ON THE ENERGY CRITERION FOR THE STABILITY OF CONTINUOUS ELASTIC STRUCTURES

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SOMMARIO: Si presentano i risultati di una ricerca volta a fornire una sicura base al criterio energetico come metodo di indagine sulla stabilità e l'instabilità dei modelli ingegneristici non lineari di strutture elastiche continue quali travi, piastre e volte soggette a carichi conservativi.

Si mostra come, nel caso di modelli strutturali con comportamento lineare nella fase precritica, sia possibile fornire una formulazione rigorosa del criterio energetico, basata su uno sviluppo in formula di Taylor del funzionale energia potenziale.

SUMMARY: We present some results of a research intended to provide a satisfactory foundation to the energy criterion as a tool for the investigation of the stability and the instability of nonlinear engineering models of continuous elastic structures such as beams, plates, and shells under conservative loading.

It is shown how, in the case of structural models with a linear prebuckling behavior, it is possible to give a rigorous formulation of the energy criterion, based on an expansion of the potential energy functional in Taylor's formula.

1. Introduction.

The stability of an equilibrium configuration of an elastic structure under conservative loading is conveniently analyzed by the energy method.

Indeed the energy criterion reduces the stability problem to a statical one based on the investigation of the behavior of the potential energy in a neighborhood of an equilibrium configuration.

An extensive and successful application of this criterion has been made in solving many engineering problems in the field of elastic stability.

Recently, however, a number of criticisms have been formulated about the validity of the energy criterion of stability when applied to continuous elastic structures.

A comprehensive discussion of the problem may be found in [1].

It turns out that the main motivation of these criticisms consists in some paradoxical results to which the criterion is shown to lead [2].

As a consequence many researchers in this fieldhave felt the need for a more rigorous approach to the problem, starting from a critical review of the very definition of the stability of an elastic structure.

Although the problem have not been solved in full generality, many valuable contributions have been made to a deeper understanding of the subject [3-5].

In this paper we present some results of a research in this direction, which specially concern the possibility of giving a satisfactory foundation to the energy criterion for the investigation of the stability and the instability of the usual nonlinear engineering models of continuous elastic structures such as beams, plates, and shells under conservative loading.

2. Generalities.

We shall take the dynamical definition of stability (according to Lyapunov's approach) as basic.

It can be roughly stated as follows:

"An equilibrium configuration of a given dynamical system is stable if, in the motion subsequent to an initial disturbance, the state of the system remains arbitrarily near to the equilibrium configuration when the initial perturbed state is sufficiently near to it."

From this definition it is apparent how the concept of stability is a topological one, since it is based on the notion of "nearness" between two states of the system.

It must be pointed out here that the set of admissible states of the system, which consists of the solution set of the nonlinear dynamical problem under a given set of initial data, is not known in the general case.

We must then make an a priori assumption regarding the state space in which the stability problem is set.

In connection with the development of an energy approach to the analysis of the stability of conservative dynamical systems, the choice of the state space and of its topology must meet the following obvious physical requirements:

i) the total energy is finite at every state of the system;

ii) the unloaded natural configuration is stable.

3. The energy criterion.

A sufficient energy criterion of stability may be obtained by a suitable generalization of the Lagrange-Dirichlet

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theorem of analytical mechanics, stating that:[2]

"If the potential energy functional is continuous in a neighborhood of an equilibrium configuration and has a positive lower bound on arbitrarily small closed surfaces around it, then the equilibrium configuration is stable in the dynamical sense."

It is worth noting that the simple positivity of the potential energy on arbitrarily small closed surfaces around the equilibrium configuration suffices for the proof of the theorem in the discrete case since the potential energy functional, being continuous, admits a positive minimum on every closed and bounded (and hence compact) set.

This conclusion does no more hold in the continuous case where closed and bounded sets of the infinite dimensional state space need not to be compact.

A sufficient energy criterion of instability cannot be proved in full generality in the context of a purely conservative theory.

However, a general result can be obtained under the additional assumption of a strictly positive dissipation.

In this case indeed the total energy functional is strictly decreasing along any trajectory which does not degenerate in an equilibrium point.

On the other hand, by the assumed continuity, the total energy functional will be constant on every limit orbit. It follows that the limit orbits must consists of equilibrium configurations.

This simple result and the existence of limit orbits of every trajectory which remains in a compact subset of the state space are the key points in the proof of the energy criterion of instability which may be roughly stated as follows:

"An isolated equilibrium configuration of a strictly dissipative dynamical system is unstable if the potential energy functional is continuous and assumes negative values in every sufficiently small neighborhood of it and each trajectory of the system lies in a compact subset of the state space."

A more rigorous statement and an explicit proof are given in [6].

We remark that in the discrete case the last assumption can be trivially dropped, since trajectories which do not lie in a compact subset of the state space are in fact unbounded.

A simple physical interpretation of the abstract compactness assumption above and a discussion of it plausibility will be given in the last section, in connection with the stability analysis of the nonlinear engineering models of continuous elastic structures.

The results of the stability and the instability theorems which have been stated in this section may be made operative if an effective method of investigating the behavior of the potential energy functional in a neighborhood of the equilibrium configuration is available.

If the potential energy functional is Fréchet differentiable a sufficient number of times in a neighborhood of the equilibrium configuration, a suitable tool is provided by an expansion in Taylor's formula.

4. Structural systems.

Let us consider a one or two-dimensional model of an elastic structure under conservative loading.

To be definite we shall refer to a two-dimensional plane elastic structure.

Let A be the regular domain which the structure occupies in its natural undeformed configuration.

The dynamical behavior of the structure will be described by the function pair $\{u, \dot{u}\}$ where u is the displacement vector field from A and $\dot{u}$ is the corresponding velocity field.

The total energy:

$$E(u, \dot{u}) = \phi(u) + K(\dot{u})$$

which is the sum of the potential energy ϕ and of the kinetic energy K is assumed to be strictly decreasing along any trajectory:

$$\begin{cases} u = u(x, t) \\ \dot{u} = \dot{u}(x, t) \end{cases} \quad x \in A \quad t \in [t_0, +\infty)$$

and hence is the most obvious candidate to the role of Lyapunov's functional of the system.

We shall consider structural models whose potential energy functional meets the following requirements:

the functional $\phi(u)$ has the expression:

$$\phi(u) = \int_A \varphi(u(x))\, dx$$

with φ function of the displacement vector u and of its derivatives up to and including that of order m, such that derivatives of u at most of order $m - 1$ do appear in the expression of φ with an exponent greater than two and with a quadratic part expressible as:

$$a(u, u) - \lambda\, b(u, u)$$

where:

$$a(u, v) = \sum_{0 \le |p|,\, |q| \le m} \int_A a_{pq}(x)\, D^p u(x) \cdot D^q v(x)\, dx$$

$$b(u, v) = \sum_{0 \le |r|,\, |s| \le m-1} \int_A b_{rs}(x)\, D^r u(x) \cdot D^s v(x)\, dx$$

with $a_{pq} = a_{qp}$ and $b_{rs} = b_{sr}$ so that the bilinear forms a and b are symmetric. The multi-index notation has been used: e. g., $p = (p_1, p_2)$ and $|p| = p_1 + p_2$ with p_1 and p_2 integers.

Here λ is the load parameter.

[2] We assume that the kinetic energy itself is chosen as a measure of "nearness in velocity" to the equilibrium state. Without loss of generality we also assume that the potential energy is zero at the equilibrium configuration under investigation.

The previous requirements are met by the potential energy functional of the usual nonlinear engineering models of beams, plates, and shells with a linear prebuckling behavior.

We shall show that a satisfactory foundation can be given to the energy method when applied to the analysis of the stability and the instability of these structural systems.

To this end let us recall some mathematical definitions and results needed in the sequel.

Consider the Sobolev space $H^m(A)$ of the displacement functions on A with square summable generalized derivatives up to and including that of order m, whith the norm:

$$\|u\|_m = \left(\sum_{0 \leq |s| \leq m} \int_A |D^s u(x)|^2 \, dx \right)^{\frac{1}{2}}$$

If rigid body displacements are ruled out, the following inequality holds:

$$a(u, u) \geqslant \gamma^2 \|u\|_m^2 \qquad \forall \, v \in V \tag{4.1}$$

where V is the closed subspace of $H^m(A)$ of the displacement functions satisfying the geometric boundary conditions.

We need the following basic result of functional analysis, known as Rellich's "selection principle":

"If the domain A is properly regular,[3] the embedding of the space $H^m(A)$ into the space $H^{m-1}(A)$ is compact."

The coerciveness condition (4.1) on $a(u, u)$, the continuity of $b(u, v)$ on $H^{m-1}(A)$ and the selection principle above, allow to prove the following existence theorem:

Consider the variational problem:

$$P) \qquad a(u, v) - \lambda \, b(u, v) = l(v) \quad \forall \, v \in V$$

with $l(v)$ bounded linear functional on V.

Problem P admits an unique solution for every l if λ doesn't belong to the spectrum of the associated eigenproblem:

$$P_\lambda) \qquad a(u, v) - \lambda \, b(u, v) = 0 \quad \forall \, v \in V$$

The spectrum of P consists of an increasing sequence of positive eigenvalues $\{\lambda_n^+\}$ and a decreasing sequence of negative eigenvalues $\{\lambda_n^-\}$,[4] each of finite multiplicity.

When λ belongs to the spectrum of P_λ, problem P admits solutions if and only if l vanish on the corresponding eigenspace.

[3] The exact definition of properly regular domain is given in [7].

[4] If the quadratic form $b(u, u)$ is positive (negative) definite the sequence $\{\lambda_n^-\}$ ($\{\lambda_n^+\}$) is empty.

The solution set is then a finite dimensional linear variety parallel to this eigenspace.

Assuming that the potential energy functional ϕ is Fréchet differentiable in a neighborhood of the equilibrium configuration (which we take to be the origin of the state space) as many times as needed, we may analyze its behavior near the origin by an expansion in Taylor's formula.

The first differential at the origin will be zero by the equilibrium condition.

Let us consider the two basic situations to be detected:

(s) the potential energy functional has a positive lower bound on any sufficiently small sphere around the origin.

(i) the potential energy functional assumes negative values in any neighborhood of the origin.

Now, if the second differential satisfies a coerciveness condition:

$$d^2\phi(0) \, u^2 \geqslant k^2 \|u\|_m^2 \qquad \forall \, u \in V \tag{4.2}$$

then (s) holds true (see, e.g., [8]).

Note that since:

$$d^2\phi(0) \, u^2 = a(u, u) - \lambda \, b(u, u) \geqslant \left(1 - \frac{\lambda}{\lambda_1^+}\right) a(u, u) \geqslant$$

$$\geqslant \gamma^2 \left(1 - \frac{\lambda}{\lambda_1^+}\right) \|u\|_m^2 \qquad \forall \, u \in V \tag{4.3}$$

the coerciveness condition (4.2) is equivalent to require that:[5]

$$0 \leqslant \lambda < \lambda_1^+$$

If $\lambda = \lambda_1^+$ (the critical load) and the associated eigenspace is one-dimensional, i.e., there exists only one "critical direction" u_1 such that:

$$d^2\phi(0) \, u_1 v = a(u_1, v) - \lambda_1^+ \, b(u_1, v) = 0 \qquad \forall \, v \in V$$

we have to look at higher order differentials.

Indeed, if:

$$d^3\phi(0) \, u_1^3 \neq 0$$

or

$$d^3\phi(0) \, u_1^3 = 0$$

and

$$d^4\phi(0) \, u_1^4 < 0$$

then (i) holds true.

[5] Henceforth we shall consider for simplicity the case when the quadratic form ($b(u, u)$) is positive definite.

In the alternative case we must look at the expression:

$$d^4\phi(0)\, u_1^4 - 3d^2\phi(0)\, u_2^2 \qquad (4.4)$$

where u_2 is any solution of the variational problem:

$$d^2\phi(0)\, u_2 v = a(u_2, v) - \lambda_1^+ b(u_2, v) = - d^3\phi(0)\, u_1^2 v \quad \forall\, v \in V$$

which, by the existence theorem stated above, admits a one-dimensional linear variety of solutions (indeed, the compatibility condition $d^3\phi(0)\, u_1^3 = 0$ is satisfied).

Now, depending on whether (4.4) is positive or negative, respectively (s) or (i) holds true.

The previous analysis yields then effective sufficient criteria of stability (s) and instability (i) of a continuous elastic structural model, under the assumption of the Fréchet differentiability of ϕ and the condition that the trajectories lie in a compact subset of the state space.

A proper choice of the state space for the structural models under investigation is then the product Hilbert space:

$$H^m(A) \times L^2(A)$$

with the norm:

$$\| \{u, \dot{u}\} \| = \left(\|u\|_m^2 + \|\dot{u}\|_0^2 \right)^{\frac{1}{2}}$$

where

$$\|\dot{u}\|_0 = \left(\int_A |\dot{u}|^2\, dx \right)^{\frac{1}{2}}$$

is the usual norm in $L^2(A)$ (equivalent to the kinetic energy).

With this choice we have that, if ϕ is Fréchet differentiable, the natural unloaded configuration is stable (in fact, by (4.3), we have stability for $0 \leqslant \lambda < \lambda_1^+$).

The explicit proof of the Fréchet differentiability of ϕ and an interpretation of the compactness assumption on the trajectories, will be given in the next section with reference to a simplest structural model.

5. A simplest example (Euler problem).

Let us consider an inextensible simply supported elastic beam of constant cross section and length l, under a compressive loading N.

The potential energy is given by:

$$\frac{1}{2} EI \int_0^l \alpha'(s)\, ds - N \int_0^l (1 - \cos \alpha(s))\, ds \qquad (5.1)$$

where E is the Young modulus, I is the lowest moment of inertia, $\alpha(s)$ is the rotation of the cross section as a function of the curvilinear abscissa s and $\alpha'(s) = d\alpha/ds$ is the curvature of the beam axis.

Denoting by $w(s)$ the deflection, we get from (5.1) the approximate expression:

$$\Phi(w) = \frac{EI}{2} \int_0^l w''^2 (1 + w'^2 - w'^4)\, ds -$$
$$- N \int_0^l \left(\frac{w'^2}{2} - \frac{w'^4}{8} + \frac{w'^6}{16} \right) ds$$

which setting:

$$u = \frac{\pi}{l} w \qquad x = \frac{\pi}{l} s$$

$$\lambda = \frac{N l^2}{\pi^2 EI} \qquad \varphi = \Phi \frac{l}{\pi EI}$$

may be expressed in adimensional form as:

$$\varphi(u) = \frac{1}{2} \int_0^\Pi u''^2 (1 + u'^2 - u'^4)\, dx -$$
$$-\lambda \int_0^\Pi \left(\frac{u'^2}{2} - \frac{u'^4}{8} + \frac{u'^6}{16} \right) dx \qquad (5.2)$$

where a prime denotes the differentiation with respect to x.

The kinetic energy may be assumed to be given by:

$$K = \frac{1}{2} \int_0^l \varrho \dot{w}^2\, ds$$

where ϱ is the linear density, and setting:

$$u = \frac{l^4}{\pi^2 EIT^2}$$

$$\tau = \frac{t}{T}$$

$$k = K \frac{\pi^3 T^2}{l^3 \varrho_0}$$

where T is a time factor and ϱ_0 is a reference linear density, we get the adimensional form:

$$k = \frac{1}{2} \int_0^\pi \mu\, \dot{u}^2\, dx \qquad (5.3)$$

Consider now the nonlinear structural model defined by the total energy:

$$E(u, \dot{u}) = \varphi(u) + k(\dot{u})$$

with φ and k given by (5.2) and (5.3) respectively.

To prove that φ is twice Fréchet differentiable we must show that, setting:

$$\varphi(u) - \frac{1}{2} d^2\phi(0)\, u^2 = \frac{1}{2} \int_0^\pi u''^2(u'^2 - u'^4)\, dx -$$

$$- \lambda \int_0^\pi \left(- \frac{u'^4}{8} + \frac{u'^6}{16}\right) dx = \omega_2(u)$$

we have that:

$$\frac{|\omega^2(u)|}{\|u\|_2^2} \to 0 \tag{5.4}$$

as

$$\|u\|_2 = \left(\int_0^\pi (u^2 + u'^2 + u''^2)\, dx\right)^{\frac{1}{2}} \to 0$$

Now by Sobolev's lemma (see, e.g., [7]) we have:

$$\max_{x \in [0,\pi]} |u'(x)| \leqslant c\, \|u\|_2$$

Hence, from (5.4) we get:

$$|\omega_2(u)| \leqslant c'\left(\|u\|_2^4 + \|u\|_2^6 + \lambda(\|u\|_2^4 + \|u\|_2^6)\right)$$

whence (5.4) follows.

By the same procedure the indefinite Fréchet differentiability of φ can be immediately proved.

Moreover, we have that:

$$d^2\varphi(0)\, u^2 = \int_0^\pi u''^2\, dx - \lambda \int_0^\pi u'^2\, dx$$

and, by the Poincaré inequality:

$$\int_0^\pi u''^2\, dx \geqslant c\, \|u\|_2^2$$

which yields the coercitivity condition corresponding to (4.1).

The results of the previous section may then be applied to the present simplest structural case.

Finally, a simple physical interpretation may be given in this context to the condition that the trajectories of the system remain in a compact subset of the state space.

Indeed, from the "selection principle," we infer that otherwise the trajectories should be unbounded in $H^3(0,\pi)$.

Since the third derivatives of the deflection are related to the shear in the beam, this would imply an unbounded shear energy in the beam.

This situation must be rejected as contradictory from the physical point of view, because a purely flexural energy has been adopted in the beam model assuming the other contributions to be negligible and hence, *a fortiori*, bounded.

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