ON THE NECESSITY OF KORN'S INEQUALITY

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Summary

The celebrated KORN's second inequality is the milestone along the way that leads to the basic existence results in continuum mechanics and linear elastostatics. An abstract result by L. TARTAR shows that KORN's inequality implies that the range of the kinematic operator is closed and that its kernel is finite dimensional. A full extension of TARTAR's lemma is provided in this paper and leads to the conclusion that conversely the closedness of the range of the kinematic operator and the finite dimensionality of its kernel are sufficient to ensure the validity of KORN's inequality.

Introduction

On reading the brilliant proof of KORN's second inequality in the book by G. DUVAUT and J. L. LIONS [4] the author realized that the peculiar form of the sym grad operator plays a basic role in the proof. More specifically he realized that the finite dimensionality of the kernel of sym grad should be a necessary property, although this condition was not appealed to explicitly in the proof. Some time later the autor became aware of a nice result by L. TARTAR concerning an abstract inequality of the KORN's type expressed in term of a bounded linear operator and a compact operator whose kernels have a trivial intersection. TARTAR proved that the inequality implies the finite dimensionality of the kernel and the closedness of the image of the bounded linear operator. The conjecture about the role of the kernel of sym grad in KORN's second inequality was thus confirmed. At this point it raised naturally the question whether conversely the finite dimensionality of the kernel of sym grad and the closedness of its image were also sufficient to assess the validity of KORN's second inequality. This converse property requires to complete TARTAR's result with the opposite implication. A full extension of TARTAR's lemma is provided in this paper and leads to the conclusion that conversely the closedness of the range of the kinematic operator and the finite dimensionality of its kernel are sufficient to ensure the validity of KORN's inequality. The main result contributed here shows that both properties are equivalent to require that a similar inequality be valid for any linear continuous operator.

Tartar's Lemma

A nice abstract result due to L. TARTAR was reported by F. BREZZI and D. MARINI in [5], lemma 4.1 and quoted by P. G. CIARLET in [6], exer. 3.1.1. Since TARTAR's lemma plays a basic role in our discussion about KORN's inequality we provide hereafter an explicit proof of this result. Preliminarily we quote that BANACH's open mapping theorem implies the following lemma (see BREZIS [8] th. II.8 and [10], th. 9.1, 9.2).

Bounded decomposition. *Let* $\mathcal{X}$ *be a* BANACH *space and* $\mathcal{A} \subseteq \mathcal{X}$, $\mathcal{B} \subseteq \mathcal{X}$ *closed linear subspaces of* $\mathcal{X}$ *such that their sum* $\mathcal{A} + \mathcal{B}$ *is closed. Then any* $\mathbf{x} \in \mathcal{A} + \mathcal{B}$ *admits a decomposition* $\mathbf{x} = \mathbf{a} + \mathbf{b}$, *with* $\mathbf{a} \in \mathcal{A}$, $\mathbf{b} \in \mathcal{B}$, *such that*

$$\|\mathbf{x}\|_{\mathcal{X}} \geq c\,\|\mathbf{a}\|_{\mathcal{X}}\,, \quad \|\mathbf{x}\|_{\mathcal{X}} \geq c\,\|\mathbf{b}\|_{\mathcal{X}}\,,$$

where $c > 0$. □

If $\mathcal{X} = \mathcal{A} + \mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \{\mathbf{o}\}$, the closed subspaces $\mathcal{A}$ and $\mathcal{B}$ are topological supplements in $\mathcal{X}$ and the projectors $\mathbf{P}_{\mathcal{A}}\mathbf{x} = \mathbf{a}$ and $\mathbf{P}_{\mathcal{B}}\mathbf{x} = \mathbf{b}$ are well defined linear bounded operators from $\mathcal{X}$ to $\mathcal{X}$.

A decomposition $\mathcal{X} = \mathcal{A} + \mathcal{B}$ of $\mathcal{X}$ into the direct sum of two topological supplementary subspaces $\mathcal{A}$ and $\mathcal{B}$ certainly exists if either $\mathcal{X}$ is a HILBERT space or at least one of them, say $\mathcal{A}$, is finite dimensional.

In the former case $\mathcal{B}$ is simply the orthogonal complement of $\mathcal{A}$ in $\mathcal{X}$. In the latter case we can take as $\mathcal{B}$ the annihilator in $\mathcal{X}$ of a subspace of $\mathcal{X}'$ generated by fixing a basis in $\mathcal{A}$, taking the dual basis in $\mathcal{A}'$ and extending its functionals to $\mathcal{X}'$ (by the HAHN-BANACH theorem).

From the **bounded decomposition**, being $\mathbf{P}_{\mathcal{A}}\,\mathbf{a} = \mathbf{a} \quad \forall\, \mathbf{a} \in \mathcal{A}$, we infer that

$$\|\, \mathbf{x} - \mathbf{a} \,\|_{\mathcal{X}} \geq c \,\|\, (\mathbf{x} - \mathbf{a}) - \mathbf{P}_{\mathcal{A}}\,(\mathbf{x} - \mathbf{a}) \,\|_{\mathcal{X}} = c \,\|\, \mathbf{x} - \mathbf{P}_{\mathcal{A}}\,\mathbf{x} \,\|_{\mathcal{X}}\,, \quad \forall\, \mathbf{a} \in \mathcal{A}\,, \quad \forall\, \mathbf{x} \in \mathcal{X}\,,$$

which is equivalent to $\|\, \mathbf{x} \,\|_{\mathcal{X}/\mathcal{A}} \geq c \,\|\, \mathbf{x} - \mathbf{P}_{\mathcal{A}}\,\mathbf{x} \,\|_{\mathcal{X}} \quad \forall\, \mathbf{x} \in \mathcal{X}$. Hence we have that

$$\|\, \mathbf{x} - \mathbf{P}_{\mathcal{A}}\,\mathbf{x} \,\|_{\mathcal{X}} \geq \|\, \mathbf{x} \,\|_{\mathcal{X}/\mathcal{A}} \geq c \,\|\, \mathbf{x} - \mathbf{P}_{\mathcal{A}}\,\mathbf{x} \,\|_{\mathcal{X}}\,, \quad \forall\, \mathbf{x} \in \mathcal{X}\,.$$

Tartar's Lemma. *Let H be a reflexive BANACH space, E, F be normed linear spaces and $\mathbf{A} \in \mathrm{Lin}\,\{H, E\}$ a bounded linear operator. If there exists a bounded linear operator $\mathbf{L}_o \in \mathrm{Lin}\,\{H, F\}$ such that*

$$\begin{cases} i) \quad \mathbf{L}_o \in \mathrm{Lin}\,\{H, F\} \quad \text{is compact}\,, \\ ii) \quad \|\, \mathbf{A}\mathbf{u} \,\|_E + \|\, \mathbf{L}_o\mathbf{u} \,\|_F \geq \alpha \,\|\, \mathbf{u} \,\|_H \quad \forall\, \mathbf{u} \in H\,, \end{cases}$$

then we have that

$$\begin{cases} a) \quad \dim\,(\,\mathrm{Ker}\,\mathbf{A}\,) < +\infty\,, \\ b) \quad \|\, \mathbf{A}\mathbf{u} \,\|_E \geq c_{\mathbf{A}} \,\|\, \mathbf{u} \,\|_{H/\mathrm{Ker}\,\mathbf{A}} \quad \forall\, \mathbf{u} \in H\,. \end{cases}$$

Proof. Let's prove that the closed linear subspace $\mathrm{Ker}\,\mathbf{A} \subset H$ is finite dimensional. We first note that $ii)$ implies that

$$\|\, \mathbf{L}_o\mathbf{u} \,\|_F \geq \alpha \,\|\, \mathbf{u} \,\|_H \quad \forall\, \mathbf{u} \in \mathrm{Ker}\,\mathbf{A}\,.$$

On the other hand, denoting by $\overset{w}{\rightarrow}$ the weak convergence in H, the compactness property $i)$ implies that

$$\left.\begin{array}{l} \{\mathbf{u}_n\} \subset \mathrm{Ker}\,\mathbf{A}\,, \\ \mathbf{u}_n \overset{w}{\rightarrow} \mathbf{u}_\infty \quad \text{in} \quad H\,, \end{array}\right\} \Rightarrow \|\, \mathbf{L}_o(\mathbf{u}_n - \mathbf{u}_\infty) \,\|_F \rightarrow 0 \;\Rightarrow\; \|\, \mathbf{u}_n - \mathbf{u}_\infty \,\|_H \rightarrow 0\,,$$

We may then conclude that every weakly convergent sequence in $\mathrm{Ker}\,\mathbf{A}$ is strongly convergent. Hence, by the reflexivity of $\mathcal{H}$ ([8] III.2, remark 4) we must have $\dim\,(\,\mathrm{Ker}\,\mathbf{A}\,) < \infty$ and $a)$ is proved. Then $\mathrm{Ker}\,\mathbf{A}$ admits a topological supplement $\mathcal{S}$ and we can consider the bounded linear operator $\mathbf{P}_{\mathbf{A}} \in \mathrm{Lin}\,\{H, H\}$ which is the projector on $\mathrm{Ker}\,\mathbf{A}$ subordinated to the decomposition $H = \mathrm{Ker}\,\mathbf{A} + \mathcal{S}$. Let us now suppose that $b)$ is false. There would exists a sequence $\{\mathbf{u}_n\} \subset H$ such that $\|\, \mathbf{A}\mathbf{u}_n \,\|_E \rightarrow 0$ and $\|\, \mathbf{u}_n \,\|_{H/\mathrm{Ker}\,\mathbf{A}} = 1$. By the inequality $\|\, \mathbf{u}_n \,\|_{H/\mathrm{Ker}\,\mathbf{A}} \geq c \,\|\, \mathbf{u}_n - \mathbf{P}_{\mathcal{A}}\,\mathbf{u}_n \,\|_H$ the sequence $\mathbf{u}_n - \mathbf{P}_{\mathbf{A}}\mathbf{u}_n$ is bounded in H. Hence the compactness of the operator $\mathbf{L}_o \in \mathrm{Lin}\,\{H, F\}$ ensures that we can extract from the sequence $\mathbf{L}_o(\mathbf{u}_n - \mathbf{P}_{\mathbf{A}}\mathbf{u}_n)$ a CAUCHY subsequence $\mathbf{L}_o(\mathbf{u}_k - \mathbf{P}_{\mathbf{A}}\mathbf{u}_k)$ in F. The sequence $\mathbf{A}\mathbf{u}_k$ is convergent in E by assumption and hence we infer from $ii)$ that $\mathbf{u}_k - \mathbf{P}_{\mathbf{A}}\mathbf{u}_k$ is a CAUCHY sequence which by the completeness of H converges to an element $\mathbf{u}_\infty \in H$. Since $\mathbf{A}\mathbf{u}_k$ converges to zero in E the boundedness of $\mathbf{A} \in \mathrm{Lin}\,\{H, E\}$ ensures that $\mathbf{u}_\infty \in \mathrm{Ker}\,\mathbf{A}$ so that also $\mathbf{P}_{\mathbf{A}}\mathbf{u}_k + \mathbf{u}_\infty \in \mathrm{Ker}\,\mathbf{A}$. Finally from $ii)$ we get that

$$\alpha \,\|\, \mathbf{u}_k \,\|_{H/\mathrm{Ker}\,\mathbf{A}} \leq \|\, \mathbf{A}\mathbf{u}_k \,\|_E + \|\, \mathbf{L}_o(\mathbf{u}_k - \mathbf{P}_{\mathbf{A}}\mathbf{u}_k - \mathbf{u}_\infty) \,\|_F \rightarrow 0\,,$$

and this is absurd since $\|\, \mathbf{u}_k \,\|_{H/\mathrm{Ker}\,\mathbf{A}} = 1$. $\square$

Remark. TARTAR's lemma is quoted in [6] referring to [5] for the proof of the statement. Although in [5] and [6] the space H was assumed to be a (non reflexive) BANACH space, property $a)$ cannot be inferred in this general context. A well-known counterexample is provided by the space l^1 of absolutely convergent real sequences. In fact SHUR'S theorem states that in this infinite dimensional BANACH space every weakly convergent sequence is also strongly convergent (see [3] V.1 theorem 5 and [8] III.2, remark 4). We also note that the proof of property $b)$, as developed in [5], requires the existence of a weakly convergent subsequence of a bounded sequence and hence, by the EBERLEIN-SHMULYAN theorem, the BANACH space H should be reflexive. The proof of property $b)$ proposed here is instead based on a completeness argument which does not require the reflexivity of the BANACH space H (private communication by RENATO FIORENZA). ■

Inverse Lemma

Let us now face the question whether TARTAR's lemma can be completed by assessing the converse implication. A positive answer needs an existence result. We have in fact to prove that properties $a)$ and $b)$ in TARTAR's lemma imply the existence of a compact operator $\mathbf{L}_o \in \text{Lin}\{H, F\}$ fulfilling property $ii)$. Firstly we observe that $ii)$ implies that $\text{Ker}\,\mathbf{A} \cap \text{Ker}\,\mathbf{L}_o = \{\mathbf{o}\}$. Our strategy consists in relaxing the requests on $\mathbf{L}_o$ by considering at its place any operator $\mathbf{L} \in \text{Lin}\{H, F\}$. We then try to establish the inequality

$$\|\mathbf{Au}\|_E + \|\mathbf{Lu}\|_F \geq \alpha_{\mathbf{L}} \|\mathbf{u}\|_{H/(\text{Ker}\,\mathbf{A} \cap \text{Ker}\,\mathbf{L})} \quad \forall\, \mathbf{u} \in H$$

for any $\mathbf{L} \in \text{Lin}\{H, F\}$. Once this goal has been achieved we can choose $\mathbf{L}$ to be compact and such that $\text{Ker}\,\mathbf{A} \cap \text{Ker}\,\mathbf{L} = \{\mathbf{o}\}$. We need some preliminary results. From the **bounded decomposition** we infer the next proposition.

Distance inequalities. *Let $\mathcal{X}$ be a* BANACH *space and $\mathcal{A} \subseteq \mathcal{X}$, $\mathcal{B} \subseteq \mathcal{X}$ closed linear subspaces of $\mathcal{X}$ such that their sum $\mathcal{A} + \mathcal{B}$ is closed. Then, setting $k = c^{-1} > 0$ we have*

$$i) \quad \|\mathbf{x}\|_{\mathcal{X}/\mathcal{A}\cap\mathcal{B}} \leq \|\mathbf{x} - \mathbf{a}\|_{\mathcal{X}} + k\,\|\mathbf{a} + \mathbf{b}\|_{\mathcal{X}}, \quad \mathbf{x} \in \mathcal{X}, \quad \forall\, \{\mathbf{a}, \mathbf{b}\} \in \mathcal{A} \times \mathcal{B}.$$

If $\mathcal{A}$ admits a topological supplement $\mathcal{S}$ so that $\mathcal{X} = \mathcal{A} \dotplus \mathcal{S}$ then we infer that

$$ii) \quad \|\mathbf{x}\|_{\mathcal{X}/\mathcal{A}\cap\mathcal{B}} \leq \|\mathbf{x} - \mathbf{P}_{\mathcal{A}}\mathbf{x}\|_{\mathcal{X}} + k\,\|\mathbf{P}_{\mathcal{A}}\mathbf{x}\|_{\mathcal{X}/\mathcal{B}}, \quad \mathbf{x} \in \mathcal{X}.$$

where $\mathbf{P}_{\mathcal{A}}$ is the projector on $\mathcal{A}$ subordinated to the direct sum decomposition of $\mathcal{X}$.

Proof. The **bounded decomposition** ensures that for every $\mathbf{x} \in \mathcal{X}$, $\mathbf{a} \in \mathcal{A}$, $\mathbf{b} \in \mathcal{B}$ there exists a $\boldsymbol{\rho} \in \mathcal{A} \cap \mathcal{B}$ such that $\|\mathbf{a} + \boldsymbol{\rho}\|_{\mathcal{X}} \leq k\,\|\mathbf{a} + \mathbf{b}\|_{\mathcal{X}}$. Hence we infer $i)$:

$$\|\mathbf{x}\|_{\mathcal{X}/\mathcal{A}\cap\mathcal{B}} \leq \|\mathbf{x} + \boldsymbol{\rho}\|_{\mathcal{X}} \leq \|\mathbf{x} - \mathbf{a}\|_{\mathcal{X}} + \|\mathbf{a} + \boldsymbol{\rho}\|_{\mathcal{X}} \leq \|\mathbf{x} - \mathbf{a}\|_{\mathcal{X}} + k\,\|\mathbf{a} + \mathbf{b}\|_{\mathcal{X}}.$$

Setting $\mathbf{a} = \mathbf{P}_{\mathcal{A}}\mathbf{x}$ and taking the infimum with respect to $\mathbf{b} \in \mathcal{B}$ we get the inequality $ii)$. □

The following two lemmas yield the tools for the main result. The first one is a variant of a result quoted in [9] with reference to symmetric quadratic forms.

Projection inequality. *Let H be a* BANACH *space and E, F be linear normed spaces. Let moreover $\mathbf{A} \in \text{Lin}\{H, E\}$ e $\mathbf{L} \in \text{Lin}\{H, F\}$ be linear bounded operators such that*

$$\begin{cases} i) & \|\mathbf{Au}\|_E \geq c_{\mathbf{A}} \|\mathbf{u}\|_{H/\text{Ker}\,\mathbf{A}}, \quad \forall\, \mathbf{u} \in H, \\ ii) & \|\mathbf{Lu}\|_F \geq c_{\mathbf{L}} \|\mathbf{u}\|_{H/\text{Ker}\,\mathbf{L}}, \quad \forall\, \mathbf{u} \in \text{Ker}\,\mathbf{A}. \end{cases}$$

Let moreover $\operatorname{Ker}\mathbf{A}$ *admit a topological supplement* $\mathcal{S}$ *so that* $H = \operatorname{Ker}\mathbf{A} \dotplus \mathcal{S}$. *Then we have*

$$a) \quad \|\mathbf{A}\mathbf{u}\|_E + \|\mathbf{L}\,\mathbf{u}\|_F \geq \alpha\,\|\mathbf{P_A}\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{L}}\,, \quad \forall\,\mathbf{u}\in H\,.$$

where $\mathbf{P_A} \in \operatorname{Lin}\{H,H\}$ *is the projector on* $\operatorname{Ker}\mathbf{A}$ *subordinated to the decomposition* $H = \operatorname{Ker}\mathbf{A} \dotplus \mathcal{S}$.

Proof. If $a)$ would be false we could find a sequence $\{\mathbf{u}_n\}\subset H$ such that

$$\|\mathbf{P_A}\mathbf{u}_n\|_{H/\operatorname{Ker}\mathbf{L}} = 1\,, \quad \|\mathbf{A}\mathbf{u}_n\|_E \to 0\,, \quad \|\mathbf{L}\mathbf{u}_n\|_F \to 0\,.$$

Since $\|\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{A}} \geq c\,\|\mathbf{u} - \mathbf{P}_{\mathcal{A}}\mathbf{u}\|_H \quad \forall\,\mathbf{u}\in H$ we infer from $i)$ that

$$\|\mathbf{A}\mathbf{u}_n\|_E \to 0 \;\Rightarrow\; \|\mathbf{u}_n - \mathbf{P_A}\mathbf{u}_n\|_H \to 0\,.$$

Moreover we have

$$\begin{cases} \|\mathbf{L}\|\,\|\mathbf{u}_n - \mathbf{P_A}\mathbf{u}_n\|_H \geq \|\mathbf{L}(\mathbf{u}_n - \mathbf{P_A}\mathbf{u}_n)\|_F\,, \\ \|\mathbf{L}\mathbf{P_A}\mathbf{u}_n\|_F \leq \|\mathbf{L}(\mathbf{u}_n - \mathbf{P_A}\mathbf{u}_n)\|_F + \|\mathbf{L}\mathbf{u}_n\|_F\,. \end{cases}$$

Hence $\|\mathbf{L}\mathbf{P_A}\mathbf{u}_n\|_F \to 0$ and from $ii)$ we get

$$\|\mathbf{L}\mathbf{P_A}\mathbf{u}_n\|_F \geq c_{\mathbf{L}}\,\|\mathbf{P_A}\mathbf{u}_n\|_{H/\operatorname{Ker}\mathbf{L}} \;\Rightarrow\; \|\mathbf{P_A}\mathbf{u}_n\|_{H/\operatorname{Ker}\mathbf{L}} \to 0\,,$$

which is absurd since $\|\mathbf{P_A}\mathbf{u}_n\|_{H/\operatorname{Ker}\mathbf{L}} = 1$. □

Abstract inequality. *Let* H *be a* BANACH *space and* E, F *be linear normed spaces. Let moreover* $\mathbf{A}\in\operatorname{Lin}\{H,E\}$ *e* $\mathbf{L}\in\operatorname{Lin}\{H,F\}$ *be linear bounded operators such that*

$$\begin{cases} i) \quad \|\mathbf{A}\mathbf{u}\|_E \geq c_{\mathbf{A}}\,\|\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{A}}\,, \quad \forall\,\mathbf{u}\in H\,, \\ ii) \quad \|\mathbf{L}\mathbf{u}\|_F \geq c_{\mathbf{L}}\,\|\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{L}}\,, \quad \forall\,\mathbf{u}\in \operatorname{Ker}\mathbf{A}\,, \\ iii) \quad \operatorname{Ker}\mathbf{A} + \operatorname{Ker}\mathbf{L} \quad \text{closed in} \quad H\,. \end{cases}$$

Let moreover $\operatorname{Ker}\mathbf{A}$ *admit a topological supplement* $\mathcal{S}$ *so that* $H = \operatorname{Ker}\mathbf{A} \dotplus \mathcal{S}$. *Then we have*

$$c) \quad \|\mathbf{A}\mathbf{u}\|_E + \|\mathbf{L}\mathbf{u}\|_F \geq \alpha\,\|\mathbf{u}\|_{H/(\operatorname{Ker}\mathbf{A}\cap\operatorname{Ker}\mathbf{L})}\,.$$

Proof. Summing up the inequalities $a)$ and $i)$ in proposition **projection inequality** we get

$$\|\mathbf{A}\mathbf{u}\|_E + \|\mathbf{L}\,\mathbf{u}\|_F \geq \alpha_o\Big(\|\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{A}} + \|\mathbf{P_A}\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{L}}\Big)\,, \quad \forall\,\mathbf{u}\in H\,.$$

Moreover by assumption $iii)$ the proposition **distance inequalities** implies that

$$\|\mathbf{u} - \mathbf{P_A}\mathbf{u}\|_H + k\,\|\mathbf{P_A}\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{L}} \geq c\,\|\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{A}\cap\operatorname{Ker}\mathbf{L}}\,, \quad \forall\,\mathbf{u}\in H\,.$$

Recalling that $\|\mathbf{u}\|_{H/\operatorname{Ker}\mathbf{A}} \geq c\,\|\mathbf{u} - \mathbf{P}_{\mathcal{A}}\mathbf{u}\|_H \quad \forall\,\mathbf{u}\in H$ we get the result. □

The next lemma yields the crucial result for our analysis.

Inverse lemma. *Let* H *be a* BANACH *space and* E *,* F *be linear normed spaces. Let moreover* $\mathbf{A} \in \mathrm{Lin}\,\{H, E\}$ *be a linear bounded operator such that*

$$\begin{cases} a) & \dim \mathrm{Ker}\,\mathbf{A} < +\infty\,, \\ b) & \|\,\mathbf{Au}\,\|_E \geq c_{\mathbf{A}}\,\|\,\mathbf{u}\,\|_{H/\mathrm{Ker}\,\mathbf{A}}\,, \quad \forall\,\mathbf{u} \in H\,. \end{cases}$$

Then for any $\mathbf{L} \in \mathrm{Lin}\,\{H, F\}$ *we have*

$$i) \quad \|\,\mathbf{Au}\,\|_E + \|\,\mathbf{Lu}\,\|_F \geq \alpha\,\|\,\mathbf{u}\,\|_{H/(\mathrm{Ker}\,\mathbf{A} \cap \mathrm{Ker}\,\mathbf{L})}\,, \quad \forall\,\mathbf{u} \in H\,.$$

Proof. It suffices to observe that any finite dimensional subspace admits a topological supplement in H and that condition $a)$ implies the validity of $ii)$ and $iii)$ of the **abstract inequality** for any $\mathbf{L} \in \mathrm{Lin}\,\{H, F\}$. □

Now we recall that

- any continuous projection operator on a finite dimensional subspace is compact.

It follow that if $\dim \mathrm{Ker}\,\mathbf{A} < +\infty$ there exists at least a compact operator $\mathbf{L}_o \in \mathrm{Lin}\,\{H, F\}$ such that $\mathrm{Ker}\,\mathbf{A} \cap \mathrm{Ker}\,\mathbf{L}_o = \{\mathbf{o}\}$. Indeed we can set $\mathbf{L}_o = \mathbf{P}_{\mathbf{A}} \in \mathrm{Lin}\,\{H, H\}$, the projection operator on the finite dimensional subspace $\mathrm{Ker}\,\mathbf{A} \subset H$ defined by a direct sum decomposition $H = (\mathrm{Ker}\,\mathbf{A}) \dotplus \mathcal{S}$ with $\mathcal{S}$ topological supplement of $\mathrm{Ker}\,\mathbf{A}$.

We can now provide a full extension of TARTAR's lemma by including the converse implication and the equivalence to a new property.

Equivalent inequalities. *Let* H *be a reflexive* BANACH *space,* E *,* F *be normed linear spaces and* $\mathbf{A} \in \mathrm{Lin}\,\{H, E\}$ *a bounded linear operator. Then the following propositions are equivalent:*

$$\mathbb{P}_1) \quad \boxed{\begin{cases} \dim \mathrm{Ker}\,\mathbf{A} < +\infty\,, \\ \|\,\mathbf{Au}\,\|_E \geq c_{\mathbf{A}}\,\|\,\mathbf{u}\,\|_{H/\mathrm{Ker}\,\mathbf{A}}\,, \quad \forall\,\mathbf{u} \in H\,, \end{cases}}$$

$$\mathbb{P}_2) \quad \boxed{\begin{array}{l} \text{There exists } \mathbf{L}_o \in \mathrm{Lin}\,\{H, F\} \text{ compact} \\ \text{such that } \mathrm{Ker}\,\mathbf{A} \cap \mathrm{Ker}\,\mathbf{L}_o = \{\mathbf{o}\} \text{ and} \\ \|\,\mathbf{Au}\,\|_E + \|\,\mathbf{L}_o\mathbf{u}\,\|_F \geq \alpha\,\|\,\mathbf{u}\,\|_H\,, \quad \forall\,\mathbf{u} \in H\,, \end{array}}$$

$$\mathbb{P}_3) \quad \boxed{\begin{cases} \dim \mathrm{Ker}\,\mathbf{A} < +\infty\,, \\ \|\,\mathbf{Au}\,\|_E + \|\,\mathbf{Lu}\,\|_F \geq \alpha\,\|\,\mathbf{u}\,\|_{H/(\mathrm{Ker}\,\mathbf{A} \cap \mathrm{Ker}\,\mathbf{L})}\,, \quad \forall\,\mathbf{u} \in H\,, \quad \forall\,\mathbf{L} \in \mathrm{Lin}\,\{H, F\}\,, \end{cases}}$$

Proof. $\mathbb{P}_3 \Rightarrow \mathbb{P}_1$ setting $\mathbf{L} = \mathbf{O}$. $\mathbb{P}_3 \Rightarrow \mathbb{P}_2$ setting $\mathbf{L} = \mathbf{L}_o = \mathbf{P}_{\mathbf{A}}$. $\mathbb{P}_1 \Rightarrow \mathbb{P}_3$ by the **inverse lemma**. Finally $\mathbb{P}_2 \Rightarrow \mathbb{P}_1$ by TARTAR's lemma which is the one requiring the reflexivity of the BANACH space H . □

Korn's Inequality

In continuum mechanics the fundamental theorems concerning the variational formulation of equilibrium and compatibility are founded on the property that the kinematic operator has a closed range and a finite dimensional kernel. Theabstractframework isthefollowing. Astructuralmodel isdefined onaregularbounded domain Ω of an euclidean space and is governed by a kinematic operator $\mathbf{B}$ which is the regular part of a distributional differential operator $\mathbb{B} : \mathcal{V}(\Omega) \mapsto \mathbb{D}'(\Omega)$ of order m acting on kinematic fields $\mathbf{u} \in \mathcal{V}(\Omega)$ which are square integrable on Ω and such that the corresponding distributional linearized strain field $\mathbb{B}\mathbf{u} \in \mathbb{D}'(\Omega)$ is square integrable on a finite subdivision $\mathcal{T}_{\mathbf{u}}(\Omega)$ of Ω. The kinematic space $\mathcal{V}(\Omega)$ is a pre-HILBERT space when endowed with the topology induced by the norm

$$\| \mathbf{u} \|^2_{\mathcal{V}(\Omega)} = \| \mathbf{u} \|^2_{H(\Omega)} + \| \mathbf{B}\mathbf{u} \|^2_{\mathcal{H}(\Omega)} ,$$

where $H(\Omega)$ and $\mathcal{H}(\Omega)$ are the spaces of kinematic and linearized strain fields which are square integrable on Ω [11]. The conforming kinematisms $\mathbf{u} \in \mathcal{L}(\Omega)$ belong to a closed linear subspace $\mathcal{L}(\Omega) \subset H^m(\mathcal{T}(\Omega)) \subset \mathcal{V}(\Omega)$ of the SOBOLEV space $H^m(\mathcal{T}(\Omega))$, where $\mathcal{T}(\Omega)$ is a given finite subdivision of Ω. Thus $\mathcal{L}(\Omega) \subset H^m(\mathcal{T}(\Omega))$ is an HILBERT space and the operator $\mathbf{B}_{\mathcal{L}} \in \mathrm{Lin}\,\{\mathcal{L}(\Omega), \mathcal{H}(\Omega)\}$ defining the linearized regular strain $\mathbf{B}\mathbf{u} \in \mathcal{H}(\Omega)$ associated with the conforming kinematic field $\mathbf{u} \in \mathcal{L}(\Omega)$ is linear and continuous. The kinematic operator $\mathbf{B} \in \mathrm{Lin}\,\{\mathcal{V}(\Omega), \mathcal{H}(\Omega)\}$ is assumed to be regular in the sense that for any $\mathcal{L}(\Omega) \subset \mathcal{V}(\Omega)$ the following conditions are met [11]

$$\begin{cases} \dim \, \mathrm{Ker}\, \mathbf{B}_{\mathcal{L}} < +\infty \,, \\ \| \mathbf{B}\mathbf{u} \|_{\mathcal{H}(\Omega)} \geq c_{\mathbf{B}} \, \| \mathbf{u} \|_{\mathcal{L}(\Omega)/\mathrm{Ker}\,\mathbf{B}_{\mathcal{L}}} \,, \quad \forall \mathbf{u} \in \mathcal{L}(\Omega) \end{cases} \iff \quad \mathrm{Im}\,\mathbf{B}_{\mathcal{L}} \;\; \text{closed in} \;\; \mathcal{H}(\Omega) \,.$$

The requirement that the property must hold for any $\mathcal{L}(\Omega) \subset \mathcal{V}(\Omega)$ is motivated by the observation that in applications it is fundamental to assess that the basic existence results hold for any choice of the kinematic contraints. The regularity of $\mathbf{B} \in \mathrm{Lin}\,\{\mathcal{V}(\Omega), \mathcal{H}(\Omega)\}$ is the basic tool for the proof of the theorem of virtual powers which ensures the existence of a stress field in equilibrium with an equilibrated system of active forces.

Theorem of Virtual Powers. *Let $\mathbf{f} \in \mathcal{L}'(\Omega)$ be a system of active forces. Then*

$$\mathbf{f} \in (\,\mathrm{Ker}\,\mathbf{B}_{\mathcal{L}})^{\perp} \;\Rightarrow\; \exists\, \boldsymbol{\sigma} \in \mathcal{H}(\Omega) \;:\; \langle\, \mathbf{f} \,,\, \mathbf{v} \,\rangle = ((\, \boldsymbol{\sigma} \,,\, \mathbf{B}\mathbf{v} \,)) \,, \quad \forall \mathbf{v} \in \mathcal{L}(\Omega) \,.$$

Proof. Let $\mathbf{B}'_{\mathcal{L}} \in \mathrm{Lin}\,\{\mathcal{H}(\Omega), \mathcal{L}'(\Omega)\}$ be the equilibrium operator dual to $\mathbf{B}_{\mathcal{L}}$. By BANACH's closed range theorem we have that $\mathbf{f} \in (\,\mathrm{Ker}\,\mathbf{B}_{\mathcal{L}})^{\perp} = \mathrm{Im}\,\mathbf{B}'_{\mathcal{L}}$ and the duality relation yields the result. □

A linearized strain field $\boldsymbol{\varepsilon} \in \mathcal{H}(\Omega)$ is kinematically compatible if there exists a conforming kinematic field $\mathbf{u} \in \mathcal{L}(\Omega)$ such that $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}$. Self-equilibrated stress fields are the elements of $\mathcal{H}(\Omega)$ which belong to the kernel of the equilibrium operator $\mathbf{B}'_{\mathcal{L}} \in \mathrm{Lin}\,\{\mathcal{H}(\Omega), \mathcal{L}'(\Omega)\}$. The regularity of $\mathbf{B} \in \mathrm{Lin}\,\{\mathcal{L}(\Omega), \mathcal{H}(\Omega)\}$ provides the following variational condition.

Kinematical compatibility.

$$((\, \boldsymbol{\sigma} \,,\, \boldsymbol{\varepsilon} \,)) = 0 \quad \forall \boldsymbol{\sigma} \in \,\mathrm{Ker}\,\mathbf{B}'_{\mathcal{L}} \;\Rightarrow\; \exists\, \mathbf{u} \in \mathcal{L}(\Omega) \;:\; \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u} \,.$$

Proof. By BANACH's closed range theorem we have that $\mathrm{Im}\,\mathbf{B}_{\mathcal{L}} = (\,\mathrm{Ker}\,\mathbf{B}'_{\mathcal{L}})^{\perp}$. □

The regularity of the kinematic operator $\mathbf{B} \in \mathrm{Lin}\,\{\mathcal{V}(\Omega), \mathcal{H}(\Omega)\}$ is then a fundamental property to be assessed in a structural model. Our analysis shows that a necessary and sufficient condition is the validity of an inequality of the KORN's type

$$\| \mathbf{Bu} \|_{\mathcal{H}(\Omega)} + \| \mathbf{u} \|_{H(\Omega)} \geq \alpha \, \| \mathbf{u} \|_{H^m(\Omega)} \,, \quad \forall \mathbf{u} \in H^m(\Omega) \,,$$

Note that by RELLICH selection principle [2] the canonical immersion from $H^m(\Omega)$ into $H(\Omega) = \mathcal{L}^2(\Omega)$ is compact. If KORN's inequality holds for any $\mathbf{u} \in H^m(\Omega)$ it will hold also for any $\mathbf{u} \in H^m(\mathcal{T}(\Omega))$ and then a fortiori for any $\mathbf{u} \in \mathcal{L}(\Omega)$.

With reference to the three-dimensional continuous model we remark that KORN's first inequality can be easily derived from KORN's second inequality by appealing to the **inverse lemma**.

In fact denoting by $H^{1/2}(\partial\Omega)^3$, the space of traces of fields in $H^1(\Omega)^3$ on the boundary $\partial\Omega$ of Ω and taking $\mathbf{L}$ to be the boundary trace operator $\mathbf{\Gamma} \in \mathrm{Lin}\,\{H^1(\Omega)^3, H^{1/2}(\partial\Omega)^3\}$ we get

$$\| \mathbf{Bu} \|_{\mathcal{H}(\Omega)} + \| \mathbf{\Gamma u} \|_{H^{1/2}(\partial\Omega)^3} \geq \alpha \, \| \mathbf{u} \|_{H^1(\Omega)^3} \quad \forall \mathbf{u} \in H^1(\Omega)^3 \,,$$

and hence

$$\| \mathbf{Bu} \|_{\mathcal{H}(\Omega)} \geq \alpha \, \| \mathbf{u} \|_{H^1(\Omega)^3} \quad \forall \mathbf{u} \in H^1(\Omega)^3 \cap \mathrm{Ker}\,\mathbf{\Gamma} = H^1_0(\Omega)^3 \,,$$

which is KORN's first inequality. The original form of the second inequality as stated by KORN was in fact

$$\| \,\mathrm{sym\ grad}\, \mathbf{u} \|_{\mathcal{L}^2(\Omega)} \geq \alpha \, \| \mathbf{u} \|_{H^1(\Omega)} \quad \forall \mathbf{u} \in H^1(\Omega) \,:\, \int_{\Omega} \mathrm{emi\ grad}\, \mathbf{u} \; \mathrm{d}\mu = \mathbf{O} \,.$$

By the **inverse lemma** also this original form can be recovered simply by setting

$$\mathbf{L} \in \mathrm{Lin}\,\{H^1(\Omega)^3, \Re^6\} \,, \quad \mathbf{Lu} \,:= \int_{\Omega} \mathrm{emi\ grad}\, \mathbf{u} \; \mathrm{d}\mu \,.$$

We thus get the inequality

$$\| \,\mathrm{sym\ grad}\, \mathbf{u} \|_{\mathcal{L}^2(\Omega)} + \left\| \int_{\Omega} \mathrm{emi\ grad}\, \mathbf{u} \; \mathrm{d}\mu \right\| \geq \alpha \, \| \mathbf{u} \|_{H^1(\Omega)} \quad \forall \mathbf{u} \in H^1(\Omega) \,.$$

which immediately implies KORN's original inequality.

The proof of the converse implication is more involved and can be found in G. FICHERA's article [2], remark on page 384. A more detailed version of the proof is provided in [10], lemma 7.11.

From the **inverse lemma** we can also infer POINCARÉ inequality.
Let Ω be an open bounded connected set in $\Re^d$ with a regular boundary. We set

- $\mathbf{A} \in \mathrm{Lin}\,\{H^m(\Omega), \mathcal{L}^2(\Omega)^k\}$ continuous linear operator $\mathbf{Au} = \{D^{\mathbf{p}}\mathbf{u}\}$, with $k = \mathrm{card}\{\mathbf{p} \in \mathcal{N}^d \,:\, |\,\mathbf{p}\,| = m\}$ and $|\,\mathbf{p}\,| = m$,
- $\mathbf{L}_o \in \mathrm{Lin}\,\{H^m(\Omega), H^{m-1}(\Omega)\}$ compact identity map $\mathbf{L}_o\mathbf{u} = \mathbf{u}$,
- $\mathbf{L} \in \mathrm{Lin}\,\{H^m(\Omega), \mathcal{L}^2(\Omega)^r\}$ continuous linear operator defined by

$$\mathbf{Lu} = \left\{ \frac{1}{\sqrt{\mathrm{meas}\,\Omega}} \int_{\Omega} D^{\mathbf{p}}\mathbf{u}(\mathbf{x}) \; \mathrm{d}\mu \right\} , \quad 0 \leq |\,\mathbf{p}\,| \leq m-1 \,,$$

with $r = \mathrm{card}\{\mathbf{p} \in \mathcal{N}^d \,:\, |\,\mathbf{p}\,| < m\}$,

where $\mathbf{p}$ is a d-multi-index and $|\,\mathbf{p}\,|$ is the sum of the components of $\mathbf{p}$.

We set $H = H^m(\Omega)$, $E = \mathcal{L}^2(\Omega)^k$, $E_o = H^{m-1}(\Omega)$, $F = \mathcal{L}^2(\Omega)^r$, so that

$$\mathbf{A} \in \text{Lin}\{H, E\}\,, \quad \mathbf{L}_o \in \text{Lin}\{H, E_o\}\,, \quad \mathbf{L} \in \text{Lin}\{H, F\}\,.$$

Then property $\mathbb{P}_2$ of proposition **equivalent inequalities** is fulfilled since

$$\begin{cases} \|\,\mathbf{Au}\,\|_E^2 + \|\,\mathbf{L}_o\mathbf{u}\,\|_{E_o}^2 = \|\,\mathbf{u}\,\|_H^2\,, \\ \mathbf{L}_o \in \text{Lin}\{H, E_o\} \quad \text{is compact}\,. \end{cases}$$

We remark that $\text{Ker}\,\mathbf{A} = P_{m-1}(\Omega)$ is the finite dimensional linear subspace of polynomials of total degree not greater than $m-1$ so that $\dim P_{m-1}(\Omega) = (m-1+d)!/(d!\,(m-1)!)$. Moreover we have that

$$\text{Ker}\,\mathbf{A} \cap \text{Ker}\,\mathbf{L} = \{\mathbf{o}\}\,,$$

and hence property $\mathbb{P}_3$ of proposition **equivalent inequalities** yields

$$\|\,\mathbf{Au}\,\|_E + \|\,\mathbf{Lu}\,\|_F \geq \alpha\,\|\,\mathbf{u}\,\|_H \quad \forall\,\mathbf{u} \in H\,,$$

or explicitly

$$\sum_{|\,\mathbf{p}\,|=m} \int_\Omega |\,D^{\mathbf{p}}\mathbf{u}(\mathbf{x})\,|^2\, \mathrm{d}\mu + \sum_{|\,\mathbf{p}\,|<m} \left|\, \int_\Omega D^{\mathbf{p}}\mathbf{u}(\mathbf{x})\, \mathrm{d}\mu \,\right|^2 \geq \alpha\,\|\,\mathbf{u}\,\|_{H^m(\Omega)}^2\,, \quad \forall\,\mathbf{u} \in H^m(\Omega)\,,$$

which is POINCARÉ inequality.

While proof-reading this paper the author became aware of a result, quoted by ROGER TEMAM in [7], section I.1, which is a special case of the inverse lemma. This result was not explicitly proved in [7] and was resorted to in deriving a proof of KORN's inequality from the property that the distributional operator grad $\in \text{Lin}\{\mathcal{L}^2(\Omega)^n, H^{-1}(\Omega)^{n\times n}\}$ has a closed range and a one-dimensional kernel consisting of the constant fields on Ω (see [10] for an explicit proof). This property is in turn a direct consequence of a fundamental inequality due to J. NECAS [1].

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