Basic issues in convex homogenization

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Summary. The basic results in homogenization theory are revisited in the abstract context of continuum mechanics in which the constitutive behaviour and the kinematic constraints are governed by pairs of conjugate convex potentials. The theory and the methods of this generalized elastic model are briefly recalled and applied to extend the classical linear theory of homogenization to the nonlinear and possibly multivalued constitutive framework.

1 Prolegomena

The fundamentals of homogenization theory are here revisited with reference to an abstract structural model whose constitutive properties are characterized by monotone conservative multivalued laws governed by closed convex potentials.

The theory of such constitutive behaviour, termed generalized elasticity, was developed by the first author and his co-workers in a number of papers (see [10], [12]) and is illustrated in detail in [17].

The topic of nonlinear homogenization theory was investigated by Talbot, Willis and Toland in a series of papers [7], [8], [9]. Their approach was based on the theory of conjugate convex problems has developed in [5]. The present approach makes direct reference to an abstract structural problem and is carried out under the guidelines of the theory of generalized elasticity.

2 The continuum model

In continuum mechanics the fundamental theorems concerning the variational formulations of equilibrium and of tangent compatibility are founded on the property that the tangent kinematic operator has a closed range and a finite dimensional kernel at every configuration in the admissible manifold.

The abstract framework is the following. Let V and D be the finite dimensional linear spaces of local values of tangent (virtual) displacememts (also referred to as kinematic fields) and tangent strains respectively. Let further S be the linear space of local values of stress fields, the dual space of D.

A continuous structural model, defined on a regular bounded connected domain Ω of an n-dimensional euclidean space E^n, is governed by a kinematic operator $\mathbf{B}$. This operator is the regular part of a distributional differential operator $\mathbb{B} : \mathcal{V}_\Omega \mapsto \mathbb{D}'_S$ of order m acting on GREEN-regular kinematic fields $\mathbf{u} \in \mathcal{V}_\Omega$ and ranging in the space of tangent strain distributions $\mathbb{B}\mathbf{u} \in \mathbb{D}'_S$ in Ω. Tangent strain distributions are linear functionals, defined on the linear space $\mathbb{D}_S = \mathrm{C}^\infty_o(\Omega\,;\, S)$ of test stress field with compact support in Ω, and continuous according to the uniform topology on compact subsets of Ω (see e.g. [4], [16]).

Piecewise GREEN-regular kinematic fields $\mathbf{u} \in \mathcal{V}_\Omega$ are square integrable fields $\mathbf{u} \in H_{\mathrm{V}} = \mathcal{L}^2(\Omega\,;\, \mathrm{V})$ such that the corresponding distributional tangent strain fields $\mathbb{B}\mathbf{u} \in \mathbb{D}'_S$ are square integrable on a finite subdivision $\mathcal{T}_{\mathbf{u}}(\Omega)$ of Ω (see [16], [18], [19]). The kinematic space $\mathcal{V}_\Omega$ is a pre-HILBERT space when endowed with the topology induced by the norm

$$\| \mathbf{u} \|^2_{\mathcal{V}_\Omega} = \| \mathbf{u} \|^2_{H_{\mathrm{V}}} + \| \mathbf{B}\mathbf{u} \|^2_{\mathcal{H}_D} \,,$$

where $\mathcal{H}_D = \mathcal{L}^2(\Omega\,;\, D)$ is the space of square integrable tangent strain fields on Ω. The subdivision $\mathcal{T}_{\mathbf{u}}(\Omega)$ is said to be a support of regularity of the kinematic field $\mathbf{u} \in \mathcal{V}_\Omega$.

The kinematic constrains on a continuum are imposed by a sequence of two requirements. The first is a regularity requirement on the tangent displacements and is expresseed by considering a basic finite subdivision $\mathcal{T}(\Omega)$ of Ω and by imposing that the tangent displacements must have $\mathcal{T}(\Omega)$ as a support of regularity. The closed linear subspace $\mathcal{V}(\mathcal{T}(\Omega)) \subset \mathcal{V}_\Omega$ of $\mathcal{T}(\Omega)$-regular tangent displacements is an HILBERT space for the topology of $\mathcal{V}_\Omega$.

The second requirement is that tangent displacements must belong to a conformity subspace, a closed linear subspace $\mathcal{L} = \mathcal{L}(\mathcal{T}(\Omega)) \subset \mathcal{V}(\mathcal{T}(\Omega))$.

In boundary value problems the HILBERT space $\mathcal{L}$ is the kernel of a bounded linear operator which prescribes an additional linear constraint on the boundary values of the tangent displacements $\mathbf{u} \in \mathcal{V}(\mathcal{T}(\Omega))$.

The operator $\mathbf{B}_{\mathcal{L}} \in BL(\mathcal{L}\,;\, \mathcal{H}_D)$, which yields the regular tangent strain $\mathbf{B}\mathbf{u} \in \mathcal{H}_D$ corresponding to a conforming tangent displacement $\mathbf{u} \in \mathcal{L}$ is linear and continuous.

The tangent kinematic operator $\mathbf{B} \in BL(\mathcal{V}_\Omega\,;\, \mathcal{H}_D)$ is assumed to be KORN-regular in the sense that for any conformity subspace $\mathcal{L} \subset \mathcal{V}_\Omega$ the following conditions are met [13], [14]:

$$\begin{cases} \dim \operatorname{Ker} \mathbf{B}_{\mathcal{L}} = \dim \left(\operatorname{Ker} \mathbf{B} \cap \mathcal{L} \right) < +\infty \,, \\ \| \mathbf{B}\mathbf{u} \|_{\mathcal{H}} \geq c_{\mathbf{B}} \, \| \mathbf{u} \|_{\mathcal{L}/\operatorname{Ker} \mathbf{B}_{\mathcal{L}}} \,, \quad \forall\, \mathbf{u} \in \mathcal{L} \quad \Longleftrightarrow \quad \operatorname{Im} \mathbf{B}_{\mathcal{L}} \ \text{ closed in } \ \mathcal{H}_D \,. \end{cases}$$

The requirement that these properties must hold for any conformity subspace $\mathcal{L} \subset \mathcal{V}_\Omega$ is motivated by the requirement that in engineering structural models the equilibrium condition can be imposed by a finite number of scalar equations and that the existence results must hold for any choice of linear kinematic constraints. The KORN-regularity of $\mathbf{B} \in BL(\mathcal{V}_\Omega\,;\,\mathcal{H}_D)$ is the basic tool for the proof of the theorem of virtual powers [16] which ensures the existence of a stress field $\boldsymbol{\sigma} \in \mathcal{H}_S = \mathcal{L}^2(\Omega\,;\,S)$ in equilibrium with an equilibrated system of active forces, i.e. bounded linear functional $\mathbf{f} \in \mathcal{L}'$ such that $\langle\, \mathbf{f}\,,\,\mathbf{v}\,\rangle = 0$ for all $\mathbf{v} \in \operatorname{Ker}\mathbf{B} \cap \mathcal{L}$. It can be shown [14] that a necessary and sufficient condition for the KORN-regularity of $\mathbf{B} \in BL(\mathcal{V}_\Omega\,;\,\mathcal{H}_D)$ is the validity of an inequality of the KORN's type

$$\|\,\mathbf{B}\mathbf{u}\,\|_{\mathcal{H}_D} + \|\,\mathbf{u}\,\|_H \geq \alpha\,\|\,\mathbf{u}\,\|_m\,, \quad \forall\,\mathbf{u} \in H^m(\Omega\,;\,\mathrm{V})\,,$$

where $H^m(\Omega\,;\,\mathrm{V})$ is the SOBOLEV space of tangent displacements which are square integrable on Ω with their distributional derivative up to the order m. The formal adjoint of $\mathbf{B} \in BL(\mathcal{V}_\Omega\,;\,\mathcal{H}_D)$ is the distributional differential operator $\mathbb{B}'_o : \mathcal{H}_S \mapsto \mathbb{D}'_{\mathrm{V}}$ of order m defined by the identity

$$\langle\, \mathbb{B}'_o\boldsymbol{\sigma}\,,\,\mathbf{v}\,\rangle := (\!(\,\boldsymbol{\sigma}\,,\,\mathbf{B}\mathbf{v}\,)\!)\,, \qquad \forall\,\mathbf{v} \in \mathbb{D}'_{\mathrm{V}}\,, \quad \forall\,\boldsymbol{\sigma} \in \mathcal{H}(\Omega)\,.$$

The space $\mathcal{S}_\Omega$ of piecewise GREEN-regular stress fields on Ω is then defined as the linear space of stress fields $\boldsymbol{\sigma} \in \mathcal{H}_S$ such that the corresponding body force distributions $\mathbb{B}'_o\boldsymbol{\sigma} \in \mathbb{D}'_{\mathrm{V}}$, are square integrable on a finite subdivision $\mathcal{T}_{\boldsymbol{\sigma}}(\Omega)$ of Ω (see [16], [18]). The space $\mathcal{S}_\Omega$ is a pre-HILBERT space when endowed with the induced norm

$$\|\,\boldsymbol{\sigma}\,\|^2_{\mathcal{S}_\Omega} = \|\,\boldsymbol{\sigma}\,\|^2_{\mathcal{H}_S} + \|\,\mathbf{B}'_o\boldsymbol{\sigma}\,\|^2_{H_F}\,,$$

where $\mathbf{B}'_o \in BL(\mathcal{S}_\Omega\,;\,\mathcal{H}_S)$ is the regular part of the distributional differential operator $\mathbb{B}'_o : \mathcal{H}_S \mapsto \mathbb{D}'_{\mathrm{V}}$. Any pair of GREEN-regular tangent displacement fields $\mathbf{v} \in \mathcal{V}_\Omega$ and GREEN-regular stress fields $\boldsymbol{\sigma} \in \mathcal{S}_\Omega$ fulfil the GREEN's formula for the operator $\mathbf{B} \in BL(\mathcal{V}_\Omega, \mathcal{H}_D)$ [15]:

$$(\!(\,\boldsymbol{\sigma}\,,\,\mathbf{B}\mathbf{v}\,)\!) = (\,\mathbf{B}'_o\boldsymbol{\sigma}\,,\,\mathbf{v}\,) + \langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}\,,\,\boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle\,, \quad \forall\,\mathbf{v} \in \mathcal{V}_\Omega, \quad \forall\,\boldsymbol{\sigma} \in \mathcal{S}_\Omega,$$

where by definition

$$(\!(\,\boldsymbol{\sigma}\,,\,\mathbf{B}\mathbf{v}\,)\!) := \int_\Omega \boldsymbol{\sigma} : \mathbf{B}\mathbf{v}\;\mathrm{d}\mu\,, \qquad (\,\mathbf{B}'_o\boldsymbol{\sigma}\,,\,\mathbf{v}\,) := \int_\Omega \mathbf{B}'_o\boldsymbol{\sigma}\cdot\mathbf{v}\;\mathrm{d}\mu\,,$$

and the duality pairing $\langle\!\langle\, \mathbf{N}\boldsymbol{\sigma}\,,\,\boldsymbol{\Gamma}\mathbf{v}\,\rangle\!\rangle$ is the extension by continuity of a sum of boundary integrals over $\partial\mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega) = \cup\,\partial\Omega_e,\ e = 1, \ldots, \mathrm{n_{elements}}$:

$$\int_{\partial\mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega)} \mathbf{N}\boldsymbol{\sigma}\cdot\boldsymbol{\Gamma}\mathbf{v}\;\mathrm{d}ß\,.$$

where $\mathcal{T}_{\mathbf{v}\boldsymbol{\sigma}}(\Omega) = \mathcal{T}_{\mathbf{v}}(\Omega) \vee \mathcal{T}_{\boldsymbol{\sigma}}(\Omega)$ is finer than $\mathcal{T}_{\mathbf{v}}(\Omega)$ and $\mathcal{T}_{\boldsymbol{\sigma}}(\Omega)$.

The trace $\boldsymbol{\Gamma}$ and the flux $\mathbf{N}$ are differential operators, with order ranging between 0 and $m-1$, associated to the operator $\mathbf{B}$ and defined by m subsequent applications of the rule of integration by parts.

2.1 Averaging operators

Let $\mathbf{M}_\Omega \in BL(\mathcal{H}_D\,;\,D)$ and $\textsc{med} \in BL(\mathcal{H}_D\,;\,D)$ be the surjective averaging operators defined by

$$\mathbf{M}_\Omega(\boldsymbol{\varepsilon}) := \int_\Omega \boldsymbol{\varepsilon}(\mathbf{x})\,\mathtt{d}\mu\,, \qquad \textsc{med}_\Omega = \frac{1}{\mathrm{vol}\,(\Omega)}\,\mathbf{M}_\Omega\,.$$

The dual operator $\mathbf{M}^*_\Omega \in BL(S\,;\,\mathcal{H}_S)$ of $\mathbf{M}_\Omega \in BL(\mathcal{H}_D\,;\,D)$ is defined by the identity

$$\langle\, \mathbf{M}^*_\Omega(\mathbf{T})\,,\,\boldsymbol{\varepsilon}\,\rangle = \langle\,\mathbf{T}\,,\,\mathbf{M}_\Omega(\boldsymbol{\varepsilon})\,\rangle\,, \quad \forall\,\mathbf{T}\in S \quad \boldsymbol{\varepsilon}\in\mathcal{H}_D\,.$$

When applied to $\mathbf{T}\in S$ the operator $\mathbf{M}^*_\Omega \in BL(S\,;\,\mathcal{H}_S)$ provides the constant field in $\mathcal{H}_S = \mathcal{L}^2(\Omega\,;\,S)$ given by

$$(\mathbf{M}^*_\Omega(\mathbf{T}))(\mathbf{x}) = \mathbf{T}\,, \quad \forall\,\mathbf{x}\in\Omega\,.$$

Note that the roles of the spaces D and S may be interchanged in the preceeding definitions. We remark that $\textsc{med}_\Omega \in BL(\mathcal{H}_S\,;\,S)$ is a left inverse of $\mathbf{M}^*_\Omega \in BL(S\,;\,\mathcal{H}_S)$ since

$$(\textsc{med}_\Omega \circ \mathbf{M}^*_\Omega)(\mathbf{T}) = \mathbf{T}\,, \quad \forall\,\mathbf{T}\in S\,.$$

The surjectivity of $\mathbf{M}_\Omega \in BL(\mathcal{H}_D\,;\,D)$ yields

$$\mathrm{Im}\,\mathbf{M}^*_\Omega = (\,\mathrm{Ker}\,\mathbf{M}_\Omega)^\perp\,,$$

which implies that a square integrable field, orthogonal to any square integrable field with vanishing mean value, must be constant. Trivially we also have that

$$\mathrm{Ker}\,\mathbf{M}^*_\Omega = (\,\mathrm{Im}\,\mathbf{M}_\Omega)^\perp = \{\,0\,\}\,.$$

To symplify the notations we shall denote by the same symbols $\mathbf{M}_\Omega$ and $\mathbf{M}^*_\Omega$ also the operators $\mathbf{M}_\Omega \in BL(\mathcal{L}^1(\Omega\,;\,\mathcal{R})\,;\,\mathcal{R})$, $\mathbf{M}^*_\Omega \in BL(\mathcal{R}\,;\,\mathcal{L}^\infty(\Omega\,;\,\mathcal{R}))$ where $\mathcal{L}^1(\Omega\,;\,\mathcal{R})$ is the space of real valued integrable functions on Ω and $\mathcal{L}^\infty(\Omega\,;\,\mathcal{R})$ is the dual space of essentially bounded functions on Ω.

2.2 Conjugate convex potentials

A structural model is defined by considering a subdivision $\mathcal{T}(\Omega)$ of the domain Ω and the associated Hilbert space $\mathcal{V} = \mathcal{V}(\mathcal{T}(\Omega), \mathrm{V})$ of $\mathcal{T}(\Omega)$-regular displacements defined as those giving rise to distributional tangent

strain fields which are square integrable in each element of the subdivision. Force systems are the bounded linear functionals of the dual space $\mathcal{F} = BL(\mathcal{V}(\mathcal{T}(\Omega), \mathrm{V}); \mathcal{R})$.

The model is further characterized by a bounded linear tangent kinematic operator $\mathbf{B} \in BL(\mathcal{V}; \mathcal{H})$ which provides the tangent strain field corresponding to any $\mathcal{T}(\Omega)$-regular tangent displacement field. The operator $\mathbf{B} \in BL(\mathcal{V}; \mathcal{H})$ is assumed to fulfill an inequality of KORN's type so that the kernel $\operatorname{Ker} \mathbf{B} \subset \mathcal{V}$ is finite dimensional and, for any set of linear constraints defining a closed linear subspace $\mathcal{L} \subset \mathcal{V}$ of conforming displacements, the image $\mathbf{B}\mathcal{L}$ is closed in $\mathcal{H}$. The dual equilibrium operator $\mathbf{B}' \in BL(\mathcal{H}; \mathcal{F})$ is defined by the identity

$$\langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{v}\,\rangle = \langle\, \mathbf{B}'\boldsymbol{\sigma}\,,\, \mathbf{v}\,\rangle\,, \qquad \forall\, \mathbf{v} \in \mathcal{V}\,, \quad \forall\, \boldsymbol{\sigma} \in \mathcal{H}_S = \mathcal{L}^2(\Omega; S)\,.$$

The constitutive properties of the elastic material are described, according to GREEN [**?**], by a field of local potentials $\varphi_{\mathbf{e}} : D \times \Omega \mapsto \overline{\mathcal{R}}$, where $\overline{\mathcal{R}} = \mathcal{R} \cup \{+\infty\}$ is the upper-extended real line. We consider a generalized GREEN elasticity in which at each $\mathbf{x} \in \Omega$ the local potential is assumed to be proper, convex and everywhere subdifferentiable on its domain $\operatorname{dom} \varphi_{\mathbf{e}}(\cdot, \mathbf{x}) \subset D$. Convex analysis provides the mathematical tools to deal with such problems [3], [5], [6], [12]. In this context a potential theory for monotone conservative multivalued operators was developed by the first author and his coworkers, [10], [17].

The convex global constitutive potential $\Phi_{\mathbf{e}} : \mathcal{H} \mapsto \overline{\mathcal{R}}$ is a function of the (small) strain fields $\boldsymbol{\varepsilon} \in \mathcal{H}$ and is defined by the integral

$$\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) \;:= \int_{\Omega} (\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}))(\mathbf{x})\, \mathtt{d}\mu\,,$$

where the potential $\Phi_{\mathbf{e}} : \mathcal{H} \mapsto \mathcal{L}^2(\Omega; \overline{\mathcal{R}})$ is given by

$$(\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}))(\mathbf{x}) \;:= \varphi_{\mathbf{e}}(\boldsymbol{\varepsilon}(\mathbf{x}), \mathbf{x})\,,$$

and $\mathtt{d}\mu$ is the volume form on Ω.

We consider a general nondecreasing monotone and conservative stress-strain relation $\mathcal{G} \subset \mathcal{H}_S \times \mathcal{H}_D$. Monotonicity means that

$$\langle\, \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\,,\, \boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1\,\rangle \geq 0 \qquad \forall\, \{\boldsymbol{\sigma}_1\,, \boldsymbol{\varepsilon}_1\} \in \mathcal{G}, \quad \forall\, \{\boldsymbol{\sigma}_2\,, \boldsymbol{\varepsilon}_2\} \in \mathcal{G}\,,$$

and conservativity means that

$$\oint_{\Pi_{\boldsymbol{\varepsilon}}} \langle\, \mathcal{E}(\boldsymbol{\varepsilon})\,,\, d\boldsymbol{\varepsilon}\,\rangle = 0 \quad \Longleftrightarrow \quad \oint_{\Pi_{\boldsymbol{\sigma}}} \langle\, \mathcal{E}^{-1}(\boldsymbol{\sigma})\,,\, d\boldsymbol{\sigma}\,\rangle = 0\,,$$

where $\Pi_{\boldsymbol{\varepsilon}} \subset \mathcal{H}_S$, $\Pi_{\boldsymbol{\sigma}} \subset \mathcal{H}_D$ are closed polylines and $\mathcal{E} : \mathcal{H}_D \mapsto \mathcal{H}_S$, $\mathcal{E}^{-1} : \mathcal{H}_S \mapsto \mathcal{H}_D$ are the left and right multivalued maps associated with the relation $\mathcal{G}$ and defined by

$$\mathcal{E}(\varepsilon) := \{\, \boldsymbol{\sigma} \in \mathcal{L}^2(\Omega\,;\, S) \mid \{\boldsymbol{\sigma}\,, \varepsilon\} \in \mathcal{G}\,\}\,,$$
$$\mathcal{E}^{-1}(\boldsymbol{\sigma}) := \{\, \boldsymbol{\varepsilon} \in \mathcal{L}^2(\Omega\,;\, D) \mid \{\boldsymbol{\sigma}\,, \boldsymbol{\varepsilon}\} \in \mathcal{G}\,\}\,.$$

The domains $\operatorname{dom} \mathcal{E} \in \mathcal{L}^2(\Omega\,;\, D)\,,\ \operatorname{dom} \mathcal{E}^{-1} \in \mathcal{L}^2(\Omega\,;\, S)$, the loci where the images $\mathcal{E}(\boldsymbol{\varepsilon})$ and $\mathcal{E}^{-1}(\boldsymbol{\sigma})$ are non-empty, are assumed to be convex sets.

It can be shown that the integrals along segments are independent of the special representative in the sets $\mathcal{E}(\boldsymbol{\varepsilon})$ and $\mathcal{E}^{-1}(\boldsymbol{\sigma})$ choosen to evaluate the integrands [10]. A multivalued monotone and conservative relation is governed by a pair of conjugate convex potentials $\Phi_{\mathbf{e}} : \mathcal{H}_D \mapsto \overline{\mathcal{R}}$ and $\Phi^*_{\mathbf{e}} : \mathcal{H}_S \mapsto \overline{\mathcal{R}}$ related by the involutive relation

$$\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) := \sup_{\boldsymbol{\varepsilon} \in \mathcal{H}_D} \{\, \langle\, \boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,\rangle - \Phi(\boldsymbol{\varepsilon})\,\}\,,$$
$$\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) := \sup_{\boldsymbol{\sigma} \in \mathcal{H}_S} \{\, \langle\, \boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,\rangle - \Phi^*(\boldsymbol{\sigma})\,\}\,.$$

The conjugate convex potentials $\Phi_{\mathbf{e}} : \mathcal{H}_D \mapsto \overline{\mathcal{R}}$ and $\Phi^*_{\mathbf{e}} : \mathcal{H}_S \mapsto \overline{\mathcal{R}}$ can be evaluated by direct integration of the multivalued maps along a segment or by the conjugacy relations above.

The effective domains $\operatorname{dom} \Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) \subset \mathcal{H}_D$ and $\operatorname{dom} \Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) \subset \mathcal{H}_S$ are the convex sets where the potentials $\Phi_{\mathbf{e}} : \mathcal{H}_D \mapsto \overline{\mathcal{R}}$ and $\Phi^*_{\mathbf{e}} : \mathcal{H}_S \mapsto \overline{\mathcal{R}}$ assume finite values in $\mathcal{R}$. The convex potentials $\Phi_{\mathbf{e}} : \mathcal{H}_D \mapsto \overline{\mathcal{R}}$ and $\Phi^*_{\mathbf{e}} : \mathcal{H}_S \mapsto \overline{\mathcal{R}}$ are subdifferentiable in their domains. The subdifferentials are the convex sets defined by [3], [6], [11]

$$\partial\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) := \{\, \boldsymbol{\sigma} \in \mathcal{H}_S \mid \Phi_{\mathbf{e}}(\overline{\boldsymbol{\varepsilon}}) - \Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) \geq \langle\, \boldsymbol{\sigma}\,,\, \overline{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}\,\rangle\,\}\,,$$
$$\partial\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) := \{\, \boldsymbol{\varepsilon} \in \mathcal{H}_D \mid \Phi^*_{\mathbf{e}}(\overline{\boldsymbol{\sigma}}) - \Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) \geq \langle\, \overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,\rangle\,\}\,.$$

The global generalized elastic law is expressed by the subdifferential maps:

$$\boldsymbol{\sigma} \in \partial\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon})\,, \quad \boldsymbol{\varepsilon} \in \partial\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma})\,.$$

By definition we have that

$$\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) + \Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) \geq \langle\, \boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,\rangle\,, \qquad \forall\, \boldsymbol{\varepsilon} \in \mathcal{H}_D \quad \forall\, \boldsymbol{\sigma} \in \mathcal{H}_S\,,$$
$$\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) + \Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) = \langle\, \boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,\rangle \iff \boldsymbol{\sigma} \in \partial\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}) \iff \boldsymbol{\varepsilon} \in \partial\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma})\,.$$

Recalling that the elastic law is pointwise defined, we remark that the convex conjugate $\varphi^*_{\mathbf{e}} : S \times \Omega \mapsto \overline{\mathcal{R}}$ of the local potential $\varphi_{\mathbf{e}} : D \times \Omega \mapsto \overline{\mathcal{R}}$ is given by

$$\varphi^*_{\mathbf{e}}(\mathbf{T}, \mathbf{x}) := \sup_{\mathbf{D} \in D} \{\, \langle\, \mathbf{T}\,,\, \mathbf{D}\,\rangle - \varphi_{\mathbf{e}}(\mathbf{D}, \mathbf{x})\,\}\,.$$

The global convex potential $\Phi^*_{\mathbf{e}} : \mathcal{H}_S \mapsto \overline{\mathcal{R}}$, convex conjugate to $\Phi_{\mathbf{e}} : \mathcal{H}_D \mapsto \overline{\mathcal{R}}$, can then be evaluated by each one of the following procedures [17]:

$$\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) := \sup_{\boldsymbol{\eta} \in \mathcal{H}_D} \{\, \langle\, \boldsymbol{\sigma}\,,\, \boldsymbol{\eta}\,\rangle - \Phi_{\mathbf{e}}(\boldsymbol{\eta})\,\}\,,$$

$$\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) := \int_{\Omega} (\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}))(\mathbf{x}) \, \mathtt{d}\mu \, .$$

In an analogous way, kinematic constraints are described by a conservative multivalued monotone nonincreasing relation $\mathcal{G} \subset \mathcal{F} \times \mathcal{V}$ and by the pair of conjugate proper superdifferentiable concave functional $J : \mathcal{V} \mapsto \underline{\mathcal{R}}$ and $J^* : \mathcal{F} \mapsto \underline{\mathcal{R}}$ where $\underline{\mathcal{R}} := \mathcal{R} \cup \{ -\infty \}$ is the lower extended real line [17].

We remark that kinematic constraint conditions are global in character and accordingly $J : \mathcal{V} \mapsto \underline{\mathcal{R}}$ and $J^* : \mathcal{F} \mapsto \underline{\mathcal{R}}$ are global functionals which may not be defined as integrals of local functionals.

The constraint map is nonincreasing since it provides the relation between the diplacement fields of the constraint and the force systems that the constraint applies to the structure, that is the opposite of the force systems acted by the structure on the constraint. This change in sign turns the monotone nondecreasing constitutive map into a nonincreasing one.

Multivaluedness of the constraint relations is the rule rather then the exception: also the simplest linear frictionless bilateral kinematic constraint relation is described by multivalued maps. if $\mathcal{L}$ is the subspace of conforming virtual displacements the constraint relation is

$$\mathcal{G} := \{ \{\mathbf{r}, \mathbf{v}\} \in \mathcal{F} \times \mathcal{V} \mid \mathbf{v} \in \mathcal{L}, \quad \mathbf{r} \in \mathcal{L}^{\perp} \},$$

Both the left and right map are constant:

$$\mathcal{M}(\mathbf{v}) := \mathcal{L}^{\perp}, \quad \mathcal{M}^{-1}(\mathbf{r}) := \mathcal{L}.$$

In general reactive force systems are conjugate to the displacements with respect to the concave functional $J : \mathcal{V} \mapsto \underline{\mathcal{R}}$:

$$\mathbf{r} \in \partial J(\mathbf{u}) \quad \Longleftrightarrow \quad J(\mathbf{v}) - J(\mathbf{u}) \leq \langle \, \mathbf{r}, \mathbf{v} - \mathbf{u} \, \rangle \qquad \forall \, \mathbf{u} \in \mathcal{V}.$$

The inverse multivalued law is expressed by

$$\mathbf{u} \in \partial J^*(\mathbf{r}) \quad \Longleftrightarrow \quad J^*(\bar{\mathbf{r}}) - J^*(\mathbf{r}) \leq \langle \, \bar{\mathbf{r}} - \mathbf{r}, \mathbf{u} \, \rangle \qquad \forall \, \bar{\mathbf{r}} \in \mathcal{F}.$$

By definition we have that

$$\begin{aligned} &J(\mathbf{v}) + J^*(\mathbf{r}) \leq \langle \, \mathbf{r}, \mathbf{v} \, \rangle, \qquad \forall \, \mathbf{v} \in \mathcal{V} \quad \forall \, \mathbf{r} \in \mathcal{F}, \\ &J(\mathbf{u}) + J^*(\mathbf{r}) = \langle \, \mathbf{r}, \mathbf{u} \, \rangle \quad \Longleftrightarrow \quad \mathbf{r} \in \partial J(\mathbf{u}) \quad \Longleftrightarrow \quad \mathbf{u} \in \partial J^*(\mathbf{r}). \end{aligned}$$

The concave conjugate potentials $J : \mathcal{V} \mapsto \underline{\mathcal{R}}$ and $J^* : \mathcal{F} \mapsto \underline{\mathcal{R}}$ are related by

$$\begin{aligned} J^*(\mathbf{r}) &:= \inf_{\mathbf{u} \in \mathcal{V}} \{ \langle \, \mathbf{r}, \mathbf{u} \, \rangle - J(\mathbf{u}) \}, \\ J(\mathbf{u}) &:= \inf_{\mathbf{r} \in \mathcal{F}} \{ \langle \, \mathbf{r}, \mathbf{u} \, \rangle - J^*(\mathbf{r}) \}. \end{aligned}$$

2.3 Variational formulations

Let us consider a convex structural problem governed by a kinematic operator $\mathbf{B} \in BL(\mathcal{V};\mathcal{H})$ under the constitutive law defined by a convex potential $\Phi : \mathcal{H} \mapsto \overline{\mathcal{R}}$ and the constraint condition defined by a concave potential $J : \mathcal{V} \mapsto \underline{\mathcal{R}}$, according to the rules

$$\begin{cases} \mathbf{B}\mathbf{u} = \boldsymbol{\varepsilon}\,, \\ \mathbf{B}'\boldsymbol{\sigma} = \mathbf{f}\,, \end{cases} \qquad \begin{cases} \boldsymbol{\sigma} \in \partial\Phi(\boldsymbol{\varepsilon})\,, \\ \mathbf{f} \in \partial J(\mathbf{u})\,, \end{cases}$$

which respectively impose the kinematic compatibility, the equilibrium, the global stress-strain law and the force-displacement law.

The stress-strain law is multivalued and monotone nondecreasing while the force-displacement law is multivalued and monotone nonincreasing.

Recalling the duality between the equilibrium operator $\mathbf{B}' \in BL(\mathcal{H}_S;\mathcal{F})$ and the kinematic operator $\mathbf{B} \in BL(\mathcal{V};\mathcal{H}_D)$:

$$\langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{v}\,\rangle = \langle\, \mathbf{B}'\boldsymbol{\sigma}\,,\, \mathbf{v}\,\rangle\,, \qquad \forall\, \mathbf{u} \in \mathcal{V}\,, \quad \forall\, \boldsymbol{\sigma} \in \mathcal{H}_S = \mathcal{L}^2(\Omega\,;\, S)\,,$$

the equilibrium condition $\mathbf{B}'\boldsymbol{\sigma} = \mathbf{f}$ may be rewritten in variational terms by the virtual work principle

$$\langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{v}\,\rangle = \langle\, \mathbf{f}\,,\, \mathbf{v}\,\rangle\,, \qquad \forall\, \mathbf{v} \in \mathcal{V}\,,$$

or explicitly

$$\int_{\Omega} \langle\, \boldsymbol{\sigma}(\mathbf{x})\,,\, (\mathbf{B}\mathbf{v})(\mathbf{x})\,\rangle\, \mathtt{d}\mu = \langle\, \mathbf{f}\,,\, \mathbf{v}\,\rangle\,, \qquad \forall\, \mathbf{v} \in \mathcal{V}\,.$$

The convex structural problem defined above can be associated with a family of ten basic functionals whose stationarity points are the solutions of the structural problem [17]. By introducing the product HILBERT spaces

$$\mathcal{H} = \mathcal{V} \times \mathcal{H}_S \times \mathcal{H}_D \times \mathcal{F}\,,$$
$$\mathcal{H}' = \mathcal{F} \times \mathcal{H}_D \times \mathcal{H}_S \times \mathcal{V}\,,$$

the operator $\mathbf{A} : \mathcal{H} \mapsto \mathcal{H}'$ governing the structural problem is given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{B}' & \mathbf{O} & -\mathbf{I}_{\mathcal{F}} \\ \mathbf{B} & \mathbf{O} & -\mathbf{I}_{\mathcal{D}} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_{\mathcal{S}} & \partial\Phi & \mathbf{O} \\ -\mathbf{I}_{\mathcal{V}} & \mathbf{O} & \mathbf{O} & \partial J^* \end{bmatrix}$$

The operator $\mathbf{A} : \mathcal{H} \mapsto \mathcal{H}'$ is apparently self adjoint and hence, by integrating along a ray in $\mathcal{H}$, we get the potential

$$\mathcal{L}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{f}) = \Phi(\boldsymbol{\varepsilon}) + J^*(\mathbf{f}) + \langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u}\,\rangle - \langle\, \boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,\rangle - \langle\, \mathbf{f}\,,\, \mathbf{u}\,\rangle\,,$$

which is convex in $\boldsymbol{\varepsilon}$, concave in $\mathbf{f}$ and linear in $\mathbf{u}$ and $\boldsymbol{\sigma}$.

A solution $\{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}, f\}$ is then a minimum point with respect to $\boldsymbol{\varepsilon}$, a maximum point with respect to f and a stationarity point with respect to $\mathbf{u}$ and $\boldsymbol{\sigma}$. A progressive elimination of state variables based on the conjugacy relations, leads to a family of potentials according to the tree-shaped scheme:

$$\begin{array}{ccccccc} & & & \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{f}\} & & & \\ & & \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}\} & & \{\boldsymbol{\sigma}, \mathbf{u}, \mathbf{f}\} & & \\ & \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}\} & & \{\boldsymbol{\sigma}, \mathbf{u}\} & & \{\mathbf{u}, \mathbf{f}\} & \\ \{\boldsymbol{\varepsilon}\} & & \{\boldsymbol{\sigma}\} & & \{\mathbf{u}\} & & \{\mathbf{f}\} \end{array}$$

The family is composed by the ten basic functionals:

$$\begin{aligned} L(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{f}) &= \Phi(\boldsymbol{\varepsilon}) + J^*(\mathbf{f}) + \langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u} - \boldsymbol{\varepsilon}\,\rangle - \langle\, \mathbf{f}\,,\, \mathbf{u}\,\rangle, \\ H_1(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \mathbf{u}) &= \Phi(\boldsymbol{\varepsilon}) - J(\mathbf{u}) + \langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u} - \boldsymbol{\varepsilon}\,\rangle, \\ H_2(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{f}) &= -\Phi^*(\boldsymbol{\sigma}) + J^*(\mathbf{f}) + \langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u}\,\rangle - \langle\, \mathbf{f}\,,\, \mathbf{u}\,\rangle, \\ R_1(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) &= \Phi(\boldsymbol{\varepsilon}) + J^*(\mathbf{B}'\boldsymbol{\sigma}) - \langle\, \boldsymbol{\sigma}\,,\, \boldsymbol{\varepsilon}\,\rangle, \\ R_2(\mathbf{u}, \boldsymbol{\sigma}) &= -\Phi^*(\boldsymbol{\sigma}) - J(\mathbf{u}) + \langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u}\,\rangle, \\ R_3(\mathbf{u}, \mathbf{f}) &= \Phi(\mathbf{B}\mathbf{u}) + J^*(\mathbf{f}) - \langle\, \mathbf{f}\,,\, \mathbf{u}\,\rangle, \\ P(\boldsymbol{\varepsilon}) &= \Phi(\boldsymbol{\varepsilon}) - (J^* \circ \mathbf{B}')^*(\boldsymbol{\varepsilon}), \\ G(\boldsymbol{\sigma}) &= -\Phi^*(\boldsymbol{\sigma}) + J^*(\mathbf{B}'\boldsymbol{\sigma}), \\ F(\mathbf{u}) &= \Phi(\mathbf{B}\mathbf{u}) - J(\mathbf{u}), \\ \mathrm{Q}(\mathbf{f}) &= -(\Phi \circ \mathbf{B})^*(\mathbf{f}) + J^*(\mathbf{f}). \end{aligned}$$

All ten functionals of the family do have the same value at a solution.

Assuming that the solution $\{\mathbf{u}\,, \boldsymbol{\sigma}\} \in \mathcal{V} \times \mathcal{H}_S$ of the structural problem be unique, it can be detected as the minimum point of the extremum problem

$$F(\mathbf{u}) = \min_{\mathbf{v} \in \mathcal{V}} F(\mathbf{v}) = \min_{\mathbf{v} \in \mathcal{V}} \{\, \Phi(\mathbf{B}\mathbf{v}) - J(\mathbf{v})\,\}\,,$$

or as the maximum point of the extremum problem

$$G(\boldsymbol{\sigma}) = \max_{\mathbf{s} \in \mathcal{H}_S} G(\mathbf{s}) = \max_{\mathbf{s} \in \mathcal{H}_S} \{\, J^*(\mathbf{B}'\mathbf{s}) - \Phi^*(\mathbf{s})\,\}\,.$$

Moreover at the solution we have that

$$\max_{\mathbf{s} \in \mathcal{H}_S} G(\mathbf{s}) = G(\boldsymbol{\sigma}) = F(\mathbf{u}) = \min_{\mathbf{v} \in \mathcal{V}} F(\mathbf{v})\,.$$

This relation provides a basis for bounding techiques which will be applied in the sequel to the effective response of homogenized media.

3 Periodic homogenization

Let $\mathcal{C}$ be a periodicity cell (a parallelepiped in E^n) and $\mathbf{u}_\sharp \in \mathcal{L}^2(E^n\,;\mathrm{V})$ the $\mathcal{C}$-periodic extension of a vector field $\mathbf{u} \in H_\mathrm{V} = \mathcal{L}^2(\mathcal{C}\,;\mathrm{V})$, defined by

$$\mathbf{u}_\sharp(\mathbf{x} + k\,\mathbf{h}_i) := \mathbf{u}(\mathbf{x})\,, \quad \forall\,\mathbf{x} \in \mathcal{C}\,,$$

for any integer k and each oriented side $\mathbf{h}_i\,, i = 1, .., n$ of the periodicity cell.

Let us then consider a convex structural problem in the cell $\mathcal{C}$ with kinematic constraints imposing that conforming displacement fields $\mathbf{u} \in \mathcal{L}_{\mathrm{PER}}(\mathcal{C})$, belonging to a conformity linear subspace $\mathcal{L}_{\mathrm{PER}}(\mathcal{C})$, be such that the corresponding $\mathcal{C}$-periodic extension $\mathbf{u}_\sharp \in \mathcal{L}^2(E^n, \mathrm{V})$ be Green regular, that is such that:

$$\int_\omega \| \mathbf{u}_\sharp(\mathbf{x}) \|_V^2 + \| (\mathbf{B}\mathbf{u}_\sharp)(\mathbf{x}) \|_D^2 \, \mathtt{d}\mu < +\infty\,,$$

for any compact subset ω in the euclidean space E^n.

The fulfilment of this condition means that there are no jumps of the boundary traces of the $\mathcal{C}$-periodic extension displacement field across the interfaces of a regular mesh of repetitive periodicity cells. This condition is equivalent to require that the boundary traces of the displacement be equal on opposite faces of the cell. It follows that the mean value of the corresponding strain field vanishes, since

$$\mathrm{MED}_\mathcal{C}\,(\mathbf{B}\mathbf{u}) = \mathrm{sym} \int_{\partial\mathcal{C}} \boldsymbol{\Gamma}\mathbf{u} \otimes \mathbf{n} \, \mathtt{d}S = 0\,.$$

Homogenization can be performed by solving the direct structural problem of the cell under the action of a constant strain field $\boldsymbol{\varepsilon} = \mathrm{Im}\,\mathbf{M}_\mathcal{C}^* \subset \mathcal{H}(\mathcal{C})$ so that $\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{D} \in D$ for almost all $\mathbf{x} \in \mathcal{C}$. Setting $\Omega = \mathcal{C}$ and $\mathcal{T}(\Omega) = \{\,\mathcal{C}\,\}$ we denote by $\mathcal{V}(\mathcal{C}\,;\mathrm{V}) \subset \mathcal{V}_\mathcal{C}$ the kinematic space of displacements fields which are Green-regular in $\mathcal{C}$.

Conforming displacements fields belong to the closed linear subspace $\mathcal{L}_{\mathrm{PER}}(\mathcal{C}) \subset \mathcal{V}(\mathcal{C}\,;\mathrm{V})$. The problem is well posed since strain fields corresponding to conforming displacements have null mean value and hence any constant strain field is effective as an imposed strain. The homogenized local constitutive law is the one that relates the mean value of the elastic stress field to the imposed constant strain field.

3.1 Orthogonal decomposition

A basic property of conforming displacements considered in periodic homogenization problems is that they have a null mean value:

$$\mathcal{L}_{\mathrm{PER}} \subset \mathrm{Ker}\,\mathbf{M}_\mathcal{C} = (\,\mathrm{Im}\,\mathbf{M}_\mathcal{C}^*)^\perp\,,$$

where $\mathcal{L}_{\rm PER}$ stands for $\mathcal{L}_{\rm PER}(\mathcal{C})$. Let us then cosider the closed linear subspace of displacement fields which can be expressed as the sum of a conforming one and of a constant-strain one:

$$\mathcal{L} := \{\, \mathbf{v} \in \mathcal{V}(\mathcal{C}\,;\,\mathrm{V}) \mid \mathbf{B}\mathbf{v} \in \mathbf{B}\mathcal{L}_{\rm PER} \dotplus \operatorname{Im}\mathbf{M}^*_{\mathcal{C}} \,\}\,.$$

Then the following relations hold:

$$\mathbf{B}\mathcal{L}_{\rm PER} = \mathbf{B}\mathcal{L} \cap \operatorname{Ker}\mathbf{M}_{\mathcal{C}}\,, \qquad (\mathbf{B}\mathcal{L}_{\rm PER})^{\perp} = (\mathbf{B}\mathcal{L})^{\perp} \dotplus \operatorname{Im}\mathbf{M}^*_{\mathcal{C}}\,,$$

$$\mathbf{B}\mathcal{L} = \mathbf{B}\mathcal{L}_{\rm PER} \dotplus \operatorname{Im}\mathbf{M}^*_{\mathcal{C}}\,, \qquad (\mathbf{B}\mathcal{L})^{\perp} = (\mathbf{B}\mathcal{L}_{\rm PER})^{\perp} \cap \operatorname{Ker}\mathbf{M}_{\mathcal{C}}\,.$$

By Korn's inequality the linear subspace $\mathbf{B}\mathcal{L}$ is closed in $\mathcal{H}_D$ and hence the following direct sum decomposition holds

$$\mathcal{H}_D = \mathbf{B}\mathcal{L} \dotplus (\mathbf{B}\mathcal{L})^{\perp}\,.$$

It follows that the Hilbert space $\mathcal{H}_D$ can be decomposed into the following direct sum of three mutually orthogonal subspaces

$$\begin{aligned}\mathcal{H}_D &= \operatorname{Im}\mathbf{M}^*_{\mathcal{C}} \dotplus \mathbf{B}\mathcal{L}_{\rm PER} \dotplus (\mathbf{B}\mathcal{L})^{\perp} \\ &= \operatorname{Im}\mathbf{M}^*_{\mathcal{C}} \dotplus \mathbf{B}\mathcal{L}_{\rm PER} \dotplus (\mathbf{B}\mathcal{L}_{\rm PER})^{\perp} \cap \operatorname{Ker}\mathbf{M}_{\mathcal{C}}\,.\end{aligned}$$

This direct sum decomposition in orthogonal complements plays a basic role in the subsequent developments.

3.2 Conjugate potentials for the cell problem

The stress-strain law is assumed to be expressed by a generalized elastic law governed by two regular conjugate global convex potentials $\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon})$ and $\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma})$. The conjugate potentials governing the kinematic constraint for the cell problem are given by

$$\begin{aligned}J(\mathbf{u}) &:= \sqcap_{\mathcal{L}_{PER}}(\mathbf{u} - \mathbf{u}_D)\,,\\ J^*(\mathbf{f}) &:= \sqcap_{\mathcal{L}^{\perp}_{PER}}(\mathbf{f}) + \langle\, \mathbf{f}\,,\, \mathbf{u}_D \,\rangle\,,\end{aligned}$$

where $\mathbf{u}_D \in \mathcal{L}$ is a dislacement field such that $(\mathbf{B}\mathbf{u}_D)(\mathbf{x}) = \mathbf{D}$ for all $\mathbf{x} \in \mathcal{C}$. Then $\mathbf{B}\mathbf{u}_D \in \operatorname{Im}\mathbf{M}^*_{\mathcal{C}}$. The symbol $\sqcap_{\mathcal{A}}$ denotes the concave indicator of the set $\mathcal{A}$, defined by

$$\sqcap_{\mathcal{A}}(\mathbf{x}) := \begin{cases} 0 & \mathbf{x} \in \mathcal{A}\,,\\ -\infty & \mathbf{x} \notin \mathcal{A}\,.\end{cases}$$

The functionals

$$\begin{aligned}F(\mathbf{u}) &= \Phi(\mathbf{B}\mathbf{u}) - J(\mathbf{u})\,, && \mathbf{u} \in \mathcal{V}\,,\\ G(\boldsymbol{\sigma}) &= J^*(\mathbf{B}'\boldsymbol{\sigma}) - \Phi^*(\boldsymbol{\sigma})\,, && \boldsymbol{\sigma} \in \mathcal{H}\,,\end{aligned}$$

take the explicit form

$$F(\mathbf{u}) = \Phi_{\mathbf{e}}(\mathbf{B}\mathbf{u})\,, \qquad \mathbf{u} \in \mathbf{u}_D + \mathcal{L}_{\mathrm{PER}}\,,$$
$$G(\boldsymbol{\sigma}) = \langle\, \boldsymbol{\sigma}\,,\, \mathbf{B}\mathbf{u}_D \,\rangle - \Phi^*_{\mathbf{e}}(\boldsymbol{\sigma})\,, \qquad \boldsymbol{\sigma} \in (\mathbf{B}\mathcal{L}_{\mathrm{PER}})^{\perp}\,,$$

Recalling the orthogonal decomposition $(\mathbf{B}\mathcal{L}_{\mathrm{PER}})^{\perp} = \operatorname{Im}\mathbf{M}^*_{\mathcal{C}} \dot{+} (\mathbf{B}\mathcal{L})^{\perp}$ we can conveniently rewrite

$$F_{\mathbf{D}}(\mathbf{v}) = \Phi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{D} + \mathbf{B}\mathbf{v})\,, \qquad \mathbf{v} \in \mathcal{L}_{\mathrm{PER}}\,,$$
$$G_{\mathbf{D}}(\mathbf{s},\mathbf{T}) = \langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\, \mathbf{M}^*_{\mathcal{C}}\mathbf{D} \,\rangle - \Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{T} + \mathbf{s})\,, \qquad \mathbf{s} \in (\mathbf{B}\mathcal{L})^{\perp}\,, \quad \mathbf{T} \in S\,.$$

3.3 Effective response

The global effective potential of the homogenized constitutive law is defined by

$$\begin{aligned}\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}) &= \min\{\, F_{\mathbf{D}}(\mathbf{v}) \mid \mathbf{v} \in \mathcal{L}_{\mathrm{PER}} \,\} \\ &= \max\{\, G_{\mathbf{D}}(\mathbf{s},\mathbf{T}) \mid \mathbf{s} \in (\mathbf{B}\mathcal{L})^{\perp}\,, \mathbf{T} \in S \,\}\,,\end{aligned}$$

or explicitly

$$\begin{aligned}\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}) &= \min\{\, \Phi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D} + \boldsymbol{\eta}) \mid \boldsymbol{\eta} \in \mathbf{B}\mathcal{L}_{\mathrm{PER}} \,\} \\ &= \max\{\, \langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\, \mathbf{M}^*_{\mathcal{C}}\mathbf{D} \,\rangle - \Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{T} + \mathbf{s}) \mid \mathbf{s} \in (\mathbf{B}\mathcal{L})^{\perp}\,, \mathbf{T} \in S \,\}\,.\end{aligned}$$

The global effective potential is convex, being the inf-convolution of the two convex functionals. Indeed we have that

$$\begin{aligned}\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}) &= \min\{\, \Phi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D} - \boldsymbol{\eta}) \mid \boldsymbol{\eta} \in \mathbf{B}\mathcal{L}_{\mathrm{PER}} \,\} \\ &= \min\{\, \Phi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D} - \boldsymbol{\eta}) + \sqcup_{\mathbf{B}\mathcal{L}_{PER}}(\boldsymbol{\eta}) \,\} \\ &= (\Phi_{\mathbf{e}} \,\square\, \sqcup_{\mathbf{B}\mathcal{L}_{PER}})(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D})\,.\end{aligned}$$

We recall that the epigraph of the inf-convolution of two convex functionals is the convex sum of the two convex epigraphs and that

$$\Phi_{\mathbf{e}} \,\square\, \sqcup_{\mathbf{B}\mathcal{L}_{PER}} = (\Phi^*_{\mathbf{e}} + \sqcup_{(\mathbf{B}\mathcal{L}_{PER})^{\perp}})^*\,.$$

The local potential of the homogenized constitutive law is then defined as

$$\varphi_H(\mathbf{D}) = \frac{1}{\operatorname{vol}(\mathcal{C})}\,(\Phi_H \circ \mathbf{M}^*_{\mathcal{C}})(\mathbf{D}) = \frac{1}{\operatorname{vol}(\mathcal{C})}\left[(\Phi_{\mathbf{e}} \,\square\, \sqcup_{\mathbf{B}\mathcal{L}_{PER}}) \circ \mathbf{M}^*_{\mathcal{C}}\right](\mathbf{D})\,.$$

Observing that $\Phi_{\mathbf{e}} = \mathbf{M}_{\mathcal{C}}\,\varphi_{\mathbf{e}}$ and that

$$\langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\, \mathbf{M}^*_{\mathcal{C}}\mathbf{D} \,\rangle = \operatorname{vol}(\mathcal{C})\,\langle\, \mathbf{T}\,,\, \mathbf{D} \,\rangle\,,$$

we get the following expression for the local homogenized potential:

$$\begin{aligned}\varphi_H(\mathbf{D}) &= \min\{\, \mathrm{MED}_{\mathcal{C}}(\varphi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D} - \boldsymbol{\eta})) \mid \boldsymbol{\eta} \in \mathbf{B}\mathcal{L}_{\mathrm{PER}} \,\} \\ &= \max\{\, \langle\, \mathbf{T}\,,\, \mathbf{D} \,\rangle - \mathrm{MED}_{\mathcal{C}}(\varphi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{T} + \mathbf{s})) \mid \mathbf{s} \in (\mathbf{B}\mathcal{L})^{\perp}\,, \mathbf{T} \in S \,\} \\ &= \left[\max_{\mathbf{T}\in S}\left\{\, \langle\, \mathbf{T}\,,\, \mathbf{D} \,\rangle - \frac{1}{\operatorname{vol}(\mathcal{C})}(\Phi^*_{\mathbf{e}} \,\square\, \sqcup_{(\mathbf{B}\mathcal{L})^{\perp}}) \circ \mathbf{M}^*_{\mathcal{C}}(\mathbf{T}) \right\}\right](\mathbf{D})\,.\end{aligned}$$

Hence, setting

$$\psi_H(\mathbf{T}) = \left[\frac{1}{\mathrm{vol}\,(\mathcal{C})}(\Phi^*_{\mathbf{e}} \,\square\, \sqcup_{(\mathbf{B}\mathcal{L})^\perp}) \circ \mathbf{M}^*_{\mathcal{C}}\right](\mathbf{T})\,,$$

we get the conjugacy relation

$$\varphi_H = (\psi_H)^*\,.$$

By the properties of the inf-convolution we know that setting

$$\begin{aligned}\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}) &= \min\{\,\Phi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D} - \boldsymbol{\eta}) \mid \boldsymbol{\eta} \in \mathbf{B}\mathcal{L}_{\mathrm{PER}}\,\} = \Phi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D} - \boldsymbol{\varepsilon}_{\mathbf{D}}) \\ &= (\Phi_{\mathbf{e}} \,\square\, \sqcup_{\mathbf{B}\mathcal{L}_{PER}})(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D})\,,\end{aligned}$$

with $\boldsymbol{\varepsilon}_{\mathbf{D}} = \mathbf{B}\mathbf{u}_{\mathbf{D}}$ and $\mathbf{u}_{\mathbf{D}} \in \mathcal{L}_{\mathrm{PER}}$, we have that

$$\begin{cases}\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D} - \boldsymbol{\varepsilon}_{\mathbf{D}} \in \partial\Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}_{\mathbf{D}})\,, \\ \boldsymbol{\varepsilon}_{\mathbf{D}} \in \partial \sqcup_{(\mathbf{B}\mathcal{L}_{PER})^\perp}(\boldsymbol{\sigma}_{\mathbf{D}})\,,\end{cases}$$

where

$$\boldsymbol{\sigma}_{\mathbf{D}} \in \partial(\Phi_{\mathbf{e}} \,\square\, \sqcup_{\mathbf{B}\mathcal{L}_{PER}})(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D}) = \partial\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D})\,,$$

is the stress solution of the direct problem [17].

By the chain rule of subdifferential calculus we have that

$$\partial(\Phi_H \circ \mathbf{M}^*_{\mathcal{C}})(\mathbf{D}) = \mathbf{M}_{\mathcal{C}}\,\partial\Phi_H(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D})\,,$$

and from the definition of φ_H we eventually get the relation

$$\mathrm{MED}(\boldsymbol{\sigma}_{\mathbf{D}}) \in \partial\varphi_H(\mathbf{D})\,,$$

that justifies the homogenization role played by the potential φ_H.

3.4 Inverse effective response

An alternative procedure to perform the homogenization process, consists in solving the inverse structural problem of the cell under the action of a constant stress field $\boldsymbol{\sigma} = \mathrm{Im}\,\mathbf{M}^*_{\mathcal{C}} \subset \mathcal{H}_S(\mathcal{C}) = \mathcal{L}^2(\mathcal{C}\,;S)$ so that $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{T} \in S$ for almost all $\mathbf{x} \in \mathcal{C}$. Setting $\Omega = \mathcal{C}$ and $\mathcal{T}(\Omega) = \{\,\mathcal{C}\,\}$ we denote by $\mathcal{V}(\mathcal{C}\,;\mathrm{V})$ the kinematic space of displacements fields which are Green-regular in $\mathcal{C}$. Conforming displacements fields are assumed to belong to the subspace $\mathcal{L}_{\mathrm{PER}}(\mathcal{C}) \subset \mathcal{V}(\mathcal{C},\mathrm{V})$. Selfequilibrated stresses then belong to the linear subspace $\mathcal{L}^\perp_{\mathrm{PER}}(\mathcal{C})$. The problem is well posed if the stress fields are assumed to be the sum of the prescribed constant one and any selfequilibrated field with zero mean value. Indeed in this case any constant stress field is effective as an imposed stress.

According to the inverse homogenization procedure the homogenized local constitutive law is the one that relates the mean value of the strain field to the imposed constant stress field.

The conjugate pairs of convex potentials governing the monotone stress-strain and force-displacement relations are given by

$$\begin{aligned}
\Phi^*(\boldsymbol{\sigma}) &:= \Phi^*_{\mathbf{e}}(\boldsymbol{\sigma}) + \sqcup_{\operatorname{Ker}\mathbf{M}_{\mathcal{C}}}(\boldsymbol{\sigma} - \mathbf{M}^*_{\mathcal{C}}\mathbf{T})\,,\\
\Phi(\boldsymbol{\varepsilon}) &:= \Big(\Phi_{\mathbf{e}} \,\square\, (\sqcup_{\operatorname{Im}\mathbf{M}^*_{\mathcal{C}}} + \langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\,\cdot\,\rangle\Big)(\boldsymbol{\varepsilon})\,,\\
J^*(\mathbf{f}) &:= \sqcap_{\mathcal{L}^{\perp}_{PER}}(\mathbf{f})\,,\\
J(\mathbf{u}) &:= \sqcap_{\mathcal{L}_{PER}}(\mathbf{u})\,,
\end{aligned}$$

Recalling that $(\mathbf{B}\mathcal{L})^{\perp} = (\mathbf{B}\mathcal{L}_{PER})^{\perp} \cap \operatorname{Ker}\mathbf{M}_{\mathcal{C}}$ and setting

$$\boldsymbol{\sigma} = \mathbf{M}^*_{\mathcal{C}}\mathbf{T} + \mathbf{s} \quad \text{with} \quad \mathbf{s} \in (\mathbf{B}\mathcal{L})^{\perp}\,,$$

The functionals

$$\begin{aligned}
F(\mathbf{u}) &= \Phi(\mathbf{B}\mathbf{u}) - J(\mathbf{u})\,, \qquad && \mathbf{u} \in \mathcal{V}\,,\\
G(\boldsymbol{\sigma}) &= J^*(\mathbf{B}'\boldsymbol{\sigma}) - \Phi^*(\boldsymbol{\sigma})\,, && \boldsymbol{\sigma} \in \mathcal{H}\,,
\end{aligned}$$

take the explicit forms

$$\begin{aligned}
F_{\mathbf{T}}(\mathbf{v},\mathbf{D}) &= \inf_{\mathbf{D}\in D}\big\{\Phi_{\mathbf{e}}(\mathbf{B}\mathbf{v} + \mathbf{M}^*_{\mathcal{C}}\mathbf{D}) - \langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\,\mathbf{M}^*_{\mathcal{C}}\mathbf{D}\,\rangle\big\} - \sqcap_{\mathcal{L}_{PER}}(\mathbf{v})\,,\\
G_{\mathbf{T}}(\mathbf{s}) &= -\Big(\Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{T} + \mathbf{s}) + \sqcup_{(\mathbf{B}\mathcal{L})^{\perp}}(\mathbf{s})\Big)\,,
\end{aligned}$$

The global effective potential of the homogenized medium is the convex functional $\Psi_H : \mathcal{H} \mapsto \overline{\mathcal{R}}$ defined by one of the equivalent relations

$$\begin{aligned}
-\Psi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{T}) &:= \min_{\mathbf{v}\in\mathcal{L}_{PER}} F_{\mathbf{T}}(\mathbf{v}) = \min_{\mathbf{v}\in\mathcal{L}}\{\,\Phi_{\mathbf{e}}(\mathbf{B}\mathbf{v}) - \langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\,\mathbf{B}\mathbf{v}\,\rangle\,\}\,,\\
-\Psi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{T}) &:= \max_{\mathbf{s}\in(\mathbf{B}\mathcal{L})^{\perp}} G_{\mathbf{T}}(\mathbf{s}) = \max_{\mathbf{s}\in(\mathbf{B}\mathcal{L})^{\perp}}\{\,-\Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{T} + \mathbf{s})\,\}\,,\\
&= -\min_{\mathbf{s}\in(\mathbf{B}\mathcal{L})^{\perp}}\{\,\Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{T} - \mathbf{s}) + \sqcup_{(\mathbf{B}\mathcal{L})^{\perp}}(\mathbf{s})\,\}\,,\\
&= -(\Phi^*_{\mathbf{e}} \,\square\, \sqcup_{(\mathbf{B}\mathcal{L})^{\perp}})(\mathbf{M}^*_{\mathcal{C}}\mathbf{T})\,.
\end{aligned}$$

The local potential of the homogenized constitutive law is then defined as

$$\psi_H(\mathbf{T}) := \frac{1}{\operatorname{vol}(\mathcal{C})}\,(\Psi_H \circ \mathbf{M}^*_{\mathcal{C}})(\mathbf{T})\,.$$

Recall that the corresponding convex potential for the direct problem is defined by the equivalent relations

$$\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}) := \min_{\mathbf{v}\in\mathcal{L}_{PER}} F_{\mathbf{D}}(\mathbf{v}) = \min_{\mathbf{v}\in\mathcal{L}_{PER}} \Phi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}+\mathbf{B}\mathbf{v})\,,$$
$$\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}) := \max_{\mathbf{s}\in(\mathbf{B}\mathcal{L}_{PER})^\perp} G_{\mathbf{D}}(\mathbf{s}) = \max_{\mathbf{s}\in(\mathbf{B}\mathcal{L}_{PER})^\perp} \{\, \langle\, \mathbf{s}\,,\, \mathbf{M}^*_{\mathcal{C}}\mathbf{D}\,\rangle - \Phi^*_{\mathbf{e}}(\mathbf{s})\,\}\,.$$

Hence, being $(\mathbf{B}\mathcal{L}_{\rm PER})^\perp = (\mathbf{B}\mathcal{L})^\perp \dotplus \operatorname{Im}\mathbf{M}^*_{\mathcal{C}}$, we have that

$$\begin{aligned}\Phi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{D}) &= \max_{\mathbf{s}\in(\mathbf{B}\mathcal{L}_{PER})^\perp} \{\, \langle\, \mathbf{s}\,,\, \mathbf{M}^*_{\mathcal{C}}\mathbf{D}\,\rangle - \Phi^*_{\mathbf{e}}(\mathbf{s})\,\}\,,\\ &= \max_{\mathbf{T}\in S}\max_{\mathbf{s}\in(\mathbf{B}\mathcal{L})^\perp} \big\{\langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\, \mathbf{M}^*_{\mathcal{C}}\mathbf{D}\,\rangle - \Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\mathbf{T}+\mathbf{s})\big\}\\ &= \max_{\mathbf{T}\in S}\big\{\langle\, \mathbf{M}^*_{\mathcal{C}}\mathbf{T}\,,\, \mathbf{M}^*_{\mathcal{C}}\mathbf{D}\,\rangle - \Psi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{T})\big\}\\ &= (\Psi_H)^*(\mathbf{M}^*_{\mathcal{C}}\mathbf{D})\,.\end{aligned}$$

By the properties of the inf-convolution we have also that

$$\begin{aligned}\Psi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{T}) &= \min\{\, \Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T}-\mathbf{s}) \mid \mathbf{s}\in(\mathbf{B}\mathcal{L})^\perp\,\} = \Phi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T}-\mathbf{s}_{\mathbf{T}})\\ &= (\Phi^*_{\mathbf{e}}\,\square\sqcup_{(\mathbf{B}\mathcal{L})^\perp})(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T})\,,\end{aligned}$$

with $\mathbf{s}_{\mathbf{T}}\in(\mathbf{B}\mathcal{L})^\perp$ and

$$\begin{cases}\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T}-\mathbf{s}_{\mathbf{T}}\in\partial\Phi_{\mathbf{e}}(\boldsymbol{\varepsilon}_{\mathbf{T}})\,,\\ \boldsymbol{\sigma}_{\mathbf{T}}\in\partial\sqcup_{\mathbf{B}\mathcal{L}}(\boldsymbol{\varepsilon}_{\mathbf{T}})\,,\end{cases}$$

where

$$\boldsymbol{\varepsilon}_{\mathbf{T}}\in\partial(\Phi^*_{\mathbf{e}}\,\square\sqcup_{(\mathbf{B}\mathcal{L})^\perp})(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T}) = \partial\Psi_H(\mathbf{M}^*_{\mathcal{C}}\mathbf{T})\,,$$

is the strain solution of the inverse problem [17].

By the chain rule of subdifferential calculus we infer that

$$\partial(\Psi_H\circ\mathbf{M}^*_{\mathcal{C}})(\mathbf{T}) = \mathbf{M}_{\mathcal{C}}\,\partial\Psi_H(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T})\,,$$

and from the definition of ψ_H we eventually get the relation

$$\text{MED}(\boldsymbol{\varepsilon}_{\mathbf{T}})\in\partial\psi_H(\mathbf{T})\,,$$

that justifies the homogenization role played by the potential ψ_H.

Remark 1. The conjugacy relation between the potentials of the direct and the inverse cell problems can be also revealed by applying the following conjugacy rules:

$$\begin{aligned}&(\alpha\, f)^*(\mathbf{x}^*) = \alpha\, f^*(\frac{1}{\alpha}\,\mathbf{x}^*), \qquad \forall\,\alpha>0\,,\\ &(f\circ\mathbf{L})^*(\mathbf{x}^*) = \inf\{\; f^*(\mathbf{y}^*)\mid \mathbf{L}'(\mathbf{y}^*)=\mathbf{x}^*\,\}\,,\\ &(f\,\square\, g)^*(\mathbf{x}^*) = \inf\{\; f^*(\mathbf{x}^*_1)+g^*(\mathbf{x}^*_2)\mid \mathbf{x}^*_1+\mathbf{x}^*_2=\mathbf{x}^*\,\}\,,\end{aligned}$$

which hold under reasonable global regularity conditions of the involved potentials [3], [5], [6]. Less stringent local condition were contributed in [12].

Indeed we have that

$$
\begin{aligned}
(\varphi_H)^*(\mathbf{T}) &= \left[\frac{1}{\mathrm{vol}\,(\mathcal{C})}(\Phi_{\mathbf{e}}\,\square\,\sqcup_{\mathbf{B}\mathcal{L}_{PER}})\circ\mathbf{M}^*_{\mathcal{C}}\right]^*(\mathbf{T})\\
&= \frac{1}{\mathrm{vol}\,(\mathcal{C})}\left[(\Phi_{\mathbf{e}}\,\square\,\sqcup_{\mathbf{B}\mathcal{L}_{PER}})\circ\mathbf{M}^*_{\mathcal{C}}\right]^*(\mathrm{vol}\,(\mathcal{C})\,\mathbf{T})\\
&= \frac{1}{\mathrm{vol}\,(\mathcal{C})}\inf\left\{(\Phi_{\mathbf{e}}\,\square\,\sqcup_{\mathbf{B}\mathcal{L}_{PER}})^*(\boldsymbol{\sigma})\mid\mathbf{M}_{\mathcal{C}}(\boldsymbol{\sigma})=\mathrm{vol}\,(\mathcal{C})\,\mathbf{T}\right\}\\
&= \frac{1}{\mathrm{vol}\,(\mathcal{C})}\inf\left\{(\Phi^*_{\mathbf{e}}\,+\,\sqcup_{(\mathbf{B}\mathcal{L}_{PER})^\perp})(\boldsymbol{\sigma})\mid\mathbf{M}_{\mathcal{C}}(\boldsymbol{\sigma})=\mathrm{vol}\,(\mathcal{C})\,\mathbf{T}\right\}\\
&= \frac{1}{\mathrm{vol}\,(\mathcal{C})}\inf\left\{(\Phi^*_{\mathbf{e}}(\mathbf{M}^*\mathbf{T}+\mathbf{s})\,+\,\sqcup_{(\mathbf{B}\mathcal{L})^\perp}(\mathbf{s})\right\}\\
&= \left[\frac{1}{\mathrm{vol}\,(\mathcal{C})}(\Phi^*_{\mathbf{e}}\,\square\,\sqcup_{(\mathbf{B}\mathcal{L})^\perp})\circ\mathbf{M}^*_{\mathcal{C}}\right](\mathbf{T})=\psi_H(\mathbf{T})\,.
\end{aligned}
$$

Note that from the relation $(\mathbf{B}\mathcal{L}_{\mathrm{PER}})^\perp=(\mathbf{B}\mathcal{L})^\perp\,\dot{+}\,\mathrm{Im}\,\mathbf{M}^*_{\mathcal{C}}$ we have argued that the conditions

$$
\boldsymbol{\sigma}\in(\mathbf{B}\mathcal{L}_{PER})^\perp\,,\qquad\mathbf{M}_{\mathcal{C}}(\boldsymbol{\sigma})=\mathrm{vol}\,(\mathcal{C})\,\mathbf{T}\,,
$$

are equivalent to assume that $\boldsymbol{\sigma}=\mathbf{M}^*_{\mathcal{C}}\mathbf{T}+\mathbf{s}$ with $\mathbf{s}\in(\mathbf{B}\mathcal{L})^\perp$.

3.5 Bounds on the effective response

In computing the local potential of the homogenized constitutive law we can get a rough estimate by taking respectively $\boldsymbol{\eta}=0$ and $\mathbf{s}=0$ in the expressions to be minimized and maximized as reported in section 3.3. The upper and lower bounds so obtained are the generalized Voigt (upper) and Reuss (lower) bounds for the effective potential of the homogenized medium:

$$
\max\{\,\langle\,\mathbf{T}\,,\,\mathbf{D}\,\rangle-\mathrm{MED}_{\mathcal{C}}(\varphi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T}))\mid\mathbf{T}\in S\,\}\le\varphi_H(\mathbf{D})\le\mathrm{MED}_{\mathcal{C}}(\varphi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D}))\,.
$$

To get the Voigt bound we consider a constant strain field $\mathbf{M}^*_{\mathcal{C}}\,\mathbf{D}$, evaluate the corresponding local potential φ at any point of the cell and take its mean value. In this way an *aritmetic mean* approximation is performed.

In the linear elastic case the Voigt approximation amount to perform the composition of the local elastic stiffnesses by a parallel scheme of elastic springs and the effective elastic stiffness is given by the average of the local stiffnesses.

To get the Reuss bound we consider a constant stress field $\mathbf{M}^*_{\mathcal{C}}\,\mathbf{T}$, evaluate the corresponding conjugate local potential φ^* at any point of the cell, take the mean value and evaluate the conjugate local potential. In this way an *armonic mean* approximation is performed.

In the linear elastic case the REUSS approximation amount to perform the composition of the local elastic stiffnesses by a serial scheme of elastic springs and the effective compliance is given by the average of the local compliances.

Better bounds can be found by computing approximate solutions of the cell problem either in the direct way, in terms of conforming displacements with zero mean strain, to get upper bounds, or in the complementary way, in terms of selfstresses with zero mean value, to get lower bounds.

Another approach to the problem of bounding the effective properties of the homogenized medium is provided by polarization techniques which have been first applied to elasticity problems by HASHIN and SHTRIKMAN in 1962 [1], [2] and then extended and generalized to the nonlinear setting by TALBOT and WILLIS in 1985 [7], WILLIS and TOLAND-WILLIS in 1989 [8], [9].

3.6 Uniform local bounds

Let us now assume that the field of local potentials $\varphi_{\mathbf{e}}$ is uniformly bounded from above and from below:

$$\varphi^- \le \varphi_{\mathbf{e}} \le \varphi^+ ,$$

where $\varphi^-, \varphi^+ : D \mapsto \overline{\mathcal{R}}$ are convex functions. From VOIGT-REUSS inequalities

$$\max\{ \langle \mathbf{T}, \mathbf{D} \rangle - \mathrm{MED}_{\mathcal{C}}(\varphi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}} \mathbf{T})) \mid \mathbf{T} \in S \} \le \varphi_H(\mathbf{D}) \le \mathrm{MED}_{\mathcal{C}}(\varphi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}} \mathbf{D})) ,$$

being

$$(\varphi^+)^* \le \varphi^*_{\mathbf{e}} \le (\varphi^-)^* ,$$

$$\mathrm{MED}_{\mathcal{C}}(\varphi_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}} \mathbf{D})) \le \varphi^+(\mathbf{D}) ,$$

$$\mathrm{MED}_{\mathcal{C}}(\varphi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}} \mathbf{T})) \le (\varphi^-)^*(\mathbf{T}) ,$$

$$\begin{aligned} \varphi^-(\mathbf{D}) &= \max\{ \langle \mathbf{T}, \mathbf{D} \rangle - (\varphi^-)^*(\mathbf{T}) \mid \mathbf{T} \in S \} \\ &\le \max\{ \langle \mathbf{T}, \mathbf{D} \rangle - \mathrm{MED}_{\mathcal{C}}(\varphi^*_{\mathbf{e}}(\mathbf{M}^*_{\mathcal{C}} \mathbf{T})) \mid \mathbf{T} \in S \} , \end{aligned}$$

we infer that the same bounds hold for the local potential of the homogenized constitutive law, that is

$$\varphi^- \le \varphi_{\mathbf{e}} \le \varphi^+ \quad \Longrightarrow \quad \varphi^- \le \varphi_H \le \varphi^+ .$$

3.7 Geometric constraints

We remark that the analysis carried out above relies only on the property that conforming displacements belonging to the subspace $\mathcal{L}_{\mathrm{PER}}$ have a zero mean value, that is that $\mathcal{L}_{\mathrm{PER}} \subset \mathrm{Ker}\, \mathbf{M}_{\mathcal{C}}$.

We could thus also choose the conforming subspace

$$\mathcal{L}_{o}(\mathcal{C}) := \{\, \mathbf{v} \in \mathcal{V}(\mathcal{C}) \mid \boldsymbol{\Gamma}\mathbf{v} = 0 \,\} = \operatorname{Ker} \boldsymbol{\Gamma} \subset \operatorname{Ker} \mathbf{M}_{\mathcal{C}} \,,$$

instead of $\mathcal{L}_{\mathrm{PER}}(\mathcal{C})$. Since $\mathcal{L}_{o}(\mathcal{C}) \subset \mathcal{L}_{\mathrm{PER}}(\mathcal{C})$, denoting by φ_H^o and ψ_H^o the direct and inverse local effective potentials under the constraints defined by $\mathcal{L}_{o}(\mathcal{C})$, we get the inequalities

$$\varphi_H \leq \varphi_H^o \,, \qquad \psi_H \geq \psi_H^o \,.$$

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