The general law of dynamics in nonlinear manifolds and Noether's theorem

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Summary. The theory of continuous dynamical systems, undergoing motions in a nonlinear configuration manifold, is formulated and developed with a coordinate-free variational approach. The starting point is a new formulation of Hamilton's action principle in the velocity-time phase-space in which piecewise regularity of the trajectory is assumed and testing variations which are infinitesimal isometries of the trajectory are allowed for. It is shown that localization yields a new general differential law of dynamics and the related jump conditions at singular points. An extended version of Noether's theorem follows as a simple corollary. The Lagrange's law is recovered by assuming a torsion-free connection on the configuration manifold.

1 Introduction

Calculus on manifolds is the suitable mathematical tool in the dynamics of continuous systems undergoing motions in a nonlinear configuration manifold. The basic concepts, due to Marius Sophus Lie, Henri Poincaré and Elie Cartan, are extension of the theory originated by Euler, Legendre, Lagrange, Poisson, Hamilton and Jacobi, inspired by earlier ideas of Fermat and Huygens in optics. We premise an abstract general statement of the action principle on a manifold and the relevant basic localization results, in which Reynolds transport theorem, the Ampère-Hankel-Kelvin transform, usually dubbed Stokes's formula, and its expression in terms of differential forms due to Poincaré, Cartan's magic formula and Palais' formula for the exterior derivative of a differential one-form, are the playmates. Attention is then turned towards continuum dynamics, according to the lagrangian description, in the velocity-time phase-space. The starting point is a new statement of Hamilton's action principle in which arbitrary infinitesimal isometries of the trajectory are allowed for. On this basis, the localization result provided for the abstract action principle is applied to get the differential condition in terms of the exterior derivative of the lagrangian one-form and the related jump conditions at singular points on the trajectory. The differential law of dynamics in a nonlinear configuration manifold is derived by specializing to the dynamical context the abstract result which provides the explicit expression of the exterior derivative of the one-form. The key property is PALAIS' formula [2] which, by the tensoriality of the exterior derivative of a differential form, may be applied by envisaging an expedient extension of the time-speed of the trajectory at the actual configuration-velocity point in the velocity phase-space. The differential law of dynamics stated here is not quoted in the literature and provides the most general formulation of the governing law in terms of the lagrangian of the system. A generalized version of EMMY NOETHER's theorem [1] on symmetry of the Lagrangian and invariance along the trajectory is implied as a simple corollary. The references, on dynamics of abstract systems in nonlinear manifolds, most strictly related to the present approach, are the books [4], [7], [9], [10] and the article [6]. In all these treatments the subject is developed in modern geometrical form but in the spirit of classical rigid-body dynamics, the context in which the basic principles where originarily developed by the old masters, and recourse to coordinates is always made in the decisive steps. A main innovative feature of the analysis developed in the present paper is the explicit introduction of the rigidity constraint from the very beginning. This is in the spirit of the definition of dynamical equilibrium as stated by JOHANN BERNOULLI in 1717 in a famous letter to VARIGNON. To take account of the rigidity constraint, it is compelling to state principles and laws of dynamics in variational form and this leads, in addition, to develop a completely general and coordinate-free theory. We consider continuous systems under potential force systems so that the dynamics is completely described in terms the Lagrangian functional.

2 Calculus on manifolds

Let us consider a differentiable manifold $\mathbb{M}$ modeled on a BANACH's space E. The basic theory can be found in [5], [7], [8], [9], [10], [13], [15]. The collection of tangent spaces $\mathbb{T}_{\mathbf{x}}\mathbb{M}$ is the tangent bundle $\mathbb{TM}$ to $\mathbb{M}$. The cotangent bundle $\mathbb{T}^*\mathbb{M}$ to $\mathbb{M}$ is the collection of the dual cotangent spaces $\mathbb{T}^*_{\mathbf{x}}\mathbb{M}$. An exterior k-form is an alternating k-linear scalar-valued function defined on a tangent space to a nD manifold, with $n > k$. Differential k-forms are differentiable fields of exterior k-forms Volume-forms are n-forms on a nD manifold. On a differentiable manifold integrals of n-forms over compact nD submanifolds can be performed. The contraction of a k-form $\boldsymbol{\omega}^k$ with a vector $\mathbf{v}$ is the $(k-1)$-form $\boldsymbol{\omega}^k\mathbf{v}$ defined by taking $\mathbf{v}$ as the first argument of the form $\boldsymbol{\omega}^k$. Vectors in distinct tangent spaces can be compared if a connection and the related parallel transport is defined. We denote by $\boldsymbol{\varphi}\!\uparrow$ and $\boldsymbol{\varphi}\!\downarrow$ the push-forward and its inverse, the pull-back, of scalar, vector and tensor fields due to a diffeomorphism $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$. A dot $\cdot$ and a crochet $\langle\,,\rangle$ denote respectively linear dependence on the subsequent argument and the duality pairing. The variational analysis performed in this paper is based on the following tools of calculus on manifolds. The first tool is the POINCARÉ-

STOKES' formula which states that the integral of a differential $(k-1)$-form $\boldsymbol{\omega}^{k-1}$ on the boundary chain $\partial\Sigma$ of a kD submanifold Σ of $\mathbb{M}$ is equal to the integral of its exterior derivative $d\boldsymbol{\omega}^{k-1}$, a differential k-form, on Σ i.e.

$$\int_{\Sigma} d\boldsymbol{\omega}^{k-1} = \oint_{\partial\Sigma} \boldsymbol{\omega}^{k-1} .$$

The second tool is LIE's derivative of a vector field $\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ along a flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ with velocity $\mathbf{v} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$:

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} = \partial_{\lambda=0}\,(\boldsymbol{\varphi}_\lambda{\downarrow}\mathbf{w})\,,$$

which is equal to the antisymmetric LIE-bracket: $\mathcal{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v},\mathbf{w}] = -[\mathbf{w},\mathbf{v}]$ defined by: $d_{[\mathbf{v},\mathbf{w}]}f = d_{\mathbf{v}}d_{\mathbf{w}}f - d_{\mathbf{w}}d_{\mathbf{v}}f$, for any $f \in \mathrm{C}^2(\mathbb{M}\,;\mathcal{R})$. The LIE derivative of a differential form $\boldsymbol{\omega}^k \in \mathrm{C}^1(\mathbb{M}\,;\Lambda^k(\mathbb{TM}))$ is similarly defined by $\mathcal{L}_{\mathbf{v}}\boldsymbol{\omega}^k = \partial_{\lambda=0}\,(\boldsymbol{\varphi}_\lambda{\downarrow}\boldsymbol{\omega}^k)$. The third tool is the extrusion formula

$$\partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_{\Sigma} (d\boldsymbol{\omega}^k)\mathbf{v} + \int_{\partial\Sigma} \boldsymbol{\omega}^k\mathbf{v}\,,$$

which by REYNOLDS' transport formula:

$$\int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_{\Sigma} \boldsymbol{\varphi}_\lambda{\downarrow}\boldsymbol{\omega}^k \quad\Longrightarrow\quad \partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_{\Sigma} \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k\,,$$

yields CARTAN's magic formula (or homotopy formula): $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k = (d\boldsymbol{\omega}^k)\mathbf{v} + d(\boldsymbol{\omega}^k\mathbf{v})$. The homotopy formula may be readily inverted to get PALAIS formula for the exterior derivative. Indeed, by LEIBNIZ rule for the LIE derivative, we have that, for any two vector fields $\mathbf{v},\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$:

$$d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{w} = (\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^1)\cdot\mathbf{w} - d(\boldsymbol{\omega}^1\mathbf{v})\cdot\mathbf{w} = d_{\mathbf{v}}\,(\boldsymbol{\omega}^1\mathbf{w}) - \boldsymbol{\omega}^1\cdot[\mathbf{v},\mathbf{w}] - d_{\mathbf{w}}\,(\boldsymbol{\omega}^1\mathbf{v})\,.$$

The expression at the r.h.s. of PALAIS formula fulfills the tensoriality criterion, as quoted in [8], [15] and thus its value at a point depends only on the values of the argument vector fields at that point. The exterior derivative of a differential one-form is thus a differential two-form. The same algebra may be repeatedly applied to deduce PALAIS formula for a k-form.

3 The abstract action principle

Let a status of the system be described by a point of the *phase space* $\mathbb{M}$, a differentiable manifold. According to HAMILTON's point of view, the evolution of the system is governed by a variational condition on the signed-length of the trajectory $\boldsymbol{\Gamma} \in \mathrm{C}^1(I\,;\mathbb{M})$, evaluated according to a differential one-form $\boldsymbol{\omega}^1 \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}^*\mathbb{M})$ dubbed the *action one-form.*

The signed-length $\int_{\boldsymbol{\Gamma}} \boldsymbol{\omega}^1$ is called the *action integral* at $\boldsymbol{\Gamma}$. To provide a general statement of the action principle, we call *virtual flows* in $\mathbb{M}$ the elements of a suitably defined one-parameter subfamily of $\mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$. Vector fields which are velocities of virtual flows are dubbed *virtual velocities*. The test-subbundle $V_{\mathrm{RIG}} \subset \mathbb{TM}$ is a subbundle of the bundle of virtual velocities, and $V_{\mathrm{RIG}}(\boldsymbol{\Gamma})$ denotes the restriction of the test-subbundle to $\boldsymbol{\Gamma}$.

Proposition 1 (Action principle). *At a trajectory $\boldsymbol{\Gamma} \subset \mathbb{M}$ of the system the action integral meets the variational condition:*

$$\partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega}^1 = \int_{\partial \boldsymbol{\Gamma}} \boldsymbol{\omega}^1 \cdot \mathbf{v} \,,$$

for all virtual flows $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ with initial velocity $\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\Gamma}\,; V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))$.

This means that the initial rate of increase of the $\boldsymbol{\omega}^1$-length of the trajectory $\boldsymbol{\Gamma}$ along a rigid virtual flow is equal to the outward flux of the virtual velocities at the end points. Denoting by $\mathbf{x}_1$ and $\mathbf{x}_2$ the initial and final end points of $\boldsymbol{\Gamma}$, we have that $\partial \boldsymbol{\Gamma} = \mathbf{x}_2 - \mathbf{x}_1$ (a 0-chain) and the boundary integral may be written as $\int_{\partial \boldsymbol{\Gamma}} \boldsymbol{\omega}^1 \cdot \mathbf{v} = (\boldsymbol{\omega}^1 \cdot \mathbf{v})(\mathbf{x}_2) - (\boldsymbol{\omega}^1 \cdot \mathbf{v})(\mathbf{x}_1)$. The stationarity of the action integral is a problem of *calculus of variations* on a nonlinear manifold. Necessary and sufficient local conditions for a path to be a trajectory are provided by the next proposition which considers piecewise regular paths with non-fixed end points on a nonlinear manifold and is stated in coordinate-free terms. The classical local result of Euler and Lagrange is formulated in coordinates and considers regular paths with fixed end points. We will denote by $\mathcal{T}(\boldsymbol{\Gamma})$ an open regularity partition of $\boldsymbol{\Gamma}$ and by $\mathcal{I}(\boldsymbol{\Gamma})$ the corresponding set of singularity interfaces.

Proposition 2 (Local conditions). *A path $\boldsymbol{\Gamma} \subset \mathbb{M}$ is a trajectory if and only if the tangent vector field $\mathbf{v}_{\boldsymbol{\Gamma}} \in \mathrm{C}^1(\mathcal{T}(\boldsymbol{\Gamma})\,;\mathbb{T}\boldsymbol{\Gamma})$ meets, in each element of a regularity partition $\mathcal{T}(\boldsymbol{\Gamma})$, the pointwise differential condition*

$$d\boldsymbol{\omega}^1 \cdot \mathbf{v}_{\boldsymbol{\Gamma}} \cdot \mathbf{v} = 0 \,, \quad \forall\, \mathbf{v} \in \mathrm{C}^0(\boldsymbol{\Gamma}\,; V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))\,,$$

and, at the singularity interfaces $\mathcal{I}(\boldsymbol{\Gamma})$, the jump conditions

$$[[\boldsymbol{\omega}^1 \mathbf{v}]] = 0 \,, \quad \forall\, \mathbf{v} \in \mathrm{C}^0(\boldsymbol{\Gamma}\,; V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))\,.$$

Proof. By applying the extrusion formula to each element of the regularity partition, we get

$$\partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega}^1 - \int_{\partial \boldsymbol{\Gamma}} \boldsymbol{\omega}^1 \mathbf{v} = \int_{\mathcal{T}(\boldsymbol{\Gamma})} (d\boldsymbol{\omega}^k)\mathbf{v} - \int_{\mathcal{I}(\boldsymbol{\Gamma})} [[\boldsymbol{\omega}^1 \mathbf{v}]] \,,$$

so that the action principle writes

$$\int_{\mathcal{T}(\boldsymbol{\Gamma})} (d\boldsymbol{\omega}^1)\mathbf{v} = \int_{\mathcal{I}(\boldsymbol{\Gamma})} [[\boldsymbol{\omega}^1 \mathbf{v}]] \,, \quad \forall\, \mathbf{v} \in \mathrm{C}^0(\boldsymbol{\Gamma}\,; V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))\,.$$

Let the path $\boldsymbol{\Gamma}$ be parametrized by $s \in I$ and $\mathbf{v}_{\boldsymbol{\Gamma}} \in \mathrm{C}^1(\boldsymbol{\Gamma}\,;\mathbb{T}\boldsymbol{\Gamma})$ be the velocity field along the path. Then:

$$\int_{\mathcal{T}(\boldsymbol{\Gamma})} (d\boldsymbol{\omega}^1)\mathbf{v} - \int_{\mathcal{I}(\boldsymbol{\Gamma})} [[\boldsymbol{\omega}^1\mathbf{v}]] = \int_{\mathcal{T}(I)} d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{v}_{\boldsymbol{\Gamma}}\, ds - \int_{\mathcal{I}(\boldsymbol{\Gamma})} [[\boldsymbol{\omega}^1\mathbf{v}]]\,.$$

If the differential and jump conditions are fulfilled, the action principle holds. Conversely, if $d\boldsymbol{\omega}^1 \cdot \mathbf{v}_{\boldsymbol{\Gamma}} \cdot \mathbf{v} \neq 0$ at a point inside an element of the regularity partition, by continuity of $d\boldsymbol{\omega}^1 \cdot \mathbf{v}_{\boldsymbol{\Gamma}} \cdot \mathbf{v}$, we could take $\mathbf{v} \in \mathrm{C}^0(\boldsymbol{\Gamma}\,;V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))$ such that $d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{v}_{\boldsymbol{\Gamma}} > 0$ on an open segment $U_{\boldsymbol{\Gamma}}$ around that point and $d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{v}_{\boldsymbol{\Gamma}} = 0$ on $\boldsymbol{\Gamma}\backslash U_{\boldsymbol{\Gamma}}$. Hence $\int_{\mathcal{T}(\boldsymbol{\Gamma})} (d\boldsymbol{\omega}^1)\mathbf{v} > 0$, contrary to the assumption. The vanishing of the jumps follows by a simple argument.

The next two results are due to the first author.

Proposition 3 (Palais-Romano condition). *The differential condition fulfilled by a trajectory* $\boldsymbol{\Gamma} \subset \mathbb{M}$ *may equivalently be written as*

$$d_{\mathbf{v}_{\boldsymbol{\Gamma}}}(\boldsymbol{\omega}^1 \cdot \mathbf{v}) = d_{\mathbf{v}}(\boldsymbol{\omega}^1 \cdot \mathbf{v}_{\boldsymbol{\Gamma}})\,, \quad \forall\, \mathbf{v} \in \mathrm{C}^0(\boldsymbol{\Gamma}\,;V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))\,,$$

where $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{M})$ *is an extension of the virtual velocity* $\mathbf{v} \in \mathrm{C}^0(\boldsymbol{\Gamma}\,;V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))$ *and* $\mathbf{v}_{\boldsymbol{\Gamma}} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{M})$ *is the extension of* $\mathbf{v}_{\boldsymbol{\Gamma}} \in \mathrm{C}^0(\boldsymbol{\Gamma}\,;V_{\mathrm{RIG}}(\boldsymbol{\Gamma}))$ *performed by pushing it along the flow* $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ *generated by* $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{M})$.

Proof. The result follows from proposition 2 by a direct application of Palais formula: $d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{v}_{\boldsymbol{\Gamma}} = d_{\mathbf{v}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v}_{\boldsymbol{\Gamma}}) - d_{\mathbf{v}_{\boldsymbol{\Gamma}}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v}) - \boldsymbol{\omega}^1 \cdot [\mathbf{v}, \mathbf{v}_{\boldsymbol{\Gamma}}]$. Indeed, by tensoriality, the r.h.s. is independent of the extensions of $\mathbf{v}$ and $\mathbf{v}_{\boldsymbol{\Gamma}}$. Moreover the special extension of $\mathbf{v}_{\boldsymbol{\Gamma}}$ implies that $[\mathbf{v}, \mathbf{v}_{\boldsymbol{\Gamma}}] = 0$.

As a simple corollary we infer that:

Proposition 4 (Abstract Noether's theorem). *If the action one-form* $\boldsymbol{\omega}^1 \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}^*\mathbb{M})$ *enjoys the property:* $d_{\mathbf{v}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v}_{\boldsymbol{\Gamma}}) = 0$ *, then the functional* $\boldsymbol{\omega}^1 \cdot \mathbf{v}$ *is constant along the trajectory* $\boldsymbol{\Gamma} \subset \mathbb{M}$ *.*

4 Continuum vs rigid-body dynamics

The theory developed for the abstract action principle, may be applied to continuum mechanics by envisaging a suitable phase-space. A continuous body is identified with an open, connected, reference manifold $\mathbb{B} \subset \mathbb{S}$ embedded in the euclidean space $\{\mathbb{S}\,, \mathbf{g}\}$ with metric tensor $\mathbf{g} \in BL\,(\mathbb{TS}^2\,;\mathcal{R})$ and canonical connection ∇ .

The configurations $\boldsymbol{\chi} \in \mathrm{C}^1(\mathbb{B};\mathbb{S})$ of the continuous body $\mathbb{B} \subset \mathbb{S}$ are injective maps with the property of being diffeomorphic transformations onto their ranges. The configuration-space $\mathbb{C}$ is assumed to be a differentiable manifold endowed with the topology inherited by a model Banach space.

The velocity phase-space is the tangent bundle $\mathbb{TC}$ and the momentum phase-space is the cotangent bundle $\mathbb{T}^*\mathbb{C}$. The velocity-time phase-space is $\mathbb{TC} \times I$, is the cartesian product of the velocity-space $\mathbb{TC}$ and an open time interval I, and the momentum-time phase-space is $\mathbb{T}^*\mathbb{C} \times I$. These two phase-spaces are respectively adopted in the Lagrangian and Hamiltonian descriptions of dynamics. Vectors tangent to the velocity-time phase-space $\mathbb{TC} \times I$ are in the bundle $\mathbb{TTC} \times \mathbb{T}I$ whose elements are pairs $\{\delta\mathbf{v}, \delta t\} \in \mathbb{T}_{\mathbf{v}}\mathbb{TC} \times \mathbb{T}_t I$. Denoting by $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{C})$ the projector on the base manifold, the velocity of the configuration $\boldsymbol{\pi}(\mathbf{v}) \in \mathbb{C}$, corresponding to the tangent vector $\delta\mathbf{v} \in \mathbb{T}_{\mathbf{v}}\mathbb{TC}$ is provided by the tangent to the projector: $\boldsymbol{\pi}{\uparrow}\delta\mathbf{v} := d\boldsymbol{\pi}(\mathbf{v}) \cdot \delta\mathbf{v} \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C}$.

Rigidity constraint

Two configurations $\boldsymbol{\chi}_1 \in \mathrm{C}^1(\mathbb{B}; \mathbb{S})$ and $\boldsymbol{\chi}_2 \in \mathrm{C}^1(\mathbb{B}; \mathbb{S})$ are metric-equivalent if $\boldsymbol{\chi}_2{\downarrow}\mathbf{g} = \boldsymbol{\chi}_1{\downarrow}\mathbf{g}$. Here $\boldsymbol{\chi}{\downarrow}\mathbf{g}$ is the pull back along $\boldsymbol{\chi} \in \mathrm{C}^1(\boldsymbol{\chi}(\mathbb{B}); \mathbb{S})$ of the metric tensor: $(\boldsymbol{\chi}{\downarrow}\mathbf{g})(\mathbf{a}, \mathbf{b}) = \mathbf{g}(\boldsymbol{\chi}{\uparrow}\mathbf{a}, \boldsymbol{\chi}_\lambda{\uparrow}\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{TS}$.

Then the diffeomorphic map $\boldsymbol{\chi}_2 \circ \boldsymbol{\chi}_1^{-1} \in \mathrm{C}^1(\boldsymbol{\chi}_1(\mathbb{B}); \boldsymbol{\chi}_2(\mathbb{B}))$ is a metric-preserving (or rigid) transformation of the configuration $\boldsymbol{\chi}_1 \in \mathrm{C}^1(\mathbb{B}; \mathbb{S})$ into the configuration $\boldsymbol{\chi}_2 \in \mathrm{C}^1(\mathbb{B}; \mathbb{S})$. By the metric-equivalence relation so introduced, the manifold $\mathbb{C}$ is partitioned into a family of disjoint connected rigidity-classes $\mathbb{C}_R$ which are submanifolds of $\mathbb{C}$.

The elements of the tangent space $\mathbb{T}_{\boldsymbol{\chi}}\mathbb{C}_R$ to a rigidity-class $\mathbb{C}_R$ at $\boldsymbol{\chi} \in \mathbb{C}_R$ are the infinitesimal isometries, that is, the vector fields $\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\chi}(\mathbb{B}); \mathbb{S})$ fulfilling the EULER-KILLING condition [15]: $\mathcal{L}_{\mathbf{v}}\mathbf{g} = 2\,\mathbf{g}\,(\mathrm{sym}\,\nabla\mathbf{v}) = 0$.

The LIE derivative of the metric tensor is defined by: $\mathcal{L}_{\mathbf{v}}\mathbf{g} := \partial_{\lambda=0}\,\boldsymbol{\chi}_\lambda{\downarrow}\mathbf{g}$ where $\boldsymbol{\chi}_\lambda \in \mathrm{C}^1(\boldsymbol{\chi}(\mathbb{B}); \mathbb{S})$ is the flow generated by $\mathbf{v} = \partial_{\lambda=0}\,\boldsymbol{\chi}_\lambda$. In rigid-body dynamics the body is assumed to evolve in a fixed rigidity-class $\mathbb{C}_R$ so that at each configurarion test vector fields and trajectory velocities belong to the same tangent space $\mathbb{T}_{\boldsymbol{\chi}}\mathbb{C}_R$.

5 Hamilton's action principle

In the *lagrangian description*, the phase-space is the *velocity phase space*, that is, the tangent bundle $\mathbb{TC}$ to the configuration manifold. The state variables are then pairs formed by a configuration and a velocity vector based at that configuration. The projector $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{C})$ maps the velocity phase space onto the configuration space so that $\mathbf{v} = \{\boldsymbol{\chi}, \mathbf{v}_{\boldsymbol{\chi}}\} \in \mathbb{TC}$ and $\boldsymbol{\pi}(\mathbf{v}) = \boldsymbol{\chi} \in \mathbb{C}$. The *Lagrangian* of the system is a time-dependent functional $\mathrm{L}_t \in \mathrm{C}^1(\mathbb{TC}; \mathcal{R})$ on the velocity phase space.

The usual expression of the Lagrangian is $\mathrm{L}_t = K_t \circ \mathrm{DIAG} + P_t \circ \boldsymbol{\pi}$ where $\mathrm{DIAG} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{TC}^2)$ is the diagonal map, defined by $\mathrm{DIAG}(\mathbf{v}) := \{\mathbf{v}, \mathbf{v}\}$, $K_t(\mathbf{v}, \mathbf{v}) \in \mathrm{C}^1(\mathbb{TC}^2; \mathcal{R})$ is the positive definite quadratic kinetic energy and $P_t(\boldsymbol{\pi}(\mathbf{v})) \in \mathrm{C}^1(\mathbb{C}; \mathcal{R})$ is the force potential.

The fiber-derivative $d_{\mathrm{F}}\mathrm{L}_t \in \mathrm{C}^1(\mathbb{T}\mathbb{C}\,;\mathbb{T}^*\mathbb{C})$ of the Lagrangian associates to any velocity $\mathbf{v} \in \mathbb{T}\mathbb{C}$ the one-form $d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}) := \partial_{\lambda=0}\mathrm{L}_t(\boldsymbol{\psi}_\lambda(\mathbf{v}))$ where $\boldsymbol{\psi}_\lambda \in \mathrm{C}^1(\mathbb{T}\mathbb{C}\,;\mathbb{T}\mathbb{C})$ is a configuration-preserving flow, that is such that $\boldsymbol{\pi}(\boldsymbol{\psi}_\lambda(\mathbf{v})) = \boldsymbol{\pi}(\mathbf{v})\,, \quad \forall\, \lambda \in \mathcal{R}$. In the the tangent bundle $\mathbb{T}\mathbb{C}$ to the configuration manifold, the fiber-derivative plays the role of the partial derivative with respect to the base point due to the linearity of the tangent fiber. No analogue of the partial derivative with respect to the vectorial part of tangent vectors is available in a nonlinear configuration manifold, unless a connection is defined. Being $\mathrm{L}_t(\mathbf{v}) = K_t(\mathbf{v},\mathbf{v}) + P_t(\boldsymbol{\pi}(\mathbf{v}))$, the fiber-derivative of the Lagrangian and of the kinetic energy are equal and have the mechanical meaning of kinetic momentum. Let us now consider an open time interval I, a time-parametrized path $\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{C})$ in the configuration space. The velocity $\mathbf{v}_t = \{\boldsymbol{\gamma}(t)\,, \partial_{\tau=t}\boldsymbol{\gamma}(\tau)\} \in \mathbb{T}_{\boldsymbol{\gamma}(t)}\boldsymbol{\gamma}$ along $\boldsymbol{\gamma}$ at time $t \in I$ spans a lifted trajectory $\boldsymbol{\Gamma} \in \mathrm{C}^1(I\,;\mathbb{T}\mathbb{C})$ in the phase-space. To provide the classical statement of HAMILTON's principle, the kinetic energy $K_t \in \mathrm{C}^1(\mathbb{T}\mathbb{C}\,;\mathcal{R})$, which makes sense only on the trajectory $\boldsymbol{\Gamma} \in \mathrm{C}^1(I\,;\mathbb{T}\mathbb{C})$, must be extended to a functional on the velocity phase space $\mathbb{T}\mathbb{C}$, at least in a neighbourhood of the trajectory. In continuum dynamics, although never stated explicitly, the extension is performed by assuming that the mass-form be dragged by the virtual flow. Let us denote by $\mathbf{m} = \rho\,\boldsymbol{\mu}$ the mass-form, with ρ the density and $\boldsymbol{\mu}$ the volume-form in $\{\mathbb{S}\,,\mathbf{g}\}$. Under the action of a flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$, a configuration $\boldsymbol{\chi}_t \in \mathbb{C}$ is changed into $\boldsymbol{\varphi}_\lambda \circ \boldsymbol{\chi}_t \in \mathbb{C}$ and the kinetic energy $K_t \in \mathrm{C}^1(\mathbb{T}\mathbb{C}\,;\mathcal{R})$ at the point $\mathbf{v}_t \in \mathbb{T}_{\boldsymbol{\chi}_t}\mathbb{C}$ on the trajectory is transformed into

$$K_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t) := \frac{1}{2}\int_{(\boldsymbol{\varphi}_\lambda\circ\boldsymbol{\chi}_t)(\mathbb{B})} \|\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t\|^2\, \boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{m}\,,$$

where $(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{m})(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{a}_1, \boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{a}_2, \boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{a}_3) = \mathbf{m}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ for all $\mathbf{a}_i \in \mathbb{T}_{\mathbf{x}}\mathbb{B}$.

Accordingly, the standard statement of HAMILTON's principle in the dynamics of continuous bodies is the following.

Proposition 5 (Standard form of Hamilton's principle). *A dynamical trajectory of a continuous mechanical system in the configuration manifold is a time-parametrized path* $\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{C})$ *fulfilling the stationarity condition*

$$\partial_{\lambda=0}\int_I L_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t)\,dt = 0\,.$$

for any flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *in the configuration manifold whose velocity field* $\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{T}\mathbb{C})$ *is an infinitesimal isometry at each point of* $\boldsymbol{\gamma}$ *and vanishes at the end points of the path.*

In the literature HAMILTON's principle is usually stated in the special context of rigid body dynamics [7], [10], [12]. Then the stationarity condition is required to hold for any flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}_R\,;\mathbb{C}_R)$, in a rigidity class $\mathbb{C}_R \subset \mathbb{C}$ of the configuration manifold, whose velocity field $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(\mathbb{C}_R\,;\mathbb{T}\mathbb{C}_R)$ vanishes at the end points of the path.

The basic step towards the formulation of a general law of dynamics consists in a suitable modification of the statement of Hamilton's principle to drop out the condition that the virtual velocity fields vanish at the end points of the path. The proper way to perform the modification is suggested by the discussion of the action principle illustrated in section 3, when specialized to velocity phase space of lagrangian dynamics.

6 The action one-form

A new formulation of HAMILTON's action principle is inferred from the abstract theory of section 3. To this end we have to express HAMILTON's principle in terms of the integral of an action one-form over the trajectory $\boldsymbol{\Gamma}_I$ in the *velocity-time phase-space*, the cartesian product $\mathbb{TC} \times I$. Let us consider a time-parametrized path $\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{C})$ in the configuration manifold $\mathbb{C}$ and the corresponding lifted path $\boldsymbol{\Gamma} := \mathbb{T}\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{TC})$ in the velocity phase-space, setting $\mathbf{v}_t = \{\boldsymbol{\gamma}(t)\,,\dot{\boldsymbol{\gamma}}(t)\} \in \mathbb{T}\boldsymbol{\gamma} \subset \mathbb{TC}$. We denote by $\boldsymbol{\psi}_\lambda := \boldsymbol{\varphi}_\lambda\!\uparrow = d\boldsymbol{\varphi}_\lambda \circ \boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{TC}\,;\mathbb{TC})$ the tangent flow induced, in the velocity phase-space, by the flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ in the configuration manifold, so that $\boldsymbol{\pi} \circ \boldsymbol{\psi}_\lambda = \boldsymbol{\varphi}_\lambda \circ \boldsymbol{\pi}$ and

$$\boldsymbol{\psi}_\lambda(\mathbf{v}) := d\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v})) \cdot \mathbf{v}\,, \quad \forall\, \mathbf{v} \in \mathbb{TC}\,.$$

Denoting by $d\boldsymbol{\pi}(\mathbf{v}) \in BL(\mathbb{T}_{\mathbf{v}}\mathbb{TC}\,;\mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C})$ the differential of the projector $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{TC}\,;\mathbb{C})$ and setting $\mathbf{v}_{\boldsymbol{\psi}} = \partial_{\lambda=0}\,\boldsymbol{\psi}_\lambda$ we have that

$$\begin{aligned} d\boldsymbol{\pi}(\mathbf{v}) \cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}) &= d\boldsymbol{\pi}(\mathbf{v}) \cdot \partial_{\lambda=0}\,\boldsymbol{\psi}_\lambda(\mathbf{v}) = \partial_{\lambda=0}\,(\boldsymbol{\pi} \circ \boldsymbol{\psi}_\lambda)(\mathbf{v}) \\ &= \partial_{\lambda=0}\,(\boldsymbol{\varphi}_\lambda \circ \boldsymbol{\pi})(\mathbf{v}) = \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}))\,, \quad \forall\, \mathbf{v} \in \mathbb{TC}\,. \end{aligned}$$

Taking the time-derivative of $\boldsymbol{\pi}(\mathbf{v}_t) = \boldsymbol{\gamma}(t)$ and denoting by $\{\dot{\mathbf{v}}_t\,,1\} \in \mathbb{T}_{\mathbf{v}_t}\mathbb{TC} \times \mathbb{T}_t I$ the time-speed along of the trajectory $\boldsymbol{\Gamma}_I$ at the point $\{\mathbf{v}_t\,,t\} \in \boldsymbol{\Gamma}_I$, we get the relation: $d\boldsymbol{\pi}(\mathbf{v}_t) \cdot \dot{\mathbf{v}}_t = \mathbf{v}_t$ which is a special case of the one reported above. The basic tool to reach the goal is LEGENDRE transform which defines the *energy* of the system $E_t \in \mathrm{C}^1(\mathbb{TC}\,;\mathcal{R})$ as the conjugate of the *Lagrangian*, according to the relation:

$$\mathrm{L}_t(\mathbf{v}_t) + E_t(\mathbf{v}_t) = \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \mathbf{v}_t \rangle = \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), d\boldsymbol{\pi}(\mathbf{v}_t) \cdot \dot{\mathbf{v}}_t \rangle\,,$$

and the differential one-form $\boldsymbol{\theta}_{\mathrm{L}_t} \in \mathrm{C}^1(\mathbb{TC} \times I\,;\mathbb{TTC} \times \mathbb{T}I)$ by the identity

$$\boldsymbol{\theta}_{\mathrm{L}_t}(\{\mathbf{v}\,,t\}) \cdot \{\delta\mathbf{v}\,,\delta t\} := \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), d\boldsymbol{\pi}(\mathbf{v}) \cdot \delta\mathbf{v} \rangle\,,$$

for all $\{\delta\mathbf{v}\,,\delta t\} \in \mathbb{T}_{\mathbf{v}}\mathbb{TC} \times \mathbb{T}_t I$. Then, noting that

$$E_t(\mathbf{v})dt \cdot \{\delta\mathbf{v}\,,\delta t\} = E_t(\mathbf{v})\langle dt, \delta t \rangle\,,$$

and defining the differential one-form $\boldsymbol{\omega}^1_{\mathrm{L}} \in \mathrm{C}^1(\mathbb{TC} \times I\,;\mathbb{TTC} \times \mathbb{T}I)$ by

$$\boldsymbol{\omega}^1_{\mathrm{L}}(\{\mathbf{v}\,,t\}) = \boldsymbol{\theta}_{\mathrm{L}_t}(\{\mathbf{v}\,,t\}) - E(\mathbf{v},t)dt\,,$$

we have that

$$\boldsymbol{\omega}^1_{\mathrm{L}}(\{\mathbf{v}_t\,,t\}) \cdot \{\dot{\mathbf{v}}_t\,,1\} = \langle d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \mathbf{v}_t\rangle - E_t(\mathbf{v}_t)\langle dt, 1\rangle = \mathrm{L}_t(\mathbf{v}_t)\,,$$

and also

$$\boldsymbol{\omega}^1_{\mathrm{L}}(\{\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t\,,t\}) \cdot \{\boldsymbol{\psi}_\lambda\!\uparrow\!\dot{\mathbf{v}}_t\,,1\} = \mathrm{L}_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t)$$

$$\begin{aligned}\boldsymbol{\omega}^1_{\mathrm{L}}(\{\mathbf{v}_t\,,t\}) \cdot \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,,0\} &= \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), d\boldsymbol{\pi}(\mathbf{v}_t)\cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,\rangle \\ &= \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,.\end{aligned}$$

We may thus conclude that

$$\int_{\boldsymbol{\varphi}_\lambda\uparrow\boldsymbol{\Gamma}_I} \boldsymbol{\omega}^1_{\mathrm{L}} = \int_I \boldsymbol{\omega}^1_{\mathrm{L}}(\{\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t\,,t\}) \cdot \{\boldsymbol{\psi}_\lambda\!\uparrow\!\dot{\mathbf{v}}_t\,,1\}\,dt = \int_I \mathrm{L}_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t)\,dt\,,$$

and

$$\int_{\partial\boldsymbol{\Gamma}_I} \boldsymbol{\omega}^1_{\mathrm{L}}(\{\mathbf{v}_t\,,t\}) \cdot \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,,0\} = \int_{\partial I} \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,dt\,.$$

The action principle for the action one-form $\boldsymbol{\omega}^1_{\mathrm{L}} \in \mathrm{C}^1(\mathbb{TC}\times I\,;\mathbb{TTC}\times\mathbb{T}I)$ in the *velocity-time phase-space* is then expressed by the variational condition:

$$\partial_{\lambda=0}\int_{\boldsymbol{\varphi}_\lambda\uparrow\boldsymbol{\Gamma}_I} \boldsymbol{\omega}^1_{\mathrm{L}} = \int_{\partial\boldsymbol{\Gamma}_I} \boldsymbol{\omega}^1_{\mathrm{L}} \cdot \{\mathbf{v}_{\boldsymbol{\psi}}\,,0\}\,,$$

for any flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\boldsymbol{\gamma}\,;\mathbb{C})$ in the configuration manifold whose velocity field $\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\boldsymbol{\gamma}\,;\mathbb{TC})$ is an infinitesimal isometry at each point of $\boldsymbol{\gamma}$. The boundary term vanishes if $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) = 0$ at the end points of $\boldsymbol{\Gamma}_I$, which means that the initial and final configurations are hold fixed by the flow. This is the assumption made in all the previous literature on dynamics in formulating the action principle (see e.g. [7]). The abstract action principle for the action one-form $\boldsymbol{\omega}^1_{\mathrm{L}} \in \mathrm{C}^1(\mathbb{TC}\times I\,;\mathbb{TTC}\times\mathbb{T}I)$ leads to the following statement of HAMILTON's principle.

Proposition 6 (New statement of Hamilton's principle). *A dynamical trajectory of a continuous mechanical system in the configuration manifold is a time-parametrized path* $\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{C})$ *fulfilling the variational condition*

$$\partial_{\lambda=0}\int_I L_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t)\,dt = \int_{\partial I} \langle\, d_{\mathrm{F}}L_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,dt\,.$$

for any flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *in the configuration manifold whose velocity field* $\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{TC})$ *is an infinitesimal isometry at each point of* $\boldsymbol{\gamma}$.

This general statement of HAMILTON's principle is equivalent to the standard one (proposition 5) in the case of regular motions. Indeed both principles are equivalent to the same differential law of dynamics.

6.1 Differential and jump conditions

By theorem 2, the action principle in the *velocity-time phase-space* is equivalent to the differential condition:

$$d\boldsymbol{\omega}^1_{\mathrm{L}_t}(\mathbf{v}_t, t) \cdot \{\, \dot{\mathbf{v}}_t, 1 \,\} \cdot \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,, 0\} = 0\,,$$

at regular points and to the jump condition

$$[[\boldsymbol{\omega}^1_{\mathrm{L}_t}(\mathbf{v}_t, t)]] \cdot \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,, 0\} = 0\,,$$

at singular points along the trajectory. Recalling the definition of the one-form $\boldsymbol{\omega}^1_{\mathrm{L}_t}$, the jump condition writes: $\langle [[d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t)]], \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \rangle = 0$ and the differential condition takes the expression:

$$d\boldsymbol{\theta}_{\mathrm{L}_t}(\mathbf{v}_t, t) \cdot \dot{\mathbf{v}}_t \cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t) = d(E(\mathbf{v}_t, t)\, dt) \cdot \{\, \dot{\mathbf{v}}_t, 1 \,\} \cdot \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,, 0\}\,.$$

To perform the exterior derivatives we apply Palais's formula, extending the vector $\dot{\mathbf{v}}_t \in \mathbb{T}_{\mathbf{v}_t}\boldsymbol{\Gamma}$ to a vector field $\dot{\mathbf{v}} \in \mathrm{C}^1(\mathbb{TC}\,; \mathbb{TTC})$ by pushing it along the phase-flow $\boldsymbol{\psi}_\lambda \in \mathrm{C}^1(\mathbb{TC}\,; \mathbb{TC})$, so that $\dot{\mathbf{v}}(\boldsymbol{\psi}_\lambda(\mathbf{v}_t)) := \boldsymbol{\psi}_\lambda\!\uparrow\!\dot{\mathbf{v}}_t$. Then the r.h.s. becomes

$$\begin{aligned} d(E_t(\mathbf{v}_t)\, dt) \cdot \{\dot{\mathbf{v}}_t\,, 1\} \cdot \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,, 0\} &= d_{\{\dot{\mathbf{v}}_t\,,1\}} \langle\, E_t(\mathbf{v}_t)\, dt, \{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,, 0\} \rangle \\ &\quad - d_{\{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)\,,0\}} \langle\, E_t(\mathbf{v}_t)\, dt, \{\, \dot{\mathbf{v}}_t, 1\,\} \rangle \\ &\quad + \langle\, E_t(\mathbf{v}_t)\, dt, \{(\mathcal{L}_{\mathbf{v}_{\boldsymbol{\psi}}} \dot{\mathbf{v}})(\mathbf{v}_t)\,, 0\} \rangle \\ &= \; -d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} E_t(\mathbf{v}_t)\,. \end{aligned}$$

Accordingly, the differential condition writes:

$$d\boldsymbol{\theta}_{\mathrm{L}_t}(\mathbf{v}_t, t) \cdot \dot{\mathbf{v}}_t \cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t) = -d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} E_t(\mathbf{v}_t)\,.$$

Applying again Palais formula and taking into account that the Lie derivative $\mathcal{L}_{\mathbf{v}_{\boldsymbol{\psi}}} \dot{\mathbf{v}}$ vanishes identically, the l.h.s. becomes:

$$\begin{aligned} d\boldsymbol{\theta}_{\mathrm{L}_t}(\mathbf{v}_t, t) \cdot \dot{\mathbf{v}}_t \cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t) &= d_{\dot{\mathbf{v}}_t}(\boldsymbol{\theta}_{\mathrm{L}_t} \cdot \mathbf{v}_{\boldsymbol{\psi}})(\mathbf{v}_t) - d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)}(\boldsymbol{\theta}_{\mathrm{L}_t} \cdot \dot{\mathbf{v}})(\mathbf{v}_t) \\ &\quad + (\boldsymbol{\theta}_{\mathrm{L}_t} \cdot \mathcal{L}_{\mathbf{v}_{\boldsymbol{\psi}}} \dot{\mathbf{v}})(\mathbf{v}_t) \\ &= d_{\dot{\mathbf{v}}_t}(\boldsymbol{\theta}_{\mathrm{L}_t} \cdot \mathbf{v}_{\boldsymbol{\psi}})(\mathbf{v}_t) - d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)}(\boldsymbol{\theta}_{\mathrm{L}_t} \cdot \dot{\mathbf{v}})(\mathbf{v}_t)\,, \end{aligned}$$

with

$$\begin{aligned} d_{\dot{\mathbf{v}}_t}(\boldsymbol{\theta}_{\mathrm{L}_t} \cdot \mathbf{v}_{\boldsymbol{\psi}})(\mathbf{v}_t) &= \partial_{\tau=t}\, \langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), d\boldsymbol{\pi}(\mathbf{v}_\tau) \cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_\tau) \rangle \\ &= \partial_{\tau=t}\, \langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau)) \rangle\,, \end{aligned}$$

and

$$\begin{aligned} d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)}(\boldsymbol{\theta}_{\mathrm{L}_t} \cdot \dot{\mathbf{v}})(\mathbf{v}_t) &= d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), d\boldsymbol{\pi}(\mathbf{v}) \cdot \dot{\mathbf{v}} \rangle = d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}), \mathbf{v} \rangle \\ &= d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} \mathrm{L}_t(\mathbf{v}_t) + d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} E_t(\mathbf{v}_t)\,. \end{aligned}$$

Hence we get the general expression of the law of dynamics. The differential part is a specialization to mechanical systems of the abstract condition provided in proposition 3.

Theorem 1 (The law of dynamics). *A trajectory of the system is a time-parametrized path* $\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{C})$ *in the configuration manifold* $\mathbb{C}$*, fulfilling the differential condition:*

$$\partial_{\tau=t} \, \langle d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau)) \rangle = d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} L_t(\mathbf{v}_t)\,,$$

and the jump conditions

$$\langle \, [[d_{\mathrm{F}} L_t(\mathbf{v}_t)]], \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \rangle = 0\,,$$

for all flows $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *whose velocity* $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))$ *at the actual configuration* $\boldsymbol{\pi}(\mathbf{v}_t) \in \mathbb{C}$ *is an admissible infinitesimal isometry. The r.h.s. of the differential condition may also be written as*

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\boldsymbol{\psi}}} L_t(\mathbf{v}_t) &= \partial_{\lambda=0} \, L_t(\boldsymbol{\psi}_\lambda(\mathbf{v}_t)) \\ &= \partial_{\lambda=0} \, \langle d_{\mathrm{F}} L_t(\boldsymbol{\psi}_\lambda(\mathbf{v}_t)), \boldsymbol{\psi}_\lambda(\mathbf{v}_t) \rangle - d_{\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t)} E_t(\mathbf{v}_t)\,. \end{aligned}$$

The differential law of dynamics states that the time-rate of increase of the virtual power of the momentum along the trajectory is equal to the rate of variation of the Lagrangian along any flow whose velocity at the actual configuration fulfils the rigidity property. In the authors' knowledge, the differential and jump laws of dynamics in a non-linear configuration manifold contributed above, are not quoted in the literature. They provide the most general formulation of the rules of dynamics in terms of the Lagrangian.

Remark 1. In expressing the differential law of dynamics, it is compelling to assign the flows $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ at least in a neighborhood of $\boldsymbol{\pi}(\mathbf{v}_t) \in \boldsymbol{\gamma}$ and not just the initial velocity $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))$ at the actual configuration $\boldsymbol{\pi}(\mathbf{v}_t) \in \boldsymbol{\gamma}$. By tensoriality, the flows $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ leading to the same value of $\mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}_t) \in \mathbb{T}_{\mathbf{v}_t}\mathbb{T}\mathbb{C}$ are equivalent.

Remark 2. The new statement of Hamilton's principle of proposition 6, which takes into account arbitrary variations, permits a direct derivation of the differential and jump law of dynamics. Indeed, by applying the fundamental theorem of integral calculus, the principle may be rewritten as

$$\int_I \partial_{\lambda=0} \, \mathrm{L}_t(\boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t) \, dt = \int_{\mathcal{T}(I)} \partial_{\tau=t} \, \langle d_{\mathrm{F}} \mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau)) \rangle \, dt\,.$$

Then, the arbitrarity of the flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ and the piecewise continuity of the integrands, yield the result.

Remark 3. In the variational expression of the law of dynamics the test fields $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(\boldsymbol{\gamma}\,;\mathbb{TC})$ are assumed to be infinitesimal isometries at each point of the trajectory $\boldsymbol{\gamma}$. This rigidity constraint has a basic physical meaning since it reveals that the dynamical equilibrium at a given configuration is independent of the material properties of the body.

The evaluation of the equilibrium configuration requires in general to take into account the constitutive properties of the material and hence to get rid of the rigidity constraint. This task can be accomplished in complete generality by the method of LAGRANGE multipliers, which, in continuum mechanics, provide the stress field in the body [14].

Remark 4. In analogy to the abstract result of proposition 4, the general expression of the law of dynamics implies, as a trivial corollary, a statement which extends to continuum dynamics EMMY NOETHER's theorem as formulated in [1], [7], [9] in the context of rigid-body dinamics and finite dimensional configuration spaces. Indeed from the law of dynamics we infer that

$$\partial_{\lambda=0}\,\mathrm{L}_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t) = 0 \quad \Longrightarrow \quad \partial_{\tau=t}\,\langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle = 0\,,$$

while NOETHER's theorem consists in the weaker statement:

$$\mathrm{L}_t(\boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t) = \mathrm{L}_t(\mathbf{v}_t) \quad \Longrightarrow \quad \partial_{\tau=t}\,\langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle = 0\,,$$

for all flows $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ whose velocity $\mathbf{v}_{\boldsymbol{\varphi}} := \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda$ is an infinitesimal isometry at each point of $\boldsymbol{\gamma}$.

7 Dynamics in a manifold with a connection

Let us assume that the configuration manifold $\mathbb{C}$ be endowed with an affine connection ∇ and with the associated parallel transport. We denote by $\mathbf{c}_{\tau,t}\!\Uparrow$ the parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1(I\,;\mathbb{C})$ from the point $\mathbf{c}(t) \in \mathbb{C}$ to the point $\mathbf{c}(\tau) \in \mathbb{C}$, setting $\mathbf{c}_{t,\tau}\!\Downarrow\, := \mathbf{c}_{\tau,t}\!\Uparrow$. The covariant derivative of a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{TC})$ is expressed in terms of parallel transport as:

$$\nabla_{\dot{\mathbf{c}}_t}\mathbf{v} = \partial_{\tau=t}\,\mathbf{c}_{\tau,t}\!\Downarrow\mathbf{v}(\mathbf{c}(\tau))\,.$$

The parallel transport of a covector field $\boldsymbol{\omega} \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{T}^*\mathbb{C})$ is defined so that the duality-pairing be invariant:

$$\langle\, \mathbf{c}_{\tau,t}\!\Uparrow\boldsymbol{\omega}(\mathbf{c}(t)), \mathbf{v}(\mathbf{c}(\tau))\,\rangle = \langle\, \boldsymbol{\omega}(\mathbf{c}(t)), \mathbf{c}_{\tau,t}\!\Downarrow\mathbf{v}(\mathbf{c}(\tau))\,\rangle\,, \quad \forall\, \mathbf{v}(\mathbf{c}(\tau)) \in \mathbb{T}_{\mathbf{c}(\tau)}\mathbb{C}\,.$$

Accordingly, the covariant derivative of a covector field $\boldsymbol{\omega} \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{T}^*\mathbb{C})$ is given by

$$\begin{aligned}\langle\, \nabla_{\dot{\mathbf{c}}_t}\boldsymbol{\omega}, \mathbf{v}_t\,\rangle &= \partial_{\tau=t}\,\langle\, \mathbf{c}_{\tau,t}\!\Downarrow\boldsymbol{\omega}(\mathbf{c}(\tau)), \mathbf{v}_t\,\rangle \\ &= \partial_{\tau=t}\,\langle\, \boldsymbol{\omega}(\mathbf{c}(\tau)), \mathbf{c}_{\tau,t}\!\Uparrow\mathbf{v}_t\,\rangle\,, \quad \forall\, \mathbf{v}_t \in \mathbb{T}_{\mathbf{c}(t)}\mathbb{C}\,.\end{aligned}$$

Let us then consider the vector field $\mathbf{v}_\gamma \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{TC})$, extension of the velocity $\mathbf{v}_t := \partial_{t=0}\boldsymbol{\gamma}(t)$ of the trajectory by dragging it along the flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^2(\mathbb{C}\,;\mathbb{C})$: $\mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v}_t))) := \boldsymbol{\varphi}_\lambda\!\uparrow\!\mathbf{v}_t$ so that $\mathbf{v}_\gamma(\boldsymbol{\pi}(\mathbf{v}_t)) = \mathbf{v}_t$.

We observe that

$$\boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t = \boldsymbol{\varphi}_\lambda \!\Uparrow\! \boldsymbol{\varphi}_\lambda \!\Downarrow\! \boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t = \boldsymbol{\varphi}_\lambda \!\Uparrow\! \boldsymbol{\varphi}_\lambda \!\Downarrow\! \mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v}_t)))\,.$$

The *base derivative* of a functional $f \in \mathrm{C}^1(\mathbb{TC}\,;\mathcal{R})$ at $\mathbf{v} \in \mathbb{TC}$ along a vector $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v})) \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C}$ is then defined by:

$$\langle\, d_{\mathrm{B}} f(\mathbf{v}), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}))\,\rangle := \partial_{\lambda=0}\, f(\boldsymbol{\varphi}_\lambda \!\Uparrow\! \mathbf{v})\,,$$

where $\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda$. The definition is well-posed since the r.h.s. depends linearly on $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v})) \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C}$ for any fixed $\mathbf{v} \in \mathbb{TC}$. The base derivative provides the rate of change of the functional $f \in \mathrm{C}^1(\mathbb{TC}\,;\mathcal{R})$ when the base point $\boldsymbol{\pi}(\mathbf{v}) \in \mathbb{C}$ is dragged by the flow while the velocity $\mathbf{v} \in \mathbb{TC}$ is parallel transported along the flow. Let $\textsc{Tors} \in BL\,(\mathbb{TC}\,; BL\,(\mathbb{TC}\,;\mathbb{TC}))$ be the linear operator defined by

$$\textsc{Tors}(\mathbf{v})\cdot\mathbf{u} = \textsc{tors}(\mathbf{v},\mathbf{u})\,,\quad \forall\,\mathbf{u},\mathbf{v} \in \mathbb{T}_{\mathbf{x}}\mathbb{C}\,,$$

where $\textsc{tors}(\mathbf{v},\mathbf{u}) = \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v},\mathbf{u}] \in \mathbb{TC}$ is the evaluation of the torsion tensor in the connection ∇. The definition is well-posed by tensoriality of the torsion. The next result provides the special form taken by the differential law of dynamics when the configuration manifold is endowed with a connection.

Proposition 7 (Special form of the law of dynamics). *In a configuration manifold $\mathbb{C}$ endowed with an affine connection ∇ the differential law of dynamics takes the special form*

$$\langle\, \partial_{\tau=t}\, d_{\mathrm{F}} L_\tau(\mathbf{v}_t) + \nabla_{\mathbf{v}_t} d_{\mathrm{F}} L_t + d_{\mathrm{F}} L_t(\mathbf{v}_t)\textsc{Tors}(\mathbf{v}_t) - d_{\mathrm{B}} L_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle = 0\,,$$

for any virtual velocity field $\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\boldsymbol{\gamma}\,;\mathbb{TC})$ which is an admissible infinitesimal isometry at the configuration $\boldsymbol{\pi}(\mathbf{v}_t)$. If the connection ∇ is torsion-free, we get Lagrange*'s differential condition:*

$$\langle\, \partial_{\tau=t}\, d_{\mathrm{F}} L_\tau(\mathbf{v}_t) + \nabla_{\mathbf{v}_t} d_{\mathrm{F}} L_t - d_{\mathrm{B}} L_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle = 0\,.$$

Proof. Being $\partial_{\lambda=0}\,\mathrm{L}_t(\boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t) = \partial_{\lambda=0}\,\mathrm{L}_t(\boldsymbol{\varphi}_\lambda \!\Uparrow\! \boldsymbol{\varphi}_\lambda \!\Downarrow\! \mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v}_t))))$, by the Leibniz rule we get:

$$\partial_{\lambda=0}\,\mathrm{L}_t(\boldsymbol{\varphi}_\lambda \!\uparrow\! \mathbf{v}_t) = \partial_{\lambda=0}\,\mathrm{L}_t(\boldsymbol{\varphi}_\lambda \!\Uparrow\! \mathbf{v}_t) + \partial_{\lambda=0}\,\mathrm{L}_t(\boldsymbol{\varphi}_\lambda \!\Downarrow\! \mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\mathbf{v}_t)))\,,$$

and, by definition of the covariant derivative in terms of the parallel transport:

$$\nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}\mathbf{v}_\gamma(\boldsymbol{\pi}(\mathbf{v}_t)) := \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \!\Downarrow\! \mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v}_t)))\,,$$

being $\boldsymbol{\varphi}_\lambda \!\Downarrow\! \mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v}_t))) \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v}_t)}\mathbb{C}$, we have that

$$\partial_{\lambda=0}\,\mathrm{L}_t(\boldsymbol{\varphi}_\lambda \!\Downarrow\! \mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v}_t)))) = \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}\mathbf{v}_\gamma(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,.$$

Hence, by definition of the *base derivative* $d_{\mathrm{B}}\mathrm{L}_t(\mathbf{v}_t) \in \mathrm{C}^1(\mathbb{TC}\,;\mathbb{T}^*\mathbb{C})$, we get

$$\partial_{\lambda=0}\, \mathrm{L}_t(\mathbf{v}_\gamma(\boldsymbol{\varphi}_\lambda(\boldsymbol{\pi}(\mathbf{v}_t)))) = \langle\, d_{\mathrm{B}}\mathrm{L}_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle + \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}\mathbf{v}_\gamma(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,.$$

On the other hand, denoting by $\boldsymbol{\chi}_{\tau,t} := \boldsymbol{\gamma}_\tau \circ \boldsymbol{\gamma}_t^{-1} \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ the flow along the trajectory, we may write

$$\partial_{\tau=t}\,\langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle = \partial_{\tau=t}\,\langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \boldsymbol{\chi}_{\tau,t}\!\Uparrow \boldsymbol{\chi}_{\tau,t}\!\Downarrow \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle\,,$$

and applying the Leibniz rule:

$$\begin{aligned}\partial_{\tau=t}\,\langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle &= \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \partial_{\tau=t}\,\boldsymbol{\chi}_{\tau,t}\!\Downarrow \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle \\ &\quad + \partial_{\tau=t}\,\langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \boldsymbol{\chi}_{\tau,t}\!\Uparrow \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,.\end{aligned}$$

Finally, by definition of the covariant derivatives, we have:

$$\begin{aligned}\langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \partial_{\tau=t}\,\boldsymbol{\chi}_{\tau,t}\!\Downarrow \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle &= \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \nabla_{\mathbf{v}_\gamma}\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,, \\ \partial_{\tau=t}\,\langle\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau), \boldsymbol{\chi}_{\tau,t}\!\Uparrow \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle &= \langle\, \partial_{\tau=t}\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle \\ &\quad + \langle\, \nabla_{\mathbf{v}_t} d_{\mathrm{F}}\mathrm{L}_t, \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,.\end{aligned}$$

The general law of dynamics may then be written as

$$\begin{aligned}&\langle\, \partial_{\tau=t}\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_t) + \nabla_{\mathbf{v}_t} d_{\mathrm{F}}\mathrm{L}_t, \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle + \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \nabla_{\mathbf{v}_\gamma}\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle \\ &= \langle\, d_{\mathrm{B}}\mathrm{L}_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle + \langle\, d_{\mathrm{F}}\mathrm{L}_t(\mathbf{v}_t), \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}\mathbf{v}_\gamma(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,.\end{aligned}$$

From the expression: $\mathrm{TORS}(\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_\gamma) := \nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}\mathbf{v}_\gamma - \nabla_{\mathbf{v}_\gamma}\mathbf{v}_{\boldsymbol{\varphi}} - [\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_\gamma]$ being $[\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_\gamma] = 0$ by definition of the vector field $\mathbf{v}_\gamma \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{TC})$, we infer that $\nabla_{\mathbf{v}_{\boldsymbol{\varphi}}}\mathbf{v}_\gamma = \nabla_{\mathbf{v}_\gamma}\mathbf{v}_{\boldsymbol{\varphi}}$ if the torsion vanishes, and the statements are proven.

The standard connection associated with the distant parallel transport by translation in the model Banach space is torsion-free. Indeed any pair of vector fields generated by translation of a given pair of tangent vectors at a point, has a vanishing Lie bracket and vanishing covariant derivatives of each of them with respect to the other one. Since the torsion tensor is natural with respect to diffeomorphic transformations, the connection induced on the manifold $\mathbb{C}$ by a system of coordinates is still torsion-free. It follows that the differential law of dynamics takes Lagrange's form when written in coordinates. In rigid body dynamics, all virtual velocities being rigid, the differential law reduces to Lagrange's equation of dynamics:

$$\partial_{\tau=t}\, d_{\mathrm{F}}\mathrm{L}_\tau(\mathbf{v}_\tau) = d_{\mathrm{B}}\mathrm{L}_t(\mathbf{v}_t)\,.$$

The special forms of the differential law of dynamics hide however the direct implication of Emmy Noether's theorem.

8 Conclusions

About two centuries have passed away since Lagrange's and Hamilton's genial discoveries, while an extended formulation of Hamilton's action principle was at hand to provide a general form of the differential law of dynamics and of the jump conditions at singular points of the trajectory. One century after the publication of Emmy Noether's celebrated theorem, the extension of Hamilton's principle contributed in this paper reveals that Noether's theorem follows, as a direct corollary from the differential law of dynamics. In short Noether's theorem states that *if* $\mathbf{a} = 0$ *then* $\mathbf{b} = 0$, while the differential law of dynamics states that $\mathbf{a} = \mathbf{b}$. It should be noted that the translation of Hamilton's action principle in geometrical differential terms, gave us the idea of how to rewrite it and opened the way for the direct proof of the general law of dynamics. Remarkably, the proof of this more general result is definitely simpler than the classical one of the special Lagrange's law of dynamics in coordinates.

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