline{s}_2}^{\overline{s}_2} \phi_2'(s)\,ds = \int_{\underline{s}_3}^{\overline{s}_3} \phi_3'(s)\,ds,$$

by formula (2.8) and the symmetry of the curve γ_3 with respect to the straight line through the origin and the point $\gamma_3(s_3^\circ)$ of minimum distance, we have

$$\mathcal{I}(\mathcal{F}_3) = \int_{s_3^*}^{s_3^\circ} \phi_3'(s)\,ds = \int_{s_3^\circ}^{s_3^{**}} \phi_3'(s)\,ds < \frac{\alpha}{2}$$

while, since γ_2 does not contain any interior point of maximum radius,

$$\mathcal{I}(\mathcal{F}_2) > \max\left\{\int_{\underline{s}_2}^{s_2^\circ} \phi_2'(s)\,ds,\ \int_{s_2^\circ}^{\overline{s}_2} \phi_2'(s)\,ds\right\} \geq \frac{\alpha}{2}.$$

Thus, $\mathcal{I}(\mathcal{F}_2) > \mathcal{I}(\mathcal{F}_3)$ and $\mathcal{F}_2 > \mathcal{F}_3$, which is in contradiction with the monotonicity of the function $\mathcal{I}$ given by Proposition 2.2. Hence, also this case can be excluded.

Since we excluded both cases, our hypothetical Θ–shrinker cannot exist and we are done with the proof of Theorem 1.1.

4 A step into higher dimension

We are currently working on the classification of self–shrinking *clusters* of surfaces ("bubbles") in the three–dimensional Euclidean space. The equation of a surface $\Sigma \subset \mathbb{R}^3$ self–shrinking toward the origin has formally the same structure as the one for the planar case, i.e. such a surface must satisfy the structural equation

$$\overline{\mathrm{H}} + \Sigma^\perp = 0, \tag{4.1}$$

where $\overline{\mathrm{H}}$ is the *mean curvature vector* and $\Sigma^\perp$ is the normal component of the position vector Σ. A major difficulty arises since equation (4.1) is no longer an ODE, as in the planar case, but it is a PDE that has to be solved by the unknown surface Σ. There are some known embedded surfaces solving equation (4.1), some of which can be easily found by direct inspection. In particular, known solutions are:

- any plane through the origin;
- any cylinder of radius 1 about an axis through the origin;
- the sphere of radius $\sqrt{2}$ centered at the origin;
- given any axis of symmetry ℓ through the origin, an embedded surface $\Sigma \subset \mathbb{R}^3$ homeomorphic to the torus $\mathbb{T}^2$ and cylindrically symmetric with respect to the axis ℓ (see [2]).

Contrary to the planar case, it is not known whether these are the *only* complete, embedded, self–similarly shrinking surfaces in $\mathbb{R}^3$ or not.

Concerning self–shrinking clusters of surfaces, it is again natural to restrict our attention to clusters satisfying a suitable "energetic" condition. Precisely, we call an embedded self–shrinking cluster *regular* if intersections occur only between three surfaces, along regular curves, with the three concurring surfaces forming angles of 120 degrees, or between four surfaces, at isolated points where four of the previous intersection curves come together and the tangent vectors of the four curves at the intersection point have a tetrahedral symmetry (that is, they form equal angles like the heights of a regular tetrahedon at their intersection point).

Trivial examples of regular embedded self–shrinking clusters are:

- three planes having in common a line through the origin, forming angles of 120 degrees;
- four planes through the origin, with a tetrahedral symmetry.

Our aim is to investigate about the existence or non–existence of some non–trivial examples of regular self–shrinking clusters of surfaces in $\mathbb{R}^3$. As a starting point, it seems very reasonable to look for shrinkers enjoying several symmetries, both because this might simplify the analysis and because all the known examples of self–shrinkers (either planar or non–planar) are very symmetric. In particular, we are going to look for clusters that enjoy a rotational symmetry about the z–axis: in this way, the equation that has to be satisfied by the profile of the surface (a curve γ in the xz–plane) is an ODE, even though it is slightly more complicated than the one we had to deal with in the planar case.

Introducing the arclength parameter s on the curve γ that defines the profile of the shrinking surface in the xz–plane, we get the equation

$$k + \frac{\langle \gamma_s | e_2 \rangle}{\langle \gamma | e_1 \rangle} + \langle \gamma | \nu \rangle = 0, \tag{4.2}$$

where e_1 is the unit vector parallel to the x–axis, e_2 is the unit vector parallel to the z–axis and, as before, k is the scalar curvature of the curve γ and ν is the unit normal vector of the curve γ. Equation (4.2) looks like equation (1.1) for a planar self–shrinking curve, except for the presence of the term $\frac{\langle \gamma_s | e_2 \rangle}{\langle \gamma | e_1 \rangle}$, which comes from the curvature of the surface Σ in the direction orthogonal to the plane xz. The presence of such a term is rather unpleasant, both because it introduces a singularity of the equation as the curve γ approaches the z–axis (which is expected, since γ has to describe the profile of a smooth surface of rotation around the z–axis), and because it destroys the conservation of the *Energy* (2.1), which was a crucial ingredient in the analysis of planar shrinkers.

We are currently focusing our attention on the question of the existence of an embedded curve γ solving equation (4.2) and such that:

1. it satisfies the initial condition $\gamma(0) = (\alpha, 0)$ for some $\alpha > 0$, $\gamma_s(0) = (\pm 1/2, \sqrt{3}/2)$;
2. it satisfies the "boundary condition" $\gamma(s) \to (0, \beta)$ for some $\beta > 0$ and $\gamma_s(s) \to (-1, 0)$, as s tends to some value $s_0 > 0$;
3. the curve $\gamma((0, s_0))$ has no intersections with the x–axis or with the z–axis.

In the case $\gamma_s(0) = (1/2, \sqrt{3}/2)$, the existence of such a curve would imply the existence of a symmetric regular self–shrinking cluster of surfaces with a shape of a "three–dimensional double bubble". In the case $\gamma_s(0) = (-1/2, \sqrt{3}/2)$, the existence of such a curve would imply the existence of a symmetric regular unbounded self–shrinking cluster of surfaces with a shape of a "three–dimensional lens".

At present, we have not completed a rigorous proof of existence or non–existence in either case. However, numerical evidence suggests that the "three–dimensional lens" shrinker exists, while the "three–dimensional double bubble" shrinker does not exist.

Concerning the "three–dimensional lens", we have elaborated a reasonable strategy to prove its existence (work in progress): the idea is to exploit on one hand the presence of the explicit solution where γ runs on the circle of radius $\sqrt{2}$, which intersects the x–axis with an angle of 90 degrees, on the other hand, we aim at proving that there exist solutions satisfying the "boundary condition" (2) and the "non–degeneracy condition" (3) with very small $\beta > 0$, intersecting the x–axis with a very low angle: this fact is both heuristically reasonable if one makes a linear approximation for very small values of $\beta > 0$ and confirmed by numerical experiments. This, combined with the continuity of the angle of intersection between γ and the x–axis with respect to the parameter β, would imply the existence of a curve gamma satisfying (2)-(3) and $\gamma_s(0) = (-1/2, \sqrt{3}/2)$, i.e. intersecting the x–axis with an angle of 60 degrees.

Concerning the "three–dimensional double bubble", it would be natural to try to compare (4.2) with (1.1). In fact, planar Θ-shrinkers do not exist because solutions to (1.1) "do not bend enough" and the additional term $\frac{\langle \gamma_s | e_2 \rangle}{\langle \gamma | e_1 \rangle}$ in (4.2) seems to force the curve to "bend even less", which might be a reason for which self–shrinking double–bubbles do not exist. However, we have not found yet a proper way to make this comparison rigorous.

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