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The Covariance Paradigm in Nonlinear Continuum Mechanics

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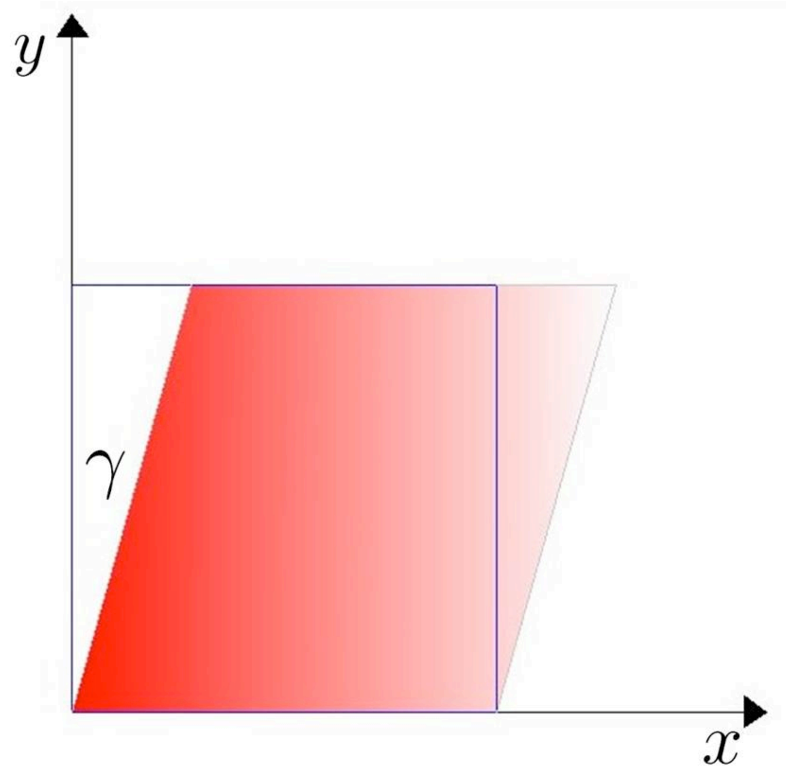
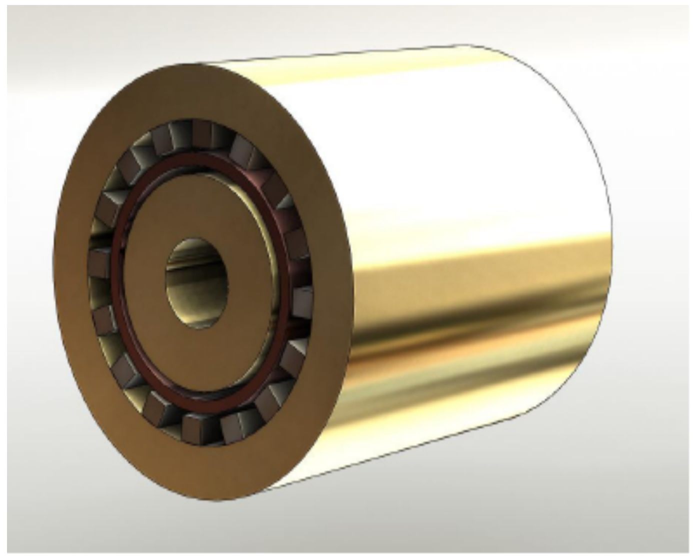
Two paradigmatic examples of the role of differential geometry in classical physics

Rate laws of material behavior

and

Electromagnetic induction

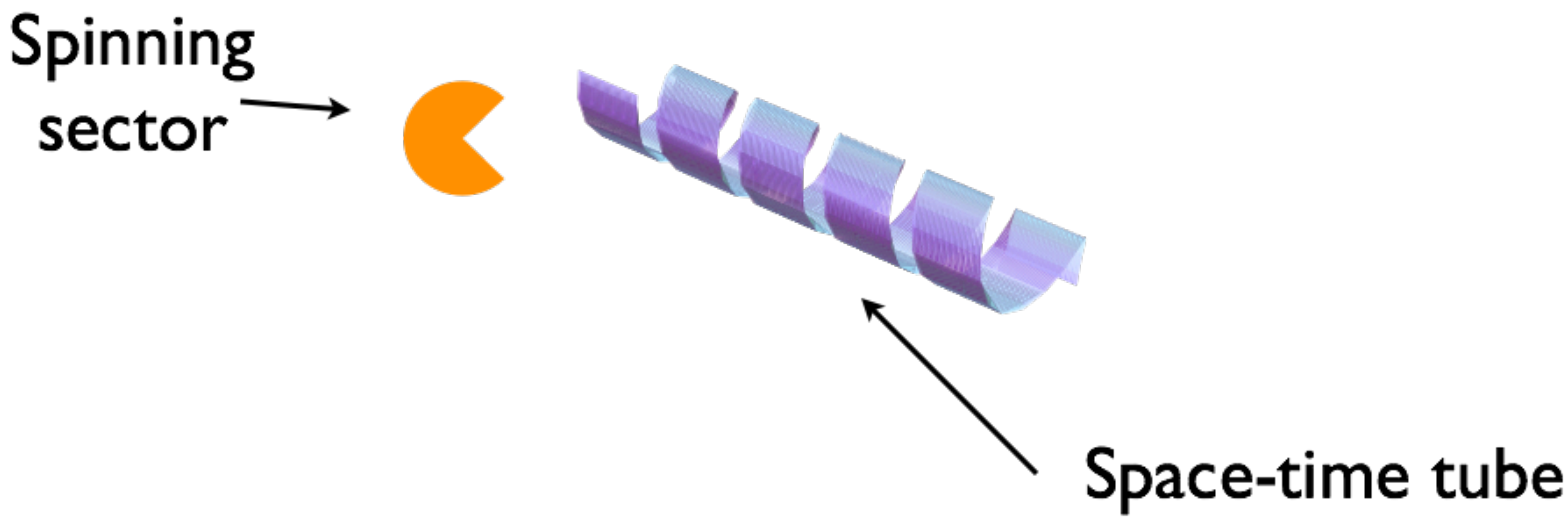




PART I

The Covariance Paradigm

in material behavior



Continuum kinematics

Described by the following differentiable geometric structures:

- The ambient space $\mathcal{S}$, in which motions take place, is a finite dimensional Riemann manifold without boundary, endowed with a metric tensor field $\hat{\mathbf{g}}$;
- The material body $\mathcal{B}$ is a finite dimensional manifold with boundary, of dimension less than or equal to the one of the ambient space;
- The observation time interval I an open, connected subset of the reals;
- The configurations manifold $\mathbb{C}$, an infinite dimensional manifold of maps which are C^1-diffeomorphisms of the body manifold onto submanifolds of the ambient space manifold.

Motions and displacements

A *motion* is described by a map $\hat{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,; \mathcal{S})$ from the manifold $\mathcal{B} \times I$ of *material events* into the ambient space manifold $(\mathcal{S}\,, \hat{\mathbf{g}})$.

To a motion there correspond at each time $t \in I$ a *material configuration* map $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,; \boldsymbol{\Omega}_t)$ which is a diffeomorphisms of the body manifold $\mathcal{B}$ onto the placement manifold $\boldsymbol{\Omega}_t$.

The *material displacement* from a source placement $\boldsymbol{\Omega}_t = \boldsymbol{\varphi}_t(\mathcal{B})$ to the target placement $\boldsymbol{\Omega}_\tau = \boldsymbol{\varphi}_\tau(\mathcal{B})$, is the diffeomorphism

$$\boldsymbol{\varphi}_{\tau,t} := \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,; \boldsymbol{\Omega}_\tau)\,,$$

providing the position in $\boldsymbol{\Omega}_\tau$ at time $\tau \in I$ of the particle which occupies the given position in $\boldsymbol{\Omega}_t$ at time $t \in I$.

The inclusion map

To emphasize the distinction between material fields and spatial fields, it is expedient to consider the inclusion map:

$$\mathbf{i}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathcal{S})$$

We denote by:

- $\hat{\varphi}_t = \mathbf{i}_{\varphi,t} \circ \boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\mathbf{i}_{\varphi,t}(\boldsymbol{\Omega}_t))$ → the spatial configuration map,
- $\hat{\varphi}_{\tau,t} \in \mathrm{C}^1(\mathbf{i}_{\varphi,t}(\boldsymbol{\Omega}_t)\,;\mathbf{i}_{\varphi,\tau}(\boldsymbol{\Omega}_\tau))$ → the spatial displacement map,
- $\boldsymbol{\varphi}_{\tau,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\boldsymbol{\Omega}_\tau)$ → the material displacement map.

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\hat{\varphi}_{\tau,t}"] & \mathcal{S} \\ \boldsymbol{\Omega}_t \arrow[u, "\mathbf{i}_{\varphi,t}"] \arrow[r, "\boldsymbol{\varphi}_{\tau,t}"'] & \boldsymbol{\Omega}_\tau \arrow[u, "\mathbf{i}_{\varphi,\tau}"'] \end{tikzcd} \iff \hat{\varphi}_{\tau,t} \circ \mathbf{i}_{\varphi,t} := \mathbf{i}_{\varphi,\tau} \circ \boldsymbol{\varphi}_{\tau,t}\,.$$

$$\begin{tikzcd} \mathbb{T}\mathcal{S} \arrow[r, "T\hat{\varphi}_{\tau,t}"] & \mathbb{T}\mathcal{S} \\ \mathbb{T}\boldsymbol{\Omega}_t \arrow[u, "T\mathbf{i}_{\varphi,t}"] \arrow[r, "T\boldsymbol{\varphi}_{\tau,t}"'] & \mathbb{T}\boldsymbol{\Omega}_\tau \arrow[u, "T\mathbf{i}_{\varphi,\tau}"'] \end{tikzcd} \iff T\hat{\varphi}_{\tau,t} \circ T\mathbf{i}_{\varphi,t} = T\mathbf{i}_{\varphi,\tau} \circ T\boldsymbol{\varphi}_{\tau,t}\,.$$

Continuum Mechanics is a field theory aimed to describe the evolution of a material body in the physical ambient space.

A treatment, in the spirit of differential geometry on manifolds, induces to underline the need for a careful distinction between the various typologies of fields involved in the analysis:

- Spatial fields;
- Material fields;
- Spatial-valued material fields;
- Material inductions of spatial fields;
- Spatial descriptions of spatial-valued material fields.

Manifolds and Fibre Bundles

A manifold $\mathbb{M}$ is the generalization of the notion of a curve or a surface in the Euclidean space.

A fibre bundle $(\mathrm{BUN}(\mathbb{M}), \boldsymbol{\pi}, \mathbb{M})$ is a geometrical construction which is useful to provide a clear mathematical description of many basic items in mechanics and other physical sciences.
It may be naïvely described as a base manifold with a fibre-manifold attached at each of its points. Each fibre is a diffeomorphic image of a given manifold called the typical fibre.
The surjective map π , which associates with, to each point of the bundle, the base point of the relevant fibre, is called the projection.

Spatial fields

- Spatial tensors are multilinear maps over a tangent space to the space manifold.

- Spatial fields are defined at each point of the ambient space manifold and at any time. Their values are spatial tensors based at that point, independently of whether there is a body particle crossing it or not.

A spatial field is a section $\hat{\mathbf{s}}_t \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{BUN}(\mathcal{S}))$ of the tensor bundle $(\mathrm{BUN}(\mathcal{S}), \boldsymbol{\pi}, \mathcal{S})$:

$$\begin{tikzcd} & \mathrm{BUN}(\mathcal{S}) \arrow[d, "\boldsymbol{\pi}"] \\ \mathcal{S} \arrow[ur, "\hat{\mathbf{s}}_t"] \arrow[r, "\mathrm{ID}_{\mathcal{S}}"] & \mathcal{S} \end{tikzcd} \qquad \Longleftrightarrow \qquad \boldsymbol{\pi} \circ \hat{\mathbf{s}}_t = \mathrm{ID}_{\mathcal{S}}$$

The twice covariant metric tensor field $\hat{\mathbf{g}}$ is a spatial field.

Material fields

- Material tensors are multilinear maps that operate, at each time instant, over a tangent space at a point of the body's placement along the motion.

- Material fields are defined, at each time instant, at particles of the body manifold and their values are material tensors based at the particle location evolving in the motion.

A material field at time $t \in I$ is a section $\check{\mathrm{s}}_{\varphi,t} \in \mathrm{C}^1(\mathcal{B}\,; \mathrm{BUN}(\boldsymbol{\Omega}_t))$ of the bundle: $(\mathrm{BUN}(\boldsymbol{\Omega}_t), \boldsymbol{\pi}, \boldsymbol{\Omega}_t)$ along the motion:

$$\begin{tikzcd}
 & \mathrm{BUN}(\boldsymbol{\Omega}_t) \arrow[d, shift right, "\boldsymbol{\pi}"'] \\
\mathcal{B} \arrow[ur, "\check{\mathrm{s}}_{\varphi,t}"] \arrow[r, "\boldsymbol{\varphi}_t"'] & \boldsymbol{\Omega}_t \arrow[u, shift right, "\mathrm{s}_{\varphi,t}"']
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{aligned}
&\boldsymbol{\pi} \circ \check{\mathrm{s}}_{\varphi,t} = \boldsymbol{\varphi}_t \,, \\
&\mathrm{s}_{\varphi,t} = \check{\mathrm{s}}_{\varphi,t} \circ \boldsymbol{\varphi}_t^{-1} \,, \\
&\boldsymbol{\pi} \circ \mathrm{s}_{\varphi,t} = \mathrm{ID}_{\,\boldsymbol{\Omega}_t} \,.
\end{aligned}$$

Most fields of interest in continuum mechanics are material fields, for instance, stretch, stretching, stress, stressing, temperature, heat flow, entropy, thermodynamical potentials.

Spatial-valued material fields

- Spatial-valued material fields are defined, at any instant of time, at particles of the body manifold, their values being spatial tensors based at the particle location evolving in the motion.

A spatial-valued material field is a section $\hat{\hat{\mathbf{s}}}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\mathcal{B}\,; \mathrm{BUN}(\mathcal{S}))$ of the bundle $(\mathrm{BUN}(\mathcal{S}), \boldsymbol{\pi}, \mathcal{S})$ along $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,; \boldsymbol{\Omega}_t)$:

$$\begin{tikzcd} & \mathrm{BUN}(\mathcal{S})_{\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t)} \arrow[d, shift right, "\boldsymbol{\pi}"'] \\ \mathcal{B} \arrow[ur, "\hat{\hat{\mathbf{s}}}_{\boldsymbol{\varphi},t}"] \arrow[r, "\boldsymbol{\varphi}_t"'] & \boldsymbol{\Omega}_t \arrow[u, shift right, "\hat{\mathbf{s}}_{\boldsymbol{\varphi},t}"'] \end{tikzcd} \iff \begin{aligned} &\boldsymbol{\pi} \circ \hat{\hat{\mathbf{s}}}_{\boldsymbol{\varphi},t} = \boldsymbol{\varphi}_t\,, \\ &\hat{\mathbf{s}}_{\boldsymbol{\varphi},t} = \hat{\hat{\mathbf{s}}}_{\boldsymbol{\varphi},t} \circ \boldsymbol{\varphi}_t^{-1}\,, \\ &\boldsymbol{\pi} \circ \hat{\mathbf{s}}_{\boldsymbol{\varphi},t} = \mathrm{ID}_{\boldsymbol{\Omega}_t}\,, \end{aligned}$$

In Continuum Dynamics:
velocity, force and kinetic momentum are spatial-valued material fields.

Material inductions of covariant spatial fields

A covariant spatial tensor field $\hat{\mathbf{s}}_t \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{COV}(\mathcal{S}))$ *at time* $t \in I$ *induces, at the configuration* $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_t)$, *a spatial-valued material field* $\hat{\mathbf{s}}_{\boldsymbol{\varphi},t} = \hat{\mathbf{s}}_t \circ \mathbf{i}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t, \mathrm{COV}(\mathcal{S}))$ *and, by co-restriction, the material fields* $\mathbf{s}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\mathcal{B}\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$ *and* $\check{\mathbf{s}}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$, *according to the commutative diagram:*

$$\begin{tikzcd}
 & \mathrm{COV}(\boldsymbol{\Omega}_t) \arrow[d, shift right, "\boldsymbol{\pi}"'] & \mathrm{COV}(\mathcal{S})_{\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t)} \arrow[l, "\mathbf{i}_{\boldsymbol{\varphi},t}\downarrow"'] \arrow[d, shift right, "\boldsymbol{\pi}"'] \\
\mathcal{B} \arrow[ur, "\check{\mathbf{s}}_{\boldsymbol{\varphi},t}"] \arrow[r, "\boldsymbol{\varphi}_t"'] & \boldsymbol{\Omega}_t \arrow[u, shift right, "\mathbf{s}_{\boldsymbol{\varphi},t}"'] \arrow[ur, "\hat{\mathbf{s}}_{\boldsymbol{\varphi},t}"'] \arrow[r, "\mathbf{i}_{\boldsymbol{\varphi},t}"'] & \mathcal{S} \arrow[u, shift right, "\hat{\mathbf{s}}_t"']
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{cases}
\boldsymbol{\pi} \circ \hat{\mathbf{s}}_t = \mathrm{ID}_{\mathcal{S}} \,, \\
\hat{\mathbf{s}}_t \circ \mathbf{i}_{\boldsymbol{\varphi},t} = \hat{\mathbf{s}}_{\boldsymbol{\varphi},t} \,, \\
\mathbf{i}_{\boldsymbol{\varphi},t}\!\downarrow\!\hat{\mathbf{s}}_t = \mathbf{s}_{\boldsymbol{\varphi},t} \,, \\
\boldsymbol{\pi} \circ \mathbf{s}_{\boldsymbol{\varphi},t} = \mathrm{ID}_{\boldsymbol{\Omega}_t} \,.
\end{cases}$$

The bundle $\mathrm{COV}(\mathcal{S})_{\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t)}$ *denotes the restriction of* $\mathrm{COV}(\mathcal{S})$ *to the base* $\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t) \subset \mathcal{S}$. *The pull-back* $\mathbf{i}_{\boldsymbol{\varphi},t}\!\downarrow \in \mathrm{C}^1(\mathrm{COV}(\mathcal{S})_{\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t)}\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$ *between covariant tensor bundles is defined in terms of the inclusion map* $\mathbf{i}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathcal{S})$ *and of the push-forward* $\mathbf{i}_{\boldsymbol{\varphi},t}\!\uparrow \in \mathrm{C}^1(\mathbb{T}\boldsymbol{\Omega}_t\,;\mathbb{T}\mathcal{S})$ *between tangent bundles, by:*

$$\mathbf{s}_{\boldsymbol{\varphi},t}(\mathbf{a}_{\boldsymbol{\varphi},t}, \mathbf{b}_{\boldsymbol{\varphi},t}) := \hat{\mathbf{s}}_{\boldsymbol{\varphi},t}(\mathbf{i}_{\boldsymbol{\varphi},t}\!\uparrow\!\mathbf{a}_{\boldsymbol{\varphi},t}, \mathbf{i}_{\boldsymbol{\varphi},t}\!\uparrow\!\mathbf{b}_{\boldsymbol{\varphi},t}) \,,$$

for all $\mathbf{a}_{\boldsymbol{\varphi},t}, \mathbf{b}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathbb{T}\boldsymbol{\Omega}_t)$.

Spatial descriptions of spatial-valued material fields

A spatial-valued material field $\hat{\hat{\mathrm{s}}}_{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,; \mathrm{BUN}(\mathcal{S}))$ admits in the trajectory manifold $\mathcal{T}(\mathcal{B}, \hat{\varphi})$ a spatial description:

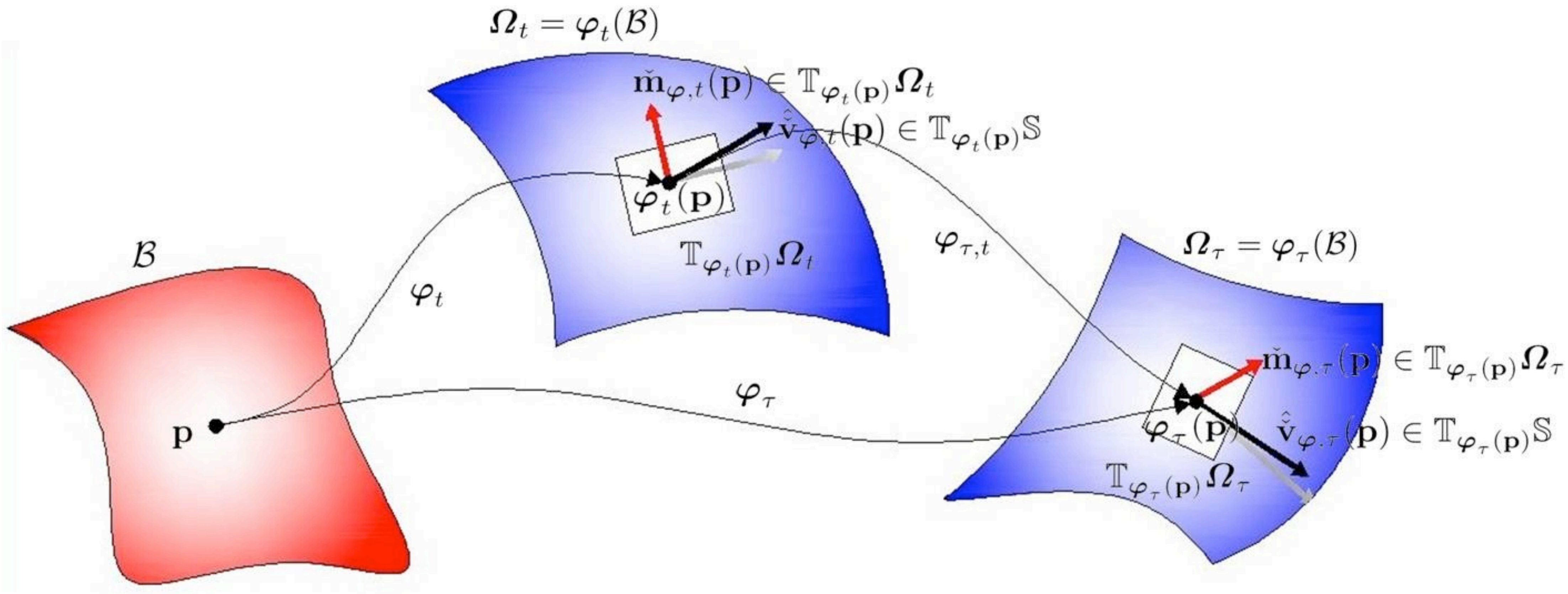
$\hat{\mathrm{s}}_{\mathcal{T}(\mathcal{B},\hat{\varphi})} \in \mathrm{C}^1(\mathcal{T}(\mathcal{B}, \hat{\varphi}), \mathrm{BUN}(\mathcal{S}))\,,$ according the diagram:

$$
\begin{tikzcd}
\mathcal{B} \times I \arrow[r, "(\hat{\varphi}\,,\, \mathrm{PR}_I)"] \arrow[rr, bend left, "\hat{\hat{\mathrm{s}}}_{\varphi}"] \arrow[dr, "\hat{\varphi}"'] & \mathcal{T}(\mathcal{B}, \hat{\varphi}) \arrow[r, "\hat{\mathrm{s}}_{\mathcal{T}(\mathcal{B},\hat{\varphi})}"] \arrow[d, "\mathrm{PR}_{\mathcal{S}}"] & \mathrm{BUN}(\mathcal{S}) \arrow[dl, "\pi"] \\
& \mathcal{W}(\mathcal{B}, \hat{\varphi}) &
\end{tikzcd}
\quad \Longrightarrow \quad \hat{\mathrm{s}}_{\mathcal{T}(\mathcal{B},\hat{\varphi})} := \hat{\hat{\mathrm{s}}}_{\varphi} \circ (\varphi\,, \mathrm{PR}_I)^{-1}
$$

$$
\begin{cases}
\mathrm{PR}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{S} \times I\,; \mathcal{S}) \ \text{and} \ \mathrm{PR}_I \in \mathrm{C}^1(\mathcal{S} \times I\,; I) \\
\hat{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,; \mathcal{S}) \\
\mathcal{W}(\mathcal{B}, \hat{\varphi}) := \hat{\varphi}(\mathcal{B} \times I) \subset \mathcal{S} \\
(\hat{\varphi}\,, \mathrm{PR}_I) \in \mathrm{C}^1(\mathcal{B} \times I\,; \mathcal{S} \times I) \\
\mathcal{T}(\mathcal{B}, \hat{\varphi}) := (\hat{\varphi}\,, \mathrm{PR}_I)(\mathcal{B} \times I)
\end{cases}
$$

- $\mathrm{PR}_{\mathcal{S}}$, PR_I → cartesian projectors,
- $\hat{\varphi}$ → motion of the body in the ambient space,
- $\mathcal{W}(\mathcal{B}, \hat{\varphi})$ → wake manifold,
- $(\hat{\varphi}\,, \mathrm{PR}_I)$ → events map,
- $\mathcal{T}(\mathcal{B}, \hat{\varphi})$ → trajectory manifold.

Material and spatial-valued material fields on a membrane



In most presentations of continuum mechanics, material fields and spatial-valued material fields are not distinguished.

The basic distinction is usually hidden by the context, in which a 3-D dimensional body manifold is considered embedded in a 3-D ambient space manifold.

Geometric tools for comparing:

- Tangent vectors at different points of the space manifold.
- Tangent vectors at the same particle for different placements of a body.
- Tangent vectors to different particles in the same body placement.

Parallel transport along a curve: a transformation, in the ambient space manifold, which takes a tangent vector to this manifold, that is a velocity of a curve in the space manifold, into another such tangent vector.

Push transformation by the material displacement map:

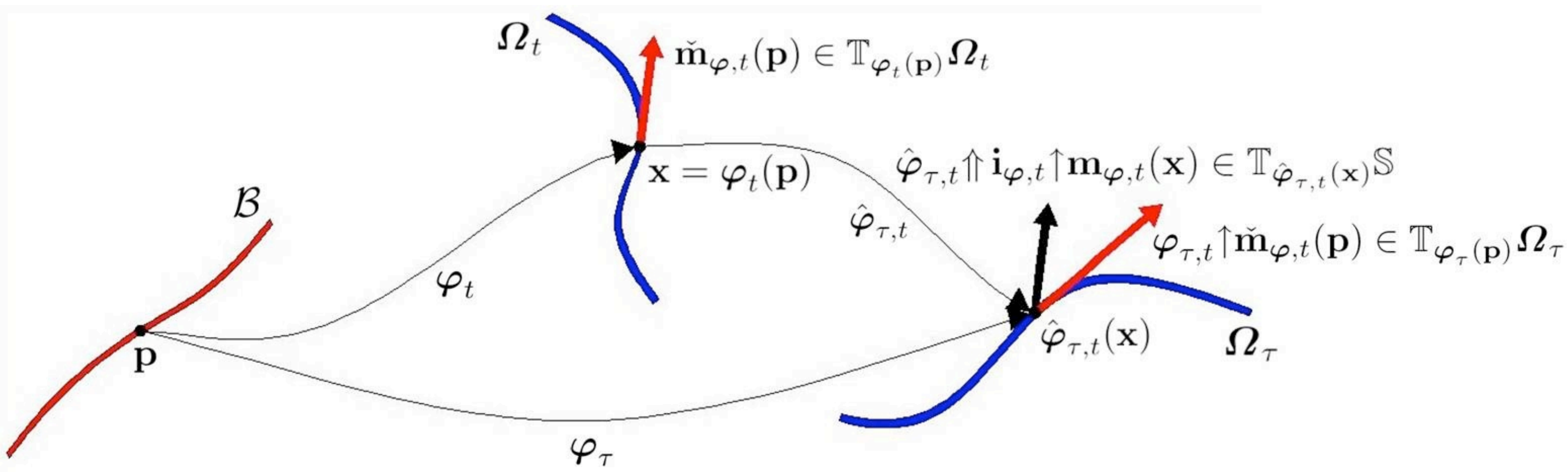
a transformation which takes a material tangent vector, that is a velocity of a curve in a body placement, into a material tangent vector in a displaced placement.

It is the suitable transformation tool to compare material tensors based at different configurations.

A material tangent vector is not in the domain of the parallel transport along a path. Even if a material vector in a 3-D body is improperly identified with its spatial immersion, the parallel transported vector will depend on the chosen connection, a dependence which should be taken into account in the description of a material behavior.

In a lower dimensional body, the image by a parallel transport along a path will, in general, no more be the spatial immersion of a material vector (see below).

Push of a material vector tangent to a wire
and parallel transport of its spatial immersion



Parallel transport in space is applicable only to spatial-valued vectors. Material vectors can be only tranformed by push along diffeomorphic material displacement maps as sketched in the above figure.

The covariance paradigm

Material fields, pertaining to the same material body at different configurations, must be compared according to the transformation by push along the material displacement diffeomorphism.

The time rate of variation of a material tensor field

$$\mathbf{s}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{BUN}(\boldsymbol{\Omega}_t))$$

is the convective time-derivative along the motion $\hat{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,;\mathcal{S})$:

$$\boxed{\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{s}_{\boldsymbol{\varphi}} := \partial_{\tau=t}\,\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{s}_{\boldsymbol{\varphi},\tau}}$$

Continuum Mechanics

The spatial metric tensor field $\hat{\mathbf{g}} \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{COV}(\mathcal{S}))$ induces at the configuration $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_t)$ the material metric tensor field:

$$\mathbf{g}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$$

defined by $\mathbf{g}_{\varphi,t} = \mathbf{i}_{\varphi,t}{\downarrow}\hat{\mathbf{g}}_{\varphi,t}$ where $\hat{\mathbf{g}}_{\varphi,t} = \hat{\mathbf{g}} \circ \mathbf{i}_{\varphi,t}$

and explicitly:

$$\mathbf{g}_{\varphi,t}\,(\mathbf{a}_{\varphi,t}, \mathbf{b}_{\varphi,t}) := \hat{\mathbf{g}}_{\varphi,t}\,(\mathbf{i}_{\varphi,t}{\uparrow}\mathbf{a}_{\varphi,t}, \mathbf{i}_{\varphi,t}{\uparrow}\mathbf{b}_{\varphi,t})\,.$$

To any pair of configurations $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_t)$ and $\boldsymbol{\varphi}_\tau \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_\tau)$ there corresponds a Green strain (STRETCH) tensor field:

$$\boxed{\tfrac{1}{2}(\boldsymbol{\varphi}_{\tau,t}{\downarrow}\mathbf{g}_{\varphi,\tau} - \mathbf{g}_{\varphi,t}) \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}(\boldsymbol{\Omega}_t))}$$

The strain rate (STRETCHING) is the material tensor field defined by:

$$\boxed{\boldsymbol{\varepsilon}_{\varphi,t} := \tfrac{1}{2}\mathcal{L}_{\varphi,t}\,\mathbf{g}_{\varphi} = \partial_{\tau=t}\;\tfrac{1}{2}(\boldsymbol{\varphi}_{\tau,t}{\downarrow}\mathbf{g}_{\varphi_\tau} - \mathbf{g}_{\varphi,t}) \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}(\boldsymbol{\Omega}_t))}$$

The STRESS field $\boldsymbol{\sigma}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}^*(\boldsymbol{\Omega}_t))$ is a section of the bundle $\mathrm{SYM}^*(\boldsymbol{\Omega}_t)$ of symmetric contravariant material tensor fields, defined by duality with the stretching:

$$\langle \boldsymbol{\sigma}_{\varphi,t}, \boldsymbol{\varepsilon}_{\varphi,t} \rangle := J_1(\boldsymbol{\sigma}_{\varphi,t} \circ \boldsymbol{\varepsilon}_{\varphi,t}) \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{FUN}(\boldsymbol{\Omega}_t))$$

Material field of linear invariants of the mixed tensor field:

$$\boldsymbol{\sigma}_{\varphi,t} \circ \boldsymbol{\varepsilon}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{MIX}(\boldsymbol{\Omega}_t))$$

In Continuum Mechanics, material stress fields, whose duality pairing with the covariant stretching provide the virtual power per unit volume in the actual configuration, are contravariant Cauchy stress fields.

The ones providing the virtual power per unit mass, or per unit reference volume, are contravariant Kirchhoff stress fields. The mixed form of the covariant stretching tensor is provided by the symmetric part of the material covariant derivative of the velocity field, an outcome of Euler’s formula.

Their mixed forms are Cauchy true stress and Kirchhoff true stress fields, respectively, the adjective true stemming from the fact that the boundary flux of mixed stress tensor fields provides the boundary tractions field.

The stress rate (STRESSING) is the material tensor field defined by:

$$\boxed{\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi} := \partial_{\tau=t}\ \varphi_{\tau,t}\!\downarrow\!\boldsymbol{\sigma}_{\varphi,\tau} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}^*(\boldsymbol{\Omega}_t))\,.}$$

The other tool provided by Differential Geometry for spatial-valued material fields is the covariant time-derivative along a motion $\hat{\varphi} \in \mathrm{C}^1(\mathcal{B}\times I\,;\mathcal{S})$

$$\boxed{\nabla_{\varphi,t}\,\hat{\mathbf{s}}_{\varphi} := \partial_{\tau=t}\ \hat{\varphi}_{\tau,t}\!\Downarrow\hat{\mathbf{s}}_{\varphi,\tau}}$$

$\hat{\varphi}_{\tau,t}\!\Downarrow$ $\longrightarrow$ parallel transport from the spatial placement $\mathbf{i}_{\varphi,\tau}(\boldsymbol{\Omega}_\tau)$
to the spatial placement $\mathbf{i}_{\varphi,t}(\boldsymbol{\Omega}_t)$.

In Continuum Dynamics, acceleration is the covariant time-derivative of the velocity, along the motion.

Remark The convective time-derivative of a material field is well-defined, while the covariant time-derivative is defined only for spatial-valued material tensor fields.

A review

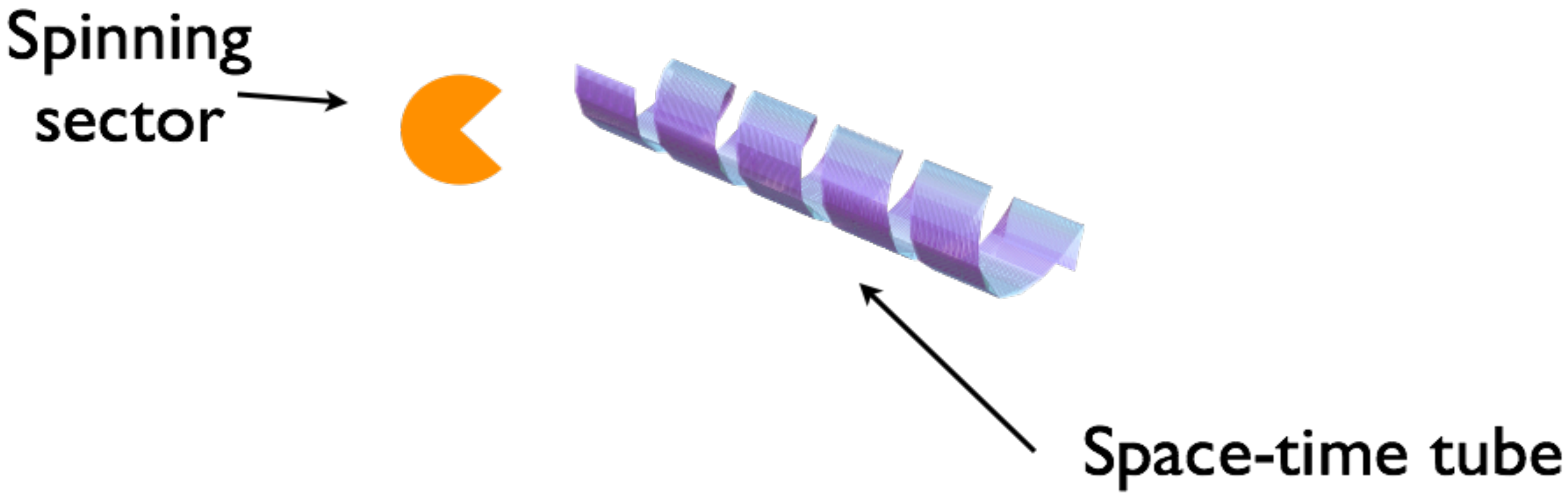
This excursus on fundamentals of field theories provides arguments for a critical analysis of most treatments in which a recourse to spatial descriptions of material fields, was made, by treating material fields as if they were spatial-valued material fields.

This geomeric flaw opened the door to difficulties and lasting, hopeless debates on basic issues and related computational procedures.

Most troubles originated from the unwary intention of differentiating, in terms of cartesian components, the material stress tensor field with respect to time along the motion, an operation which is forbidden in the geometric context of continuum kinematics.

While the impossibility is apparent for body dimension less than the space dimension, when the body dimension is equal to the space dimension, the procedure could look like as performable, at first sight. A careful inspection reveals however a confusion between material fields and spatial-valued material fields and furthermore an irregular dependence of the spatial description on time makes this description deprived of usefulness, in general.

The spatial description of a spatial-valued material field is highly irregular in time, with the exception of very special instances.



The stressing is the convective time-derivative of the material stress tensor and the stretching is the convective time-derivative of the material metric tensor along the motion. If the covariant time-derivative would be adopted to evaluate the rate of change of the metric tensor, a vanishing derivative would be got, since the standard Euclid connection is metric preserving. So, why try to use the covariant time-derivative for the stress rate?

An easy-to-follow explanation of the difficulty may be provided by considering that the material stress tensor is referred to a translated basis, while the material metric tensor is referred to a basis dragged by the motion (or rotated according to a rigid body motion with the same local spin – i.e. co-rotational)

If the covariant time-derivative of the material stress tensor field is related to the material stretching field by a constitutive relation, material frame indifference is violated, in the sense that, in rigid body motions, a non-vanishing stress rate could correspond to a vanishing stretching.

The shortcomings consequent to this incongruence were detected long ago:

Zaremba, S.: Sur une forme perfectionée de la théorie de la relaxation. Bull. Int. Acad. Sci. Cracovie, 594-614 (1903).
Jaumann, G.: Geschlossenes system physikalischer und chemischer differentialgesetze. Akad. Wiss. Wien Sitzber. **IIa**, 385-530 (1911).

The resulting objective rate is called the co-rotational Zaremba-Jaumann derivative.

Many different proposals of objective rates have been made in the relevant literature, see e.g.:

Bruhns, O.T., Xiao, H., Meyers, A.: Self-consistent Eulerian rate type elasto-plasticity models based upon the logarithmic stress rate. Int. J. Plasticity **15**, 479-520 (1999).
Liu, C.S.: Lie symmetries of finite strain elastic-perfectly plastic models and exactly consistent schemes for numerical integrations. Int. J. Solids Struct. **41**, 1823-1853 (2004).

Some stress rates proposed in literature

Truesdell rate of Cauchy stress $\longrightarrow$ $\boxed{\overset{\circ}{\boldsymbol{\sigma}} = \phi_{t*}(J^{-1}\dot{\boldsymbol{S}})}$

This stress rate is the Piola transformation of the material time derivative of the symmetric Piola-Kirchhoff stress $\boldsymbol{S} = J\boldsymbol{\phi}_t^*(\boldsymbol{\sigma})$

For contravariant components it has the form $\rightarrow$ $\boxed{\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{t}} + \boldsymbol{\sigma}\,\mathrm{tr}(\boldsymbol{d})}$

$\boldsymbol{l} = \nabla\boldsymbol{v}$ $\rightarrow$ spatial velocity gradient tesor.

Jaumann rate of Cauchy stress $\longrightarrow$ $\boxed{\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\boldsymbol{\sigma}}$

$\boldsymbol{\omega}$ $\rightarrow$ spin rate or vorticity tensor.

Jaumann rate of Kirchhoff stress $\longrightarrow$ $\boxed{\overset{\triangledown}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\boldsymbol{\tau}}$

Green and Naghdi rate of Cauchy stress $\longrightarrow$ $\boxed{\hat{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma}}$

$\boldsymbol{\Omega}$ $\rightarrow$ rate of rotation tensor related to the rotation tensor $\boldsymbol{R}$ by $\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\cdot\boldsymbol{R}^{\mathrm{t}}$.

The adoption of the convective time-derivative along the motion was foreshadowed by Zaremba (1903) and proposed in:

Oldroyd, J.G.: On the formulation of rheological equations of state. Proc. R. Soc. London A **200**, 523-541 (1950).

Oldroyd, J.G.: Finite strains in an anisotropic elastic continuum. Proc. R. Soc. London A **202**, 345-358 (1950).

Truesdell, C.: Hypo-elasticity. J. Rational Mech. Anal. **4**, 83-133, 1019-1020 (1955).

A rationale for the formulation of objective rates, based on the expression of Lie derivative in terms of covariant derivative, for different alterations of the stress tensor, was proposed in:

Marsden, J.E., Hughes, T.J.R.: Mathematical foundations of elasticity. Prentice-Hall, Redwood City, Cal (1983).

In all these proposals, however, expressions of the convective time-derivative in terms of spatial covariant time-derivatives of stress tensor fields were taken *ab initio*, so that covariance and its basic theoretic implications were not even explored.

Comments

The remedies to the lack of objectivity adopted in literature have been eventually ineffective, because the primary cause of ill-posedness was neither detected nor avoided, in the absence of a working covariance paradigm.

In this respect it is to be underlined that, although for three-dimensional bodies covariant time-derivatives of spatial immersions of material tensor fields are sometimes performable, this tool should be treated at most as a special computational mean and not as a basic definition.

The evaluation according the Leibniz rule is subject to stringent regularity requirements and, in addition, covariant differentations are forbidden by the geometry of continuum mechanics for lower dimensional bodies (such as wires or membranes).

Failure of Leibniz rule

Proposition 1 (Fibration) *The trajectory manifold* $\mathcal{T}(\mathcal{B}, \hat{\varphi})$ *is endowed with a structure of fibred manifold over the wake manifold by the cartesian projection* $\mathrm{PR}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{T}(\mathcal{B}, \hat{\varphi}) ; \mathcal{W}(\mathcal{B}, \hat{\varphi}))$ *. Since it associates the location* $\mathbf{x} \in \mathcal{S}$ *with a spatial event* $(\mathbf{x}, t) \in \mathcal{T}(\mathcal{B}, \hat{\varphi})$ *, the fibre at* $\mathbf{x} \in \mathcal{W}(\mathcal{B}, \hat{\varphi})$ *is the set* $I_{\mathbf{x}}$ *of time instants at which a particle of the body crosses the spatial location* $\mathbf{x} \in \mathcal{S}$ *, which is a nonempty and in general neither open nor connected set.*

For instance, the nonempty set $I_{\mathbf{x}}$ may be a nonconnected union of connected intervals or also a set of isolated points, since the body may cross a point of the trajectory at nonconsecutive time instants.

Then the time-fibre $I_{\mathbf{x}}$ fails to be a differentiable manifold and differentiations of spatial descriptions, with respect to time, lose significance.

This is the rule in Solid Mechanics and in Newton’s Particle Mechanics.

Accordingly, neither the convective time-derivative of a material field, nor the covariant time-derivative of a spatial-valued material tensor field along a motion, can be evaluated by Leibniz rule.

The question to be properly answered consists in finding out how to compare the expressions of a rate constitutive behaviour of the material at the same particle in different configurations of the body and at different points in the same placement.

This question appears to be unanswerable in a non suitably geometrized context.

There is however a clear evidence that a definite comparison is needed to give to a mathematical formula the proper meaning of analytical model of material behaviour.

The covariance paradigm provides, in a natural way, a definite answer to all these basic questions.

Synopsis

- Push transformation and convective time-differentiation of a material tensor field along a motion in the space manifold are allowed.

- Parallel transport and covariant time-differentiation of a material tensor field along a curve in the space manifold are forbidden operations for lower dimensional bodies.

- Parallel transport and covariant time-differentiation of a spatial-valued material tensor field along a curve in the space manifold are allowed.

- Push transformation and convective time-differentiation of a spatial-valued material tensor field along a motion in the space manifold are allowed.

- The split according to Leibniz rule, of the convective time-differentiation of a material tensor field along a motion in the space manifold, is not performable.

- The split according to Leibniz rule, of the convective or covariant time-differentiation of a spatial-valued material tensor field along a motion in the space manifolf, is not performable.

- The split according to Leibniz rule, of the convective or covariant time-differentiation of the spatial description of a spatial-valued material tensor field along a motion in the space manifold, is performable only under stringent regularity assumptions which are admissible in many modelings proper to fluid dynamics but are not likely to be fulfilled in solid mechanics.

Covariant hypo-elasticity

A nonlinear hypo-elastic response of a body undergoing a motion under the action of a time-dependent system of forces, imposed distorsions and kinematical control parameters, is expressed, at each configuration, by a morphism from the Whitney product of the stress tensor bundle times itself, to the dual bundle of stretching fields:

$$\boxed{\tfrac{1}{2}\mathcal{L}_{\varphi,t}\,\mathbf{g}_{\varphi} = \mathbf{H}_{\varphi_t}(\boldsymbol{\sigma}_{\varphi,t}\,,\,\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi}) = \mathbf{H}_{\varphi_t}(\boldsymbol{\sigma}_{\varphi,t})\cdot\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi}\,.}$$

The hypo-elastic response is assumed to depend in a nonlinear way on the material stress field and relates the convective time-derivative along the motion of the material metric field to the convective time-derivative of the dual material stress field.

Covariance paradigm provides the hypo-elastic law at any displaced material configuration $(\zeta \circ \varphi)_t \in \mathrm{C}^1(\mathcal{B}\,;\delta\boldsymbol{\Omega}_t)$:

$$\boxed{\tfrac{1}{2}\mathcal{L}_{\zeta\circ\varphi,t}\,\zeta\!\uparrow\!\mathbf{g}_{\varphi} = \mathbf{H}_{(\zeta\circ\varphi)_t}(\zeta_t\!\uparrow\!\boldsymbol{\sigma}_{\varphi,t}\,,\;\mathcal{L}_{\zeta\circ\varphi,t}\,\zeta\!\uparrow\!\boldsymbol{\sigma}_{\varphi})}$$

The convective time-derivative of a pushed tensor field along the pushed motion fulfils:

$$\mathcal{L}_{\zeta\circ\varphi,t}\,(\zeta\!\uparrow\!\boldsymbol{\alpha}_{\varphi}) = \zeta_t\!\uparrow\!(\mathcal{L}_{\varphi,t}\,\boldsymbol{\alpha}_{\varphi})$$

and hence the relevant responses are related by the covariance property:

$$\boxed{\mathbf{H}_{(\zeta\circ\varphi)_t}\circ\zeta_t\!\uparrow\; = \zeta_t\!\uparrow\circ\,\mathbf{H}_{\varphi_t}}$$

$$\begin{array}{ccc} \delta\boldsymbol{\Omega}_t & \xrightarrow{(\zeta\circ\varphi)_{\tau,t}} & \delta\boldsymbol{\Omega}_\tau \\ \zeta_t\uparrow & & \uparrow\zeta_\tau \\ \boldsymbol{\Omega}_t & \xrightarrow{\varphi_{\tau,t}} & \boldsymbol{\Omega}_\tau \end{array} \quad \Longleftrightarrow \quad (\zeta\circ\varphi)_{\tau,t}\circ\zeta_t := \zeta_\tau\circ\varphi_{\tau,t}\,.$$

Push forward of the material displacement $\varphi_{\tau,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\boldsymbol{\Omega}_\tau)$

Implications of covariance of hypo-elastic law

As a first implication, if the transformation $\boldsymbol{\zeta}_t \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,; \delta\boldsymbol{\Omega}_t)$ is an isometry at $t \in I$, viz. $\boldsymbol{\zeta}_t \!\uparrow\! \mathbf{g}_{\boldsymbol{\varphi},t} = \mathbf{g}_{\boldsymbol{\varphi},t}$, the covariance property implies the fulfillment of the principle of material frame indifference. In fact it states that the hypo-elastic law transforms by push according to the relative isometric motion, when measured by two observers. In treatments of the principle of material frame indifference, inspired to the one in (Truesdell and Noll, 1965), the response acting at the configuration $\boldsymbol{\zeta}_t \circ \boldsymbol{\varphi}_t$, was considered equal to $\mathbf{H}_{\boldsymbol{\varphi}_t}$, by tacitly performing a translation to relate the material tangent spaces $\mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t$ and $\mathbb{T}_{\boldsymbol{\zeta}_t(\mathbf{x})}\boldsymbol{\zeta}_t(\boldsymbol{\Omega}_t)$. This does not comply with covariance.

Truesdell, C., Noll, W., 1965. The non-linear field theories of mechanics, Handbuch der Physik, Springer, 1-591.

A second implication is got, by chosing $\boldsymbol{\zeta}_t \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\delta\boldsymbol{\Omega}_t)$ to be a θ-time translation along the trajectory: $\boldsymbol{\zeta}_t = \boldsymbol{\varphi}_{t+\theta,t} = \boldsymbol{\varphi}_{t+\theta} \circ \boldsymbol{\varphi}_t^{-1}$. The covariance property assures that the hypo-elastic constitutive relation, once defined at a given configuration, may be reproduced at every other configuration by pushing forward according to the relevant displacement map. The push according to a chain of material displacements is equal to the composition of the pushes according to each component displacement. By this property, the covariant constitutive relation is independent of the chain of displacements followed to reach any other configuration. The hypo-elastic law so formulated defines in a proper way a *hypo-elastic material.* In this respect, we remark that the other hypo-elastic constitutive laws proposed in literature do not fulfill this basic requirement.

All constitutive properties of an hypo-elastic material are most simply investigated in terms of a reference natural configuration $\boldsymbol{\chi}_o \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_o)$, the results being however independent of a particular choice in the class of natural configurations. According to the diagram

$$\begin{tikzcd} \boldsymbol{\Omega}_o \arrow[dr, rightsquigarrow, "\boldsymbol{\xi}_t"] & \\ & \boldsymbol{\Omega}_t \\ \mathcal{B} \arrow[uu, "\boldsymbol{\chi}_o"] \arrow[ur, "\boldsymbol{\varphi}_t"'] & \end{tikzcd} \qquad \Longleftrightarrow \qquad \boldsymbol{\xi}_t \circ \boldsymbol{\chi}_o = \boldsymbol{\varphi}_t \,,$$

by the properties of convective time-derivatives

$$\begin{aligned} \boldsymbol{\xi}_t{\downarrow}\mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\sigma}_{\boldsymbol{\varphi}} &= \boldsymbol{\xi}_t{\downarrow}\partial_{\tau=t}\, \boldsymbol{\varphi}_{\tau,t}{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau} = \partial_{\tau=t}\, \boldsymbol{\xi}_\tau{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau} \,, \\ \boldsymbol{\xi}_t{\downarrow}\mathcal{L}_{\boldsymbol{\varphi},t}\, \mathbf{g}_{\boldsymbol{\varphi}} &= \boldsymbol{\xi}_t{\downarrow}\partial_{\tau=t}\, \boldsymbol{\varphi}_{\tau,t}{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},\tau} = \partial_{\tau=t}\, \boldsymbol{\xi}_\tau{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},\tau} \,, \end{aligned}$$

the pull-back of the hypo-elastic law at configuration $\boldsymbol{\chi}_o \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_o)$ is given by

$$\tfrac{1}{2}\partial_{\tau=t}\; \boldsymbol{\xi}_\tau{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},\tau} = \mathbf{H}_o(\boldsymbol{\xi}_t{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}\,,\; \partial_{\tau=t}\; \boldsymbol{\xi}_\tau{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau})\,, \quad \mathbf{H}_o := \mathbf{H}_{\boldsymbol{\chi}_o} = \mathbf{H}_{\boldsymbol{\xi}_t^{-1}\circ\boldsymbol{\varphi}_t} \,.$$

Evaluation of the stress field

In most computational algorithms, what has to be evaluated is the constitutive response in terms of stress in a given motion. Then, a basic question concerns the evaluation of the stress field along the motion by means of the inverse hypo-elastic response according to the formula:

$$\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi} = \bar{\mathbf{H}}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t}\,,\,\tfrac{1}{2}\mathcal{L}_{\varphi,t}\,\mathbf{g}_{\varphi})\,.$$

This highly nonlinear implicit relation is best solvable by means of iterative algorithms formulated in a reference configuration with reference to the pulled back inverse hypo-elastic law:

$$\partial_{\tau=t}\;\boldsymbol{\xi}_{\tau}\!\downarrow\!\boldsymbol{\sigma}_{\varphi,\tau} = \bar{\mathbf{H}}_{o}(\boldsymbol{\xi}_{t}\!\downarrow\!\boldsymbol{\sigma}_{\varphi,t}\,,\,\tfrac{1}{2}\partial_{\tau=t}\;\boldsymbol{\xi}_{\tau}\!\downarrow\!\mathbf{g}_{\varphi,\tau})\,.$$

Accordingly, the stress increment in a time interval $[t_o, t]$ is evaluated by time-integration along the motion:

$$\boxed{\boldsymbol{\xi}_{t}\!\downarrow\!\boldsymbol{\sigma}_{\varphi,t} - \boldsymbol{\xi}_{t_o}\!\downarrow\!\boldsymbol{\sigma}_{\varphi,t_o} = \int_{t_o}^{t} \bar{\mathbf{H}}_{o}(\boldsymbol{\xi}_{s}\!\downarrow\!\boldsymbol{\sigma}_{\varphi,s}\,,\,\tfrac{1}{2}\partial_{\tau=s}\;\boldsymbol{\xi}_{\tau}\!\downarrow\!\mathbf{g}_{\varphi,\tau})\,ds\,.}$$

The converse problem of evaluating the strain field along the motion according to the direct hypo-elastic law:

$$\tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\varphi}} = \mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}\,,\;\mathcal{L}_{\boldsymbol{\varphi},t}\,\boldsymbol{\sigma}_{\boldsymbol{\varphi}})\,,$$

is readily solved by the integral formula

$$\boxed{\tfrac{1}{2}\boldsymbol{\xi}_t{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},t} = \tfrac{1}{2}\boldsymbol{\xi}_{t_o}{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},t_o} + \int_{t_o}^{t}\mathbf{H}_o(\boldsymbol{\xi}_s{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},s}\,,\;\partial_{\tau=s}\,(\boldsymbol{\xi}_\tau{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}))\,ds\,.}$$

The isotropy property

In the literature it has been sustained that the fulfillment of the principle of material frame indifference requires that the hypo-elastic response be an isotropic map.

This implication, which is cannot be accepted on a physical ground, is contradicted by the covariance paradigm.

The covariance axiom requires that the response to a pushed cause acting on a pushed specimen should provide the pushed effect.

Material frame indifference is the more special requirement that the response to a rotated cause acting on a rotated specimen should provide the rotated effect.

Isotropy consist instead in the property that the response to a rotated cause acting on an unrotated specimen should provide the rotated effect.

Definition of isotropy

A hypo-elastic body is $\mathbf{g}$-*isotropic* at $\mathbf{x} \in \mathbf{\Omega}_o$, if the invariance property:

$$\boldsymbol{\rho}_o\!\uparrow\!\mathbf{H}_o(\boldsymbol{\sigma}_{\mathbf{x}}) \cdot \delta\boldsymbol{\sigma}_{\mathbf{x}} = \mathbf{H}_o(\boldsymbol{\rho}_o\!\uparrow\!\boldsymbol{\sigma}_{\mathbf{x}}\,,\, \boldsymbol{\rho}_o\!\uparrow\!\delta\boldsymbol{\sigma}_{\mathbf{x}})\,, \quad \forall\, \delta\boldsymbol{\sigma}_{\mathbf{x}} \in \mathrm{SYM}^*(\mathbf{\Omega}_o)_{\mathbf{x}}\,,$$

holds for any linear automorphism $\mathbf{Q} \in BL\,(\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_o\,;\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_o)$, which is $\mathbf{g}$-isometric, i.e. such that:

$$\boldsymbol{\rho}_o\!\uparrow\!\mathbf{g}_{\mathbf{x}} = \mathbf{g}_{\mathbf{x}}\,.$$

A linear automorphism $\mathbf{Q} \in BL\,(\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_o\,;\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_o)$ induces the following isomorphisms acting on the tensors $\mathbf{g}_{\mathbf{x}} \in \mathrm{SYM}(\mathbf{\Omega}_o)_{\mathbf{x}}$, $\boldsymbol{\varepsilon}_{\mathbf{x}} \in \mathrm{SYM}(\mathbf{\Omega}_o)_{\mathbf{x}}$, $\boldsymbol{\sigma}_{\mathbf{x}} \in \mathrm{SYM}^*(\mathbf{\Omega}_o)_{\mathbf{x}}$:

$$\boldsymbol{\rho}_o\!\uparrow\!\mathbf{g}_{\mathbf{x}}(\mathbf{a},\mathbf{b}) := \mathbf{g}_{\mathbf{x}}(\mathbf{Q}^{-1}\mathbf{a},\mathbf{Q}^{-1}\mathbf{b}) = \langle\,\mathbf{Q}^{-*}\mathbf{g}_{\mathbf{x}}\mathbf{Q}^{-1}\mathbf{a},\mathbf{b}\,\rangle\,, \quad \forall\,\mathbf{a},\mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_o\,,$$

$$\boldsymbol{\rho}_o\!\uparrow\!\boldsymbol{\varepsilon}_{\mathbf{x}}(\mathbf{a},\mathbf{b}) := \boldsymbol{\varepsilon}_{\mathbf{x}}(\mathbf{Q}^{-1}\mathbf{a},\mathbf{Q}^{-1}\mathbf{b}) = \langle\,\mathbf{Q}^{-*}\boldsymbol{\varepsilon}_{\mathbf{x}}\mathbf{Q}^{-1}\mathbf{a},\mathbf{b}\,\rangle\,, \quad \forall\,\mathbf{a},\mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_o\,,$$

$$\boldsymbol{\rho}_o\!\uparrow\!\boldsymbol{\sigma}_{\mathbf{x}}(\mathbf{a}^*,\mathbf{b}^*) := \boldsymbol{\sigma}_{\mathbf{x}}(\mathbf{Q}^*\mathbf{a}^*,\mathbf{Q}^*\mathbf{b}^*) = \langle\,\mathbf{Q}\boldsymbol{\sigma}_{\mathbf{x}}\mathbf{Q}^*\mathbf{a}^*,\mathbf{b}^*\,\rangle\,, \quad \forall\,\mathbf{a}^*,\mathbf{b}^* \in \mathbb{T}^*_{\mathbf{x}}\mathbf{\Omega}_o\,,$$

that is: $\boldsymbol{\rho}_o\!\uparrow\!\mathbf{g}_{\mathbf{x}} := \mathbf{Q}^{-*}\mathbf{g}_{\mathbf{x}}\mathbf{Q}^{-1}$, $\quad \boldsymbol{\rho}_o\!\uparrow\!\boldsymbol{\varepsilon}_{\mathbf{x}} := \mathbf{Q}^{-*}\boldsymbol{\varepsilon}_{\mathbf{x}}\mathbf{Q}^{-1}$, $\quad \boldsymbol{\rho}_o\!\uparrow\!\boldsymbol{\sigma}_{\mathbf{x}} := \mathbf{Q}\boldsymbol{\sigma}_{\mathbf{x}}\mathbf{Q}^*$.

The property of $\mathbf{g}$-isotropy means that a relative rotation between the tangent fibre to the body at a point and the pair of stress and stress-rate tensors, implies that the corresponding strain-rate is equally rotated. Recalling that $\mathbf{Q}^A := \mathbf{g}^*_{\mathbf{x}} \circ \mathbf{Q}^* \circ \mathbf{g}_{\mathbf{x}}$, we have that $\mathbf{Q}^A = \mathbf{Q}^{-1}$ and $\mathbf{Q}^{-*} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{Q} \circ \mathbf{g}^*_{\mathbf{x}}$. Hence:

$$\mathbf{g}^*_{\mathbf{x}} \circ (\mathbf{Q}^{-*} \delta\boldsymbol{\varepsilon}_{\mathbf{x}} \mathbf{Q}^{-1}) = \mathbf{g}^*_{\mathbf{x}} \circ (\mathbf{g}_{\mathbf{x}} \circ \mathbf{Q} \circ \mathbf{g}^*_{\mathbf{x}}) \circ \delta\boldsymbol{\varepsilon}_{\mathbf{x}} \circ \mathbf{Q}^A = \mathbf{Q} \circ \mathbf{D}_{\mathbf{x}} \circ \mathbf{Q}^A ,$$

$$(\mathbf{Q}\, \delta\boldsymbol{\sigma}_{\mathbf{x}} \mathbf{Q}^*) \circ \mathbf{g}_{\mathbf{x}} = \mathbf{Q} \circ \delta\boldsymbol{\sigma}_{\mathbf{x}} \circ (\mathbf{g}_{\mathbf{x}} \circ \mathbf{Q}^A \circ \mathbf{g}^*_{\mathbf{x}}) \circ \mathbf{g}_{\mathbf{x}} = \mathbf{Q} \circ \delta\boldsymbol{\Sigma}_{\mathbf{x}} \circ \mathbf{Q}^A ,$$

In terms of the mixed stretching tensor $\mathbf{D}_{\mathbf{x}} := \mathbf{g}^*_{\mathbf{x}} \circ \frac{1}{2} \mathcal{L}_{\boldsymbol{\varphi},t}\, \mathbf{g}_{\boldsymbol{\varphi}}$ and of the mixed stressing tensor $\boldsymbol{\Sigma}_{\mathbf{x}} := \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\sigma}_{\boldsymbol{\varphi}} \circ \mathbf{g}_{\mathbf{x}}$, the $\mathbf{g}_{\mathbf{x}}$-isotropy property

$$\boldsymbol{\rho}_o{\uparrow}\delta\boldsymbol{\varepsilon}_{\mathbf{x}} = \mathbf{H}_o(\boldsymbol{\rho}_o{\uparrow}\boldsymbol{\sigma}_{\mathbf{x}} ,\, \boldsymbol{\rho}_o{\uparrow}\delta\boldsymbol{\sigma}_{\mathbf{x}}) , \quad \forall\, \delta\boldsymbol{\sigma}_{\mathbf{x}} \in \mathrm{SYM}^*_{\mathbf{x}} ,$$

takes the usual form $\mathbf{Q}\, \mathbf{D}_{\mathbf{x}} \mathbf{Q}^A = \mathbf{H}_{\boldsymbol{\Sigma}}(\mathbf{Q}\boldsymbol{\Sigma}_{\mathbf{x}}\mathbf{Q}^A) \cdot (\mathbf{Q}\delta\boldsymbol{\Sigma}_{\mathbf{x}}\mathbf{Q}^A)$.

By the covariance paradigm:

The $\mathbf{g}$-isotropy at a configuration $\boldsymbol{\chi}_o \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_o)$ of a hypo-elastic body implies the $\boldsymbol{\xi}{\uparrow}\mathbf{g}$-isotropy at $\boldsymbol{\xi}$-displaced configurations.

The homogeneity property

A hypo-elastic body is $\mathbf{g}$-*homogeneous* at a configuration $\boldsymbol{\chi}_o \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_o)$, if for any pair of points $\mathbf{x}, \mathbf{y} \in \boldsymbol{\Omega}_o$ there exists a linear isomorphism $\mathbf{Q}_{\mathbf{y},\mathbf{x}} \in BL\,(\mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_o\,;\mathbb{T}_{\mathbf{y}}\boldsymbol{\Omega}_o)$ which is $\mathbf{g}$-isometric, i.e.:

$$\boldsymbol{\rho}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\mathbf{g}_{\mathbf{x}} = \mathbf{g}_{\mathbf{y}} \quad \Longleftrightarrow \quad \mathbf{g}_{\mathbf{y}}(\mathbf{Q}_{\mathbf{y},\mathbf{x}}(\mathbf{a}), \mathbf{Q}_{\mathbf{y},\mathbf{x}}(\mathbf{b})) = \mathbf{g}_{\mathbf{x}}(\mathbf{a}, \mathbf{b})\,,$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_o$, and such that:

$$\boldsymbol{\rho}_{\mathbf{y},\mathbf{x}}\!\uparrow\!(\mathbf{H}_{\mathbf{x}}(\boldsymbol{\sigma}_{\mathbf{x}}) \cdot \delta\boldsymbol{\sigma}_{\mathbf{x}}) = \mathbf{H}_{\mathbf{y}}(\boldsymbol{\rho}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\boldsymbol{\sigma}_{\mathbf{x}}, \boldsymbol{\rho}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\delta\boldsymbol{\sigma}_{\mathbf{x}})\,, \quad \forall\, \delta\boldsymbol{\sigma}_{\mathbf{x}} \in \mathrm{SYM}^*(\boldsymbol{\Omega}_o)_{\mathbf{x}}\,.$$

The push $\boldsymbol{\rho}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\boldsymbol{\alpha} \in \mathbb{T}^*_{\mathbf{y}}\boldsymbol{\Omega}_o$ of a covector $\boldsymbol{\alpha} \in \mathbb{T}^*_{\mathbf{x}}\boldsymbol{\Omega}_o$ is defined by duality:

$$\langle \boldsymbol{\rho}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\boldsymbol{\alpha}, \mathbf{Q}_{\mathbf{y},\mathbf{x}}(\mathbf{h}) \rangle_{\mathbf{y}} := \langle \boldsymbol{\alpha}, \mathbf{h} \rangle_{\mathbf{x}}\,, \quad \forall\, \mathbf{h} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_o\,,$$

and similarly for other tensors.

By the covariance paradigm:

The property of $\mathbf{g}$-homogeneity at a configuration $\boldsymbol{\chi}_o \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_o)$ of a hypo-elastic body implies $\boldsymbol{\xi}\!\uparrow\!\mathbf{g}$-homogeneity at $\boldsymbol{\xi}$-displaced configurations.

Integrability of a linear hypo-elastic law

Let us provide an answer to the question about the integrability of a linear hypo-elastic response, formulated as a tangent compliance. The available mathematical tool is the standard symmetry lemma of potential theory in linear spaces, which in its modern form concerns with the differential of an operator from a BANACH space to its dual

In a covariant theory the symmetry lemma may be applied by considering the pull back of the hypo-elastic law to a fixed reference configuration, according to the displacement maps associated with the motion.

The integrability may be investigated on the pulled response which, for each particle, is defined on the linear fibre of stress-stressing pairs and takes values into the dual stretching linear fibre based at the reference position.

The results so obtained do not agree with the ones provided by the analysis of integrability performed in

Bernstein, B.: Hypo-elasticity and elasticity. Arch. Rat. Mech. Anal. **6**, 90-104 (1960).

which all the subsequent relevant literature have made reference to.

In the sequel, C^k_F means k-fold continuous fibre-differentiability of fibre-preserving maps.

Definition: integrability

A linear hypo-elastic response is integrable to an elastic response if there exists a strain-valued hypo-elastic stress-potential, that is a bundle morphism:

$$\boldsymbol{\Phi}_{\varphi,t} \in \mathrm{C}^1_F(\mathrm{SYM}^*(\boldsymbol{\Omega}_t)\,;\mathrm{SYM}(\boldsymbol{\Omega}_t))\,,$$

such that $\mathbf{H}_{\varphi_t} = d_F\boldsymbol{\Phi}_{\varphi_t}$, so that

$$\tfrac{1}{2}\mathcal{L}_{\varphi,t}\,\mathbf{g}_{\varphi} = d_F\boldsymbol{\Phi}_{\varphi_t}(\boldsymbol{\sigma}_{\varphi,t}) \cdot \mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi}\,.$$

It is integrable to an hyper-elastic response if it exists a scalar-valued stress-potential $E^*_{\varphi,t} \in \mathrm{C}^2_F(\mathrm{SYM}^*(\boldsymbol{\Omega}_t)\,;\mathrm{FUN}(\boldsymbol{\Omega}_t))$ such that $\mathbf{H}_{\varphi_t} = d^2_F E^*_{\varphi,t}$, so that

$$\tfrac{1}{2}\mathcal{L}_{\varphi,t}\,\mathbf{g}_{\varphi} = d^2_F E^*_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t}) \cdot \mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi}\,.$$

As a consequence of the covariance paradigm, the integrability property may be formulated in terms of the pull-back of the constitutive law to a fixed reference configuration.

This is the basic property that opens the way to the application of the standard symmetry lemma of potential theory in linear spaces.

Volterra, V.: Leçons sur les fonctions de lignes, professées à la Sorbonne en 1912. Gauthier-Villars, Paris (1913).

Vainberg, M.M.: Variational methods for the study of nonlinear operators. Holden-Day, Inc., San Francisco (1964).

Hereafter d_F denotes the fibre-derivative, in the bundle $(\mathrm{SYM}^*(\boldsymbol{\Omega}_o), \boldsymbol{\pi}_{\mathrm{CON}}, \boldsymbol{\Omega}_o)$ taken by holding the base point fixed.

Proposition: Integrability

A linear hypo-elastic response is integrable to an elastic response if there exist a reference configuration $\boldsymbol{\chi}_o \in \mathrm{C}^1(\mathcal{B}\,;\mathcal{S})$ and a strain-valued stress-potential $\boldsymbol{\Phi}_o \in \mathrm{C}^1_F(\mathrm{SYM}^*(\boldsymbol{\Omega}_o)\,;\mathrm{SYM}(\boldsymbol{\Omega}_o))$, such that $\mathbf{H}_o = d_F\boldsymbol{\Phi}_o$, i.e.

$$\tfrac{1}{2}\partial_{\tau=t}\ \boldsymbol{\xi}_\tau{\downarrow}\mathbf{g}_{\boldsymbol{\varphi}_\tau} = d_F\boldsymbol{\Phi}_o(\boldsymbol{\xi}_t{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \cdot \partial_{\tau=t}\ \boldsymbol{\xi}_\tau{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}\,,$$

which integrated provides the elastic response

$$\tfrac{1}{2}(\boldsymbol{\xi}_t{\downarrow}\mathbf{g}_{\boldsymbol{\varphi}_t} - \boldsymbol{\xi}_{t_o}{\downarrow}\mathbf{g}_{\boldsymbol{\varphi}_{t_o}}) = \boldsymbol{\Phi}_o(\boldsymbol{\xi}_t{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},t} - \boldsymbol{\xi}_{t_o}{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi}_{t_0}})\,.$$

The linear hypo-elastic response is said integrable to an hyperlastic response if there exists a scalar-valued potential $E^*_o \in \mathrm{C}^2_F(\mathrm{SYM}^*(\boldsymbol{\Omega}_o)\,;\mathrm{FUN}(\boldsymbol{\Omega}_o))$, such that $\mathbf{H}_o = d^2_F E^*_o$, i.e.

$$\tfrac{1}{2}\partial_{\tau=t}\ \boldsymbol{\xi}_\tau{\downarrow}\mathbf{g}_{\boldsymbol{\varphi}_\tau} = d^2_F E^*_o(\boldsymbol{\xi}_t{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \cdot \partial_{\tau=t}\ \boldsymbol{\xi}_\tau{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}\,,$$

which integrated provides the hyper-elastic response

$$\tfrac{1}{2}(\boldsymbol{\xi}_t{\downarrow}\mathbf{g}_{\boldsymbol{\varphi}_t} - \boldsymbol{\xi}_{t_o}{\downarrow}\mathbf{g}_{\boldsymbol{\varphi}_{t_o}}) = d_F E^*_o(\boldsymbol{\xi}_t{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},t} - \boldsymbol{\xi}_{t_o}{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi}_{t_0}})\,,$$

with hyper-elastic stress-potential $E^*_o \in \mathrm{C}^2_F(\mathrm{SYM}^*(\boldsymbol{\Omega}_o)\,;\mathrm{FUN}(\boldsymbol{\Omega}_o))$.

Proposition: Integrability conditions

The hypo-elastic response is integrable to an elastic response if and only if, in each linear fibre, the following symmetry condition holds:

$$\langle d_F \mathbf{H}_o(\boldsymbol{\sigma}_{\mathbf{x}}) \cdot \delta\boldsymbol{\sigma}_{\mathbf{x}} \cdot \delta_1\boldsymbol{\sigma}_{\mathbf{x}}, \delta_2\boldsymbol{\sigma}_{\mathbf{x}} \rangle = \langle d_F \mathbf{H}_o(\boldsymbol{\sigma}_{\mathbf{x}}) \cdot \delta\boldsymbol{\sigma}_{\mathbf{x}} \cdot \delta_2\boldsymbol{\sigma}_{\mathbf{x}}, \delta_1\boldsymbol{\sigma}_{\mathbf{x}} \rangle \,,$$

for all $\delta\boldsymbol{\sigma}_{\mathbf{x}}, \delta_1\boldsymbol{\sigma}_{\mathbf{x}}, \delta_2\boldsymbol{\sigma}_{\mathbf{x}} \in \mathrm{SYM}^*(\boldsymbol{\Omega}_o)_{\mathbf{x}}$. The further symmetry condition:

$$\langle \mathbf{H}_o(\boldsymbol{\sigma}_{\mathbf{x}}) \cdot \delta_1\boldsymbol{\sigma}_{\mathbf{x}}, \delta_2\boldsymbol{\sigma}_{\mathbf{x}} \rangle = \langle \mathbf{H}_o(\boldsymbol{\sigma}_{\mathbf{x}}) \cdot \delta_2\boldsymbol{\sigma}_{\mathbf{x}}, \delta_1\boldsymbol{\sigma}_{\mathbf{x}} \rangle \,,$$

ensures that the elastic response is hyper-elastic.

This integrability condition is trivially verified if the hypo-elastic response in a reference configuration is independent of the stress state.

This independence, if verified at a given reference configuration, will also hold at any other one. The following chain of inclusions then holds true:

$$\boxed{\text{linear hypo-elasticity} \supset \text{elasticity} \supset \text{hyper-elasticity}}$$

Proposition: reference configuration independence

The integrability properties are independent of the choice of a reference configuration.

The following proposition assures that integrability is independent of the kind of dual tensors chosen to formalize the hypo-elastic response.

Proposition: independence of alteration

Integrability of a linear hypo-elastic response expressed in terms of covariant stretching and contravariant stress and stressing, is equivalent to integrability of the linear hypo-elastic response expressed in terms of mixed tensors, provided that alteration is performed at each configuration by means of the metric tensor pushed according to the displacement map. The relevant potentials are related by:

$$\mathbf{H}_o = d_F \boldsymbol{\Phi}_o \quad \Longleftrightarrow \quad \mathbf{H}_o^{\mathrm{MIX}} = d_F \boldsymbol{\Phi}_o^{\mathrm{MIX}} \,,$$

$$\boldsymbol{\Phi}_o = d_F E_o^* \quad \Longleftrightarrow \quad \boldsymbol{\Phi}_o^{\mathrm{MIX}} = d_F E_o^{*\mathrm{MIX}} \,,$$

The relationships between the potentials are the following:

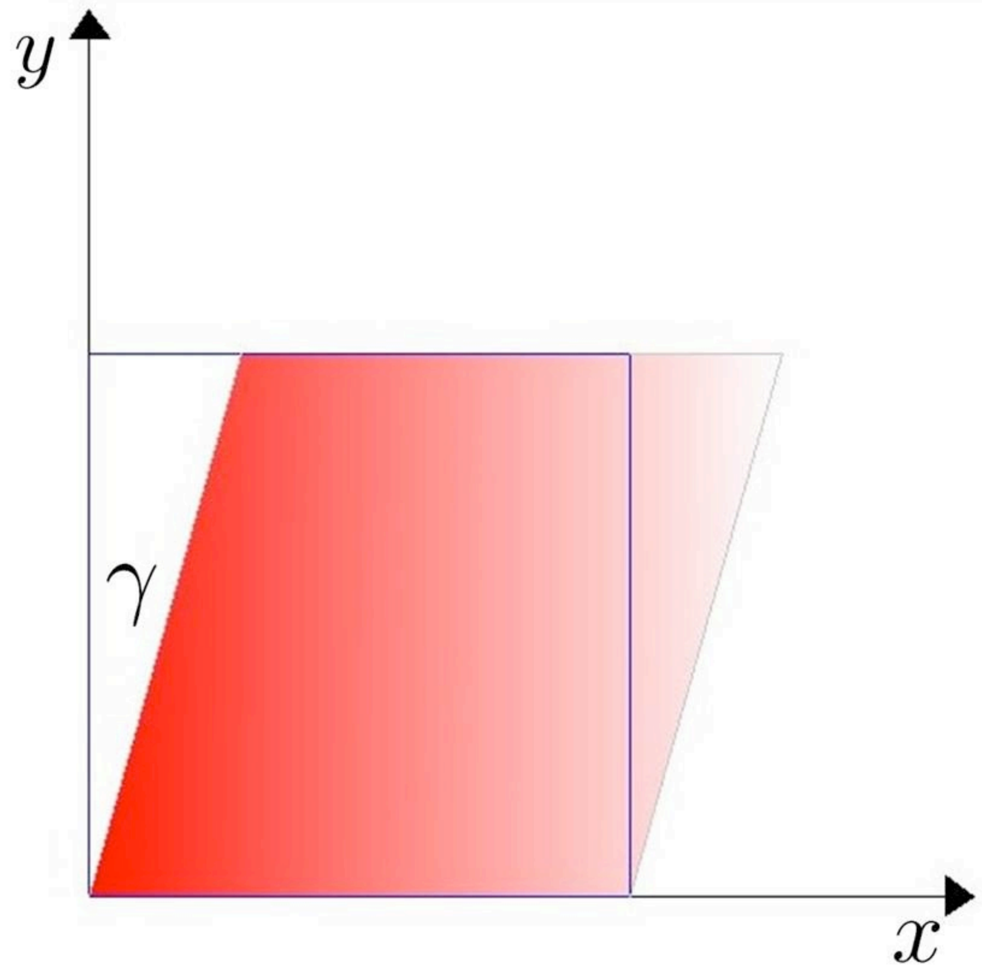
$$\boldsymbol{\Phi}_o^{\mathrm{MIX}}(\boldsymbol{\sigma}_{\mathbf{x}}^{\mathrm{MIX}}) = \mathbf{g}^*_{\boldsymbol{\chi}_o} \circ \boldsymbol{\Phi}_o(\boldsymbol{\sigma}_{\mathbf{x}}^{\mathrm{MIX}} \circ \mathbf{g}^*_{\boldsymbol{\chi}_o}) \,,$$

$$E_o^{\mathrm{MIX}}(\boldsymbol{\sigma}_{\mathbf{x}}^{\mathrm{MIX}}) = E_o(\boldsymbol{\sigma}_{\mathbf{x}}^{\mathrm{MIX}} \circ \mathbf{g}^*_{\boldsymbol{\chi}_o}) \,.$$

Simple isochoric shearing

Let us consider a unit cube as a natural stress-free configuration of a body and a cartesian reference system. A simple shear is a one parameter displacement whose expression in the cartesian reference system is given by

$$\boxed{\boldsymbol{\varphi}_\gamma(x, y, z) = (x + \gamma\, y)\, \mathbf{e}_1 + y\, \mathbf{e}_2 + z\, \mathbf{e}_3\,.}$$



Let us recall that, given the matrix of a linear map with respect to an orthonormal basis, its transpose is the matrix of the dual linear map with respect to the dual basis. The matrices of the relevant tangent map, of its inverse and of the dual and its inverse, taken with respect to the orthonormal basis $\{\, \mathbf{e}_i \,\}$ and to the dual basis $\{\, \mathbf{e}^i \,\}$, with $\mathbf{e}^i = \mathbf{g}_{\boldsymbol{\chi}_o}(\mathbf{e}_i)$, are then:

$$[T\boldsymbol{\varphi}_\gamma] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [T\boldsymbol{\varphi}_\gamma^{-1}] = \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[T^*\boldsymbol{\varphi}_\gamma] = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [T^*\boldsymbol{\varphi}_\gamma^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix $[\mathbf{E}_o]$ of Green's strain

$$\mathbf{E}_o(\gamma) = \tfrac{1}{2}(\boldsymbol{\varphi}_\gamma{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},\gamma} - \mathbf{g}_{\boldsymbol{\chi}_o}) = \tfrac{1}{2}(T^*\boldsymbol{\varphi}_\gamma \circ \mathbf{g}_{\boldsymbol{\varphi},\gamma} \circ T\boldsymbol{\varphi}_\gamma - \mathbf{g}_{\boldsymbol{\chi}_o})\,,$$

with respect to the basis $\{\,\mathbf{e}_i\,\}$ is given by

$$[\mathbf{E}_o](\gamma) = \tfrac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,.$$

From the constitutive law:

$$\frac{1}{\mu}\boldsymbol{\varphi}_\gamma{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},t} = \mathbf{g}^*_{\boldsymbol{\chi}_o}\circ(\boldsymbol{\varphi}_\gamma{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},\gamma}-\mathbf{g}_{\boldsymbol{\chi}_o})\circ\mathbf{g}^*_{\boldsymbol{\chi}_o}+\frac{\nu}{1-2\,\nu}\,J_1(\mathbf{g}^*_{\boldsymbol{\chi}_o}\circ(\boldsymbol{\varphi}_\gamma{\downarrow}\mathbf{g}_{\boldsymbol{\varphi},\gamma}-\mathbf{g}_{\boldsymbol{\chi}_o}))\circ\mathbf{g}^*_{\boldsymbol{\chi}_o}\,,$$

we get the expression of the matrix of the referential Cauchy true stress $\mathbf{T}_o = \boldsymbol{\varphi}_\gamma{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\gamma}\circ\mathbf{g}_{\boldsymbol{\chi}_o}$ with respect to the basis $\{\,\mathbf{e}_i\,\}$ and the dual $\{\,\mathbf{e}^i\,\}$:

$$\frac{1}{\mu}\,[\mathbf{T}_o](\gamma) \;=\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\frac{\nu}{1-2\,\nu}\,\gamma^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\,.$$

Being

$$\begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the matrix of the Cauchy true stress $\mathbf{T} = T\boldsymbol{\varphi}_\gamma \circ \mathbf{T}_o \circ T\boldsymbol{\varphi}_\gamma^{-1}$ is given by

$$\frac{1}{\mu}\,[\mathbf{T}](\gamma) \;=\begin{bmatrix} \gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\nu}{1-2\,\nu}\,\gamma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

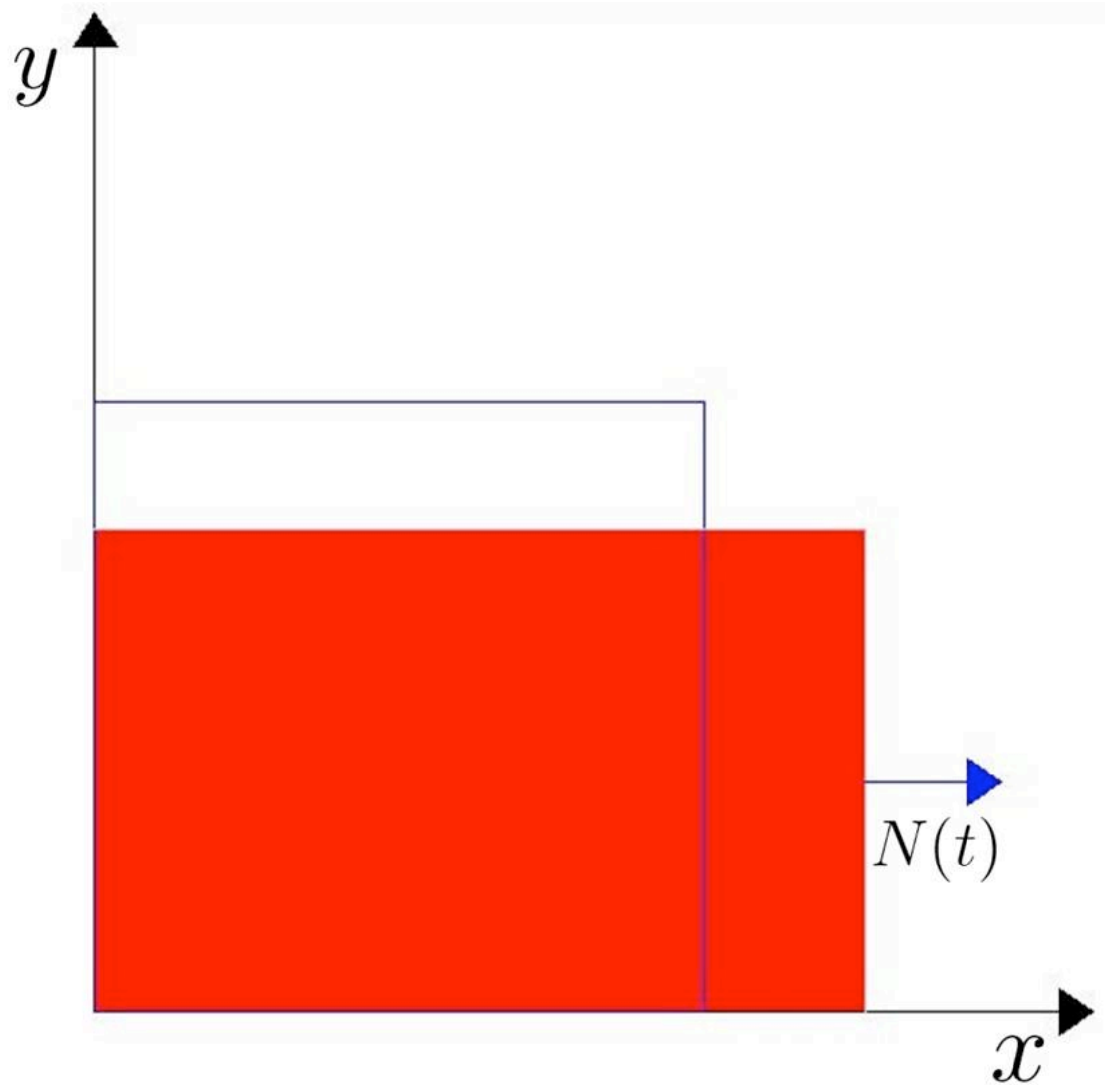
The initial linearized law is expressed by the usual linear relation between shearing stress and strains, i.e.

$$d_{\gamma=0}\,[\mathbf{T}](\gamma) = \mu \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Homogeneous extension

A homogeneous extension is got by a one parameter displacement of a unitary cube:

$$\boxed{\varphi_t(x,y,z) = \alpha\, t\, x\, \mathbf{e_x} + \beta\, t\, y\, \mathbf{e_y} + \, z\, \mathbf{e_z}}$$



$$[T\boldsymbol{\varphi}_t] = \begin{bmatrix} \alpha\, t & 0 & 0 \\ 0 & \beta\, t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [\mathbf{E}_o](t) = \tfrac{1}{2}\begin{bmatrix} \alpha^2\, t^2 - 1 & 0 & 0 \\ 0 & \beta^2\, t^2 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Tangent map

Green strain $\mathbf{E}_o(t) = \frac{1}{2}(T^*\boldsymbol{\varphi}_t \circ \mathbf{g}_{\boldsymbol{\varphi},t} \circ T\boldsymbol{\varphi}_t - \mathbf{g}_{\boldsymbol{\chi}_o})$

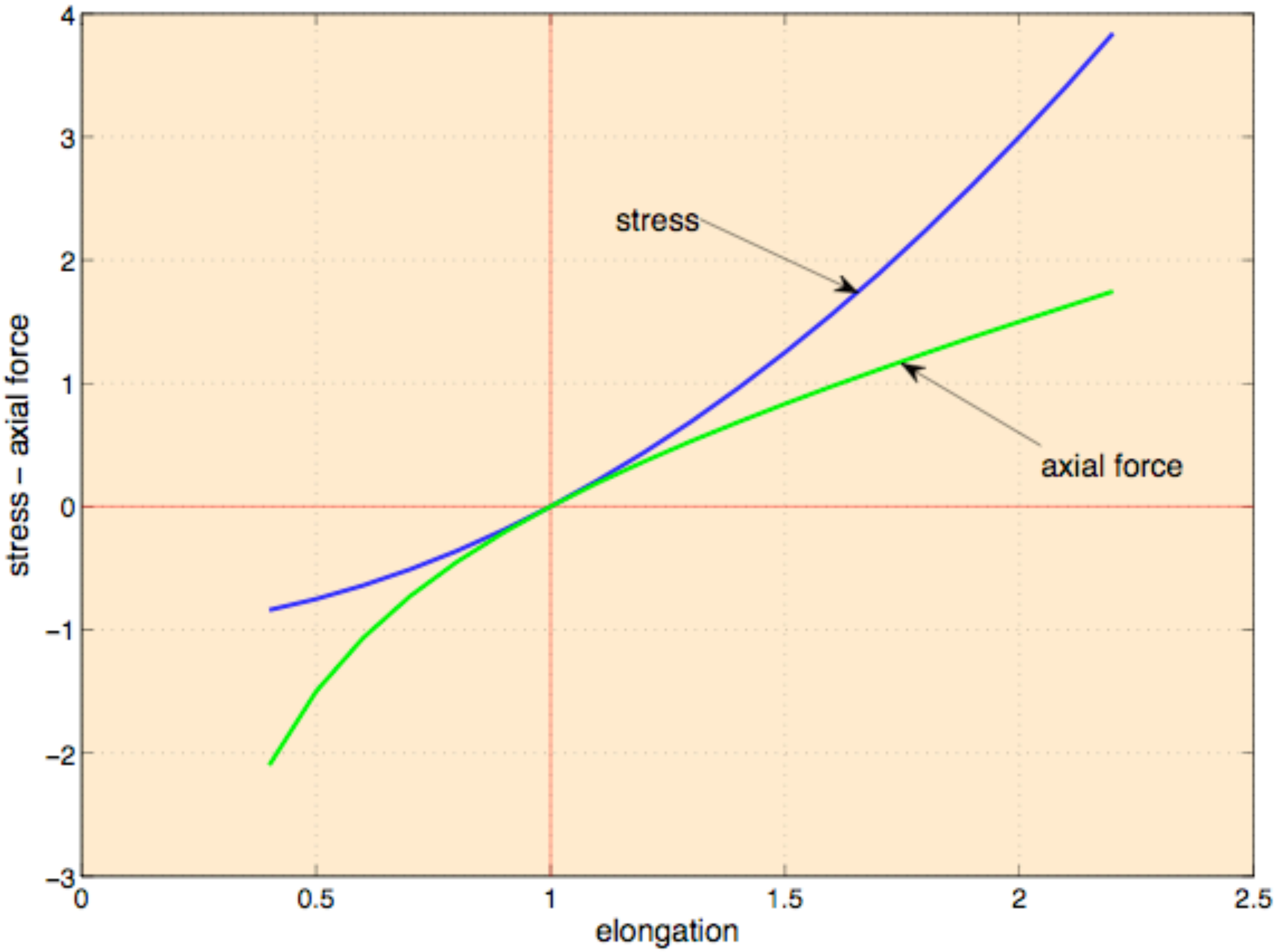
According to the simplest rate law, the matrix of the referential Cauchy true stress $\mathbf{T}_o(t)$, setting $k(t) = \frac{\nu\,((\alpha^2+\beta^2)\,t^2-2)}{1-2\,\nu}$, is given by

$$\frac{1}{\mu}[\mathbf{T}_o](t) = \begin{bmatrix} \alpha^2\, t^2 - 1 & 0 & 0 \\ 0 & \beta^2\, t^2 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - k(t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that for the Cauchy true stress $\mathbf{T} = T\boldsymbol{\varphi}_\alpha \circ \mathbf{T}_o \circ T\boldsymbol{\varphi}_\alpha^{-1}$ we get

$$\frac{1}{\mu}[\mathbf{T}](t) = \begin{bmatrix} \alpha^2\, t^2 - 1 & 0 & 0 \\ 0 & \beta^2\, t^2 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - k(t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Assuming $\nu = 0$ and $\beta\, t = (\alpha\, t)^{-1}$, which corresponds to a vanishing Poisson effect and to an isochoric displacement, the normal stress $\mathbf{T}_{11}(t)$ and the resultant axial force $N(t) = A(t)\,\mathbf{T}_{11}(t) = \mu\,(\alpha\, t - 1/(\alpha\, t))$, where $A(t) = 1/(\alpha\, t)$ is the transversal area, are plotted in



Concluding remarks

- The property of covariance is formulated as variance by push instead of invariance under push.

- The principle of material frame indifference is accordingly correctly reformulated and shown to be trivially satisfied by any (covariant) material response.

- Spurious results, such as that material frame indifference should imply isotropy of the hypo-elastic response and of plastic yield functions, are eliminated. Accordingly, treatments devoted to recover a description of anisotropic behaviors of elastic and plastic responses should be reconsidered.

- Homogeneity and isotropy of the material are properly defined and shown to be consistent with the covariant transformation of the material response at different configurations.

- Formulations in terms of different alterations of the relevant tensors and push to other configurations may be interchanged without affecting the result, thus restoring a sound physical basis to the constitutive theory.

- The integration needed for the evaluation of the stress may be performed on the time dependent pull-backs of the stressing to a fixed reference configuration, the result being got by a subsequent push-forward to the actual configuration, in a way independent of the chosen configuration.

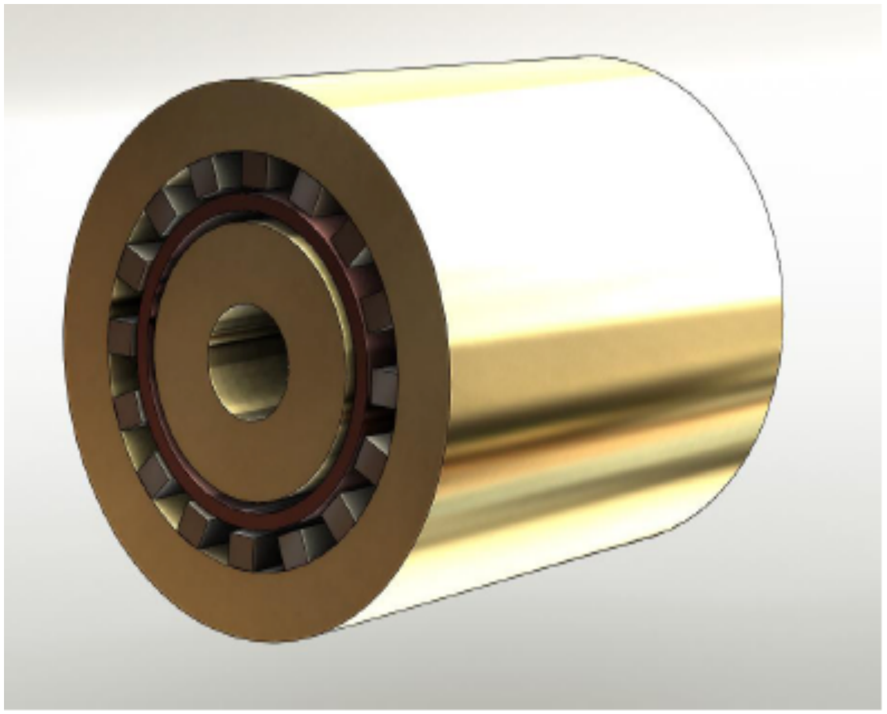
- The integrability conditions of the hypo-elastic behaviour may be checked at any fixed reference configuration and the relevant potentials may be readily computed, still in a way independent of the chosen reference configuration.

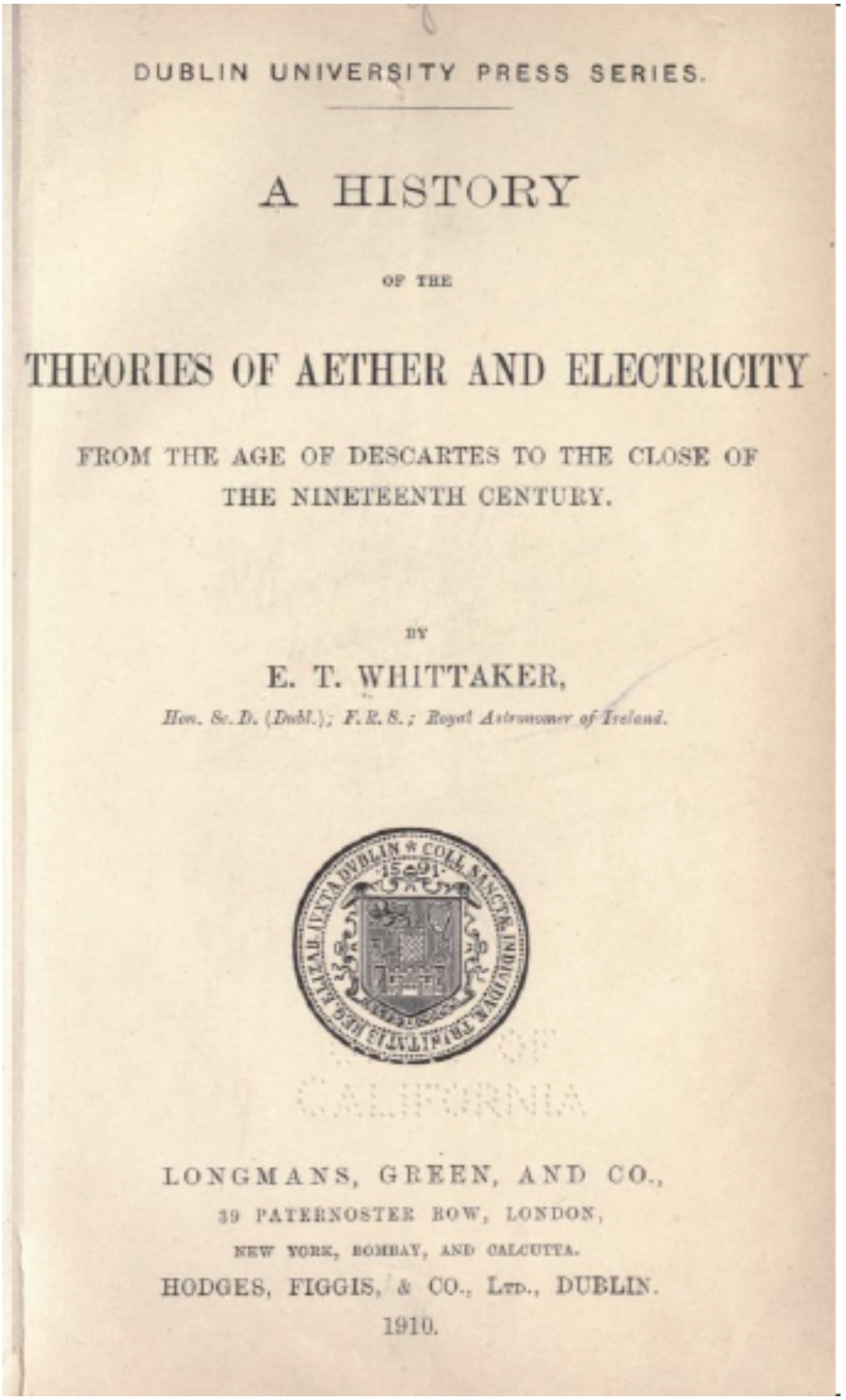
These basic implications of the covariance paradigm require to review most existing theoretical and computational approaches.

PART II

The Covariance Paradigm

in electromagnetic induction




DUBLIN UNIVERSITY PRESS SERIES.

A HISTORY

OF THE

THEORIES OF AETHER AND ELECTRICITY

FROM THE AGE OF DESCARTES TO THE CLOSE OF THE NINETEENTH CENTURY.

BY

E. T. WHITTAKER,

Hon. Sc.D. (Dubl.); F.R.S.; Royal Astronomer of Ireland.

LONGMANS, GREEN, AND CO.,
39 PATERNOSTER ROW, LONDON,
NEW YORK, BOMBAY, AND CALCUTTA.
HODGES, FIGGIS, & CO., LTD., DUBLIN.
1910.

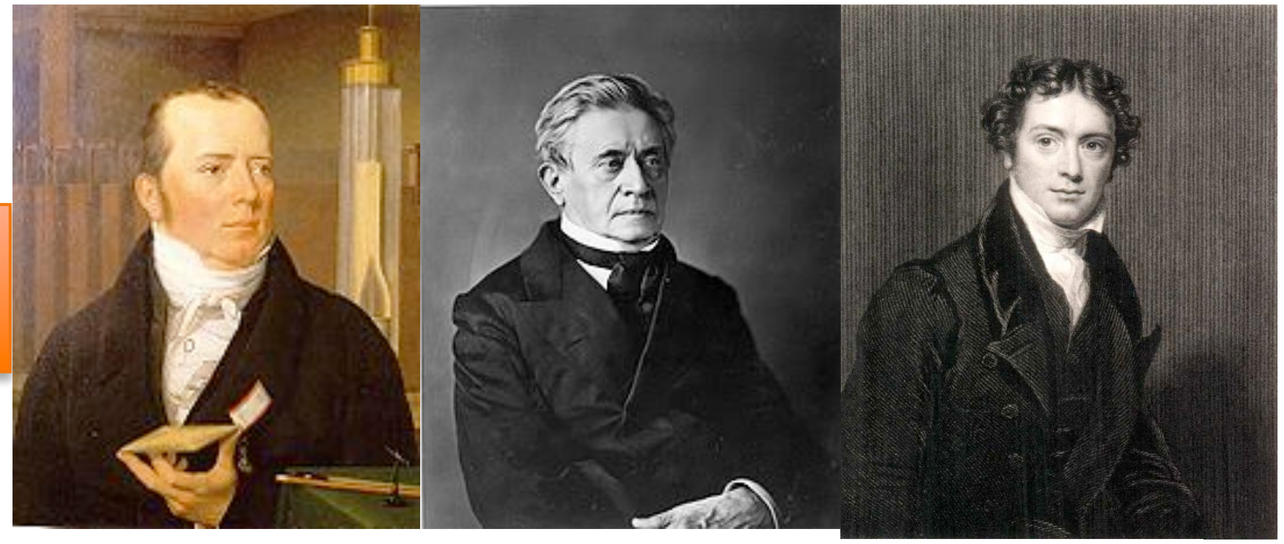

Edmund Whittaker

Darrigol, O. (2000). Electrodynamics from Ampère to Einstein. Oxford University Press. ISBN 0-198-50593-0

Classical Formulation of the laws of electromagnetism as introduced in most modern textbooks

HANS CHRISTIAN ØRSTED (1820)

Henry-Faraday Law (1826-1831)



$$\oint_{\partial \boldsymbol{\Sigma}_t} \mathbf{gE} = -\int_{\boldsymbol{\Sigma}_t} \boldsymbol{\mu}\dot{\mathbf{B}}$$

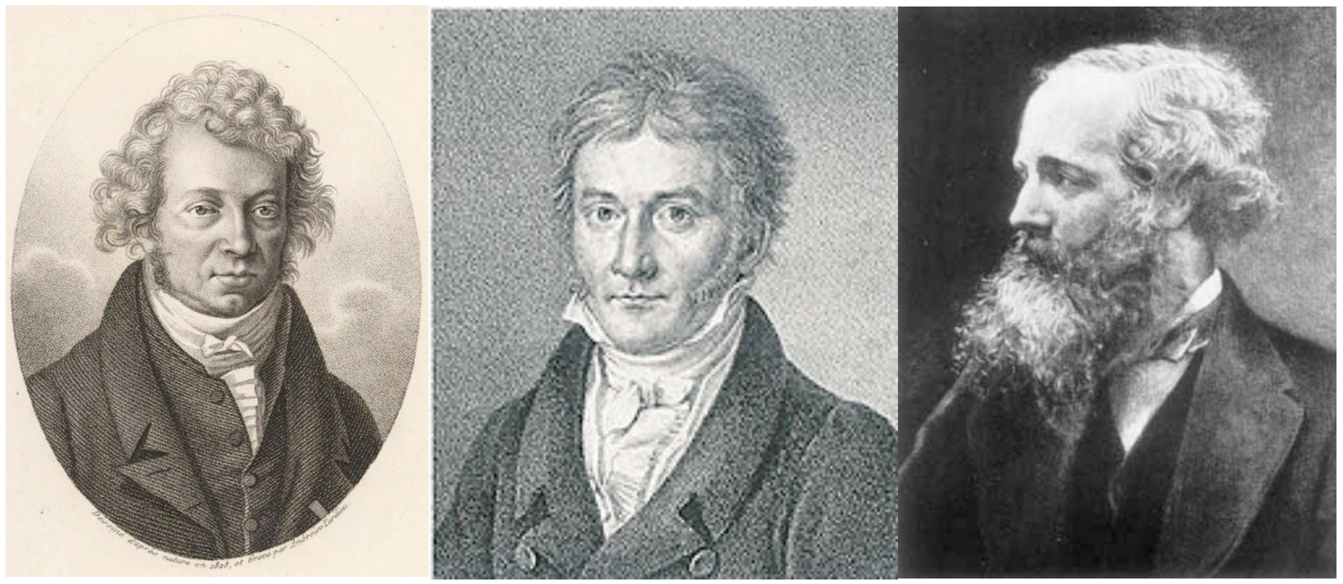
MAXWELL(1881)-HENRY(1831)-FARADAY(1831)

$$\oint_{\partial \boldsymbol{\Omega}} \boldsymbol{\mu}\mathbf{D} = \int_{\boldsymbol{\Omega}} \rho_{\mathbf{E}}\boldsymbol{\mu}$$

GAUSS(1835)

$$\oint_{\partial \boldsymbol{\Omega}} \boldsymbol{\mu}\mathbf{B} = 0$$

GAUSS(1831)



$$\oint_{\partial \boldsymbol{\Sigma}_t} \mathbf{gH} = \int_{\boldsymbol{\Sigma}_t} \boldsymbol{\mu}(\dot{\mathbf{D}} + \mathbf{J}_{\mathbf{E}})$$

MAXWELL(1861)-AMPÈRE(1826)

Maxwell-Ampère Law (1820-1861)

Abraham R., Marsden J.E., Ratiu T.,
Manifolds, Tensor Analysis, and Applications (2002)

9.3 Electromagnetism

Classical electromagnetism is governed by Maxwell's field equations. The form of these equations depends on the physical units chosen, and changing these units introduces factors like 4π, $c =$ the speed of light, $\epsilon_0 =$ the dielectric constant and $\mu_0 =$ the magnetic permeability. The discussion in this section assumes that ϵ_0, μ_0 are constant; the choice of units is such that the equations take the simplest form; thus $c = \epsilon_0 = \mu_0 = 1$ and factors 4π disappear. We also do not consider Maxwell's equations in a material, where one has to distinguish $\mathbf{E}$ from $\mathbf{D}$, and $\mathbf{B}$ from $\mathbf{H}$.

Let $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{J}$ be time dependent C^1-vector fields on $\mathbb{R}^3$ and $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ a scalar. These are said to satisfy ***Maxwell's equations*** with ***charge density*** ρ and ***current density*** $\mathbf{J}$ when the following hold:

$$\operatorname{div} \mathbf{E} = \rho \quad (\textit{Gauss's law}) \tag{9.3.1}$$

$$\operatorname{div} \mathbf{B} = 0 \quad (\textit{no magnetic sources}) \tag{9.3.2}$$

$$\operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad (\textit{Faraday's law of induction}) \tag{9.3.3}$$

$$\operatorname{curl} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (\textit{Ampère's law}) \tag{9.3.4}$$

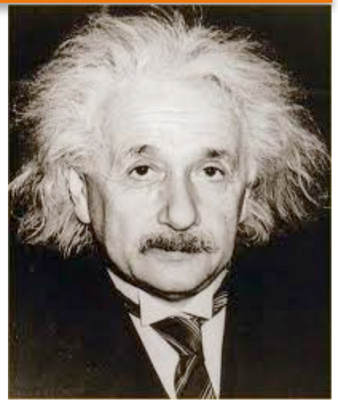
$\mathbf{E}$ is called the ***electric field*** and $\mathbf{B}$ the ***magnetic field***.

By Stokes' theorem, equation (9.3.3) is equivalent to

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{ds} = \int_S (\operatorname{curl} \mathbf{E}) \cdot \mathbf{n}\, dS = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n}\, dS \tag{9.3.7}$$

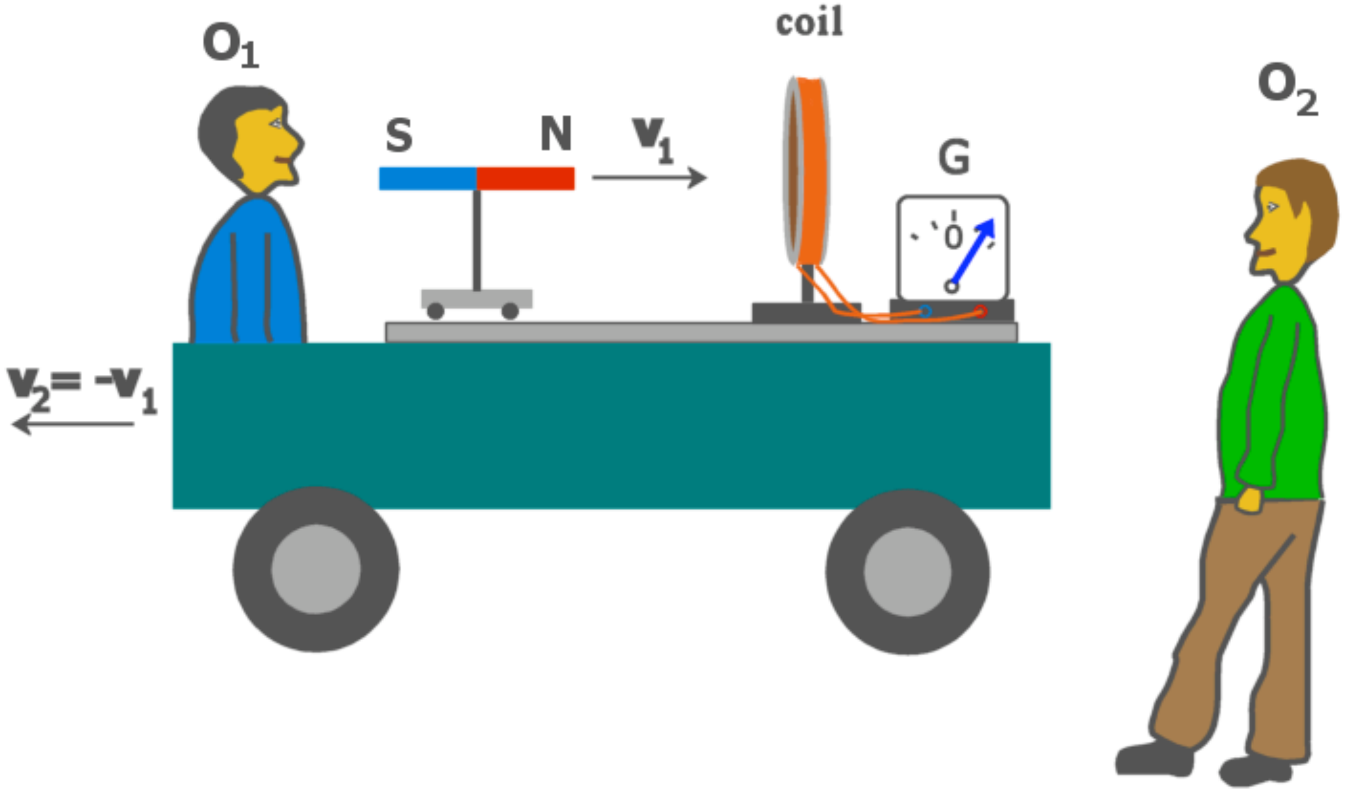
for any closed loop ∂S bounding a surface S. The quantity $\int_{\partial S} \mathbf{E} \cdot \mathbf{ds}$ is called the ***voltage*** around ∂S. Thus, Faraday's law of induction equation (9.3.3), says that *the voltage around a loop equals the negative of the rate of change of the magnetic flux through the loop.*

Electrodynamics



Moving truck

Faraday law



Lorentz force

According to the most formulation of electrodynamics, the man in **blue sweater** explains the turning of the galvanometer needle by the Faraday law of magnetic induction, while the green fellow explains the same phenomenon by the Lorentz force.

Jackson, D.J. Classical Electrodynamics (1999)

When Einstein began to think about these matters
There existed several possibilities:

1. The Maxwell equations were incorrect.
   The proper theory of electromagnetism was invariant
   under Galilean transformations.

2. Galilean relativity applied to classical mechanics,
   but electromagnetism had a preferred reference frame,
   the frame in which the luminiferous ether was at rest.

3. There existed a relativity principle for classical mechanics
   and electromagnetism, but it was not Galilean relativity.
   This would imply that the laws of mechanics were in need of
   modification.

The first possibility was hardly viable.

Albert Einstein 1905

ON THE ELECTRODYNAMICS OF MOVING BODIES

By A. EINSTEIN

June 30, 1905

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

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Sadiku, M.N.O., 2010. Elements of Electromagnetics (Fifth ed.). Oxford University Press. USA. ISBN-13: 9780195387759

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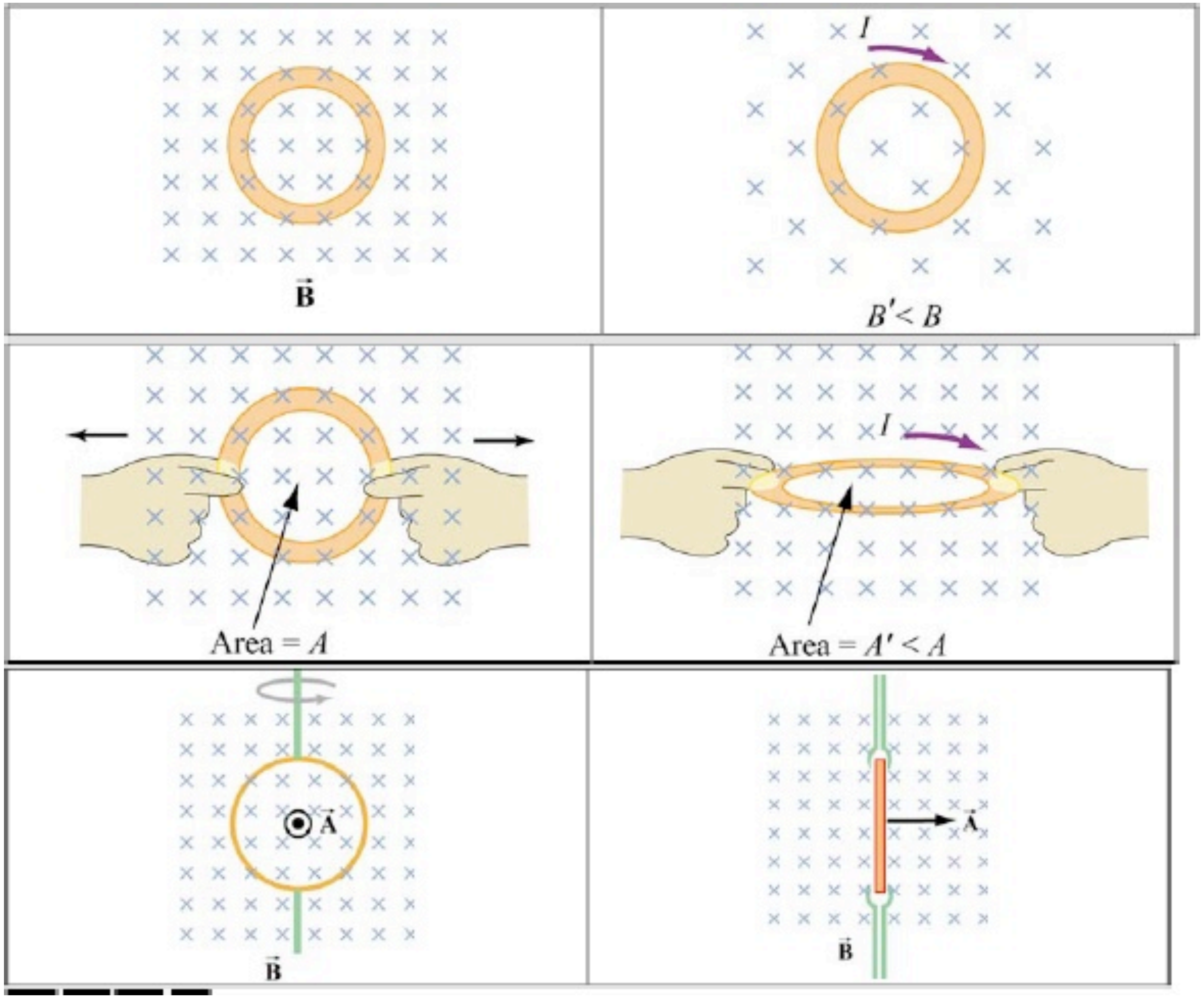
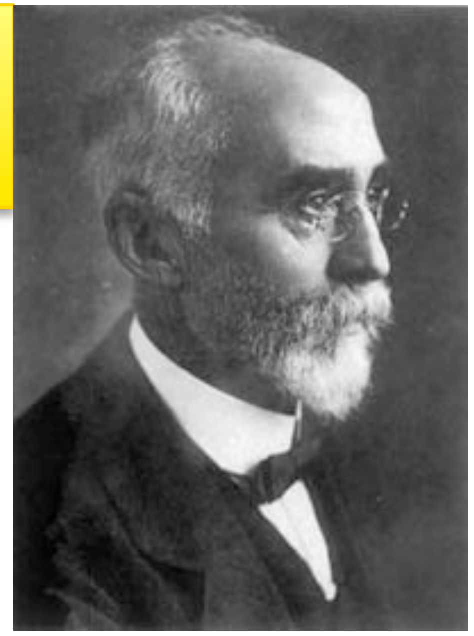
Faraday law of induction: examples



Lorentz force (1892)

Richard Phillips Feynman



Hendrik Antoon Lorentz

FEYNMAN:

We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathrm{rot}\, \mathbf{E}_t = -\partial_{\tau=t}\, \mathbf{B}_\tau$$

When we said that the magnetic force on a charge was proportional to its velocity, you may have wondered: "What velocity? With respect to which reference frame?" It is,in fact, clear from the definition of B given at the beginning of this chapter that what this vector is will depend on what we choose as a reference frame for our specification of the velocity of charges. But we have said nothing about which is the proper frame for specifying the magnetic field.

Benjamin Crowell, 2010.

Electricity and Magnetism

Book 4 in the Light and Matter series

Experiments show that the magnetic force on a moving charged particle has a magnitude given by

$$|F| = q|v||B| \sin \theta ,$$

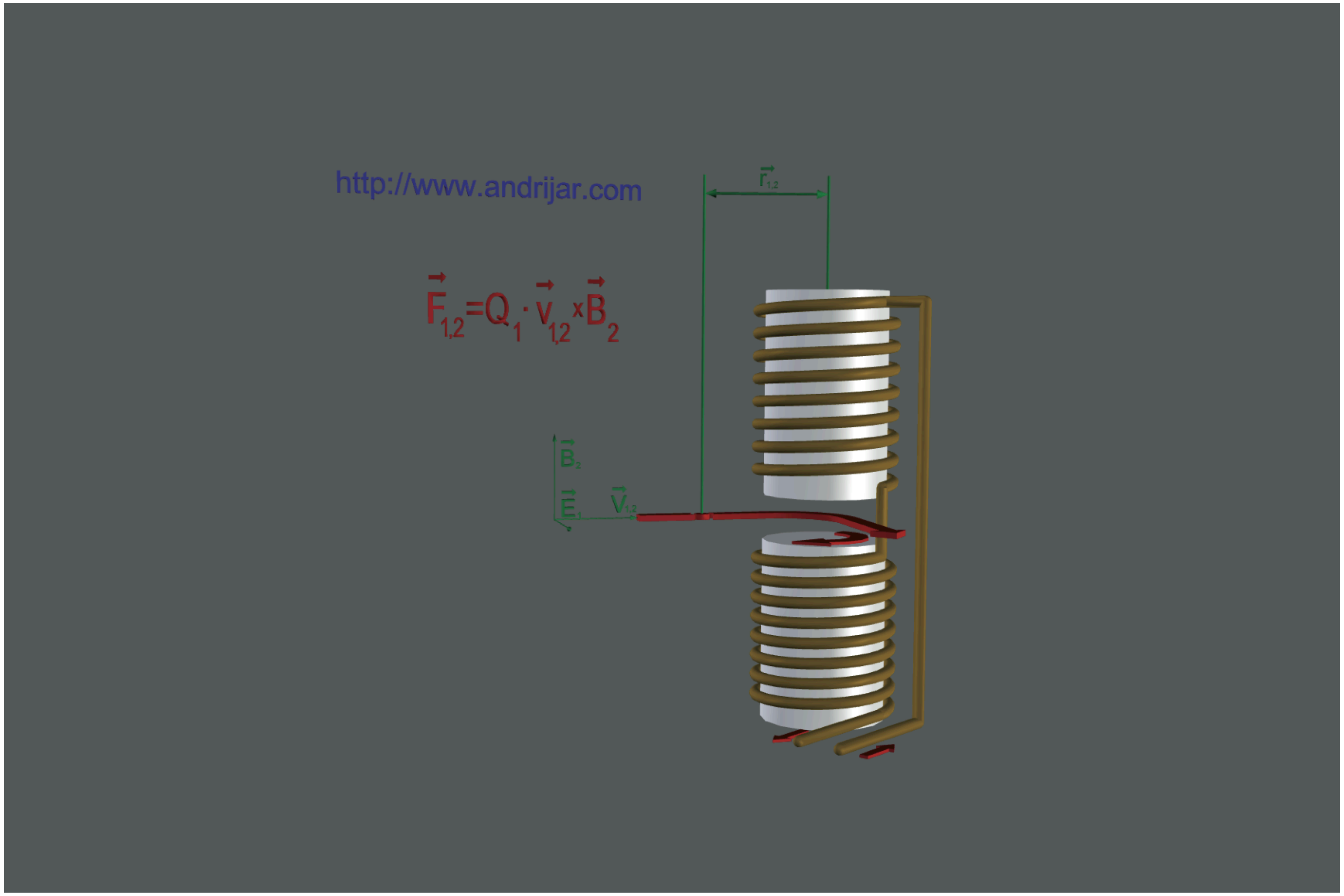
where

v is the velocity vector of the particle, and

θ is the angle between the v and B vectors.

Unlike electric and gravitational forces, magnetic forces do not lie along the same line as the field vector.

Lorentz force



Lorentz Force

A force on moving charges particles in the presence of magnetic **B** and electric fields **E**. The electromagnetic field tensor in MKS is given by

$$E'_x = \gamma(E_x - vB_y) \quad (1)$$

$$B'_x = \gamma\left(B_x + \frac{v}{c^2}E_y\right) \quad (2)$$

$$E'_y = \gamma(E_y + vB_x) \quad (3)$$

$$B'_y = \gamma\left(B_y - \frac{v}{c^2}E_x\right) \quad (4)$$

$$E'_z = E_z \quad (5)$$

$$B'_z = B_z, \quad (6)$$

so the Lorenz force is

$$\mathbf{F} \equiv q\mathbf{E}' = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (7)$$

$$= q\left[-\nabla\phi - \frac{d\mathbf{A}}{dt} + \nabla(\mathbf{A} \cdot \mathbf{v})\right], \quad (8)$$

where q is the charge on a particle, **v** is its velocity, ϕ is the electric potential, and **A** is the magnetic vector potential.

SEE ALSO: Canonical Momentum, Electric Force, Magnetic Field, Magnetic Force

In the introduction and survey of (Jackson, 1999, p.3) it is said: *Also essential for consideration of charged particle motion is the Lorentz force equation,* $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ *, which gives the force acting on a point charge* q *in the presence of electromagnetic fields.* In dealing with FARADAY's law of induction, in (Jackson, 1999, p.210) it is further said: *It is important to note, however, that the electric field* $\mathbf{E}'$ *is the electric field at* $d\boldsymbol{l}$ (an infinitesimal piece of circuit) *in the coordinate system or medium in which* $d\boldsymbol{l}$ *is at rest, since is that field that causes current to flow if a circuit is actually present.* And a little bit later (Jackson, 1999, p.211) the following formula is claimed: $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ where $\mathbf{E}$ is *the electric field in the laboratory* and $\mathbf{E}'$ is *the electric field at* $d\boldsymbol{l}$ *in its rest frame of coordinates.*

In (Sadiku, 2010, chapter 9.5) it is said that: *it is worthwhile to mention other equations that go hand in hand with Maxwell's equations. The* LORENTZ *force equation* $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ *is associated with Maxwell's equations. Also the equation of continuity is implicit in Maxwell's equations.* No mention is made of the way the observer measuring the velocity is to be selected, in writing the LORENTZ force equation.

Geometric Formulation of the laws of electromagnetism

Faraday Law for a moving body

$$d\boldsymbol{\omega}^2_{\mathbf{B}} = 0 \qquad \textsc{Gauss}(1831)$$

$$-\oint_{\partial \boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\boldsymbol{\Sigma}_t)} \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{B}}$$

Maxwell-Ampère Law for a moving body

$$d\boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\rho}_{\mathbf{E}} \qquad \textsc{Gauss}(1835)$$

$$\oint_{\partial \boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{H}} = \partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\boldsymbol{\Sigma}_t)} \boldsymbol{\omega}^2_{\mathbf{D}} + \int_{\boldsymbol{\Sigma}_t} \boldsymbol{\omega}^2_{\mathbf{J_E}} = \int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J_E}}$$

Romano, G.: *The laws of Electromagnetism for moving bodies and related questions (2010)*

Well-posedness of Faraday law

For any control-window $\mathbf{C}_t$:

$$\int_{\partial \mathbf{C}_t} \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\mathbf{C}_t} d(\mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}}) = \int_{\mathbf{C}_t} \mathcal{L}_{\varphi,t}\, (d\boldsymbol{\omega}^2_{\mathbf{B}}) = 0 \,.$$

Well-posedness of Maxwell-Ampère law

$$\oint_{\partial \mathbf{C}_t} (\mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}}) = \int_{\mathbf{C}_t} d(\mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}}) = \int_{\mathbf{C}_t} (\mathcal{L}_{\varphi,t}\, d\boldsymbol{\omega}^2_{\mathbf{D}} + d\boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}}) = 0 \,,$$

equivalent to the property of *electric charge conservation*:

$$\mathcal{L}_{\varphi,t}\, \boldsymbol{\rho}_\mathbf{E} + d\boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0 \,,$$

or in the equivalent integral form:

$$\partial_{\tau=t} \int_{\varphi_{\tau,t}(\mathbf{C}_t)} \boldsymbol{\rho}_\mathbf{E} + \oint_{\partial \mathbf{C}_t} \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0 \,.$$

Romano, G.: *The laws of Electromagnetism for moving bodies and related questions (2010)*

Let $\hat{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,;\mathcal{S})$ be a motion of a body $\mathcal{B}$ in the ambient space and $\hat{\gamma} \in \mathrm{C}^1(\mathcal{S} \times I\,;\mathcal{S})$ be a time-dependent automorphism of the ambient space onto itself which we will call a *relative motion*. The pushed motion $\hat{\gamma}\!\uparrow\!\hat{\varphi} \in \mathrm{C}^1(\mathcal{S} \times I\,;\mathcal{S})$ according to the relative motion $\hat{\gamma} \in \mathrm{C}^1(\mathcal{S} \times I\,;\mathcal{S})$ is defined by the composition:

$$(\hat{\gamma}\!\uparrow\!\hat{\varphi})_\tau := \hat{\gamma}_\tau \circ \hat{\varphi}_\tau \,,$$

and the corresponding displacement from time $t \in I$ to time $\tau \in I$ along the pushed motion is given by:

$$(\hat{\gamma}\!\uparrow\!\hat{\varphi})_{\tau,t} = (\hat{\gamma}_\tau \circ \hat{\varphi}_\tau) \circ (\hat{\gamma}_t \circ \hat{\varphi}_t)^{-1} = \hat{\gamma}_\tau \circ \hat{\varphi}_{\tau,t} \circ \hat{\gamma}_t^{-1} \,.$$

Definition 5.1 (Invariance). *The invariance of a time-dependent spatial tensor field $\hat{\boldsymbol{\alpha}}$ under the action of a relative motion $\hat{\gamma} \in \mathrm{C}^1(\mathcal{S} \times I\,;\mathcal{S})$ is expressed by the drag condition*

$$(\hat{\gamma}\!\uparrow\!\hat{\boldsymbol{\alpha}})_t := \hat{\gamma}_t\!\uparrow\!\hat{\boldsymbol{\alpha}}_t = \hat{\boldsymbol{\alpha}}_t \,, \quad \forall\, t \in I \,.$$

For twice covariant tensors, the invariance property is written, explicitly:

$$(\hat{\gamma}\!\downarrow\!\hat{\boldsymbol{\alpha}})_t(\mathbf{a}\,,\mathbf{b}) := (\hat{\gamma}_t\!\downarrow\!\hat{\boldsymbol{\alpha}}_t)(\mathbf{a}\,,\mathbf{b}) = \hat{\boldsymbol{\alpha}}_t(\hat{\gamma}_t\!\uparrow\!\mathbf{a}\,,\hat{\gamma}_t\!\uparrow\!\mathbf{b})\circ\gamma_t = \hat{\boldsymbol{\alpha}}_t(\mathbf{a}\,,\mathbf{b}) \,, \qquad \forall\, \tau \in I \,,$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{T}\mathcal{S}$.

The basic result concerning invariance is provided by the next Lemma.

Lemma 5.2 (Invariance of convective time-derivatives). *Invariance of a time-dependent spatial tensor field $\hat{\boldsymbol{\alpha}}$ with respect to a relative motion, implies invariance of its convective time-derivativive, expressed by:*

$$\mathcal{L}_{(\hat{\gamma}\uparrow\hat{\varphi}),t}\, \hat{\boldsymbol{\alpha}} = \hat{\gamma}_t\!\uparrow\!\mathcal{L}_{\varphi,t}\, \hat{\boldsymbol{\alpha}} \,.$$

Covariance of electromagnetic induction laws under relative motion

$$\oint_{\partial \boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{E}} = \oint_{\partial \hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \hat{\gamma}_t {\uparrow} \boldsymbol{\omega}^1_{\mathbf{E}}$$

$$\int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \hat{\gamma}_t {\uparrow} \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \mathcal{L}_{(\hat{\gamma}\uparrow\hat{\varphi}),t}\, (\hat{\gamma}{\uparrow}\boldsymbol{\omega}^2_{\mathbf{B}})\,.$$

Invariance of electric field and magnetic flux

$$\gamma_t {\uparrow} \boldsymbol{\omega}^1_{\mathbf{E},t} = \boldsymbol{\omega}^1_{\mathbf{E},t}$$

$$\gamma_t {\uparrow} \boldsymbol{\omega}^2_{\mathbf{B},t} = \boldsymbol{\omega}^2_{\mathbf{B},t}$$

implies invariance of Faraday law

$$\oint_{\partial \boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{E}} = -\int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} \quad \Longleftrightarrow$$

$$\oint_{\partial \hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \boldsymbol{\omega}^1_{\mathbf{E}} = -\int_{\hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \mathcal{L}_{(\hat{\gamma}\uparrow\hat{\varphi}),t}\, \boldsymbol{\omega}^2_{\mathbf{B}}\,.$$

Romano, G.: *The laws of Electromagnetism for moving bodies and related questions (2010)*

Differential formulation of Faraday law

$$-d\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^2_{\mathbf{B}}$$

$$\mathrm{rot}\,\mathbf{E}_t = -\partial_{\tau=t}\,\mathbf{B}_\tau + \mathrm{rot}\,(\mathbf{v}_{\varphi,t} \times \mathbf{B}_t)$$

$$\boldsymbol{\omega}^2_{\mathbf{B}} = d\boldsymbol{\omega}^1_{\mathbf{F}}$$ Faraday potential

$$d\boldsymbol{\omega}^2_{\mathbf{B}} = 0$$ Gauss law

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^1_{\mathbf{F}} + dV_{\mathbf{E},t}\,,}$$

$$\mathcal{L}_{\varphi,t}\,d\,\boldsymbol{\omega}^1_{\mathbf{F}} = d\,\mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^1_{\mathbf{F}}$$

$$\boxed{\mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^1_{\mathbf{F}} = \partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} + \mathcal{L}_{\mathbf{v}_{\varphi,t}}\,\boldsymbol{\omega}^1_{\mathbf{F},t} = \partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} + d(\boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}) + (d\boldsymbol{\omega}^1_{\mathbf{F},t})\cdot\mathbf{v}_{\varphi,t}\,.}$$

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t}\cdot\mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.}$$

$$\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - \boldsymbol{\omega}^2_{\mathbf{B},t}\cdot\mathbf{v}_{\varphi,t} + dU_{\mathbf{E},t}$$

$$U_{\mathbf{E},t} = V_{\mathbf{E},t} - \boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}$$

$$\mathbf{E}_t = -\partial_{\tau=t}\,\mathbf{F}_\tau + \mathbf{v}_{\varphi,t} \times \mathbf{B}_t + \nabla U_{\mathbf{E},t}$$

Lorentz force

Maxwell, J.C., 1861. On Physical Lines of Force. The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science Fourth series, Part I, II, III, IV.

Electric field in a body in translational motion across a region of spatially uniform magnetic flux.

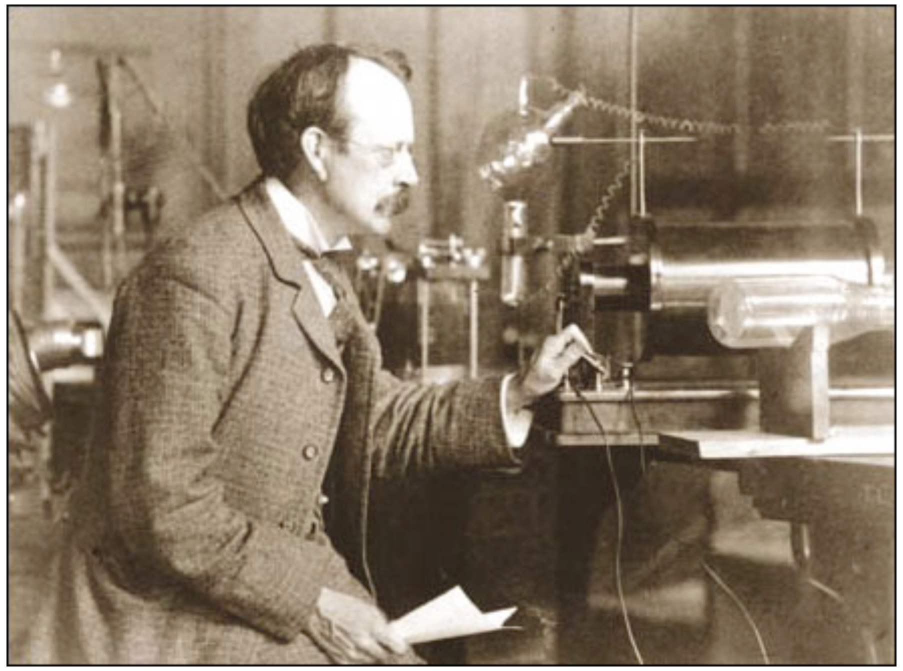
Lemma 8.1 (Linear Faraday potential). *In the* Euclid *space with the standard connection, the linear field*

$$\boldsymbol{\omega}^1_{\mathbf{F},t} := \tfrac{1}{2}\boldsymbol{\mu} \cdot \mathbf{B}_t \cdot \mathbf{r} = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{r}\,,$$

where $\mathbf{r}(\mathbf{x}) := \mathbf{x}$, *provides a* Faraday *potential for the spatially constant magnetic flux, viz.* $d\boldsymbol{\omega}^1_{\mathbf{F},t} = \boldsymbol{\omega}^2_{\mathbf{B},t}$.

Proposition 8.1 (Electric field in a translating body). *A body in translational motion across a region of spatially uniform magnetic flux experiences an electric field given by:*

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.}$$



Joseph John Thomson

Cavendish Professors

* James Clerk Maxwell (1871 – 1879)
* Lord Rayleigh (1879 – 1884)
* J.J. Thomson (1884 – 1919)
* Lord Rutherford (1919 – 1937)
* William Lawrence Bragg (1938 – 1953)
* Nevill Mott (1954 – 1971)
* Brian Pippard (1971 – 1984)
* Sam Edwards (1984 – 1995)
* Richard Friend (1995 –)

J.J. Thomson was the first to apply the concept of fields to determine the electromagnetic forces on an object in terms of its properties and of external fields.
Interested in determining the electromagnetic behavior of the charged particles in cathode rays, J.J. Thomson published a paper in **1881** wherein he gave the force on the particles due to an external magnetic field as

$$\tfrac{1}{2}\, q\, \mathbf{v} \times \mathbf{B}.$$

J.J. Thomson was able to arrive at the correct basic form of the formula, but, because of some miscalculations and an incomplete description of the displacement current, included an incorrect numerical coefficient in front of the formula.
It was Oliver Heaviside, who had invented the modern vector notation and applied them to Maxwell's field equations, that was able to correctly derive in 1885 and 1889 the correct form of the magnetic force on a charged particle [9]. Finally, in 1892, Hendrik Antoon Lorentz derived the modern day form of the formula for the electromagnetic force.

Darrigol, O. (2000). Electrodynamics from Ampère to Einstein. Oxford University Press. ISBN 0-198-50593-0

NOTES

ON

RECENT RESEARCHES IN

ELECTRICITY AND MAGNETISM

INTENDED AS A SEQUEL TO

PROFESSOR CLERK-MAXWELL'S TREATISE
ON ELECTRICITY AND MAGNETISM

BY

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Oxford
AT THE CLARENDON PRESS
1893

In the course of Maxwell's investigation of the values of X, Y, Z due to induction, the terms

$$-\frac{d}{dx}(Fu+Gv+Hw), \quad -\frac{d}{dy}(Fu+Gv+Hw),$$
$$-\frac{d}{dz}(Fu+Gv+Hw)$$

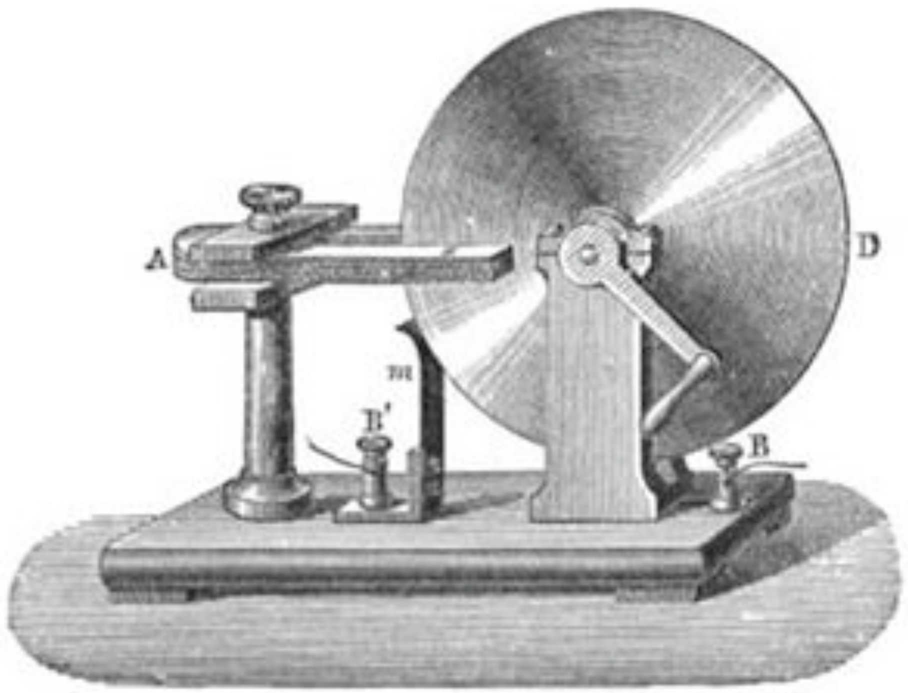
respectively in the final expressions for X, Y, Z are included under the Ψ terms. We shall find it clearer to keep these terms separate and write the expressions for X, Y, Z as

$$\left.\begin{aligned} X &= cv - bw - \frac{dF}{dt} - \frac{d}{dx}(Fu+Gv+Hw) - \frac{d\phi}{dx}, \\ Y &= aw - cu - \frac{dG}{dt} - \frac{d}{dy}(Fu+Gv+Hw) - \frac{d\phi}{dy}, \\ Z &= bu - av - \frac{dH}{dt} - \frac{d}{dz}(Fu+Gv+Hw) - \frac{d\phi}{dz}. \end{aligned}\right\} \qquad (1)$$

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t}\cdot\mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.}$$

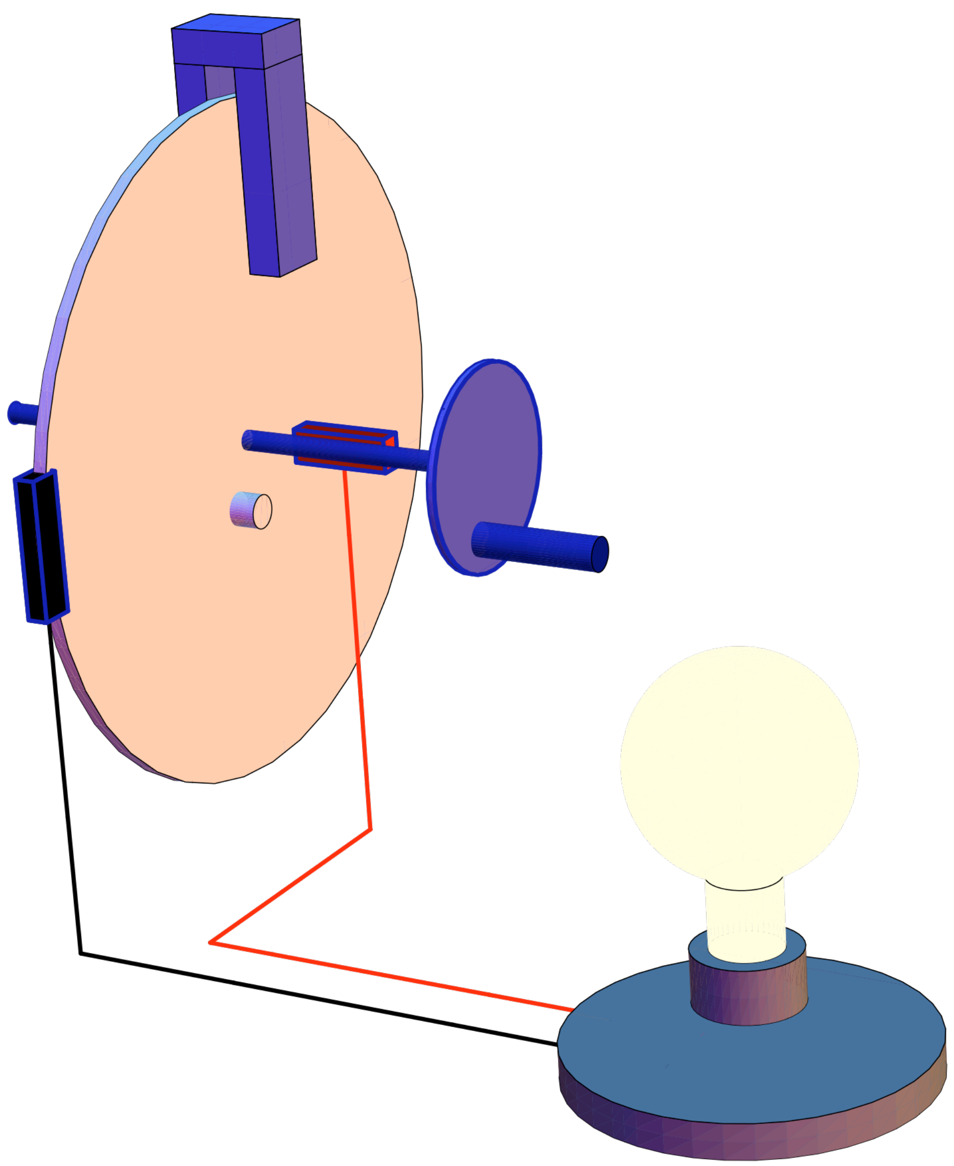
Electromagnetic induction

Faraday Disk
Dynamo



Mathematica Demonstration Project

Faraday Disk Dynamo



Vai alla Faraday disk Cerca immagini

Torna ai risultati illustrati

Mostra immagine a dimensione intera

312 × 475 - 41 kB - jpg - users.erols.com/ iri/homopolar.jpg

L'immagine potrebbe essere soggetta a copyright.

L'immagine di seguito si trova all'indirizzo users.erols.com/ iri/homopolar.html Rimuovi frame

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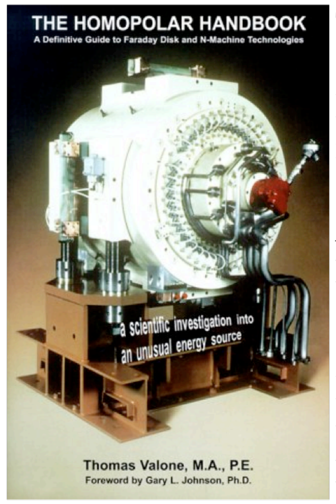
THE HOMOPOLAR HANDBOOK:
A Definitive Guide to Faraday Disk and N-Machine Technologies
Foreword by Gary Johnson, Ph.D.



The rotating disk dynamo has mystified every scientist since Faraday's 1831 discovery. Also called a unipolar generator (or N-Machine by Bruce DePalma), its efficiency is often known to above 95% in commercial models. Nikola Tesla's *Notes on a Unipolar Dynamo*, Einstein and Laub's article on a rotating magnetic dielectric, Inomata's new *Paradigm and N-Machine*, a list of homopolar patents, and more are included. Can the homopolar generator become a self-running free energy machine? Facts are here so the reader can reach an informed conclusion. (192-page book)

Investigating the Paulsen UFO story and the DePalma claims of overunity, the author began an earnest scientific endeavor in 1980 to build a homopolar generator and test for the elusive "back torque" which had never been measured before. The project helped complete his Master's degree in Physics at SUNY at Buffalo. Only afterwards did the connection to John R. R. Searl's energy and propulsion invention become apparent. (Each roller magnet in the Searl device is a small homopolar generator and the entire set of rollers create a radial Lorentz force too.)

...CONTENTS:

- Historical Development of the Field Rotation Paradox
- History of the Torque Controversy
- Classical Theory of the Faraday Disk Dynamo
- Unipolar Induction is Fundamentally a Relativistic Effect
- General Relativistic Approach for Rigorous Scientists
- The Theory of Armature Reaction and Resulting Back Torque
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- I- Homopolar Reference Material
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- III- Wilhelm Report: The Stelle Homopolar Machine
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- V - Sunburst Material
- VI - DePalma Material
- VII - Miscellaneous Papers and Letters

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In the FARADAY experiments described in Section 8.1, the spatial magnetic induction flux is time-independent, so that the GALILEI observer sitting on the support of the disk axis will measure a time-independent FARADAY potential, so that: $\partial_{\tau=t}\,\omega^1_{\mathbf{F},\tau} = 0$ and a velocity field of the spinning disk given by:

$$\mathbf{v}_{\varphi,t}(\mathbf{x}) = \mathbf{\Omega}_t \cdot \mathbf{r}(\mathbf{x})$$

with $\mathbf{r}(\mathbf{x}) := \mathbf{x}$ and $\mathbf{x}$ a radius vector with origin at the disk axis. Then $\nabla \mathbf{v}_{\varphi,t} = \mathbf{\Omega}_t$. Assuming that the magnetic flux $\omega^2_{\mathbf{B},t}$ is spatially constant in the disk, i.e. $\nabla\,\omega^2_{\mathbf{B},t} = 0$, in terms of the potential $\omega^1_{\mathbf{F},t} = \frac{1}{2}\omega^2_{\mathbf{B},t} \cdot \mathbf{r}$, we have:

$$\begin{aligned}\mathcal{L}_{\mathbf{v}_{\varphi,t}}\,\omega^1_{\mathbf{F},t} &= \nabla_{\mathbf{v}_{\varphi,t}}\omega^1_{\mathbf{F},t} + \omega^1_{\mathbf{F},t} \circ \nabla \mathbf{v}_{\varphi,t} \\ &= \nabla_{\mathbf{v}_{\varphi,t}}\,\omega^1_{\mathbf{F},t} + \tfrac{1}{2}(\omega^2_{\mathbf{B},t} \cdot \mathbf{r}) \circ \mathbf{\Omega}_t\,,\end{aligned}$$

with the covariant derivative of the magnetic potential given by:

$$2\,\nabla_{\mathbf{v}_{\varphi,t}}\,\omega^1_{\mathbf{F},t} = \omega^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} + \nabla_{\mathbf{v}_{\varphi,t}}\omega^2_{\mathbf{B},t} \cdot \mathbf{r}\,.$$

For an arbitrary vector field $\mathbf{h}$ in the disk plane, we have that:

$$\begin{aligned}2\,\langle \mathcal{L}_{\mathbf{v}_{\varphi,t}}\,\omega^1_{\mathbf{F},t}, \mathbf{h}\rangle &= 2\,\langle \nabla_{\mathbf{v}_{\varphi,t}}\,\omega^1_{\mathbf{F},t}, \mathbf{h}\rangle + \omega^2_{\mathbf{B},t}(\mathbf{r}, \mathbf{\Omega}_t \cdot \mathbf{h}) \\ &= \omega^2_{\mathbf{B},t}(\mathbf{\Omega}_t \cdot \mathbf{r}, \mathbf{h}) + \langle \nabla_{\mathbf{v}_{\varphi,t}}\omega^2_{\mathbf{B},t} \cdot \mathbf{r}, \mathbf{h}\rangle + \omega^2_{\mathbf{B},t}(\mathbf{r}, \mathbf{\Omega}_t \cdot \mathbf{h}) = 0\,,\end{aligned}$$

being $\nabla\,\omega^2_{\mathbf{B},t} = 0$ by assumption and

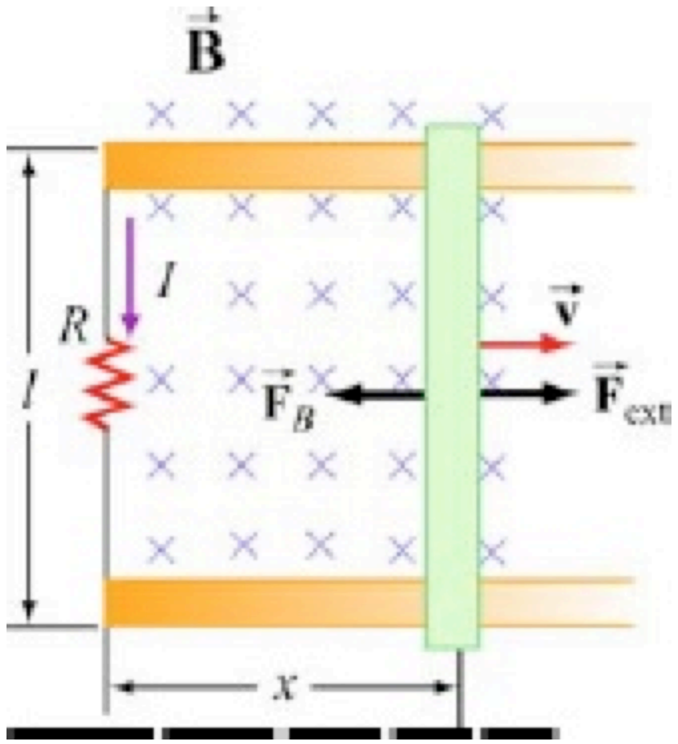
$$\omega^2_{\mathbf{B},t}(\mathbf{\Omega}_t \cdot \mathbf{r}, \mathbf{h}) = \omega^2_{\mathbf{B},t}(\mathbf{\Omega}_t \cdot (\mathbf{\Omega}_t \cdot \mathbf{r}), \mathbf{\Omega}_t \cdot \mathbf{h}) = -\omega^2_{\mathbf{B},t}(\mathbf{r}, \mathbf{\Omega}_t \cdot \mathbf{h})\,.$$

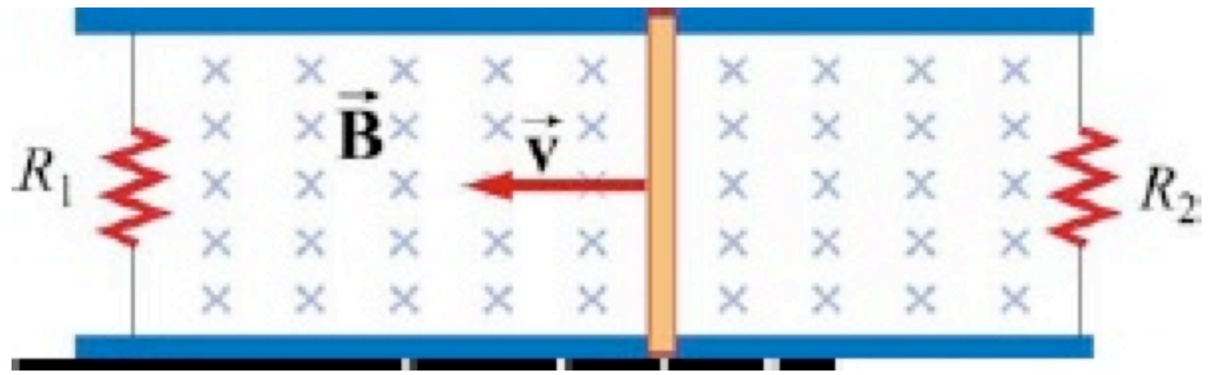
The analysis reveals that the magnetically induced electric vector field in the disk vanishes identically if the magnetic flux in the disk is spatially uniform. However, to compute the electromotive force in the circuit we should take into account the discontinuity points of the velocity at the axis and at the rib brush contacts, which provide concentrated contributions to the *emf* whose sum is equal to:

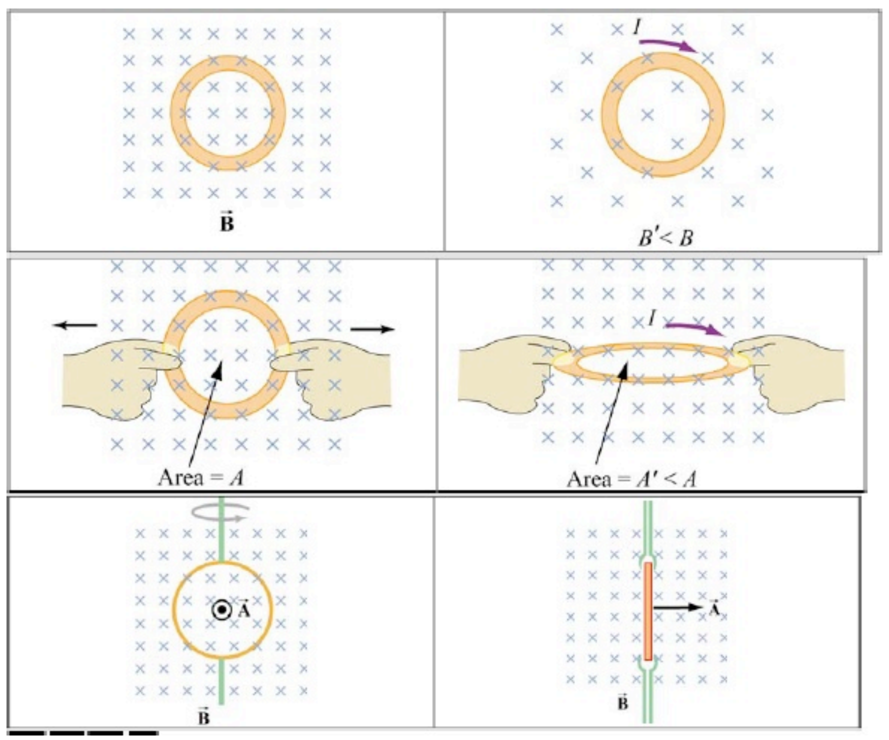
$$\begin{aligned}&-\omega^1_{\mathbf{F},t}(\mathbf{x}_1) \cdot (\mathbf{\Omega}_t \cdot \mathbf{x}_1) + \omega^1_{\mathbf{F},t}(\mathbf{x}_2) \cdot (\mathbf{\Omega}_t \cdot \mathbf{x}_2) \\ &= -\tfrac{1}{2}\omega^2_{\mathbf{B},t} \cdot \mathbf{x}_1 \cdot (\mathbf{\Omega}_t \cdot \mathbf{x}_1) + \tfrac{1}{2}\omega^2_{\mathbf{B},t} \cdot \mathbf{x}_2 \cdot (\mathbf{\Omega}_t \cdot \mathbf{x}_2)\,.\end{aligned}$$

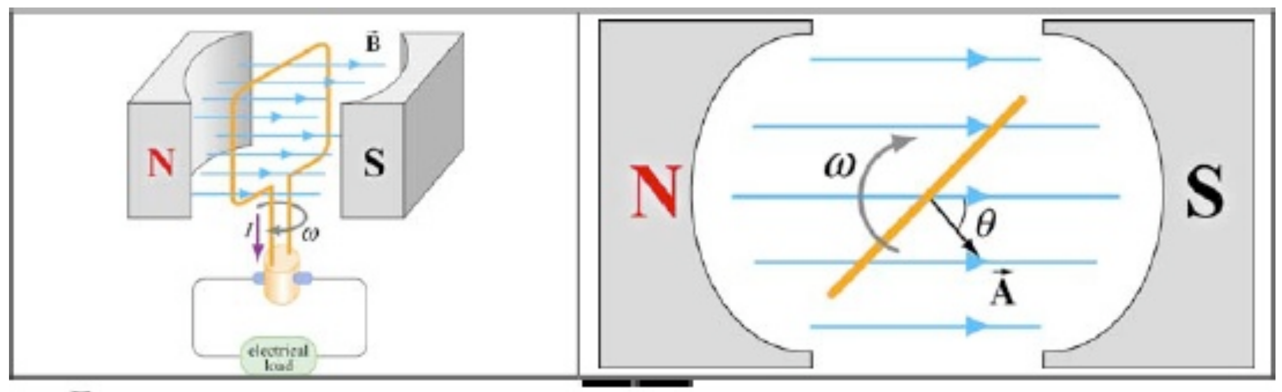
The global *emf* is thus again coincident with the one evaluated by the integral flux formula of FARADAY in which the spinning velocity of the disk radius closing the circuit is taken into account.

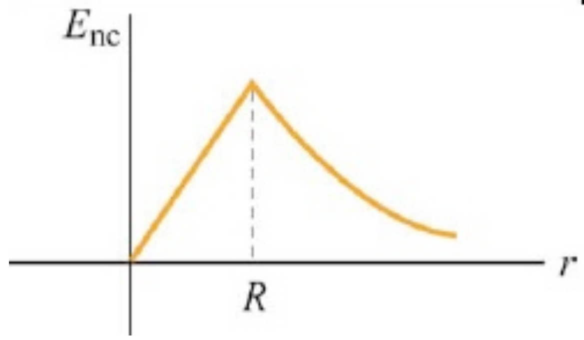
Faraday law of induction: examples











Let us consider the problem concerning the electromotive force *(emf)* generated in a conductive bar sliding on two fixed parallel rails under a transverse magnetic field which is spatially uniform and time-independent. An observer sitting on the rails measures a time independent FARADAY potential field and may thus evaluate the *emf* due to the electric field distributed along the bar is found by integration along the line from $\mathbf{x}_1$ to $\mathbf{x}_2$:

$$\boldsymbol{\omega}^1_{\mathbf{E},t} \cdot \mathbf{l} = -\tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}\,.$$

On the other hand, by the integral formula of FARADAY, the total *emf* in a circuit closed by another fixed bar is evaluated to be:

$$\oint \boldsymbol{\omega}^1_{\mathbf{E}} = -\oint \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} = -\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}\,.$$

So one-half of the total *emf* is lost as a result of the evaluation of the contribution provided by the electric field distributed along the bar. To resolve this puzzling result we have to consider that, in this example, the velocity field is no more uniform in space. Moreover, being uniform in the bar and vanishing in the rails, it presents two points of discontinuities at the sliding contacts. Then, the observer sitting on the rails measures the distributed electric field in the bar, as evaluated before, plus two impulses of *emf* concentrated at the sliding contacts, whose sum is given by

$$-(\boldsymbol{\omega}^1_{\mathbf{F},t}(\mathbf{x}_1) - \boldsymbol{\omega}^1_{\mathbf{F},t}(\mathbf{x}_2)) \cdot \mathbf{v}_{\varphi,t} = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{l} \cdot \mathbf{v}_{\varphi,t} = -\tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}$$

where $\mathbf{x}_1, \mathbf{x}_2$ are the positions of the sliding contacts and $\mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1$. Indeed the velocity jumps, in going from 1 to 2, are $\mathbf{v}_{\varphi,t}$ and $-\mathbf{v}_{\varphi,t}$ respectively. Thus, the two impulses of *emf* concentrated at the sliding contacts provide just the lost one-half of the total *emf* in the translating bar and in the sliding contacts, which therefore amounts to $-\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}$ and is equal to the one previously computed in one stroke by the integral flux rule of FARADAY.

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t} \cdot \mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.}$$

Romano, G.: *The laws of Electromagnetism for moving bodies and related questions (2010)*

Concluding remarks

- Covariance means variance by push and ensures in particular Galilei and Euclid invariance of the constitutive relations involving invariant tensors.

- Homogeneity and isotropy of the electromagnetic material properties are consistent with the covariant transformation of the material response at different configurations.

- Formulations in terms of different alterations of the relevant tensors and push to other configurations may be interchanged without affecting the result, thus restoring a sound physical basis to the constitutive theory.

- The induction laws are covariantly formulated so that a natural variance under push due to relative motion holds.