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AN INTERNAL VARIABLE THEORY OF INELASTIC BEHAVIOUR DERIVED FROM THE UNIAXIAL RIGID–PERFECTLY PLASTIC LAW†

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Abstract—In the general framework provided by the internal variable theories of associated inelastic behaviour the formulation of constitutive relations is addressed in this paper. Attention is focused on the basic properties of the evolution relation involving rates of internal variables and dual thermodynamic forces. It is shown that a suitable generalization of the uniaxial rigid–perfectly plastic law can be performed by introducing the definition of step-shaped constitutive maps. This definition allows us to derive a general theory of associated inelastic behaviour with its characteristic properties: convexity of the elastic locus, normality rule, existence of a sublinear dissipation functional and of a canonical yield functional. Finally the formulation of the constitutive relation in terms of yield functionals and related inelastic multipliers is discussed. The analysis is performed on the basis of a chain rule of subdifferential calculus, recently contributed by the authors, which provides an effective tool to develop the theory of Kuhn–Tucker vectors in optimization problems.

1. INTRODUCTION

In recent times a relevant interest has been devoted to the formulation of constitutive theories in which the inelastic behaviour is described in terms of internal variables [1–10].

The basic properties to be fulfilled by the evolution relation between the rates of the internal variables and the corresponding thermodynamic forces have been explored in a recent paper by Eve *et al.* [11] using concepts and methods of convex analysis.

In particular their contribution has been devoted to the analysis of the connections existing among Hill's principle of maximum dissipation, the existence of a dissipation functional and the normality rule to a convex elastic locus.

The main theorem proved in [11] concerns the conditions to be imposed on the set-valued map expressing the relation between the rates of internal variables and the corresponding thermodynamic forces in order to ensure the existence of a nonnegative lower semicontinuous sublinear potential, having the physical meaning of dissipation functional.

Maximal responsiveness is assumed to be the characteristic property of the constitutive operator; it reproduces Hill's principle of maximum plastic dissipation.

The approach and the ideas contributed in [11] are quite interesting and deserve a special attention. Incidentally we remark that some slips have been detected by a careful reading of the paper; a detailed discussion on this point is reported in Appendix C.

A different and, in the authors' opinion, more attractive approach is proposed in this paper.

A background of convex analysis is preliminarily reported and results from the potential theory of monotone multi-valued maps developed in [12] are recalled.

We start from the consideration that the rigid–perfectly plastic behaviour in a ideal uniaxial test is represented by a step function relating the uniaxial stress to the plastic elongation rate.

In a general model of inelastic behaviour the constitutive relation involves rates of internal state parameters and dual thermodynamic forces. The following question arises then in a natural way: "which are the essential features that the graph of the constitutive relation directly inherits from the simplest uniaxial step function?"

The answer is straightforward and leads to the definition of the class of **step-shaped** constitutive maps; these are maximal monotone set-valued maps whose value at zero coincides with the entire image set of the map.

† Dedicated to the memory of Professor MANFREDI ROMANO.

The image set of the constitutive map is the closed convex set of admissible thermodynamic forces and will be referred to as the elastic locus.

The theory of step-shaped maps is developed in detail by means of a number of theorems. To make the exposition more plain these results are collected in Appendix A.

The proofs of further results which are relevant to the analysis performed in the paper are reported in Appendix B.

It is proved that step-shaped maps are precisely those ones which admit a sublinear potential; this potential turns out to be the support functional of the elastic locus and assumes the meaning of dissipation functional.

It is further shown that the constitutive map is step-shaped if and only if the rates of internal variables fulfill the normality rule to the elastic locus.

The relations among step-shaped, responsive and monotone conservative maps are then analyzed and it is proved that step-shapedness and maximal responsiveness are equivalent properties for a multi-valued map.

In the formulation of specific models of inelastic behaviour the elastic locus is often conveniently assigned as the level set of a convex yield functional rather than as the image of a multi-valued map. The normality rule is accordingly reformulated in terms of a complementarity relation involving the yield functional and the relevant inelastic multiplier. The treatment is based upon a chain-rule of subdifferential calculus, recently contributed by the authors, which provides an effective tool for the development of the theory of Kuhn–Tucker vectors in optimization theory.

For metallic materials the concept of *classical yield functionals* is then introduced. Their distinctive feature is that of having level sets all proportional. This property expresses the mechanical requirement that the elastic locus changes proportionally to the value of the nominal yield stress.

It is shown that a classical yield functional can always be represented as the composition of a non-constant Young function and of a *canonical yield functional.* Such a functional has been first defined in [11] as the polar of the dissipation functional.

Canonical yield functionals turn out to be inherently non-differentiable at the origin. A direct application of the chain-rule quoted above shows however that regularization at the origin can always be achieved by means of a suitable smoothing Young function.

The theory developed in the paper covers a broad range of applications from associated elasto-plasticity to unilateral problems in mechanics, no-tension or no-compression materials, frictionless contact and so on.

As final remark it is pointed out that the definitions and the results here presented are amenable to an extension to more general models of inelastic behaviours in which the evolution relation is defined in product spaces of state variables.

2. SOME PRELIMINARY RESULTS

We recall some basic definitions and properties of convex analysis and of potential theory for monotone multi-valued operators which will be useful in the sequel.

2.1 A background of convex analysis

A comprehensive treatment of the subject can be found in [13–16].

Let (X, X') be a pair of locally convex topological vector spaces (l.c.t.v.s.) placed in separating duality by a bilinear form $\langle .\,, . \rangle$.

Let us consider a **convex** functional $f: X \mapsto \Re \cup \{+\infty\}$ with a nonempty effective domain, which is the convex set on which it assumes finite values:

$$\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}.$$

The Fenchel's **conjugate** $f^*: X' \mapsto \Re \cup \{+\infty\}$ of f is defined as:

$$f^*(x^*) = \sup_{y \in X} \{\langle x^*, y \rangle - f(y)\},$$

so that the following Fenchel's inequality holds:

$$f(y) + f^*(x^*) \geq \langle x^*, y \rangle, \qquad \forall y \in X, \qquad \forall x^* \in X'.$$

A relevant example of conjugate functionals is provided by the **indicator** functional of a convex set C:

$$\sqcup_C(x) = \begin{cases} 0, & \text{if } x \in C \\ +\infty, & \text{otherwise,} \end{cases}$$

and by its **support** functional:

$$\sqcup_C^*(x^*) = \sup_{x \in C} \langle x^*, x \rangle.$$

The pairs (x, x^*) for which Fenchel's inequality holds as an equality are said to be conjugate and are related by the **subdifferential** multi-valued operator ∂, defined by:

$$x^* \in \partial f(x) \overset{\text{def}}{\Leftrightarrow} f(z) - f(x) \geq \langle x^*, z - x \rangle \qquad \forall z \in X.$$

If the closed convex subdifferential set $\partial f(x)$ is nonempty, the functional f is said to be subdifferentiable at x and each $x^* \in \partial f(x)$ is called a subgradient of f at x.

For any convex function f, it turns out to be:

$$x^* \in \partial f(x) \Rightarrow x \in \partial f^*(x^*),$$

while the converse implication holds if f is lower-semicontinuous (l.s.c.) [14]:

$$\liminf_{z \to x} f(z) \geq f(x) \qquad \forall x \in X.$$

In particular we recall that the subdifferential of the indicator functional of a convex set C at a point $x \in C$ coincides with the **normal cone** to C at x:

$$\partial \sqcup_C(x) = N_C(x) \overset{\text{def}}{=} \{x^* \in X' : \langle x^*, y - x \rangle \leq 0 \qquad \forall y \in X\}.$$

A functional $f: X \mapsto \Re \cup \{+\infty\}$ is said to be **sublinear** if it is positively homogeneous and subadditive; in formulas:

$$\begin{cases} f(\alpha x) = \alpha f(x) & \forall \alpha \geq 0 \\ f(x_1) + f(x_2) \geq f(x_1 + x_2) & \forall x_1, x_2 \in X. \end{cases}$$

Under these conditions f is convex and its epigraph is a convex cone in $X \times \Re \cup \{+\infty\}$; the cone is closed if f is l.s.c.

2.2 Monotone multi-valued maps

To make the paper reasonably self-contained we briefly report, without proofs, the basic results of the potential theory for monotone multi-valued maps developed in [12].

A **graph** G is a nonempty subset of the product space: $G \subseteq X \times X'$. Two multi-valued maps are naturally associated with a graph G: the right map $M: X \mapsto X'$, and the left map $M^{-1}: X' \mapsto X$, defined by:

$$M(x) = \{x^* \in X' \mid (x, x^*) \in G\} \subseteq X',$$
$$M^{-1}(x^*) = \{x \in X \mid (x, x^*) \in G\} \subseteq X.$$

A graph $G \subseteq X \times X'$ is said to be **monotone** if:

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geq 0 \qquad \forall (x_i, x_i^*) \in G; \qquad i = 1, 2.$$

The maps M and M^{-1} themselves are then said to be monotone.

A graph G is **cyclically monotone** [16] if it results:

$$\sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0 \Leftrightarrow \sum_{i=0}^{n} \langle x_{i+1}^* - x_i^*, x_{i+1} \rangle \geq 0,$$

for every $(x_i, x_i^*) \in G$ with $i = 1, \ldots, n, n+1 \equiv 0$.

A graph $G_{ex} \subseteq X \times X'$ is an **extension** of a graph G if $G \subseteq G_{ex}$; the extension is **proper** if $G_{ex} \neq G$. Whenever G and G_{ex} are monotone, G_{ex} will be called a **monotone extension** of G.

A graph $G \subseteq X \times X'$ is said to be **maximal** in a given family if it is not properly included in any other graph of the family. The maps M and M^{-1} are then said to be maximal.

It has been proved in [12] that for a monotone multi-valued map $M: X \mapsto X'$ the integral along line segments and polylines in its domain can be unambiguously defined.

Given an oriented line segment with extremes $a, b \in \operatorname{dom} M$ and parametric representation $\hat{x}(t) = a + th$, where $h = b - a$ and $0 \leq t \leq 1$, the **line** integral of M along the segment $a, b \in \operatorname{dom} M$ is then well-defined by:

$$\int_a^b \langle M(x), \mathrm{d}x \rangle = \int_0^1 \langle M(\hat{x}(t)), h \rangle \,\mathrm{d}t \stackrel{\text{def}}{=} \int_0^1 \langle \bar{M}(\hat{x}(t)), h \rangle \,\mathrm{d}t,$$

the last integral being independent upon the choice of $\bar{M}(\hat{x}(t)) \in M(\hat{x}(t))$ [12].

A monotone multi-valued map $M: X \mapsto X'$ is said to be **conservative** if:

$$\oint_\pi \langle M(x), \mathrm{d}x \rangle = 0,$$

for every closed polyline $\pi \subseteq \operatorname{dom} M$.

A potential functional $f: X \mapsto \Re \cup \{+\infty\}$ can be associated with a conservative map $M: X \mapsto X'$ having a convex domain. It is defined on $\operatorname{dom} M$ by the formula:

$$f(x) - f(x_o) \stackrel{\text{def}}{=} \int_{x_o}^x \langle M(z), \mathrm{d}z \rangle = \int_0^1 \langle M(\hat{x}(t)), h \rangle \,\mathrm{d}t,$$

and it is assumed to be $+\infty$ outside $\operatorname{dom} M$. It has been proved in [12] that the potential $f(x)$ turns out to be the restriction to $\operatorname{dom} M$ of a proper convex functional and that cyclical monotonicity and conservativity are equivalent properties for a map M with a convex domain.

3. THE CONSTITUTIVE MODEL

It is nowadays widely accepted that internal variable theories provide a suitable framework for the formulation of constitutive relations of inelastic behaviours of continuous media [4, 5, 10, 13, 17].

In an internal variable theory the central role is played by the evolution relation between the rates of internal variables and the conjugate thermodynamic forces. The physical meaning of internal parameters depends on the particular constitutive model under consideration.

In a recent paper by Eve *et al.* [11], the following question has been addressed: which are the minimal assumptions to be made on the evolution relation in order to derive an associated model of inelastic behaviour?

The basic property assumed in [11] is that of maximality in the class of the evolution relations satisfying Hill's principle of maximum dissipation.

The same topic will be dealt with in this paper starting from a different and, in authors' opinion, more attractive approach. The key idea is to envisage a suitable generalization of the characteristic properties of the one-dimensional rigid–perfectly plastic law relating the uniaxial tension σ to the plastic strain rate $\dot{\varepsilon}_p$.

The characteristic features of the multi-valued map $\sigma \in M(\dot{\varepsilon}_p)$ can be summarized as follows:

(a) the map M is monotone;

(b) the entire image set of M coincides with its value at zero;

(c) the map M is maximal in the class of the maps satisfying properties (a) and (b).

In a general theory of inelastic behaviour the internal variables α and the dual thermodynamic forces χ are supposed to belong to a pair of dual l.c.t.v.s. X and X'. The multi-valued map M will now relate a set of thermodynamic forces to a rate of internal variables:

$$\chi \in M(\dot{\alpha}).$$

The family of maps M which meet properties (a), (b) above is named according to the following:

DEFINITION 3.1. *Step-like maps.* A map $M : X \mapsto X'$ is said to be **step-like** if:

(i) $G = \text{graph}\,(M)$ is monotone,

(ii) $M(0) = \text{Im}\, M$ (step property). △

The class of maps which fulfill also property (c) is then introduced:

DEFINITION 3.2. *Step-shaped maps.* A map $M : X \mapsto X'$ is said to be **step-shaped** if it is maximal in the family of step-like maps. △

An example of step-like maps is sketched in Fig. 1(a). The simple one-dimensional example of Fig. 1(b) stresses the common properties shared by step-shaped maps and rigid–perfectly plastic laws.

We will show that Definition 3.2 can be properly assumed as starting point of an internal variable formulation of associated inelastic models.

To this end a detailed analysis of the basic properties of step-shaped maps has been performed. The relevant proofs are collected in Appendix A to make the exposition more plain.

The properties which play a basic role in the description of the mechanical model are proved in Lemma A.1 and Theorems A.2, A.3 and are summarized hereafter. These properties allow us to state that the following alternative formulations of the evolution relation are equivalent:

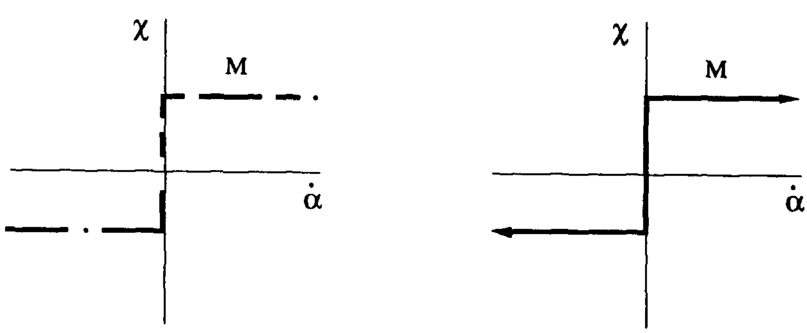
(i) $\chi \in M(\dot{\alpha})$ with M step-shaped,

(ii) $\chi \in \partial D(\dot{\alpha})$ with D l.s.c. and sublinear,

(iii) $\dot{\alpha} \in N_K(\chi)$ with K nonempty, closed, convex,

where:

$$K = \text{Im}\, M = M(0) = \{\bar{\chi} \in X' : D(\dot{\alpha}) \geq \langle \bar{\chi}, \dot{\alpha} \rangle \quad \forall \dot{\alpha} \in X\},$$



a) Step-like map b) Step-shaped map

Fig. 1. A one-dimensional example of step-like and step-shaped maps.

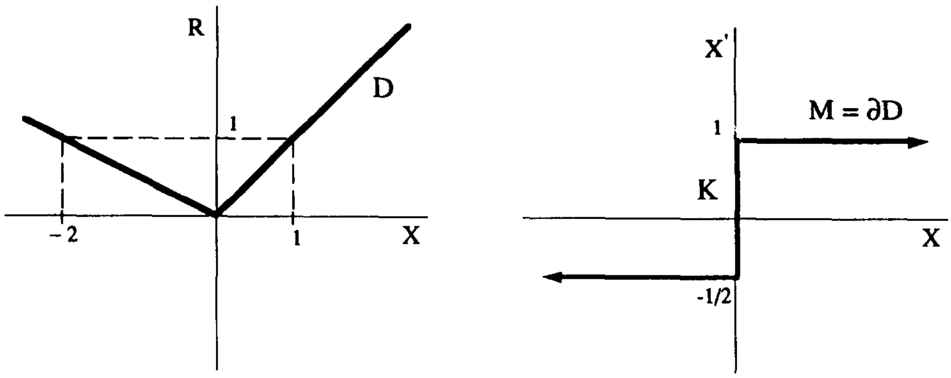


Fig. 2. Sublinear functional and related step-shaped map.

and

$$M = \partial D; \qquad D(\dot{\alpha}) = \sqcup_K^*(\dot{\alpha}) = \sup_{\bar{\chi} \in K} \langle \bar{\chi}, \dot{\alpha} \rangle.$$

A simple example is sketched in Fig. 2.

In the model of inelastic behaviour the convex set K is the locus of admissible thermodynamic forces and its boundary defines the yield surface. The law $\dot{\alpha} \in N_K(\chi)$ embodies the normality rule which is characteristic of associated inelastic behaviours.

In fact non-vanishing rates of internal variables must necessarily correspond to thermodynamic forces belonging to the boundary of K since, as well known, it results:

$$\begin{cases} \chi \in \operatorname{int} K \Leftrightarrow N_K(\chi) = \{0\}, \\ \chi \in \operatorname{bnd} K \Leftrightarrow \exists \dot{\alpha} \in X - \{0\} : \dot{\alpha} \in N_K(\chi). \end{cases}$$

The support functional D of the set K provides the value of the dissipation associated with a given rate of internal variables.

The second principle of thermodynamics requires that the dissipation must be non-negative. This is accomplished by assuming that the null thermodynamic force is admissible:

$$0 \in K \Leftrightarrow D(\dot{\alpha}) \geq 0 \qquad \forall \dot{\alpha} \in X.$$

An example is provided in Fig. 3 with reference to no-tension materials with limited strength in compression, a model often used for concrete and masonry.

A mechanical model characterized by a strict dissipative behaviour requires the more stringent assumption that the null thermodynamic force belongs to the interior of the set K:

$$0 \in \operatorname{int} K \Leftrightarrow D(\dot{\alpha}) > 0 \qquad \forall \dot{\alpha} \in X - \{0\}.$$

The two equivalences above are proved in Theorem B.1.

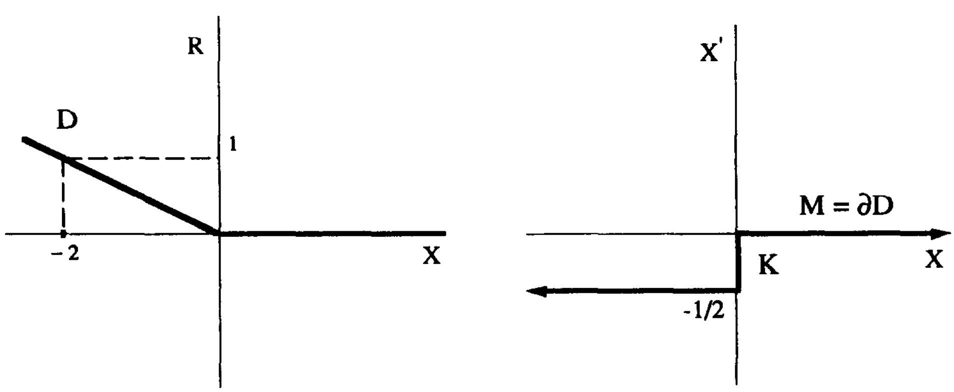


Fig. 3. An example of non-strict dissipative behaviour.

Let us now show that a theory of inelastic behaviour can be developed by postulating, as alternative starting points, either the validity of Hill's principle of maximum dissipation or the step-shapedness of the constitutive operator.

The statement of Hill's principle in terms of the constitutive operator has been introduced in [11] and is based on the following:

DEFINITION 3.3. *Responsive maps.* A map $M:X \mapsto X'$ is said to be **responsive** if:

$$\begin{cases} (\dot{\alpha}_1, \chi_1) \in G \\ (\dot{\alpha}_2, \chi_2) \in G \end{cases} \Rightarrow \begin{cases} \langle \chi_1, \dot{\alpha}_1 \rangle \geq \langle \chi_2, \dot{\alpha}_1 \rangle \\ \langle \chi_2, \dot{\alpha}_2 \rangle \geq \langle \chi_1, \dot{\alpha}_2 \rangle \end{cases},$$

where $G = \text{graph}\,(M)$. $\triangle$

It can be easily verified that the definition above is equivalent to the following property:

$$(\dot{\alpha}, \chi) \in G = \text{graph}\,(M) = \Rightarrow \langle \chi, \dot{\alpha} \rangle = \max\{\langle \bar{\chi}, \dot{\alpha} \rangle \; : \; \bar{\chi} \in \text{Im}\, M\},$$

which is the formal statement of Hill's principle.

The constitutive map M will be fully characterized by Hill's principle if the converse of the implication above holds true that is any pair $(\dot{\alpha}, \chi)$ which provides a maximum value of the dissipation must belong to the graph of M. This property is expressed in mathematical terms by imposing that the graph of the map M must be maximal in the class of the responsive ones.

The equivalence of Hill's principle and of the step-shapedness of the constitutive operator follows then from Remark A.9 and Theorem A.10 which provide the proof that step-shapedness and maximal responsiveness are equivalent properties for a multi-valued map.

Incidentally we notice that the original definition of responsiveness given in [11] includes the further assumption that $(0, 0) \in G$, that is $0 \in K$. This hypothesis is claimed by the requirement of the second principle of thermodynamics but plays no role in the formalization of Hill's principle.

Two further significant results are proved in Appendix A. According to the first one (see Remark A.6) step-like and responsive maps are monotone and conservative. The second result (see Remarks A.9 and A.11) shows that the maximality of step-like and of responsive maps in the corresponding families is equivalent to their maximality in the broader class of monotone maps.

The hierarchic nesting of the families of the maps introduced above is symbolically represented in Fig. 4.

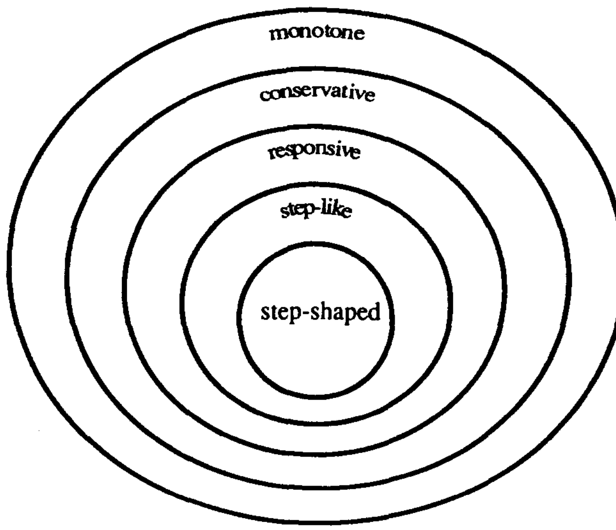


Fig. 4. Nested families of monotone maps.

A summary of the main results proved in Appendix A is reported hereafter:

$$\text{step-like} \Rightarrow \text{responsive} \Rightarrow \text{monotone and conservative},$$

$$\text{step-shaped} \Leftrightarrow \text{maximal responsive},$$

$$\text{step-shaped} \Leftrightarrow \text{step-like and maximal monotone},$$

$$\text{maximal responsive} \Leftrightarrow \text{responsive and maximal monotone}.$$

The form of the evolution relation considered in this paper is similar to the one adopted by Martin [6, 7] in the context of elastoplasticity. In this model an evolutive relation is assumed between the rates of internal variables and the dual thermodynamic forces; the hardening behaviour is modelled by means of an appropriate form of the Helmholtz free energy.

Starting from the contribution by Halphen and Nguyen [4] a number of different models of associated elastoplasticity have been proposed in literature [1–3, 9, 10].

The theoretical treatment developed in [4] is based on the following idea: a generalized associated behaviour is formulated by means of an evolution relation stated with reference to an elastic domain which is defined in the product space of the pairs (σ, χ) of stresses and thermodynamic forces χ.

The normality rule is expressed in terms of the corresponding rates $(\dot{\varepsilon}_p, \dot{\alpha})$ of plastic strains and internal variables. The hardening behaviour is accounted for by considering, depending on the value of the thermodynamic force, different sections of the elastic domain projected into the stress space.

It can be shown [18] that Martin's model can be recovered by assuming a cylindrical shape for the generalized elastic domain proposed in [4].

The definitions and the results contributed in the present paper can be immediately extended to these more general models of inelastic behaviour by properly identifying the dual parameters involved in the evolution relation.

4. YIELD FUNCTIONALS

Although the possibility of assigning an elastic locus as the image set of a multi-valued map could appear intriguing, when dealing with applications of the theory it is usually more convenient to define it as the level set of a convex functional $f: X' \mapsto \Re \cup \{+\infty\}$. The choice of this functional depends upon the particular yield criterion adopted for the material.

Assuming that the minimum value of f is not zero, the elastic domain K is defined by:

$$K = \{\chi \in X' \; : \; f(\chi) \leq 0\}.$$

The normality rule can then be re-formulated in terms of the functional f. To this end we preliminarily notice that:

$$\sqcup_K(\chi) = \sqcup_{\Re^-}[f(\chi)],$$

where $\Re^-$ is the convex cone of non-positive reals.

Assuming f continuous on K, a chain-rule of subdifferential calculus, recently contributed in [19], yields:

$$\partial \sqcup_K(\chi) = \partial(\sqcup_{\Re^-} \circ f)(\chi) = \partial \sqcup_{\Re^-}[f(\chi)] \, \partial f(\chi) \qquad \forall \chi \in K.$$

The normality rule can then be re-written in the following equivalent forms:

(i) $\dot{\alpha} \in N_K(\chi) = \partial \sqcup_K(\chi)$,

(ii) $\dot{\alpha} \in \partial \sqcup_{\Re^-}[f(\chi)] \, \partial f(\chi) = N_{\Re^-}[f(\chi)] \, \partial f(\chi)$,

(iii) $\dot{\alpha} \in \lambda \, \partial f(\chi)$; $\lambda \in N_{\Re^-}[f(\chi)]$.

The last inclusion shows that λ is zero when $f(\chi) < 0$ and non-negative when $f(\chi) = 0$, so that we can write:

$$\dot{\alpha} \in \lambda\, \partial f(\chi), \qquad \lambda \geq 0, \qquad f(\chi) \leq 0, \qquad \lambda f(\chi) = 0,$$

which is the explicit expression of the normality rule in terms of a complementarity relation.

The functional f is often conveniently generated by means of a finite family of differentiable yield modes f_i according to the formula:

$$f(\chi) = \sup_{i=1,\dots,n} f_i(\chi),$$

so that the elastic domain turns out to be the intersection of a finite number of convex domains:

$$K = \bigcap_{i=1}^{n} K_i \quad \text{with} \quad K_i = \{\chi \in X' \ : \ f_i(\chi) \leq 0\},$$

and the normality rule can be written as:

$$\dot{\alpha} \in \sum_{i \in I} \lambda_i \, \mathrm{d}f_i(\chi), \quad \text{with} \quad \begin{cases} \lambda_i \geq 0, \\ f_i(\chi) \leq 0, \\ \lambda_i f_i(\chi) = 0, \end{cases}$$

where I is the numerical set of the indices associated with the active modes (see Fig. 5).

For metallic materials the yield modes are usually expressed in the form:

$$f_i(\chi) = h_i(\chi) - a,$$

where the parameter a is related to the nominal yield stress.

A familiar example is provided by the classical yield criterions proposed by von Mises and Tresca [20]. The deviatoric part of the stress tensor σ plays here the role of thermodynamic force, that is $\chi = \operatorname{dev} \sigma = \sigma - \frac{1}{3}(\operatorname{tr} \sigma)I$.

Von Mises criterion is obtained by setting:

$$f(\chi) = \|\chi\| - a \quad \text{with} \quad a = \sqrt{2/3}\, \sigma_y,$$

where $\|\cdot\|$ denotes the norm of a tensor and σ_y is the nominal yield stress of the material determined in a uniaxial test.

Tresca criterion follows in turn by considering the yield functional:

$$f(\chi) = \sup\{\chi_i - \chi_j - a; i, j = 1, 2, 3\},$$

where χ_i are the principal values of the deviatoric stress χ and the parameter a is the nominal yield stress of the material.

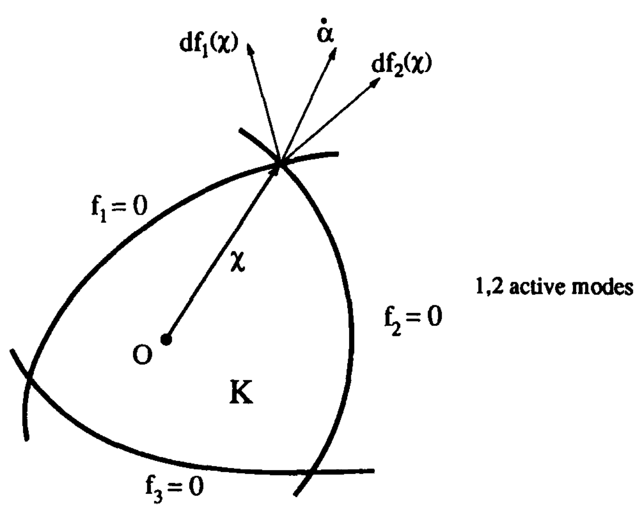


Fig. 5. An inelastic model with three yield modes.

Experimental tests for metallic materials sharing the same qualitative yield properties show that the elastic loci corresponding to different nominal yield stresses can all be expressed as positive scalar multiples of a single set. This mechanical requirement can be fulfilled by endowing this class of materials with a yield criterion governed by a functional whose level sets are all proportional.

It is then natural to put the following question: which convex functionals do have the property that their level sets are all proportional?

To answer this question we recall some preliminary results.

A *canonical yield functional* $Y: X' \mapsto \Re \cup \{+\infty\}$ can be associated with any elastic locus K containing the origin, according to the Minkowski formula:

$$Y(\chi) = \inf\{\gamma \geq 0 \ : \ \chi \in \gamma K\}.$$

Its characteristic property is that the unitary level set yields back the assigned elastic locus, that is:

$$K = \{\chi \in X' : Y(\chi) \leq 1\}.$$

The functional Y turns out to be sublinear, l.s.c. and non-negative. In fact, denoting by K° the closed convex set polar of K:

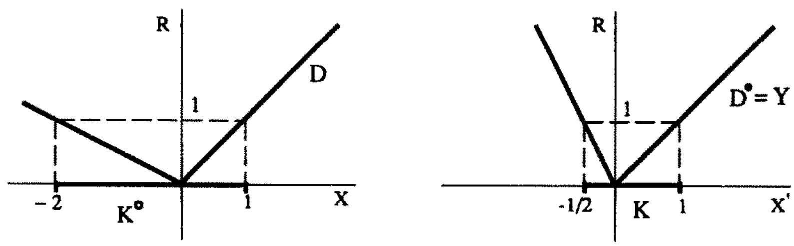
$$K^{\circ} = \{\dot{\alpha} \in X : \langle \chi, \dot{\alpha} \rangle \leq 1, \forall \chi \in K\},$$

it turns out to be

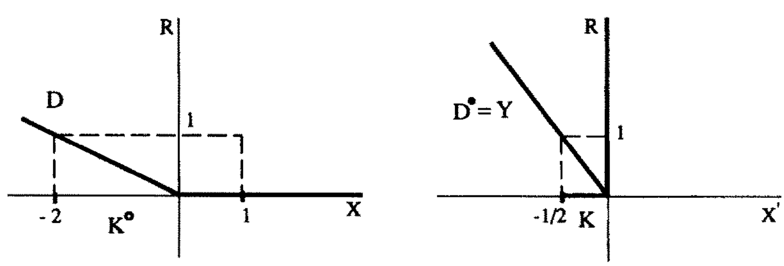
$$Y(\chi) = \sqcup^{*}_{K^{\circ}}(\chi) = \sup_{\dot{\alpha} \in K^{\circ}} \langle \chi, \dot{\alpha} \rangle.$$

The dissipation and the canonical yield functionals are then support functionals of polar sets [see Figs. 6(a) and 6(b)] and by Theorem B.3 the following inequalities hold:

$$D^{*}(\chi) + D(\dot{\alpha}) \geq D(\dot{\alpha}) Y(\chi) \geq \langle \chi, \dot{\alpha} \rangle, \qquad \forall \dot{\alpha} \in X, \qquad \forall \chi \in X'.$$



a) Strict dissipative behaviour



b) Non-strict dissipative behaviour

Fig. 6. Canonical yield functionals for dissipative behaviours.

When the pair $(\dot{\alpha}, \chi)$ satisfies the evolution relation, the expression above turns out to be an equality. Consequently $D(\dot{\alpha})$ will vanish when $Y(\chi) < 1$. According to Theorem B.2 the assumption of a strict dissipative behaviour ensures that $D(\dot{\alpha})$ vanishes if and only if $\chi \in \operatorname{int} K$. Under the less stringent assumption of a non-strict dissipative behaviour a null dissipation can in general occur also when $\chi \in \operatorname{bnd} K$.

We further recall that a **Young** function is an extended real-valued function on $\Re \cup \{+\infty\}$ which is monotone, l.s.c., convex and non-negative with $g(0) = 0$.

We are now ready to answer properly the question put forth above, by invoking a result proved in [16]: the level sets of a yield functional $\Pi: X' \mapsto \Re \cup \{+\infty\}$ are all proportional if and only if Π can be represented as the composition of a canonical yield functional Y and of a non-constant Young function g:

$$\Pi = g \circ Y.$$

The yield functionals which meet this property will be called *classical yield functionals.*

To show that the level sets of Π are all proportional we notice that, given any $a > 0$, the a-level set of Π is given by:

$$\begin{aligned} K_a &= \{\chi \in X' : \Pi(\chi) \le a\} = \{\chi \in X' : (g \circ Y)(\chi) \le a\} \\ &= \{\chi \in X' : Y(\chi) \le g^{-1}(a)\} = [g^{-1}(a)]K. \end{aligned}$$

Under suitable assumptions the normality rule to the set K_a can be expressed in terms of the classical yield functional Π.

Assuming that 0 belongs to the interior of K and that $\operatorname{dom} g = [0, +\infty[$, the domain of Π is the whole space X' and then $K_a \subset \operatorname{dom} \Pi$. Since $a > 0 = \inf \Pi$ the chain-rule of subdifferential calculus contributed in [19] can be applied to obtain the following relation:

$$\partial\Pi(\chi) = \partial g[Y(\chi)]\, \partial Y(\zeta).$$

Hence we can write:

$$\dot{\alpha} \in N_{K_a}(\chi) = \partial \sqcup_{\Re^-}[\Pi(\chi) - a]\, \partial\Pi(\chi),$$

or equivalently:

$$\dot{\alpha} \in \lambda\, \partial\Pi(\chi), \qquad \lambda \ge 0, \qquad \Pi(\chi) \le a, \qquad \lambda[\Pi(\chi) - a] = 0.$$

It has to be pointed out that a canonical yield functional, being sublinear and non-negative, is inherently non-differentiable at the origin. However, when the origin is the only point at which Y is non-differentiable, the associated classical yield functional Π turns out to be differentiable at the origin if and only if $\mathrm{d}g(0) = 0$.

The result follows by observing that the set $\partial\Pi(\chi)$ is given by the product of the interval $\partial g[Y(\chi)]$ and of the set $\partial Y(\chi)$. Hence $\partial\Pi(0) = 0$ if and only if $\partial g[Y(0)] = \mathrm{d}g(0) = 0$.

Yield functionals are often choosen to be positively homogeneous of degree m, with $m \ge 1$ to ensure convexity. A representation theorem [16] states that the most general form of such a functional is the following:

$$\Pi(\chi) = \frac{1}{m}[Y(\chi)]^m,$$

which turns out to be a classical yield functional with $g(\cdot) = (1/m)(\cdot)^m$. It is apparent that positively homogeneous yield functionals of degree $m > 1$ are differentiable at the origin. In fact, being $\mathrm{d}g(\cdot) = (\cdot)^{m-1}$, it results:

$$\begin{cases} \mathrm{d}g(0) = 0 & \text{if } m > 1 \\ \mathrm{d}g(0) = 1 & \text{if } m = 1 \\ \mathrm{d}g(0) = +\infty & \text{if } m < 1. \end{cases}$$

A familiar example of a differentiable classical yield functional is provided by the expression of

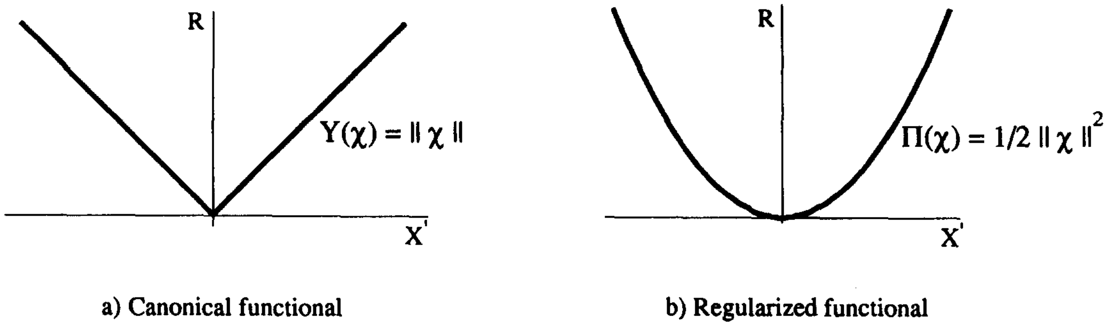


Fig. 7. Yield functionals for von Mises criterion.

the von Mises criterion in terms of the functional $\Pi(\chi) = \frac{1}{2}\|\chi\|^2$ (see Fig. 7). This expression can be immediately derived from the representation formula of Π as the composition of the parabolic Young function $\frac{1}{2}(\cdot)^2$ and the canonical yield functional $Y(\chi) = \|\chi\|$.

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APPENDIX A

We present here a detailed development of the theory of step-shaped maps which has been referred to in the previous sections.

Given a family of reponsive (step-like) maps, we shall say that the graph of a monotone map M is maximal responsive (step-like) if graph (M) is maximal in the corresponding family.

We preliminarily prove the following:

LEMMA A.1. *The image of a step-shaped map* $M: X \mapsto X'$ *is the nonempty closed convex set:*

$$K \overset{\text{def}}{=} \{x^* \in X' : \langle x^*, y\rangle \leq \langle y^*, y\rangle \quad \forall (y, y^*) \in G\},$$

where $G = \text{graph}\,(M)$.

PROOF. By definition of step-like maps, any $(x, x^*) \in G$ is such that $x^* \in \operatorname{Im} M = M(0)$. Moreover the monotonicity of G implies that:

$$\langle x^* - y^*, -y \rangle \geq 0 \qquad \forall (y, y^*) \in G,$$

and then $M(0) = \operatorname{Im} M \subseteq K$.

To prove the opposite inclusion, let us consider an arbitrary $x^* \in K$. Adding the pair $(0, x^*)$ to G we obtain a monotone extension $G_{ex} = \operatorname{graph}(M_{ex})$ of G such that the map M_{ex} is step-like. Since M is step-shaped its graph can not be properly included in the graph of any other step-like map; hence $G_{ex} = G$ and $x^* \in M(0)$. □

The next two theorems show that a map M is step-shaped if and only if it is the subdifferential of the support functional of a closed convex set, that is:

$$M \text{ step-shaped} \Leftrightarrow M = \partial \sqcup_K^* \quad \text{with} \quad K = \operatorname{Im} M.$$

THEOREM A.2. *The subdifferential of a lower-semicontinuous sublinear functional* $f: X \mapsto \Re \cup \{+\infty\}$ *is a step-shaped map.*

PROOF. Let f be a l.s.c. sublinear functional on X. A classical result of convex analysis [15] ensures that the closed convex set:

$$K = \partial f(0) = \{x^* \in X' \ : \ f(x) \geq \langle x^*, x \rangle \qquad \forall x \in X\},$$

is nonempty and that f is the support functional of K:

$$f(x) = \sqcup_K^*(x).$$

Since f is l.s.c. the following equivalence holds:

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*) = N_K(x^*),$$

so that the normal cone to K at x^* is nonempty. It follows that $x^* \in K$ and:

$$\operatorname{Im} \partial f \subseteq K = \partial f(0).$$

Since the opposite inclusion holds trivially, the subdifferential ∂f is a step-like map. To prove the maximality of ∂f in the family of step-like maps let us consider the graph G_{ex} obtained by adding to $G = \operatorname{graph}(\partial f)$ a pair $(x, x^*) \in X \times X'$. The maximality property amounts to stating that, if G_{ex} turns out to be step-like, then (x, x^*) must belong to G. In formula:

$$(x, x^*) \in X \times X' : \begin{cases} \text{(i)} \ \langle x^* - y^*, x - y \rangle \geq 0 & \forall (y, y^*) \in \operatorname{graph}(\partial f) \\ \text{(ii)} \ x^* \in \partial f(0) = K \end{cases} \Rightarrow (x, x^*) \in \operatorname{graph}(\partial f).$$

The monotonicity property (i) implies that:

$$\langle y^* - x^*, x \rangle \leq 0 \qquad \forall (0, y^*) \in \operatorname{graph}(\partial f)$$

and the step property (ii) ensures that the normal cone $N_K(x^*)$ is nonempty. We can then conclude that $x \in N_K(x^*)$ and the result is proved. □

THEOREM A.3. *A step-shaped map* $M: X \mapsto X'$ *is the subdifferential of a l.s.c. sublinear functional* $f: X \mapsto \Re \cup \{+\infty\}$.

PROOF. By Lemma A.1, $K = M(0) = \operatorname{Im} M$ is a nonempty closed convex set and, by Theorem A.2, the subdifferential of the support functional $f(x) = \sqcup_K^*(x)$ is a step-shaped map. Since:

$$x^* \in M(x) \Rightarrow x^* \in M(0) = K,$$

the set $N_K(x^*)$ is nonempty. By monotonicity of M we have:

$$\langle y^* - x^*, x \rangle \leq 0 \qquad \forall y^* \in K \Leftrightarrow x \in N_K(x^*) \Leftrightarrow x^* \in \partial f(x),$$

so that $\operatorname{graph}(M) \subseteq \operatorname{graph}(\partial f)$. The opposite inclusion follows by the maximality of M in the class of step-like maps. □

We present hereafter a number of results concerning the relations existing among step-like, responsive and monotone conservative maps. Further we provide an alternative proof of Theorem A.3 based upon the potential theory of monotone operators recently developed in [12] and briefly recalled in Section 2.2. Finally the analysis allows us to establish the equivalence of step-shapedness and maximal responsiveness for a multi-valued map.

LEMMA A.4. *A* **step-like** *map* $M: X \mapsto X'$ *is* ***responsive***.

PROOF. Setting $G = \operatorname{graph}(M)$ we have, by monotonicity:

$$(x, x^*) \in G \Rightarrow \{x^* - y^*, x\rangle \geq 0 \qquad \forall y^* \in M(0) = \operatorname{Im} M \Leftrightarrow \langle x^*, x \rangle = \max\{\langle y^*, x \rangle : y^* \in \operatorname{Im} \mathrm{M}\},$$

and then M is responsive. □

An important feature of responsive maps is illustrated hereafter.

LEMMA A.5. *A* **responsive** *map* $M: X \mapsto X'$ *is* **cyclically monotone**.

PROOF. Let M be a responsive map and G the associated graph. For every choice $(x_i, x_i^*) \in G$, with $i = 0, \cdots n, n+1 \equiv 0$, we have:

$$\begin{cases} \langle x_1^* - x_o^*, x_1 \rangle \geq 0 \\ \langle x_2^* - x_1^*, x_2 \rangle \geq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \langle x_{i+1}^* - x_i^*, x_{i+1} \rangle \geq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \langle x_o^* - x_n^*, x_o \rangle \geq 0, \end{cases}$$

and summing up:

$$\sum_{i=0}^{n} \langle x_{i+1}^* - x_i^*, x_{i+1} \rangle \geq 0,$$

which is the characteristic property of a cyclically monotone graph. □

REMARK A.6. The two lemmas above imply that any step-like map M is cyclically monotone. Since cyclic monotonicity is a property of the graph of M and implies conservativity [12], it turns out that both M and M^{-1} are conservative. △

Let us now present an alternative proof of Theorem A.3, based upon the results of the potential theory of monotone multi-valued maps [12], which provide an explicit formula for the potential of the map M^{-1}.

ALTERNATIVE PROOF OF THEOREM A.3. *A step-shaped map $M: X \mapsto X'$ is the subdifferential of a l.s.c. sublinear functional $f: X \mapsto \Re \cup \{+\infty\}$.*

PROOF. Let us first notice that, by Remark A.6, M^{-1} is conservative and that, by Lemma A.1, $K = \operatorname{dom} M^{-1} = \operatorname{Im} M$ is closed and convex. Given a line segment $[x_o^*, x^*]$ in K with parametric representation $z^*(t) = x_o^* + th^*$, where $h^* = x^* - x_o^*$ and $t \in [0, 1]$, the potential associated with the inverse map M^{-1} is given by the following expression:

$$p(x^*) - p(x_o^*) = \int_{x_o^*}^{x^*} \langle M^{-1}(z^*(t)), h^* \rangle \, dt \qquad \forall x^* \in \operatorname{dom} M^{-1}.$$

Assuming that $p(x_o^*) = 0$ and recalling that the integral above does not depend upon the choice of $\bar{M}^{-1}(z^*(t)) \in M^{-1}(z^*(t))$, we can set $\bar{M}^{-1}(z^*(t)) = 0$. Hence, the potential p turns out to be zero on dom M^{-1} and, by definition, $+\infty$ outside dom M^{-1} so that:

$$p(x^*) = \sqcup_K(x^*).$$

Denoting by f the conjugate of p we get:

$$f(x) = p^*(x) = \sqcup_K^*(x).$$

It remains to prove that $M = \partial f$. Since p is l.s.c., ∂f is the inverse map of ∂p and hence:

$$\operatorname{graph}(\partial p) = \operatorname{graph}(\partial f).$$

Being G included in the graph of the potential of M^{-1} [12], it results $G \subseteq \operatorname{graph}(\partial p) = \operatorname{graph}(\partial f)$. By Theorem A.2 the map ∂f is step-shaped and hence the maximality of G in the family of step-like maps implies that:

$$G = \operatorname{graph}(\partial f).$$

The equality above allows us to infer:

$$M = \partial f \quad \text{and} \quad M^{-1} = \partial f^*,$$

which proves the theorem. □

The following results show that a step-shaped map admits no proper extension in the broader class of monotone maps; i.e. step-shaped maps are step-like and maximal monotone.

LEMMA A.7. *The image of every monotone extension of a step-shaped map $M: X \mapsto X'$ coincides with* $\operatorname{Im} M = M(0)$.

PROOF. Let us consider a monotone extension $G_{ex} = \operatorname{graph}(M_{ex})$ obtained by adding a point $(x, x^*) \in X \times X'$ to $G = \operatorname{graph}(M)$.

If $x^* \notin \operatorname{Im} M = M(0)$ the pair (x, x^*) would not satisfy the monotonicity property for all $(y, y^*) \in G$ (see Lemma A.1). The lemma is thus proved. □

THEOREM A.8. *There is no proper monotone extension of a step-shaped map $M: X \mapsto X'$.*

PROOF. Let M_{ex} be a monotone extension of M and $G_{ex} = \operatorname{graph}(M_{ex})$ and $G = \operatorname{graph}(M)$ the corresponding graphs.

The map M_{ex}^{-1} turns out to be conservative [12] since it is a monotone extension of the conservative map M^{-1} having the same domain by Lemma A.7. Hence its potential coincides with the potential p of M^{-1} and:

$$G_{ex} \subseteq \operatorname{graph}(\partial p) = G.$$

Since the opposite inclusion is trivial we have $G_{ex} = G$ and the lemma is proved. □

REMARK A.9. From the theorem above we infer that a step-shaped map is maximal in the family of monotone maps and hence *a fortiori* in the subfamily of responsive maps. △

The converse result is the argument of the following:

THEOREM A.10. *A* maximal responsive *map* $M: X \mapsto X'$ *is* **step-shaped**.

PROOF. Setting $G = \text{graph}(M)$ the maximal responsiveness of M can be explicitly stated as follows:

$$(x, x^*) \in X \times X' \ : \ \langle x^*, x\rangle = \max\{\langle y^*, x\rangle \ : \ y^* \in \operatorname{Im} M\} \Rightarrow (x, x^*) \in G.$$

Hence, observing that trivially:

$$\langle z^* - y^*, 0\rangle = 0 \qquad \forall y^*, \ z^* \in \operatorname{Im} M,$$

we infer that $(0, z^*) \in G \quad \forall z^* \in \operatorname{Im} M$, i.e.

$$\operatorname{Im} M = M(0).$$

The map M is then step-like and, being maximal in the class of responsive maps, it will be also maximal in the subclass of step-like maps. □

REMARK A.11. By Theorems A.8 and A.10 it follows that a responsive map is maximal responsive if and only if it is responsive and maximal in the class of monotone maps. △

APPENDIX B

We collect here some results of convex analysis [16] which have been directly referred to in the previous sections.

THEOREM B.1. *Let K be a closed convex set with a nonempty interior. The support functional $D: X \mapsto \Re \cup \{+\infty\}$ of K is non-negative if and only if $0 \in K$ and turns out to be strictly positive if and only if $0 \in \operatorname{int} K$.*

PROOF. The first statement is proved as follows.

The implication:

$$0 \in K \ \Rightarrow \ D(x) \geq 0 \qquad \forall x \in X - \{0\},$$

follows the definition of $D(x) = \sup\{\langle x^*, x\rangle \ : \ x^* \in K\}$ by observing that the numerical set $\{\langle x^*, x\rangle : x^* \in K\}$ includes the zero.

The converse implication:

$$D(x) \geq 0 \qquad \forall x \in X - \{0\} \Rightarrow 0 \in K,$$

is proved *per absurdum*. In fact if $0 \notin K$ the Hahn–Banach theorem [21] would ensure that $\exists \bar{x} \in X - \{0\} : \langle x^*, \bar{x}\rangle < 0 \quad \forall x^* \in K$ and hence $D(\bar{x}) < 0$, in contrast with the assumption.

To prove the second statement we first consider the implication:

$$0 \in \operatorname{int} K < \Rightarrow D(x) > 0 \qquad \forall x \in X - \{0\}.$$

Now if $0 \in \operatorname{int} K$ there exists a neighbourhood $\mathcal{N}$ of the origin included in the set K. Being X' a locally convex linear topological space, the open set $\mathcal{N}$ will include a convex, balanced and absorbing open set I [21]. Since the spaces X and X' are in separating duality, it follows that:

$$\forall x \in X - \{0\} \qquad \exists x^* \in X' \ : \ \langle x^*, x\rangle \neq 0;$$

whence, being the set I balanced and absorbing, we infer that:

$$\forall x \in X - \{0\} \qquad \exists x^* \in K \ : \ \langle x^*, x\rangle > 0,$$

and then it results $D(x) > 0$.

The converse implication:

$$D(x) > 0 \qquad \forall x \in X - \{0\} \Rightarrow 0 \in \operatorname{int} K$$

is proved *per absurdum*.

In fact if $0 \notin \operatorname{int} K$ the Hahn–Banach theorem ensures that:

$$\exists x \in X - \{0\} : \langle x^*, x\rangle \leq 0 \qquad \forall x^* \in K.$$

Hence $D(x) \leq 0$ in contrast with the assumption. □

THEOREM B.2. *If $Y: X' \mapsto \Re \cup \{+\infty\}$ is the Minkowski functional of a nonempty closed convex set $K \subseteq X'$ and $0 \in \operatorname{int} K$, we have:*

$$\chi \in \operatorname{bnd} K \Leftrightarrow Y(\chi) = 1.$$

PROOF. First we prove that when $0 \in \operatorname{int} K$ every ray in X' meets the boundary of K at most at one point.

In fact let $\chi \in \operatorname{bnd} K$ and R be the ray passing through χ:

$$R = \{\chi_t \in X' \ : \ \chi_t = t\chi; t \geq 0\}.$$

The segment $S = \{\chi_t \in X' : \chi_t = t\chi; 0 \leq t < 1\}$ will then belong to the interior of the convex set K.

To prove uniqueness we observe that any intersection between R and $\operatorname{bnd} K$ other than χ should correspond to a value $t > 1$; this would in turn imply that $\chi \in \operatorname{int} K$, in contrast with the assumption.

Let us now prove the statement of the theorem. The value of the Minkowski functional Y is less than one at all interior points of K and is greater then unity outside K. Since $Y(t\chi)$ is a continuous function of $t>0$, it will be of unitary value for $t=1$. □

In order to prove the next theorem we recall some further results of convex analysis. Given a Young function g and a non-negative, sublinear, l.s.c. functional k, the conjugate of $g \circ k$ is the functional:

$$(g \circ k)^*(x^*) = g^*(k^\circ(x^*)).$$

By virtue of the polarity inequality:

$$k^\circ(x^*)k(x) \geq \langle x^*, x \rangle,$$

and of Fenchel's inequality, we get:

$$f^*(x^*) + f(x) = g^*(k^\circ(x^*)) + g(k(x)) \geq k^\circ(x^*)k(x) \geq \langle x^*, x \rangle \qquad \forall x \in X, \qquad \forall x^* \in X'.$$

It follows that if (x, x^*) is a conjugate pair with respect to f and f^* it turns out to be also a polar pair with respect to k and k°.

We can now prove the following:

THEOREM B.3. *Given a non-negative lower-semicontinuous sublinear functional* $D: X \mapsto \Re \cup \{+\infty\}$, *the following inequalities hold:*

$$D^*(x^*) + D(x) \geq D^\circ(x^*)D(x) \geq \langle x^*, x \rangle.$$

PROOF. Let us consider a Young function g. Being:

$$(gD)^*(x^*) = (g^*D^\circ)(x^*),$$

and:

$$(g^*D^\circ)(x^*) + (gD)(x) \geq D^\circ(x^*)D(x) \geq \langle x^*, x \rangle,$$

the result follows by choosing as g the identity function, so that $gD = D$ and $g^*D^\circ = (gD)^* = D^*$. □

APPENDIX C

A careful reading of the recent paper by Eve *et al.* [11] has revealed some flaws in the presentation of the subject and in the proof of some results.

We list here the main observations to be made with the spirit of stimulating the discussion and of contributing to a deeper understanding of the subject.

(i) Formula (3.13) of [11] reads $p = F(X)$, but clearly should be $p \in F(X)$ since there is no reason to rule out the absence of corners on the yield surface (notice that in our notation $p = \dot{\alpha}$ and $X = \chi$).

(ii) Figures 2.3, and 6.1c–6.4c are wrongly drawn; the subdifferential set is not convex!

(iii) The proof of the basic Lemma 5.1, which deals with the existence of the plastic multiplier is not completely satisfactory. In fact the assumptions that x belongs to the interior of $\operatorname{dom} g$ and that $g(x) > 0$ are essential for the validity of the result but play no role in the proof. Moreover there is no evidence that the maximum appearing at the end of the proof must be actually attained.