ON THE MINIMUM OF THE POTENTIAL ENERGY FUNCTIONAL AT A CRITICAL POINT

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SOMMARIO: Si analizza la stabilità di una configurazione d'equilibrio critico di un sistema elastico conservativo sulla base del metodo energetico, come esistenza di un minimo stretto del funzionale energia potenziale (f.e.p.).

Si fa riferimento a sistemi discreti che vengono studiati con i metodi dell'algebra lineare.

L'analisi originale di Koiter è riformulata con una trattazione più compatta.

Nel caso generale di un punto critico multiplo si sviluppa un procedimento alternativo consistente nell'indagine del comportamento del f.e.p. lungo i percorsi critici.

In un punto critico semplice lo sviluppo dell'autoproblema critico consente di fornire una semplice ed interessante interpretazione meccanica delle usualmente complesse espressioni il cui segno è decisivo per la valutazione della stabilità.

Un semplice esempio illustra l'analisi generale sviluppata.

SUMMARY: The stability of a critical equilibrium configuration of a conservative elastic system is analyzed, on the basis of the energy method, as the existence of a strict minimum of the potential energy functional (p.e.f.).

Finite dimensional systems are investigated in the framework of a linear algebraic approach.

Koiter's original analysis of the subject is reviewed with a more compact formulation.

An alternative procedure is developed by looking at the behaviour of the p.e.f. along the critical paths, in the general case of a multiple critical point. At a simple critical point the expansion of the critical eigenproblem allows to give a simple and interesting mechanical interpretation of the usually involved expressions whose sign is decisive for stability.

A simple example illustrates the general analysis developed.

1. Introduction.

In the mechanics of conservative systems the concept of the potential energy functional (p.e.f.) plays a fundamental role.

Equilibrium configurations are characterized as stationary points of the p.e.f. of the system and their stability or instability is defined according to whether there is a strict local minimum of the p.e.f. or not.

The connection between this definition of stability and the dynamical approach (according to Lyapounov's definition) is still a main open problem in the theory of conservative systems, although many valuable results in this direction are available, especially for finite dimensional systems.

According to the well-known Lagrange-Dirichlet theorem of analytical mechanics, a strict local minimum of the p.e.f. at an equilibrium configuration ensures its dynamical stability.

The inverse implication has been proved only under some special assumptions concerning the structure of the p.e.f. [1].

A complete converse of the Lagrange-Dirichlet theorem is however available if an arbitrary positive definite velocity dependent dissipation is assumed [2], [3].

The situation is much more difficult in the case of infinite dimensional systems where some basic existence results are lacking due to the non-compactness of the configuration space.

A discussion of the difficulties involved may be found in [4].

In spite of this situation the energy definition is usually adopted in the technical literature on elastic stability and its equivalence to the dynamical definition is assumed, both for discrete and continuous systems [5].

A strict minimum of a functional at a stationary point is usually discussed by means of a Taylor's expansion in a neighbourhood of the point.

While the analysis is generally stopped at the inspection of the second differential of the functional, the so called "doubtful case", in which the second differential is non-negative definite, occurs in the most interesting situations in the investigation of the stability of a conservative elastic system: when a lack of uniqueness appears in the incremental static equilibrium solution.

This situation will be referred to as a critical state of the system or as a critical point of the p.e.f.

A general procedure has been first developed, to the author's knowledge, by Koiter in his pioneering work on the stability of continuous elastic systems [6], [7].

The existence of a strict minimum of the p.e.f. at a critical point depends upon the positiveness of a forth order homogeneous functional involving second and forth order differentials of the p.e.f.

An original approach to the problem is developed in this paper and interesting and useful interpretations of this condition are given.

Reference is made to finite dimensional systems where a simple general analysis can be performed.

Indeed, while a formal extension to the infinite dimensional case is straightforward, a rigorous approach would require a much deeper mathematical treatment.

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It must be pointed out to this regard that most "continuum" analyses consist in purely notational changes from the discrete case, while basic conceptual problems as existence and regularity are completely disregarded.

The analysis is carried out from the point of view and in the language of abstract linear algebra which allows to emphasize the geometric interpretations of the problem.

This is a distinguishing feature, with respect to previous treatments of the subject, [8], [9].

Some background definitions and results from linear algebra, which are referred to in the sequel, are first recalled to make the exposition reasonably self-contained.

A detailed discussion of the basic minimum problem which occurs in the analysis and of its equivalent formulations is given in a separate section.

Koiter's approach, founded upon a Taylor's expansion of the functional at the critical point, is then reviewed with a very compact formulation. The general case of a multiple critical point is considered.

By investigating the behaviour of the restrictions of the p.e.f. along any critical path, it is shown that a strict minimum of the p.e.f. at a critical point exists if and only if the same is true for the restrictions of the p.e.f. along any critical path.

In the case of a simple critical point, the expansion of the critical eigenvalue problem along the critical path allows to give an interesting new interpretation of the expressions whose sign is decisive for stability. It is shown that if the lowest stiffness of the system, which is zero at the critical point, has a minimum along this path, the critical point is stable and unstable otherwise. The critical path is moreover characterized as the envelope whose tangent at any point is the principal direction of minimal stiffness of the system. This point of view allows to establish in a direct way the connexion between the stability of the critical point itself and the postcritical behaviour of the system [10].

2. Some results from linear algebra.

Let us recall some notions and results of linear and multilinear algebra which will be needed in the sequel.

Let V be a finite dimensional inner product linear space on the real field R. The inner product of two vectors $\mathbf{x}, \mathbf{y} \in V$ will be denoted by $\mathbf{x} \cdot \mathbf{y}$ and the induced norm by $\|\mathbf{x}\|$.

To every linear functional l and every bilinear functional b on V there correspond biunivocally a vector $\mathbf{f}$ and a couple of linear operators on V, $\mathbf{T}$ and $\tilde{\mathbf{T}}$ such that:

$$l(\mathbf{x}) = \mathbf{f} \cdot \mathbf{x} \qquad \forall\, \mathbf{x} \in V \tag{2.1}$$

$$b(\mathbf{x}, \mathbf{y}) = \mathbf{T}\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \tilde{\mathbf{T}}\mathbf{y} \qquad \forall\, (\mathbf{x}, \mathbf{y}) \in V^2 \tag{2.2}$$ (1)

The operators $\mathbf{T}$ and $\tilde{\mathbf{T}}$ are said to be adjoint each other. If:

$$b(\mathbf{x}, \mathbf{y}) = b(\mathbf{y}, \mathbf{x}) \qquad \forall\, \mathbf{x}, \mathbf{y} \in V$$

(1) V^n denotes the cartesian product $V \times V \times \ldots \times V$ n times. Obviously $V^1 = V$.

we have $\mathbf{T} = \tilde{\mathbf{T}}$ and b and $\mathbf{T}$ are said to be symmetric.

Analogously we say that a multilinear functional on V is symmetric if it is independed of any permutation of its arguments.

If $b(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is a symmetric n - linear functional on V denoting by:

$$b_n(\mathbf{x}) = b(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) \tag{2.3}$$

the associated n-th order functional on V, the following identity holds:

$$b(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \frac{1}{n!\, 2^{(n-1)}} \sum_{\mathbf{x} \in C} \sigma(\mathbf{x}) b_n(\mathbf{x}) \tag{2.4}$$ (2)

where C is the set of all distinct linear combinations of the form:

$$\mathbf{x} = \pm\, \mathbf{x}_1 \pm \mathbf{x}_2 \pm \ldots \pm \mathbf{x}_n \tag{2.5}$$

such that $\mathbf{x} \in C \Rightarrow -\mathbf{x} \notin C$.

The set C is finite and contains $2^{(n-1)}$ vectors.

If we call even or odd an element of C depending on whether there is an even or odd number of minus in (2.5), the functional $\sigma(\mathbf{x})$ in (2.4) is defined by:

$$\sigma(\mathbf{x}) = \begin{cases} +1 & \text{if } \mathbf{x} \text{ is even} \\ -1 & \text{if } \mathbf{x} \text{ is odd}. \end{cases}$$

In the simplest case of a symmetric bilinear functional b and the associated quadratic functional q, formula (2.4) reduces to the usual polar identity:

$$b(\mathbf{x}, \mathbf{y}) = \frac{1}{4} \left[q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x} - \mathbf{y}) \right]. \tag{2.6}$$

A simple but useful consequence of (2.4) is that:

$$\begin{array}{ll} b_n(\mathbf{x}) = 0 & \forall\, \mathbf{x} \in V \\ \Updownarrow & \\ b(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = 0 & \forall\, (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \in V^n. \end{array} \tag{2.7}$$

Let us define the following sets:

$$N(q) = \{\mathbf{x} \in V\colon\ q(\mathbf{x}) = 0\} \qquad \text{null set of } q \tag{2.8}$$

$$N(b) = \{\mathbf{x} \in V\colon\ b(\mathbf{x}, \mathbf{y}) = 0\ \ \forall\, \mathbf{y} \in V\} \qquad \text{null set of } b \tag{2.9}$$

$$N(l) = \{\mathbf{x} \in V\colon\ l(\mathbf{x}) = 0\} \qquad \text{null set of } l \tag{2.10}$$

$$N(\mathbf{T}) = \{\mathbf{x} \in V\colon\ \mathbf{T}\mathbf{x} = 0\} \qquad \text{null set of } \mathbf{T} \tag{2.11}$$

$$R(\mathbf{T}) = \{\mathbf{x} \in V\colon\ \mathbf{x} = \mathbf{T}\mathbf{y},\ \mathbf{y} \in V\} \qquad \text{range of } \mathbf{T}. \tag{2.12}$$

It is easily verified that $N(b)$, $N(\mathbf{T})$, $N(l)$ and $R(\mathbf{T})$ are linear subspaces of V.

For every linear operator $\mathbf{T}$ on V we have:

$$N(\tilde{\mathbf{T}}) = R(\mathbf{T})^{\perp} \tag{2.13}$$

$$R(\tilde{\mathbf{T}}) = N(\mathbf{T})^{\perp} \tag{2.14}$$

(2) Formula (2.4), which is not found in usual textbooks, has been derived by the author.

where $R(\mathbf{T})^{\perp}$ and $N(\mathbf{T})^{\perp}$ denote the orthogonal complements of $R(\mathbf{T})$ and $N(\mathbf{T})$ respectively.

From (2.13) taking orthogonal complements we have:

$$R(\mathbf{T}) = N(\tilde{\mathbf{T}})^{\perp} \tag{2.15}$$

Hence the solution set of the linear equation: $\mathbf{Tx} = \mathbf{a}$ $\mathbf{a} \in V$, will be non empty iff: (3)

$$\mathbf{a} \in R(\mathbf{T}) = N(\tilde{\mathbf{T}})^{\perp}. \tag{2.16}$$

If the operator $\mathbf{T}$ is symmetric condition (2.16) becomes:

$$\mathbf{a} \in N(\mathbf{T})^{\perp} \Leftrightarrow \mathbf{a} \cdot \mathbf{e} = 0 \quad \forall\, \mathbf{e} \in N(\mathbf{T}). \tag{2.17}$$

If b is symmetric and q is the associated quadratic functional from definition (2.8), (2.9) and (2.11) we have:

$$N(\mathbf{T}) = N(b) \subset N(q). \tag{2.18}$$

If moreover q is non negative definite on V, i. e.:

$$q(\mathbf{x}) \geqslant 0 \qquad\qquad \forall\, \mathbf{x} \in V$$

we have:

$$N(b) = N(q) \tag{2.19}$$

Indeed by Schwarz inequality:

$$|b(\mathbf{x}, \mathbf{y})|^2 \leqslant q(\mathbf{x})q(\mathbf{y})$$

it follows that:

$$N(b) \supset N(q). \tag{2.20}$$

Hence from (2.18) we have (2.19).

If the operator $\mathbf{T}$ is symmetric, there exists an orthonormal set $\{\mathbf{e}_i\}$ of eigenvectors of $\mathbf{T}$:

$$\mathbf{Te}_i = \tau_i \mathbf{e}_i \qquad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\,(4) \qquad i, j = 1, 2, \dots, n$$

which is a basis for V.

The eigenvalues τ_i are real and we assume the set of eigenvectors $\{\mathbf{e}_i\}$ to be ordered so that:

$$i < j \Rightarrow \tau_i \leqslant \tau_j \qquad i, j = 1, 2, \dots, n.$$

If $\tau_i = \tau_{i+1} = \dots = \tau_{i+m_i-1}$ the eigenvalue τ_i is said to be of multiplicity m_i. The invariant subspace spanned by the set $\{\mathbf{e}_p\}$, $p = i, i+1, \dots, i+m_i-1$ is called the (m_i - dimensional) eigenspace of τ_i and is denoted by S_i.

The r-th eigenvalue problem:

$$\mathbf{Te}_r = \tau_r \mathbf{e}_r$$

is equivalent to the minimum problem:

$$\tau_r = \min \{\mathbf{Tx} \cdot \mathbf{x};\ \|\mathbf{x}\| = 1,\ \mathbf{x} \cdot \mathbf{e}_i = 0,\ i = 1, 2, \dots, r-1\}$$
$$= \mathbf{Te}_r \cdot \mathbf{e}_r,\ \|\mathbf{e}_r\| = 1,\ \mathbf{e}_r \in S_r.$$

(3) Here and in the sequel "iff" stands for "if and only if".

(4) δ_{ij} is the Kronecker symbol.

3. A basic minimum problem.

Let us consider the functional:

$$p(\mathbf{x}) = \frac{1}{2} q(\mathbf{x}) - l(\mathbf{x}) \tag{3.1}$$

with $q(\mathbf{x})$ non-negative definite, and denote by b, $\mathbf{T}$ and $\mathbf{f}$ the bilinear symmetric functional, the symmetric operator and the vector associated to q and l respectively.

The following discussion will be basic in the sequel:

THEOREM 3.1.

The solution set X of the equivalent problems:

$$\begin{array}{lllll} \text{i)} & \min\{p(\mathbf{x});\ \mathbf{x} \in V\} & & & \\ \text{ii)} & b(\mathbf{x}, \mathbf{z}) = l(\mathbf{z}), & \mathbf{x} \in V, & \forall\, \mathbf{z} \in V & (3.2) \\ \text{iii)} & \mathbf{Tx} = \mathbf{f} & \mathbf{x} \in V & & \end{array}$$

is non-empty iff:

$$N(l) \supset N(q) \tag{3.3}$$

or equivalently:

$$\mathbf{f} \in R(\mathbf{T}) = N(q)^{\perp}$$

If (3.3) is satisfied we have:

$$X = \mathbf{v}_0 + N(q) \tag{3.4}$$

where $\mathbf{v}_0 \in R(\mathbf{T})$ is the unique solution of the equivalent problems:

$$\begin{array}{llll} \text{i)} & \min\{p(\mathbf{v}); & v \in R(\mathbf{T})\} & \\ \text{ii)} & b(\mathbf{v}, \mathbf{z}) = l(\mathbf{z}) & \mathbf{v} \in R(\mathbf{T}),\ \forall\, \mathbf{z} \in V & (3.5) \\ \text{iii)} & \mathbf{Tv} = \mathbf{f} & \mathbf{v} \in R(\mathbf{T}) & \end{array}$$

and moreover:

$$\min\{p(\mathbf{x}); \mathbf{x} \in V\} = \min\{p(\mathbf{v}); \mathbf{v} \in R(\mathbf{T})\} \tag{3.6}$$

Most of the proof follows directly from well-known results of linear algebra.

We shall only hint that setting:

$$\mathbf{x} = \mathbf{e} + \mathbf{v} \qquad \mathbf{e} \in N(q),\ \mathbf{v} \in N(q)^{\perp} = R(\mathbf{T})$$

we have:

$$p(\mathbf{x}) = p(\mathbf{e} + \mathbf{v}) = p(\mathbf{v}) + \frac{1}{2} q(\mathbf{e}) + b(\mathbf{e}, \mathbf{v}) + l(\mathbf{e}) = p(\mathbf{v})$$

since $N(q) = N(b)$ by (2.19), and $N(l) \supset N(q)$ by (3.3).

Moreover, if $\mathbf{v}_0$ is the unique solution of problems (3.5) ii) or iii) setting $\mathbf{v}^* = \mathbf{v} - \mathbf{v}_0$ it follows:

$$p(\mathbf{v}) = p(\mathbf{v}_0 + \mathbf{v}^*) = \frac{1}{2} q(\mathbf{v}^*) + b(\mathbf{v}_0, \mathbf{v}^*) - l(\mathbf{v}^*) +$$
$$+ p(\mathbf{v}_0) \geqslant p(\mathbf{v}_0).$$

Finally we let us note that if (3.3) does not hold:

$$\inf\{p(\mathbf{x});\ \mathbf{x} \in V\} \leqslant \inf\{p(\mathbf{e});\ \mathbf{e} \in N(q)\} = \\ = \inf\{l(\mathbf{e});\ \mathbf{e} \in N(q)\} = -\infty\,.$$

At a minimal point $\mathbf{x}_0 \in X$ we have:

$$q(\mathbf{x}_0) = l(\mathbf{x}_0)$$

and the minimum value in (3.6) is the non-positive number:

$$p(\mathbf{x}_0) = -\frac{1}{2}\,q(\mathbf{x}_0) = -\frac{1}{2}\,l(\mathbf{x}_0)\,. \tag{3.7}$$

Let $\dim N(\mathbf{T}) = m$, then: $\dim R(\mathbf{T}) = n - m$. If $\{\mathbf{a}_k\}$ $k = m+1, \ldots, n$ is an orthonormal basis of $R(\mathbf{T})$ we have the following explicit expressions for the minimal solution of (3.5) and for the minimal value (3.6):

$$\mathbf{v}_0 = \sum_{k=m+1}^{n}\left(\frac{\mathbf{f}\cdot\mathbf{a}_k}{\mathbf{T}\mathbf{a}_k\cdot\mathbf{a}_k}\right)\mathbf{a}_k = \sum_{k=m+1}^{n}\frac{l(\mathbf{a}_k)}{q(\mathbf{a}_k)}\,\mathbf{a}_k \tag{3.8}$$

$$p(\mathbf{v}_0) = -\frac{1}{2}\sum_{k=m+1}^{n}\frac{(\mathbf{f}\cdot\mathbf{a}_k)^2}{\mathbf{T}\mathbf{a}_k\cdot\mathbf{a}_k} = -\frac{1}{2}\sum_{k=m+1}^{n}\frac{l(\mathbf{a}_k)^2}{q(\mathbf{a}_k)} \tag{3.9}$$

If the orthonormal basis of $R(\mathbf{T})$ consists of eigenvectors of $\mathbf{T}$:

$$\mathbf{T}\mathbf{e}_i = \tau_i\mathbf{e}_i \qquad i = m+1, \ldots, n$$

formulas (3.8) and (3.9) may be written as:

$$\mathbf{v}_0 = \sum_{k=m+1}^{n}\frac{\mathbf{f}\cdot\mathbf{e}_k}{\tau_k}\,\mathbf{e}_k = \sum_{k=m+1}^{n}\frac{l(\mathbf{e}_k)}{\tau_k}\,\mathbf{e}_k \tag{3.10}$$

$$p(\mathbf{v}_0) = -\frac{1}{2}\sum_{k=m+1}^{n}\frac{(\mathbf{f}\cdot\mathbf{e}_k)^2}{\tau_k} = -\frac{1}{2}\sum_{k=m+1}^{n}\frac{l(\mathbf{e}_k)^2}{\tau_k} \tag{3.11}$$

4. The critical point.

Let En be an euclidean n-dimensional space and V a linear inner product space of translations for En. For a fixed origin 0 in En the following corrispondence holds:

$$P = 0 + \mathbf{x} \qquad P \in En \qquad \mathbf{x} \in V.$$

It is worth noting that the choice of a scalar product or another is inessential in what follows due to the equivalence of any two normed linear finite dimensional spaces.
Let $\phi(\mathbf{x})$ be an m-times Fréchet differentiable functional on V. Taylor's formula for ϕ at $\mathbf{0}$ holds:

$$\phi(\mathbf{x}) = \sum_{k=0}^{m}\frac{1}{k!}\,D^k\phi(\mathbf{0})\mathbf{x}^k + \omega(\mathbf{x}^m) \tag{4.1}$$

where $D^k\phi(\mathbf{0})\mathbf{x}^k$ is the k-th Fréchet differential of ϕ at $\mathbf{0}$ along $\mathbf{x}$ and $\omega(\mathbf{x}^m)$ is such that:

$$\lim_{\|x\|\to 0}\frac{\omega(\mathbf{x}^m)}{\|\mathbf{x}\|^m} = 0\,.$$

The following simplified notation will be used:

$$\phi_k = \frac{1}{k!}\,D^k\phi(\mathbf{0}) \tag{4.2}$$

and more generally:

$$\phi_{k_1\ldots k_r} = \frac{1}{k_1!\,\ldots\,k_r!}\,D^{k_1+\ldots+k_r}\phi(\mathbf{0})\,. \tag{4.3}$$

Taylor's formula (4.1) may then be written:

$$\phi(\mathbf{x}) = \sum_{k=0}^{m}\phi_k\mathbf{x}^k + \omega(\mathbf{x}^m)\,. \tag{4.4}$$

Setting $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_r$ from (4.3) and (4.4) we have:

$$\phi(\mathbf{x}) = \sum_{k_1+k_2+\ldots+k_r=0}^{m}\phi_{k_1\ldots k_r}\mathbf{x}^{k_1}\ldots\mathbf{x}^{k_r} + \omega(\mathbf{x}^m)\,. \tag{4.5}$$

Let us now suppose the origin 0 of En to be a stationary point for ϕ:

$$\phi_1 = 0\,. \tag{4.6}$$

Taylor's formula (4.4) for $m = 2$ yields:

$$\phi(\mathbf{x}) = \phi_2\mathbf{x}^2 + \omega(\mathbf{x}^2) \tag{4.7}$$

hence a sufficient condition for a strict minimum of ϕ at $\mathbf{0}$ will be:

$$\min\{\phi_2(\mathbf{x});\ \|\mathbf{x}\| = 1\} > 0 \tag{4.8}$$

and a necessary condition is that the minimum in (4.8) be greater than or equal to zero. An indecisive case is got if:

$$\min\{\phi_2\mathbf{x}^2;\ \|\mathbf{x}\| = 1\} = 0 \tag{4.9}$$

i. e. if the quadratic functional ϕ_2 is non-negative definite.
The minimum problem (4.9) is equivalent to the following equations:

$$\phi_{11}\mathbf{x}\mathbf{z} = 0 \qquad \forall\,\mathbf{z} \in V \tag{4.10}$$

$$\mathbf{K}\mathbf{x} = \mathbf{0} \tag{4.11}$$

where: $\mathbf{K}$ is the symmetric operator associated to the quadratic functional $2\phi_2 = D^2\phi(\mathbf{0})$.
In this case the origin will be called a critical point for the functional ϕ_2. The operator $\mathbf{K}$ is singular, the null-space of $\mathbf{K}$ is called the critical subspace and its dimension is the multiplicity of the critical point. If the multiplicity is equal to one the critical point is said to be simple, otherwise multiple.
The analysis of the behaviour of the functional ϕ about a critical point is the subject of the following sections.

5. Taylor expansion about the critical point.

Let the quadratic functional ϕ_2 be non-negative definite. A Taylor expansion of ϕ in the critical subspace gives the following necessary conditions for a strict minimum of ϕ at $\mathbf{0}$:

$$\phi_3\mathbf{e}^3 = 0 \qquad \forall\ \mathbf{e} \in N(\mathbf{K}) \tag{5.1}$$

$$\phi_4\mathbf{e}^4 \geqslant 0 \qquad \forall\ \mathbf{e} \in N(\mathbf{K})\,. \tag{5.2}$$

If conditions (5.1) and (5.2) are satisfied, we fix a given $\mathbf{e} \in N(\mathbf{K})$ and set:

$$\mathbf{x} = \mathbf{e} + \mathbf{y} \tag{5.3}$$

Taylor's formula (4.5) gives:

$$\phi(\mathbf{x}) \doteq \phi_4\mathbf{e}^4 + \phi_{21}\mathbf{e}^2\mathbf{y} + \phi_2\mathbf{y}^2 \tag{5.4}$$

where the symbol $\doteq$ means that, in a sufficiently small neighbourhood of $\mathbf{0}$, both members of (5.4) have the same sign.

Let us now look for the minimum value of the second member of (5.4) for the given $\mathbf{e} \in N(\mathbf{K})$.

By theorem (3.1) we must solve one of the equivalent problems:

$$\min\ \{\phi_2\mathbf{y}^2 + \phi_{21}\mathbf{e}^2\mathbf{y};\ \mathbf{y} \in V\} \tag{5.5}$$

$$\phi_{11}\mathbf{y}\mathbf{z} = -\,\phi_{21}\mathbf{e}^2\mathbf{z} \quad \forall\ \mathbf{z} \in V\,. \tag{5.6}$$

Problems (5.5) and (5.6) are well-posed for every given $\mathbf{e} \in N(\mathbf{K})$ iff:

$$\phi_{21}\mathbf{e}^2\mathbf{e}^* = 0 \qquad \forall \mathbf{e}^* \in N(\mathbf{K}) \tag{5.7}$$

which is satisfied since, by (2.7), it is equivalent to (5.1).

The solution set of (5.5) and (5.6) is the linear variety:

$$X = \mathbf{v}_0 + N(\mathbf{K}) \tag{5.8}$$

where $\mathbf{v}_0$ is the unique solution of the equivalent problems:

$$\min\ \{\phi_2\mathbf{v}^2 + \phi_{21}\mathbf{e}^2\mathbf{v};\ \mathbf{v} \in R(\mathbf{T})\} \tag{5.9}$$

$$\phi_{11}\mathbf{v}\mathbf{z} = -\ \phi_{21}\mathbf{e}^2\mathbf{z} \qquad \forall\ \mathbf{z} \in V \tag{5.10}$$

If $\{\mathbf{a}_k\}$ and $\{\mathbf{e}_k\}$ $k = m+1, \ldots, n$ are the orthonormal bases defined in Sec. 3, with $\mathbf{T} = \mathbf{K}$, the minimal vector $\mathbf{v}_0$ may be expressed as:

$$\mathbf{v}_0 = -\sum_{k=m+1}^{n}\left(\frac{\phi_{21}\mathbf{e}^2\mathbf{a}_k}{\phi_2\mathbf{a}^2{}_k}\right)\mathbf{a}_k = -\sum_{k=m+1}^{n}\left(\frac{\phi_{21}\mathbf{e}^2\mathbf{e}_k}{\tau_k}\right)\mathbf{e}_k\,. \tag{5.11}$$

The minimum in (5.5) or (5.9) is the non positive value:

$$-\,\phi_2\mathbf{x}_0^2 = \frac{1}{2}\,\phi_{21}\mathbf{e}^2\mathbf{x}_0 \qquad \mathbf{x}_0 \in X \tag{5.12}$$

which is an homogeneous functional of the fourth order in $\mathbf{e}$, since by (5.11) we have:

$$\frac{1}{2}\,\phi_{21}\mathbf{e}^2\mathbf{v}_0 = -\frac{1}{2}\sum_{k=m+1}^{n}\frac{(\phi_{21}\mathbf{e}^2\mathbf{a}_k)^2}{\phi_2\mathbf{a}^2{}_k} = -\frac{1}{2}\sum_{k=m+1}^{n}\frac{(\phi_{21}\mathbf{e}^2\mathbf{e}_k)^2}{\tau_k}\,.$$

The minimum value of the second member of (5.4) is then:

$$\phi_4\mathbf{e}^4 - \phi_2\mathbf{x}_0^2 = \phi_4\mathbf{e}^4 + \frac{1}{2}\,\phi_{21}\mathbf{e}^2\mathbf{x}_0 \quad \mathbf{x}_0 \in X\,. \tag{5.13}$$

A sufficient condition for a strict minimum of ϕ at $\mathbf{0}$ will then be:

$$\min\ \{\phi_4\mathbf{e}^4 - \phi_2\mathbf{x}_0^2\ ;\ \|\mathbf{e}\| = 1,\ \mathbf{e} \in N(\mathbf{K})\} > 0 \tag{5.14}$$

while a necessary condition is a non-negative minimum (5.14).

If the functional (5.13) is non-negative definite, the analysis must be continued looking for the directions along which it vanishes.

The situation is more complex than in the previous discussion. This further investigation is however straightforward in the simplest case of a simple critical point in which the minimum problem (5.14) is trivial.

6. Expansion along the critical paths.

An interesting interpretation may given to the results of the previous section by investigating the behaviour of the restrictions of the functional $\phi(\mathbf{x})$ along any path emerging from the origin:

$$\mathbf{x} = \mathbf{x}(t)\,, \qquad \mathbf{x}(0) = 0\,. \tag{6.1}$$

Let us denote by $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$, ..., $\mathbf{x}^{(n)}$ respectively the first, second, ..., n-th derivatives of $\mathbf{x}(t)$ with respect to t at $t = 0$. Setting: $\mathbf{x}_n = (1/n!)\mathbf{x}^{(n)}$ we have the following expansion of (6.1):

$$\mathbf{x}(t) = \mathbf{x}_1 t + \mathbf{x}_2 t^2 + \ldots + \mathbf{x}_n t^n + \ldots \tag{6.2}$$

Analogously setting:

$$\varphi(t) = \phi(\mathbf{x}(t)) \tag{6.3}$$

we have the expansion:

$$\varphi(t) = \varphi_1 t + \varphi_2 t^2 + \ldots + \varphi_n t^n + \ldots \tag{6.4}$$

The following explicit expressions will be needed:

$$\varphi_1 = \phi_1\mathbf{x}_1 \tag{6.5}$$

$$\varphi_2 = \phi_2\mathbf{x}_1^2 + \phi_1\mathbf{x}_2 \tag{6.6}$$

$$\varphi_3 = \phi_3\mathbf{x}_1^3 + \phi_{11}\mathbf{x}_1\mathbf{x}_2 + \phi_1\mathbf{x}_3 \tag{6.7}$$

$$\varphi_4 = \phi_4\mathbf{x}_1^4 + \phi_{21}\mathbf{x}_1\mathbf{x}_2 + \phi_{11}\mathbf{x}_1\mathbf{x}_3 + \phi_2\mathbf{x}_2^2 + \phi_1\mathbf{x}_4\,. \tag{6.8}$$

If the origin is a critical point we have:

$$\varphi_1 = \phi_1\mathbf{x}_1 = 0 \qquad \forall\ \mathbf{x}_1 \in V \tag{6.9}$$

$$\min\ \{\varphi_2 = \phi_2\mathbf{x}_1^2\ ;\ \|\mathbf{x}_1\| = 1\} = 0\,. \tag{6.10}$$

The minimum problem (6.10) is equivalent to the equation:

$$\phi_{11}\mathbf{x}_1\mathbf{z} = 0 \qquad \forall\, \mathbf{z} \in V \tag{6.11}$$

and the solution set is $N(\mathbf{K})$.

The paths (6.1) satisfying (6.11) will be referred to as critical paths.

From (6.7), taking into account (6.9) and (6.10), we have that a necessary condition for a strict minimum at $\mathbf{0}$ of the restrictions of ϕ along any critical path is:

$$\varphi_3 = \phi_3\mathbf{e}^3 = 0 \qquad \forall\, \mathbf{e} \in N(\mathbf{K})\,. \tag{6.12}$$

If (6.12) is satisfied, a sufficient condition will be given by a positive minimum of:

$$\varphi_4 = \phi_4\mathbf{e}^4 + \phi_2\mathbf{x}_2^2 + \phi_{21}\mathbf{e}^2\mathbf{x}_2 \tag{6.13}$$

as a functional of $\mathbf{x}_2$, for every fixed $\mathbf{e} \in N(\mathbf{K})$.

A non negative minimum of (6.13) will be a necessary condition.

We are thus led to the study of the minimum problem:

$$\min\,\{\phi_2\mathbf{x}_2^2 + \phi_{21}\mathbf{e}^2\mathbf{x}_2\,;\ \mathbf{x}_2 \in V\} \tag{6.14}$$

which is identical to (5.5).

We remark that (6.14) is equivalent to:

$$\phi_{11}\mathbf{x}_2\mathbf{z} = -\,\phi_{21}\mathbf{e}^2\mathbf{z} \qquad \forall\, \mathbf{z} \in V \tag{6.15}$$

so that the critical path along which the minimum (6.14) is attained may be written, to the second order:

$$\mathbf{x}(t) = \mathbf{e}t + \mathbf{x}_0 t^2 \quad \text{with} \quad \mathbf{x}_0 \in X = \mathbf{v}_0 + N(\mathbf{K}) \tag{6.16}$$

and will be referred to as the minimal path.

We remark that the indeterminacy of $\mathbf{e}$ and $\mathbf{x}_0$ is due to the arbitrariness in the choice of the parameter t.

It is worth noting that (6.16) is the second order expansion of the equilibrium paths defined by:

$$\phi_1(\mathbf{x}(t)) = 0 \tag{6.17}$$

Since (6.11) and (6.15) are the first and second order perturbations of (6.17).

We remark that, if the sufficient condition for a strict minimum (5.14) is satisfied, the third order expansion of the equilibrium path does not exist, for every $\mathbf{e} \in N(\mathbf{K})$.

Indeed, we have:

$$\phi_{11}\mathbf{x}_3\mathbf{z} = -\,(\phi_{31}\mathbf{x}_1^3\mathbf{z} + \phi_{111}\mathbf{x}_1\mathbf{x}_2\mathbf{z}) \qquad \forall\, \mathbf{z} \in V \tag{6.18}$$

which admits solutions iff:

$$\phi_{31}\mathbf{e}^3\mathbf{e}^* + \phi_{111}\mathbf{e}\mathbf{e}^*\mathbf{x}_0 = 0 \qquad \forall\, \mathbf{e}^* \in N(\mathbf{K})$$

while by (5.14):

$$\phi_4\mathbf{e}^4 + \frac{1}{2}\,\phi_{21}\mathbf{e}^2\mathbf{x}_0 = \phi_4\mathbf{e}^4 - \phi_2\mathbf{x}_0^2 > 0 \qquad \forall\, \mathbf{e} \in N(\mathbf{K})\,.$$

This is not surprising: if the functional ϕ has a strict minimum at $\mathbf{0}$, the existence of an equilibrium path (a path along which ϕ is constant (by (6.17)) is ruled out.

7. Expansion of the critical eigenproblem.

Let us consider the revelant special case in which the origin is a simple critical point for the functional $\phi(\mathbf{x})$, i. e.:

$$\tau_1 = 0\,; \qquad \tau_2 > 0\,.$$

If we consider the eigenvalue problems:

$$\mathbf{K}(\mathbf{x})\mathbf{e}_i(\mathbf{x}) = \tau_i(\mathbf{x})\mathbf{e}_i(\mathbf{x})$$

since the functions $\tau_i(\mathbf{x})$ are continuous, it will exist a neighbourhood of the origin where

$$\tau_1(\mathbf{x}) < \tau_2(\mathbf{x})$$

and hence the eigenspace associated with the critical eigenvalue τ_1 is still one-dimensional.

Setting $\tau(\mathbf{x}) = \tau_1(\mathbf{x})$ and $\mathbf{e}(\mathbf{x}) = \mathbf{e}_1(\mathbf{x})$ we have:

$$\mathbf{K}(\mathbf{x})\mathbf{e}(\mathbf{x}) = \tau(\mathbf{x})\mathbf{e}(\mathbf{x}) \tag{7.1}$$

and

$$\mathbf{K}\mathbf{e} = \tau\mathbf{e} = \mathbf{0} \qquad (\tau = 0) \tag{7.2}$$

where functions with no argument are intended to be evaluated at the origin.

Let us now consider, in a neighbourhood of the origin, the path defined by:

$$\mathbf{x}(t) = \int_0^t \mathbf{e}(\mathbf{x}(\theta))\, d\theta \tag{7.3}$$

or equivalently

$$\dot{\mathbf{x}}(t) = \mathbf{e}(\mathbf{x}(t))\,, \qquad \mathbf{x}(0) = 0\,. \tag{7.4}$$

If we expand the eigenproblem (7.1) along the path (7.3) we have:

$$\mathbf{K}\mathbf{e} = \tau\mathbf{e} = \mathbf{0} \qquad (\tau = 0) \tag{7.5}$$

$$\mathbf{K}\dot{\mathbf{e}} = \dot{\tau}\mathbf{e} + \tau\dot{\mathbf{e}} - D\mathbf{K}\mathbf{e}^2 \tag{7.6}$$

$$\mathbf{K}\ddot{\mathbf{e}} = \ddot{\tau}\mathbf{e} + 2\dot{\tau}\dot{\mathbf{e}} + \tau\ddot{\mathbf{e}} - D^2\mathbf{K}\mathbf{e}^3 - 3D\mathbf{K}\mathbf{e}^2\,. \tag{7.7}$$

The first order perturbation (7.6), since $\tau = 0$, yields the compatibility condition:

$$\dot{\tau} = D\mathbf{K}\mathbf{e}^3\,. \tag{7.8}$$

A comparison with (5.1) shows that a necessary condition for a strict minimum of ϕ at the origin is

$$\dot{\tau} = 0\,. \tag{7.9}$$

If (7.9) is verified Eq. (7.6) becomes:

$$\mathbf{K}\dot{\mathbf{e}} = -D\mathbf{K}\mathbf{e}^3 \tag{7.10}$$

whose solution set is the one dimensional linear variety:

$$\dot{\mathbf{e}}_0 + N(\mathbf{K}) \tag{7.11}$$

where $\mathbf{e}_0 \in R(\mathbf{K})$ is given by:

$$\dot{\mathbf{e}}_0 = -\sum_{k=2}^{n} \left(\frac{D\mathbf{K}\mathbf{e}^2\mathbf{e}_k}{\tau_k}\right) \mathbf{e}_k \tag{7.12}$$

The second order perturbation (7.7), taking into account that $\tau = \dot{\tau} = 0$, yields the compatibility condition:

$$\ddot{\tau} = D^2\mathbf{K}\mathbf{e}^4 + 3D\mathbf{K}\mathbf{e}^2\dot{\mathbf{e}}_0 \tag{7.13}$$

or equivalently by (7.10):

$$\ddot{\tau} = D^2\mathbf{K}\mathbf{e}^4 - 3\mathbf{K}\dot{\mathbf{e}}_0^2 \,. \tag{7.14}$$

Now a simple algebra shows that:

$$\dot{\mathbf{e}}_0 = 2\mathbf{v}_0$$

and

$$D^2\mathbf{K}\mathbf{e}^4 - 3\mathbf{K}\dot{\mathbf{e}}_0^2 = 24(\phi_4\mathbf{e}^4 - \phi_2\mathbf{v}_0^2)$$

and hence a comparison with (5.13) shows that a sufficient condition for a strict minimum is given by:

$$\ddot{\tau} > 0 \tag{7.15}$$

and a necessary condition by:

$$\ddot{\tau} \leqslant 0 \,. \tag{7.16}$$

Under the assumption $\dot{\tau} = 0$ the second order expression of the path (7.3) about the origin will then be:

$$\mathbf{x}(t) = \mathbf{e}t + \frac{1}{2}\mathbf{e}_0 t^2 + \ldots \tag{7.17}$$

The analysis developed shows how, to look for a strict minimum of ϕ at the origin, is equivalent to investigate about a strict minimum of $\tau(\mathbf{x})$ along the "critical path" (7.3).

This result has a simple and interesting structural interpretation which will be exploited in the next section.

8. Structural stability.

If we assume $\phi(\mathbf{x})$ to be the potential energy of a discrete conservative structural system S, the energy criterion for stability states that:

"An equilibrium configuration of S is stable if the potential energy functional has there a strict minimum".

The analysis developed in the previous paragraphs gives then necessary and sufficient conditions for the stability in terms of the subsequent differentials of the potential energy.

In particular the procedure of Sec. 7 yields, in the case of a simple critical point, an interesting interpretation of the expressions which are decisive for stability. These are shown to be the subsequent derivatives of the lowest eigenvalue of the second differential $D^2\phi(\mathbf{x})$ of the potential energy along the critical path defined by (7.3).

Noting that the operator $K(\mathbf{x}) = D^2\phi(\mathbf{x})$ has the meaning of the incremental stiffness of the structural system, we may significantly call its eigenvalue and eigenvectors respectively the principal stiffnesses and the principal directions of the system. The analysis of the stability on an equilibrium configuration of a structural system may be carried out by looking at the behaviour of the lowest principal stiffness along the critical path. A comparison between (7.17) and (6.16) shows that this path can be characterized as the path along which the system "tends to have an adjacent equilibrium configuration".

The vector tangent at any point of the critical path is the principal direction of minimal stiffness, as shown by (7.3) and (7.4).

At a simple critical point the system may be displaced infinitesimally along the critical direction without any effort. Depending on whether the critical stiffness has a minimum at the critical configuration, and hence tends to became positive when the system is displaced, or doesn't have a minimum and hence there is a direction in which the stiffness becomes negative, the critical configuration itself will be respectively stable or unstable. This result gives a quite clear picture of the connection, first emphasized by Koiter [6], between the stability of the critical configuration and the kind of postcritical behaviour of the system [10].

The general analysis developed until now is applied to a simple example in the next section.

9. A simple example.

Let us investigate the stability of the undeflected position of a two degrees of freedom structural system characterized by the potential energy functional:

$$\phi(\mathbf{u}) = \frac{1}{4}(u_1^4 + u_2^4) + k u_1^2 u_2 + \frac{1}{2} h u_2^2 \tag{9.1}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ is the displacement vector}$$

the first and the second gradients of (9.1) are:

$$D\phi(\mathbf{u}) = \begin{bmatrix} u_1^3 + 2k u_1 u_2 \\ u_2^3 + h u_2 + k u_1^2 \end{bmatrix}$$

$$D^2\phi(\mathbf{u}) = \begin{bmatrix} 3u_1^2 + 2k u_2 & 2k u_1 \\ 2k u_1 & 3u_2^2 + h \end{bmatrix}. \tag{9.2}$$

Hence, following the analysis of Sec. 5, we have:

$$D^2\phi(\mathbf{0}) = \begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \qquad \mathbf{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$D^2\phi(\mathbf{u})\mathbf{e}^2 = 3u_1^2 + 2ku_2$$

$$D^3\phi(\mathbf{u})\mathbf{e}^2 = \begin{bmatrix} 6u_1 \\ 2k \end{bmatrix} \qquad D^3\phi(\mathbf{0})\mathbf{e}^2 = \begin{bmatrix} 0 \\ 2k \end{bmatrix}$$

$$D^3\phi(\mathbf{u})\mathbf{e}^3 = 6u_1 \qquad D^3\phi(\mathbf{0})\mathbf{e}^3 = 0$$

$$D^4\phi(\mathbf{u})\mathbf{e}^4 = 6 > 0\,.$$

Hence the necessary conditions for minimum at $u = 0$ are then satisfied. The solution of the problem:

$$D^2\phi(\mathbf{0})\dot{\mathbf{e}} = -\,D^3\phi(\mathbf{0})\mathbf{e}^2$$

i. e.:

$$\begin{bmatrix} 0 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ 2k \end{bmatrix}$$

gives:

$$\dot{\mathbf{e}} = \begin{bmatrix} 0 \\ -\,2k/h \end{bmatrix}$$

where the condition $\dot{\mathbf{e}} \cdot \mathbf{e} = 0$ has been imposed.

The sufficient condition for a strict minimum will then be:

$$D^4\phi(\mathbf{0})\mathbf{e}^4 - 3D^2\phi(\mathbf{0})\dot{\mathbf{e}}^2 = 6 - 12\,\frac{k^2}{h} > 0$$

The situation is depicted in Fig. 1:

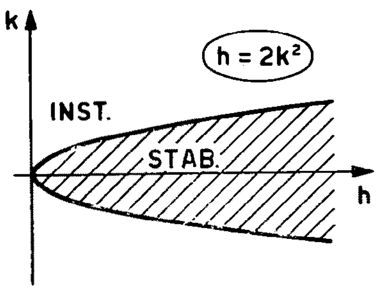

Fig. 1.

where the dashed region denotes the points of the plane $(h,\ k)$ where stability occurs.

It can be interesting to refind the previous results by a direct investigation of the critical eigenvalue problem, in the spirit of the analysis developed in Sec. 7.

Setting

$$\mathbf{K}(\mathbf{u}) = D^2\phi(\mathbf{u})$$

the characteristic equation of $\mathbf{K}(\mathbf{u})$ is:

$$\tau^2(\mathbf{u}) - \tau(\mathbf{u})\ \mathrm{tr}\ \mathbf{K}(\mathbf{u}) + \det \mathbf{K}(\mathbf{u}) = 0 \tag{9.3}$$

where:

$$\mathrm{tr}\ \mathbf{K}(\mathbf{u}) = 3(u_1^2 + u_2^2) + 2ku_2 + h$$

$$\det \mathbf{K}(\mathbf{u}) = (3u_1^2 + 2ku_2)(3u_2^2 + h) - \tau k^2 u_1^2\,.$$

Solving the homogeneous equation associated to (9.2), we find:

$$\mathbf{e}(\mathbf{u}) = \begin{bmatrix} 1 + \dfrac{3}{h}\,u_2^2 \\ -\,\dfrac{2k}{h}\,u_1 \end{bmatrix} \quad \text{with} \qquad \mathbf{e}(\mathbf{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now the first order perturbation of (9.3) along the critical path (7.3), setting $\tau = \tau(\mathbf{0}) = 0$, yields:

$$\dot{\tau}(\mathrm{tr}\ \mathbf{K}) = (\det \mathbf{K})^{\cdot}$$

with

$$(\det \mathbf{K})^{\cdot} = D(\det \mathbf{K}) \cdot \mathbf{e} = 0\ ;\ \mathrm{tr}\ \mathbf{K} = h$$

hence

$$\dot{\tau} = 0$$

the second order perturbation gives:

$$\ddot{\tau}(\mathrm{tr}\ \mathbf{K}) = (\det \mathbf{K})^{\cdot\cdot}$$

where

$$\begin{aligned} (\det \mathbf{K})^{\cdot\cdot} &= D^2(\det \mathbf{K})\mathbf{e} \cdot \mathbf{e} + D(\det \mathbf{K}) \cdot \dot{\mathbf{e}} = \\ &= 6h - 8k^2 + 2k\left(-\frac{2k}{h}\right) = 6h - 12k^2 \end{aligned}$$

and hence finally

$$\ddot{\tau} = 6 - \frac{12k^2}{h}$$

in accordance with the previous analysis.

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