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# Variational Principles
# and Constraints
# in Continuum Dynamics

*Giovanni Romano, Raffaele Barretta*

*Dept. of Structural Engineering*

*University of Naples Federico II*

## 1 Plan of the presentation

(1) Abstract Action Principle with no fixed-boundary conditions.
(2) EULER extremality conditions, Symmetry Lemma, NOETHER Theorem.
(3) Extremal principles for Lagrangians, Hamilton-Jacobi eikonal equation.
(4) Law of motion in the configuration manifold.
(5) Linear connections in the configuration manifold.
(6) LAGRANGE and HAMILTON laws of motion for connections with torsion.
(7) Locality and criticism of vakonomic Dynamics.
(8) Configuration manifold versus ambient manifold representations.
(9) Extension of kinetic energy: Ansatz of virtual mass conservation.
(10) Nonlinear constraints and multivalued monotone relations.

## 2 Variational approach

A variational approach to Dynamics, in alternative to a differential one [1], is mainly motivated by the viability of a general geometric treatment and by the effectiveness of direct methods in computational Dynamics. The basic tool is the Ampère-Gauss-Green-Ostrogradski-Kelvin-Poincaré integral transformation formula

$$\int_{\Sigma} d\boldsymbol{\omega}^{k-1} = \oint_{\partial\Sigma} \boldsymbol{\omega}^{k-1} \,,$$

(the so-called Stokes-formula) and on the related notion of exterior derivative of volume forms on a $k$-manifold $\Sigma$. A variational approach has the advantage of being independent of the introduction of a connection, leading to more general expressions for the extremality condition. The pleasant flavour of perfection of a law of nature expressed as an extremality property is appealing, but will not be mentioned here.

[1] R. Abraham, J.E. Marsden: Foundations of Mechanics, second edition, the Benjamin/Cummings Publishing Company, Reading Massachusetts (1978)

## 3 Action principle and extremality conditions

- An extremality principle is meant to be a property to be fulfilled by the action integral:

$$\boxed{\int_{\Sigma} \boldsymbol{\omega}^k}$$

of a governing volume form on an evolving $k$-dimensional trajectory manifold $\Sigma$ flying in the (infinite dimensional) container manifold $\mathbb{M}$.

Contrary to most statements, the variational condition of extremality does not express the stationarity of a functional. Rather it requires that, when the trajectory manifold is drifted by a flow, the gap between rate of variation of the action integral and the outward flux of the drifting velocity through the trajectory boundary be equal to the integral of the virtual power of the force form over the trajectory. Denoting by $\mathbf{v}_{\boldsymbol{\Phi}} := \partial_{\lambda=0} \boldsymbol{\Phi}_{\lambda}$, the virtual velocity, the Action Principle states the extremality property as a variational balance law:

$$\boxed{\partial_{\lambda=0} \int_{\boldsymbol{\Phi}_{\lambda}(\Sigma)} \boldsymbol{\omega}^k - \oint_{\partial\Sigma} \boldsymbol{\omega}^k \cdot \mathbf{v}_{\boldsymbol{\Phi}} = \int_{\Sigma} \boldsymbol{\alpha}^{(k+1)} \cdot \mathbf{v}_{\boldsymbol{\Phi}} \,.}$$

This definition is the generalization of the property of a geodesic line on a surface: when the line is drifted by a flow, the rate of change of its length is only due to the lack of equiprojectivity of the flow velocity at the end-points.

STOKES formula, REYNOLDS formula and FUBINI's theorem may be combined to yield the so-called extrusion formula:

$$\boxed{\partial_{\lambda=0} \int_{\mathbf{\Phi}_\lambda(\Sigma)} \boldsymbol{\omega}^k - \oint_{\partial\Sigma} \boldsymbol{\omega}^k \cdot \mathbf{v}_{\mathbf{\Phi}} = \int_\Sigma d\boldsymbol{\omega}^k \cdot \mathbf{v}_{\mathbf{\Phi}} \,.}$$

A localization argument leads to the equivalence between the Action Principle and the generalized EULER's differential condition of extremality which is a homogeneous condition expressed in terms of exterior derivative:

$$\boxed{(d\boldsymbol{\omega}^k - \boldsymbol{\alpha}^{(k+1)}) \cdot \mathbf{v}_\Sigma \cdot \mathbf{v}_{\mathbf{\Phi}} = 0 \,, \qquad \mathbf{v}_\Sigma \in \text{TRIAL}(\Sigma) \,, \quad \forall\, \mathbf{v}_{\mathbf{\Phi}} \in \text{TEST}(\Sigma) \,.}$$

$\text{TRIAL}(\Sigma) \subseteq \mathbb{T}_\Sigma\mathbb{M}$ and $\text{TEST}(\Sigma) \subseteq \mathbb{T}_\Sigma\mathbb{M}$, restriction of $\mathbb{TM}$ to $\Sigma$, with

$$\text{TRIAL}(\Sigma) \supseteq \text{TEST}(\Sigma) \,.$$

Palais' formula for the exterior derivative

$$d\boldsymbol{\omega}^k \cdot \mathbf{v}_\Sigma \cdot \mathbf{v}_{\boldsymbol{\Phi}} = d_{\mathbf{v}_\Sigma}(\boldsymbol{\omega}^k \cdot \mathbf{v}_{\boldsymbol{\Phi}}) - d_{\mathbf{v}_{\boldsymbol{\Phi}}}(\boldsymbol{\omega}^k \cdot \hat{\mathbf{v}}_\Sigma) - \boldsymbol{\omega}^k \cdot [\hat{\mathbf{v}}_\Sigma\,, \hat{\mathbf{v}}_{\boldsymbol{\Phi}}]\,,$$

leads to the equivalent formulation:

- Symmetry Lemma[2]:

$$\boxed{d_{\mathbf{v}_\Sigma}(\boldsymbol{\omega}^k \cdot \mathbf{v}_{\boldsymbol{\Phi}}) = d_{\mathbf{v}_{\boldsymbol{\Phi}}}(\boldsymbol{\omega}^k \cdot \hat{\mathbf{v}}_\Sigma) + \boldsymbol{\alpha}^{(k+1)} \cdot \mathbf{v}_\Sigma \cdot \mathbf{v}_{\boldsymbol{\Phi}}\,,}$$

  where the vector field $\hat{\mathbf{v}}_\Sigma \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM})$ is the transversal extension of the natural frame $\mathbf{v}_\Sigma = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) \in \mathrm{C}^1(\Sigma\,; (\mathbb{T}_{\mathbf{x}}\Sigma)^k)$ performed by pushing the frame along the flow $\mathbf{Fl}_\lambda^{\hat{\mathbf{v}}_{\boldsymbol{\Phi}}} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{M})$ generated by the transversal field $\hat{\mathbf{v}}_{\boldsymbol{\Phi}} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM})$ extension of the transversal virtual velocity field $\mathbf{v}_{\boldsymbol{\Phi}} \in \mathrm{C}^1(\Sigma\,; \mathrm{TEST}(\Sigma))$ to a tubular neighbourhood of $\Sigma$,

In this context, Emmy Noether's theorem is a corollary of the Lemma.

[2] G. Romano, M. Diaco, R. Barretta: The general law of dynamics in nonlinear manifolds and Noether's theorem, in Mathematical Physics Models and Engineering Sciences, Acc. Sc. Fis. Mat. Liguori, Napoli, (2008) 439-453.

## 4 Paradigms of extremal length principles

Paradigmatic extremality principles characterizing paths in riemannian geometry, light rays in geometrical Optics and trajectories in geometrical Dynamics, are (extremality becomes minimality for sufficiently short paths):

- Principle of minimal length (Geodesics),
- Fermat's principle of minimal optical length,
- Maupertuis' principle of least action (principle of minimal dynamical length).

The configuration manifold $\mathbb{C}$ is a riemannian manifold endowed with a metric tensor field $\mathbf{g}$ having, in the various physical contexts, the meaning of

- length metric tensor field,
- optical index tensor field,
- mass metric tensor field.

The extremality principle for the parametrization-independent length of a path $\gamma \in \mathrm{C}^1(I\,;\mathbb{C})$ :

$$\textsc{Lenght}(\gamma) := \int_I \|\mathbf{v}_t\|_{\mathbf{g}}\, dt\,,$$

whose velocity $\mathbf{v}_t := \partial_{\tau=t}\,\gamma(\tau) \in \textsc{test}(\boldsymbol{\gamma})$ is conforming to imposed linear constraints, is expressed by the variational condition:

$$\boxed{\partial_{\lambda=0} \int_I \|T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t)\|_{\mathbf{g}}\, dt = \oint_{\partial I} \langle \frac{\mathbf{g}\mathbf{v}_t}{\|\mathbf{v}_t\|_{\mathbf{g}}}, \delta\mathbf{v}_t \rangle\, dt}$$

where $\boldsymbol{\gamma} = \gamma(I)$ is the path image, $\mathbf{v}_{\boldsymbol{\varphi}} := \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \textsc{test}(\boldsymbol{\gamma})$ is the conforming virtual velocity, $\delta\mathbf{v}_t = \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\tau}(\mathbf{v}_t)) = \mathbf{v}_{\boldsymbol{\varphi}}(\gamma(t))$ and $T$ is the tangent functor.

The Lagrangian is the sublinear functional

$$L(\mathbf{v}) := \sqrt{\mathbf{g}(\mathbf{v}\,,\mathbf{v})} = \|\mathbf{v}\|_{\mathbf{g}}\,,$$

whose fiber-derivative is

$$d_{\mathrm{F}}L(\mathbf{v}) = \frac{\mathbf{g}\mathbf{v}}{\|\mathbf{v}\|_{\mathbf{g}}}\,.$$

The extremality principle for the lenght of a path takes the form of HAMILTON's extremality principle for the action integral associated with a Lagrangian:

$$\textsc{Action}(\gamma) := \int_I L_t(\mathbf{v}_t)\, dt \,,$$

expressed by the variational condition:

$$\boxed{\partial_{\lambda=0} \int_I L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t))\, dt = \oint_{\partial I} \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \delta\mathbf{v}_t \rangle \, dt}$$

The proper mathematical context for the discussion of the extremality principle for the lenght of a path is Convex Analysis [3] .The Hamiltonian is the FENCHEL-LEGENDRE convex conjugate function, that is the indicator of the unit ball in $\mathbb{T}^*\mathbb{C}$ according to the metric $\mathbf{g}^{-1}$ :

$$H(\mathbf{v}^*) := \sup\{\langle \mathbf{v}^*, \mathbf{v} \rangle - \|\mathbf{v}\|_{\mathbf{g}} \mid \mathbf{v} \in \mathbb{T}\mathbb{C}\} = \sqcup_{B^1(\mathbb{T}^*\mathbb{C}, \mathbf{g}^{-1})}(\mathbf{v}^*) \,.$$

[3] G. Romano: New Results in Subdifferential Calculus with Applications to Convex Optimization, Appl. Math. Optim. 32, 213-234 (1995).

The eikonal functional $J_t \in \mathrm{C}^1(\mathbb{C}\,;\Re)$ associated with a central field of trajectories starting at $(\mathbf{x}_0\,, t_0)$ in the configuration-time manifold $\mathbb{C} \times I$, is

$$J(\mathbf{x}, t) := \int_\gamma \mathrm{L}_t(\dot{\gamma}(t))\, dt = \int_{\mathbf{\Gamma}^*_I} \boldsymbol{\omega}^1\,,$$

where $\mathbf{x} = \gamma(t)$, $\mathbf{\Gamma}^*_I$ is the lifted trajectory in the cotangent-time manifold $\mathbb{T}^*\mathbb{C} \times I$ and $\boldsymbol{\omega}^1 := \boldsymbol{\theta} - H dt$ is the fundamental one-form on $\mathbb{T}^*\mathbb{C} \times I$, with the LIOUVILLE one-form $\boldsymbol{\theta}$ on $\mathbb{T}^*\mathbb{C}$ defined by

$$\boldsymbol{\theta}(\mathbf{v}^*) \cdot \mathbf{Y}(\mathbf{v}^*) := \langle\, \mathbf{v}^*, T_{\mathbf{x}}\boldsymbol{\tau}^* \cdot \mathbf{Y}(\mathbf{v}^*)\,\rangle\,, \quad \forall\, \mathbf{Y}(\mathbf{v}^*) \in \mathbb{T}_{\mathbf{v}^*}\mathbb{T}^*\mathbb{C}\,.$$

Setting $\mathbf{v}^*_t = d_{\mathrm{F}} L_t(\dot{\gamma}(t))$, the differential of the action functional is:

$$dJ(\mathbf{x}, t) = \mathbf{v}^*_t - H_t(\mathbf{v}^*_t)dt \in \mathbb{T}^*_{(\mathbf{x},t)}(\mathbb{C} \times I) \iff \begin{cases} dJ_t(\mathbf{x}) = \mathbf{v}^*_t\,, \\ \partial_{\tau=t}\; J_\tau(\mathbf{x}) = -H_t(\mathbf{v}^*_t)\,, \end{cases}$$

which gives the HAMILTON-JACOBI equation for the eikonal functional:

$$\partial_{\tau=t}\; J_\tau(\gamma_t) + H(dJ_t(\gamma_t)) = 0\,.$$

Being $H = \sqcup_{B^1(\mathbb{T}^*\mathbb{C}\mathbf{g}^{-1})}$ , the HAMILTON-JACOBI equation splits into

$$\partial_{\tau=t}\ J_\tau(\gamma_t) = 0\,, \qquad dJ_t(\gamma_t) \in B^1_{\gamma_t}(\mathbb{T}^*\mathbb{C}, \mathbf{g}^{-1})\,,$$

which implies that eikonal inequality $\|dJ(\gamma_t)\|_{\mathbf{g}^{-1}} \leq 1$ .

In components with respect to dual natural bases $\partial q_i$ and $dq^i$ , the expression of the differential of the eikonal functional and of the fundamental one-form

$$dJ(\mathbf{x}, t) = \mathbf{v}^*_t - H_t(\mathbf{v}^*_t)\, dt \qquad \in \mathbb{T}^*_{(\mathbf{x},t)}(\mathbb{C} \times I)\,,$$

$$\boldsymbol{\omega}^1(\mathbf{v}^*, t) = \boldsymbol{\theta}(\mathbf{v}^*_t) - H_t(\mathbf{v}^*_t)\, dt \in \mathbb{T}^*_{(\mathbf{v}^*,t)}(\mathbb{T}^*\mathbb{C} \times I)\,.$$

setting $p_i = d_{\dot{q}_i} L(q, \dot{q})$ , are given by

$$dJ(q, t) = p_i\, dq^i - H_t(q, p)\, dt\,,$$

$$\boldsymbol{\omega}^1(q, p, t) = \{p_i\, dq^i\,, 0\, \partial q_j\} - H_t(q, p)\, dt\,,$$

◇ The fiber-subdifferential of the Hamiltonian (unit ball indicator) at the point $\mathbf{v}^* \in \mathbb{T}^*\mathbb{C}$ is the convex outward normal cone, so that

$$\mathbf{v} \in \mathcal{N}_{B^1(\mathbb{T}^*\mathbb{C},\mathbf{g}^{-1})}(\mathbf{v}^*)\,,$$

to the unit ball $B^1(\mathbb{T}^*\mathbb{C},\mathbf{g}^{-1})$. Hence the trajectory speed is in this cone.

- If $\|\mathbf{v}^*\|_{\mathbf{g}^{-1}} < 1$ then $\mathbf{v}^* \in \mathbb{T}^*\mathbb{C}$ is internal to the unit ball and the normal cone degenerates to the null vector.
- If $\|\mathbf{v}^*\|_{\mathbf{g}^{-1}} = 1$ then $\mathbf{v}^* \in S^1_{\mathbf{x}}(\mathbb{T}^*\mathbb{C},\mathbf{g}^{-1})$ and the normal cone at $\mathbf{v}^* \in \mathbb{T}^*\mathbb{C}$ is the half-line generated by $\mathbf{g}^{-1}(\mathbf{v}^*) \in \mathbb{T}\mathbb{C}$.

It follows that, during propagation, the eikonal equation holds:

$$\|dJ(\gamma_t)\|_{\mathbf{g}^{-1}} = 1\,.$$

A comparison with standard treatments [4] [5] in which the Hamiltonian is said to vanish identically, instead of being an indicator, should be made.

[4] F. John: Partial differential equations, Mathematics Applied to Physics, pp. 229-347, Ed. Roubine É., Springer-Verlag, Berlin (1970).

[5] Y. Choquet-Bruhat. Géométrie Différentielle et Systèmes extérieurs, Travaux et recherches matématiques, Collège de France, Dunod, Paris, (1970).

## 5 Extremal energy principle

The kinetic energy is the quadratic functional $E(\mathbf{v}) := \frac{1}{2}\mathbf{g}(\mathbf{v}\,,\mathbf{v})$.

A path $\gamma \in \mathrm{C}^1(I\,;\mathbb{C})$ fulfils the extremal length principle if and only if, when parametrized with a constant kinetic energy, it fulfils the extremality principle for the energy:

$$\partial_{\lambda=0} \int_I \tfrac{1}{2}\,(\boldsymbol{\varphi}_\lambda{\downarrow}\mathbf{g})(\mathbf{v}_t, \mathbf{v}_t)\,dt = \oint_{\partial I} \mathbf{g}_{\boldsymbol{\tau}(\mathbf{v}_t)}\,(\mathbf{v}_t, \delta\mathbf{v}_t)\,dt\,,$$

which is the action principle for the quadratic Lagrangian

$$L(\mathbf{v}) := \tfrac{1}{2}\mathbf{g}(\mathbf{v}\,,\mathbf{v})\,,$$

with fiber-derivative $d_{\mathrm{F}}L(\mathbf{v}) = \mathbf{g}\mathbf{v}$.

## 6 Geometric Action Principles

Let $\boldsymbol{\theta}_L \in \mathbb{T}^*_{\mathbf{v}}\mathbb{T}\mathbb{C}$ be the Poincaré-Cartan one-form, subordinated to a fiber-differentialble Lagrangian $L_t$ :

$$\langle \boldsymbol{\theta}_L(\mathbf{v}), \mathbf{Y}(\mathbf{v}) \rangle = \langle d_{\mathrm{F}} L(\mathbf{v}), T_{\mathbf{v}} \boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}(\mathbf{v}) \rangle \,, \quad \forall\, \mathbf{Y}(\mathbf{v}) \in \mathbb{T}_{\mathbf{v}}\mathbb{T}\mathbb{C}\,,$$

with $\Gamma \in \mathrm{C}^1(I\,;\mathbb{T}\mathbb{C})$ projecting on the path $\gamma \in \mathrm{C}^1(I\,;\mathbb{C})$ .

The action functional is defined, by Legendre transform, as:

$$\mathcal{A}(\mathbf{v}) := \langle d_{\mathrm{F}} L(\mathbf{v}), \mathbf{v} \rangle = L(\mathbf{v}) + E(\mathbf{v}) \,, \quad \forall\, \mathbf{v} \in \mathbb{T}\mathbb{C}\,.$$

Forces at the configuration $\gamma_t \in \mathbb{C}$ are covectors $\mathbf{f}_t \in \mathbb{T}^*_{\gamma_t}\mathbb{C}$ . On the lifted trajectory $\Gamma = T\gamma \cdot 1$ in the tangent bundle $\mathbb{T}\mathbb{C}$ forces are well-defined as horizontal one-forms $\mathbf{F}_t \in \mathbb{T}^*_{\Gamma_t}\mathbb{T}\mathbb{C}$ by:

$$\langle \mathbf{F}_t(\mathbf{v}_t^{\mathbb{C}}), \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \rangle = \langle \mathbf{f}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}})), T_{\mathbf{v}_t^{\mathbb{C}}} \boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \rangle \,, \quad \forall\, \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \in \mathbb{T}_{\mathbf{v}_t^{\mathbb{C}}}\mathbb{T}\mathbb{C}\,.$$

The action principle for the Lagrangian is equivalent [6] to the following energy constrained action principle:

$$\boxed{\partial_{\lambda=0} \int_{\boldsymbol{\Phi}_\lambda(\boldsymbol{\Gamma}_I)} \boldsymbol{\theta}_L = \oint_{\partial \boldsymbol{\Gamma}_I} \boldsymbol{\theta}_L \cdot \mathbf{v}_{T\boldsymbol{\varphi}} \,,}$$

to hold for any phase-velocity $\mathbf{v}_{T\boldsymbol{\varphi}} := \partial_{\lambda=0} \, T\boldsymbol{\varphi}_\lambda \in \mathrm{TEST}(\boldsymbol{\Gamma})$ fulfilling along $\boldsymbol{\Gamma}$ the constraint of virtual energy balance:

$$\boxed{\langle dE_t - \mathbf{F}_t, \mathbf{v}_{T\boldsymbol{\varphi}} \rangle = 0 \,.}$$

The energy constrained action principle is written explicitly as

$$\boxed{\partial_{\lambda=0} \int_I \mathcal{A}_t(\boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t) \, dt = \oint_{\partial I} \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \delta \mathbf{v}_t \rangle \, dt \,.}$$

[6] G. Romano, R. Barretta, A. Barretta: On Maupertuis principle in dynamics, Reports on Mathematical Physics 63, 3 (2009) 331-346.

The related EULER's differential condition of extremality is:

$$d\boldsymbol{\theta}_L \cdot \dot{\mathbf{v}}_t \cdot \mathbf{v}_{T\boldsymbol{\varphi}}(\mathbf{v}_t) = 0\,, \quad \dot{\mathbf{v}}_t \in \text{TRIAL}(\boldsymbol{\Gamma})\,, \quad \forall\, \mathbf{v}_{T\boldsymbol{\varphi}}(\mathbf{v}_t) \in \text{TEST}(\boldsymbol{\Gamma}) \cap \mathbf{ker}(dE_{\mathbf{g}}(\mathbf{v}_t) - \mathbf{F}(\mathbf{v}_t))\,.$$

LAGRANGE's multiplier theorem assures that the parametrization of the path $\Gamma \in \mathrm{C}^1(I\,;\mathbb{TC})$ can be fixed so that energy balance is fulfilled along the path and that EULER's differential condition of extremality is equivalent to:

$$d\boldsymbol{\theta}_L \cdot \dot{\mathbf{v}}_t \cdot \mathbf{v}_{T\boldsymbol{\varphi}}(\mathbf{v}_t) = (\mathbf{F} - dE)(\mathbf{v}_t) \cdot \mathbf{v}_{T\boldsymbol{\varphi}}(\mathbf{v}_t)\,, \quad \forall\, \mathbf{v}_{T\boldsymbol{\varphi}}(\mathbf{v}_t) \in \text{TEST}(\boldsymbol{\Gamma})\,.$$

In the velocity-time manifold we introduce the action one-form

$$\boldsymbol{\omega}^1_L(\mathbf{v}_t\,, t) := \boldsymbol{\theta}_L(\mathbf{v}_t) - E(\mathbf{v}_t)dt\,,$$

and the force two-form $\boldsymbol{\alpha}^2 := dt \wedge \mathbf{F}$. EULER's differential condition for the trajectory $\boldsymbol{\Gamma}_I(t) := (\boldsymbol{\Gamma}(t)\,, t)$ writes

$$\boxed{(d\boldsymbol{\omega}^1_L - \boldsymbol{\alpha}^2) \cdot (\dot{\mathbf{v}}_t\,, 1_t) \cdot (\mathbf{v}_{\boldsymbol{\Phi}}(\mathbf{v}_t)\,, 0) = 0\,,}$$

with $(\dot{\mathbf{v}}_t\,, 1_t) \in \text{TRIAL}(\boldsymbol{\Gamma}_I)$, for any phase-velocity field $\mathbf{v}_{\boldsymbol{\Phi}} \in \text{TEST}(\boldsymbol{\Gamma}) \subset \mathbb{TTC}$ which projects to a virtual velocity field $\mathbf{v}_{\boldsymbol{\varphi}} \in \text{CONF} \subseteq \mathbb{TC}$: $T\boldsymbol{\tau} \circ \mathbf{v}_{\boldsymbol{\Phi}} = \mathbf{v}_{\boldsymbol{\varphi}} \circ \boldsymbol{\tau}$.

This is the extremality condition for the synchronous geometric action principle for the trajectory $\mathbf{\Gamma}_I$ in the velocity-time bundle $\mathbb{TC} \times I$ :

$$\boxed{\partial_{\lambda=0} \int_{\mathbf{\Phi}_\lambda(\mathbf{\Gamma}_I)} \boldsymbol{\omega}^1_L = \oint_{\partial\mathbf{\Gamma}_I} \boldsymbol{\omega}^1_L \cdot (\mathbf{v_\Phi}\,, 0) + \int_{\mathbf{\Gamma}_I} \boldsymbol{\alpha}^2 \cdot (\mathbf{v_\Phi}\,, 0)\,,}$$

equivalent to the asynchronous action principle which, in the covelocity-time bundle $\mathbb{T}^*\mathbb{C} \times I$ , writes:

$$\boxed{\partial_{\lambda=0} \int_{\mathbf{\Phi}^*_\lambda(\mathbf{\Gamma}^*_I)} \boldsymbol{\omega}^1 = \oint_{\partial\mathbf{\Gamma}^*_I} \boldsymbol{\omega}^1 \cdot (\mathbf{v}_{\mathbf{\Phi}^*}\,, \Theta) + \int_{\mathbf{\Gamma}^*_I} \boldsymbol{\alpha}^2 \cdot (\mathbf{v}_{\mathbf{\Phi}^*}\,, \Theta)\,,}$$

with $\Theta$ a time-change speed and $\boldsymbol{\omega}^1 := \boldsymbol{\theta} - H\,dt$ .

In applications to Dynamics, the action one-form (and the generating Lagrangian) is defined only along the dynamical trajectory. The kinetic energy is in fact defined only along the trajectory. In performing the variations, the Lagrangian should then be extended in a proper way outside the trajectory. This extension is a basic assumption to be explicitly declared in the Action Principle.

## 7 Laws of dynamics in the configuration manifold

The extremality principle in the configuration manifold is:

$$\oint_{\partial I} \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \delta\mathbf{v}_t \rangle - \partial_{\lambda=0} \int_I L_t \circ T\boldsymbol{\varphi}_\lambda \circ \mathbf{v} \circ \gamma \, dt = \int_I \langle \mathbf{f}_t, \delta\mathbf{v}_t \rangle \, dt \,,$$

for any virtual flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\boldsymbol{\gamma}\,; \mathbb{C})$ with velocity $\delta\mathbf{v}_t = \mathbf{v}_{\boldsymbol{\varphi}}(\gamma_t) \in \mathrm{TEST}_{\gamma_t}\mathbb{C}$, being $\mathbf{v}_t = \mathbf{v}_{\boldsymbol{\gamma}}(\gamma_t)$. The law of motion in the configuration manifold writes:

$$\partial_{\tau=t} \langle d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \delta\mathbf{v}_\tau \rangle - \partial_{\lambda=0} L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t)) = \langle \mathbf{f}_t(\gamma_t), \delta\mathbf{v}_t \rangle \,, \quad \forall\, \delta\mathbf{v}_t \in \mathrm{CONF}(\gamma_t) \cap \mathrm{RIG}(\gamma_t) \,,$$

with the drifting flow defined in a neighbourhood of the trajectory.

A basic result in Dynamics states that it is only the virtual velocity at the current configuration to be involved in the extremality condition.

The proof of this result requires a linear connection and the related covariant derivative $\nabla$ to be given in the configuration manifold $\mathbb{C}$.

## 8 Laws of dynamics in terms of a connection

The tangent to the composition $L \circ \mathbf{v} \in \mathrm{C}^1(\mathbb{C}\,;\Re)$ splits as:

$$T(L \circ \mathbf{v}) = TL \circ T\mathbf{v} = d_{\mathrm{F}} L(\mathbf{v}) \cdot \nabla \mathbf{v} + d_{\mathrm{B}} L(\mathbf{v})\,.$$

with the base derivative given by: $d_{\mathrm{B}} L(\mathbf{v_x}) \cdot \mathbf{w_x} = \partial_{\lambda=0}\, L(\mathbf{Fl}^{\mathbf{w}}_{\lambda} \Uparrow \mathbf{v_x})$ .Then

$$\begin{aligned}\partial_{\lambda=0}\,(L_t \circ T\boldsymbol{\varphi}_{\lambda} \circ \mathbf{v})(\gamma_t) &= T(L_t \circ \hat{\mathbf{v}}_{\boldsymbol{\gamma}}) \cdot \delta\mathbf{v}_t \\ &= d_{\mathrm{F}} L_t(\mathbf{v}_t) \cdot \nabla_{\delta\mathbf{v}_t} \hat{\mathbf{v}}_{\boldsymbol{\gamma}} + d_{\mathrm{B}} L_t(\mathbf{v}_t) \cdot \delta\mathbf{v}_t\,,\end{aligned}$$

where $\hat{\mathbf{v}}_{\boldsymbol{\gamma}} \circ \mathbf{Fl}^{\mathbf{v}_{\varphi}}_{\lambda} := \mathbf{Fl}^{\mathbf{v}_{\varphi}}_{\lambda} \uparrow \mathbf{v}_{\boldsymbol{\gamma}}$, so that $[\hat{\mathbf{v}}_{\boldsymbol{\gamma}}\,, \mathbf{v}_{\boldsymbol{\varphi}}] = 0$. Moreover

$$\begin{aligned}\partial_{\tau=t}\,\langle d_{\mathrm{F}} L_{\tau}(\mathbf{v}_{\tau}), \delta\mathbf{v}_{\tau}\rangle &= \partial_{\tau=t}\,\langle \gamma_{t,\tau}\Uparrow d_{\mathrm{F}} L_{\tau}(\mathbf{v}_{\tau}), \gamma_{t,\tau}\Uparrow \delta\mathbf{v}_{\tau}\rangle \\ &= \langle \partial_{\tau=t}\,\gamma_{t,\tau}\Uparrow d_{\mathrm{F}} L_{\tau}(\mathbf{v}_{\tau}), \delta\mathbf{v}_t\rangle + \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \partial_{\tau=t}\,\gamma_{t,\tau}\Uparrow \delta\mathbf{v}_{\tau}\rangle \\ &= \langle \partial_{\tau=t}\, d_{\mathrm{F}} L_{\tau}(\mathbf{v}_{\tau}), \gamma_{\tau,t}\Uparrow \delta\mathbf{v}_t\rangle + \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \nabla_{\mathbf{v}_t}\mathbf{v}_{\boldsymbol{\varphi}}\rangle\,.\end{aligned}$$

## 9 Generalized Lagrange and Hamilton laws of dynamics

An analysis based on tensoriality of the torsion [7] shows that the extremality condition may be written as a LAGRANGE's law of Dynamics in the form:

$$\begin{cases} \langle \partial_{\tau=t}\, d_{\mathrm{F}} L_{\tau}(\mathbf{v}_{\tau}), \gamma_{\tau,t}\Uparrow \delta\mathbf{v}_t \rangle - \langle d_{\mathrm{B}} L_t(\mathbf{v}_t), \delta\mathbf{v}_t \rangle + \langle \mathbf{v}_t^*, \mathrm{TORS}(\mathbf{v}_t, \delta\mathbf{v}_t) \rangle = \langle \mathbf{f}_t(\gamma_t), \delta\mathbf{v}_t \rangle \,, \\ \mathbf{v}_t^* = d_{\mathrm{F}} L_t(\mathbf{v}_t)\,, \quad \forall\, \delta\mathbf{v}_t \in \mathrm{CONF}(\gamma_t) \cap \mathrm{RIG}(\gamma_t)\,. \end{cases}$$

LAGRANGE's law may be transformed by means of a generalized version of DONKIN's theorem (1854):

$$d_{\mathrm{B}} H_t(\mathbf{v}_t^*) + d_{\mathrm{B}} L_t(d_{\mathrm{F}} H_t(\mathbf{v}_t^*)) = 0\,.$$

to get HAMILTON's law of Dynamics in the form:

$$\begin{cases} \langle \partial_{\tau=t}\, \mathbf{v}_{\tau}^*, \gamma_{\tau,t}\Uparrow \delta\mathbf{v}_t \rangle + \langle d_{\mathrm{B}} H_t(\mathbf{v}_t^*), \delta\mathbf{v}_t \rangle + \langle \mathbf{v}_t^*, \mathrm{TORS}(\mathbf{v}_t, \delta\mathbf{v}_t) = \langle \mathbf{f}_t(\gamma_t), \delta\mathbf{v}_t \rangle \rangle \,, \\ \mathbf{v}_t = d_{\mathrm{F}} H_t(\mathbf{v}_t^*)\,, \quad \forall\, \delta\mathbf{v}_t \in \mathrm{CONF}(\gamma_t) \cap \mathrm{RIG}(\gamma_t)\,. \end{cases}$$

[7] G. Romano, R. Barretta and M. Diaco, On the general form of the law of dynamics, Int. J. Non-Linear Mech., 44, (2009) 689-695.

## 10 Criticism of vakonomic mechanics

As a rule, forces and constraints on dynamical systems are defined only along the trajectory and are known only till the actual time. The property inferred from the generalized LAGRANGE's and HAMILTON's laws of Dynamics is then of fundamental importance and may be stated as follows:

| |
|---|
| Locality property: All flows sharing the same virtual velocity along the trajectory in the configuration manifold are equivalent in testing the extremality. |

| |
|---|
| Data: The dynamical equilibrium condition depends on the constraint relation between the force systems acting of the body and the velocity fields pertaining to configurations along the dynamical trajectory. |

Integrability issues concerning the constraints are therefore in most cases either not applicable or inessential and then also whether constraints are holonomic or nonholonomic is of no concern. According to these considerations, the recent emphasis on nonholonomic dynamics appears to be overemphasized and vakonomic mechanics should be considered as a proposal to be rejected.

## 11 Vakonomic mechanics

In [8] the variational principle of vakonomic mechanics is stated as follows.

The trajectory of a vakonomic dynamical system in the configuration manifold is a regular path $\gamma \in C^1(I\,;\mathbb{C})$ fulfilling the extremality principle:

$$\int_I \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_{\boldsymbol{\gamma}}, \nabla_{\mathbf{v}_{\gamma}} \mathbf{v}_{\boldsymbol{\varphi}} \rangle \, dt + \int_I \langle d_{\mathrm{B}} L_t \circ \mathbf{v}_{\boldsymbol{\gamma}}, \mathbf{v}_{\boldsymbol{\varphi}} \rangle \, dt = 0 \,,$$

for all virtual velocities $\mathbf{v}_{\boldsymbol{\varphi}} \in C^1(\boldsymbol{\gamma}\,;\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C})$ vanishing at the boundary points of the interval $I$ and fulfilling the symmetry condition:

$$\boxed{\mathbf{W}(\mathbf{v}_{\boldsymbol{\gamma}}, \mathbf{v}_{\boldsymbol{\varphi}}) = \mathbf{W}(\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_{\boldsymbol{\gamma}})}$$

for the Weingarten map $\mathbf{W}(\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_{\boldsymbol{\gamma}}) := \mathbf{\Pi}^{\perp} \nabla_{\mathbf{v}_{\varphi}} \hat{\mathbf{v}}_{\boldsymbol{\gamma}}$ with $\hat{\mathbf{v}}_{\boldsymbol{\gamma}} \in C^1(\mathbb{C}\,;\mathbb{T}\mathbb{C})$ extension of $\mathbf{v}_{\boldsymbol{\gamma}} \in C^1(\boldsymbol{\gamma}\,;\boldsymbol{\Delta}_{\boldsymbol{\gamma}})$ and $\nabla$ a torsion-free linear connection.

[8] I. Kupka, W.M. Oliva: The Non-Holonomic Mechanics, J. Differential Equations, 169, 169-189 (2001).

It can be proved that [9], if the linear connection is torsion-free and the trajectory speed fulfils the linear constraint i.e. $\mathbf{v}_{\boldsymbol{\gamma}} \in \mathrm{C}^1(\boldsymbol{\gamma}\,; \boldsymbol{\Delta}_{\boldsymbol{\gamma}})$, then

$$\boxed{\mathbf{W}(\mathbf{v}_{\gamma}, \mathbf{v}_{\varphi}) = \mathbf{W}(\mathbf{v}_{\varphi}, \mathbf{v}_{\gamma})}$$

for any virtual velocity $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(\boldsymbol{\gamma}\,; \mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C})$. So the condition put on the virtual velocity field results to be identically satisfied.

Questions to KOZLOV:

How to analyze the dynamical behavior of a byke or of a car driven by a controller who applies forces and kinematical constraints?

Which are the constraints and the forces in a varied configuration?

How to prove the existence of a field of lagrangian multipliers?

Vakonomic mechanics is neither mathematically consistent nor physically sound.

[9] G. Romano, R. Barretta: Variational Principles and Constraints in Continuum Dynamics, research report, University of Naples Federico II (2008)

## 12 The missing link

The link between the formulations of Dynamics in the ambient manifold $\mathcal{S}$ and in the configuration manifold $\mathbb{C}$ is most conveniently described by means of the position map [10] :

> $\mathrm{POS}_{\mathbf{p}} \in \mathrm{C}^1(\mathbb{C}\,;\mathcal{S})$ is a surjective submersion providing the position
>
> $$\mathrm{POS}_{\mathbf{p}}(\boldsymbol{\xi}) := \boldsymbol{\xi}(\mathbf{p}) \in \boldsymbol{\xi}(\mathcal{B})$$
>
> of a particle $\mathbf{p} \in \mathcal{B}$ at the configuration $\boldsymbol{\xi} \in \mathrm{C}^1(\mathcal{B}\,;\mathcal{S})$ .

To any $\mathbf{p} \in \mathcal{B}$ there corresponds a fiber bundle $(\mathbb{C}\,, \mathrm{POS}_{\mathbf{p}}\,, \mathcal{S})$, whose fiber over the position $\boldsymbol{\xi}(\mathbf{p}) \in \mathcal{S}$ is the class of all configurations $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{B}\,;\mathcal{S})$ mapping the particle into that position. The surjective tangent map $T_{\boldsymbol{\xi}}\mathrm{POS}_{\mathbf{p}} \in BL(\mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}\,;\mathbb{T}_{\mathrm{POS}_{\mathbf{p}}(\boldsymbol{\xi})}\mathcal{S})$ induces a linear correspondence: $\mathbf{v}_{\mathrm{POS}_{\mathbf{p}}(\boldsymbol{\xi})} = T_{\boldsymbol{\xi}}\mathrm{POS}_{\mathbf{p}} \cdot \mathbf{v}^{\mathbb{C}}_{\boldsymbol{\xi}}$ , between the tangent spaces, being $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\xi}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$ and $\mathbf{v}_{\mathrm{POS}_{\mathbf{p}}(\boldsymbol{\xi})} \in \mathbb{T}_{\mathrm{POS}_{\mathbf{p}}(\boldsymbol{\xi})}\mathcal{S}$ .

[10] G. Romano, R. Barretta: On Continuum Dynamics, to appear in the October issue of Journal of Mathematical Physics (2009).

Let a tangent vector field $\mathbf{u}$ be given on a placement $\boldsymbol{\xi}(\mathcal{B}) \subset \mathcal{S}$ with $\mathbf{u} \in \mathbb{T}_{\mathrm{POS}_{\mathbf{p}}(\boldsymbol{\xi})}\mathcal{S}$, for any $\mathbf{p} \in \mathcal{B}$. Then the tangent vector $\mathbf{u}^{\mathbb{C}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$ is well-defined by the $\mathrm{POS}_{\mathbf{p}}$-relatedness:

$$T\mathrm{POS}_{\mathbf{p}} \circ \mathbf{u}^{\mathbb{C}} = \mathbf{u} \circ \mathrm{POS}_{\mathbf{p}} \,, \quad \forall\, \mathbf{p} \in \mathcal{B} \,.$$

There is a natural way of endowing $\mathbb{C}$, an infinite dimensional manifold of maps, with a connection induced by one in the finite dimensional space $\mathcal{S}$. [11] [12]

Indeed, a parallel transport in $\mathcal{S}$ yields the parallel transport $\mathbb{C}$ defined by $\mathrm{POS}_{\mathbf{p}}$-relatedness: $T\mathrm{POS}_{\mathbf{p}} \circ \mathbf{Fl}_{\lambda}^{\mathbf{v}^{\mathbb{C}}}\Uparrow \mathbf{u}^{\mathbb{C}} = \mathbf{Fl}_{\lambda}^{\mathbf{v}}\Uparrow \mathbf{u} \circ \mathrm{POS}_{\mathbf{p}} \,, \quad \forall\, \mathbf{p} \in \mathcal{B}$, so that

$$\begin{aligned}
T\mathrm{POS}_{\mathbf{p}} \circ \nabla^{\mathbb{C}}_{\mathbf{v}^{\mathbb{C}}} \mathbf{u}^{\mathbb{C}} &= \nabla_{\mathbf{v}}\mathbf{u} \circ \mathrm{POS}_{\mathbf{p}} \,, \\
T\mathrm{POS}_{\mathbf{p}} \circ [\mathbf{v}^{\mathbb{C}} , \mathbf{u}^{\mathbb{C}}] &= [\mathbf{v} , \mathbf{u}] \circ \mathrm{POS}_{\mathbf{p}} \,, \\
T\mathrm{POS}_{\mathbf{p}} \circ \mathrm{TORS}^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}}, \mathbf{u}^{\mathbb{C}}) &= \mathrm{TORS}(\mathbf{v}, \mathbf{u}) \circ \mathrm{POS}_{\mathbf{p}} \,, \\
T\mathrm{POS}_{\mathbf{p}} \circ \mathrm{CURV}^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}}, \mathbf{u}^{\mathbb{C}})(\mathbf{w}^{\mathbb{C}}) &= \mathrm{CURV}(\mathbf{v}, \mathbf{u})(\mathbf{w}) \circ \mathrm{POS}_{\mathbf{p}} \,.
\end{aligned}$$

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[11] H.I. Eliasson, Geometry of Manifolds of Maps, J. Diff. Geom. **1**, 169-194 (1967)
[12] R.S. Palais, *Foundations of Global Non-Linear Analysis*, (Benjamin, New York 1968)

Setting $\boldsymbol{\Omega}_{\boldsymbol{\xi}} = \boldsymbol{\xi}(\mathbb{B})$, the metric of the riemannian ambient manifold $\{\mathcal{S}\,,\mathbf{g}\}$ induces in the configuration manifold a metric defined by

$$\mathbf{g}^{\mathbb{C}}(\mathbf{u}^{\mathbb{C}},\mathbf{v}^{\mathbb{C}}) := \int_{\boldsymbol{\Omega}_{\boldsymbol{\xi}}} \mathbf{g}(\mathbf{u},\mathbf{v})\,\mathbf{m}_{\boldsymbol{\xi}}\,,$$

and the mass form $\mathbf{m}_{\boldsymbol{\xi}}$ is such that displacements preserve the mass. The lagrangian $L_t^{\mathbb{C}} \in \mathrm{C}^0(\mathbb{TC}\,;\Re)$ on the velocity manifold is defined by:

$$(L_t^{\mathbb{C}} \circ \mathbf{v}_{\boldsymbol{\gamma}}^{\mathbb{C}})(\gamma_t) := \int_{\boldsymbol{\Omega}_t} (L_t \circ \mathbf{v}_t)\,\mathbf{m}_t\,.$$

It is assumed that the mass-form is drifted by virtual flows:

$$\mathcal{L}_{\delta\mathbf{v}_t}\mathbf{m}_t = 0 \iff \partial_{\lambda=0} \int_{\boldsymbol{\varphi}_{\lambda,t}(\mathcal{P})} \mathbf{m}_t = 0\,, \quad \forall\,\mathcal{P} \subseteq \boldsymbol{\Omega}_t\,.$$

Setting $\mathrm{T{\scriptstyle ORS}}(\mathbf{v})\cdot\mathbf{u} = \mathrm{{\scriptstyle TORS}}(\mathbf{v},\mathbf{u})\,, \quad \forall\,\mathbf{v},\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ it is:

$$\mathcal{L}_{\delta\mathbf{v}_t}\mathbf{m}_t = \nabla_{\delta\mathbf{v}_t}\,\mathbf{m}_t + \mathrm{tr}(\nabla\delta\mathbf{v}_t + \mathrm{T{\scriptstyle ORS}}(\delta\mathbf{v}_t))\,\mathbf{m}_t\,,$$

so that virtual conservation of mass involves only the virtual velocity at the actual placement.

## 13 The law of motion in the ambient manifold

Let the force form be defined by a body force field (per unit volume) and a boundary traction field (per unit surface):

$$\langle \mathbf{f}_t, \delta\mathbf{v}^{\mathbb{C}} \rangle := \int_{\boldsymbol{\Omega}_\xi} \langle \mathbf{b}_t, \delta\mathbf{v} \rangle \, \boldsymbol{\mu} + \int_{\partial\boldsymbol{\Omega}_\xi} \langle \mathbf{t}_t, \delta\mathbf{v} \rangle \, \partial\boldsymbol{\mu} \,,$$

The ansatz of virtual conservation of mass and the generalized LAGRANGE law in the configuration manifold, yield the generalized EULER's law of motion in the ambient manifold:

$$\partial_{\tau=t} \int_{\gamma_{\tau,t}(\boldsymbol{\Omega}_t)} \langle d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \gamma_{\tau,t}\Uparrow \delta\mathbf{v}_t \rangle \, \mathbf{m}_\tau - \int_{\boldsymbol{\Omega}_t} \langle d_{\mathrm{B}} L_t(\mathbf{v}_t), \delta\mathbf{v}_t \rangle \, \mathbf{m}_t$$

$$+ \int_{\boldsymbol{\Omega}_t} \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \mathrm{TORS}(\mathbf{v}_t, \delta\mathbf{v}_t) \rangle \, \mathbf{m}_t = \int_{\boldsymbol{\Omega}_t} \langle \mathbf{b}_t, \delta\mathbf{v}_t \rangle \, \boldsymbol{\mu} + \int_{\partial\boldsymbol{\Omega}_t} \langle \mathbf{t}_t, \delta\mathbf{v}_t \rangle \, \partial\boldsymbol{\mu} \,,$$

for any conforming and rigid virtual velocity field $\delta\mathbf{v}_t \in \mathrm{CONF}(\boldsymbol{\Omega}_t) \cap \mathrm{RIG}(\boldsymbol{\Omega}_t)$ .

## 14 Constraints

### *14.1 Linear Constraints*

A most useful concept in Mechanics is that of perfect, bilateral constraints (ideal constraints), introduced in its modern formulation by d'Alembert.

The mathematical modeling consists in assuming that, at any constrained configuration, conforming velocities belong to a linear subspace CONF of the tangent fibre and that reactive force systems belong to its annihilator $(\text{CONF})^{\circ}$ in the cotangent fibre.

The theoretical and computational merits of this concept are far reaching. It allows to perform the (exact or approximate) evaluation of a solution to a dynamical problem either by solving a linear problem or a sequence of global linear trials followed by nonlinear local corrections, in an algorithmic iterative scheme. The evaluation of reactive forces requires, as a rule, that a constitutive elastic behaviour be defined for the material.

## 14.2 Nonlinear Constraints

In many important engineering problems nonlinear behaviours play an essential role and their simulation is most conveniently described by maximal monotone potential relations between dual variables [13]. These relations are called constitutive laws and play a basic role in describing the essential aspect of inherently nonlinear material behaviours such as plasticity, friction, phase transformations etc. To monotone relations there correspond a right and a left multivalued map. Linear (or affine) constraints are special cases in this class of constitutive laws characterized by the property that the maximal monotone multivalued maps are constant. This is the only case in which the knowledge of domain and codomain completely describes the relation.

Nonlinear constraints must instead be properly described as maps between dual spaces. This observation should help in resolving the longly debated issue of nonlinear kinematical constraints in Dynamics.

[13] G. Romano, L. Rosati, F. Marotti de Sciarra, P. Bisegna: A potential theory for monotone multi-valued operators, Quart. Appl. Math. 51 (4) 613-631 (1993).

## 15 Maximal monotone potential relations

A fairly general formulation [14] of this kind of constraints is got by considering in each linear fibre of the Whitney product $\mathbb{T}\mathbb{C} \times_{\mathbb{C}} \mathbb{T}^*\mathbb{C}$ a monotone maximal and potential graph:

$$\begin{aligned}
&(\mathbf{v}\,,\mathbf{v}^*) \in \mathcal{G} \iff \mathbf{v}^* \in \boldsymbol{\lambda}(\mathbf{v}) \iff \mathbf{v} \in \boldsymbol{\rho}(\mathbf{v}^*) \\
&\langle\, \mathbf{v}_2^* - \mathbf{v}_1^*, \mathbf{v}_2 - \mathbf{v}_1 \,\rangle \geq 0\,, \quad \forall\,(\mathbf{v}_1\,,\mathbf{v}_1^*),(\mathbf{v}_2\,,\mathbf{v}_2^*) \in \mathcal{G} \\
&\oint_{\mathbf{c}} \boldsymbol{\lambda}(\mathbf{v}) = 0 \iff \oint_{\mathbf{c}^*} \boldsymbol{\rho}(\mathbf{v}^*) = 0\,,
\end{aligned}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ are the left and right multivalued maps associated with the graph $\mathcal{G}$ and $\mathbf{c} \subset \mathbb{T}_{\mathbf{x}}\mathbb{C}$ ( $\mathbf{c}^* \subset \mathbb{T}^*_{\mathbf{x}}\mathbb{C}$ ) is an arbitrary closed polyline in the domain of $\boldsymbol{\lambda}$ ( $\boldsymbol{\rho}$ ). Potentiality, expressed by the vanishing of the circuital integrals, means that left $\boldsymbol{\lambda}$ and right $\boldsymbol{\rho}$ multivalued maps associated with the graph $\mathcal{G}$ are subdifferentials of convex conjugate potentials.

[14] G. Romano: Continuum Mechanics on Manifolds, Lecture notes, University of Naples Federico II, Italy, URL http://wpage.unina.it/romano/ (2007-2009).

## 16 Conclusions (features) in ten points

(1) Variational statements are written explicitly in terms of trial and test fields.
(2) The unspecified and/or undefined variation symbol $\delta$ is never adopted in expressing extremality properties.
(3) The fixed-ends assumption in action principles has been eliminated.
(4) Covariant derivatives are introduced only when useful and connection are left as general as possible.
(5) Extensions of the Lagrangian outside the trajectory are explicitly declared.
(6) Convex Analysis concepts and methods are adopted when fibrewise non-differentiable Lagrangians are considered.
(7) An explicit statement of the relations between ambient and configuration manifolds is made.
(8) Component expressions are avoided in general treatments, are limited to specific formulations and properly stated.
(9) Linear constraints to be fulfilled by virtual displacement fields are required to be defined only at the actual placement along the trajectory.
(10) Nonlinear constraints are expressed as multivalued monotone relations between dual variables.