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VARIATIONAL PRINCIPLES WITH SINGULARITIES IN GEODESICS, OPTICS AND DYNAMICS

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Extremum principles

Pierre de Fermat
1601 - 1665

Fermat principle in optics

Jacob Bernoulli
1654 - 1705

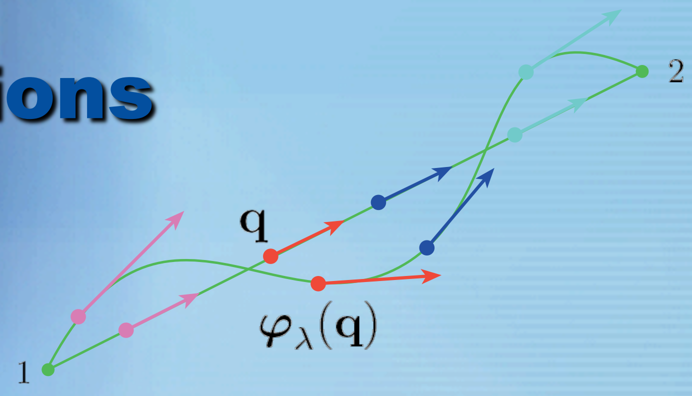
Brachistochrone

Pierre Louis Moreau de Maupertuis
1698 - 1759

Least action principle

Johann Bernoulli
1667 - 1748

Calculus of variations



Leonhard Euler
1707 - 1783

$$\delta \int_I \mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}})\,dt := \partial_{\lambda=0} \int_I \mathrm{L}_t(\boldsymbol{\varphi}_\lambda(\mathbf{q})\,, d\boldsymbol{\varphi}_\lambda(\mathbf{q}) \cdot \dot{\mathbf{q}})\,dt = 0$$

$$\int_I (d_{\mathbf{q}}\mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}}) \cdot \mathbf{v}_{\boldsymbol{\varphi}}(\mathbf{q}) + d_{\dot{\mathbf{q}}}\mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}}) \cdot \dot{\mathbf{v}}_{\boldsymbol{\varphi}}(\mathbf{q}))\,dt =$$

$$\int_I (d_{\mathbf{q}}\mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}}) - \partial_{\tau=t}\, d_{\dot{\mathbf{q}}}\mathrm{L}_\tau(\mathbf{q}_\tau\,,\dot{\mathbf{q}}_\tau)) \cdot \mathbf{v}_{\boldsymbol{\varphi}}(\mathbf{q})\,dt + \int_{\partial I} d_{\dot{\mathbf{q}}}\mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}}) \cdot \mathbf{v}_{\boldsymbol{\varphi}}(\mathbf{q})\,dt = 0$$

Euler's equation

$$\partial_{\tau=t}\, d_{\dot{\mathbf{q}}}\mathrm{L}_\tau(\mathbf{q}\,,\dot{\mathbf{q}}) = d_{\mathbf{q}}\mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}})$$

Laplace told his students, "Liesez Euler, Liesez Euler, c'est notre maître à tous"
("Read Euler, read Euler, he is our master in everything")

$$\mathbf{F} = m\,\mathbf{a}$$

Sir Isaac Newton
1643 - 1727

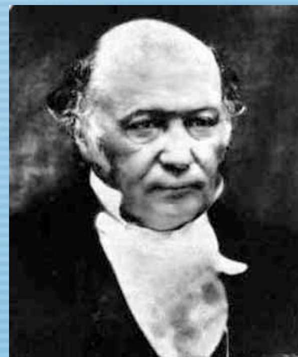
Force potential

Lagrangian ← $\mathrm{L} = K + P$

Kinetic energy

Hamilton's principle

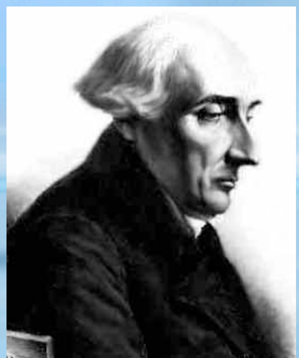
$$\delta \int_I \mathrm{L}(\mathbf{q}\,,\dot{\mathbf{q}}\,,t)\,dt = 0$$



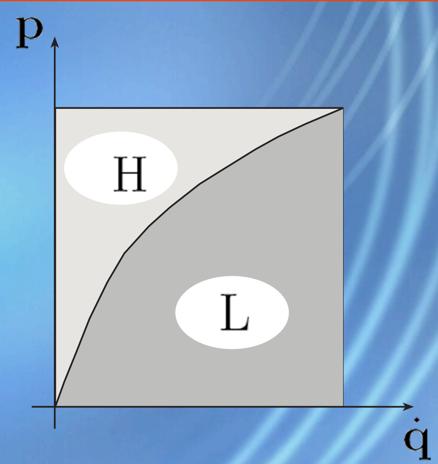
Sir William Rowan
Hamilton 1805 - 1865

Lagrange equation

$$\partial_{\tau=t}\, d_{\dot{\mathbf{q}}}\mathrm{L}_\tau(\mathbf{q}\,,\dot{\mathbf{q}}) = d_{\mathbf{q}}\mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}})$$



Joseph-Louis Lagrange
1736 - 1813



$$H(\mathbf{q},\mathbf{p}) = \dot{\mathbf{q}}\,\mathbf{p} - L(\mathbf{q},\dot{\mathbf{q}})$$

$$\begin{cases} \mathbf{p} = d_{\dot{\mathbf{q}}} L_t(\mathbf{q},\dot{\mathbf{q}}) \\ \dot{\mathbf{q}} = d_{\mathbf{p}} H_t(\mathbf{q},\mathbf{p}) \end{cases}$$

Adrien-Marie Legendre
1752 - 1833

Donkin theorem (1854): $\boxed{d_{\mathbf{q}} H_t(\mathbf{q},\mathbf{p}) = -d_{\mathbf{q}} L_t(\mathbf{q},\dot{\mathbf{q}})}$ *indeed:*

$$\begin{aligned} d_{\mathbf{q}} H_t(\mathbf{q},\mathbf{p}) &= d_{\mathbf{q}}(\mathbf{p}\, d_{\mathbf{p}} H_t(\mathbf{q},\mathbf{p}) - L_t(\mathbf{q}, d_{\mathbf{p}} H_t(\mathbf{q},\mathbf{p}))) \\ &= \mathbf{p}\, d_{\mathbf{q}}\, d_{\mathbf{p}} H_t(\mathbf{q},\mathbf{p}) - d_{\mathbf{q}} L_t(\mathbf{q},\dot{\mathbf{q}}) - d_{\dot{\mathbf{q}}} L_t(\mathbf{q},\dot{\mathbf{q}})\, d_{\mathbf{q}}\, d_{\mathbf{p}} H_t(\mathbf{q},\mathbf{p}) \\ &= -d_{\mathbf{q}} L_t(\mathbf{q},\dot{\mathbf{q}}) \end{aligned}$$

$$\partial_{\tau=t}\, d_{\dot{\mathbf{q}}} \mathrm{L}_\tau(\mathbf{q}\,,\dot{\mathbf{q}}) = d_{\mathbf{q}} \mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}})$$ ← *Lagrange*

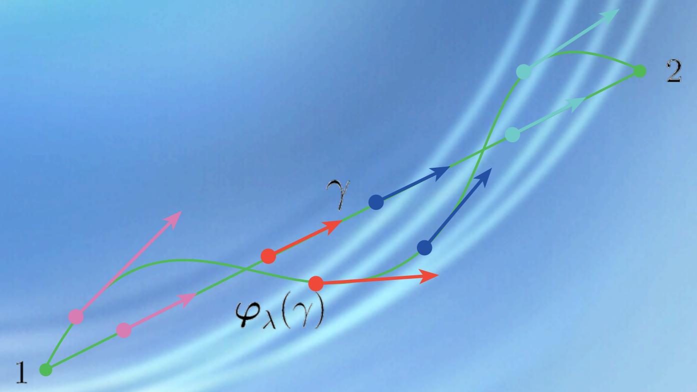
$$\mathbf{p} = d_{\dot{\mathbf{q}}} \mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}})$$ ← *Legendre*

$$d_{\mathbf{q}} \mathrm{H}_t(\mathbf{q}\,,\mathbf{p}) = -d_{\mathbf{q}} \mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}})$$ ← *Donkin*

↕

$$\begin{cases} \dot{\mathbf{q}} = d_{\mathbf{p}} \mathrm{H}_t(\mathbf{q}\,,\mathbf{p}) \\ -\dot{\mathbf{p}} = d_{\mathbf{q}} \mathrm{H}_t(\mathbf{q}\,,\mathbf{p}) \end{cases}$$ ← *Hamilton equations*

Geodesics



Riemannian manifold $\{\mathbb{M}\,;\mathbf{g}\}$

$$\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{M}) \qquad \mathbf{v} \in \mathrm{C}^1(\boldsymbol{\gamma}\,;\mathbb{T}\boldsymbol{\gamma})$$

$$\mathbf{v}(\boldsymbol{\gamma}(t)) = \mathbf{v}_t := \partial_{t=0}\boldsymbol{\gamma}(t)$$

$$\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$$

$$\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$$

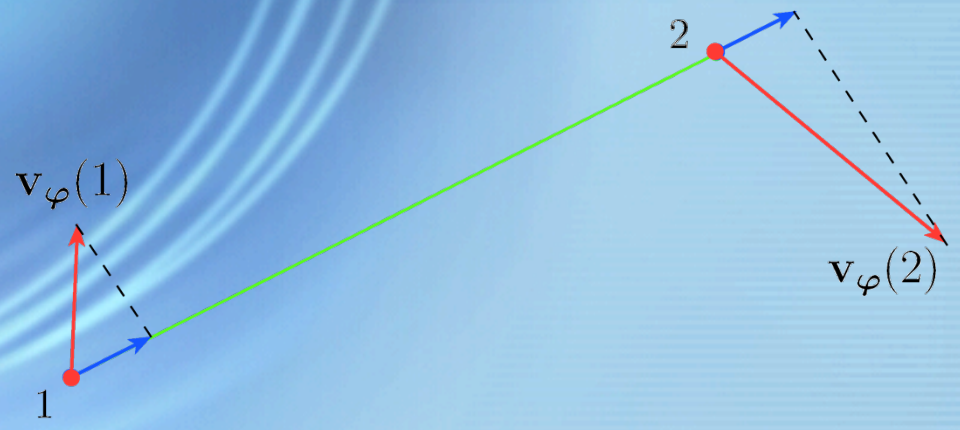
Length of a curve →

$$\ell(\boldsymbol{\gamma}) := \int_I \sqrt{\mathbf{g}\,(\mathbf{v}_t,\mathbf{v}_t)}\,dt$$

Geodesics are curves with stationary length:

$$\delta \int_I \sqrt{\mathbf{g}(\mathbf{v}_t,\mathbf{v}_t)}\,dt = 0 \quad \textit{that is} \quad \partial_{\lambda=0}\int_I \sqrt{\mathbf{g}(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t,\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t)}\,dt = 0$$

The length of a curve is stationary if its rate-of-change due to any flow is equal to the gap of equiprojectivity of the flow velocity at the boundary points.



$$\partial_{\lambda=0} \int_I \sqrt{\mathbf{g}(\boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t, \boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t)}\, dt = \int_{\partial I} \mathbf{g}(\mathbf{v}_t, \mathbf{v}_\varphi)(\mathbf{g}(\mathbf{v}_t, \mathbf{v}_t))^{-\frac{1}{2}}\, dt$$

A constant-speed curve is a geodesic if and only if :

$$\partial_{\lambda=0} \int_I \tfrac{1}{2}\, \mathbf{g}(\boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t, \boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t)\, dt = \int_{\partial I} \mathbf{g}(\mathbf{v}_t, \mathbf{v}_\varphi(\boldsymbol{\pi}(\mathbf{v}_t)))\, dt$$

Observing that, for any flow:

$$\int_{\partial I} \mathbf{g}(\mathbf{v}_t, \mathbf{v}_{\varphi}(\boldsymbol{\pi}(\mathbf{v}_t)))\,dt = \int_I \partial_{\tau=t}\,\mathbf{g}(\mathbf{v}_{\tau}, \mathbf{v}_{\varphi}(\boldsymbol{\pi}(\mathbf{v}_{\tau})))\,dt$$

we get the differential condition:

$$\partial_{\lambda=0}\,\tfrac{1}{2}\,\mathbf{g}(\boldsymbol{\varphi}_{\lambda}\!\uparrow\mathbf{v}_t, \boldsymbol{\varphi}_{\lambda}\!\uparrow\mathbf{v}_t) = \partial_{\tau=t}\,\mathbf{g}(\mathbf{v}_{\tau}, \mathbf{v}_{\varphi}(\boldsymbol{\pi}(\mathbf{v}_{\tau})))$$

In terms of a connection:

$$\nabla_{\mathbf{v}_t}(\mathbf{g}_{\boldsymbol{\pi}(\mathbf{v})}\mathbf{v}) = \tfrac{1}{2}\,d_{\mathrm{B}}q_{\mathbf{g}}(\mathbf{v}_t) + (\mathbf{g}_{\boldsymbol{\pi}(\mathbf{v}_t)}\mathbf{v}_t)\,\mathrm{TORS}(\mathbf{v}_t)$$

where:

$$\langle d_{\mathrm{B}}q_{\mathbf{g}}(\mathbf{v}), \mathbf{w}\rangle := \partial_{\lambda=0}\,\mathbf{g}_{\mathbf{c}(\lambda)}(\mathbf{c}(\lambda)\Uparrow\mathbf{v}, \mathbf{c}(\lambda)\Uparrow\mathbf{v}), \quad \forall\,\mathbf{w} \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{M}$$

→ *Base derivative*

For the Levi-Civita connection:

$$\nabla_{\mathbf{v}_t}\mathbf{v} = 0$$

Abstract action principle

action integral

$$\int_{\Gamma} \omega^1$$

Γ ⟵ *path in* $\mathbb{M}$

$\int_{\Gamma} \omega^1$ ⟵ *signed - lenght of* Γ

A trajectory Γ *of the system fulfills the stationarity condition:*

$$\partial_{\lambda=0} \int_{\varphi_\lambda(\Gamma)} \boldsymbol{\omega}^1 = \int_{\partial\Gamma} \boldsymbol{\omega}^1 \cdot \mathbf{v}$$

$\varphi_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ ⟵ *virtual flow*

$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ ⟵ *virtual velocity*

Localization: differential and jump conditions

By the extrusion formula

$$\partial_{\lambda=0} \int_{\psi_\lambda(\Gamma)} \boldsymbol{\omega}^1 = \int_{\partial\Gamma} \boldsymbol{\omega}^1 \cdot \mathbf{v} + \int_\Gamma (d\boldsymbol{\omega}^1) \cdot \mathbf{v}$$

the action principle writes

$$\int_{\mathcal{T}(\boldsymbol{\Gamma})} (d\boldsymbol{\omega}^1)\mathbf{v} = \int_{\mathcal{I}(\boldsymbol{\Gamma})} [[\boldsymbol{\omega}^1 \mathbf{v}]]$$

Localization:

$$\int_I (d\boldsymbol{\omega}^1) \cdot \mathbf{v} \cdot \mathbf{v}_\Gamma \, ds = 0 \quad \Longleftrightarrow \quad d\boldsymbol{\omega}^1 \cdot \mathbf{v}_\Gamma \cdot \mathbf{v} = 0 \,,$$

$$\mathbf{v}_\Gamma \in \mathrm{C}^1(\Gamma\,; \mathbb{T}\Gamma)$$

A symmetry condition

Let $\mathbf{v}, \mathbf{v}_\Gamma \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$

be such that $\mathcal{L}_{\mathbf{v}} \mathbf{v}_\Gamma = [\mathbf{v}, \mathbf{v}_\Gamma] = 0$

*By **Palais** formula*

$$(d\boldsymbol{\omega}^1) \cdot \mathbf{v}_\Gamma \cdot \mathbf{v} = d_{\mathbf{v}_\Gamma}(\boldsymbol{\omega}^1 \cdot \mathbf{v}) - d_{\mathbf{v}}(\boldsymbol{\omega}^1 \cdot \mathbf{v}_\Gamma) - \boldsymbol{\omega}^1 \cdot [\mathbf{v}_\Gamma, \mathbf{v}]$$

Differential condition of stationarity

$$d_{\mathbf{v}}\mathrm{L} := d_{\mathbf{v}}(\boldsymbol{\omega}^1 \cdot \mathbf{v}_\Gamma) = d_{\mathbf{v}_\Gamma}(\boldsymbol{\omega}^1 \cdot \mathbf{v})$$

*which implies **Noether's** theorem*

Continuum dynamics

Hamilton's principle in the velocity-time phase-space $\mathbb{T}\mathbb{C} \times I$

Configuration manifold ⟶ $\mathbb{C}$

Time-parametrized path ⟶ $\gamma \in C^1(I\,;\mathbb{C})$

Velocity ⟶ $\mathbf{v}_t = \{\gamma(t)\,,\dot{\gamma}(t)\} \in \mathbb{T}\gamma \subset \mathbb{T}\mathbb{C}$

Lifted path in the velocity phase space ⟶ $\mathbf{\Gamma} := \mathbb{T}\gamma \in C^1(I\,;\mathbb{T}\mathbb{C})$

Lagrangian of the system ⟶ $L_t \in C^1(\mathbb{T}\mathbb{C}\,;\mathcal{R})$

Fiber-derivative ⟶ $d_F L_t(\mathbf{v}) := \partial_{\lambda=0} L_t(\psi_\lambda(\mathbf{v})) \in C^1(\mathbb{T}\mathbb{C}\,;\mathbb{T}^*\mathbb{C})$

$\psi_\lambda \in C^1(\mathbb{T}\mathbb{C}\,;\mathbb{T}\mathbb{C})$ *configuration-preserving flow:*

$$\boldsymbol{\pi}(\psi_\lambda(\mathbf{v})) = \boldsymbol{\pi}(\mathbf{v})\,, \quad \forall\,\lambda \in \mathcal{R}\,.$$

Energy of the system, conjugate of the Lagrangian:

$$E_t(\mathbf{v}_t) := \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \mathbf{v}_t \rangle - \mathrm{L}_t(\mathbf{v}_t)$$

Differential one-form $\boldsymbol{\theta}_{\mathrm{L}_t} \in \mathrm{C}^1(\mathbb{TC} \times I\,; \mathbb{T}^*\mathbb{TC})$

$$\boldsymbol{\theta}_{\mathrm{L}_t}(\{\mathbf{v}\,, t\}) \cdot \{\delta\mathbf{v}\,, \delta t\} := \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), d\boldsymbol{\pi}(\mathbf{v}) \cdot \delta\mathbf{v} \rangle$$

for all $\{\delta\mathbf{v}\,, \delta t\} \in \mathbb{T}_{\mathbf{v}}\mathbb{TC} \times \mathbb{T}_t I$

Noting that $E_t(\mathbf{v})dt \cdot \{\delta\mathbf{v}\,, \delta t\} = E_t(\mathbf{v})\langle dt, \delta t \rangle$, *the action one-form is*

$$\boldsymbol{\omega}^1_{\mathrm{L}}(\{\mathbf{v}\,, t\}) := \boldsymbol{\theta}_{\mathrm{L}_t}(\{\mathbf{v}\,, t\}) - E(\mathbf{v}, t)dt$$

so that

$$\boldsymbol{\omega}^1_{\mathrm{L}}(\{\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t\,, t\}) \cdot \{\boldsymbol{\psi}_\lambda\!\uparrow\!\dot{\mathbf{v}}_t\,, 1\} = \mathrm{L}_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t)$$

and

$$\begin{aligned}\boldsymbol{\omega}^1_{\mathrm{L}}(\{\mathbf{v}_t\,, t\}) \cdot \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,, 0\} &= \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), d\boldsymbol{\pi}(\mathbf{v}_t) \cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t) \rangle \\ &= \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \rangle\,.\end{aligned}$$

The action principle is expressed by the variational condition:

$$\partial_{\lambda=0} \int_{\psi_\lambda(\boldsymbol{\Gamma}_I)} \boldsymbol{\omega}^1_{\mathrm{L}} = \int_{\partial \boldsymbol{\Gamma}_I} \boldsymbol{\omega}^1_{\mathrm{L}} \cdot \{\mathbf{v}_\psi\,, 0\}\,,$$

for any differential flow $\psi_\lambda = \varphi_\lambda\!\uparrow\, \in \mathrm{C}^1(\mathbb{TC}\,;\mathbb{TC})$ *such that the velocity field* $\mathbf{v}_\varphi = \partial_{\lambda=0}\,\varphi_\lambda \in \mathrm{C}^1(\gamma\,;\mathbb{TC})$ *is an infinitesimal isometry of* $\gamma \in \mathbb{C}$

Action principle in terms of the Lagrangian:

The trajectory of a continuous dynamical system in the configuration manifold is a time-parametrized path $\gamma \in \mathrm{C}^1(I\,;\mathbb{C})$ *fulfilling the variational condition*

$$\partial_{\lambda=0} \int_I L_t(\varphi_\lambda\!\uparrow\mathbf{v}_t)\,dt = \int_{\partial I} \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \mathbf{v}_\varphi(\boldsymbol{\pi}(\mathbf{v}_t))\rangle\,dt\,,$$

for any flow $\varphi_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *in the configuration manifold whose velocity field* $\mathbf{v}_\varphi = \partial_{\lambda=0}\,\varphi_\lambda \in \mathrm{C}^1(\gamma\,;\mathbb{TC})$ *is an infinitesimal isometry of* γ.

Differential stationarity condition

The action principle in the velocity-time phase-space is equivalent to the differential condition:

$$d\boldsymbol{\omega}^1_{\mathrm{L}_t}(\mathbf{v}_t, t)\cdot\{\,\dot{\mathbf{v}}_t, 1\,\}\cdot\{\mathbf{v}_\psi(\mathbf{v}_t)\,, 0\} = 0$$

recalling that: $\boldsymbol{\omega}^1_{\mathrm{L}_t}(\{\mathbf{v}\,, t\}) = \boldsymbol{\theta}_{\mathrm{L}_t}(\{\mathbf{v}\,, t\}) - E(\mathbf{v}, t)dt$

we get

$$d\boldsymbol{\theta}_{\mathrm{L}_t}(\{\mathbf{v}_t\,, t\})\cdot\dot{\mathbf{v}}_t\cdot\mathbf{v}_\psi(\mathbf{v}_t) = d(E(\mathbf{v}_t, t)\,dt)\cdot\{\,\dot{\mathbf{v}}_t, 1\,\}\cdot\{\mathbf{v}_\psi(\mathbf{v}_t)\,, 0\}$$

which, applying Palais formula to the r.h.s., becomes:

$$d\boldsymbol{\theta}_{\mathrm{L}}(\{\mathbf{v}_t\,, t\})\cdot\dot{\mathbf{v}}_t\cdot\mathbf{v}_\psi(\mathbf{v}_t) = -d_{\mathbf{v}_\psi(\mathbf{v}_t)}E_t(\mathbf{v}_t)$$

Applying Palais formula to the l.h.s we get:

The law of dynamics

The trajectory of a continuous dynamical system in the configuration manifold is a time-parametrized path $\gamma \in \mathrm{C}^1(I\,;\mathbb{C})$ *fulfilling the variational condition*

$$\partial_{\tau=t}\, \langle d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau)) \rangle = \partial_{\lambda=0}\, L_t(\boldsymbol{\psi}_\lambda(\mathbf{v}_t))$$

for any flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *in the configuration manifold whose velocity field* $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))$ *at the actual configuration is an admissible infinitesimal isometry.*

Special forms

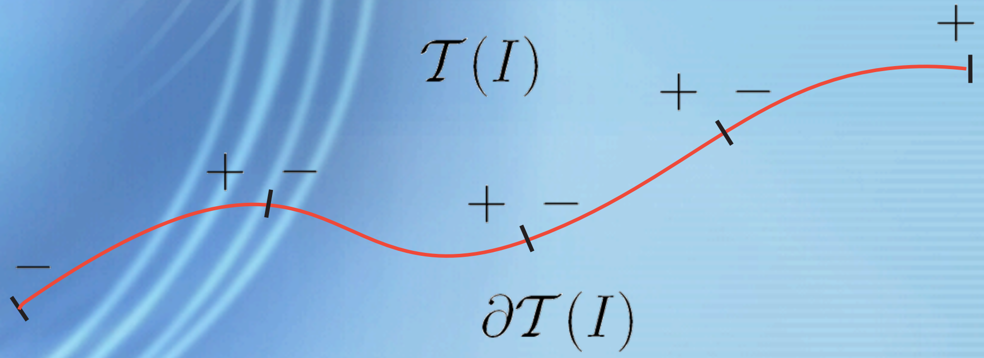
Affine connection:

$$\langle \partial_{\tau=t}\, d_{\mathrm{F}} L_\tau(\mathbf{v}_t) + \nabla_{\mathbf{v}_t}(d_{\mathrm{F}} L_t \circ \mathbf{v}_t) - d_{\mathrm{B}} L_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \rangle = \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \mathrm{TORS}(\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_{\boldsymbol{\gamma}})(\boldsymbol{\pi}(\mathbf{v}_t)) \rangle$$

Torsion-free connection: Lagrange law

$$\langle \partial_{\tau=t}\, d_{\mathrm{F}} L_\tau(\mathbf{v}_t) + \nabla_{\mathbf{v}_t}(d_{\mathrm{F}} L_t \circ \mathbf{v}_t) - d_{\mathrm{B}} L_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \rangle = 0$$

Reproducibility axiom



Action principle: $\partial_{\lambda=0} \int_I L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t))\,dt = \int_{\partial I} \langle d_{\mathrm{F}} \mathrm{L}_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\rangle\,dt$

Integrating by parts on each regularity interval in $\mathcal{T}(I)$ *, we have:*

$$\int_{\mathcal{T}(I)} \partial_{\lambda=0} L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t))\,dt = \int_{\mathcal{T}(I)} \partial_{\tau=t} \langle d_{\mathrm{F}} \mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\rangle\,dt + \int_{\mathcal{I}(I)} \langle [[d_{\mathrm{F}} \mathrm{L}_t(\mathbf{v}_t)]], \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\rangle\,dt\,.$$

$$\partial_{\lambda=0} \int_I L_t(\boldsymbol{\varphi}_\lambda \uparrow \mathbf{v}_t)\, dt = \int_{\partial I} \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\rangle \, dt$$

↓

$$\partial_{\lambda=0} \int_I \mathrm{L}_t(\boldsymbol{\varphi}_\lambda \uparrow \mathbf{v}_t)\, dt =$$

$$\int_I \partial_{\tau=t} \langle d_{\mathrm{F}} \mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\rangle \, dt$$

↓

Law of dynamics

$$\partial_{\lambda=0} \mathrm{L}_t(\boldsymbol{\varphi}_\lambda \uparrow \mathbf{v}_t) = \partial_{\tau=t} \langle d_{\mathrm{F}} \mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\rangle$$

$$\langle [[d_{\mathrm{F}} \mathrm{L}_t(\mathbf{v}_t)]], \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\rangle = 0$$

Giovanni Romano

Hamilton's law of dynamics

Hamilton's law is deduced from Lagrange's law by a translation in terms of covectors by means of Legendre's transform and of the next result, whose special case in linear spaces is referred to as Donkin's theorem (1854) by Gantmacher.

Base derivatives of Legendre transforms

In a manifold with an affine connection the following relation holds:

$$d_{\mathrm{B}} H_t(\mathbf{v}^*) + d_{\mathrm{B}} L_t(d_{\mathrm{F}} H_t(\mathbf{v}^*)) = 0$$

and

Hamilton's canonical equations

$$\begin{cases} \langle \partial_{\tau=t}\, \mathbf{v}_\tau^* + \nabla_{\mathbf{v}_t} \mathbf{v}^* + d_{\mathrm{B}} H_t(\mathbf{v}_t^*), \mathbf{v}_\varphi(\boldsymbol{\pi}^*(\mathbf{v}_t^*)) \rangle = \langle \mathbf{v}_t^*, \mathrm{TORS}(\mathbf{v}_\varphi, \mathbf{v}_t)(\boldsymbol{\pi}^*(\mathbf{v}_t^*)) \rangle\,, \\ \mathbf{v}_t = d_{\mathrm{F}} H_t(\mathbf{v}_t^*)\,. \end{cases}$$

Hamilton-Jacobi equation

If thru any point in a neighbourhood of a point $\{\mathbf{x}\,, t\} \in \mathbb{C} \times I$ *there is a unique trajectory starting from a fixed point, the action integral defines an action functional* $J \in \mathrm{C}^1(\mathbb{C} \times I\,; \mathcal{R})$ *according to the relation:*

$$J(\mathbf{x}, t) := \int_{\gamma} \mathrm{L}_t(\dot{\boldsymbol{\gamma}}(t))\, dt$$

Differential of the action functional

$$dJ(\mathbf{x}, t) = \mathbf{v}_t^* - H_t(\mathbf{v}_t^*)dt \in \mathbb{T}^*_{(\mathbf{x},t)}(\mathbb{C} \times I) \iff \begin{cases} dJ_t(\mathbf{x}) = \mathbf{v}_t^*\,, \\ \partial_{\tau=t}\, J_\tau(\mathbf{x}) = H_t(\mathbf{v}_t^*)\,. \end{cases}$$

Hamilton-Jacobi equation

The action functional fulfils the Hamilton-Jacobi equation:

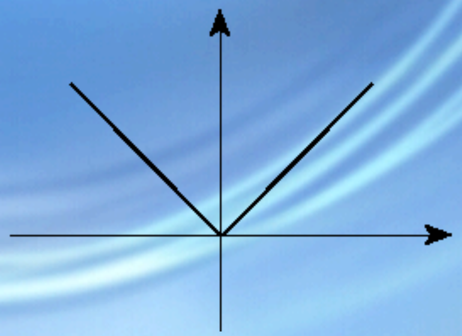
$$\partial_{\tau=t}\, J_\tau + H_t \circ dJ_t = 0$$

Geodesics and Optics

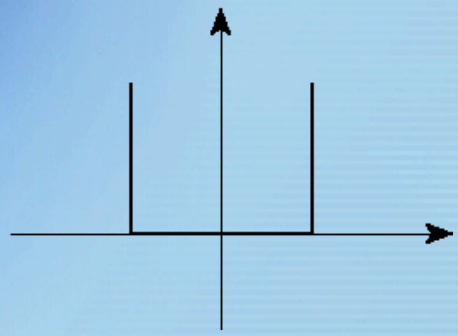
The action integral is the lenght of the path.

Accordingly:

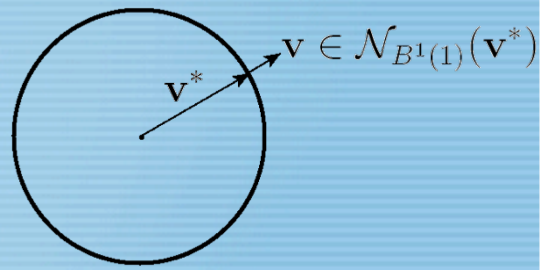
$L_t(\mathbf{v}) = \|\mathbf{v}\|_{\mathrm{g}}$



$H_t \in \mathrm{C}^1(\mathbb{T}^*\mathbb{C}\,;\mathcal{R})$



$B^1(\mathbb{T}^*\mathbb{C}, \mathbf{g}^{-1})$



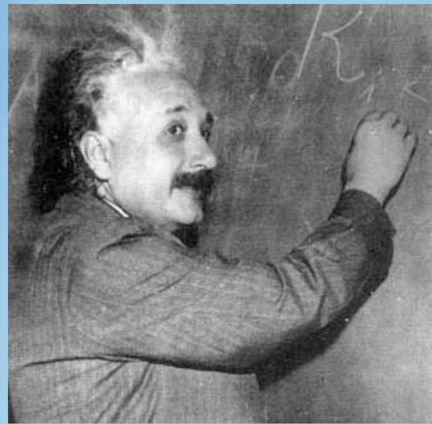
Then $\boxed{\partial_{\tau=t}\, J_\tau = 0}$ *and* $\boxed{\|dJ_t(\mathbf{x})\|_{\mathbf{g}^{-1}} = \|\mathbf{v}_t^*\|_{\mathbf{g}^{-1}} = 1}$

The gradient $\nabla J(\mathbf{x}) := \mathrm{g}^{-1} dJ_t(\mathbf{x})$ *fulfils the eikonal equation:*

$$\boxed{\|\nabla J(\mathbf{x})\|_{\mathrm{g}} = 1}$$

Symmetries and Conservation laws

David Hilbert
1862 - 1943



Albert Einstein
1879 - 1955

Emmy Amalie Noether
1882 - 1935

Lagrange's Equation (1788)

$$\partial_{\tau=t}\, d_{\dot{\mathbf{q}}} \mathrm{L}_\tau(\mathbf{q}\,,\dot{\mathbf{q}}) = d_{\mathbf{q}} \mathrm{L}_t(\mathbf{q}\,,\dot{\mathbf{q}})$$

Noether's Theorem (1918)

$$\mathrm{L}(\mathbf{q}\,,\dot{\mathbf{q}}\,,t) = \mathrm{L}(\boldsymbol{\varphi}_\lambda(\mathbf{q})\,, d\boldsymbol{\varphi}_\lambda(\mathbf{q})\cdot\dot{\mathbf{q}}\,,t)$$

$$\longrightarrow \quad \partial_{\tau=t}\, d_{\dot{\mathbf{q}}} \mathrm{L}_\tau(\mathbf{q}_\tau\,,\dot{\mathbf{q}}_\tau)\cdot \mathbf{v}_{\boldsymbol{\varphi}_\tau}(\mathbf{q}_\tau) = 0$$

Noether's theorem
Rigid body dynamics (in coordinates)

$$\mathrm{L}(\mathbf{q}\,,\dot{\mathbf{q}}\,,t) = \mathrm{L}(\boldsymbol{\varphi}_\lambda(\mathbf{q})\,, d\boldsymbol{\varphi}_\lambda(\mathbf{q})\cdot\dot{\mathbf{q}}\,,t)$$

$$\longrightarrow \quad \partial_{\tau=t}\, d_{\dot{\mathbf{q}}}\mathrm{L}_\tau(\mathbf{q}_\tau\,,\dot{\mathbf{q}}_\tau)\cdot\mathbf{v}_{\boldsymbol{\varphi}_\tau}(\mathbf{q}_\tau) = 0$$

Law of dynamics

$$\partial_{\tau=t}\,\langle d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\rangle = \partial_{\lambda=0}\, L_t(\boldsymbol{\psi}_\lambda(\mathbf{v}_t))$$

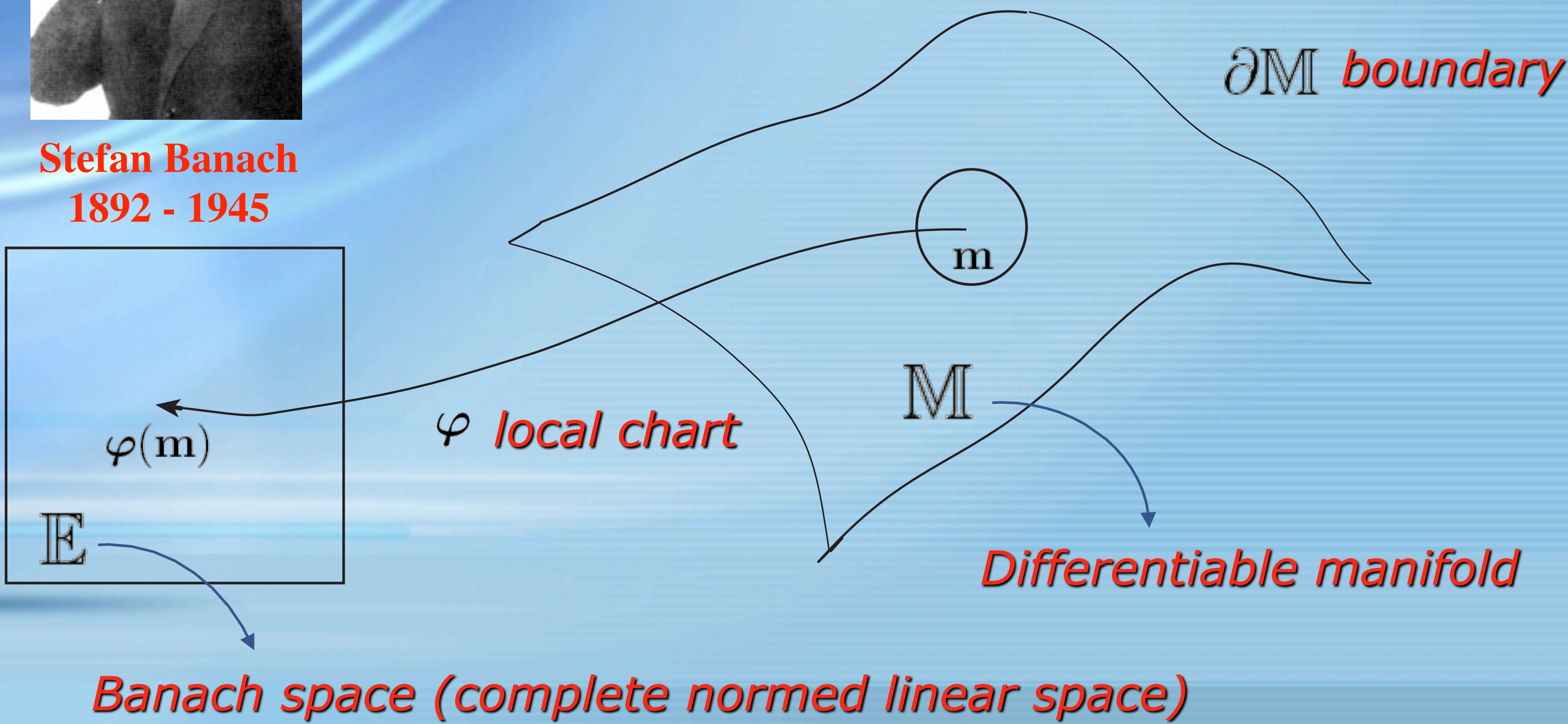
$$\downarrow$$

Extended NOETHER's theorem:

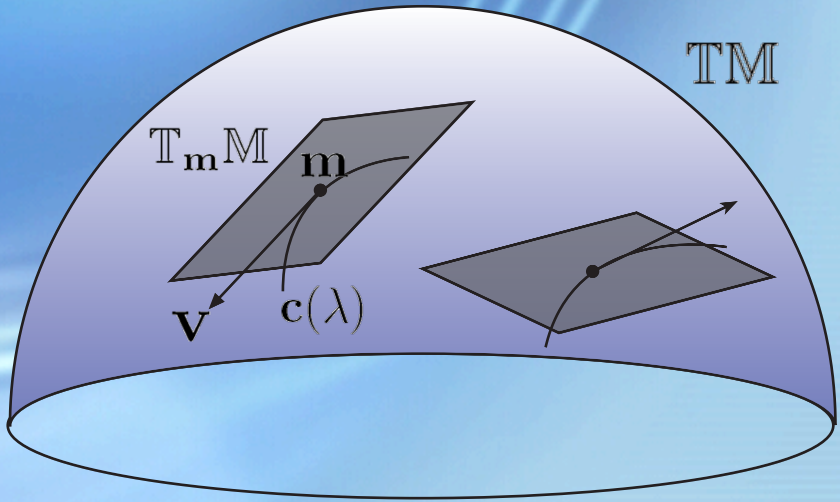
$$\partial_{\lambda=0}\, L_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t) = 0 \quad \Longrightarrow \quad \partial_{\tau=t}\,\langle d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\rangle = 0$$

Manifolds, tangent and cotangent bundles

Stefan Banach
1892 - 1945



TANGENT BUNDLE



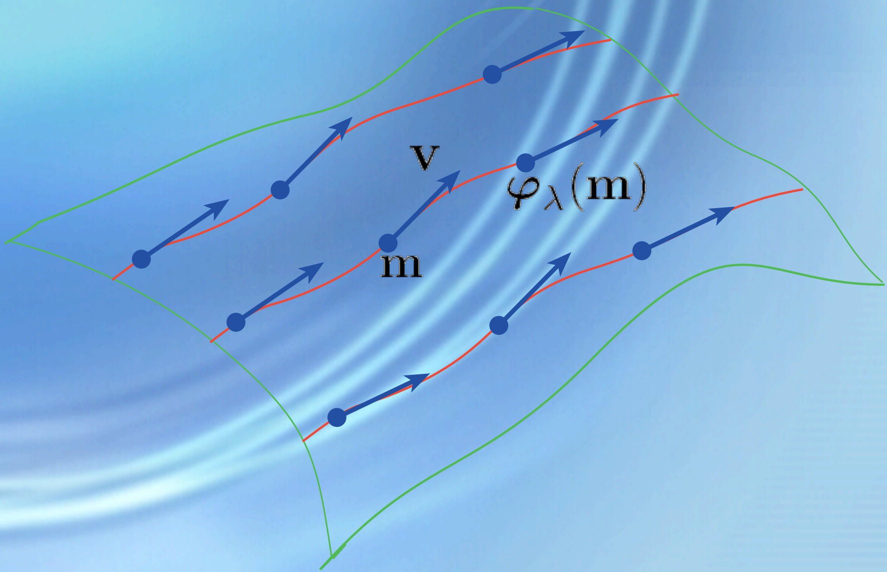
- $\mathbb{T}^*_{\mathbf{m}}\mathbb{M}$ *dual space of* $\mathbb{T}_{\mathbf{m}}\mathbb{M}$
- $\mathbb{T}^*\mathbb{M}$ *cotangent bundle*
- $\mathbf{v} \in \mathbb{T}_{\mathbf{m}}\mathbb{M}$; $\boldsymbol{\pi}(\mathbf{v}) := \mathbf{m}$

tangent vector

$$\mathbf{v}f = \partial_{\mathbf{v}} f := \partial_{\lambda=0}\,(f \circ \mathbf{c})(\lambda)$$

$$f \in \mathrm{C}^1(\mathbb{M}\,;\mathcal{R})$$

Flows of vector fields



flow *velocity*

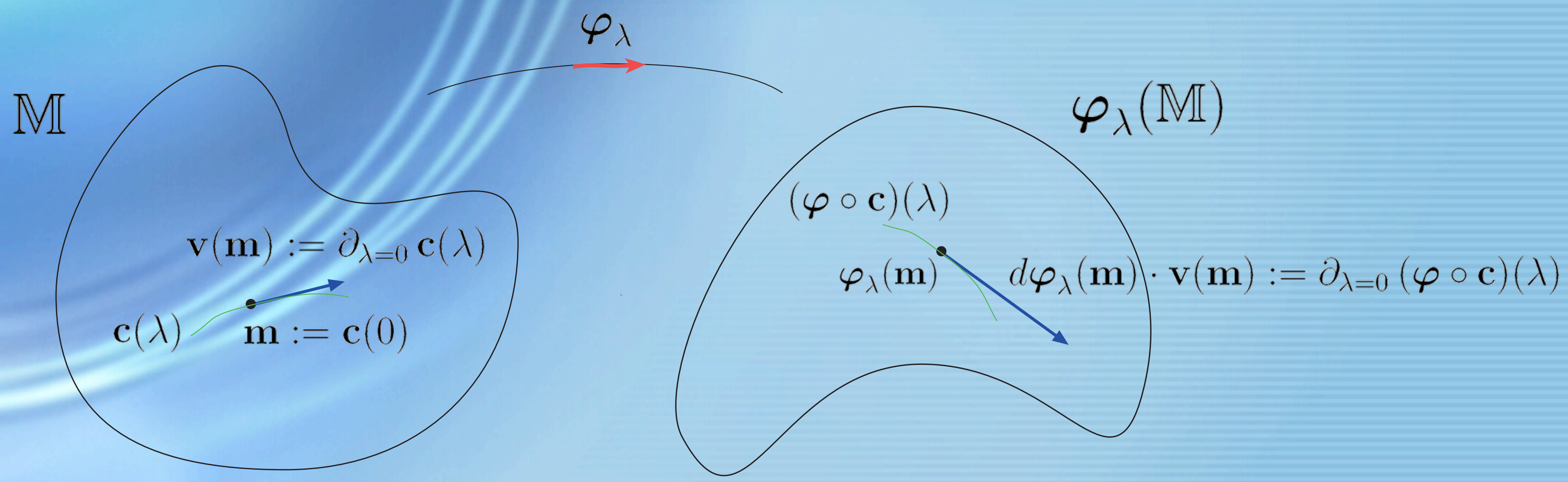
$$\partial_{\lambda=0}\varphi_\lambda(\mathbf{m}) = \mathbf{v}(\mathbf{m})$$

Sydney Chapman
1888 -1970

Andrey Nikolaevich
Kolmogorov
1903-1987

$$\varphi_{\lambda+\alpha} = \varphi_\lambda \circ \varphi_\alpha \quad \longleftrightarrow \quad \textit{Chapman - Kolmogorov law}$$

Push of a vector field



$$(\varphi_\lambda\!\uparrow\!\mathbf{v})(\mathbf{m}) := d\varphi_\lambda(\mathbf{m}) \cdot \mathbf{v}(\mathbf{m})$$

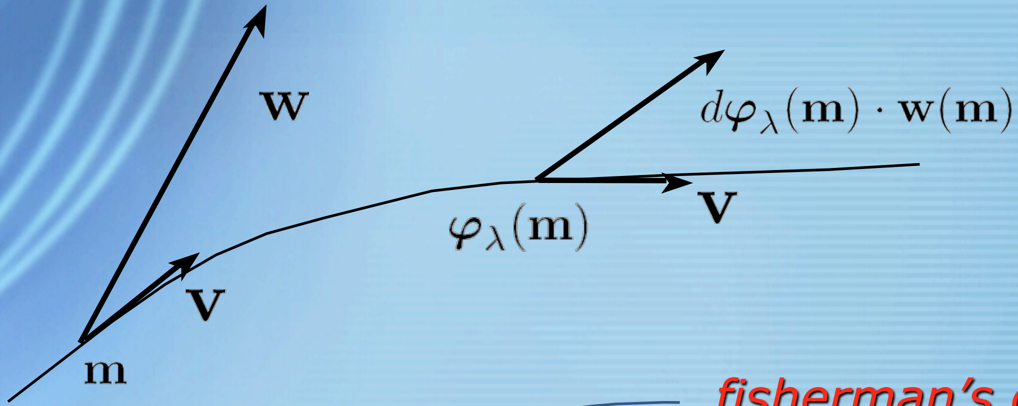
Push-forward

Pull-back

$$\varphi_\lambda\!\downarrow\, := \varphi_{-\lambda}\!\uparrow\, = \varphi_\lambda^{-1}\!\uparrow$$

Lie derivative

Marius Sophus Lie
1842 - 1899



fisherman's derivative

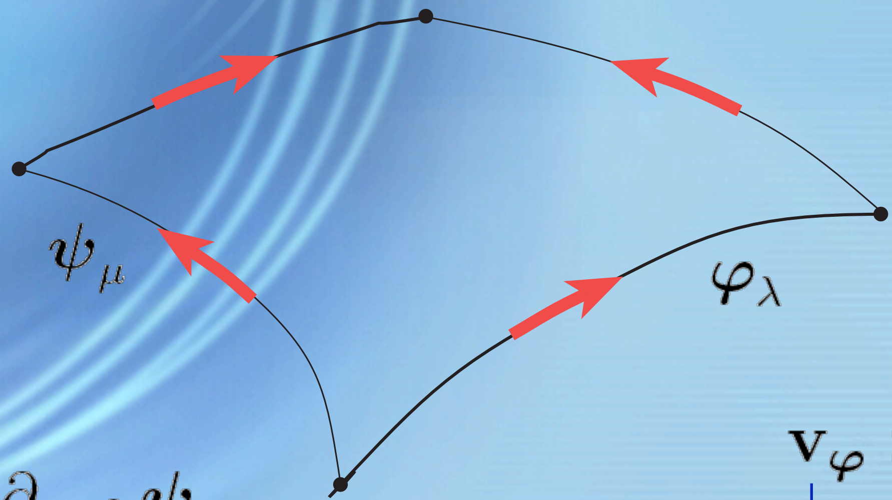
$$(\mathcal{L}_{\mathbf{v}}\mathbf{w})(\mathbf{m}) := \partial_{\lambda=0}\,(\varphi_\lambda\!\downarrow\!\mathbf{w})(\mathbf{m})$$

Lie bracket

$$(\mathcal{L}_{\mathbf{v}}\,\mathbf{u})\,f = [\mathbf{v},\mathbf{u}]\,f \;:= \mathbf{v}\,(\mathbf{u}\,f) - \mathbf{u}\,(\mathbf{v}\,f)\,, \quad \forall\, f \in \mathrm{C}^2(\mathbb{M}\,;\mathcal{R})$$

$\mathbf{v}, \mathbf{u}$ ← *vector fields on* $\mathbb{M}$

Flow commutation



$$\mathbf{v}_\varphi := \partial_{\lambda=0}\,\varphi_\lambda$$

↓

Velocity of the flow φ_λ

$$\mathbf{v}_\psi := \partial_{\mu=0}\,\psi_\mu$$

↓

Velocity of the flow ψ_μ

$$[\mathbf{v}_\varphi, \mathbf{v}_\psi] = 0 \quad \Longleftrightarrow \quad \varphi_\lambda \circ \psi_\mu = \psi_\mu \circ \varphi_\lambda$$

Differential forms

$$\omega^k(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$$

multilinear aternating k-form

it is a signed volume of the parallelepiped

with sides $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$

$$(\mathcal{L}_{\mathbf{v}}\omega^1)(\mathbf{w}) := \partial_{\lambda=0}\,(\varphi_\lambda \downarrow \omega^1)(\mathbf{w}) := \partial_{\lambda=0}\,\omega^1(d\varphi_\lambda \cdot \mathbf{w})$$

Lie derivative of a one-form

Exterior derivative and Stokes formula

*The exterior derivative is an operation on differential k-forms which is uniquely defined by **Stokes** formula:*

$$\int_{\mathbb{M}} d\boldsymbol{\omega}^{n-1} = \oint_{\partial\mathbb{M}} \boldsymbol{\omega}^{n-1} \qquad (dim\,\mathbb{M} = n)$$

The exterior derivative is the natural extension of the fundamental theorem of calculus for functions (0-forms) to integration of n-forms on compact n-dimensional chains.

*This celebrated formula is named the **Newton - Leibniz - Gauss - Green - Ostrogradski - Stokes - Poincaré** formula in:*

Arnold V.I.: Mathematical methods of classical mechanics, Springer Verlag, New York (1989).

*The generalized version in terms of exterior derivative of forms is due to **Poincaré**.*

cross product:	$\mathbf{u} \times \mathbf{v}$	$= \boldsymbol{\mu}_{\mathrm{g}} \mathbf{u}\mathbf{v}\,,$	$\dim \mathbb{S} = 2$
cross product:	$\mathrm{g}(\mathbf{u} \times \mathbf{v})$	$= \boldsymbol{\mu}_{\mathrm{g}} \mathbf{u}\mathbf{v}\,,$	$\dim \mathbb{S} = 3$
gradient:	$d\,f$	$= \mathrm{g}\nabla f\,,$	$\dim \mathbb{S} = n$
curl:	$d(\mathrm{g}\mathbf{v})$	$= (\mathrm{rot}\,\mathbf{v})\,\boldsymbol{\mu}_{\mathrm{g}}\,,$	$\dim \mathbb{S} = 2$
curl:	$d(\mathrm{g}\mathbf{v})$	$= \boldsymbol{\mu}_{\mathrm{g}}(\mathrm{rot}\,\mathbf{v})\,,$	$\dim \mathbb{S} = 3$
divergence:	$d(\boldsymbol{\mu}_{\mathrm{g}}\mathbf{v})$	$= (\mathrm{div}\,\mathbf{v})\boldsymbol{\mu}_{\mathrm{g}}\,,$	$\dim \mathbb{S} = n$

$\{\mathbb{S}\,,\mathrm{g}\}$

↑

riemannian manifold

Historical notes are reported in:

Ericksen J.: Tensor Fields, Handbuch der Physik vol. III/1, Springer-Verlag, Berlin (1960).

*which suggests that the classical Stokes theorem should be named the **Ampère - Kelvin - Hankel** transform.*

Sir Isaac Newton
1643 - 1727

Gottfried Wilhelm von Leibniz
1646 - 1716

André Marie Ampère
1775 - 1836

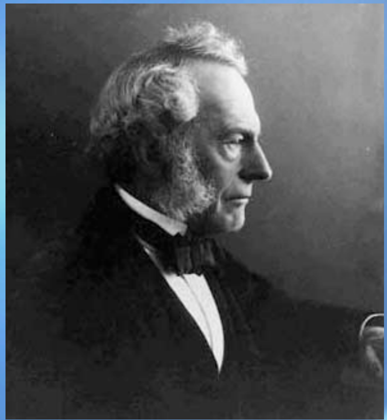
Johann Carl Friedrich Gauss
1777 - 1855

?

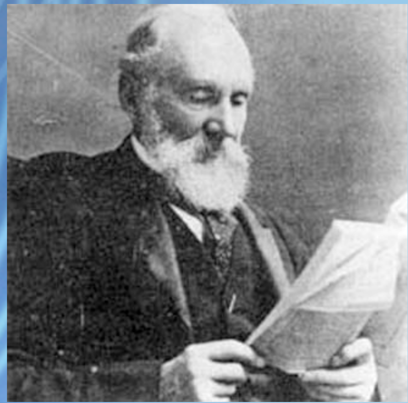
There may be no portrait of George Green, but he has a memorial stained-glass window in the Hall of the Cambridge college of which he was a Fellow.
It depicts Green's Theorem.

George Green
1793 - 1841

Mikhail Vasilevich Ostrogradski
1801 - 1862

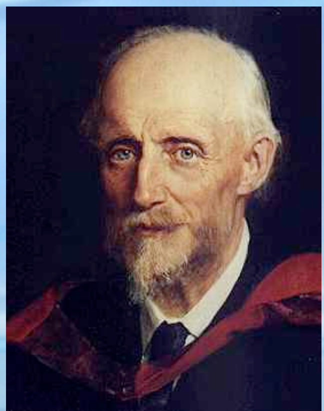


George Gabriel Stokes
1819 - 1903



William Thomson
(Lord Kelvin)
1824 - 1907

Hermann Hankel
1839 - 1873



Osborne Reynolds
1842 - 1912

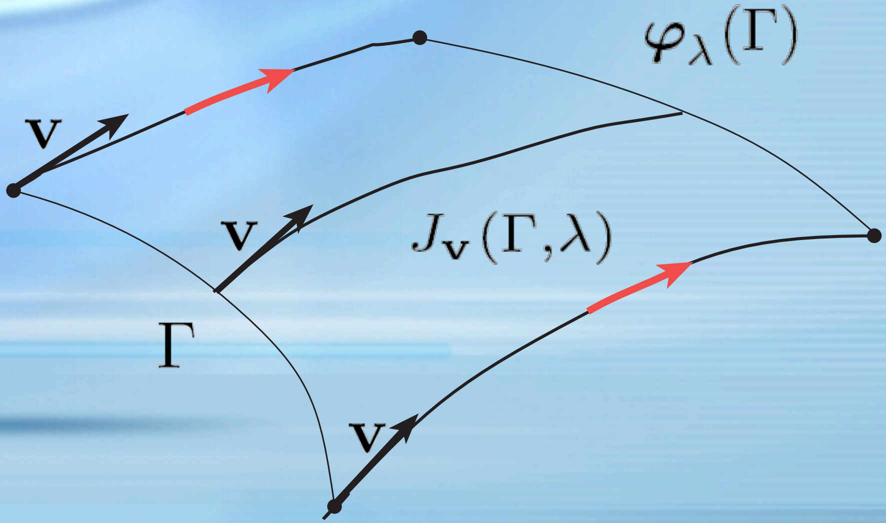
Jules Henri Poincaré
1854 - 1912

Guido Fubini
1879 - 1943

Reynolds theorem

$$\int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_\Sigma \boldsymbol{\varphi}_\lambda\!\downarrow\!\boldsymbol{\omega}^k \quad\Longrightarrow\quad \partial_{\lambda=0}\int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_\Sigma \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k$$

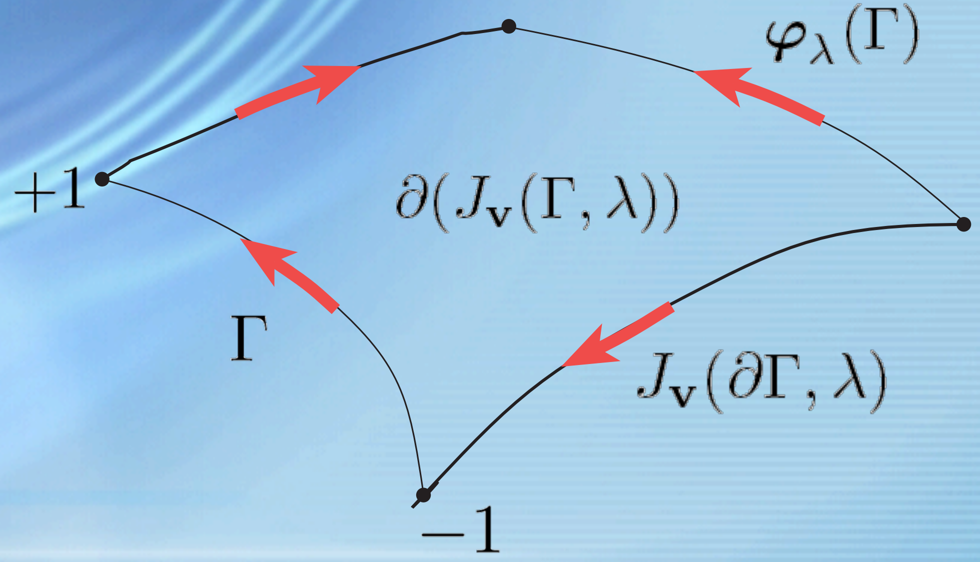
Fubini theorem



$$\partial_{\lambda=0}\int_{J_{\mathbf{v}}(\Gamma,\lambda)} \boldsymbol{\omega}^k = \int_\Gamma \boldsymbol{\omega}^k\mathbf{v}$$

Homotopy formula

Geometric homotopy formula:



$$\partial(J_{\mathbf{v}}(\Gamma,\lambda)) = \varphi_\lambda(\Gamma) - \Gamma - J_{\mathbf{v}}(\partial\Gamma,\lambda)$$

By the geometric homotopy formula we infer:

$$\int_{\varphi_\lambda(\Gamma)} \boldsymbol{\omega}^k = \int_{\partial(J_{\mathbf{v}}(\Gamma,\lambda))} \boldsymbol{\omega}^k + \int_{J_{\mathbf{v}}(\partial\Gamma,\lambda)} \boldsymbol{\omega}^k + \int_{\Gamma} \boldsymbol{\omega}^k$$

Applying Stokes and Fubini's formulas

$$\partial_{\lambda=0} \int_{\partial(J_{\mathbf{v}}(\Gamma,\lambda))} \boldsymbol{\omega}^k = \int_{\Gamma} (d\boldsymbol{\omega}^k)\mathbf{v} \qquad \partial_{\lambda=0} \int_{J_{\mathbf{v}}(\partial\Gamma,\lambda)} \boldsymbol{\omega}^k = \int_{\Gamma} d(\boldsymbol{\omega}^k \mathbf{v})$$

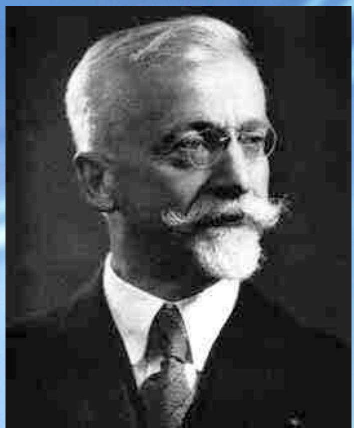
By Reynolds theorem we have

$$\partial_{\lambda=0} \int_{\varphi_\lambda(\Gamma)} \boldsymbol{\omega}^k = \int_{\Gamma} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^k$$

and summing up we have

$$\int_{\Gamma} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^k = \int_{\Gamma} (d\boldsymbol{\omega}^k)\mathbf{v} + \int_{\Gamma} d(\boldsymbol{\omega}^k \mathbf{v})$$

By the arbitraryness of the k-dimensional submanifold $\Gamma \subset \mathbb{M}$ we infer the differential homotopy formula also known as ***CARTAN's Magic Formula****:*



Elie Joseph Cartan
1869 - 1951

$$\mathcal{L}_{\mathbf{v}}\,\omega^k = (d\omega^k)\mathbf{v} + d(\omega^k\mathbf{v})$$

This formula provides the relation between the ***Lie*** *and the exterior derivative of a differential form.*

Palais formula

$$\mathcal{L}_{\mathbf{v}}\,\omega^k = (d\omega^k)\mathbf{v} + d(\omega^k\mathbf{v})$$

→ *Homotopy formula*

$$\mathcal{L}_{\mathbf{v}_\Gamma}(\omega^k(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k)) = (\mathcal{L}_{\mathbf{v}_\Gamma}\omega^k)(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k) + \omega^k(\mathcal{L}_{\mathbf{v}_\Gamma}\cdot\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k) + \ldots + \omega^k(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathcal{L}_{\mathbf{v}_\Gamma}\cdot\mathbf{v}_k)$$

→ *Leibniz rule*

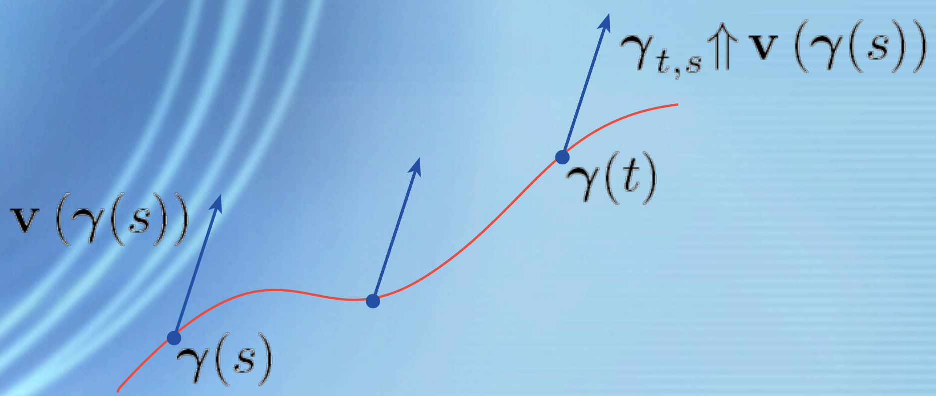
$$(d\omega^1)\cdot\mathbf{v}_\Gamma\cdot\mathbf{v} = d_{\mathbf{v}_\Gamma}(\omega^1\cdot\mathbf{v}) - d_{\mathbf{v}}(\omega^1\cdot\mathbf{v}_\Gamma) - \omega^1\cdot[\mathbf{v}_\Gamma,\mathbf{v}]$$

Richard Palais

With my wife and frequent co-author, Chuu-lian Terng at the dedication of a memorial bust of Sophus Lie, at Lie's birthplace in Nordfjord, Norway.

Palais, R. Definition of the exterior derivative in terms of the Lie derivative. Proc. Am. Math. Soc. 1954; 5: 902-908.

Connection and Parallel transport



Affine connection

$$\begin{cases} \nabla_{(\alpha\,\mathbf{u}+\beta\,\mathbf{w})}\mathbf{v} = \alpha\,\nabla_{\mathbf{u}}\mathbf{v} + \beta\,\nabla_{\mathbf{w}}\mathbf{v} \\ \nabla(\mathbf{v}_1 + \mathbf{v}_2) = \nabla\mathbf{v}_1 + \nabla\mathbf{v}_2 \\ \nabla_{\mathbf{u}}(f\,\mathbf{v}) = (\partial_{\mathbf{u}} f)\,\mathbf{v} + f\,(\nabla_{\mathbf{u}}\mathbf{v}) \end{cases}$$

Covariant derivative

$$\nabla_{\dot{\gamma}(t)}\,\mathbf{v}\,(\gamma(t)) = \partial_{s=t}\,\gamma_{t,s} \Uparrow \mathbf{v}\,(\gamma(s))$$

$$\nabla^2_{\mathbf{vu}}\,\mathbf{w} := \nabla_{\mathbf{v}}\,\nabla_{\mathbf{u}}\,\mathbf{w} - \nabla_{(\nabla_{\mathbf{v}}\mathbf{u})}\,\mathbf{w}$$

Torsion of a connection

$$(\nabla\partial)_{\mathbf{vu}}\,f - (\nabla\partial)_{\mathbf{uv}}\,f = -\partial_{\mathrm{TORS}(\mathbf{v},\mathbf{u})}\,f$$

$$\mathrm{TORS}(\mathbf{v},\mathbf{u}) := (\nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v}) - [\mathbf{v},\mathbf{u}]$$

Riemannian manifolds

Differentiable manifold $\longrightarrow$ $\mathbb{M}$

metric tensor field $\longrightarrow$ $\mathbf{g} \in \mathrm{C}^1(\mathbb{M}\,; BL\,(\mathbb{TM}^2\,; \mathcal{R}))$

$\{\,\mathbb{M}\,;\mathbf{g}\,\} \longrightarrow$ *Riemannian manifold*

Georg Friedrich Bernhard Riemann 1826-1866

Levi-Civita connection

is the torsion-free and metric-preserving connection:

$$i)\quad \mathrm{TORS}(\mathbf{v},\mathbf{u}) = \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v},\mathbf{u}] = 0\,,$$

$$ii)\ \nabla\,\mathbf{g} = 0\,.$$