On Electromagnetic Entanglements under Changes of Frame

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Presentata nell'Adunanza del 1 febbraio, 2013.

Key words: electromagnetic fields, differential forms, frame-changes, relativistic effects, entanglements, Lorentz force law.

Abstract - The theory of electromagnetic induction, developed in terms of differential forms in space-time, assesses that frame-invariance of space-time electromagnetic fields and induction laws implies invariance of their spatial counterparts, under any change of frame. Application to VOIGT-LORENTZ frame-transformations reveals that relativistic scaling effects and entanglements in longitudinal components of electromagnetic fields do occur but vanish in the non-relativistic limit. The new transformation rules correct previous statements and deprive LORENTZ *force* law of theoretical support.

Riassunto - La teoria dell'induzione elettromagnetica, sviluppata in termini di forme differenziali nello spazio-tempo, stabilisce che l'invarianza dal riferimento, dei campi elettromagnetici e delle leggi di induzione, implica l'invarianza delle loro controparti spaziali, per un qualsiasi cambiamento di riferimento. L'applicazione alla trasformazione di VOIGT-LORENTZ rivela che effetti di scalatura relativistica e intrecci delle componenti longitudinali dei campi electromagnetici avvengono ma svaniscono al limite non relativistico. Le nuove regole di trasformazione correggono precedenti affermazioni, privando la legge di forza di LORENTZ di supporto teorico.

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1. EVENTS MANIFOLD AND OBSERVERS

The events manifold $\mathcal{E}$ is a 4-dimensional star-shaped orientable manifold without boundary. The exterior derivative in the events manifold $\mathcal{E}$ will be denoted by d. A framing consists in a criterion for simultaneity of events and a field of time-arrows $\mathbf{Z} \in \mathrm{C}^1(\mathcal{E}\,;\mathbb{T}\mathcal{E})$ and is described in geometrical terms (Whiston, 1974; Marmo and Preziosi, 2006) by a field of rank-one projectors $\mathbf{R} := dt \otimes \mathbf{Z}$ with $t \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{Z})$ time-function and $\mathcal{Z}$ time-line.[2] Idempotency $\mathbf{R}^2 = \mathbf{R}$ is equivalent to tuning $\langle dt, \mathbf{Z} \rangle = 1$.

Lemma 1.1 (Space-time splitting). *A framing $\mathbf{R} := dt \otimes \mathbf{Z}$ induces a univocal splitting of tangent vectors $\mathbf{X} \in \mathbb{T}\mathcal{E}$ into spatial and temporal components by means of complementary projectors $\mathbf{R}, \mathbf{P}$ with*

$$\mathbf{P} := \mathbf{I} - \mathbf{R}\,, \quad \mathbf{P}^2 = \mathbf{P}\,, \quad \mathbf{PR} = \mathbf{RP} = \mathbf{0}\,,$$

so that $dt \circ \mathbf{P} = \mathbf{0}\,, \quad \mathbf{RZ} = \mathbf{Z}\,, \quad \mathrm{Ker}\, dt = \mathrm{Im}\,\mathbf{P}\,.$

Proof. Being $\mathbf{R}(\mathbf{X}) = (dt \otimes \mathbf{Z}) \cdot \mathbf{X} = \langle dt, \mathbf{X} \rangle\, \mathbf{Z}$ all properties are verified by a direct calculation. ■

Under the action of a framing $\mathbf{R} := dt \otimes \mathbf{Z}$ the tangent bundle $\mathbb{T}\mathcal{E}$ splits into a WHITNEY bundle $\mathbb{V}\mathcal{E} \times_{\mathcal{E}} \mathbb{H}\mathcal{E}$ of time-vertical and time-horizontal tangent vectors with $\mathbb{V}\mathcal{E} = \mathrm{Im}\,\mathbf{P}$ and $\mathbb{H}\mathcal{E} = \mathrm{Im}\,\mathbf{R}$. The 3-D fibers of $\mathbb{V}\mathcal{E}$ are in the kernel of $dt \in \mathbf{\Lambda}^1(\mathbb{T}\mathcal{E})$ while the 1-D fibers of $\mathbb{H}\mathcal{E}$ are generated by the time-arrow $\mathbf{Z} \in \mathrm{C}^1(\mathcal{E}\,;\mathbb{T}\mathcal{E})$. Both subbundles of $\mathbb{T}\mathcal{E}$ are integrable. Indeed FROBENIUS involutivity condition is trivially fulfilled by any $1D$ subbundle and for the kernel subbunble of a form $\boldsymbol{\omega} \in \mathbf{\Lambda}^1(\mathbb{T}\mathcal{E})$ reduces to

$$\boldsymbol{\omega} \cdot \mathbf{X} = \mathbf{0}, \quad \boldsymbol{\omega} \cdot \mathbf{Y} = \mathbf{0} \quad \Longrightarrow \quad d\boldsymbol{\omega} \cdot \mathbf{X} \cdot \mathbf{Y} = \mathbf{0}\,,$$

which is also fulfilled since $d\boldsymbol{\omega} = ddt = \mathbf{0}$. A framing generates then in $\mathcal{E}$ two transversal families of submanifolds, a 3D quotient manifold of 1D time-lines and a 1D quotient manifold of 3D space-slices.

Definition 1.1 (Time-vertical space-time forms). *A space-time form is time-vertical if it vanishes when any of its arguments belongs to the time-horizontal bundle $\mathbb{H}\mathcal{E}$. To any space-time form $\mathbf{\Omega}^k \in \mathbf{\Lambda}^k(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ there corresponds a time-vertical restriction $\mathbf{P}{\downarrow}\mathbf{\Omega}^k \in \mathbf{\Lambda}^k(\mathbb{V}\mathcal{E}\,;\mathcal{R})$ defined by*

$$\langle \mathbf{P}{\downarrow}\mathbf{\Omega}^k, \mathbf{X} \rangle := \langle \mathbf{\Omega}^k, \mathbf{PX} \rangle\,, \quad \forall\, \mathbf{X} \in \mathrm{C}^1(\mathcal{E}\,;\mathbb{T}\mathcal{E}^k)\,,$$

[2] The symbol $\mathcal{Z}$ is taken from the German word *Zeit* for *Time*.

where $\mathbf{X} = \{\, \mathbf{X}_1, \dots, \mathbf{X}_k \,\} \in \mathbb{T}\mathcal{E}^k$ *and* $\mathbf{PX} = \{\, \mathbf{PX}_1, \dots, \mathbf{PX}_k \,\} \in \mathbb{V}\mathcal{E}^k$.

The integral manifolds of the vertical distribution $\mathbb{V}\mathcal{E}$ define a *time-bundle* projection $\boldsymbol{\pi}_{\mathcal{Z},\mathcal{E}} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{Z})$ by fixing the time instant $t \in \mathcal{Z}$ corresponding to any given spatial slice $\mathcal{E}(t)$.

Definition 1.2 (Spatial bundle). *The spatial bundle* $\mathcal{S}$ *is the fiber bundle over* $\mathcal{Z}$ *whose fibre* $\mathcal{S}(t)$ *is a* 3*-D manifolds with canonical isomorphism* $\mathbf{i}(t) \in \mathrm{C}^1(\mathcal{S}(t)\,;\mathcal{E}(t))$ *onto the* 3*-D submanifold* $\mathcal{E}(t)$ *of the* 4*-D space-time manifold* $\mathcal{E}$ *.*

For any fixed $t \in \mathcal{Z}$, the isomorphism $\mathbf{i}(t) \in \mathrm{C}^1(\mathcal{S}(t)\,;\mathcal{E}(t))$ may be acted upon by the tangent functor to provide a fibrewise defined *space-time extension*, $\mathbf{i}\!\uparrow \in \mathrm{C}^1(\mathbb{V}\mathcal{S}\,;\mathbb{V}\mathcal{E})$ which is a global bundle isomorphism but not the tangent map of a morphism.[3] The inverse morphism $\mathbf{i}\!\downarrow \in \mathrm{C}^1(\mathbb{V}\mathcal{E}\,;\mathbb{V}\mathcal{S})$ is the *spatial restriction*.

Vectors in $\mathbb{V}\mathcal{E}$, henceforth denoted by capital letters, have four space-time components in a space-time frame, while vectors in $\mathbb{V}\mathcal{S}$, denoted by small letters, have three spatial components in a space frame. In an adapted space-time frame $\{\, \mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \,\}$ with $\mathbf{X}_0 = \mathbf{Z}$, vectors in $\mathbb{V}\mathcal{E}$ will have a zero first component.

Definition 1.3 (Spatial forms). *A spatial form* $\boldsymbol{\omega}^k \in \boldsymbol{\Lambda}^k(\mathbb{V}\mathcal{S}\,;\mathcal{R})$ *is a form defined on the spatial bundle. To any space-time form* $\boldsymbol{\Omega}^k \in \boldsymbol{\Lambda}^k(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ *there corresponds a spatial form got by spatial restriction*

$$\boldsymbol{\omega}^k := \mathbf{i}\!\downarrow\!\boldsymbol{\Omega}^k \quad \Longleftrightarrow \quad \boldsymbol{\omega}^k(\mathbf{a}) := \boldsymbol{\Omega}^k(\mathbf{i}\!\uparrow\!\mathbf{a}) = (\mathbf{i}\!\downarrow\!\boldsymbol{\Omega}^k)(\mathbf{a})\,,$$

for all $\mathbf{a} = \{\, \mathbf{a}_1, \dots, \mathbf{a}_k \,\} \in \mathbb{V}\mathcal{S}^k$.

2. TRAJECTORY AND MOTION

The trajectory $\mathcal{T}$ is a non-linear manifold characterized by an injective immersion $\mathbf{i}_{\mathcal{E},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{E})$ such that the immersed trajectory $\mathcal{T}_{\mathcal{E}} := \mathbf{i}_{\mathcal{E},\mathcal{T}}(\mathcal{T}) \subset \mathcal{E}$ is a submanifold of the events manifold.[4]

[3] The push-pull notation is however still adopted for simplicity.

[4] Events in the trajectory are labeled by coordinates in that manifold whose dimensionality may in general be lower than the one of the events manifold. Events in the immersed trajectory are instead labeled by coordinates in the events manifold.

Definition 2.1 (Material bundle). *The* material bundle $\mathcal{M}$ *is the fiber bundle over time-line* $\mathcal{Z}$ *generated by the* time-bundle *projection* $\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} = \boldsymbol{\pi}_{\mathcal{Z},\mathcal{E}} \circ \mathbf{i}_{\mathcal{E},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{Z})$ *. The fibres* $\mathcal{T}(t)$ *are called trajectory slices.*

The motion detected in a given framing, is a one-parameter family of automorphisms [5] $\boldsymbol{\varphi}_\theta \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ of the trajectory time-bundle over the time shift $\mathrm{SH}_\theta \in \mathrm{C}^1(\mathcal{Z}\,;\mathcal{Z})$, defined by $\mathrm{SH}_\theta(t) := t + \theta$ with $t \in \mathcal{R}$ *time-instant* and $\theta \in \mathcal{Z}$ *time-lapse*, described by the commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\boldsymbol{\varphi}_\theta} & \mathcal{T} \\ {\scriptstyle \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}} \\ \mathcal{Z} & \xrightarrow{\mathrm{SH}_\theta} & \mathcal{Z} \end{array} \quad \iff \quad \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} \circ \boldsymbol{\varphi}_\theta = \mathrm{SH}_\theta \circ \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}\,,$$

which expresses the simultaneity preservation property of motion.

Events related by the space-time motion along the trajectory, i.e.

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{E} \mid \exists\, \theta \in \mathcal{R} \,:\, \mathbf{e}_2 = \boldsymbol{\varphi}_\theta(\mathbf{e}_1)\,,$$

form a class of equivalence and the equivalence relation foliates the trajectory manifold (Romano and Barretta, 2011, 2012, 2013).

A material particle is a line (a one-dimensional manifold) whose elements are motion-related events in the trajectory.

The body is the disjoint union of the trajectory material particles, a quotient manifold induced by the foliation of the trajectory manifold.

A body placement is a fibre of simultaneous trajectory-events. The placement at time $t \in I$ is then the trajectory slice $\mathcal{T}(t)$.

The space-time *trajectory velocity* $\mathbf{V} \in \mathrm{C}^1(\mathcal{T}\,;\mathbb{T}\mathcal{T})$ is the vector field defined by $\mathbf{V} := \partial_{\theta=0}\, \boldsymbol{\varphi}_\theta$. Since motion is time-parametrized, we have that

$$\langle\, dt, \mathbf{V}\,\rangle = 1\,, \quad \mathbf{V} = \mathbf{Z} + \mathbf{PV}\,, \quad \mathbf{RV} = \mathbf{Z}\,.$$

Spatial velocity is related to space-time velocity by $\mathbf{v} = \mathbf{i}{\downarrow}(\mathbf{PV}) \in \mathbb{V}\mathcal{S}$. For simplicity, we will here consider the case in which the trajectory manifold $\mathcal{T}$ is four-dimensional.

[5] An *automorphism* is an invertible morphism from a fibre-bundle onto itself.

3. SPACE-TIME SPLITTING OF FORMS

The splitting formula provided in the next Lemma 3.1 extends the notion first introduced by É. Cartan (1924) under the special assumption of vanishing spatial velocity, and thenceforth taken as standard reference in literature on electrodynamics, to take into account body motions. Detailed proofs of results exposed hereafter are provided in (G. Romano, 2013).

Lemma 3.1 (Splitting of forms). *A framing* $\mathbf{R} := dt \otimes \mathbf{Z}$ *induces a representation formula for space-time forms* $\mathbf{\Omega}^k \in \mathbf{\Lambda}^k(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ *in terms of time-vertical restrictions and of the time differential*

$$\mathbf{\Omega}^k = \mathbf{P}{\downarrow}\mathbf{\Omega}^k + dt \wedge (\mathbf{P}{\downarrow}(\mathbf{\Omega}^k \cdot \mathbf{V}) - (\mathbf{P}{\downarrow}\mathbf{\Omega}^k) \cdot \mathbf{V})\,.$$

Lemma 3.2 (Spatialization of exterior derivatives). *The exterior derivative* d *in the events manifold and the spatial exterior derivative* $d_{\mathcal{S}}$ *in a fibre of the time-vertical bundle* $\mathbb{V}\mathcal{S}$ *fulfill, with the spatial restriction* $\mathbf{i}{\downarrow}$, *the commutative diagram*

$$\begin{array}{ccc} \mathbf{\Lambda}^k(\mathbb{V}\mathcal{E}\,;\mathcal{R}) & \xrightarrow{\;d\;} & \mathbf{\Lambda}^{k+1}(\mathbb{V}\mathcal{E}\,;\mathcal{R}) \\ {\scriptstyle \mathbf{i}\downarrow}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{i}\downarrow} \\ \mathbf{\Lambda}^k(\mathbb{V}\mathcal{S}\,;\mathcal{R}) & \xrightarrow{\;d_{\mathcal{S}}\;} & \mathbf{\Lambda}^{k+1}(\mathbb{V}\mathcal{S}\,;\mathcal{R}) \end{array} \quad \Longleftrightarrow \quad d_{\mathcal{S}} \circ \mathbf{i}{\downarrow} = \mathbf{i}{\downarrow} \circ d\,.$$

Lemma 3.3 (Spatialization of Lie derivatives). *The* LIE *derivatives* $\mathcal{L}_{\mathbf{V}}$ *along the motion and its spatial restriction* $\mathcal{L}^{\mathcal{S}}_{\mathbf{V}}$ *fulfill, with the spatial restriction* $\mathbf{i}{\downarrow}$, *the commutative diagram*

$$\begin{array}{ccc} \mathbf{\Lambda}^k(\mathbb{T}\mathcal{E}\,;\mathcal{R}) & \xrightarrow{\;\mathcal{L}_{\mathbf{V}}\;} & \mathbf{\Lambda}^{k}(\mathbb{T}\mathcal{E}\,;\mathcal{R}) \\ {\scriptstyle \mathbf{i}\downarrow}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{i}\downarrow} \\ \mathbf{\Lambda}^k(\mathbb{V}\mathcal{S}\,;\mathcal{R}) & \xrightarrow{\;\mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\;} & \mathbf{\Lambda}^{k}(\mathbb{V}\mathcal{S}\,;\mathcal{R}) \end{array} \quad \Longleftrightarrow \quad \mathcal{L}^{\mathcal{S}}_{\mathbf{V}} \circ \mathbf{i}{\downarrow} = \mathbf{i}{\downarrow} \circ \mathcal{L}_{\mathbf{V}}\,.$$

4. SPACE-TIME FORMULATION OF ELECTROMAGNETICS

Space-time formulation of electromagnetic induction laws, was first proposed by Bateman (1910) on the basis of earlier work by Hargreaves (1908) on invariant integral forms, as quoted in the treatise (Truesdell and Toupin, 1960, Ch. F). An early treatment in terms of differential forms was formulated in (É. Cartan, 1924, p. 17-19). A detailed revisitation in the context of relativity theory can be found in (Misner, Thorne, Wheeler, 1973). A brand new approach is adopted here on the basis of the space-time splitting introduced in Lemma 3.1. Electric and magnetic induction rules take their natural and most elegant form when expressed, in the space-time manifold $\mathcal{E}$, in terms of FARADAY and AMPÈRE electromagnetic space-time two- and three-forms

$$\mathbf{\Omega}^2_{\mathbf{F}}, \mathbf{\Omega}^2_{\mathbf{A}} \in \mathbf{\Lambda}^2(\mathbb{T}\mathcal{E}\,;\mathcal{R})\,, \quad \mathbf{\Omega}^3_{\mathbf{F}}, \mathbf{\Omega}^3_{\mathbf{A}} \in \mathbf{\Lambda}^3(\mathbb{T}\mathcal{E}\,;\mathcal{R})\,.$$

The treatment here developed extends the classical one introduced by ÉLIE CARTAN, in which body motion was not taken into account.

The GAUSS-HENRY-FARADAY induction law is expressed, in terms of FARADAY forms $\mathbf{\Omega}^2_{\mathbf{F}} \in \mathbf{\Lambda}^2(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ and $\mathbf{\Omega}^3_{\mathbf{F}} \in \mathbf{\Lambda}^3(\mathbb{T}\mathcal{E}\,;\mathcal{R})$, by the condition

$$\oint_{\partial \mathbf{C}^3} \mathbf{\Omega}^2_{\mathbf{F}} = \int_{\mathbf{C}^3} \mathbf{\Omega}^3_{\mathbf{F}} \quad \Longleftrightarrow \quad d\,\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{\Omega}^3_{\mathbf{F}}\,.$$

In the same way, the GAUSS-AMPÈRE-MAXWELL induction law is expressed, in terms of AMPÈRE forms $\mathbf{\Omega}^2_{\mathbf{A}} \in \mathbf{\Lambda}^2(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ and $\mathbf{\Omega}^3_{\mathbf{A}} \in \mathbf{\Lambda}^3(\mathbb{T}\mathcal{E}\,;\mathcal{R})$, by the condition

$$\oint_{\partial \mathbf{C}^3} \mathbf{\Omega}^2_{\mathbf{A}} = \int_{\mathbf{C}^3} \mathbf{\Omega}^3_{\mathbf{A}} \quad \Longleftrightarrow \quad d\,\mathbf{\Omega}^2_{\mathbf{A}} = \mathbf{\Omega}^3_{\mathbf{A}}\,.$$

Above $\mathbf{C}^3$ is any 3-D control manifold with boundary and equivalences hold by STOKES formula.

Since the events manifold $\mathcal{E}$ is star-shaped, POINCARÉ Lemma assures that GAUSS-HENRY-FARADAY and GAUSS-AMPÈRE-MAXWELL induction laws are equivalent to the closure properties $d\,\mathbf{\Omega}^3_{\mathbf{F}} = \mathbf{0}$ and $d\,\mathbf{\Omega}^3_{\mathbf{A}} = \mathbf{0}$ and, by STOKES formula, to the integral conditions

$$\oint_{\partial \mathbf{C}^4} \mathbf{\Omega}^3_{\mathbf{F}} = \mathbf{0}\,, \qquad \oint_{\partial \mathbf{C}^4} \mathbf{\Omega}^3_{\mathbf{A}} = \mathbf{0}\,,$$

respectively expressing conservation of electric and magnetic space-time charges in an arbitrary 4-D control manifold with boundary $\mathbf{C}^4$.

5. INDUCTION LAWS IN SPACE-TIME

5.1. Faraday law

In standard electromagnetic theory it is assumed that $\mathbf{\Omega}^3_{\mathbf{F}} = \mathbf{0}$, a condition inferred from the experimental fact that magnetic monopoles and currents are still undiscovered. Recalling that $\mathbf{V} := \partial_{\theta=0}\,\boldsymbol{\varphi}_\theta \in \mathrm{C}^1(\mathcal{T}\,;\mathbb{T}\mathcal{T})$ is the trajectory velocity, from Lemma 3.1 we infer the next statement.

Definition 5.1 (Electric field and magnetic vortex). *The magnetic vortex and the electric circulation are even* [6] *spatial forms got from the electromagnetic space-time two-form* $\mathbf{\Omega}^2_{\mathbf{F}}$ *by the spatial restrictions*

$$\boldsymbol{\omega}^2_{\mathbf{B}} = \mathbf{i}{\downarrow}\mathbf{\Omega}^2_{\mathbf{B}} \in \mathbf{\Lambda}^2(\mathbb{V}\mathcal{S}\,;\mathcal{R})\,, \qquad \textit{magnetic vortex}$$

$$\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{i}{\downarrow}\mathbf{\Omega}^1_{\mathbf{E}} \in \mathbf{\Lambda}^1(\mathbb{V}\mathcal{S}\,;\mathcal{R})\,, \qquad \textit{electric field}$$

with $\mathbf{\Omega}^2_{\mathbf{B}} := \mathbf{P}{\downarrow}\mathbf{\Omega}^2_{\mathbf{F}}\,,\ -\mathbf{\Omega}^1_{\mathbf{E}} := \mathbf{P}{\downarrow}(\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V})$ *and the representation formula*

$$\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{\Omega}^2_{\mathbf{B}} - dt \wedge (\mathbf{\Omega}^1_{\mathbf{E}} + \mathbf{\Omega}^2_{\mathbf{B}} \cdot \mathbf{V})\,.$$

Proposition 5.1 (Gauss-Faraday law). *Closedness of* FARADAY *two-form in the trajectory manifold is equivalent to the spatial* GAUSS *law for the magnetic vortex and to the spatial* GAUSS-HENRY-FARADAY *induction law, i.e.*

$$d\,\mathbf{\Omega}^2_{\mathbf{F}} = 0 \quad \Longleftrightarrow \quad \begin{cases} d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = \mathbf{0}\,, \\ \mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{0}\,, \end{cases}$$

and to the integral formulation

$$\partial_{\theta=0} \int_{\boldsymbol{\varphi}^{\mathcal{S}}_{\theta}(\Sigma_{\text{IN}})} \boldsymbol{\omega}^2_{\mathbf{B}} = - \oint_{\partial\Sigma_{\text{IN}}} \boldsymbol{\omega}^1_{\mathbf{E}}\,,$$

for any inner-oriented surface Σ_{IN} *in a material slice.*

Proof. Recalling the commutativity properties in Lemmata 3.2,3.3 and the homotopy formula $(d\,\mathbf{\Omega}^2_{\mathbf{F}}) \cdot \mathbf{V} = \mathcal{L}_{\mathbf{V}}\,\mathbf{\Omega}^2_{\mathbf{F}} - d\,(\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V})$ from Lemma 3.1 we get

$$\begin{cases} \mathbf{i}{\downarrow}(d\,\mathbf{\Omega}^2_{\mathbf{F}}) = d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}\mathbf{\Omega}^2_{\mathbf{F}}) = d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}}\,, \\ \mathbf{i}{\downarrow}(d\,\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) = \mathbf{i}{\downarrow}(\mathcal{L}_{\mathbf{V}}\,\mathbf{\Omega}^2_{\mathbf{F}} - d\,(\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V})) \\ \qquad = \mathcal{L}^{\mathcal{S}}_{\mathbf{V}}(\mathbf{i}{\downarrow}\mathbf{\Omega}^2_{\mathbf{F}}) - d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}(\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V})) = \mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}}\,. \end{cases}$$

Hence the implication $\Longrightarrow$ follows. The converse implication $\Longleftarrow$ is inferred from the representation formula

$$d\,\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{P}{\downarrow}d\,\mathbf{\Omega}^2_{\mathbf{F}} + dt \wedge (\mathbf{P}{\downarrow}(d\,\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) - (\mathbf{P}{\downarrow}d\,\mathbf{\Omega}^2_{\mathbf{F}}) \cdot \mathbf{V})\,,$$

because $\mathbf{i}{\downarrow}d\,\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{0}$ and $\mathbf{i}{\downarrow}(d\,\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) = \mathbf{0}$ imply that $\mathbf{P}{\downarrow}d\,\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{0}$ and $\mathbf{P}{\downarrow}(d\,\mathbf{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) = \mathbf{0}$ and hence that $d\,\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{0}$. ■

[6] *Inner* and *outer* oriented manifolds and *even* and *odd* forms are treated in (Schouten, 1951; Tonti, 1995; Marmo et al., 2005). *Odd* forms change sign under change of orientation while *even* forms do not. Even forms represent circulations and vortices, odd forms have the meaning of sources, winding around and flux through (G. Romano, 2012).

5.1.1. Electromagnetic potentials

The *magnetic potential* even one-form $\boldsymbol{\omega}^1_{\mathbf{B}} \in \boldsymbol{\Lambda}^1(\mathbb{V}\mathcal{S}\,;\mathcal{R})$ and to the *electric potential* even zero-form $\boldsymbol{\omega}^0_{\mathbf{E}} \in \boldsymbol{\Lambda}^0(\mathbb{V}\mathcal{S}\,;\mathcal{R})$ are related to the space-time Faraday one-form $\boldsymbol{\Omega}^1_{\mathbf{F}} \in \boldsymbol{\Lambda}^1(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ by the spatial restrictions

$$\boldsymbol{\omega}^1_{\mathbf{B}} = \mathbf{i}{\downarrow}\boldsymbol{\Omega}^1_{\mathbf{B}}\,, \qquad \boldsymbol{\omega}^0_{\mathbf{E}} = \mathbf{i}{\downarrow}\boldsymbol{\Omega}^0_{\mathbf{E}}\,,$$

with $\boldsymbol{\Omega}^1_{\mathbf{B}} = \mathbf{P}{\downarrow}\boldsymbol{\Omega}^1_{\mathbf{F}}$ and $-\boldsymbol{\Omega}^0_{\mathbf{E}} = \mathbf{P}{\downarrow}(\boldsymbol{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V})$ and the representation formula

$$\boldsymbol{\Omega}^1_{\mathbf{F}} = \boldsymbol{\Omega}^1_{\mathbf{B}} - dt \wedge (\boldsymbol{\Omega}^0_{\mathbf{E}} + \boldsymbol{\Omega}^1_{\mathbf{B}} \cdot \mathbf{V})\,.$$

The Gauss-Henry-Faraday law of electromagnetic induction is equivalent to the potentiality property expressed by Poincaré Lemma

$$\mathbf{0} = d\,\boldsymbol{\Omega}^2_{\mathbf{F}} \quad \Longleftrightarrow \quad \boldsymbol{\Omega}^2_{\mathbf{F}} = d\,\boldsymbol{\Omega}^1_{\mathbf{F}}\,.$$

In terms of spatial differential forms we get the following result.

Proposition 5.2 (Electric field in terms of potentials). *In terms of the magnetic potential one-form* $\boldsymbol{\omega}^1_{\mathbf{B}}$ *and of the electric potential zero-form* $\boldsymbol{\omega}^0_{\mathbf{E}}$*, the* Gauss-Henry-Faraday *induction law is expressed by*

$$\boldsymbol{\Omega}^2_{\mathbf{F}} = d\,\boldsymbol{\Omega}^1_{\mathbf{F}} \quad \Longleftrightarrow \quad \begin{cases} \boldsymbol{\omega}^2_{\mathbf{B}} = d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{B}}\,, \\ -\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\,\boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}}\,\boldsymbol{\omega}^0_{\mathbf{E}}\,. \end{cases}$$

Proof. Assuming $\boldsymbol{\Omega}^2_{\mathbf{F}} = d\,\boldsymbol{\Omega}^1_{\mathbf{F}}$ by homotopy $(d\,\boldsymbol{\Omega}^1_{\mathbf{F}})\cdot\mathbf{V} = \mathcal{L}_{\mathbf{V}}\,\boldsymbol{\Omega}^1_{\mathbf{F}} - d\,(\boldsymbol{\Omega}^1_{\mathbf{F}}\cdot\mathbf{V})$ we infer that

$$\begin{cases} \boldsymbol{\omega}^2_{\mathbf{B}} = \mathbf{i}{\downarrow}\boldsymbol{\Omega}^2_{\mathbf{F}} = \mathbf{i}{\downarrow}d\,\boldsymbol{\Omega}^1_{\mathbf{F}} = d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}\boldsymbol{\Omega}^1_{\mathbf{F}}) = d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{B}}\,, \\ -\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{i}{\downarrow}(\boldsymbol{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) = \mathbf{i}{\downarrow}(d\,\boldsymbol{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V}) = \mathbf{i}{\downarrow}(\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\Omega}^1_{\mathbf{F}} - d\,(\boldsymbol{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V})) \\ \qquad = \mathcal{L}^{\mathcal{S}}_{\mathbf{V}}(\mathbf{i}{\downarrow}\boldsymbol{\Omega}^1_{\mathbf{F}}) - d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}(\boldsymbol{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V})) = \mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\,\boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}}\,\boldsymbol{\omega}^0_{\mathbf{E}}\,. \end{cases}$$

Hence the implication $\Longrightarrow$ follows. The converse implication $\Longleftarrow$ is inferred from the representation formulae

$$\begin{aligned} d\,\boldsymbol{\Omega}^1_{\mathbf{F}} &= \mathbf{P}{\downarrow}d\,\boldsymbol{\Omega}^1_{\mathbf{F}} + dt \wedge (\mathbf{P}{\downarrow}(d\,\boldsymbol{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V}) - (\mathbf{P}{\downarrow}d\,\boldsymbol{\Omega}^1_{\mathbf{F}}) \cdot \mathbf{V}) \\ &= d_{\mathcal{S}}\,(\mathbf{P}{\downarrow}\boldsymbol{\Omega}^1_{\mathbf{F}}) + dt \wedge (\mathbf{P}{\downarrow}(d\,\boldsymbol{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V}) - (d_{\mathcal{S}}\,(\mathbf{P}{\downarrow}\boldsymbol{\Omega}^1_{\mathbf{F}}) \cdot \mathbf{V})\,, \\ \boldsymbol{\Omega}^2_{\mathbf{F}} &= \mathbf{P}{\downarrow}\boldsymbol{\Omega}^2_{\mathbf{F}} + dt \wedge (\mathbf{P}{\downarrow}(\boldsymbol{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) - (\mathbf{P}{\downarrow}\boldsymbol{\Omega}^2_{\mathbf{F}}) \cdot \mathbf{V}))\,, \end{aligned}$$

because the conditions $\mathbf{i}{\downarrow}\boldsymbol{\Omega}^2_{\mathbf{F}} = \mathbf{i}{\downarrow}d\,\boldsymbol{\Omega}^1_{\mathbf{F}}$ and $\mathbf{i}{\downarrow}(\boldsymbol{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) = \mathbf{i}{\downarrow}(d\,\boldsymbol{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V})$ imply that $\boldsymbol{\Omega}^2_{\mathbf{F}} = d\boldsymbol{\Omega}^1_{\mathbf{F}}$. ■

The differential conditions in Prop.5.2 are expressed in integral form by

$$\int_{\Sigma_{\rm IN}} \boldsymbol{\omega}^2_{\mathbf{B}} = \oint_{\partial\Sigma_{\rm IN}} \boldsymbol{\omega}^1_{\mathbf{B}} \,, \qquad -\int_{\mathbf{L}_{\rm IN}} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\theta=0} \int_{\boldsymbol{\varphi}_\theta(\mathbf{L}_{\rm IN})} \boldsymbol{\omega}^1_{\mathbf{B}} + \oint_{\partial\mathbf{L}_{\rm IN}} \boldsymbol{\omega}^0_{\mathbf{E}} \,,$$

for any inner oriented material line $\mathbf{L}_{\rm IN}$ and surface $\Sigma_{\rm IN}$. The latter states that the electromotive force along a path is given by the sum of the decrease of the scalar electric potential from the start to the end point plus the time-rate of decrease of the integral magnetic potential along the motion.

The treatment of AMPÈRE law of electromagnetic induction may be carried out along the same lines of approach and will here be dropped for brevity.

6. CHANGES OF FRAME

A *change of frame* is an automorphism $\boldsymbol{\zeta} \in C^1(\mathcal{E}\,;\mathcal{E})$ of the events manifold. A *trajectory transformation* $\boldsymbol{\zeta}_{\mathcal{T}} \in C^1(\mathcal{T}\,;\mathcal{T}_{\boldsymbol{\zeta}})$ is a diffeomorphism between trajectory manifolds, induced by a change of frame according to the commutative diagram

$$\begin{tikzcd} \mathcal{E} & \mathcal{T} \arrow[l, "\mathbf{i}_{\mathcal{E},\mathcal{T}}"'] \arrow[r, shift left, "\boldsymbol{\zeta}_{\mathcal{T}}"] & \mathcal{T}_{\boldsymbol{\zeta}} \arrow[l, shift left, "\boldsymbol{\zeta}_{\mathcal{T}}^{-1}"] \arrow[r, "\mathbf{i}_{\mathcal{E},\mathcal{T}_{\boldsymbol{\zeta}}}"] & \mathcal{E} \arrow[from=1-1, to=1-4, bend left, "\boldsymbol{\zeta}"] \end{tikzcd} \iff \quad \mathbf{i}_{\mathcal{E},\mathcal{T}_{\boldsymbol{\zeta}}} \circ \boldsymbol{\zeta}_{\mathcal{T}} = \boldsymbol{\zeta} \circ \mathbf{i}_{\mathcal{E},\mathcal{T}} \,,$$

with $\mathbf{i}_{\mathcal{E},\mathcal{T}} \in C^1(\mathcal{T}\,;\mathcal{E})$ and $\mathbf{i}_{\mathcal{E},\mathcal{T}_{\boldsymbol{\zeta}}} \in C^1(\mathcal{T}_{\boldsymbol{\zeta}}\,;\mathcal{E})$ injective immersions.

Lemma 6.1 (Pushed framings). *Under a change of frame according to an automorphism $\boldsymbol{\zeta} \in C^1(\mathcal{E}\,;\mathcal{E})$, a framing $\mathbf{R} = dt \oplus \mathbf{Z}$ is pushed to a framing*

$$\boldsymbol{\zeta}{\uparrow}\mathbf{R} = \boldsymbol{\zeta}{\uparrow}(dt \oplus \mathbf{Z}) = (\boldsymbol{\zeta}{\uparrow}dt) \oplus (\boldsymbol{\zeta}{\uparrow}\mathbf{Z}) = dt_{\boldsymbol{\zeta}} \oplus \mathbf{Z}_{\boldsymbol{\zeta}} \,,$$

with $t_{\boldsymbol{\zeta}} := t \circ \boldsymbol{\zeta}^{-1}$ and $\mathbf{Z}_{\boldsymbol{\zeta}} := \boldsymbol{\zeta}{\uparrow}\mathbf{Z}$.

Proof. Setting $t_{\boldsymbol{\zeta}} = t \circ \boldsymbol{\zeta}^{-1}$ we have that $d\,t_{\boldsymbol{\zeta}} = d\,(t \circ \boldsymbol{\zeta}^{-1}) = \boldsymbol{\zeta}{\uparrow}(dt)$. Persistence of tuning follows from $\langle dt_{\boldsymbol{\zeta}}, \mathbf{Z}_{\boldsymbol{\zeta}} \rangle = \boldsymbol{\zeta}{\uparrow}\langle dt, \mathbf{Z} \rangle = 1$. ■

Trajectories and motions $\boldsymbol{\varphi}_\theta \in C^1(\mathcal{T}\,;\mathcal{T})$ and $(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\varphi})_\theta \in C^1(\mathcal{T}_{\boldsymbol{\zeta}}\,;\mathcal{T}_{\boldsymbol{\zeta}})$, evaluated in frames inducing a trajectory transformation $\boldsymbol{\zeta} \in C^1(\mathcal{T}\,;\mathcal{T}_{\boldsymbol{\zeta}})$, are related by the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\boldsymbol{\zeta}} \arrow[r, "(\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi})_\theta"] & \mathcal{T}_{\boldsymbol{\zeta}} \\ \mathcal{T} \arrow[u, "\boldsymbol{\zeta}"] \arrow[r, "\boldsymbol{\varphi}_\theta"] & \mathcal{T} \arrow[u, "\boldsymbol{\zeta}"'] \end{tikzcd} \iff \quad (\boldsymbol{\zeta}{\uparrow}\boldsymbol{\varphi})_\theta \circ \boldsymbol{\zeta} = \boldsymbol{\zeta} \circ \boldsymbol{\varphi}_\theta \,.$$

Definition 6.1 (Space-time frame-invariance). *A tensor field on the trajectory manifold* $\mathbf{s} \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{T\small{ENS}}(\mathbb{T}\mathcal{T}))$ *is frame-invariant under the action of a trajectory transformation* $\boldsymbol{\zeta}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T}_{\boldsymbol{\zeta}})$ *if it varies by push*

$$\mathbf{s}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}_{\mathcal{T}}{\uparrow}\mathbf{s}\,.$$

A relation involving tensor fields is frame-invariant if it transforms by push, the pushed relation being defined by the property that is it fulfilled by tensor fields if and only if their pull-back fulfill the original relation.

Lemma 6.2 (Frame-invariance of trajectory velocity). *The trajectory velocity is frame-invariant:* $\mathbf{V}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}_{\mathcal{T}}{\uparrow}\mathbf{V}$ *.*

Proof. Being $\mathbf{V} := \partial_{\theta=0}\,\boldsymbol{\varphi}_\theta$ so that $\boldsymbol{\varphi}_\theta = \mathbf{Fl}^{\mathbf{V}}_\theta$ and being $\mathbf{V}_{\boldsymbol{\zeta}} := \partial_{\theta=0}\,(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\varphi})_\theta$, the direct computation

$$\mathbf{V}_{\boldsymbol{\zeta}} = \partial_{\theta=0}\,(\boldsymbol{\zeta}_{\mathcal{T}} \circ \mathbf{Fl}^{\mathbf{V}}_\theta \circ \boldsymbol{\zeta}_{\mathcal{T}}^{-1}) = T\boldsymbol{\zeta}_{\mathcal{T}} \circ \mathbf{V} \circ \boldsymbol{\zeta}_{\mathcal{T}}^{-1} = \boldsymbol{\zeta}_{\mathcal{T}}{\uparrow}\mathbf{V}\,,$$

gives the formula. ■

Lemma 6.3 (Immersion and push of vector fields). *Spatial vectors according to a framing* $\mathbf{R}$ *are still spatial vectors in a pushed framing* $\boldsymbol{\zeta}{\uparrow}\mathbf{R}$ *as expressed by the commutative diagram*

$$\begin{tikzcd} \mathbb{T}\mathcal{E} \arrow[r, "\boldsymbol{\zeta}\uparrow"] & \mathbb{T}\mathcal{E} \\ \mathbb{V}\mathcal{S} \arrow[u, "\mathbf{i}\uparrow"] \arrow[r, "\boldsymbol{\zeta}_{\mathcal{S}}\uparrow"] & \mathbb{V}\mathcal{S}_{\boldsymbol{\zeta}} \arrow[u, "\mathbf{i}_{\boldsymbol{\zeta}}\uparrow"'] \end{tikzcd} \iff \mathbf{i}_{\boldsymbol{\zeta}}{\uparrow} \circ \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow} = \boldsymbol{\zeta}{\uparrow} \circ \mathbf{i}{\uparrow}\,.$$

The spatial bundle isomorphism $\boldsymbol{\zeta}_{\mathcal{S}}{\uparrow} \in \mathrm{C}^1(\mathbb{V}\mathcal{S}\,;\mathbb{V}\mathcal{S}_{\boldsymbol{\zeta}})$ *is induced by the space-time push* $\boldsymbol{\zeta}{\uparrow} \in \mathrm{C}^1(\mathbb{T}\mathcal{E}\,;\mathbb{T}\mathcal{E})$ *according to a change of frame* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{E})$ *. The inverse isomorphism is* $\boldsymbol{\zeta}_{\mathcal{S}}{\downarrow} \in \mathrm{C}^1(\mathbb{V}\mathcal{S}_{\boldsymbol{\zeta}}\,;\mathbb{V}\mathcal{S})$ *.*[7]

Proof. The push of forms is defined by invariance

$$\langle \boldsymbol{\zeta}{\uparrow}dt, \boldsymbol{\zeta}{\uparrow}\mathbf{X} \rangle = \boldsymbol{\zeta}{\uparrow}\langle dt, \mathbf{X} \rangle\,, \quad \forall\, \mathbf{X} \in \mathrm{C}^1(\mathcal{E}\,;\mathbb{T}\mathcal{E})\,,$$

and hence $\langle dt, \mathbf{X} \rangle = 0 \quad \Longrightarrow \quad \langle \boldsymbol{\zeta}{\uparrow}dt, \boldsymbol{\zeta}{\uparrow}\mathbf{X} \rangle = 0\,.$ ■

[7] The isomorphism $\boldsymbol{\zeta}_{\mathcal{S}}{\uparrow} \in \mathrm{C}^1(\mathbb{V}\mathcal{S}\,;\mathbb{V}\mathcal{S}_{\boldsymbol{\zeta}})$ is not the tangent map to an automorphism of the manifold $\mathcal{E}$, unless restriction to a spatial slice is considered, see the proof of Lemma 6.4. The push-pull notation is however adopted for simplicity.

Lemma 6.4 (Simultaneity preservation). *Frame-changes in space-time transform simultaneous events according to the initial framing* $\mathbf{R}$ *into simultaneous events according to the pushed framing* $\mathbf{R}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{R}$ *.*

Proof. The integral manifolds of time-vertical fields $\boldsymbol{\zeta}\!\uparrow\!\mathbf{X} \in \mathrm{C}^1(\mathcal{E}\,;\mathbb{V}\mathcal{E}_{\boldsymbol{\zeta}})$ are space-slices got as $\boldsymbol{\zeta}$-images of the integral manifolds of time-vertical fields $\mathbf{X} \in \mathrm{C}^1(\mathcal{E}\,;\mathbb{V}\mathcal{E})$. Then frame-changes transform simultaneous events in the initial frame into simultaneous events according to the pushed framing. It follows that the restriction of $\boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow$ to a space-slice is equal to the push according to $\boldsymbol{\zeta}_{\mathcal{S}}$ transformation between spatial slices, induced by the $\boldsymbol{\zeta}$-transformation. ■

Lemma 6.5 (Spatialization and push of differential forms). *Push of a form according to a change of frame* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{E})$ *and spatial restriction according to related framings* $\mathbf{R}$ *and* $\mathbf{R}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{R}$ *fulfill the commutative diagram*

$$\begin{array}{ccc} \mathbf{\Lambda}^k(\mathbb{T}\mathcal{E}\,;\mathcal{R}) & \xrightarrow{\boldsymbol{\zeta}\uparrow} & \mathbf{\Lambda}^k(\mathbb{T}\mathcal{E}_{\boldsymbol{\zeta}}\,;\mathcal{R}) \\ {\scriptstyle \mathbf{i}\downarrow}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{i}_{\boldsymbol{\zeta}}\downarrow} \\ \mathbf{\Lambda}^k(\mathbb{V}\mathcal{S}\,;\mathcal{R}) & \xrightarrow{\boldsymbol{\zeta}_{\mathcal{S}}\uparrow} & \mathbf{\Lambda}^k(\mathbb{V}\mathcal{S}_{\boldsymbol{\zeta}}\,;\mathcal{R}) \end{array} \quad \Longleftrightarrow \quad \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow \circ\, \boldsymbol{\zeta}\!\uparrow = \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow \circ\, \mathbf{i}\!\downarrow .$$

Proof. Let $\mathbf{\Omega}^k \in \mathbf{\Lambda}^k(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ be a form in the space-time manifold. Assuming $k = 2$ and $\mathbf{a}, \mathbf{b} \in \mathrm{C}^1(\mathcal{E}\,;\mathbb{V}\mathcal{S})$, we get

$$\begin{aligned} (\boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\mathbf{i}\!\downarrow\!\mathbf{\Omega}^k)(\mathbf{a}\,,\mathbf{b}) = \mathbf{\Omega}^k(\mathbf{i}\!\uparrow\!\boldsymbol{\zeta}_{\mathcal{S}}\!\downarrow\!\mathbf{a}\,,\mathbf{i}\!\uparrow\!\boldsymbol{\zeta}_{\mathcal{S}}\!\downarrow\!\mathbf{b}) &= \mathbf{\Omega}^k(\boldsymbol{\zeta}\!\downarrow\!\mathbf{i}_{\boldsymbol{\zeta}}\!\uparrow\!\mathbf{a}\,,\boldsymbol{\zeta}\!\downarrow\!\mathbf{i}_{\boldsymbol{\zeta}}\!\uparrow\!\mathbf{b}) \\ &= (\mathbf{i}_{\boldsymbol{\zeta}}\!\uparrow\!\boldsymbol{\zeta}\!\downarrow\!\mathbf{\Omega}^k)(\mathbf{a}\,,\mathbf{b})\,, \end{aligned}$$

where the result in Lemma 6.3 has been resorted to. ■

Lemma 6.6 (Pull and spatial exterior derivative). *Pull back due to a change of frame* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{E})$ *and exterior derivatives of spatial restrictions according to related framings* $\mathbf{R}$ *and* $\mathbf{R}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{R}$ *fulfill the commutative diagram*

$$\begin{array}{ccc} \mathbf{\Lambda}^{k+1}(\mathbb{V}\mathcal{S}\,;\mathcal{R}) & \xrightarrow{\boldsymbol{\zeta}_{\mathcal{S}}\downarrow} & \mathbf{\Lambda}^{k+1}(\mathbb{V}\mathcal{S}_{\boldsymbol{\zeta}}\,;\mathcal{R}) \\ {\scriptstyle d_{\mathcal{S}}}\big\uparrow & & \big\uparrow{\scriptstyle (d_{\mathcal{S}})_{\boldsymbol{\zeta}}} \\ \mathbf{\Lambda}^k(\mathbb{V}\mathcal{S}\,;\mathcal{R}) & \xrightarrow{\boldsymbol{\zeta}_{\mathcal{S}}\downarrow} & \mathbf{\Lambda}^k(\mathbb{V}\mathcal{S}_{\boldsymbol{\zeta}}\,;\mathcal{R}) \end{array} \quad \Longleftrightarrow \quad (d_{\mathcal{S}})_{\boldsymbol{\zeta}} \circ \boldsymbol{\zeta}_{\mathcal{S}}\!\downarrow = \boldsymbol{\zeta}_{\mathcal{S}}\!\downarrow \circ\, d_{\mathcal{S}}\ .$$

Proof. The proof follows along the same lines of the one in Lemma 3.2, but expressed in terms of the $\boldsymbol{\zeta}_{\mathcal{S}}$ transformation between spatial slices, defined in Lemma 6.4, instead of immersions. ■

7. FRAME INVARIANCE OF ELECTROMAGNETICS

Proposition 7.1 (Space-time frame-invariance of induction laws). *The space-time frame invariance of* FARADAY *and* AMPÈRE *electromagnetic two-forms and of the current three-form*

$$(\boldsymbol{\Omega}^2_{\mathbf{F}})_{\zeta} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}} \,, \quad (\boldsymbol{\Omega}^2_{\mathbf{A}})_{\zeta} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{A}} \,, \quad (\boldsymbol{\Omega}^3_{\mathbf{A}})_{\zeta} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^3_{\mathbf{A}} \,,$$

imply the space-time frame invariance of the laws of induction

$$\begin{aligned} d\,\boldsymbol{\Omega}^2_{\mathbf{F}} = \mathbf{0} \quad &\iff \quad d\,(\boldsymbol{\Omega}^2_{\mathbf{F}})_{\zeta} = \mathbf{0} \,, \\ d\,\boldsymbol{\Omega}^2_{\mathbf{A}} = \boldsymbol{\Omega}^3_{\mathbf{A}} \quad &\iff \quad d\,(\boldsymbol{\Omega}^2_{\mathbf{A}})_{\zeta} = (\boldsymbol{\Omega}^3_{\mathbf{A}})_{\zeta} \,. \end{aligned}$$

Proof. The result is a direct consequence of the commutativity between exterior derivative and push by a diffeomorphism. Indeed

$$d\,(\boldsymbol{\Omega}^2_{\mathbf{F}})_{\zeta} = d\,(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}}) = \boldsymbol{\zeta}{\uparrow}(d\,\boldsymbol{\Omega}^2_{\mathbf{F}}) \,,$$

and similarly for the second equivalence. ■

Electromagnetic space-time forms are assumed to be invariant under any change of frame. This means that they change in the only possible natural way, by push according to the transformation defining the change of frame. Neither *special relativity theory*, nor MINKOWSKI pseudo-metric, play any role in this general treatment of frame-transformations.

To state the invariance result, we consider a frame-change $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{E})$ and push-related framings $\mathbf{R}$ and $\mathbf{R}_{\zeta} = \boldsymbol{\zeta}{\uparrow}\mathbf{R}$.

Proposition 7.2 (Frame-invariance of fields and induction laws). *Frame-invariance of* FARADAY *and* AMPÈRE *space-time electromagnetic two-forms* $\boldsymbol{\Omega}^2_{\mathbf{F}}, \boldsymbol{\Omega}^2_{\mathbf{A}} \in \boldsymbol{\Lambda}^2(\mathbb{T}\mathcal{E}\,;\mathcal{R})$ *and of* AMPÈRE *three-form* $\boldsymbol{\Omega}^3_{\mathbf{A}} \in \boldsymbol{\Lambda}^3(\mathbb{T}\mathcal{E}\,;\mathcal{R})$, *is equivalent to spatial frame-invariance of the corresponding spatial forms*

$$\begin{cases} (\boldsymbol{\Omega}^2_{\mathbf{F}})_{\zeta} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}} \\ (\boldsymbol{\Omega}^2_{\mathbf{A}})_{\zeta} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{A}} \\ (\boldsymbol{\Omega}^3_{\mathbf{A}})_{\zeta} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^3_{\mathbf{A}} \end{cases} \iff \begin{cases} (\boldsymbol{\omega}^1_{\mathbf{E}})_{\zeta} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^1_{\mathbf{E}} \\ (\boldsymbol{\omega}^2_{\mathbf{B}})_{\zeta} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^2_{\mathbf{B}} \\ (\boldsymbol{\omega}^1_{\mathbf{H}})_{\zeta} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^1_{\mathbf{H}} \end{cases} \begin{cases} (\boldsymbol{\omega}^2_{\mathbf{D}})_{\zeta} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^2_{\mathbf{D}} \\ (\boldsymbol{\omega}^2_{\mathbf{J}})_{\zeta} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^2_{\mathbf{J}} \\ (\boldsymbol{\omega}^3_{\boldsymbol{\rho}})_{\zeta} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^3_{\boldsymbol{\rho}} \end{cases}$$

Frame-invariance of spatial laws of electromagnetic induction also holds.

Proof. Let us assume space-time frame-invariance of FARADAY two-form expressed by $(\boldsymbol{\Omega}^2_{\mathbf{F}})_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}}$. Then, by space-time frame invariance of the trajectory speed $\mathbf{V}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\mathbf{V}$, stated in Lemma 6.2, and by commutativity property in Lemma 6.5, we infer the spatial-frame invariance of the electric field one-form $\boldsymbol{\omega}^1_{\mathbf{E}}$, since

$$\begin{aligned}(\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}} &= \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}((\boldsymbol{\Omega}^2_{\mathbf{F}})_{\boldsymbol{\zeta}} \cdot \mathbf{V}_{\boldsymbol{\zeta}}) = \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}} \cdot \boldsymbol{\zeta}{\uparrow}\mathbf{V}) = \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}\boldsymbol{\zeta}{\uparrow}(\boldsymbol{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V})\\ &= \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\mathbf{i}{\downarrow}(\boldsymbol{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V}) = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^1_{\mathbf{E}}\,.\end{aligned}$$

Spatial frame-invariance of the magnetic vortex two-form $\boldsymbol{\omega}^2_{\mathbf{B}}$ follows by a similar evaluation

$$(\boldsymbol{\omega}^2_{\mathbf{B}})_{\mathbf{R}_{\boldsymbol{\zeta}}} = \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}(\boldsymbol{\Omega}^2_{\mathbf{F}})_{\boldsymbol{\zeta}} = \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\mathbf{i}{\downarrow}\boldsymbol{\Omega}^2_{\mathbf{F}} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^2_{\mathbf{B}}\,.$$

The same procedure leads to the conclusion that space-time frame-invariance of AMPÈRE two and three-forms implies spatial frame-invariance of magnetic winding $\boldsymbol{\omega}^1_{\mathbf{H}}$, electric flux $\boldsymbol{\omega}^2_{\mathbf{D}}$, electric current flux $\boldsymbol{\omega}^2_{\mathbf{J}}$, and electric charge $\boldsymbol{\omega}^3_{\boldsymbol{\rho}}$. Frame-invariance of the spatial laws of electromagnetic induction is inferred from the push naturality property of LIE derivatives and the commutativity property of Lemmata 3.3,6.3,6.5, as explicated below

$$\begin{aligned}\mathcal{L}^{\mathcal{S}}_{\mathbf{V}_{\boldsymbol{\zeta}}}(\boldsymbol{\omega}^2_{\mathbf{B}})_{\boldsymbol{\zeta}} &= \mathcal{L}^{\mathcal{S}}_{\mathbf{V}_{\boldsymbol{\zeta}}}(\mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}(\boldsymbol{\Omega}^2_{\mathbf{F}})_{\boldsymbol{\zeta}}) = \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}(\mathcal{L}_{\mathbf{V}_{\boldsymbol{\zeta}}}(\boldsymbol{\Omega}^2_{\mathbf{F}})_{\boldsymbol{\zeta}}) = \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}(\mathcal{L}_{(\boldsymbol{\zeta}{\uparrow}\mathbf{V})}(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}})\\ &= \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}\boldsymbol{\zeta}{\uparrow}(\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\Omega}^2_{\mathbf{F}}) = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\mathbf{i}{\downarrow}(\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\Omega}^2_{\mathbf{F}}) = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}(\mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\,\mathbf{i}{\downarrow}\boldsymbol{\Omega}^2_{\mathbf{F}})\\ &= \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}(\mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\,\boldsymbol{\omega}^2_{\mathbf{B}})\,.\end{aligned}$$

Being moreover by Lemma 6.6 $(d_{\mathcal{S}})_{\boldsymbol{\zeta}}\,(\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}} = (d_{\mathcal{S}})_{\boldsymbol{\zeta}}\,(\boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\boldsymbol{\omega}^1_{\mathbf{E}}) = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}(d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}})$, we get the equality expressing frame-invariance of the spatial FARADAY law of induction

$$\mathcal{L}^{\mathcal{S}}_{\mathbf{V}_{\boldsymbol{\zeta}}}(\boldsymbol{\omega}^2_{\mathbf{B}})_{\boldsymbol{\zeta}} + (d_{\mathcal{S}})_{\boldsymbol{\zeta}}\,(\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}(\mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}})\,.$$

Analogous proofs hold for all other spatial laws of induction. ■

If the splitting of the pushed space-time forms is performed according to the unpushed framing $\mathbf{R}$, the result will in general depend on the special frame-transformation considered, as exemplified in Sect.8.

Definition 7.1 (Adapted frames). *A frame is a set of tangent vector fields which gives a basis at each point. A frame is* adapted *to a framing if one family of coordinate lines is envelop of the time-arrow field and the other three families define coordinate systems in the spatial slicings.*

8. RELATIVISTIC ELECTRODYNAMICS

Let us consider a space-time frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ adapted to a framing $\mathbf{R} = dt \oplus \mathbf{Z}$ with the first vector given by $\mathbf{X}_0 = \mathbf{Z}$ and the tangent vector fields $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ got by immersion of a frame $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in spatial slices. Coordinates are classically denoted by $\{t, x, y, z\}$. Running indexes are $i, j, k = 1, 2, 3$. Then $\mathbf{X}_i = \mathbf{i}\!\uparrow\!\mathbf{x}_i$. According to relativity principle, a change of frame for a translational motion with relative spatial velocity $\mathbf{w} = w\,\mathbf{X}_1$ in the x direction, is governed by a Voigt-Lorentz transformation [8] with associated Jacobi matrix

$$\boldsymbol{\zeta} : \begin{cases} t \mapsto \gamma(t - x\,(w/c^2)) \\ x \mapsto \gamma(x - w\,t) \\ y \mapsto y \\ z \mapsto z \end{cases} , \quad [T\boldsymbol{\zeta}] = \begin{bmatrix} \gamma & -\gamma\,(w/c^2) & & \\ -\gamma\,w & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

where $\gamma := (1 - w^2/c^2)^{-1/2}$. The inverse transformation is got by changing w into $-w$. Recalling that $\mathbf{X}_0 = \mathbf{Z}$, the basis vectors are changed by the transformation into

$$\begin{cases} T\boldsymbol{\zeta} \cdot \mathbf{X}_0 = \gamma\,\mathbf{Z} - \gamma\,w\,\mathbf{X}_1\,, \\ T\boldsymbol{\zeta} \cdot \mathbf{X}_1 = -\gamma\,(w/c^2)\,\mathbf{Z} + \gamma\,\mathbf{X}_1\,, \\ T\boldsymbol{\zeta} \cdot \mathbf{X}_\alpha = \mathbf{X}_\alpha\,, \quad \text{for} \quad \alpha = 2, 3\,. \end{cases}$$

The theory developed in the previous sections and the data concerning Voigt-Lorentz transformations allow us to deduce in a direct way the transformation rules for all electromagnetic fields.

To perform a comparison with standard treatments in literature, the following representation in terms of vector and scalar fields will be adopted, with $\mathbf{g}$ metric field in the spatial bundle and $\boldsymbol{\mu}$ associated volume form.

$$\begin{aligned} &\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{g} \cdot \mathbf{E}\,, \quad \boldsymbol{\omega}^2_{\mathbf{B}} = \boldsymbol{\mu} \cdot \mathbf{B}\,, \quad \boldsymbol{\omega}^1_{\mathbf{H}} = \mathbf{g} \cdot \mathbf{H}\,, \quad \boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\mu} \cdot \mathbf{D}\,, \\ &\boldsymbol{\omega}^1_{\mathbf{B}} = \mathbf{g} \cdot \mathbf{A}\,, \quad \boldsymbol{\omega}^2_{\mathbf{J}} = \boldsymbol{\mu} \cdot \mathbf{J}\,, \quad \boldsymbol{\omega}^3_{\boldsymbol{\rho}} = \boldsymbol{\mu} \cdot \rho\,, \quad \boldsymbol{\omega}^0_{\mathbf{E}} = V_{\mathbf{E}}\,. \end{aligned}$$

[8] According to Minkowski (1908), the transformation introduced by Lorentz (1904) and by Einstein (1905) was first conceived by Voigt (1887).

8.1. Electric field and magnetic vortex

Frame-transformation formulae for spatial electric field $\boldsymbol{\omega}^1_{\mathbf{E}}$ and magnetic vortex $\boldsymbol{\omega}^2_{\mathbf{B}}$ are got by pushing FARADAY two-form $\boldsymbol{\Omega}^2_{\mathbf{F}}$ and space-time velocity $\mathbf{V}$. Being $\boldsymbol{\Omega}^2_{\mathbf{F}} = \boldsymbol{\Omega}^2_{\mathbf{B}} - dt \wedge (\boldsymbol{\Omega}^1_{\mathbf{E}} + \boldsymbol{\Omega}^2_{\mathbf{B}} \cdot \mathbf{V})$ we get

$$
\begin{aligned}
&\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{V}) = (\boldsymbol{\Omega}^1_{\mathbf{E}} \cdot \mathbf{V})\, dt - \boldsymbol{\Omega}^1_{\mathbf{E}} \qquad && \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_0) - (\boldsymbol{\Omega}^1_{\mathbf{E}} + \boldsymbol{\Omega}^2_{\mathbf{B}} \cdot \mathbf{V})\,, \\
&\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{V}, \mathbf{X}_i) = -\boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{x}_i)\,, && \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_0, \mathbf{X}_i) = -(\boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{x}_i) + \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{v}, \mathbf{x}_i))\,, \\
&\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{V}, \mathbf{X}_0) = \boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{v})\,, && \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_i, \mathbf{X}_j) = \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{x}_i, \mathbf{x}_j)\,.
\end{aligned}
$$

8.1.1. Electric field

The frame-transformation formula for the longitudinal component $\boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{x}_1)$ is given by

$$
\begin{aligned}
-(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}})(\boldsymbol{\zeta}{\uparrow}\mathbf{V}, \mathbf{X}_1) &= -\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{V}, T\boldsymbol{\zeta}^{-1} \cdot \mathbf{X}_1) \\
&= -\gamma\, \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{V}, \mathbf{X}_1) - \gamma\, (w/c^2)\, \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{V}, \mathbf{X}_0) \\
&= \gamma\, (\boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{x}_1) - (w/c^2)\, \boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{v}))\,,
\end{aligned}
$$

while for the transversal components of the electric field $\boldsymbol{\omega}^1_{\mathbf{E}}$ along $\mathbf{x}_\alpha$ with $\alpha = 2, 3$ are given by

$$
-(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}})(\boldsymbol{\zeta}{\uparrow}\mathbf{V}, \mathbf{X}_\alpha) = -\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{V}, \mathbf{X}_\alpha) = \boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{x}_\alpha)\,.
$$

8.1.2. Magnetic vortex

The frame-transformation formula for the component of the magnetic vortex $\boldsymbol{\omega}^2_{\mathbf{B}}$ in the longitudinal planes $\{\mathbf{x}_1, \mathbf{x}_\alpha\}$, with $\alpha = 2, 3$, writes

$$
\begin{aligned}
(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}})(\mathbf{X}_1, \mathbf{X}_\alpha) &= \boldsymbol{\Omega}^2_{\mathbf{F}}(T\boldsymbol{\zeta}^{-1} \cdot \mathbf{X}_1, \mathbf{X}_\alpha) = \boldsymbol{\Omega}^2_{\mathbf{F}}(\gamma\, (w/c^2)\, \mathbf{X}_0 + \gamma\, \mathbf{X}_1, \mathbf{X}_\alpha) \\
&= \gamma\, ((w/c^2)\, \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_0, \mathbf{X}_\alpha) + \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_1, \mathbf{X}_\alpha)) \\
&= \gamma\, (\boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{x}_1, \mathbf{x}_\alpha) - (w/c^2)\, (\boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{x}_\alpha) + \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{v}, \mathbf{x}_\alpha)))\,,
\end{aligned}
$$

while the component of $\boldsymbol{\omega}^2_{\mathbf{B}}$ in the transversal plane $\{\mathbf{x}_2, \mathbf{x}_3\}$ is given by

$$
(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^2_{\mathbf{F}})(\mathbf{X}_2, \mathbf{X}_3) = \boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_2, \mathbf{X}_3) = \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{x}_2, \mathbf{x}_3)\,.
$$

Table 26.3 in (Feynman, 1964, 26.3) and formulae (18.42) and (18.43) in (Panofsky and Phillips, 1962, p.330) provide the transformation rules for electric and magnetic vector fields. The latter is in agreement with our results (for $\mathbf{v} = \mathbf{0}$) but the former is not, as explicated in Sect.8.3.

8.2. Electric and magnetic potentials

Frame-transformation formulae for magnetic and electric potential are got by considering the space-time $\boldsymbol{\Omega}^1_{\mathbf{F}}$ and the spatial forms $\boldsymbol{\omega}^1_{\mathbf{B}}$ and $\boldsymbol{\omega}^0_{\mathbf{E}}$. Being $\boldsymbol{\Omega}^1_{\mathbf{F}} = \boldsymbol{\Omega}^1_{\mathbf{B}} - dt \wedge (\boldsymbol{\Omega}^0_{\mathbf{E}} + \boldsymbol{\Omega}^1_{\mathbf{B}} \cdot \mathbf{V})$ we have that

$$\boldsymbol{\Omega}^1_{\mathbf{F}}(\mathbf{V}) = -\boldsymbol{\Omega}^0_{\mathbf{E}}\,, \quad \boldsymbol{\Omega}^1_{\mathbf{F}}(\mathbf{X}_0) = -(\boldsymbol{\Omega}^0_{\mathbf{E}} + \boldsymbol{\Omega}^1_{\mathbf{B}} \cdot \mathbf{V})\,, \quad \boldsymbol{\Omega}^1_{\mathbf{F}}(\mathbf{X}_i) = \boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_i)\,.$$

8.2.1. Electric potential

The components of the electric potential $\boldsymbol{\omega}^0_{\mathbf{E}}$ transform according to the relation $-(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^1_{\mathbf{F}})(\boldsymbol{\zeta}{\uparrow}\mathbf{V}) = -\boldsymbol{\Omega}^1_{\mathbf{F}}(\mathbf{V}) = \boldsymbol{\omega}^0_{\mathbf{E}}$.

8.2.2. Magnetic potential

The frame-transformation formula for longitudinal and transversal components of the magnetic potential $\boldsymbol{\omega}^1_{\mathbf{B}}$ gives

$$\begin{aligned}(\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^1_{\mathbf{F}})(\mathbf{X}_1) &= \boldsymbol{\Omega}^1_{\mathbf{F}}(T\boldsymbol{\zeta}^{-1} \cdot \mathbf{X}_1) = \gamma\,(w/c^2)\,\boldsymbol{\Omega}^1_{\mathbf{F}}(\mathbf{X}_0) + \gamma\,\boldsymbol{\Omega}^1_{\mathbf{F}}(\mathbf{X}_1)\\ &= \gamma\,(\boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_1) - (w/c^2)\,(\boldsymbol{\omega}^0_{\mathbf{E}} + \boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v})\\ (\boldsymbol{\zeta}{\uparrow}\boldsymbol{\Omega}^1_{\mathbf{F}})(\mathbf{X}_\alpha) &= \boldsymbol{\Omega}^1_{\mathbf{F}}(T\boldsymbol{\zeta}^{-1} \cdot \mathbf{X}_\alpha) = \boldsymbol{\Omega}^1_{\mathbf{F}}(\mathbf{X}_\alpha) = \boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_\alpha)\,, \quad \alpha = 2,3\,.\end{aligned}$$

In (Feynman, 1964, 25.5) formulae (25.24_i) and (25.24_{ii}) and in (Landau and Lifshits, 1987, 24) formulae (24.1) provide transformation rules for electric and magnetic potentials. The latter agrees with our results (for $\mathbf{v} = \mathbf{0}$) but the former does not, as explicated in Sect.8.3.

8.3. Synopsis

Being $\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{g} \cdot \mathbf{E}$ the component $\mathbf{E}^{\|}$ is parallel to the longitudinal direction $\mathbf{X}_1$ of the relative spatial velocity $\mathbf{w}$, while being $\boldsymbol{\omega}^2_{\mathbf{B}} = \boldsymbol{\mu} \cdot \mathbf{B}$ the component $\mathbf{B}^{\|}$ acts in the transversal plane $(\mathbf{x}_2, \mathbf{x}_3)$. The same observation holds respectively for $\mathbf{H}^{\|}$ and $\mathbf{D}^{\|}$. Being $\boldsymbol{\omega}^2_{\mathbf{J}} = \boldsymbol{\mu} \cdot \mathbf{J}$ the component $\mathbf{J}^{\|}$ acts in the transversal plane $(\mathbf{x}_2, \mathbf{x}_3)$ and being $\boldsymbol{\omega}^1_{\mathbf{B}} = \mathbf{g} \cdot \mathbf{A}$ the component $\mathbf{A}^{\|}$ acts in the longitudinal direction $\mathbf{X}_1$.

According to the new results, relativistic effects appear only in longitudinal components and entanglements vanish in the non-relativistic limit.

To grasp the motivation for the disagreement between our results and the ones reported in literature, we assume $\mathbf{v} = \mathbf{o}$ so that $\mathbf{V} = \mathbf{Z} = \mathbf{X}_0$, as in standard treatments.

Then, the transformation rule for the electric field derived in Sect.8.1.1 is modified by mistaking $\mathbf{X}_0$ in place of $\boldsymbol{\zeta}\!\uparrow\!\mathbf{X}_0$. Observing that $T\boldsymbol{\zeta}^{-1}\cdot\mathbf{X}_0 = \gamma\,\mathbf{X}_0 + \gamma\,w\,\mathbf{X}_1$, we get

$$\begin{aligned}
-(\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\Omega}^2_{\mathbf{F}})(\mathbf{X}_0,\mathbf{X}_1) &= -\boldsymbol{\Omega}^2_{\mathbf{F}}(T\boldsymbol{\zeta}^{-1}\cdot\mathbf{X}_0, T\boldsymbol{\zeta}^{-1}\cdot\mathbf{X}_1)\\
&= -\gamma^2\,\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_0,\mathbf{X}_1) - \gamma^2\,(w^2/c^2)\,\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_1,\mathbf{X}_0)\\
&= \boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{X}_1)\,,\\
-(\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\Omega}^2_{\mathbf{F}})(\mathbf{X}_0,\mathbf{X}_\alpha) &= -\boldsymbol{\Omega}^2_{\mathbf{F}}(T\boldsymbol{\zeta}^{-1}\cdot\mathbf{X}_0, T\boldsymbol{\zeta}^{-1}\cdot\mathbf{X}_\alpha) = -\boldsymbol{\Omega}^2_{\mathbf{F}}(T\boldsymbol{\zeta}^{-1}\cdot\mathbf{X}_0, \mathbf{X}_\alpha)\\
&= -\gamma\,\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_0,\mathbf{X}_\alpha) - \gamma\,w\,\boldsymbol{\Omega}^2_{\mathbf{F}}(\mathbf{X}_1,\mathbf{X}_\alpha)\\
&= \gamma\,(\boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{x}_\alpha) + w\,\boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{x}_1,\mathbf{x}_\alpha))\,,\quad \alpha = 2,3\,.
\end{aligned}$$

In terms of vector fields, these relations express the incorrect transformation rule (Lorentz, 1904; Einstein, 1905)

$$(\mathbf{E}^{\parallel}, \mathbf{E}^{\perp}) \rightarrow (\mathbf{E}^{\parallel}, \gamma\,(\mathbf{E}^{\perp} + \mathbf{w}\times\mathbf{B}))\,.$$

All disagreements in the synoptic table are consequences of the same mistake.

9. CONCLUSIONS

Turning points outcoming from our analysis may be resumed as follows.

1. Entanglements and scaling due to frame-changes are only due to relativistic effects and accordingly vanish in the non-relativistic limit.
2. Relativistic effects acts only in longitudinal direction, similarly to the length contraction effect.

These results correct the statement that transversal components of electric and magnetic fields are affected by scaling and entanglements surviving at ordinary velocities.

The synoptic table provides a comparison of our results (*new*) with the state of art in literature (*old*). To this end we restrict ourselves to the case considered in literature, i.e. no spatial motion of test particles, so that $\mathbf{v} = \mathbf{o}$ and $\mathbf{V} = \mathbf{Z}$.

The new theory deprives the Lorentz *force law* of theoretical support. The physical evidence, that an observer will measure a force acting on a charged test particle traveling in a spatial field of magnetic potential, can be explained on the sole ground of Faraday induction law

Synoptic table ($\mathbf{v}=\mathbf{0}$)			
new		***old***	
$(\mathbf{E}^{\parallel}, \mathbf{E}^{\perp})$	$\rightarrow (\gamma\, \mathbf{E}^{\parallel}, \mathbf{E}^{\perp})$	*versus*	$(\mathbf{E}^{\parallel}, \gamma\, (\mathbf{E}^{\perp} + \mathbf{w} \times \mathbf{B}))$
$(\mathbf{B}^{\parallel}, \mathbf{B}^{\perp})$	$\rightarrow (\mathbf{B}^{\parallel}, \gamma\, (\mathbf{B}^{\perp} - (\mathbf{w}/c^2) \times \mathbf{E}))$	*idem*	
$(\mathbf{H}^{\parallel}, \mathbf{H}^{\perp})$	$\rightarrow (\gamma\, \mathbf{H}^{\parallel}, \mathbf{H}^{\perp})$	*versus*	$(\mathbf{H}^{\parallel}, \gamma\, (\mathbf{H}^{\perp} - \mathbf{w} \times \mathbf{D}))$
$(\mathbf{D}^{\parallel}, \mathbf{D}^{\perp})$	$\rightarrow (\mathbf{D}^{\parallel}, \gamma\, (\mathbf{D}^{\perp} + (\mathbf{w}/c^2) \times \mathbf{H}))$	*idem*	
$(\mathbf{J}^{\parallel}, \mathbf{J}^{\perp})$	$\rightarrow (\mathbf{J}^{\parallel}, \gamma\, \mathbf{J}^{\perp})$	*versus*	$(\gamma\, (\mathbf{J}^{\parallel} - \rho\, \mathbf{w}), \mathbf{J}^{\perp})$
ρ	$\rightarrow \gamma\, (\rho - \mathbf{g}(\mathbf{w}/c^2, \mathbf{J}))$	*idem*	
$V_{\mathbf{E}}$	$\rightarrow V_{\mathbf{E}}$	*versus*	$\gamma\, (V_{\mathbf{E}} - \mathbf{g}(\mathbf{w}, \mathbf{A}))$
$(\mathbf{A}^{\parallel}, \mathbf{A}^{\perp})$	$\rightarrow (\gamma\, (\mathbf{A}^{\parallel} - (\mathbf{w}/c^2) V_{\mathbf{E}}), \mathbf{A}^{\perp})$	*idem*	

Indeed, resorting to Prop.5.2 and to homotopy formula we get for the electric field the expression

$$
\begin{aligned}
-\boldsymbol{\omega}^1_{\mathbf{E}} &= \mathcal{L}^{\mathcal{S}}_{\mathbf{V}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}}\, \boldsymbol{\omega}^0_{\mathbf{E}} \\
&= \mathcal{L}^{\mathcal{S}}_{\mathbf{Z}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}}\, \boldsymbol{\omega}^0_{\mathbf{E}} \\
&= \mathcal{L}^{\mathcal{S}}_{\mathbf{Z}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + (d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{B}}) \cdot \mathbf{v} + d_{\mathcal{S}}\, (\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}) + d_{\mathcal{S}}\, \boldsymbol{\omega}^0_{\mathbf{E}}\,,
\end{aligned}
$$

which is coincident with the one exposed in (J.J. Thomson, 1893, ch. VII, p. 534) as reproducing the theoretical result due to (Maxwell, 1861, (77) p.342). Denoting by ∇ the Euclid connection in spatial slices and assuming that the observer measures

- a magnetic potential independent of time, $\mathcal{L}^{\mathcal{S}}_{\mathbf{Z}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = \mathbf{0}$,
- a spatially constant scalar potential, $d_{\mathcal{S}}\, \boldsymbol{\omega}^0_{\mathbf{E}} = \mathbf{0}$,
- a spatially constant magnetic vortex, $\nabla \boldsymbol{\omega}^2_{\mathbf{B}} = \mathbf{0}$,

the formula may be evaluated to give (G. Romano, 2012)

$$
-\boldsymbol{\omega}^1_{\mathbf{E}} = (d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{B}}) \cdot \mathbf{v} + d_{\mathcal{S}}\, (\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}) = \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} - \frac{1}{2}\, \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} = \frac{1}{2}\, \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}\,,
$$

or in terms of vector fields

$$\mathbf{E} = \frac{1}{2}\,\mathbf{v} \times \mathbf{B}\,,$$

which is just one-half of what is improperly called the LORENTZ *force*. This last expression seems to be in accord with early theoretical and experimental findings by J.J. Thomson (1881), see (Darrigol, 2000). An experimental verification of the formula would provide an additional and conclusive support to the physical consistency of the new mathematical theory. The implications of the new frame-transformation formulae on the interpretation of relativistic DOPPLER effect and light aberration phenomena, as described by Einstein (1905), will be discussed elsewhere.

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