Computer Methods in Applied Mechanics and Engineering 109 (1993) 293–314
North-Holland

Variational principles for a class of finite step elastoplastic problems with non-linear mixed hardening

G. Romano, L. Rosati and F. Marotti de Sciarra
Dipartimento di Scienza delle Costruzioni, Facoltà di Ingegneria, Università di Napoli Federico II, Napoli, Italy

Received 20 July 1992
Revised manuscript received 12 February 1993

A class of elastoplastic relations with non-linear mixed hardening is addressed in the framework of an internal variable theory. The relevant finite step structural problem, resulting from the time integration of the constitutive laws according to a backward difference scheme, is then formulated in a geometrically linear range. Convex analysis and potential theory for monotone multi-valued operators are shown to provide the rationale for the development of a consistent variational theory. As a special result, a convex minimum principle in the final values of displacements, plastic strains and plastic parameters is formulated and a critical comparison with an analogous result proposed in the literature is presented.

1. Introduction

In recent papers [1–5], Maier and co-workers have addressed the formulation of variational principles for finite step solutions of an elastoplastic structural problem with non-linear mixed hardening.

In the constitutive model assumed in [1–4], the yield condition is expressed by imposing that the effective stresses are bounded by limit values which are functions of suitable plastic parameters. These parameters provide a measure of the accumulated plastic strains and their time derivatives play the role of plastic multipliers. The effective stresses are assumed to be convex, positively homogeneous and twice differentiable functions of their arguments, while the reference stresses, which account for kinematic hardening, are functions of plastic strains. Both functions admit classical potentials and an associative flow rule is assumed. Performing the time integration of the constitutive law according to a backward difference scheme, a finite step structural problem is formulated in a geometrically linear range.

Under these hypotheses a minimum principle for a convex functional, defined on the product space of finite increments of displacements, plastic strains and plastic parameters, is provided; the constraint conditions are expressed by positive increments of the plastic parameters and by the normality of the plastic strain increment to the elastic domain, so that the feasible set turns out to be non-convex. Any minimum point is shown to provide a solution for the elastoplastic finite step problem and vice versa.

The variational formulation of finite step elastoplastic problems is revisited in this paper by making use of a convex analysis approach and of a recently contributed potential theory for monotone multi-valued operators [6]. In fact the problem at hand is inherently non-linear and involves multi-valued operators, so that the tools of classical differential calculus and potential theory are no longer applicable. The recourse to non-smooth analysis is thus mandatory and provides the natural mathematical setting for a constructive development of the variational theory. A generalized version of the

Correspondence to: Prof. Giovanni Romano, Dipartimento di Scienza delle Costruzioni, Facoltà di Ingegneria, Università di Napoli Federico II, 80125 Piazzale Tecchio, Napoli, Italy. Phone: +39-81-7682111; Telex: +39-81-722392-INGENA I; Telefax: +39-81-2390445.

elastoplastic constitutive relations considered in [1–4] is addressed here in the framework of an internal variable theory [7–19]. No special assumption other than convexity is imposed on the yield modes.

Following the procedure outlined in [20], the first step of the analysis consists of re-formulating the structural problem in terms of a multi-valued *structural operator* defined on the product space of all the state variables. It encompasses in a unique expression the field and the constitutive equations which describe the finite step elastoplastic problem.

Then the conservativity of the structural operator is checked and the related non-smooth potential is obtained by direct integration. Such an approach is founded upon the results recently contributed in [6]. The operator formulation of the problem is thus shown to be recovered by imposing the stationarity of the non-smooth potential. Stationarity amounts to requiring that the null vector belongs to the partial subdifferential (superdifferential) of the potential with respect to the arguments in which it results convex (concave). On the basis of the results contributed in [20], a family of variational principles for the finite step structural problem can be derived by enforcing the fulfillment of field equations or constraint conditions. In particular, the expression of a minimum principle in displacements, plastic strains and plastic parameters is provided. By assuming positively homogeneous yield modes, a special minimum principle corresponding to the one proposed in [1–4] is then recovered.

A detailed comparison between the two minimum principles reveals an essential difference in the constraint conditions. It is shown indeed that the constraints considered in [1–4], which impose normality of the plastic strain increment to the elastic domain, are unduly restrictive and define a non-convex feasible set. This kind of drawback appears to be inherent to any ad hoc analysis which, however ingeniously performed, is lacking in a systematic development.

Conversely, with the present approach, the assumptions on the elastoplastic constitutive laws are relaxed to a maximal extent. Moreover, variationally consistent principles are formulated as optimization problems for convex functionals on convex feasible sets.

The possibility of searching for the solution of the finite step elastoplastic structural problem as a minimum point of a convex functional on a convex feasible set is especially relevant from the computational point of view. Actually standard optimization algorithms can be adopted for the numerical solution of the relevant nonlinear programming problem resulting from a space discretization, e.g. by a finite element approach.

2. Some preliminary results

We briefly recall here some basic definitions and properties of convex analysis as well as the concept of stationarity for non-smooth potentials which will be useful in the sequel.

2.1. Background of convex analysis

A comprehensive treatment of the subject can be found in [21–25].

Let (X, X') be a pair of locally convex topological vector spaces (l.c.t.v.s.) placed in separating duality by a bilinear form $\langle \cdot , \cdot \rangle$ and consider the convex functional $g : X \mapsto \Re \cup \{+\infty\}$; we shall denote by $\bar{\Re}$ the set $\{-\infty\} \cup \Re \cup \{+\infty\}$.

The one-sided Gateaux derivative of g at the point $x_o \in \operatorname{dom} g$ along the direction given by the vector $x \in X$, is the functional $f : X \mapsto \bar{\Re}$ defined by

$$f(x) \overset{\text{def}}{=} \mathrm{d}g(x_o; x) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} [g(x_o + \varepsilon x) - g(x_0)] ,$$

and turns out to be *sublinear*:

$$\begin{aligned} &f(\alpha x) = \alpha f(x) && \forall \alpha \geq 0 && \text{(positive homogeneity)} , \\ &f(x_1) + f(x_2) \geq f(x_1 + x_2) && \forall x_1, x_2 \in X && \text{(subadditivity)} . \end{aligned}$$

Clearly the epigraph of f is a convex cone in $X \times \Re$.

If the sublinear functional f is proper, the *subdifferential* of the functional g is the multi-valued map, $\partial g : X \mapsto X'$, defined by

$$\partial g(x_o) \overset{\text{def}}{=} \{x^* \in X' \colon f(x) \geq \langle x^*, x \rangle \ \forall x \in X\} .$$

In particular, if the functional g is differentiable at $x_o \in X$, the subdifferential is a singleton and coincides with the usual differential.

The chain-rule concerning the composition of a convex functional and a differentiable operator as well as additivity of the subdifferential of convex functionals hold true under mild assumptions [23].

The conjugate of a convex functional g is the convex functional $g^* : X' \mapsto \Re \cup \{+\infty\}$ defined by

$$g^*(x^*) = \sup_{y \in X} \{\langle x^*, y \rangle - g(y)\} ;$$

the elements x, x^* for which the 'sup' is attained are said to be conjugate and, provided that g is closed, the following relations are equivalent:

$$g(x) + g^*(x^*) = \langle x^*, x \rangle , \quad x^* \in \partial g(x), \ x \in \partial g^*(x^*) .$$

Analogous results hold for concave functionals by interchanging the role of $+\infty, \geq$ and 'sup' with those of $-\infty, \leq$ and 'inf'; the prefix 'sub' used in the convex case has now to be replaced by 'super'. In what follows, the subdifferential (superdifferential) of a convex (concave) functional are denoted by the same symbol ∂ when no ambiguity can arise.

A relevant example of conjugate functionals is provided by the *indicator* $\sqcup_K$ and the *support* functional $\sqcup_K^*$ of a convex set K:

$$\sqcup_K(x) = \begin{cases} 0 , & \text{if } x \in K \\ +\infty, & \text{otherwise} , \end{cases} \qquad \sqcup_K^*(x^*) = \sup_{x \in K} \langle x^*, x \rangle .$$

Moreover, we recall that the subdifferential of the indicator functional of a convex set K at a point $x \in K$ coincides with the *normal cone* to K at x:

$$\partial \sqcup_K(x) = N_K(x) \overset{\text{def}}{=} \begin{cases} \{x^* \in X' \colon \langle x^*, y - x \rangle \leq 0 \ \forall y \in X\} , & x \in K , \\ \emptyset & \text{otherwise} . \end{cases}$$

A functional $k : X \times Y \mapsto \bar{\Re}$ is said to be *saddle* (concave-convex) if $k(x, y)$ is a concave functional of $x \in X$ for each $y \in Y$ and a convex functional of y for each x.

A convex function $f : X \times Y \mapsto \Re \cup \{+\infty\}$ can be associated with any saddle functional $k : X \times Y' \mapsto \bar{\Re}$ according to the formula

$$f(x, y) = \sup_{y^* \in Y'} \{\langle y^*, y \rangle - k(x, y^*)\} .$$

Given any saddle functional k, we define by $\partial_x k(x, y^*)$ the superdifferential of the concave functional $k(\cdot, y^*)$ at x and by $\partial_{y^*} k(x, y^*)$ the subdifferential of the convex functional $k(x, \cdot)$ at y^*.

The subdifferential of the saddle functional k at the point (x, y^*) is defined as follows:

$$\partial k(x, y^*) \overset{\text{def}}{=} \partial_x k(x, y^*) \times \partial_{y^*} k(x, y^*) .$$

It can be proved [24] that, given a saddle functional k and its associated convex functional f, the following equivalences hold:

$$(x^*, y^*) \in \partial f(x, y) \quad \Leftrightarrow \quad (-x^*, y) \in \partial k(x, y^*)$$

and

$$y \in \partial_{y^*} k(x, y^*) \quad \Leftrightarrow \quad f(x, y) + k(x, y^*) = \langle y^*, y \rangle .$$

The last relation stands as a generalization of Fenchel's equality.

2.2. Stationarity property

In the classical calculus of variations, the stationarity condition for a differentiable functional amounts to finding its *critical points*, that is, the points in the domain of the functional at which its derivative vanishes. When dealing with non-smooth functionals, the definition of stationarity must be suitably re-formulated.

A convex functional g is said to have a *stationary point* at $x \in \operatorname{dom} g$ if the null vector $0 \in X'$ is included in its subdifferential at x:

$$0 \in \partial g(x) \quad \Leftrightarrow \quad \mathrm{d}g(x; y) \geq 0 \quad \forall y \in X .$$

A common situation to be handled is the case of functionals which are convex with respect to some variables and concave with respect to some others. In this respect, let us consider a functional $g : X = X_1 \times X_2 \times X_3 \times X_4 \mapsto \bar{\Re}$ which is jointly convex in (x_1, x_2) and jointly concave in (x_3, x_4).

The stationarity of g is formally expressed by the condition

$$(0, 0, 0, 0) \in \partial g(x_1, x_2, x_3, x_4) ,$$

which has to be interpreted in the sense that, collecting in two groups the variables with respect to which the functional g is convex or concave, the null vector belongs to the subdifferential of g with respect to the former group and the null vector belongs to the superdifferential of g with respect to the latter. Hence, we can write

$$(0, 0) \in \partial_{(x_1, x_2)} g(x_1, x_2, x_3, x_4) \subseteq X_1' \times X_2' ,$$
$$(0, 0) \in \partial_{(x_3, x_4)} g(x_1, x_2, x_3, x_4) \subseteq X_3' \times X_4' ,$$

where $\partial_{(x_1, x_2)}$ and $\partial_{(x_3, x_4)}$ denote the subdifferential and the superdifferential in the corresponding product spaces, respectively.

Here and in the sequel, we adopt the convention of not indicating explicitly the variables with respect to which the subdifferential is performed whenever they coincide with all the variables on which the functional depends.

In general, the subdifferential of a convex functional defined in a product space is included in the Cartesian product of the corresponding partial subdifferentials [20].

Equality holds only in special cases, e.g. if the functional can be written as the sum of convex functionals, each of them being defined in a different component space.

Accordingly, if the functional g can be written as the sum of two convex functional $g_1 : X_1 \mapsto \Re \cup \{+\infty\}$ and $g_2 : X_2 \mapsto \Re \cup \{+\infty\}$ and of a concave functional $g_3 : X_3 \times X_4 \mapsto \Re \cup \{-\infty\}$, the stationarity of g can be enforced as follows:

$$0 \in \partial g_1(x_1) , \qquad 0 \in \partial g_2(x_2) , \qquad (0, 0) \in \partial g_3(x_3, x_4) .$$

The same arguments can be repeated for a concave functional.

3. Constitutive relations

Let us consider an elastoplastic medium which undergoes infinitesimal deformations in an isothermal process whose successive events are ordered by a scalar parameter t which will be referred to as time.

We remark in advance that variables appearing in constitutive relations are local entities. Accordingly, the spaces to which the variables belong will be labeled with the subscript 'l' to distinguish them from the corresponding spaces of global variables pertaining to the whole structure.

In an internal variable approach, the plastic behaviour is described in terms of internal variables and thermodynamic forces belonging to two dual linear spaces X_l and X'_l, respectively.

In such a context, the constitutive model of elastoplasticity is defined by the flow rule between the rates of internal variables and the thermodynamic forces, and by the expression of the Helmholtz free energy. A variety of models of elastoplastic behaviour has been proposed in the literature depending on the type of state variables involved in the expressions of the flow rule and of the Helmholtz free energy [7–19].

In particular, the model recently considered by Maier and co-workers [1–4] in the context of associated plasticity is revisited here and recast in the framework of an internal variable theory.

In the sequel, we denote by $\mathscr{D}_l$ the linear space of strains ε_x and by $\mathscr{S}_l$ the dual linear space of stresses σ_x where x is a point in the domain V occupied by the body; the index x is used to represent the value of the field at the point x. As usual, the total strain $\varepsilon_x \in \mathscr{D}_l$ is assumed to be the sum of an elastic strain e_x and of a plastic strain p_x:

$$\varepsilon_x = e_x + p_x \,.$$

To simplify the exposition, we first consider the model of uni-modal plasticity.

In this case the yield criterion, which defines the current yield surface, is expressed in terms of a single *yield mode* $y : X'_l \mapsto \Re \cup \{+\infty\}$ and of a corresponding *yield limit*. Strain hardening of isotropic type is modelled by assuming that the yield limit is a positive monotone function $y_o : \mathscr{L}_l \mapsto \Re \cup \{+\infty\}$ of a scalar parameter $\lambda_x \in \mathscr{L}_l \equiv \Re$. The elastic domain is then defined as the level set of the yield mode y at the value $y_o(\lambda_x)$:

$$K_x = \{\chi_x \in X'_l : y(\chi_x) \leq y_o(\lambda_x)\} \,.$$

It represents the collection of thermodynamic forces χ_x which are admissible for a given value of the parameter λ_x; essentially this parameter can be interpreted as a scalar measure of the amount of plastic strain accumulated during the process. A more precise identification of λ_x is given in the sequel.

We assume that the yield mode y is a closed convex function so that the elastic domain turns out to be convex and closed. Notice in addition that the monotonicity of y_o implies the existence of a convex potential $\pi : \mathscr{L}_l \mapsto \Re \cup \{+\infty\}$ such that

$$y_o(\lambda_x) = \mathrm{d}\pi(\lambda_x) \,.$$

In the model of plastic behaviour proposed by Maier and co-workers in [1–4], the yield mode y is assumed to be first-order positively homogeneous so that the sets K_x corresponding to different values of $y_o(\lambda_x)$ turn out to be all proportional. The shape of the elastic domain K_x thus remains fixed while its size changes proportionally to the yield limit. Accordingly the function y_o accounts for *isotropic hardening* which depends upon the amount of accumulated plastic strain through the parameter λ_x.

Actually this feature concerning the evolution of the elastic domain K_x holds for any yield mode y which is positively homogeneous of arbitrary order $m \geq 1$.

More generally [24], a convex function meets the property of having its level sets all proportional if and only if it turns out to be the composition of a non-negative sublinear function k and of a non-constant, non-decreasing convex function $m : \Re^+ \mapsto \Re \cup \{+\infty\}$:

$$y(\chi_x) = (m \circ k)(\chi_x) \,.$$

Yield functions of this type have been termed *classical yield functions* in [17], where the related mechanical implications for the formulation of constitutive theories of associated inelastic behaviour have been discussed.

In the second part of the paper, the analysis is extended to the more general case of multi-modal plasticity in which the elastic domain is defined in terms of $n \geq 1$ yield modes y_i and corresponding scalar parameters λ_{ix}.

In the theory of associated plasticity, the flow rule is classically expressed by the normality rule to the elastic domain. Maier's model can be recovered by assuming that the time derivative $\dot{p}_x$ of the plastic strain belongs to the normal cone to the elastic domain K_x at the point χ_x:

$$\dot{p}_x \in N_{K_x}(\chi_x) .$$

According to this relation the role of internal variables is played by the plastic strains p_x since they are in duality with the thermodynamic forces χ_x which then belongs to the stress space.

The following identifications are thus entailed:

$$X_1 = \mathscr{D}_1 , \qquad X_1' = \mathscr{S}_1 .$$

In what follows, the classical expression of the flow rule in terms of plastic multipliers is derived in a direct and original form; moreover, in view of the derivation of variational principles, the flow rule will be also recast in a more suitable form. In this respect, we remark that the adoption of an associated model of elastoplasticity is indispensable for the development of a related variational theory.

The normal cone $N_{K_x}(\chi_x)$ can be expressed as the subdifferential of the indicator of K_x at χ_x. Setting for convenience $h(\chi_x) = y(\chi_x) - y_o(\lambda_x)$, we have

$$K_x = \{\chi_x \in X_1' : h(\chi_x) \leq 0\} ,$$

and hence

$$N_{K_x}(\chi_x) = \partial \sqcup_{K_x}(\chi_x) = \partial(\sqcup_{\Re^-} \circ h)(\chi_x) ,$$

where $\Re^-$ is the cone of non-positive reals.

A new chain rule of subdifferential calculus, recently contributed in [26], then yields

$$\partial(\sqcup_{\Re^-} \circ h)(\chi_x) = \partial \sqcup_{\Re^-}[h(\chi_x)]\, \partial h(\chi_x) ,$$

so that

$$N_{K_x}(\chi_x) = N_{\Re^-}[h(\chi_x)]\, \partial h(\chi_x) = N_{\Re^-}[y(\chi_x) - y_o(\lambda_x)]\, \partial y(\chi_x) .$$

The flow rule can accordingly be re-written in the form

$$\dot{p}_x \in \alpha_x\, \partial y(\chi_x) , \qquad \alpha_x \in N_{\Re^-}[y(\chi_x) - y_o(\lambda_x)] .$$

The last inclusion means that α_x is zero when $y(\chi_x) < y_o(\lambda_x)$ and is non-negative when $y(\chi_x) = y_o(\lambda_x)$. The flow rule can thus be explicitly expressed in terms of a linear complementarity condition:

$$\dot{p}_x \in \alpha_x\, \partial y(\chi_x) , \qquad \alpha_x \geq 0 , \qquad y(\chi_x) - y_o(\lambda_x) \leq 0 , \qquad \alpha_x[y(\chi_x) - y_o(\lambda_x)] = 0 .$$

The model proposed by Maier and co-workers in [1–4] is recovered by assuming that the time derivative of the parameter λ_x coincides with the value of the plastic multiplier α_x, i.e. $\dot{\lambda}_x = \alpha_x$. In the sequel, this position will tacitly be assumed to hold.

Notice that the sign constraint on α_x implies that the parameter λ_x is a non-decreasing function of time t.

For future reference, we remark that the complementarity condition

$$\dot{\lambda}_x \in N_{\Re^-}[y(\chi_x) - y_o(\lambda_x)] = \partial \sqcup_{\Re^-} [y(\chi_x) - y_o(\lambda_x)] ,$$

can be re-written as

$$y(\chi_x) - y_o(\lambda_x) \in \partial \sqcup^*_{\Re^-} (\dot{\lambda}_x) = \partial \sqcup_{\Re^+} (\dot{\lambda}_x) ,$$

so that, introducing the saddle (concave-convex) function $\psi : \mathscr{L}_1 \times X'_1 \mapsto \{-\infty\} \cup \Re \cup \{+\infty\}$ defined by

$$\psi(\dot{\lambda}_x, \chi_x) \overset{\text{def}}{=} \begin{cases} \dot{\lambda}_x y(\chi_x) , & \dot{\lambda}_x \geq 0 , \\ -\infty , & \text{otherwise} , \end{cases}$$

the flow rule can be given the following equivalent expression:

$$\dot{p}_x \in \partial_{\chi_x} \psi(\dot{\lambda}_x, \chi_x) , \qquad \mathrm{d}_{\dot{\lambda}_x} \psi(\dot{\lambda}_x, \chi_x) - \mathrm{d}\pi(\lambda_x) \in \partial \sqcup_{\Re^+} (\dot{\lambda}_x) .$$

In order to complete the set of constitutive relations, the expression of the Helmholtz free energy φ must be specified.

The function φ is assumed here to be jointly convex in the total strains ε_x and in the plastic strains p_x. Stresses σ_x and thermodynamic forces χ_x are related to ε_x and p_x by means of the multi-valued relation

$$(\sigma_x, -\chi_x) \in \partial\varphi(\varepsilon_x, p_x) ,$$

where the symbol ∂ denotes the subdifferential in the product space of total and plastic strains.

We assume the Helmholtz free energy to be the sum of two convex functions; namely the elastic energy $\phi : \mathscr{D}_1 \mapsto \Re \cup \{+\infty\}$ and the hardening function $\xi : \mathscr{D}_1 \mapsto \Re \cup \{+\infty\}$:

$$\varphi(\varepsilon_x, p_x) = \phi(\varepsilon_x - p_x) + \xi(p_x) .$$

To provide an explicit expression of the subdifferential of φ with respect to the pair (ε_x, p_x), we introduce the linear operator $L : \mathscr{D}_1 \times \mathscr{D}_1 \mapsto \mathscr{D}_1$ and its dual $L' : \mathscr{S}_1 \mapsto \mathscr{S}_1 \times \mathscr{S}_1$ defined by

$$L(\varepsilon_x, p_x) \overset{\text{def}}{=} \varepsilon_x - p_x = e_x , \qquad L'\sigma_x = \sigma_x \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

The subdifferential of the elastic energy can then be achieved by means of a chain rule of subdifferential calculus in the form

$$\partial(\phi \circ L)(\varepsilon_x, p_x) = L' \, \partial\phi(L(\varepsilon_x, p_x)) = L' \, \partial\phi(e_x) = \partial\phi(e_x) \begin{bmatrix} 1 \\ -1 \end{bmatrix} ,$$

so that the constitutive relation becomes

$$\sigma_x \in \partial\phi(e_x) , \qquad \chi_x \in \partial\phi(e_x) - \partial\xi(p_x) .$$

In the case of a linear elastic behaviour, the elastic energy is expressed by

$$\phi(e_x) = \tfrac{1}{2} \|e_x\|^2_E ,$$

with the elastic operator E symmetric and positive definite. Accordingly stresses and thermodynamic forces turn out to be given by

$$\sigma_x = Ee_x\,, \qquad \chi_x \in Ee_x - \partial\xi(p_x)\,.$$

The former is the classical elastic relation, while the latter accounts for a generalized *kinematic hardening* which specializes to the classical one when the function ξ is differentiable. In fact, in this case the operator $\partial\xi(p_x)$ turns out to be single-valued and the kinematic hardening is uniquely defined for each plastic strain. In accordance, the relation

$$\sigma_x \in \mathrm{d}\xi(p_x) + K_x$$

states that the stress σ_x belongs to the translation of the elastic domain K_x by the amount $\mathrm{d}\xi(p_x)$.

4. The elasto-plastic structural model

In order to develop the variational formulation of the structural problem, the relevant relations must be written in global form, that is, in terms of quantities pertaining to the whole structure. In the sequel, such quantities are referred to as 'fields'.

For a continuous model, such fields are functions defined in the domain V occupied by the body and are assumed to be elements of a suitable functional space.

Local subdifferential relations, enforced almost everywhere in V, can be equivalently expressed in global form by integrating the relevant convex functions over the domain V.

For instance, the global elastic strain energy is defined to be the functional of the elastic strain field e obtained by integrating the specific strain energy ϕ over the whole body domain:

$$\Phi(e) \overset{\text{def}}{=} \int_V \phi(e_x)\,\mathrm{d}x\,.$$

Notice that, whenever the local functions are convex, the corresponding global ones turn out to be convex as well in the relevant fields.

The subdifferential of the global elastic energy is defined as

$$\sigma \in \partial\Phi(e) \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathrm{d}\Phi(e;\eta) \geq \langle \sigma, \eta - e\rangle \quad \forall \eta \in \mathscr{D}\,,$$

where

$$\mathrm{d}\Phi(e;\eta) \overset{\text{def}}{=} \int_V \mathrm{d}\phi(e_x;\eta_x)\,\mathrm{d}x\,, \qquad \langle \sigma, \eta - e\rangle \overset{\text{def}}{=} \int_V \sigma_x * (\eta_x - e_x)\,\mathrm{d}x\,,$$

and the symbol $*$ denotes the scalar product between the local values of dual fields.

The subdifferential of the local elastic energy is given by

$$\sigma_x \in \partial\phi(e_x) \quad \overset{\text{def}}{\Leftrightarrow} \quad \mathrm{d}\phi(e_x;\eta_x) \geq \sigma_x * (\eta_x - e_x) \quad \forall \eta_x \in \mathscr{D}_1\,,$$

for almost every (a.e.) $x \in V$ and the following equivalence can be proved [27]:

$$\sigma \in \partial\Phi(e) \quad \Leftrightarrow \quad \sigma_x \in \partial\phi(e_x) \quad \text{a.e. in } V\,.$$

In the sequel, we denote by Ξ and Π the global functionals corresponding to ξ and π, respectively.

Denoting by $\Psi : \mathscr{L} \times X' \mapsto \{-\infty\} \cup \Re \cup \{+\infty\}$ the global functional corresponding to ψ, the global form of the constitutive relation stems from the following equivalence:

$$\begin{cases} \dot{p}_x \in \partial_{\chi_x}\psi(\dot{\lambda}_x, \chi_x)\,, \\ \mathrm{d}_{\dot{\lambda}_x}\psi(\dot{\lambda}_x, \chi_x) \in \mathrm{d}\pi(\lambda_x) + \partial \sqcup_{\Re^+}(\dot{\lambda}_x)\,, \end{cases} \Leftrightarrow \quad \begin{cases} \dot{p} \in \partial_\chi \Psi(\dot{\lambda}, \chi)\,, \\ \mathrm{d}_{\dot{\lambda}}\Psi(\dot{\lambda}, \chi) \in \mathrm{d}\Pi(\dot{\lambda}) + \partial \sqcup_{\Re^+}(\dot{\lambda})\,, \end{cases}$$

where, when dealing with fields, $\Re^+$ is the convex cone of the fields $\dot{\lambda}$ which are non-negative a.e. in V.

An explicit form of the partial subdifferentials of Ψ and of the differential of Π can be given in terms of the functionals Y and Y_o defined as

$$[Y(\chi)](x) \stackrel{\text{def}}{=} y(\chi_x) , \qquad [Y_o(\dot{\lambda})](x) \stackrel{\text{def}}{=} y_o(\dot{\lambda}_x) .$$

In fact, it is easy to show that the following formulas hold:

$$\partial_x \Psi(\dot{\lambda}, \chi) = \dot{\lambda}\, \partial Y(\chi) , \qquad \mathrm{d}_{\dot{\lambda}} \Psi(\dot{\lambda}, \chi) = Y(\chi) , \qquad \mathrm{d}\Pi(\dot{\lambda}) = Y_o(\dot{\lambda}) .$$

Finally K represents the convex set of thermodynamic force fields such that $Y(\chi) \leq Y_o(\lambda)$, that is, $y(\chi_x) \leq y_o(\lambda_x)$ a.e. in V.

The description of the elastoplastic structural model is completed by specifying the field equations and the external constraints.

We make reference to structural models in which equilibrium is unaffected by geometry changes so that a linear strain measure can be adopted.

Denoting by $\mathscr{U}$ the linear space of displacement fields u and by $\mathscr{D}$ the linear space of strain fields ε, the corresponding duals will be the linear space $\mathscr{F}$ of external force fields f and the linear space $\mathscr{S}$ of stress fields σ. Compatibility and equilibrium equations are expressed by

$$\varepsilon = Tu \quad \text{and} \quad T'\sigma = f ,$$

where $T : \mathscr{U} \mapsto \mathscr{D}$ and $T' : \mathscr{S} \mapsto \mathscr{F}$ are dual linear operators [27, 28].

Writing the external forces in the form $f = l + r$, where l denotes the applied load and r the reaction of the external constraints, the external constitutive relation is given by

$$r \in \partial \Upsilon(u) ,$$

where $\Upsilon : \mathscr{U} \mapsto \Re \cup \{-\infty\}$ is a concave functional.

Such a relation provides a general model of external constraints which includes several cases of interest in Structural Mechanics such as bilateral or unilateral constraints, elastic or elastoviscoplastic foundations and so on. A survey of the particular expressions assumed by the functional Υ in each of these cases can be found in [29].

The relation between external forces and displacements can then be expressed as

$$f \in \partial \Gamma(u) \quad \text{or equivalently} \quad u \in \partial \Gamma^*(f) ,$$

where

$$\Gamma(u) = \langle l, u \rangle + \Upsilon(u) \quad \text{and} \quad \Gamma^*(f) = \Upsilon^*(f - l)$$

are concave functionals.

In the special case of external frictionless bilateral constraints with imposed displacements $\bar{u}$, the functional Υ turns out to be a concave indicator.

In fact denoting by L_o the subspace of admissible displacement fields and by $R = L_o^\perp$ the subspace of external reaction fields, it turns out to be

$$\Upsilon(u) = \sqcap_{L_o}(u - \bar{u}) .$$

Here the symbol $L_o^\perp$ represents the orthogonal complement of the subspace L_o and $\sqcap$ the concave indicator.

Accordingly the relation $r \in \partial \Upsilon(u)$ is equivalent to stating that $u \in \bar{u} + L_o$ and that $r \in L_o^\perp = R$.

The structural problem for the elastoplastic medium subject to a given load history $l(t)$ is thus governed by the following set of relations:

$f = T'\sigma$,	static equilibrium,
$\varepsilon = Tu$,	kinematic compatibility,
$\dot{p} \in N_K(\chi)$,	flow rule,
$\sigma \in \partial\Phi(\varepsilon - p)$, $\chi \in \partial\Phi(\varepsilon - p) - \partial\Xi(p)$,	Helmholtz free energy,
$u \in \partial\Gamma^*(f) = \partial Y^*(f - l)$,	external constraint,

where the explicit dependence of the state variables on time t has been dropped to simplify the notation.

4.1. *Elastoplastic finite step problem*

The evolutive analysis of the non-linear elastoplastic problem is performed by a preliminary sub-division of the load history in finite increments associated with a sequence $t_0, t_1, \ldots, t_i, \ldots, t_n$ of times. We assume that no plastic unloading can occur during any of the intervals $\Delta t_i = t_i - t_{i-1}$.

A finite step analysis of the evolutive problem amounts to evaluating the finite increments of the unknown variables corresponding to a given increment of load when their values are assigned at the beginning of the step. In the sequel, we denote by $(\cdot)_o$ the known quantities $(\cdot)$ at the beginning of each step.

In order to formulate the finite step counterpart of the flow rule $\dot{p} \in N_K(\chi)$, the time derivative of p is replaced by the finite increment ratio $(p - p_o)/\Delta t$; adopting a fully implicit time integration scheme (Euler backward difference), the flow rule is enforced at the end of the step:

$$\frac{p - p_o}{\Delta t} \in N_K(\chi) ,$$

which, as $N_K(\chi)$ is a convex cone, can also be written

$$(p - p_o) \in N_K(\chi) .$$

Using for the flow rule the same arguments as in Section 3, the finite step flow rule can be expressed in terms of the yield condition and of the increment of the plastic parameter:

$$(p - p_o) \in (\lambda - \lambda_o)\, \partial Y(\chi) , \qquad Y(\chi) - Y_o(\lambda) \in \partial \sqcup_{\Re^+} (\lambda - \lambda_o) .$$

The irreversible, path-dependent behaviour of plasticity is accounted for by updating the values of the internal variables λ at each step.

The finite step elastoplastic problem will then be conveniently formulated as follows:

$$f = T'\sigma ,$$
$$\varepsilon = Tu ,$$
$$(p - p_o) \in (\lambda - \lambda_o)\, \partial Y(\chi) ,$$
$$Y(\chi) - Y_o(\lambda) \in \partial \sqcup_{\Re^+} (\lambda - \lambda_o) ,$$
$$\sigma \in \partial\Phi(\varepsilon - p) ,$$
$$\chi \in \partial\Phi(\varepsilon - p) - \partial\Xi(p) ,$$
$$u \in \partial\Gamma^*(f) ,$$

in terms of the finite values of the fields at the end of the step.

The extremum characterization of the solution of the finite step evolutive problem is carried out in the next section.

5. Variational formulation

To prove that the finite step elastoplastic structural problem admits a variational formulation, it is convenient to recast the original problem in a suitable operator form. To this end, introducing the product space $W = \mathcal{U} \times \mathcal{S} \times \mathcal{S} \times \mathcal{L} \times \mathcal{D} \times \mathcal{D} \times \mathcal{F}$ and its dual W', we consider the multi-valued *structural operator* $\mathbf{Z} : W \mapsto W'$ defined by

$$0 \in \mathbf{Z}(w) = \hat{\mathbf{Z}}(w) + a \,,$$

or in explicit form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} 0 & T' & 0 & 0 & 0 & 0 & -I_{\mathcal{F}} \\ T & 0 & 0 & 0 & -I_{\mathcal{D}} & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\mathbf{A}} & 0 & I_{\mathcal{D}} & 0 \\ 0 & 0 & & & 0 & 0 & 0 \\ 0 & -I_{\mathcal{S}} & 0 & 0 & \multicolumn{2}{c}{\mathbf{B}} & 0 \\ 0 & 0 & I_{\mathcal{S}} & 0 & & & 0 \\ -I_{\mathcal{U}} & 0 & 0 & 0 & 0 & 0 & Z_{77} \end{bmatrix} \begin{bmatrix} u \\ \sigma \\ \chi \\ \lambda \\ \varepsilon \\ p \\ f \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -p_o \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ,$$

where $w \in W$, $a \in W'$ and the operators $I_{\mathcal{U}}, I_{\mathcal{F}}$ and $I_{\mathcal{S}}, I_{\mathcal{D}}$ are the dual pairs of identity maps in the spaces $\mathcal{U}, \mathcal{F}$ and $\mathcal{S}, \mathcal{D}$, respectively.

The multi-valued suboperator $\mathbf{A}$ in the expression of $\hat{\mathbf{Z}}$ is defined as follows:

$$\mathbf{A} : \begin{bmatrix} \chi \\ \lambda \end{bmatrix} \mapsto \begin{bmatrix} -\partial_\chi \Psi(\lambda - \lambda_o, \chi) \\ -\mathrm{d}_\lambda \Psi(\lambda - \lambda_o, \chi) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{d}\Pi(\lambda) + \partial \sqcup_{\Re^+} (\lambda - \lambda_o) \end{bmatrix} ,$$

and the multi-valued suboperators $\mathbf{B}$ and Z_{77} are given by

$$\mathbf{B} : \begin{bmatrix} \varepsilon \\ p \end{bmatrix} \mapsto \begin{bmatrix} \partial \Phi(\varepsilon - p) \\ -\partial \Phi(\varepsilon - p) \end{bmatrix} + \begin{bmatrix} 0 \\ \partial \Xi(p) \end{bmatrix} , \qquad Z_{77} : f \mapsto \partial \Gamma^*(f) \,.$$

The conservativity of the operator $\hat{\mathbf{Z}}$ is inferred from the duality existing between T and T', the duality existing between the two pairs of identity maps $I_{\mathcal{U}}, I_{\mathcal{F}}$ and $I_{\mathcal{S}}, I_{\mathcal{D}}$ and the conservativity of the other relations [6].

The existence of the potential Ω of $\mathbf{Z}$ is thus ensured and its expression can be evaluated by means of a direct integration in the product space W:

$$\Omega(w) = \int_0^w \langle \mathbf{Z}(x), \mathrm{d}x \rangle = \int_0^w \langle \hat{\mathbf{Z}}(x), \mathrm{d}x \rangle - \langle \chi, p_o \rangle \,,$$

to obtain

$$\Omega(u, \sigma, \chi, \lambda, \varepsilon, p, f) = \Phi(\varepsilon - p) + \Xi(p) + \Gamma^*(f) - \langle f, u \rangle - \langle \sigma, \varepsilon \rangle + \langle \sigma, Tu \rangle + \langle \chi, p - p_o \rangle \\ - \Psi(\lambda - \lambda_o, \chi) + \Pi(\lambda) + \sqcup_{\Re^+} (\lambda - \lambda_o) \,.$$

The functional Ω is linear in u and σ, saddle (concave-convex) with respect to the pair (χ, λ), jointly convex with respect to the pair (ε, p) and concave with respect to f.

Notice that the integral of the multi-valued operator $\hat{\mathbf{Z}}$ does not depend upon the particular choice of an element in the set $\hat{Z}(w)$. Further details concerning the mathematical aspects related to the potential theory for monotone multi-valued operators can be found in [6].

Let us now show that the stationarity of the functional Ω is equivalent to the operator formulation of the problem.

PROPOSITION 1. A vector w is a stationary point for the functional Ω if and only if it is a solution of the finite step elastoplastic structural problem.

PROOF. By virtue of the results reported in Section 2.2, the stationarity of the functional Ω at the point $w = (u, \sigma, \chi, \lambda, \varepsilon, p, f)$ is formally expressed by

$$0 \in \partial\Omega(w) ,$$

and is enforced by the following set of conditions:

$$\begin{aligned} &0 \in \partial_u\Omega(w) , \quad 0 \in \partial_\sigma\Omega(w) , \quad 0 \in \partial_\chi\Omega(w) , \\ &0 \in \partial_\lambda\Omega(w) , \quad (0,0) \in \partial_{(\varepsilon,p)}\Omega(w) , \quad 0 \in \partial_f\Omega(w) . \end{aligned}$$

Performing the subdifferentials in the corresponding spaces, we recover the operator form of the problem:

$$\begin{aligned} &0 \in \partial_u\Omega(w) \iff f = T'\sigma , \\ &0 \in \partial_\sigma\Omega(w) \iff \varepsilon = Tu , \\ &0 \in \partial_\chi\Omega(w) \iff (p - p_o) \in (\lambda - \lambda_o)\, \partial Y(\chi) , \\ &0 \in \partial_\lambda\Omega(w) \iff Y(\chi) - Y_o(\lambda) \in \partial \sqcup_{\Re^+} (\lambda - \lambda_o) , \\ &(0,0) \in \partial_{(\varepsilon,p)}\Omega(w) \iff \sigma \in \partial\Phi(\varepsilon - p) , \quad \chi \in \partial\Phi(\varepsilon - p) - \partial\Xi(p) , \\ &0 \in \partial_f\Omega(w) \iff u \in \partial\Gamma^*(f) . \end{aligned}$$

Reverting the steps above, we infer that a solution of the finite step elastoplastic problem makes the functional Ω stationary. □

6. A minimum principle

A family of functionals can be derived from the potential Ω by enforcing the fulfillment of the field equations and of the constitutive relations. All these functionals assume the same value as Ω when evaluated in correspondence of a solution w of the structural elastoplastic problem.

However, we are mainly interested in specializing the general expression of Ω to simpler forms which can result in greater interest for practical applications; more specifically, we derive from Ω a three-field functional which is jointly convex in (u, p, λ).

The interest in such a functional clearly rests on the circumstance that the solution of the elastoplastic structural problem (if any) can be obtained by numerically solving a minimization problem [30, 31].

Let us preliminarily derive an intermediate functional in the four variables (u, p, λ, χ). To this end, we recall that a pair (f, u) fulfills the constraint relation if and only if it satisfies Fenchel's equality

$$u \in \partial\Gamma^*(f) \iff \Gamma(u) + \Gamma^*(f) = \langle f, u \rangle .$$

In addition, enforcing the kinematic compatibility condition, the functional Ω specializes to

$$\Omega_1(u, p, \lambda, \chi) = \Phi(Tu - p) + \Xi(p) - \Gamma(u) + \langle \chi, p - p_0 \rangle - \Psi(\lambda - \lambda_o, \chi) + \Pi(\lambda) + \sqcup_{\Re^+}(\lambda - \lambda_o) ,$$

which is jointly convex in (u, p) and saddle (convex-concave) in (λ, χ).

In order to eliminate the explicit dependence on the thermodynamic forces χ in the expression of Ω_1, we make use of the results concerning the saddle functionals and the associated convex functionals (see Section 2.1). By definition, the convex functional $\Psi_c : \mathscr{L} \times \mathscr{D} \mapsto \Re \cup \{+\infty\}$ associated with the saddle (concave-convex) functional Ψ is given by

$$\Psi_c(\lambda - \lambda_o, p - p_o) \overset{\text{def}}{=} \sup_{\chi} \{\langle \chi, p - p_o \rangle - \Psi(\lambda - \lambda_o, \chi)\} .$$

Denoting by $\psi_c : \mathscr{L}_1 \times \mathscr{D}_1 \mapsto \Re \cup \{+\infty\}$ the convex function associated with the saddle function ψ given by

$$\psi_c(\lambda_x - \lambda_{o_x}, p_x - p_{o_x}) = \begin{cases} [(\lambda_x - \lambda_{o_x})y]^*(p_x - p_{o_x}) , & \lambda_x - \lambda_{o_x} \geq 0 , \\ +\infty , & \text{otherwise} , \end{cases}$$

the functional Ψ_c turns out to be the global functional corresponding to ψ_c (see [27]):

$$\Psi_c(\lambda - \lambda_o, p - p_o) = \int_V \psi_c(\lambda_x - \lambda_{o_x}, p_x - p_{o_x}) \, dx .$$

For any $\lambda \geq \lambda_o$, a pair $(p - p_o, \chi)$ satisfies the relation $(p - p_o) \in \partial\Psi(\lambda - \lambda_o, \chi) = (\lambda - \lambda_o) \, \partial Y(\chi)$ if and only if

$$\Psi(\lambda - \lambda_o, \chi) + \Psi_c(\lambda - \lambda_o, p - p_o) = \langle \chi, p - p_o \rangle ,$$

which substituted in the expression for the potential Ω_1 yields

$$\Omega_2(u, p, \lambda) = \Phi(Tu - p) + \Xi(p) - \Gamma(u) + \Psi_c(\lambda - \lambda_o, p - p_o) + \Pi(\lambda) + \sqcup_{\Re^+}(\lambda - \lambda_o) .$$

The functional Ω_2 turns out to be jointly convex in the three state variables (u, p, λ) and we infer the following.

PROPOSITION 2. A triplet (u, p, λ) is an absolute minimum point for the convex functional Ω_2 if and only if it is a solution of the finite step elastoplastic structural problem.

PROOF. Let us show first that the operator formulation of the problem can be inferred by the stationarity of the functional Ω_2. In fact this condition is formally expressed by the relation

$$0 \in \partial\Omega_2(u, p, \lambda) ,$$

where the subdifferential is performed with respect to (u, p, λ) jointly. The relation above can be re-written as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} C(u, p) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \partial\Xi(p) \\ 0 \end{bmatrix} - \begin{bmatrix} \partial g(u) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ D(p, \lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F(\lambda) \end{bmatrix} ,$$

where

$$C(u, p) = \partial(\Phi \circ N)(u, p) , \qquad D(\lambda, p) = \partial\Psi_c(\lambda - \lambda_o, p - p_o) ,$$

$$F(\lambda) = d\Pi(\lambda) + \partial \sqcup_{\Re^+}(\lambda - \lambda_o) .$$

The operator $N : \mathcal{U} \times \mathcal{D} \mapsto \mathcal{D}$ appearing in the expression of C is given by

$$N = [T, -I_{\mathcal{D}}] ,$$

and is introduced simply to express the subdifferential of the elastic energy with respect to the pair (u, p). The dual operator $N' : \mathcal{S} \mapsto \mathcal{F} \times \mathcal{S}$ of N is given by

$$N' = \begin{bmatrix} T' \\ -I_{\mathcal{S}} \end{bmatrix} ,$$

so that

$$\partial(\Phi \circ N)(u, p) = N' \, \partial\Phi(N(u, p)) = \begin{bmatrix} T' \, \partial\Phi(Tu - p) \\ -\partial\Phi(Tu - p) \end{bmatrix} .$$

The stationarity of Ω_2 is thus equivalently expressed by the following set of relations:

$$\exists f \in \partial\Gamma(u): \quad f \in T' \, \partial\Phi(Tu - p) ,$$

and

$$\exists \begin{bmatrix} \Lambda \\ \chi \end{bmatrix} \in \boldsymbol{D}(\lambda, p): \quad \begin{cases} -\Lambda \in \mathrm{d}\Pi(\lambda) + \partial \sqcup_{\Re^+} (\lambda - \lambda_o) , \\ -\chi \in -\partial\Phi(Tu - p) + \partial\Xi(p) , \end{cases}$$

where $\Lambda \in \mathcal{L}' \equiv \Re$ represents the dual parameter of λ.

In order to evaluate the multi-valued suboperator $\boldsymbol{D}$, we recall the relation existing between the subdifferential of the functional Ψ_c and the subdifferential of the associated saddle functional Ψ:

$$\begin{bmatrix} \Lambda \\ \chi \end{bmatrix} \in \partial\Psi_c(\lambda - \lambda_o, p - p_o) \quad \Leftrightarrow \quad \begin{bmatrix} -\Lambda \\ (p - p_o) \end{bmatrix} \in \begin{bmatrix} \mathrm{d}_\lambda \Psi(\lambda - \lambda_o, \chi) \\ \partial_\chi \Psi(\lambda - \lambda_o, \chi), \end{bmatrix} ,$$

which can be rewritten as

$$-\Lambda = Y(\chi) , \qquad (p - p_o) \in (\lambda - \lambda_o) \, \partial Y(\chi) .$$

We may then conclude that correspondingly to any minimum point (u, p, λ) of the functional Ω_2, there exists an external force satisfying the external constraint condition

$$f \in \partial\Gamma(u) ,$$

which is equilibrated by an internal stress

$$\sigma \in \partial\Phi(e) ,$$

such that the elastic strain e is associated with the pair (u, p) by the relations

$$\varepsilon = Tu , \qquad e = \varepsilon - p .$$

Moreover, there exists a thermodynamic force χ satisfying the condition

$$(p - p_o) \in (\lambda - \lambda_o) \, \partial Y(\chi) ,$$

such that

$$\chi \in \partial\Phi(e) - \partial\Xi(p) , \qquad Y(\chi) - Y_o(\lambda) \in \partial \sqcup_{\Re^+} (\lambda - \lambda_o) .$$

The whole elastoplastic finite step problem is thus recovered. The converse implication follows at once by reverting the steps above. □

The minimum principle presented in Proposition 2 can be re-written as a constrained optimization (non-linear programming) problem if we make explicit the constraint condition imposed by the indicator $\sqcup_{\Re^+}$. We can then state the following.

PROPOSITION 3. A triplet (u, p, λ) *is a solution of the convex optimization problem*

$$\min\{\hat{\Omega}_2(u, p, \lambda) \mid (\lambda - \lambda_o) \geq 0\} ,$$

with

$$\hat{\Omega}_2(u, p, \lambda) = \Phi(Tu - p) + \Xi(p) - \Gamma(u) + \Gamma_c(\lambda - \lambda_o, p - p_o) + \Pi(\lambda) ,$$

if and only if it is a solution of the finite step elastoplastic structural problem.

The functional Ω_2 can be further specialized if the yield mode y is sublinear in χ_x. Actually, for any $\lambda_x \geq \lambda_{ox}$, the function ψ turns out to be sublinear in χ_x and its conjugate with respect to χ_x turns out to be the indicator of a convex set C_x:

$$\psi_c(\lambda_x - \lambda_{o_x}, p_x - p_{o_x}) = \sqcup_{C_x}(p_x - p_{o_x}) .$$

The expression of C_x is provided in Appendix A and is given by $C_x = (\lambda_x - \lambda_{o_x}) y_o(\lambda_x) K_x^\circ$ where K_x° is the polar set of K_x.

Finally, integrating over V, we obtain

$$\Psi_c(\lambda - \lambda_o, p - p_o) = \int_V \sqcup_{C_x}(p_x - p_{o_x})\, dx = \sqcup_C (p - p_o) ,$$

where the set C is defined as

$$C = (\lambda - \lambda_o) Y_o(\lambda) K^\circ .$$

The condition $(p - p_o) \in C$ amounts then to requiring that $(p_x - p_{o_x}) \in C_x$ a.e. in V.

By substituting the expression of Ψ_c in the functional Ω_2, we obtain

$$\Omega_3(u, p, \lambda) = \Phi(Tu - p) + \Xi(p) - \Gamma(u) + \sqcup_C (p - p_o) + \Pi(\lambda) + \sqcup_{\Re^+}(\lambda - \lambda_o) ,$$

and the following statement holds.

PROPOSITION 4. If the yield mode is sublinear, a triplet (u, p, λ) *will be a solution of the convex optimization problem*

$$\min\{\hat{\Omega}_3(u, p, \lambda) \mid (\lambda - \lambda_o) \geq 0,\ (p - p_o) \in C\} ,$$

with

$$\hat{\Omega}_3(u, p, \lambda) = \Phi(Tu - p) + \Xi(p) - \Gamma(u) + \Pi(\lambda) ,$$

if and only if it is a solution of the finite step elastoplastic structural problem.

The meaning of the constraint conditions in the minimum problem can be elucidated if we consider the Von Mises yield criterion. Denoting by $\bar{\sigma}_x$ the nominal yield stress of the material determined in a uniaxial test, the elastic domain K_x is a cylinder in the stress space having a circular section of radius $y_o(\lambda_x) = \sqrt{2/3}\bar{\sigma}_x(\lambda_x)$ in the deviatoric subspace:

$$K_x = \{\chi_x \in \mathcal{S}_1 \colon \|\text{dev}\,\chi_x\| \le y_o(\lambda_x)\}\,,$$

where dev $\chi_x = \sigma_x - \text{sph}\,\sigma_x$ and sph $\sigma_x = \frac{1}{3}(\text{tr}\,\sigma_x)I$, I being the identity tensor. Here $\|\cdot\|$ denotes the norm in the space of tensors; typically $\|A\| = \sqrt{\text{tr}\,A'A}$ for any tensor A, where tr denotes the trace and A' is the transpose of A.

The polar set K_x° turns out to be the circular disc of radius $y_o^{-1}(\lambda_x)$ lying in the deviatoric subspace of the strain space $\mathcal{D}_1$:

$$K_x^\circ = \{p_x \in \mathcal{D}_1 \colon \|\text{dev}\,p_x\| \le y_o^{-1}(\lambda_x);\ \text{sph}\,p_x = 0\}\,,$$

so that the set $y_o(\lambda_x)K_x^\circ$ is the unitary disc in the deviatoric subspace:

$$y_o(\lambda_x)K_x^\circ = \{p_x \in \mathcal{D}_1 \colon \|\text{dev}\,p_x\| \le 1;\ \text{sph}\,p_x = 0\}\,,$$

and the set C_x becomes

$$C_x = \{p_x \in \mathcal{D}_1 \colon \|\text{dev}\,p_x\| \le (\lambda_x - \lambda_{o_x});\ \text{sph}\,p_x = 0\}\,.$$

Hence, the constraint conditions supplementing the minimum principle in Proposition 4 require that, a.e. in V, the tensor $(p_x - p_{xo})$ is deviatoric and its norm is not greater than the non-negative number $(\lambda_x - \lambda_{o_x})$.

REMARK. A non-convex minimum principle for finite step solutions of an elastoplastic model, which is a special case of the one considered in this paper, has been proposed by Maier and co-workers in [1–4]. In their papers, increments of the state variables in each finite step, instead of their final values, are taken as unknowns.

Classical tools of variational analysis are resorted to in the treatment given in [1–4]. The proofs of necessity and of sufficiency follow two distinct paths of reasoning and require very special assumptions such as sublinearity of the yield mode Y and its differentiability (even twice in the proof of sufficiency).

The minimum principle in (u, p, λ) proposed in [1–4], written according to our notation, reads

$$\min\{\Omega_M(u, p, \lambda) \mid \lambda \ge \lambda_o,\ (p - p_o) \in (\lambda - \lambda_o)\,\partial Y(\chi)\}\,,$$

with

$$\Omega_M(u, p, \lambda) = \tfrac{1}{2}\|Tu - p\|_E^2 + \Xi(p) - \Gamma(u) + \Pi(\lambda)\,.$$

The functional $\hat{\Omega}_3$ of our Proposition 4 coincides with Ω_M if we assume a differentiable hardening functional Ξ and a linear elastic behaviour.

The significant difference between the two minimum principles is represented by the second constraint condition. In this respect, two observations have to be made.

First, we remark that in the original version of the principle formulated in [1–4], the yield functional Y was assumed to be differentiable so that this condition was written as

$$(p - p_o) = (\lambda - \lambda_o)\,\mathrm{d}Y(\chi)\,.$$

Actually such an assumption appears to be unnecessary and even questionable. In fact the sublinear functional Y is inherently non-differentiable at the origin since its epigraph is a convex cone; accordingly the notion of differentiability must be replaced by the weaker notion of subdifferentiability.

Second, we notice that the following implication holds:

$$(p - p_o) \in (\lambda - \lambda_o)\, \partial Y(\chi) \quad \Rightarrow \quad (p - p_o) \in C = (\lambda - \lambda_o) Y_o(\lambda) K^\circ .$$

In fact, expressing Y in terms of the Minkowski functional γ_K of the set K,

$$Y(\chi) = Y_o(\lambda)\gamma_K(\chi) = Y_o(\lambda) \sqcup^*_{K^\circ}(\chi) ,$$

the left-hand side of the implication above can be re-written as

$$(p - p_o) \in (\lambda - \lambda_o) Y_o(\lambda)\, \partial \sqcup^*_{K^\circ}(\chi) .$$

The result follows then by observing that

$$\hat{p} \in \partial \sqcup^*_{K^\circ}(\chi) \quad \Leftrightarrow \quad \chi \in \partial \sqcup_{K^\circ}(\hat{p}) \Rightarrow \hat{p} \in K^\circ .$$

The analysis performed above shows that the constraint considered in [1–4] turns out to be more stringent than strictly required by the variational principle. The minimum of the objective functional Ω_M was then searched for in a non-convex subset of the convex feasible set.

A further remark must be devoted to the variational tools which are required in the analysis of non-smooth structural problems. Classical differential calculus and potential theory are no longer applicable to this kind of problem; subdifferential calculus and potential theory for monotone multi-valued operators must be invoked [6, 24] in their place.

From a computational standpoint, the implications of the present results are relevant.

Actually Propositions 3 and 4 allow us to state that the solution of the finite step elastoplastic structural problem can be achieved by minimizing a convex functional on a convex feasible set.

Algorithms of convex optimization [30, 31] can then be adopted for the numerical solution of the constrained optimization (non-linear programming) problem which arises from a suitable space discretization, e.g. by finite element modelling.

7. Multi-modal plasticity

We now address the case of multi-modal plasticity which is characterized by an elastic domain defined by multiple convex yield modes intersecting in a non-smooth way.

The yield criterion is expressed in terms of a finite family of yield modes $y_i : X'_1 \mapsto \Re \cup \{+\infty\}$ with $i = 1, \ldots, m$ and of the corresponding yield limits which are positive monotone functions $y_{o_i} : \mathscr{L}_1 \mapsto \Re \cup \{+\infty\}$ depending upon a vectorial parameter $\boldsymbol{\lambda}_x \in \mathscr{L}_1 \equiv \Re^m$.

Accordingly, the elastic domain turns out to be the intersection of a finite number of convex domains:

$$K_x = \bigcap_{i=1}^{m} K_{ix} \quad \text{with } K_{ix} = \{\chi_x \in X'_1 \colon y_i(\chi_x) \leq y_{o_i}(\boldsymbol{\lambda}_x)\} .$$

Collecting the m yield modes y_i in the vector $\boldsymbol{y}$, the m functions y_{o_i} in the vector $\boldsymbol{y}_o$ and setting for convenience $\boldsymbol{h}(\chi_x) = \boldsymbol{y}(\chi_x) - \boldsymbol{y}_o(\boldsymbol{\lambda}_x)$, the flow rule

$$\dot{p}_x \in N_{K_x}(\chi_x)$$

can be written as

$$\dot{p}_x \in \partial \sqcup_{K_x}(\chi_x) = \partial(\sqcup_{\mathscr{L}_1^-} \circ \boldsymbol{h})(\chi_x) = \partial \sqcup_{\mathscr{L}_1^-} [\boldsymbol{h}(\chi_x)]\, \partial \boldsymbol{h}(\chi_x) = N_{\mathscr{L}_1^-}[\boldsymbol{h}(\chi_x)]\, \partial \boldsymbol{h}(\chi_x) ,$$

where $\mathscr{L}_1^- \equiv (\Re^m)^-$ is the cone of real m-tuples with non-positive components.

As in the uni-modal case, we assume that the time derivative of the parameter $\boldsymbol{\lambda}_x$ coincides with the value of the plastic multiplier $\boldsymbol{\alpha}_x \in N_{\mathscr{L}_1^-}[\boldsymbol{h}(\chi_x)]$. The flow-rule is then re-written in the form

$$\dot{p}_x \in \dot{\boldsymbol{\lambda}}_x \cdot \partial \boldsymbol{y}(\chi_x) , \qquad \dot{\boldsymbol{\lambda}}_x \in N_{\mathscr{L}_1^-}[\boldsymbol{h}(\chi_x)] ,$$

and its explicit formulation in terms of a linear complementarity condition reads

$$\dot{p}_x \in \dot{\boldsymbol{\lambda}}_x \cdot \partial \boldsymbol{y}(\chi_x) , \qquad \dot{\boldsymbol{\lambda}}_x \geqslant 0 , \qquad \boldsymbol{y}(\chi_x) - \boldsymbol{y}_o(\boldsymbol{\lambda}_x) \leqslant 0 , \qquad \dot{\boldsymbol{\lambda}}_x \cdot [\boldsymbol{y}(\chi_x) - \boldsymbol{y}_o(\boldsymbol{\lambda}_x)] = 0 ,$$

where the dot $\cdot$ denotes the scalar product in $\Re^m$.

According to a Euler backward difference scheme, the finite step flow rule is given by

$$p_x - p_{o_x} \in N_{K_x}(\chi_x) ;$$

equivalently, in terms of yield modes, it turns out to be

$$p_x - p_{o_x} \in (\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_{o_x}) \cdot \partial \boldsymbol{y}(\chi_x) , \qquad \boldsymbol{y}(\chi_x) - \boldsymbol{\lambda}_{o_x} \in \partial \sqcup_{\mathscr{L}_1^+} (\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_{o_x}) ,$$

or in explicit form

$$p_x - p_{o_x} \in \sum_{i=1}^{m} (\lambda_{ix} - \lambda_{o_{ix}}) \, \partial y_i(\chi_x) ,$$
$$y_i(\chi_x) - y_{o_i}(\boldsymbol{\lambda}_x) \in \partial \sqcup_{\Re^+} (\lambda_{ix} - \lambda_{o_{ix}}) , \quad i = 1, \ldots, m .$$

In order to derive the variational formulation of the structural problem, we refer in the sequel to global variables which are defined and labeled in perfect analogy with the case of uni-modal plasticity.

The finite step evolutive problem is thus governed by the following set of relations:

$$\begin{aligned}
&f = T'\sigma , \\
&\varepsilon = Tu , \\
&(p - p_o) \in (\boldsymbol{\lambda} - \boldsymbol{\lambda}_o) \cdot \partial \boldsymbol{Y}(\chi) , \\
&\boldsymbol{Y}(\chi) - \boldsymbol{Y}_o(\boldsymbol{\lambda}) \in \partial \sqcup_{\mathscr{L}^+} (\boldsymbol{\lambda} - \boldsymbol{\lambda}_o) , \\
&\sigma \in \partial \Phi(e) , \\
&\chi \in \partial \Phi(e) - \partial \Xi(p) , \\
&u \in \partial \Gamma^*(f) ,
\end{aligned}$$

which can be encompassed in a global structural operator whose conservativity can be inferred by repeating the same arguments of Section 5.

The potential of the structural operator is then

$$\begin{aligned}
\Omega(u, \sigma, \chi, \boldsymbol{\lambda}, \varepsilon, p, f) = {} & \Phi(\varepsilon - p) + \Xi(p) + \Gamma^*(f) - \langle f, u \rangle - \langle \sigma, \varepsilon \rangle + \langle \sigma, Tu \rangle + \langle \chi, p - p_o \rangle \\
& - \Psi(\boldsymbol{\lambda} - \boldsymbol{\lambda}_o, \chi) + \boldsymbol{\Pi}(\boldsymbol{\lambda}) + \sqcup_{\mathscr{L}^+} (\boldsymbol{\lambda} - \boldsymbol{\lambda}_o) ,
\end{aligned}$$

where the convex functional $\boldsymbol{\Pi} : \mathscr{L} \mapsto \Re$ is the potential of the operator $\boldsymbol{Y}_o : \mathscr{L} \mapsto \mathscr{L}'$, that is,

$$\boldsymbol{Y}_o(\boldsymbol{\lambda}) = \mathrm{d}\boldsymbol{\Pi}(\boldsymbol{\lambda}) .$$

In analogy with the uni-modal plasticity, if each y_i is sublinear in χ_x, we have for $\lambda_i \geqslant \lambda_{o_i}$,

$$\Psi_c(\boldsymbol{\lambda} - \boldsymbol{\lambda}_o, p - p_o) = \sqcup_C (p - p_o) ,$$

where $C = \Sigma_{i=1}^{m} (\lambda_i - \lambda_{o_i}) Y_{o_i}(\boldsymbol{\lambda}) K_i^{\circ}$ (see Appendix A), and K_i° is the polar set of K_i.

Enforcing the external constraint relation, kinematic compatibility and the relation

$$\Psi(\boldsymbol{\lambda} - \boldsymbol{\lambda}_o, \chi) + \Psi_c(\boldsymbol{\lambda} - \boldsymbol{\lambda}_o, p - p_o) = \langle \chi, p - p_o \rangle ,$$

the following functional can be derived from Ω:

$$\Omega_3(u, p, \boldsymbol{\lambda}) = \Phi(Tu - p) + \Xi(p) - \Gamma(u) + \sqcup_C (p - p_o) + \boldsymbol{\Pi}(\boldsymbol{\lambda}) + \sqcup_{\mathscr{L}^+} (\boldsymbol{\lambda} - \boldsymbol{\lambda}_o) ,$$

which is jointly convex in the three state variables $(u, p, \boldsymbol{\lambda})$.

We can thus state the following.

PROPOSITION 5. If the yield modes are sublinear, a triplet $(u, p, \boldsymbol{\lambda})$ is a solution of the optimization problem

$$\min\{\hat{\Omega}_3(u, p, \boldsymbol{\lambda}) \mid (\lambda_i - \lambda_{o_i}) \geq 0,\ i = 1, \ldots, m,\ p - p_o \in \sum_{i=1}^{m} (\lambda_i - \lambda_{o_i}) Y_{o_i}(\boldsymbol{\lambda}) K_i^{\circ}\} ,$$

with

$$\hat{\Omega}_1(u, p, \boldsymbol{\lambda}) = \Phi(Tu - p) + \Xi(p) - \Gamma(u) + \boldsymbol{\Pi}(\boldsymbol{\lambda}) ,$$

if and only if it is a solution of the finite step elastoplastic structural problem.

Appendix A

We provide here the expression of the set C_x in the case of sublinear yield modes y_i. To this end, we first recall that, given the closed convex elastic domain K_x,

$$K_x = \{\chi_x \in \mathscr{S}_1 : y(\chi_x) \leq y_o(\lambda_x)\} ,$$

the Minkowski formula associates with K_x a non-negative sublinear closed function $\gamma_{K_x} : X_1' \mapsto \Re \cup \{+\infty\}$ as follows:

$$\gamma_{K_x}(\chi_x) = \inf\{\alpha \geq 0 : \chi_x \in \alpha K_x\} .$$

The characteristic property of the functional γ_{K_x} is that its unitary level set yields back the set K_x.

Denoting by K_x° the closed convex set polar of K_x,

$$K_x^{\circ} = \{p_x \in \mathscr{D}_1 : \chi_x * p_x \leq 1\ \ \forall \chi_x \in K_x\} ,$$

we have

$$\gamma_{K_x}(\chi_x) = \sqcup_{K_x^{\circ}} (\chi_x) = \sup_{p_x \in K_x^{\circ}} \chi_x * p_x .$$

For the sake of simplicity the expression of C_x is derived separately for single and multiple yield modes.

A.1. Uni-modal plasticity

Assuming a sublinear yield mode y, we prove that, for $\lambda_x \geq \lambda_{o_x}$, we have

$$[(\lambda_x - \lambda_{o_x})y]^*(p_x - p_{o_x}) = \sqcup_{C_x}(p_x - p_{o_x}) ,$$

where

$$C_x = (\lambda_x - \lambda_{o_x})y_o(\lambda_x)K_x^\circ .$$

Actually, for any given $\lambda_x \geq \lambda_{o_x}$, the function $[(\lambda_x - \lambda_{o_x})y](\chi_x)$ is convex in χ_x and its conjugate is defined as

$$[(\lambda_x - \lambda_{o_x})y]^*(p_x - p_{o_x}) = \sup_{\chi_x} \{\chi_x * (p_x - p_{o_x}) - [(\lambda_x - \lambda_{o_x})y](\chi_x)\} .$$

Expressing the function y in the form

$$y(\chi_x) = y_o(\lambda_x)\gamma_{K_x}(\chi_x) ,$$

we can write

$$[(\lambda_x - \lambda_{o_x})y]^*(p_x - p_{o_x}) = \sup_{\chi_x} \{\chi_x * (p_x - p_{o_x}) - (\lambda_x - \lambda_{o_x})y_o(\lambda_x)\gamma_{K_x}(\chi_x)\} .$$

For $\lambda_x = \lambda_{o_x}$, we trivially have

$$[(\lambda_x - \lambda_{o_x})y]^*(p_x - p_{o_x}) = \sup_{\chi_x} \chi_x * (p_x - p_{o_x}) = \sqcup_{\{0\}}(p_x - p_{o_x}) ,$$

so that the formula to be proved holds with $C_x = \{0\}$.
For $\lambda_x > \lambda_{o_x}$, we have

$$\begin{aligned}
[(\lambda_x - \lambda_{o_x})y]^*(p_x - p_{o_x}) &= \sup_{\chi_x} \left\{\chi_x * \frac{p_x - p_{o_x}}{(\lambda_x - \lambda_{o_x})y_o(\lambda_x)} - \gamma_{K_x}(\chi_x)\right\}(\lambda_x - \lambda_{o_x})y_o(\lambda_x) \\
&= \sup_{\chi_x} \left\{\chi_x * \frac{p_x - p_{o_x}}{(\lambda_x - \lambda_{o_x})y_o(\lambda_x)} - \sqcup^*_{K_x^\circ}(\chi_x)\right\}(\lambda_x - \lambda_{o_x})y_o(\lambda_x) \\
&= \sqcup_{K_x^\circ}\left[\frac{p_x - p_{o_x}}{(\lambda_x - \lambda_{o_x})y_o(\lambda_x)}\right] = \sqcup_{C_x}(p_x - p_{o_x}) ,
\end{aligned}$$

with $C_x = (\lambda_x - \lambda_{o_x})y_o(\lambda_x)K_x^\circ$.

A.2. Multi-modal plasticity

Assuming a finite family of sublinear yield modes y_i, for any $\boldsymbol{\lambda}_x \geq \boldsymbol{\lambda}_{o_x}$, the conjugate of the convex function $[(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_{o_x}) \cdot \boldsymbol{y}](\chi_x)$ is given by

$$[(\boldsymbol{\lambda} - \boldsymbol{\lambda}_{o_x}) \cdot \boldsymbol{y}]^*(p_x - p_{o_x}) = \sqcup_{C_x}(p_x - p_{o_x}) ,$$

where

$$C_x = \sum_{i=1}^{m} (\lambda_{ix} - \lambda_{o_{ix}})y_{o_i}(\boldsymbol{\lambda}_x)K_{ix}^\circ .$$

Actually, expressing the function $y_i(\chi_x)$ in the form $y_{o_i}(\boldsymbol{\lambda}_x)\gamma_{K_{ix}}(\chi_x)$, the conjugate of the function $[(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_{o_x}) \cdot \boldsymbol{y}](\chi_x)$ is defined as

$$\begin{aligned}
[(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_{o_x}) \cdot \boldsymbol{y}]^*(p_x - p_{o_x}) &= \sup_{\chi_x} \left\{\chi_x * (p_x - p_{o_x}) - \sum_{i=1}^{m} (\lambda_{ix} - \lambda_{o_{ix}})y_i(\chi_x)\right\} \\
&= \sup_{\chi_x} \left\{\chi_x * (p_x - p_{o_x}) - \sum_{i=1}^{m} [(\lambda_{ix} - \lambda_{o_{ix}})y_{o_i}(\boldsymbol{\lambda}_x)]\gamma_{K_{ix}}(\chi_x)\right\} .
\end{aligned}$$

Setting for convenience $\beta_{ix} = (\lambda_{ix} - \lambda_{o_{ix}})y_{o_i}(\boldsymbol{\lambda}_x)$, we have

$$[(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_{o_x})\cdot y]^*(p_x - p_{o_x}) = \sup_{\chi_x} \left\{\chi_x * (p_x - p_{o_x}) - \sum_{i=1}^{m} \beta_{ix}\gamma_{K_{ix}}(\chi_x)\right\} = \left(\sum_{i=1}^{m} \beta_{ix}\gamma_{K_{ix}}\right)^*(p_x - p_{o_x}) .$$

Recalling that the conjugate of the finite sum of convex functions is given by their infimal convolution [24],

$$\left(\sum_{i=1}^{m} \beta_{ix}\gamma_{K_{ix}}\right)^*(p_x - p_{o_x}) = \inf\left\{(\beta_{1x}\gamma_{K_{1x}})^*(\eta_{1x}) + \cdots + (\beta_{mx}\gamma_{K_{mx}})^*(\eta_{mx}) \,\middle|\, \sum_{i=1}^{m} \eta_{ix} = p_x - p_{o_x}\right\},$$

and observing that

$$(\beta_{ix}\gamma_{K_{ix}})^*(\eta_{ix}) = \sqcup_{(\beta_{ix}K_{ix}^{\circ})}(\eta_{ix}) ,$$

we finally obtain

$$\begin{aligned}[(\boldsymbol{\lambda}_x - \boldsymbol{\lambda}_{o_x})\cdot y]^*(p_x - p_{o_x}) &= \inf\left\{\sqcup_{(\beta_{1x}K_{1x}^{\circ})}(\eta_{1x}) + \cdots + \sqcup_{(\beta_{mx}K_{mx}^{\circ})}(\eta_{mx}) \,\middle|\, \sum_{i=1}^{m} \eta_{ix} = p_x - p_{o_x}\right\} \\ &= \sqcup_{C_x}(p_x - p_{o_x}) ,\end{aligned}$$

where

$$C_x = \sum_{i=1}^{m} \beta_{ix}K_{ix}^{\circ} .$$

Acknowledgment

The financial support of the Italian Ministry for Scientific and Technological Research is gratefully acknowledged.

References

[1] G. Maier and A. Nappi, Backward difference time integration, nonlinear programming and extremum theorems in elastoplastic analysis, SM Arch. 14 (1989) 37–64.
[2] G. Maier and G. Novati, Extremum theorems for finite-step backward-difference analysis of elastic-plastic nonlinearly hardening solids, Internat. J. Plasticity 6 (1990) 1–10.
[3] C. Comi and G. Maier, Extremum theorem and convergence criterion for an iterative solution to the finite-step problem in elastoplasticity with mixed nonlinear hardening, European J. Mech., A/Solids 9 (1990) 563–585.
[4] C. Comi, A. Corigliano and G. Maier, Extremum properties of finite-step solutions in elastoplasticity with nonlinear mixed hardening, Internat. J. Solids and Structures 27 (1991) 965–981.
[5] C. Comi, G. Maier and U. Perego, Generalized variable finite element modelling and extremum theorems in stepwise holonomic elastoplasticity with internal variables, Comput. Methods Appl. Mech. Engrg. 96 (1992) 213–237.
[6] G. Romano, L. Rosati, F. Marotti de Sciarra and P. Bisegna, A potential theory for monotone multi-valued operators, Quart. Appl. Math. (1993) in press.
[7] B. Halphen and Q.S. Nguyen, Sur les matériaux standards généralisés, J. Mech. 14 (1975) 39–63.
[8] J.B. Martin, Plasticity (MIT Press, Cambridge, MA, 1975).
[9] Q.S. Nguyen, On the elastic plastic initial-boundary value problem and its numerical integration, Internat. J. Numer. Methods Engrg. 11 (1977) 817–832.
[10] J.B. Martin, An internal variable approach to the formulation of finite element problems in plasticity, in: J. Hult and J. Lemaitre, eds., Physical Non-Linearities in Structural Analysis (Springer, Berlin, 1981).
[11] M. Ortiz and J.C. Simo, An analysis of a new class of integration algorithms for elastoplastic constitutive relations, Internat. J. Numer. Methods Engrg. 23 (1986) 353–366.
[12] J.B. Martin and B.D. Reddy, Variational principles and solution algorithms for internal variable formulations of problems in plasticity, Volume in Onore del Prof. Giulio Ceradini, Rome (1988) 465–477.

[13] J.C. Simo, A framework for finite strain elastoplasticity based on maximum plastic dissipation and multiplicative decomposition: Part I. Continuum formulation. Comput. Methods Appl. Mech. Engrg. 66 (1988) 199–219.
[14] J.C. Simo, J.J. Kennedy and S. Govindjee, Non-smooth multisurface plasticity and viscoplasticity. Loading/unloading conditions and numerical algorithms, Internat. J. Numer. Methods Engrg. 26 (1988) 2161–2185.
[15] R.A. Eve, B.D. Reddy and R.T. Rockafellar, An internal variable theory of plasticity based on the maximum plastic work inequality, Quart. Appl. Math. 68 (1990) 59–83.
[16] J.B. Martin and A. Nappi, An internal variable formulation for perfectly plastic and linear hardening relations in plasticity, European J. Mech., A/Solids 9 (1990) 107–131.
[17] G. Romano, L. Rosati and F. Marotti de Sciarra, An internal variable theory of inelastic behaviour derived from the uniaxial rigid-perfectly plastic law, Internat. J. Engrg. Sci. 31 (1992) 1105–1120.
[18] J.C. Simo and T. Honein, Variational formulations, discrete conservation laws, and path-domain independent integrals for elasto-viscoplasticity, J. Appl. Mech., Trans. ASME 57 (1990) 488–497.
[19] G. Romano, L. Rosati and F. Marotti de Sciarra, Variational principles for finite-step solutions in elasto-plasticity with internal variables, Atti XI Congr. Naz. Ass. Ital. Mec. Teo. Appl. (A.I.M.E.T.A.), Trento (1992).
[20] G. Romano, L. Rosati and F. Marotti de Sciarra, Variational formulations of non-linear and non-smooth structural problems, Internat. J. Non-Linear Mech. 28 (1993) 195–208.
[21] J.P. Aubin and I. Ekeland, Applied Nonlinear Analysis (Wiley, New York, 1984).
[22] I. Ekeland and R. Temam, Convex Analysis and Variational Problems (North-Holland, Amsterdam, 1976).
[23] A.D. Ioffe and V.M. Tihomirov, The Theory of Extremal Problems (Nauka, Moscow); English translation (North-Holland, Amsterdam, 1979).
[24] R.T. Rockafellar, Convex Analysis (Princeton Univ. Press, Princeton, NJ, 1970).
[25] J.J. Moreau, Fonctionelles Convexes, Lecture Notes, Séminaire Equationes aux dérivées partielles, Collègie de France, 1966.
[26] G. Romano, L. Rosati and F. Marotti de Sciarra, Convex Problems in Structural Mechanics, DISCO, Lecture notes, Naples, 1992.
[27] P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications (Birkhauser, Basel, 1985).
[28] G. Romano and L. Rosati, Variational principles in convex structural analysis, Volume in Onore del Prof. Giulio Ceradini, Rome (1988) 591–603.
[29] L. Rosati, Problemi di Meccanica delle Strutture con Vincoli Convessi, PhD Dissertation, Napoli, 1989.
[30] D.G. Luenberger, Introduction to Linear and Non-Linear Programming (Addison-Wesley, Reading, MA, 1973).
[31] R. Fletcher, Practical Methods of Optimization, 2nd Edition (Wiley, New Delhi, 1989).