Journal of

Mechanics of Materials and Structures

NONLINEAR SHELL THEORY: A DUALITY APPROACH

Giovanni Romano, Carmen Sellitto and Raffaele Barretta

Volume 2, Nº 7 *September 2007*

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 2, No. 7, 2007

NONLINEAR SHELL THEORY: A DUALITY APPROACH

GIOVANNI ROMANO, CARMEN SELLITTO AND RAFFAELE BARRETTA

The nonlinear kinematics of thin shells is developed in full generality according to a duality approach in which kinematics plays the basic role in the definition of the model. The Kirchhoff–Love shell model is the central issue and is discussed in detail but shear deformable and polar models are also considered and critically reviewed. The analysis is developed with a coordinate-free approach which provides a direct geometrical picture of the shell model. The finite and tangent Green strains of the foliated continuum are explicitly expressed in terms of middle surface kinematics. The new expressions contributed here do not require the splitting of the velocity into parallel and normal components to the middle surface, and provide a computationally convenient context. Finite strain measures for the shell and their tangent and secant rates are analyzed and consistency and nonredundancy properties are discussed. The relations between the finite Green strain, its tangent and secant rates and the corresponding shell strains, are provided. The differential and boundary equilibrium equations of the shell are given in variational terms, both in unsplit and split form. A new expression of the boundary equilibrium equations is contributed and its mechanical soundness with respect to the classical one is emphasized. Equilibrium in a reference placement for the shell model is briefly discussed.

1. Introduction

The theory of shells is intimately connected with the geometry of Riemannian manifolds and elements of the relevant theory are summarized in the paper. The middle surface of a shell model is a two-dimensional submanifold of the three-dimensional euclidean space (the classical physical space). The treatment of large configuration changes of the Kirchhoff–Love shell is conveniently carried out by considering the shell as a foliation of the middle surface induced by a distance function. The description of the shell kinematics in terms of that of the middle surface is carried out by considering the nonlinear projection of the points of the foliation onto the middle surface and its derivative which is a symmetric linear operator expressible in terms of the linear projector onto the tangent plane and of the middle surface shape operator. Remarkably, this result is also useful in elastoplasticity for conveniently estimating the algorithmic tangent stiffness [Romano et al. 2005a; Romano et al. 2006]. The properties of consistency and nonredundancy to be fulfilled by a well defined strain measure are briefly illustrated to provide a basis for the analysis of shell models. The nonlinear analysis of shell deformation is developed in detail with a coordinate-free exposition. The treatment is founded on a variational approach in which kinematics plays a primary role while statics is introduced by duality. Stress fields are envisaged as Lagrange multipliers of the implicit representation of the rigidity constraint provided by a strain measure. In the classical three-dimensional continuum mechanics the Green strain measure is the most natural one. Accordingly,

Keywords: strain measures, nonlinear shells, kinematics, equilibrium.
The financial support of the Italian Ministry for the University and the Scientific Research (MIUR) is gratefully acknowledged.

a shell theory requires one to provide the expression of this strain measure and its rates in terms of the shell kinematical parameters. For the Kirchhoff–Love shell an expression is also provided in terms of the membrane and curvature strains of the middle surface and the relevant properties of consistency and nonredundancy are proven. The tangent deformation of the shell is then investigated to provide explicit expressions of the membrane and curvature tangent strains in terms of the velocity field of the shell middle surface, and their relation to the tangent Green strain is provided. The presentation is nontraditional since it makes no reference to coordinate systems and does not require one to split the velocity field into tangent and normal components to the middle surface. The equilibrium of membrane and flexural shells is analyzed in variational terms by relying on the virtual work theorem and on the divergence theorem on a surface, and the relevant differential and boundary conditions are provided. New results concerning both finite transformations and rate kinematics of the shell are contributed and a new expression of the equilibrium boundary condition is shown to have a sound mechanical interpretation. In fact, for nonflat shells, the classical expression of the boundary conditions does not properly take into account the rotation rate of the transversal fibers. The models of shear deformable shells and of polar shells, often adopted in computational mechanics, are illustrated and critically discussed.

2. Distance function and shape operator

A differentiable manifold $\mathbb{M}$ is a set locally diffeomorphic by C^k-charts to an open set of a model Banach space E. The tangent space $\mathbb{T}_{\mathbb{M}}(\mathbf{x})$ at a point $\mathbf{x} \in \mathbb{M}$ is the linear space of tangent vectors $\{\mathbf{x}, \mathbf{v}\} : C^k(U_{\mathbf{x}}; \mathcal{R}) \mapsto C^{k-1}(U_{\mathbf{x}}; \mathcal{R})$ where $C^k(U_{\mathbf{x}}; \mathcal{R})$ is the germ of scalar functions which are k-times continuously differentiable in a neighborhood $U_{\mathbf{x}}$ of $\mathbf{x} \in \mathbb{M}$. Tangent vectors at a point are uniquely defined by requiring that they fulfill the formal properties of a point-derivation:

$$\begin{cases} (\mathbf{v}_1 + \mathbf{v}_2)(f) = \mathbf{v}_1(f) + \mathbf{v}_2(f), & \text{additivity,} \\ \mathbf{v}(af) = a\mathbf{v}(f), \quad a \in \mathcal{R}, & \text{homogeneity,} \\ \mathbf{v}(fg) = \mathbf{v}(f)g + f(\mathbf{v}(g)), & \end{cases} \quad \begin{matrix} \Big\} \ \mathcal{R}\text{-linearity,} \\ \\ \text{Leibniz rule,} \end{matrix}$$

where $f \in C^k(U_{\mathbf{x}}; \mathcal{R})$. The *tangent bundle* $\mathbb{T}_{\mathbb{M}}$ of $\mathbb{M}$ is the disjoint union of the pairs $\{\mathbf{x}, \mathbb{T}_{\mathbb{M}}(\mathbf{x})\}$ with $\mathbf{x} \in \mathbb{M}$. An element $\{\mathbf{x}, \mathbf{v}\} \in \{\mathbf{x}, \mathbb{T}_{\mathbb{M}}(\mathbf{x})\}$ is said to be a tangent vector applied at the base point $\mathbf{x} \in \mathbb{M}$. We denote by $\mathbb{T}_{\mathbb{M}}(\mathcal{P}) \subseteq \mathbb{T}_{\mathbb{M}}$ the disjoint union of the pairs $\{\mathbf{x}, \mathbb{T}_{\mathbb{M}}(\mathbf{x})\}$ with $\mathbf{x} \in \mathcal{P} \subseteq \mathbb{M}$. With a look at the theory of shells, we recall some basic ingredients of surface theory. More generally, we may consider a $(n-1)$-dimensional hypersurface $\mathbb{M}$ embedded in an n-dimensional Riemannian manifold $\{\mathbb{S}, \mathbf{g}\}$. Since in shell theory the manifold $\{\mathbb{S}, \mathbf{g}\}$ is the euclidean space, the relevant connection will be denoted by the symbol ∂ of ordinary derivation. A distance function from $\mathbb{M} \subset \mathbb{S}$ is a twice continuously differentiable scalar valued map $r \in C^2(\mathbb{S}; \mathcal{R})$ such that its gradient is a vector field with unitary norm: $\|\partial r(\mathbf{x})\| = 1$. To build a distance function we consider an open strip $U_{\mathbb{M}} \subset \mathbb{S}$ including $\mathbb{M}$ whose thickness is so small that every point $\mathbf{x} \in U_{\mathbb{M}}$ can be $\mathbf{g}$-orthogonally projected in an unique fashion onto a point $\mathbf{P}(\mathbf{x}) \in \mathbb{M}$. Denoting by $\mathbf{n}(\mathbf{P}(\mathbf{x}))$ the normal versor to $\mathbb{M}$ at $\mathbf{P}(\mathbf{x}) \in \mathbb{M}$, and by $\mathbf{n}(\mathbf{x})$ the normal versor at $\mathbf{x} \in U_{\mathbb{M}}$ to the parallel hypersurface $\mathbb{M}^r$ to $\mathbb{M}$ passing through the point $\mathbf{x} \in U_{\mathbb{M}}$, we have $\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{P}(\mathbf{x}))$ and hence

$$\mathbf{x} = \mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{n}(\mathbf{P}(\mathbf{x})).$$

Given $\mathbf{n}(\mathbf{x}) = \partial r(\mathbf{x})$ for all $\mathbf{x} \in U_{\mathbb{M}}$, the hessian $\partial^2 r(\mathbf{x})$ provides the variation of the normal to the hypersurface $\mathbb{M}^r$ through $\mathbf{x} \in U_{\mathbb{M}}$ and is named the shape operator of $\mathbb{M}^r$. Two basic properties are proven in the next Lemmas.

Lemma 2.1. The shape operator $\mathbf{S}(\mathbf{x}) := \partial \mathbf{n}(\mathbf{x}) = \partial^2 r(\mathbf{x})$ is symmetric.

Proof. The symmetry of $\mathbf{S}(\mathbf{x}) = \partial^2 r(\mathbf{x})$ is a direct consequence of the symmetry of the Riemannian connection of $\{\mathbb{S}, \mathbf{g}\}$. □

Lemma 2.2. At any point $\mathbf{x} \in \mathbb{M}^r$ the normal versor $\mathbf{n}(\mathbf{x}) \in \mathbb{T}_{\mathbb{S}}(\mathbf{x})$ belongs to the kernel of the shape operator $\mathbf{S}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{S}}(\mathbf{x}))$:

$$\mathbf{S}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \partial^2 r(\mathbf{x})\mathbf{n}(\mathbf{x}) = 0,$$

hence $\operatorname{Im} \mathbf{S}(\mathbf{x}) = (\operatorname{Ker} \mathbf{S}(\mathbf{x}))^{\perp} \subseteq \mathbb{T}_{\mathbb{M}^r}(\mathbf{x})$.

Proof. By the symmetry of $\mathbf{S}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{S}}(\mathbf{x}))$ we have

$$\mathbf{g}(\mathbf{S}\mathbf{n}, \mathbf{h}) = \mathbf{g}(\mathbf{S}\mathbf{h}, \mathbf{n}) = \mathbf{g}(\partial_{\mathbf{h}}\mathbf{n}, \mathbf{n}) = \frac{1}{2}\partial_{\mathbf{h}}\mathbf{g}(\mathbf{n}, \mathbf{n}) = 0,$$

for any $\mathbf{h} \in \mathbb{T}_{\mathbb{S}}(\mathbf{x})$. The last statement holds since the kernel and the image of a symmetric operator are mutual orthogonal complements.

From Lemma 2.2 it follows that the shape operator $\mathbf{S} \in BL(\mathbb{T}_{\mathbb{S}}; \mathbb{T}_{\mathbb{S}})$ may be also considered as an operator $\mathbf{S} \in BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{S}})$ or as an operator $\mathbf{S}_{\mathbb{M}} \in BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}})$, named the Weingarten operator. □

3. Nonlinear projector

Let us provide here, for subsequent use, a formula concerning the derivative of the nonlinear projector $\mathbf{P} \in \mathrm{C}^1(U_{\mathbb{M}}; \mathbb{M})$ on the hypersurface $\mathbb{M}$.

Lemma 3.1. Let $\mathbf{P} \in \mathrm{C}^1(U_{\mathbb{M}}; \mathbb{M})$ be the nonlinear projector of the points of the foliation $U_{\mathbb{M}} \subset \mathbb{S}$ on the hypersurface $\mathbb{M}$. Its derivative at $\mathbf{x} \in \mathbb{M}^r \subset U_{\mathbb{M}}$ is a symmetric linear operator

$$\partial \mathbf{P}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{M}^r}(\mathbf{P}(\mathbf{x}))),$$

related to the linear projector $\mathbf{\Pi}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{M}^r}(\mathbf{x}))$ of the vectors $\mathbf{h} \in \mathbb{T}_{\mathbb{S}}(\mathbf{x})$ on the tangent plane at $\mathbf{x} \in \mathbb{M}^r$ to the r-level folium $\mathbb{M}^r$, by the formulas

$$\mathbf{\Pi}(\mathbf{x}) = \partial \mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{S}^r(\mathbf{x}) = (\mathbf{I} + r(\mathbf{x})\mathbf{S}(\mathbf{P}(\mathbf{x})))\partial \mathbf{P}(\mathbf{x}) = \mathbf{\Pi}(\mathbf{P}(\mathbf{x})),$$

where $\mathbf{S}(\mathbf{P}(\mathbf{x}))$ is the shape operator of $\mathbb{M}$ at the point $\mathbf{P}(\mathbf{x}) \in \mathbb{M}$ and $\mathbf{S}^r(\mathbf{x})$ is the shape operator of $\mathbb{M}^r$ at the point $\mathbf{x} \in \mathbb{M}^r$. Moreover

$$\operatorname{Ker} \partial \mathbf{P}(\mathbf{x}) = \operatorname{Ker} \mathbf{\Pi}(\mathbf{x}) = \operatorname{Span} \mathbf{n}(\mathbf{x}) \qquad \text{and} \qquad \partial \mathbf{P}(\mathbf{x}) = (\mathbf{I} + r(\mathbf{x})\mathbf{S}(\mathbf{P}(\mathbf{x})))^{-1}\mathbf{\Pi}(\mathbf{x}).$$

Proof. Taking the directional derivative along any $\mathbf{h} \in \mathbb{T}_{\mathbb{S}}(\mathbf{x})$ in the formula

$$\mathbf{x} = \mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{n}(\mathbf{P}(\mathbf{x}))$$

and observing that

$$\partial \mathbf{n}(\mathbf{x}) = \mathbf{S}^r(\mathbf{x}), \quad \partial(\mathbf{n} \circ \mathbf{P})(\mathbf{x}) = \partial \mathbf{n}(\mathbf{P}(\mathbf{x}))\partial \mathbf{P}(\mathbf{x}) = \mathbf{S}(\mathbf{P}(\mathbf{x}))\partial \mathbf{P}(\mathbf{x}),$$

we get $\mathbf{h} = \partial_{\mathbf{h}}\mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{S}^r(\mathbf{x})\mathbf{h} = \partial_{\mathbf{h}}\mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{S}(\mathbf{P}(\mathbf{x}))\partial_{\mathbf{h}}\mathbf{P}(\mathbf{x})$. The symmetry of $\partial\mathbf{P}(\mathbf{x})$ follows from that of $\mathbf{S}^r(\mathbf{x})$. Since

$$\partial_{\mathbf{n}}\mathbf{P}(\mathbf{x}) = 0, \qquad \mathbf{S}^r(\mathbf{x})\mathbf{n} = 0, \qquad \mathbf{S}(\mathbf{P}(\mathbf{x}))\mathbf{n} = 0,$$

we infer that $\mathbf{n}$ is in the kernel of the symmetric linear operator

$$\partial\mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{S}^r(\mathbf{x}) = (\mathbf{I} + r(\mathbf{x})\mathbf{S}(\mathbf{P}(\mathbf{x})))\partial\mathbf{P}(\mathbf{x}).$$

By projecting on $\mathbb{T}_{\mathbb{M}^r}(\mathbf{x})$, we get the first result. To get the last formula we remark that the linear projector $\mathbf{\Pi}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{S}}(\mathbf{x}))$ is symmetric since $\mathbf{g}(\mathbf{\Pi a}, \mathbf{b}) = \mathbf{g}(\mathbf{\Pi a}, \mathbf{\Pi b})$, for all $\mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbb{S}}(\mathbf{x})$ [Yosida 1974]. Moreover, the operator $\mathbf{I} + r(\mathbf{x})\mathbf{S}^r(\mathbf{x})$ is symmetric and positive definite, and hence invertible, due to the suitably small value of the thickness of the shell ensuring that the nonlinear projector $\mathbf{P} \in C^1(U_{\mathbb{M}}; \mathbb{M})$ is well defined. □

Remark 3.1. The ranges of the operators $\mathbf{\Pi}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{S}}(\mathbf{x}))$ and $\partial\mathbf{P}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{S}}(\mathbf{P}(\mathbf{x})))$ at $\mathbf{x} \in \mathbb{M}^r$ are $\mathbb{T}_{\mathbb{M}^r}(\mathbf{x})$ and $\mathbb{T}_{\mathbb{M}}(\mathbf{P}(\mathbf{x}))$. In a euclidean space the linear subspaces $\mathbb{T}_{\mathbb{S}}(\mathbf{x})$ and $\mathbb{T}_{\mathbb{S}}(\mathbf{P}(\mathbf{x}))$ are identified by means of a translation. Accordingly, the subspaces $\mathbb{T}_{\mathbb{M}^r}(\mathbf{x})$ and $\mathbb{T}_{\mathbb{M}}(\mathbf{P}(\mathbf{x}))$ are also identified and considered as subspaces of the linear space $\mathbb{T}_{\mathbb{S}}(\mathbf{x})$.

4. First and second fundamental forms

The first fundamental form is the twice covariant tensor field $\mathbf{g}_{\mathbb{M}} \in BL(\mathbb{T}^2_{\mathbb{M}}; \mathcal{R})$ defined on $\mathbb{M}$ as the restriction of the metric $\mathbf{g} \in BL(\mathbb{T}^2_{\mathbb{S}}; \mathcal{R})$ of $\{\mathbb{S}, \mathbf{g}\}$ to the vectors of the tangent spaces $\mathbb{T}_{\mathbb{M}}(\mathbf{x})$. The pair $\{\mathbb{M}, \mathbf{g}_{\mathbb{M}}\}$ is a Riemannian manifold. The second fundamental form on $\mathbb{M}$ is the twice covariant tensor field $\mathbf{s}_{\mathbb{M}} \in BL(\mathbb{T}^2_{\mathbb{M}}; \mathcal{R})$ defined by

$$\mathbf{s}_{\mathbb{M}}(\mathbf{a}, \mathbf{b}) = \mathbf{g}_{\mathbb{M}}(\mathbf{S}_{\mathbb{M}}\mathbf{a}, \mathbf{b}), \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbb{M}}.$$

Lemma 4.1. The Weingarten operator $\mathbf{S}_{\mathbb{M}} \in BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}})$ meets the identity

$$\mathbf{g}_{\mathbb{M}}(\mathbf{S}_{\mathbb{M}}\mathbf{a}, \mathbf{b}) = -\mathbf{g}(\mathbf{n}, \partial_{\mathbf{a}}\mathbf{b}), \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbb{M}}.$$

Proof. For any vector field $\mathbf{b} \in C^1(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$ the inner product $\mathbf{g}(\mathbf{n}, \mathbf{b})$ vanishes identically on $\mathbb{M}$, so that $0 = \partial_{\mathbf{a}}\mathbf{g}(\mathbf{n}, \mathbf{b}) = \mathbf{g}_{\mathbb{M}}(\mathbf{S}_{\mathbb{M}}\mathbf{a}, \mathbf{b}) + \mathbf{g}(\mathbf{n}, \partial_{\mathbf{a}}\mathbf{b})$, for all $\mathbf{a} \in \mathbb{T}_{\mathbb{M}}$. □

Remark 4.1. The identity in Lemma 4.1 is often taken as the definition of the Weingarten operator. It is important to highlight the tensoriality property of the bilinear form $\mathbf{g}(\mathbf{n}, \partial_{\mathbf{a}}\mathbf{b})$ in spite of the apparent dependence of the derivative $\partial_{\mathbf{a}}\mathbf{b}$ on the local behavior of the field $\mathbf{b} \in C^1(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$.

5. Strain measures and constrained continua

As a premise to the discussion of shell models, we present here some general considerations that will be referred to in Section 13. A detailed account may be found in [Romano et al. 2002]. Motions of a continuous body are mathematically described by flows of a submanifold $\mathbb{B}$ in a finite dimensional ambient manifold $\mathbb{S}$, whose points are the kinematical parameters. Vector fields $\mathbf{v} \in C^1(\mathbb{B}; \mathbb{T}_{\mathbb{S}})$, defined on the body-submanifold and tangent to the ambient manifold, are called virtual displacements. The basic issue in formulating a structural model is the statement of what is understood for a rigid transformation of the body in the ambient manifold, when passing from any placement to another one. Rigidity is an essential issue in defining the elastic response of materials and in formulating the equilibrium conditions. Rigid transformations form a group under map composition. Once the group of rigid transformations has been defined, one may choose among several alternative ways to provide an implicit description of the rigidity group as a submanifold of the manifold of all configuration-changes. This description is accomplished by choosing a strain measure, that is a suitable differential operator which associates, to any configuration-change of the body from any given placement, a set of tensor-valued fields, defined on that placement, which vanish if and only if the configuration-change is a rigid transformation. Although there are many possible choices in defining a strain measure, definite rules must be fulfilled. A first rule is expressed by a nonredundancy requirement, a condition which in optimization theory is dubbed *constraint qualification*. It is expressed by the property that no set of lower dimensional tensor-valued fields can do the job in providing an implicit description of the rigidity group. A further *consistency* requirement states that a strain measure must meet a pseudo-additivity property as a function of the configuration-changes. To state this property let us consider a flow $\boldsymbol{\psi}_{\tau,t} \in \mathrm{C}^1(\mathbb{B}; \mathbb{S})$ with $\boldsymbol{\psi}_{\tau,s} = \boldsymbol{\psi}_{\tau,t} \circ \boldsymbol{\psi}_{t,s}$ and $\boldsymbol{\psi}_{s,s}$ the identity map. Denoting by $\mathbf{G}(\boldsymbol{\psi}_{\tau,s})$ the strain measure corresponding to the configuration-change $\boldsymbol{\psi}_{\tau,s}$, the pseudo-additivity property is

$$\mathbf{G}(\boldsymbol{\psi}_{\tau,s}) = \mathbf{G}(\boldsymbol{\psi}_{t,s}) + \mathbf{A}(\boldsymbol{\psi}_{t,s}, \mathbf{G}(\boldsymbol{\psi}_{\tau,t})), \qquad \text{with } \mathbf{A}(\boldsymbol{\psi}_{t,s}, 0) = 0,$$

so that the strain measure value is unaffected by a post composition of the configuration-change by a rigid transformation. The operator $\mathbf{A}$ is assumed differentiable with respect to the second argument, with an invertible derivative so that a stress field conjugate to the strain measure can be defined. Indeed a stress field is the Lagrange multiplier associated with an implicit representation of the rigidity constraint. It follows that the equilibrium of the body at any placement can be expressed in terms of the virtual work of the stress field by the time derivative of the strain measure from that placement to the actual one [Romano et al. 2002]. The three-dimensional Cauchy continuum is the standard model for experimental tests on materials and elastic stress and strain computations on a body. To define lower-dimensional structural models providing suitable approximations, the first step is to choose a simplified kinematics by specifying which configuration changes are admissible. If the Green strain measure of the continuum from a reference natural placement can be expressed as a differentiable function of the strain measure of the approximate model (by means of a strain-relating map) the elastic potential of the model is defined as the composition between the elastic potential of the continuum and the strain-relating map. Once the elastic solution of the model is found, the corresponding Green strain measure of the continuum is available and hence the approximate Cauchy stress field is recovered by means of the constitutive relations via the one-to one correspondence with the referential stress field. Often engineers, in assigning the constitutive properties of approximate models, are guided by the results to be expected for the standard three-dimensional continuum. Accordingly, in shell theories, when the transversal shell fibers are considered elastically inextensible, the elastic relations of the linear isotropic material in a plane stress state are assumed and the Poisson effect in the transversal direction is taken into account only after the elastic solution in terms of stress is found. The following basic result of three-dimensional kinematics in the euclidean space ensures that, for any diffeomorphic transformation $\boldsymbol{\psi} \in \mathrm{C}^2(\mathbb{B}; \mathbb{S})$ of a three-dimensional continuum, the difference between the pull-back of the metric tensor and the metric tensor itself is a well defined strain measure. This measure is known as the Green strain and is defined by

$$2\mathbf{g}(\mathbf{G}_{\boldsymbol{\psi}}\mathbf{h}_1, \mathbf{h}_2) := \mathbf{g}((\partial_{\mathbf{h}_1}\boldsymbol{\psi}_{\varphi}), (\partial_{\mathbf{h}_2}\boldsymbol{\psi}_{\varphi})) - \mathbf{g}(\mathbf{h}_1, \mathbf{h}_2), \qquad \text{for all } \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{T}_{\mathbb{S}}.$$

A direct computation shows that the Green strain is consistent.

Theorem 5.1 (Homogeneous finite strains). Let $\mathbb{B} \subset \mathbb{S}$ be a connected open set and $\boldsymbol{\psi} \in \mathrm{C}^2(\mathbb{B};\mathbb{S})$ be a diffeomorphism such that the associated Green strain tensor field is constant on $\mathbb{B}$. Then the differential $\partial\boldsymbol{\psi} \in \mathrm{C}^1(\mathbb{B}; BL(\mathbb{T}_{\mathbb{S}}(\mathbb{B}); \mathbb{T}_{\mathbb{S}}(\boldsymbol{\psi}(\mathbb{B}))))$ is a constant field. If the Green strain field vanishes on $\mathbb{B}$, the differential $\partial\boldsymbol{\psi}$ is a constant isometry on $\mathbb{B}$.

Proof. Let $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h} \in \mathbb{T}_{\mathbb{S}}$ be arbitrary constant fields. By assumption:

$$\partial_{\mathbf{h}}\mathbf{g}(\partial_{\mathbf{h}_1}\boldsymbol{\psi}(\mathbf{x}), \partial_{\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x})) = \mathbf{g}(\partial^2_{\mathbf{h}\mathbf{h}_1}\boldsymbol{\psi}(\mathbf{x}), \partial_{\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x})) + \mathbf{g}(\partial^2_{\mathbf{h}\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x}), \partial_{\mathbf{h}_1}\boldsymbol{\psi}(\mathbf{x})) = 0.$$

By exchanging $\mathbf{h}_1$ with $\mathbf{h}$ and $\mathbf{h}_2$ with $\mathbf{h}$ we get the relations:

$$\begin{aligned}
&\mathbf{g}(\partial^2_{\mathbf{h}\mathbf{h}_1}\boldsymbol{\psi}(\mathbf{x}), \partial_{\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x})) + \mathbf{g}(\partial_{\mathbf{h}_1}\boldsymbol{\psi}(\mathbf{x}), \partial^2_{\mathbf{h}\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x})) = 0,\\
&\mathbf{g}(\partial^2_{\mathbf{h}_1\mathbf{h}}\boldsymbol{\psi}(\mathbf{x}), \partial_{\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x})) + \mathbf{g}(\partial_{\mathbf{h}}\boldsymbol{\psi}(\mathbf{x}), \partial^2_{\mathbf{h}_1\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x})) = 0,\\
&\mathbf{g}(\partial^2_{\mathbf{h}_2\mathbf{h}_1}\boldsymbol{\psi}(\mathbf{x}), \partial_{\mathbf{h}}\boldsymbol{\psi}(\mathbf{x})) + \mathbf{g}(\partial_{\mathbf{h}_1}\boldsymbol{\psi}(\mathbf{x}), \partial^2_{\mathbf{h}_2\mathbf{h}}\boldsymbol{\psi}(\mathbf{x})) = 0.
\end{aligned}$$

From these equalities, by the symmetry of the second derivative, we infer that $\mathbf{g}(\partial^2_{\mathbf{h}_1\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x}), \partial_{\mathbf{h}}\boldsymbol{\psi}(\mathbf{x})) = 0$. Then, by the nonsingularity of $\partial\boldsymbol{\psi}(\mathbf{x})$, we have $\partial^2_{\mathbf{h}_1\mathbf{h}_2}\boldsymbol{\psi}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{B}$ and by the connectedness of $\mathbb{B}$, we get that $\partial\boldsymbol{\psi}$ is a constant field. □

Let $\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S};\mathbb{S})$ be a flow in the space $\mathbb{S}$, with velocity field $\mathbf{v} \in C^k(\mathbb{S}; \mathbb{T}_{\mathbb{S}})$, that is, $\mathbf{v} = \partial_{t=s}\mathbf{Fl}^{\mathbf{v}}_{t,s}$. The tangent Green strain at s is the time-derivative at $t = s$ of the Green strain evaluated at the configuration change from $\mathbb{B}$ to $\boldsymbol{\psi}_{t,s}(\mathbb{B})$:

$$\mathbf{g}(\dot{\mathbf{G}}(\mathbf{v})\mathbf{h}_1, \mathbf{h}_2) := \frac{1}{2}\partial_{t=s}\mathbf{g}(\partial_{\mathbf{h}_1}\mathbf{Fl}^{\mathbf{v}}_{t,s}, \partial_{\mathbf{h}_2}\mathbf{Fl}^{\mathbf{v}}_{t,s}) = \mathbf{g}((\operatorname{sym}\partial\mathbf{v})\mathbf{h}_1, \mathbf{h}_2),$$

Then the following rate counterpart of Theorem 5.1 holds.

Theorem 5.2 (Homogeneous tangent strains). Let $\mathbb{B} \subset \mathbb{S}$ be a connected open set and $\mathbf{v} \in \mathrm{C}^2(\mathbb{B}; \mathbb{T}_{\mathbb{S}})$ be a tangent displacement such that the associated tangent Green strain tensor field is constant on $\mathbb{B}$. Then the gradient field $\partial\mathbf{v} \in BL(\mathbb{T}_{\mathbb{S}}(\mathbb{B}); \mathbb{T}_{\mathbb{S}}(\mathbb{B}))$ is constant on $\mathbb{B}$. In particular, if the tangent Green strain field vanishes in $\mathbb{B}$ then the tangent displacement is rigid:

$$\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}) = \mathbf{W}(\mathbf{x} - \mathbf{y}), \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{B},$$

where $\mathbf{W} \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{S}}(\mathbf{x}))$ is a skew-symmetric tensor.

Proof. Let $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h} \in \mathbb{T}_{\mathbb{S}}$ be arbitrary constant fields. By assumption:

$$\partial_{\mathbf{h}}(\mathbf{g}(\partial_{\mathbf{h}_1}\mathbf{v}(\mathbf{x}), \mathbf{h}_2) + \mathbf{g}(\partial_{\mathbf{h}_2}\mathbf{v}(\mathbf{x}), \mathbf{h}_1)) = \mathbf{g}(\partial^2_{\mathbf{h}\mathbf{h}_1}\mathbf{v}(\mathbf{x}), \mathbf{h}_2) + \mathbf{g}(\partial^2_{\mathbf{h}\mathbf{h}_2}\mathbf{v}(\mathbf{x}), \mathbf{h}_1) = 0.$$

By exchanging $\mathbf{h}_1$ with $\mathbf{h}$ and $\mathbf{h}_2$ with $\mathbf{h}$, we get two more relations, so that

$$\mathbf{g}(\partial^2_{\mathbf{h}\mathbf{h}_1}\mathbf{v}(\mathbf{x}), \mathbf{h}_2) + \mathbf{g}(\partial^2_{\mathbf{h}\mathbf{h}_2}\mathbf{v}(\mathbf{x}), \mathbf{h}_1) = 0,$$
$$\mathbf{g}(\partial^2_{\mathbf{h}_1\mathbf{h}}\mathbf{v}(\mathbf{x}), \mathbf{h}_2) + \mathbf{g}(\partial^2_{\mathbf{h}_1\mathbf{h}_2}\mathbf{v}(\mathbf{x}), \mathbf{h}) = 0,$$
$$\mathbf{g}(\partial^2_{\mathbf{h}_2\mathbf{h}_1}\mathbf{v}(\mathbf{x}), \mathbf{h}) + \mathbf{g}(\partial^2_{\mathbf{h}_2\mathbf{h}}\mathbf{v}(\mathbf{x}), \mathbf{h}_1) = 0.$$

From these equalities, by the symmetry of the second derivative, we infer that $\mathbf{g}(\partial^2_{\mathbf{h}_1\mathbf{h}_2}\mathbf{v}(\mathbf{x}), \mathbf{h}) = 0$ and hence that $\partial^2_{\mathbf{h}_1\mathbf{h}_2}\mathbf{v}(\mathbf{x}) = 0 \iff \partial^2\mathbf{v}(\mathbf{x}) = 0$, for all $\mathbf{x} \in \mathbb{B}$. By the connectedness of $\mathbb{B}$ we infer that $\partial\mathbf{v}$ is a constant field. If its symmetric part vanishes, we get $\partial\mathbf{v} = \mathbf{W}$, a skew-symmetric tensor. □

6. Shell models

We will primarily be interested in presenting a general, coordinate free exposition of the classical shell model without shear deformation, known in the literature as the Kirchhoff–Love shell model. This basic model provides also the guidelines for the investigation of some more sophisticated shell models proposed in the discouragingly vast literature on the topic. We will not even try to be exhaustive in this respect and will limit our attention to the most popular shell models for computational purposes. These models will be briefly illustrated, with critical remarks, in Section 13.2. Also, neither constitutive relations nor computational issues will be treated in detail, since attention is focused on the basic kinematical aspects of the models.

6.1. ***Kirchhoff–Love shell model.*** In the classical Kirchhoff–Love model, a shell is described as the foliation $U_{\mathbb{M}} \subset \mathbb{S}$ of a regular surface $\mathbb{M}$ in the three-dimensional euclidean physical space $\{\mathbb{S}, \mathbf{g}\}$. To this end we consider a distance function from it and define the folii of the shell as level sets for the distance function by assuming that the thickness of the shell is small enough to ensure that a distance function of any of its points from the middle surface $\mathbb{M}$ is well defined. Let $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}; \mathbb{S})$ be a one-to-one transformation of the middle surface $\mathbb{M}$ of the shell in the physical space $\mathbb{S}$ which is a diffeomorphism from the placement $\mathbb{M} \subset \mathbb{S}$ to the placement $\boldsymbol{\varphi}(\mathbb{M}) \subset \mathbb{S}$. The differential $\partial\boldsymbol{\varphi}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{M}}(\mathbf{x}); \mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}(\boldsymbol{\varphi}(\mathbf{x})))$ is a linear isomorphism:

$$\operatorname{Im}\partial\boldsymbol{\varphi}(\mathbf{x}) = \mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}(\boldsymbol{\varphi}(\mathbf{x})),$$
$$\operatorname{Ker}\partial\boldsymbol{\varphi}(\mathbf{x}) = \{0\} \in \mathbb{T}_{\mathbb{M}}(\mathbf{x}).$$

The $\mathbf{g}$-transposed operator $\partial\boldsymbol{\varphi}^T(\boldsymbol{\varphi}(\mathbf{x})) \in BL(\mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}(\boldsymbol{\varphi}(\mathbf{x})); \mathbb{T}_{\mathbb{M}}(\mathbf{x}))$ is a linear isomorphism too, with

$$\operatorname{Im}\partial\boldsymbol{\varphi}^T(\boldsymbol{\varphi}(\mathbf{x})) = (\operatorname{Ker}\partial\boldsymbol{\varphi}(\mathbf{x}))^{\perp} = \mathbb{T}_{\mathbb{M}}(\mathbf{x}),$$
$$\operatorname{Ker}\partial\boldsymbol{\varphi}^T(\mathbf{x}) = (\operatorname{Im}\partial\boldsymbol{\varphi}(\mathbf{x}))^{\perp} = \{0\} \in \mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}(\boldsymbol{\varphi}(\mathbf{x})).$$

Performing the decomposition $\mathbf{x} = \mathbf{P}(\mathbf{x}) + r(\mathbf{x})\mathbf{n}(\mathbf{P}(\mathbf{x}))$, with $\mathbf{x} \in U_{\mathbb{M}}$, the induced transformation

$$\boldsymbol{\psi}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(U_{\mathbb{M}}; \mathbb{S})$$

of the shell is defined by

$$\boldsymbol{\psi}_{\boldsymbol{\varphi}}(\mathbf{x}) := \boldsymbol{\varphi}(\mathbf{P}(\mathbf{x})) + r(\mathbf{x})\mathbf{n}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(\mathbf{P}(\mathbf{x}))), \qquad \text{for all } \mathbf{x} \in U_{\mathbb{M}},$$

where $\mathbf{n}_{\boldsymbol{\varphi}}(\boldsymbol{\varphi}(\mathbf{P}(\mathbf{x})))$ is the normal versor to $\boldsymbol{\varphi}(\mathbb{M})$ at $\boldsymbol{\varphi}(\mathbf{P}(\mathbf{x}))$. This is the basic kinematical assumption in the Kirchhoff–Love shell model: material fibers normal to the middle surface remain normal to it and do not change their length when transformations of the shell in the physical space take place. Given $\partial_{\mathbf{h}} r = \mathbf{g}(\mathbf{h}, \mathbf{n})$, the differential $\partial\boldsymbol{\psi}_{\boldsymbol{\varphi}}$ is given by:

$$\partial_{\mathbf{h}}\boldsymbol{\psi}_{\boldsymbol{\varphi}} = (\mathbf{I} + r\mathbf{S}_{\boldsymbol{\varphi}})\partial\boldsymbol{\varphi}\partial_{\mathbf{h}}\mathbf{P} + \mathbf{g}(\mathbf{h}, \mathbf{n})\mathbf{n}_{\boldsymbol{\varphi}} \circ \boldsymbol{\varphi} \circ \mathbf{P}, \qquad \text{for all } \mathbf{h} \in \mathbb{T}_{\mathbb{S}},$$

where $\mathbf{S}_{\boldsymbol{\varphi}} := \partial\mathbf{n}_{\boldsymbol{\varphi}} \in BL(\mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}; \mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})})$ is the shape operator of the deformed middle surface $\boldsymbol{\varphi}(\mathbb{M})$. The last two terms are mutually orthogonal. If $r = 0$ and $\mathbf{h} \in \mathbb{T}_{\mathbb{M}}$ by Lemma 3.1 it is $\partial_{\mathbf{h}}\boldsymbol{\psi}_{\boldsymbol{\varphi}} = \partial_{\mathbf{h}}\boldsymbol{\varphi}$.

Proposition 6.1. The Green's strain

$$\mathbf{G}_{\boldsymbol{\varphi}} := \frac{1}{2}(\boldsymbol{\psi}_{\boldsymbol{\varphi}}^T\boldsymbol{\psi}_{\boldsymbol{\varphi}} - \mathbf{I}),$$

is expressed, in terms of the transformation $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}; \mathbb{S})$ of the shell middle surface $\mathbb{M}$, by:

$$2\mathbf{g}(\mathbf{G}_{\boldsymbol{\varphi}}\mathbf{h}_1, \mathbf{h}_2) := \mathbf{g}((\partial_{\mathbf{h}_1}\boldsymbol{\psi}_{\boldsymbol{\varphi}}), (\partial_{\mathbf{h}_2}\boldsymbol{\psi}_{\boldsymbol{\varphi}})) - \mathbf{g}(\mathbf{h}_1, \mathbf{h}_2) = \mathbf{g}((\mathbf{I} + r\mathbf{S}_{\boldsymbol{\varphi}})^2\partial\boldsymbol{\varphi}\partial_{\mathbf{h}_1}\mathbf{P}, \partial\boldsymbol{\varphi}\partial_{\mathbf{h}_2}\mathbf{P}) - \mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1, \boldsymbol{\Pi}\mathbf{h}_2),$$

for all $\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{T}_{\mathbb{S}}(\mathbf{x})$, $\mathbf{x} \in U_{\mathbb{M}}$ or, in operator form:

$$\begin{aligned} 2\mathbf{G}_{\boldsymbol{\varphi}} &= \boldsymbol{\Pi}^T((\mathbf{I} + r\mathbf{S}_{\mathbb{M}})^{-1}\partial\boldsymbol{\varphi}^T(\mathbf{I} + r\mathbf{S}_{\boldsymbol{\varphi}})^2\partial\boldsymbol{\varphi}(\mathbf{I} + r\mathbf{S}_{\mathbb{M}})^{-1} - \mathbf{I})\boldsymbol{\Pi} \\ &= \partial\mathbf{P}^T(\partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi} + 2r\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} + r^2\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}^2\partial\boldsymbol{\varphi})\partial\mathbf{P} - \boldsymbol{\Pi}^T\boldsymbol{\Pi}, \end{aligned}$$

with $\partial\mathbf{P} = (\mathbf{I} + r\mathbf{S}_{\mathbb{M}})^{-1}\boldsymbol{\Pi} = \boldsymbol{\Pi} - r\mathbf{S}^r = \partial\mathbf{P}^T$ and $\boldsymbol{\Pi}^T = \boldsymbol{\Pi}$ and $\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}$.

Proof. From the expression of $\partial\boldsymbol{\psi}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(U_{\mathbb{M}}; BL(\mathbb{T}_{\mathbb{S}}; \mathbb{T}_{\mathbb{S}}))$ we have

$$\mathbf{g}(\partial_{\mathbf{h}_1}\boldsymbol{\psi}_{\boldsymbol{\varphi}}, \partial_{\mathbf{h}_2}\boldsymbol{\psi}_{\boldsymbol{\varphi}}) = \mathbf{g}((\mathbf{I} + r\mathbf{S}_{\boldsymbol{\varphi}})\partial\boldsymbol{\varphi}\partial_{\mathbf{h}_1}\mathbf{P}, (\mathbf{I} + r\mathbf{S}_{\boldsymbol{\varphi}})\partial\boldsymbol{\varphi}\partial_{\mathbf{h}_2}\mathbf{P}) + \mathbf{g}(\mathbf{h}_1^{\perp}, \mathbf{h}_2^{\perp}).$$

Subtracting $\mathbf{g}(\mathbf{h}_1, \mathbf{h}_2) = \mathbf{g}(\mathbf{h}_1^{||}, \mathbf{h}_2^{||}) + \mathbf{g}(\mathbf{h}_1^{\perp}, \mathbf{h}_2^{\perp})$, we get the result. □

Proposition 6.2. The Green strain field in the shell vanishes if and only if the following two conditions are fulfilled:

$$\partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi} = \mathbf{I}_{\mathbb{M}}, \qquad \partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} = \mathbf{S}_{\mathbb{M}},$$

where $\mathbf{I}_{\mathbb{M}}$ is the identity in $\mathbb{T}_{\mathbb{M}}$.

Proof. To prove the *if* part of the statement we observe that from the two assumptions it follows that

$$\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} = \mathbf{S}_{\mathbb{M}} = \partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi}\mathbf{S}_{\mathbb{M}} \iff \partial\boldsymbol{\varphi}^T(\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} - \partial\boldsymbol{\varphi}\mathbf{S}_{\mathbb{M}}) = 0.$$

Since $\mathrm{Ker}\,\partial\boldsymbol{\varphi}^T(\mathbf{x}) = \{0\}$, we get $\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} = \partial\boldsymbol{\varphi}\mathbf{S}_{\mathbb{M}}$. Then

$$\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}^2\partial\boldsymbol{\varphi} = \partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}\mathbf{S}_{\mathbb{M}} = \mathbf{S}_{\mathbb{M}}^2,$$

and, substituting in the expression for the Green strain field, we get

$$2\mathbf{G}_{\boldsymbol{\varphi}} = \boldsymbol{\Pi}^T((\mathbf{I} + r\mathbf{S})^{-1}(\mathbf{I} + r\mathbf{S})^2(\mathbf{I} + r\mathbf{S})^{-1})\boldsymbol{\Pi} - \boldsymbol{\Pi}^T\boldsymbol{\Pi} = 0.$$

The proof of the *only if* part of the statement is conveniently carried out by making recourse to Theorem 5.1 which ensures that, if $\mathbf{G}_{\boldsymbol{\varphi}} = 0$, the differential $\partial\boldsymbol{\psi}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(U_{\mathbb{M}}; BL(\mathbb{T}_{\mathbb{S}}; \mathbb{T}_{\mathbb{S}}))$ of the transformation

$\boldsymbol{\psi}_{\boldsymbol{\varphi}} \in \mathrm{C}^2(U_{\mathbb{M}};\mathbb{S})$ is a constant proper isometry in $\mathbb{S}$. Then $\partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi} = \mathbf{I}_{\mathbb{M}}$ and $\mathbf{n}_{\boldsymbol{\varphi}} = \partial\boldsymbol{\psi}_{\boldsymbol{\varphi}}\cdot\mathbf{n}$. Taking the derivative along any vector in $\mathbb{T}_{\mathbb{M}}$, and observing that $\partial\boldsymbol{\psi}_{\boldsymbol{\varphi}}$ is constant, we get the equality

$$\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} = \partial\mathbf{n}_{\boldsymbol{\varphi}} = \partial\boldsymbol{\psi}_{\boldsymbol{\varphi}}\cdot\partial\mathbf{n} = \partial\boldsymbol{\varphi}\partial\mathbf{n} = \partial\boldsymbol{\varphi}\mathbf{S}.$$

Then $\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} = \partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi}\mathbf{S} = \mathbf{S}$.

To provide an explicit expression of the tangent Green's strain in terms of the shell kinematics, we consider the middle surface $\mathbb{M}$ flying in the euclidean space $\mathbb{S}$ dragged by a flow $\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S};\mathbb{S})$ associated with the velocity field $\mathbf{v} \in \mathrm{C}^1(\mathbb{S};\mathbb{T}_{\mathbb{S}})$, defined by the differential equation: $\mathbf{v} = \partial_{t=s}\mathbf{Fl}^{\mathbf{v}}_{t,s}$. The time derivative of the normal versor to the middle surface $\mathbf{Fl}^{\mathbf{v}}_{t,s}(\mathbb{M})$ is defined by

$$\dot{\mathbf{n}} := \partial_{t=s}\mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}.$$

Since $\mathbf{g}(\mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}, \mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}) = 1$ at any time t we have

$$\partial_{t=s}\mathbf{g}(\mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}, \mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}) = 2\mathbf{g}(\dot{\mathbf{n}}, \mathbf{n}) = 0.$$

We denote by $\partial\mathbf{v}_{\mathbb{M}}\boldsymbol{\Pi}$ the derivative of a vector field $\mathbf{v}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{M};\mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ along vectors in $\mathbb{T}_{\mathbb{M}}$ and by $\nabla\mathbf{v}_{\mathbb{M}} = \boldsymbol{\Pi}\partial\mathbf{v}_{\mathbb{M}}\boldsymbol{\Pi}$ the covariant derivative in $\mathbb{M}$. □

Lemma 6.1. The material time derivative of the normal versor to a surface $\mathbb{M}$ dragged by the flow $\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S};\mathbb{S})$ is given by

$$\dot{\mathbf{n}} = -(\partial\mathbf{v}_{\mathbb{M}}\boldsymbol{\Pi})^T\mathbf{n} = \mathbf{S}_{\mathbb{M}}\mathbf{v}_{\mathbb{M}} - \nabla v_{\mathbf{n}},$$

where $\mathbf{v}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{M};\mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ is the velocity of the middle surface $\mathbb{M}$ dragged by the flow $\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S};\mathbb{S})$ and $\nabla v_{\mathbf{n}} \in \mathbb{T}_{\mathbb{M}}$ is the gradient of $v_{\mathbf{n}} = \mathbf{g}(\mathbf{v}_{\mathbb{M}}, \mathbf{n}) \in \mathrm{C}^1(\mathbb{M};\mathfrak{R})$.

Proof. The first equality holds since, for all $\mathbf{a} \in \mathbb{T}_{\mathbb{M}}(\mathbf{x})$:

$$\begin{aligned}0 = \partial_{t=s}\mathbf{g}(\mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}, \partial_{\mathbf{a}}\mathbf{Fl}^{\mathbf{v}}_{t,s}) &= \mathbf{g}(\dot{\mathbf{n}}, \mathbf{a}) + \mathbf{g}(\mathbf{n}, \partial_{t=s}\partial_{\mathbf{a}}\mathbf{Fl}^{\mathbf{v}}_{t,s})\\ &= \mathbf{g}(\dot{\mathbf{n}}, \mathbf{a}) + \mathbf{g}(\mathbf{n}, \partial_{\mathbf{a}}\partial_{t=s}\mathbf{Fl}^{\mathbf{v}}_{t,s})\\ &= \mathbf{g}(\dot{\mathbf{n}}, \mathbf{a}) + \mathbf{g}(\mathbf{n}, \partial_{\mathbf{a}}\mathbf{v}_{\mathbb{M}})\\ &= \mathbf{g}_{\mathbb{M}}(\dot{\mathbf{n}}, \mathbf{a}) + \mathbf{g}_{\mathbb{M}}((\partial\mathbf{v}_{\mathbb{M}}\boldsymbol{\Pi})^T\mathbf{n}, \mathbf{a}).\end{aligned}$$

The second equality follows from Leibniz rule:

$$\mathbf{g}(\mathbf{n}, \partial_{\mathbf{a}}\mathbf{v}_{\mathbb{M}}) = \partial_{\mathbf{a}}(\mathbf{g}(\mathbf{v}_{\mathbb{M}}, \mathbf{n})) - \mathbf{g}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}, \mathbf{S}_{\mathbb{M}}\mathbf{a}) = \mathbf{g}_{\mathbb{M}}(\nabla v_{\mathbf{n}}, \mathbf{a}) - \mathbf{g}_{\mathbb{M}}(\mathbf{S}_{\mathbb{M}}\mathbf{v}_{\mathbb{M}}, \mathbf{a}).$$

□

Theorem 6.1. If the middle surface $\mathbb{M}$ is dragged by a flow $\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S};\mathbb{S})$, the tangent Green strain in the shell $\dot{\mathbf{G}}(\mathbf{v}) := \partial_{t=s}\mathbf{G}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}} = \operatorname{sym}\partial\mathbf{v}$ in terms of the middle surface velocity $\mathbf{v}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{M};\mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ is given by:

$$\mathbf{g}(\dot{\mathbf{G}}(\mathbf{v})\mathbf{h}_1, \mathbf{h}_2) = \mathbf{g}_{\mathbb{M}}((\operatorname{sym}\nabla\mathbf{v}_{\mathbb{M}})\mathbf{a}, \mathbf{b}) + r\mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\dot{\mathbf{n}}, \mathbf{b}) + r\mathbf{g}_{\mathbb{M}}(\mathbf{S}_{\mathbb{M}}\mathbf{a}, \nabla_{\mathbf{b}}\mathbf{v}_{\mathbb{M}}) + r^2\mathbf{g}_{\mathbb{M}}(\operatorname{sym}(\mathbf{S}_{\mathbb{M}}\nabla\dot{\mathbf{n}})\mathbf{a}, \mathbf{b}),$$

where $\mathbf{a} := \partial_{\mathbf{h}_1}\mathbf{P}$, $\mathbf{b} := \partial_{\mathbf{h}_2}\mathbf{P}$ and $\dot{\mathbf{n}} = -(\partial\mathbf{v}_{\mathbb{M}}\boldsymbol{\Pi})^T\mathbf{n}$ by Lemma 6.1. In operator form:

$$\dot{\mathbf{G}}(\mathbf{v}) = \partial\mathbf{P}^T(\operatorname{sym}\nabla\mathbf{v}_{\mathbb{M}} + r(\nabla\dot{\mathbf{n}} + (\mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}_{\mathbb{M}})^T) + r^2\operatorname{sym}(\mathbf{S}_{\mathbb{M}}\nabla\dot{\mathbf{n}}))\partial\mathbf{P}.$$

Proof. A direct computation shows that

$$\partial_{t=s}\mathbf{g}(\partial_{\mathbf{a}}\mathbf{Fl}^{\mathbf{v}}_{t,s}, \partial_{\mathbf{b}}\mathbf{Fl}^{\mathbf{v}}_{t,s}) = \mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\mathbf{v}_{\mathbb{M}}, \mathbf{b}) + \mathbf{g}_{\mathbb{M}}(\mathbf{a}, \nabla_{\mathbf{b}}\mathbf{v}_{\mathbb{M}}),$$
$$\partial_{t=s}\mathbf{g}(\partial_{\mathbf{a}}\mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}, \partial_{\mathbf{b}}\mathbf{Fl}^{\mathbf{v}}_{t,s}) = \mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\dot{\mathbf{n}}, \mathbf{b}) + \mathbf{g}_{\mathbb{M}}(\mathbf{S}_{\mathbb{M}}\mathbf{a}, \nabla_{\mathbf{b}}\mathbf{v}_{\mathbb{M}}),$$
$$\partial_{t=s}\mathbf{g}(\partial_{\mathbf{a}}\mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}, \partial_{\mathbf{b}}\mathbf{n}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}) = \mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\dot{\mathbf{n}}, \mathbf{S}_{\mathbb{M}}\mathbf{b}) + \mathbf{g}_{\mathbb{M}}(\mathbf{S}_{\mathbb{M}}\mathbf{a}, \nabla_{\mathbf{b}}\dot{\mathbf{n}}),$$

and the result follows from the formula in Proposition 6.1. □

Let us explicitly observe that the operator $\nabla\dot{\mathbf{n}} + (\mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}_{\mathbb{M}})^T$ is symmetric despite of the single terms $\nabla\dot{\mathbf{n}}$ and $\mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}_{\mathbb{M}}$ being not symmetric, in general.

Remark 6.1. Let a spherical shell of mean radius R be subject to a homothetical transformation

$$\boldsymbol{\psi}_{\boldsymbol{\varphi}}(\mathbf{x}) = \alpha\mathbf{x},$$

with $\mathbf{x}$ originating from the center. Then $\mathbf{S} = R^{-1}\mathbf{I}_{\mathbb{M}}$, $\partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi} = \alpha^2\mathbf{I}_{\mathbb{M}}$, and $\mathbf{n}_{\boldsymbol{\varphi}} = \mathbf{n}$, so that:

$$\partial_{\mathbf{a}}\mathbf{n}_{\boldsymbol{\varphi}} = \partial_{\mathbf{a}}\mathbf{n},$$
$$\mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi}, \partial_{\mathbf{b}}\boldsymbol{\varphi}) = \alpha\,\mathbf{g}(\mathbf{S}\mathbf{a}, \mathbf{b}),$$
$$\mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi}, \mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{b}}\boldsymbol{\varphi}) = \mathbf{g}(\mathbf{S}\mathbf{a}, \mathbf{S}\mathbf{b}).$$

Given $(1 + rR^{-1})\partial\mathbf{P} = \boldsymbol{\Pi}$, the Green strain is given by the formula

$$2(1 + rR^{-1})^2\mathbf{G}_{\boldsymbol{\varphi}} = (\alpha^2 - 1 + \alpha(2rR^{-1} + r^2R^{-2})\boldsymbol{\Pi}.$$

If $rR^{-1} \ll 1$ we may set $\mathbf{G}_{\boldsymbol{\varphi}} \approx \frac{1}{2}(\alpha^2 - 1)\boldsymbol{\Pi}$.

7. Shell strains

The result of Proposition 6.2 shows that a strain measure of the shell, associated with a configuration change $\boldsymbol{\varphi} \in C^1(\mathbb{M}; \mathbb{S})$ of the middle surface, is provided by the pair of symmetric operators

$$\mathbf{E}_{\boldsymbol{\varphi}}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{M}}(\mathbf{x}); \mathbb{T}_{\mathbb{M}}(\mathbf{x})), \qquad \mathbf{K}_{\boldsymbol{\varphi}}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{M}}(\mathbf{x}); \mathbb{T}_{\mathbb{M}}(\mathbf{x})),$$

defined by

$$2\mathbf{E}_{\boldsymbol{\varphi}} := \partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi} - \mathbf{I}_{\mathbb{M}}, \qquad \mathbf{K}_{\boldsymbol{\varphi}} := \partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} - \mathbf{S}_{\mathbb{M}},$$

where $\partial\boldsymbol{\varphi}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{M}}(\mathbf{x}); \mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}(\mathbf{x}))$ and $\partial\boldsymbol{\varphi}^T(\mathbf{x}) \in BL(\mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}(\mathbf{x}); \mathbb{T}_{\mathbb{M}}(\mathbf{x}))$. The related pair $\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}} = \mathbf{g}\mathbf{E}_{\boldsymbol{\varphi}}$ and $\boldsymbol{\chi}_{\boldsymbol{\varphi}} = \mathbf{g}\mathbf{K}_{\boldsymbol{\varphi}}$ of symmetric tensor fields $\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}(\mathbf{x}) \in BL(\mathbb{T}^2_{\mathbb{M}}(\mathbf{x}); \mathcal{R})$, $\boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{x}) \in BL(\mathbb{T}^2_{\mathbb{M}}(\mathbf{x}); \mathcal{R})$ are given by:

$$2\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}(\mathbf{a}, \mathbf{b}) := \mathbf{g}(\partial_{\mathbf{a}}\boldsymbol{\varphi}, \partial_{\mathbf{b}}\boldsymbol{\varphi}) - \mathbf{g}(\mathbf{a}, \mathbf{b}),$$
$$\boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{a}, \mathbf{b}) := \mathbf{g}(\partial_{\mathbf{a}}(\mathbf{n}_{\boldsymbol{\varphi}} \circ \boldsymbol{\varphi}), \partial_{\mathbf{b}}\boldsymbol{\varphi}) - \mathbf{g}(\partial_{\mathbf{a}}\mathbf{n}, \mathbf{b}) = \mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi}, \partial_{\mathbf{b}}\boldsymbol{\varphi}) - \mathbf{g}(\mathbf{S}\mathbf{a}, \mathbf{b}),$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbb{M}}$. The symmetric tensor fields $\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}$ and $\boldsymbol{\chi}_{\boldsymbol{\varphi}}$ are called the membrane and the curvature strains and can be expressed in terms of pull-backs of the first and of the second fundamental forms of the middle surface: $2\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}} := \boldsymbol{\varphi}\!\downarrow\!\mathbf{g}_{\boldsymbol{\varphi}(\mathbb{M})} - \mathbf{g}_{\mathbb{M}}$ and $\boldsymbol{\chi}_{\boldsymbol{\varphi}} := \boldsymbol{\varphi}\!\downarrow\!\mathbf{s}_{\boldsymbol{\varphi}(\mathbb{M})} - \mathbf{s}_{\mathbb{M}}$.

By Proposition 6.2, the shell strain fields vanish on $\mathbb{M}$ if and only if the configuration change $\boldsymbol{\psi}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(U_{\mathbb{M}};\mathbb{S})$ is a rigid transformation of the shell. Consistency of the shell strain measure follows from the relations:

$$\begin{aligned}\boldsymbol{\varepsilon}_{\boldsymbol{\psi}\circ\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}) &= \boldsymbol{\varepsilon}_{\boldsymbol{\psi}}(\partial_{\mathbf{a}}\boldsymbol{\varphi},\partial_{\mathbf{b}}\boldsymbol{\varphi}) + \boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}),\\ \boldsymbol{\chi}_{\boldsymbol{\psi}\circ\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}) &= \boldsymbol{\chi}_{\boldsymbol{\psi}}(\partial_{\mathbf{a}}\boldsymbol{\varphi},\partial_{\mathbf{b}}\boldsymbol{\varphi}) + \boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}).\end{aligned}$$

Theorem 7.1. The Green finite strain measure in the shell is expressed in terms of the membrane and the curvature strains of the middle surface and of the shape operators by:

$$\mathbf{g}(\mathbf{G}_{\boldsymbol{\varphi}}\mathbf{h}_1,\mathbf{h}_2) = \boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}) + r\,\boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}) + \frac{1}{2}r^2\boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{S}\mathbf{b}) + \frac{1}{2}r^2\mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi},(\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}-\partial\boldsymbol{\varphi}\mathbf{S})\mathbf{b}),$$

with $\mathbf{a} := \partial_{\mathbf{h}_1}\mathbf{P}$, $\mathbf{b} := \partial_{\mathbf{h}_2}\mathbf{P}$. In operator form, we have

$$\mathbf{G}_{\boldsymbol{\varphi}} = \partial\mathbf{P}^T(\mathbf{E}_{\boldsymbol{\varphi}} + r\mathbf{K}_{\boldsymbol{\varphi}} + \tfrac{1}{2}r^2(\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S})(\mathbf{I}_{\mathbb{M}}+2\mathbf{E}_{\boldsymbol{\varphi}})^{-1}(\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S}) - \tfrac{1}{2}r^2\mathbf{S}^2)\partial\mathbf{P}.$$

Note that $\mathbf{G}_{\boldsymbol{\varphi}}$ and $\partial\mathbf{P}$ are evaluated at $\mathbf{x} \in \mathbb{M}^r$ while all other tensors are evaluated at $\mathbf{P}(\mathbf{x}) \in \mathbb{M}$.

Proof. By Proposition 6.1 we have $2\mathbf{g}(\mathbf{G}_{\boldsymbol{\varphi}}\mathbf{h}_1,\mathbf{h}_2) = \mathcal{A} + 2r\mathcal{B} + r^2\mathcal{C}$ with

$$\begin{aligned}\mathcal{A} &= \mathbf{g}(\partial_{\mathbf{a}}\boldsymbol{\varphi},\partial_{\mathbf{b}}\boldsymbol{\varphi}) - \mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1,\boldsymbol{\Pi}\mathbf{h}_2),\\ \mathcal{B} &= \mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi},\partial_{\mathbf{b}}\boldsymbol{\varphi}),\\ \mathcal{C} &= \mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi},\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{b}}\boldsymbol{\varphi}).\end{aligned}$$

A little algebra shows that

$$\begin{aligned}\mathcal{A} &= 2\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}) + \mathbf{g}(\mathbf{a},\mathbf{b}) - \mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1,\boldsymbol{\Pi}\mathbf{h}_2),\\ \mathcal{B} &= \boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}) + \mathbf{g}(\mathbf{S}\mathbf{a},\mathbf{b}),\\ \mathcal{C} &= \boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{S}\mathbf{b}) + \mathbf{g}(\mathbf{S}\mathbf{a},\mathbf{S}\mathbf{b}) + \mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi},(\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}-\partial\boldsymbol{\varphi}\mathbf{S})\mathbf{b}).\end{aligned}$$

The result then follows by observing that

$$\begin{aligned}\mathbf{g}(\mathbf{a},\mathbf{b}) + r\mathbf{g}(\mathbf{S}\mathbf{a},\mathbf{b}) &= \mathbf{g}((\mathbf{I}+r\mathbf{S})\mathbf{a},\mathbf{b}) = \mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1,\mathbf{b}),\\ r\mathbf{g}(\mathbf{S}\mathbf{a},\mathbf{b}) + r^2\mathbf{g}(\mathbf{S}\mathbf{a},\mathbf{S}\mathbf{b}) &= r\mathbf{g}((\mathbf{I}+r\mathbf{S})\mathbf{a},\mathbf{S}\mathbf{b}) = r\mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1,\mathbf{S}\mathbf{b}),\\ \mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1,\mathbf{b}) + r\mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1,\mathbf{S}\mathbf{b}) &= \mathbf{g}(\boldsymbol{\Pi}\mathbf{h}_1,\boldsymbol{\Pi}\mathbf{h}_2).\end{aligned}$$

To get the operator expression we write

$$\begin{aligned}\boldsymbol{\chi}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{S}\mathbf{b}) &= \mathbf{g}(\mathbf{K}_{\boldsymbol{\varphi}}\mathbf{a},\mathbf{S}\mathbf{b}) = \mathbf{g}(\mathbf{S}\mathbf{K}_{\boldsymbol{\varphi}}\mathbf{a},\mathbf{b}),\\ \mathbf{g}(\mathbf{S}_{\boldsymbol{\varphi}}\partial_{\mathbf{a}}\boldsymbol{\varphi},(\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}-\partial\boldsymbol{\varphi}\mathbf{S})\mathbf{b}) &= \mathbf{g}(\mathbf{a},\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}(\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}-\partial\boldsymbol{\varphi}\mathbf{S})\mathbf{b}).\end{aligned}$$

Then, given $\mathbf{S}_{\boldsymbol{\varphi}} = \partial\boldsymbol{\varphi}^{-T}(\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S})\partial\boldsymbol{\varphi}^{-1}$, we have

$$\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}(\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}-\partial\boldsymbol{\varphi}\mathbf{S}) = (\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S})\partial\boldsymbol{\varphi}^{-1}\partial\boldsymbol{\varphi}^{-T}(\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S}) - (\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S})\mathbf{S}.$$

Recalling that $\partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi} = 2\mathbf{E}_{\boldsymbol{\varphi}} + \mathbf{I}_{\mathbb{M}}$ and taking the transpose, we get

$$(\partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}(\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}-\partial\boldsymbol{\varphi}\mathbf{S}))^T = (\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S})(2\mathbf{E}_{\boldsymbol{\varphi}}+\mathbf{I}_{\mathbb{M}})^{-1}(\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S}) - \mathbf{S}(\mathbf{K}_{\boldsymbol{\varphi}}+\mathbf{S}).$$

Adding the two expressions, the terms in $\mathbf{SK}_{\varphi}$ cancel and we get the result. □

Remark 7.1. For an initially flat shell (a plate) the expression of the Green strain becomes

$$\mathbf{G}_{\varphi} = \mathbf{\Pi}^T(\mathbf{E}_{\varphi} + r\mathbf{K}_{\varphi} + \frac{1}{2}r^2\mathbf{K}_{\varphi}(\mathbf{I}_{\mathbb{M}} + 2\mathbf{E}_{\varphi})^{-1}\mathbf{K}_{\varphi})\mathbf{\Pi}.$$

If the extensional strain $\mathbf{E}_{\varphi}$ vanishes, then $\mathbf{G}_{\varphi} = \mathbf{\Pi}^T((\mathbf{I}_{\mathbb{M}} + r\mathbf{K}_{\varphi})^2 - \mathbf{I}_{\mathbb{M}})\mathbf{\Pi}$ and the ratio between final and initial lengths of the principal fibers has a linear variation along the thickness.

8. Tangent shell strains

The tangent membrane strain $\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) \in BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}})$, associated with a flow $\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S}; \mathbb{S})$, is defined by

$$\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) := \partial_{t=s}\mathbf{E}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}} = \frac{1}{2}\partial_{t=s}(\partial\mathbf{Fl}^{\mathbf{v}}_{t,s})^T\,\partial\mathbf{Fl}^{\mathbf{v}}_{t,s},$$

and the tangent curvature strain $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) \in BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}})$ is given by

$$\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) := \partial_{t=s}\mathbf{K}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}} = \partial_{t=s}(\partial\mathbf{Fl}^{\mathbf{v}}_{t,s})^T\,\mathbf{S}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}\,\partial\mathbf{Fl}^{\mathbf{v}}_{t,s}.$$

We have the following results.

Theorem 8.1 (Tangent membrane strain). The tangent membrane strain $\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) \in \mathrm{C}^{k-1}(\mathbb{M}; BL(\mathbb{T}_{\mathbb{S}}; \mathbb{T}_{\mathbb{S}}))$ is expressed in terms of the velocity field $\mathbf{v}_{\mathbb{M}} \in \mathrm{C}^k(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ of the middle surface dragged by the flow by the formula:

$$\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \mathrm{sym}(\mathbf{\Pi}\,\partial\mathbf{v}_{\mathbb{M}}\mathbf{\Pi}) = \mathrm{sym}(\nabla\mathbf{v}_{\mathbb{M}}).$$

Proof. The result follows from the equality:

$$\mathbf{g}(\partial_{\mathbf{a}}\mathbf{Fl}^{\mathbf{v}}_{t,s}, \partial_{\mathbf{b}}\mathbf{Fl}^{\mathbf{v}}_{t,s}) = \mathbf{g}(\partial_{\mathbf{a}}\mathbf{v}_{\mathbb{M}}, \mathbf{b}) + \mathbf{g}(\mathbf{a}, \partial_{\mathbf{b}}\mathbf{v}_{\mathbb{M}}), \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbb{M}}.$$

It is often expedient to decompose a vector field $\mathbf{v}_{\mathbb{M}} \in \mathbb{T}_{\mathbb{S}}(\mathbb{M})$ into its parallel and orthogonal components to the tangent plane to $\mathbb{M}$: $\mathbf{v}_{\mathbb{M}} = \mathbf{v}^{||}_{\mathbb{M}} + v_{\mathbf{n}}\mathbf{n}$, with $v_{\mathbf{n}} = \mathbf{g}(\mathbf{v}_{\mathbb{M}}, \mathbf{n})$ and $\mathbf{v}^{||}_{\mathbb{M}} = \mathbf{\Pi}\mathbf{v}_{\mathbb{M}} := \mathbf{v}_{\mathbb{M}} - v_{\mathbf{n}}\mathbf{n}$, so that $\mathbf{g}(\mathbf{v}^{||}_{\mathbb{M}}, \mathbf{n}) = 0$. □

Theorem 8.2 (Tangent membrane strain — second formula). Setting $\mathbf{v}_{\mathbb{M}} = \mathbf{v}^{||}_{\mathbb{M}} + v_{\mathbf{n}}\mathbf{n}$ with $\mathbf{v}^{||}_{\mathbb{M}} \in \mathbb{T}_{\mathbb{M}}$, the tangent membrane strain is given by

$$\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \mathrm{sym}(\nabla\mathbf{v}^{||}_{\mathbb{M}}) + v_{\mathbf{n}}\mathbf{S}_{\mathbb{M}}.$$

Proof. The formula follows from the equality:

$$\begin{aligned}\mathbf{g}(\partial_{\mathbf{a}}\mathbf{v}_{\mathbb{M}}, \mathbf{b}) &= \mathbf{g}(\partial_{\mathbf{a}}(\mathbf{v}^{||}_{\mathbb{M}} + v_{\mathbf{n}}\mathbf{n}), \mathbf{b})\\ &= \mathbf{g}(\partial_{\mathbf{a}}\mathbf{v}^{||}_{\mathbb{M}}, \mathbf{b}) + v_{\mathbf{n}}\mathbf{g}(\mathbf{Sa}, \mathbf{b}) + (\partial_{\mathbf{a}}v_{\mathbf{n}})\mathbf{g}(\mathbf{n}, \mathbf{b})\\ &= \mathbf{g}(\nabla_{\mathbf{a}}\mathbf{v}^{||}_{\mathbb{M}}, \mathbf{b}) + v_{\mathbf{n}}\mathbf{g}(\mathbf{Sa}, \mathbf{b}),\end{aligned}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbb{M}}$. □

Theorem 8.3 (Tangent curvature strain). The tangent curvature strain field $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) \in BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}})$ is given by the formula

$$\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \nabla\dot{\mathbf{n}} + (\mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}_{\mathbb{M}})^T = -\nabla((\partial\mathbf{v}_{\mathbb{M}}\mathbf{\Pi})^T\mathbf{n}) + (\mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}_{\mathbb{M}})^T.$$

Proof. The result follows from Theorem 6.1. □

Theorem 8.4 (Tangent curvature strain — second formula). The tangent curvature strain is expressed in terms of the normal and parallel components of the velocity field by the formula:

$$\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \nabla(\mathbf{S}_{\mathbb{M}}\mathbf{v}^{||}_{\mathbb{M}}) - \nabla^2 v_{\mathbf{n}} + (\mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}^{||}_{\mathbb{M}})^T + v_{\mathbf{n}}\mathbf{S}^2_{\mathbb{M}}.$$

Proof. Given $\dot{\mathbf{n}} = \mathbf{S}_{\mathbb{M}}\mathbf{v}_{\mathbb{M}} - \nabla v_{\mathbf{n}}$, we infer that $\nabla\dot{\mathbf{n}} = \nabla(\mathbf{S}_{\mathbb{M}}\mathbf{v}^{||}_{\mathbb{M}}) - \nabla^2 v_{\mathbf{n}}$. Moreover, since $\mathbf{S}\mathbf{n} = 0$, we have

$$\mathbf{S}\nabla\mathbf{v}_{\mathbb{M}} = \mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}^{||}_{\mathbb{M}} + \mathbf{S}\nabla(v_{\mathbf{n}}\mathbf{n}) = \mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}^{||}_{\mathbb{M}} + \mathbf{S}(\mathbf{n}\otimes\nabla v_{\mathbf{n}} + v_{\mathbf{n}}\nabla\mathbf{n}) = \mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}^{||}_{\mathbb{M}} + v_{\mathbf{n}}\mathbf{S}^2_{\mathbb{M}},$$

Substituting in the formula of Theorem 8.3 we get the result. □

Remark 8.1. At a planar point of the shell the shape operator $\mathbf{S}$ vanishes and we get the formula $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \nabla(\mathbf{S}_{\mathbb{M}}\mathbf{v}^{||}_{\mathbb{M}}) - \nabla^2 v_{\mathbf{n}}$. In a planar shell (a plate) the normal versor $\mathbf{n} \in \mathrm{C}^2(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ is a constant field and the shape operator $\mathbf{S}$ vanishes identically so that the general formula specializes into $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = -\nabla^2 v_{\mathbf{n}}$ and the expression of the linearized curvature strain tensor field of a plate, in terms of the transversal displacement field, is recovered.

Theorem 8.5 (Tangent Green strain). The tangent Green strain in the shell $\dot{\mathbf{G}}(\mathbf{v}) := \partial_{t=s}\mathbf{G}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}} = \operatorname{sym}\partial\mathbf{v}$, in terms of the tangent membrane and curvature strains, is expressed by:

$$\mathbf{g}(\dot{\mathbf{G}}(\mathbf{v})\mathbf{h}_1, \mathbf{h}_2) = \mathbf{g}(\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}})\mathbf{a}, \mathbf{b}) + r\mathbf{g}(\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}})\mathbf{a}, \mathbf{b}) + r^2(\mathbf{g}(\operatorname{sym}(\mathbf{S}\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}))\mathbf{a}, \mathbf{b}) - \mathbf{g}(\mathbf{S}\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}})\mathbf{S}\mathbf{a}, \mathbf{b}))$$

where $\mathbf{a} := \partial_{\mathbf{h}_1}\mathbf{P}$, $\mathbf{b} := \partial_{\mathbf{h}_2}\mathbf{P}$. In operator form we may write

$$\dot{\mathbf{G}}(\mathbf{v}) = \partial\mathbf{P}^T(\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) + r\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) + r^2\operatorname{sym}(\mathbf{S}\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) - \mathbf{S}\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}})\mathbf{S})\partial\mathbf{P}.$$

Proof. The result follows by computing the derivative $\partial_{t=s}\mathbf{G}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}}$ by the formula in Theorem 7.1 and observing that by the formula for the derivative of the inverse, it is

$$\partial_{t=s}(2\mathbf{E}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}} + \mathbf{I}_{\mathbb{M}})^{-1} = -2\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}).$$

A direct proof in components notation is provided in [Ciarlet 2000, Theorem 2.6-2]. □

Remark 8.2. If a spherical balloon of mean radius R is inflated we see that $v_{\mathbf{n}}$ is a constant field and $\mathbf{v}^{||} = 0$ identically. Hence $\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}) = v_{\mathbf{n}}\mathbf{S}$ and $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}) = v_{\mathbf{n}}\mathbf{S}^2$. The tangent Green strain is then given by the formula $(1 + rR^{-1})^2\dot{\mathbf{G}}_{\varphi}(\mathbf{v}) = v_{\mathbf{n}}(\mathbf{S} + r\mathbf{S}^2)$. Since $\mathbf{S} = R^{-1}\mathbf{\Pi}$, all material fibers tangent to a shell folium have the same elongation rate $v_{\mathbf{n}}R^{-1}(1 + rR^{-1})^{-1}$. If $r \ll R$ we may assume the elongation rate to be $\approx v_{\mathbf{n}}R^{-1}$.

Theorem 8.6. The tangent Green strain of the shell vanishes if and only if the tangent membrane and curvature strains of the middle surface vanish.

Proof. That $\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = 0$, $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = 0$ implies $\dot{\mathbf{G}}(\mathbf{v}) = 0$ follows directly from Theorem 8.5. To get the converse implication we set $r = 0$ to infer that $\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = 0$ and hence, by a continuity argument for $r \to 0$, conclude that $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = 0$. That $\dot{\mathbf{G}}(\mathbf{v}) = 0$ implies $\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = 0$, $\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = 0$ may be also deduced from Theorem 5.2. Indeed, if $\dot{\mathbf{G}}(\mathbf{v}) = 0$, we infer that

$$\mathbf{v}_{\mathbb{M}}(\mathbf{x}) = \mathbf{v}_0 + \mathbf{W}\mathbf{x} \quad \text{and} \quad \partial \mathbf{v}_{\mathbb{M}}(\mathbf{x}) = \mathbf{W}.$$

Then

$$\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \mathrm{sym}(\nabla \mathbf{v}_{\mathbb{M}}) = \mathrm{sym}(\mathbf{\Pi}^T \mathbf{W} \mathbf{\Pi}) = 0,$$

and also

$$\dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = -\nabla((\partial \mathbf{v}_{\mathbb{M}} \mathbf{\Pi})^T \mathbf{n}) + (\mathbf{S} \nabla \mathbf{v}_{\mathbb{M}})^T = -\nabla((\mathbf{W}\mathbf{\Pi})^T \mathbf{n}) + (\mathbf{S}\mathbf{W}\mathbf{\Pi}^T)^T = \mathbf{\Pi}\mathbf{W}\nabla\mathbf{n} - \mathbf{\Pi}\mathbf{W}\nabla\mathbf{n} = 0.$$

□

Remark 8.3. Let $\{\mathbf{e}_\alpha \in \mathbb{T}_{\mathbb{M}}, \alpha = 1, 2\}$ be base vectors of a coordinate system on $\mathbb{M}$. Recalling that the Christoffel symbols associated with the covariant derivative on $\mathbb{M}$ are defined by the relation

$$\nabla_{\mathbf{e}_\alpha} \mathbf{e}_\beta := \mathbf{\Pi} \partial_{\mathbf{e}_\alpha} \mathbf{e}_\beta = \Gamma^\gamma_{\alpha\beta} \mathbf{e}_\gamma,$$

and that the components of the second fundamental form are given by

$$S_{\alpha\beta} := \mathbf{g}(\partial_{\mathbf{e}_\alpha} \mathbf{n}, \mathbf{e}_\beta) = \mathbf{g}(\mathbf{S}\mathbf{e}_\alpha, \mathbf{e}_\beta) = -\mathbf{g}(\partial_{\mathbf{e}_\alpha} \mathbf{e}_\beta, \mathbf{n}),$$

we get the Gauss formula

$$\partial_{\mathbf{e}_\alpha} \mathbf{e}_\beta = \mathbf{g}(\partial_{\mathbf{e}_\alpha} \mathbf{e}_\beta, \mathbf{e}^\gamma)\mathbf{e}_\gamma + \mathbf{g}(\partial_{\mathbf{e}_\alpha} \mathbf{e}_\beta, \mathbf{n})\mathbf{n} = \Gamma^\gamma_{\alpha\beta} \mathbf{e}_\gamma - S_{\alpha\beta}\mathbf{n} = \nabla_{\mathbf{e}_\alpha} \mathbf{e}_\beta - S_{\alpha\beta}\mathbf{n}.$$

By these formulas we may put all the relations above in component form.

9. Divergence theorem on a manifold

We recall here some basic definitions and results for subsequent reference [Marsden and Hughes 1983; Romano et al. 2005b]. The exterior derivative of a $(n-1)$-form is uniquely defined by Stokes formula:

$$\int_{\mathbb{M}} d\boldsymbol{\omega} = \oint_{\partial\mathbb{M}} \boldsymbol{\omega}.$$

It states the equality of the integral of a differential $(n-1)$-form $\boldsymbol{\omega}$ on the boundary $\partial\mathbb{M}$, a $(n-1)$D manifold, to the integral of the exterior derivative $d\boldsymbol{\omega}$, which is a differential n-form, on the nD manifold $\mathbb{M}$. The divergence of a vector field $\mathbf{w} \in \mathrm{C}^k(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$ on a nD manifold $\mathbb{M}$ is defined, in terms of the exterior derivative, by the formula: $d(\boldsymbol{\mu}_{\mathbb{M}}\mathbf{w}) = (\mathrm{div}\mathbf{w})\boldsymbol{\mu}_{\mathbb{M}}$, where $\boldsymbol{\mu}_{\mathbb{M}}\mathbf{w}$ is the $(n-1)$-form obtained by evaluating the n-form $\boldsymbol{\mu}_{\mathbb{M}}$ with $\mathbf{w}$ as the first argument. Substituting in Stokes formula, we get the divergence theorem

$$\int_{\mathbb{M}} (\mathrm{div}\,\mathbf{w})\boldsymbol{\mu}_{\mathbb{M}} = \oint_{\partial\mathbb{M}} \boldsymbol{\mu}_{\mathbb{M}}\mathbf{w}.$$

Let us now recall the definition of the divergence of tensor fields on a submanifold $\mathbb{M}$ of a Riemannian manifold $\{\mathbb{S}, \mathbf{g}\}$. We are interested in the definition of the divergence for two kinds of tensor fields.

(i) The divergence div maps a tensor field $\mathbf{L} \in C^k(\mathbb{M}; BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}}))$ into the vector field $\operatorname{div}\mathbf{L} \in C^{k-1}(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$ defined by the relation

$$\mathbf{g}(\operatorname{div}\mathbf{L}, \mathbf{a}) := \operatorname{div}(\mathbf{L}^T\mathbf{a}) - \mathbf{L} : \nabla\mathbf{a}, \qquad \text{for all } \mathbf{a} \in C^1(\mathbb{M}; \mathbb{T}_{\mathbb{M}}),$$

with $\mathbf{L}^T\mathbf{a} \in C^k(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$ and $\operatorname{div}(\mathbf{L}^T\mathbf{a}) := \operatorname{tr}(\nabla(\mathbf{L}^T\mathbf{a}))$ and $\mathbf{L} : (\nabla\mathbf{a}) := \operatorname{tr}(\mathbf{L}^T\nabla\mathbf{a}))$. At any $\mathbf{x} \in \mathbb{M}$, the l.h.s in the definition of $\operatorname{div}\mathbf{L}$ is an inner product which depends only on the point value $\mathbf{a}(\mathbf{x}) \in \mathbb{T}_{\mathbb{M}}(\mathbf{x})$ of the vector field $\mathbf{a} \in C^1(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$. However, the evaluation of $(\operatorname{div}\mathbf{L})(\mathbf{x})$ requires to compute the r.h.s and this requires that a field of frames be assigned on $\mathbb{M}$.

(ii) The divergence Div maps a tensor field $\mathbf{A} \in C^k(\mathbb{M}; BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{S}}(\mathbb{M})))$ into the vector field $\operatorname{Div}\mathbf{A} \in C^{k-1}(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ defined by the relation

$$\mathbf{g}(\operatorname{Div}\mathbf{A}, \mathbf{v}) := \operatorname{div}(\mathbf{A}^T\mathbf{v}) - \mathbf{A} : \partial\mathbf{v}\boldsymbol{\Pi}, \qquad \text{for all } \mathbf{v} \in C^1(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M})),$$

with $\mathbf{A}^T \in C^k(\mathbb{M}; BL(\mathbb{T}_{\mathbb{S}}(\mathbb{M}); \mathbb{T}_{\mathbb{M}}))$ so that $\mathbf{A}^T\mathbf{v} \in C^k(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$ and

$$\operatorname{div}(\mathbf{A}^T\mathbf{v}) := \operatorname{tr}(\nabla(\mathbf{A}^T\mathbf{v})), \quad \mathbf{A} : \partial_{\mathbb{M}}\mathbf{v} := \operatorname{tr}(\mathbf{A}^T\partial\mathbf{v}\boldsymbol{\Pi})).$$

At any $\mathbf{x} \in \mathbb{T}_{\mathbb{M}}(\mathbb{M})$, the l.h.s in the definition of $\operatorname{Div}\mathbf{A}$ is an inner product which depends only on the point value $\mathbf{v}(\mathbf{x}) \in \mathbb{T}_{\mathbb{M}}(\mathbf{x})$.

If the tensor field $\mathbf{A} \in C^k(\mathbb{M}; BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{S}}(\mathbb{M})))$ is such that $Im\mathbf{A} \subset \mathbb{T}_{\mathbb{M}} \subset \mathbb{T}_{\mathbb{S}}(\mathbb{M})$, then the divergence formulas yield the identity

$$\mathbf{g}(\operatorname{Div}\mathbf{A}, \mathbf{v}) := \operatorname{div}(\mathbf{A}^T\mathbf{v}) - \mathbf{A} : \nabla\mathbf{v} = \mathbf{g}(\operatorname{div}\mathbf{A}_{\mathbb{M}}, \mathbf{v}), \qquad \text{for all } \mathbf{v} \in C^1(\mathbb{M}; \mathbb{T}_{\mathbb{M}}),$$

where $\mathbf{A} = \mathbf{A}_{\mathbb{M}} \in C^k(\mathbb{M}; BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}}))$.

10. Equilibrium of a membrane shell

A membrane is a shell model whose kinematics admits any change of curvature strain in a rigid transformation. Accordingly, the strain measure reduces to the variation of the first fundamental form of the middle surface due to the transformation and the dual field is the membrane stress. The tangent membrane strain

$$\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) \in C^{k-1}(\mathbb{M}; BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}}))$$

associated with a velocity field $\mathbf{v}_{\mathbb{M}} \in C^k(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ is $\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \operatorname{sym}(\nabla\mathbf{v}_{\mathbb{M}})$ and hence the membrane stress is a symmetric tensor field $\mathbf{N}_{\mathbb{M}} \in \mathscr{L}^2(\mathbb{M}; BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{M}}))$ or equivalently a tensor field $\mathbf{N} \in \mathscr{L}^2(\mathbb{M}; BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{S}}(\mathbb{M})))$ with $Im\mathbf{N} \subset \mathbb{T}_{\mathbb{M}} \subset \mathbb{T}_{\mathbb{S}}(\mathbb{M})$. Assuming that the vector field

$$\operatorname{Div}\mathbf{N} \in \mathscr{L}^2(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M}))$$

is square integrable on $\mathbb{M}$, the divergence theorem yields the Green's formula:

$$\int_{\mathbb{M}} \mathbf{N} : (\partial\mathbf{v}_{\mathbb{M}}\boldsymbol{\Pi})\boldsymbol{\mu}_{\mathbb{M}} = \int_{\mathbb{M}} -\mathbf{g}(\operatorname{Div}\mathbf{N}, \mathbf{v}_{\mathbb{M}})\boldsymbol{\mu}_{\mathbb{M}} + \oint_{\partial\mathbb{M}} \mathbf{g}(\mathbf{N}\mathbf{n}_{\partial\mathbb{M}}, \mathbf{v}_{\mathbb{M}})\boldsymbol{\mu}_{\partial\mathbb{M}},$$

and hence the differential and boundary Cauchy's laws of equilibrium:

$$\begin{cases} -\mathrm{Div}\mathbf{N} = \mathbf{p} \in \mathscr{L}^2(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M})), & \text{body force on } \mathbb{M}, \\ \mathbf{N}\mathbf{n}_{\partial\mathbb{M}} = \mathbf{t} \in \mathscr{L}^2(\partial\mathbb{M}; \mathbb{T}_{\mathbb{M}}(\partial\mathbb{M})), & \text{boundary traction on } \partial\mathbb{M}. \end{cases}$$

Splitting the body force on the shell middle surface into the parallel and the normal component,

$$\mathbf{p} = \mathbf{p}^{||} + p_{\mathbf{n}}\mathbf{n},$$

with $\mathbf{p}^{||} \in \mathbb{T}_{\mathbb{M}}$, being $\mathbf{N}^T\mathbf{n} = 0$ identically, from the divergence formula we get

$$p_{\mathbf{n}} = \mathbf{g}(\mathbf{p}, \mathbf{n}) = -\mathbf{g}(\mathrm{Div}\mathbf{N}, \mathbf{n}) = -\mathrm{div}(\mathbf{N}^T\mathbf{n}) + \mathbf{N} : \partial_{\mathbb{M}}\mathbf{n} = \mathbf{N}_{\mathbb{M}} : \mathbf{S}_{\mathbb{M}}.$$

Given that

$$\mathbf{g}(\mathrm{Div}\mathbf{N}, \mathbf{v}_{\mathbb{M}}^{||}) = \mathbf{g}_{\mathbb{M}}(\mathrm{div}\mathbf{N}_{\mathbb{M}}, \mathbf{v}_{\mathbb{M}}^{||}),$$

we have $\mathbf{p}^{||} = -\mathrm{div}\mathbf{N}_{\mathbb{M}}$.

Equilibrium then requires that normal loads must vanish at planar points of the membrane shell and that boundary tractions must be tangent to the middle surface of the membrane shell.

11. Equilibrium of a membrane-flexural shell

The tangent strain for the Kirchhoff–Love shell model is provided by the rates of variation of the first and second fundamental forms of the middle surface along a virtual transformation. The dual stress fields are represented by tensor fields of membrane stresses and bending-torsional moments. The basic step in the analysis of the equilibrium of the shell consists again in providing the relevant Green's formula. The virtual work principle for the shell subject to a force system **f** has the expression:

$$\int_{\mathbb{M}} \mathbf{N}_{\mathbb{M}} : \dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}})\boldsymbol{\mu}_{\mathbb{M}} + \int_{\mathbb{M}} \mathbf{M}_{\mathbb{M}} : \dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}})\boldsymbol{\mu}_{\mathbb{M}} = \langle \mathbf{f}, \mathbf{v}_{\mathbb{M}} \rangle,$$

where $\mathbf{v}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M}))$ is a virtual displacement field of the actual placement of the shell. Recalling that the tangent membrane and curvature strains are given by

$$\dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \mathrm{sym}(\nabla\mathbf{v}_{\mathbb{M}}) \quad \text{and} \quad \dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \nabla\dot{\mathbf{n}} + (\mathbf{S}_{\mathbb{M}}\nabla\mathbf{v}_{\mathbb{M}})^T,$$

the appropriate divergence formulae yield

$$\mathbf{N}_{\mathbb{M}} : \dot{\mathbf{E}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \mathbf{N} : \nabla\mathbf{v}_{\mathbb{M}} = \mathrm{div}(\mathbf{N}^T\mathbf{v}_{\mathbb{M}}) - \mathbf{g}(\mathrm{Div}\mathbf{N}, \mathbf{v}_{\mathbb{M}}),$$

$$\begin{aligned}\mathbf{M}_{\mathbb{M}} : \dot{\mathbf{K}}_{\mathbb{M}}(\mathbf{v}_{\mathbb{M}}) = \mathbf{M}_{\mathbb{M}} : \nabla\dot{\mathbf{n}} + \mathbf{M}_{\mathbb{M}} : \mathbf{S}\nabla\mathbf{v}_{\mathbb{M}} = \mathbf{M}_{\mathbb{M}} : \nabla\dot{\mathbf{n}} + \mathbf{S}_{\mathbb{M}}\mathbf{M}_{\mathbb{M}} : \nabla\mathbf{v}_{\mathbb{M}} = \mathrm{div}(\mathbf{M}_{\mathbb{M}}\dot{\mathbf{n}}) - \mathbf{g}_{\mathbb{M}}(\mathrm{div}\mathbf{M}_{\mathbb{M}}, \dot{\mathbf{n}}) \\ + \mathrm{div}((\mathbf{S}_{\mathbb{M}}\mathbf{M}_{\mathbb{M}})^T\mathbf{v}_{\mathbb{M}}) - \mathbf{g}(\mathrm{Div}(\mathbf{S}\mathbf{M}_{\mathbb{M}}), \mathbf{v}_{\mathbb{M}}).\end{aligned}$$

It is convenient to define the shear vector field $\mathbf{T}_{\mathbb{M}} \in C^{k-1}(\mathbb{M}; \mathbb{T}_{\mathbb{M}})$ to be $\mathbf{T}_{\mathbb{M}} := \mathrm{div}\mathbf{M}_{\mathbb{M}}$ and its normal component at the boundary $T := \mathbf{g}_{\mathbb{M}}(\mathbf{T}_{\mathbb{M}}, \mathbf{n}_{\partial\mathbb{M}})$. Then, being $\dot{\mathbf{n}} = -(\partial\mathbf{v}_{\mathbb{M}}\mathbf{\Pi})^T\mathbf{n}$, we may write:

$$\begin{aligned} -\mathbf{g}_{\mathbb{M}}(\mathrm{div}\mathbf{M}_{\mathbb{M}}, \dot{\mathbf{n}}) &= \mathbf{g}_{\mathbb{M}}(\mathbf{T}_{\mathbb{M}}, (\partial\mathbf{v}_{\mathbb{M}}\mathbf{\Pi})^T\mathbf{n}) = \mathbf{g}_{\mathbb{M}}((\partial\mathbf{v}_{\mathbb{M}}\mathbf{\Pi})\mathbf{T}_{\mathbb{M}}, \mathbf{n}) \\ &= \mathbf{n} \otimes \mathbf{T}_{\mathbb{M}} : \partial\mathbf{v}_{\mathbb{M}}\mathbf{\Pi} \\ &= \mathrm{div}((\mathbf{T}_{\mathbb{M}} \otimes \mathbf{n})\mathbf{v}_{\mathbb{M}}) - \mathbf{g}(\mathrm{Div}(\mathbf{n} \otimes \mathbf{T}_{\mathbb{M}}), \mathbf{v}_{\mathbb{M}}) \\ &= \mathrm{div}(v_{\mathbf{n}}\mathbf{T}_{\mathbb{M}}) - \mathbf{g}(\mathrm{Div}(\mathbf{n} \otimes \mathbf{T}_{\mathbb{M}}), \mathbf{v}_{\mathbb{M}}). \end{aligned}$$

An application of the divergence theorem provides the variational form of the equilibrium condition:

$$\begin{aligned} \langle \mathbf{f}, \mathbf{v}_{\mathbb{M}} \rangle = \int_{\mathbb{M}} -\mathbf{g}((\mathrm{Div}(\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}} + \mathbf{n} \otimes \mathbf{T}_{\mathbb{M}}), \mathbf{v}_{\mathbb{M}})\boldsymbol{\mu}_{\mathbb{M}} \\ + \oint_{\partial\mathbb{M}} \mathbf{g}((\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}})\mathbf{n}_{\partial\mathbb{M}}, \mathbf{v}_{\mathbb{M}}) + \mathbf{g}_{\mathbb{M}}(\mathbf{M}_{\mathbb{M}}\mathbf{n}_{\partial\mathbb{M}}, \dot{\mathbf{n}}) + T v_{\mathbf{n}}\boldsymbol{\mu}_{\partial\mathbb{M}}, \end{aligned}$$

for any virtual displacement $\mathbf{v}_{\mathbb{M}} \in C^1(\mathbb{M}; \mathbb{T}_{\mathbb{S}}(\mathbb{M}))$.

11.1. ***Differential equilibrium equation.*** By localizing the variational form of the equilibrium condition we get the differential equilibrium equation:

$$-\mathrm{Div}(\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}} + \mathbf{n} \otimes \mathbf{T}_{\mathbb{M}}) = \mathbf{p},$$

To split this equation into tangent and normal components to the middle surface, we observe that:

$$\begin{aligned} -\mathbf{g}(\mathrm{Div}(\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}}), \mathbf{v}_{\mathbb{M}}^{||}) &= -\mathrm{div}((\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}})\mathbf{v}_{\mathbb{M}}^{||}), \\ -\mathbf{g}(\mathrm{Div}(\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}}), \mathbf{n}) &= -\mathrm{div}((\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}})^T\mathbf{n}) + (\mathbf{N}_{\mathbb{M}} + \mathbf{S}_{\mathbb{M}}\mathbf{M}_{\mathbb{M}}) : \mathbf{S}_{\mathbb{M}} \\ &= (\mathbf{N}_{\mathbb{M}} + \mathbf{S}_{\mathbb{M}}\mathbf{M}_{\mathbb{M}}) : \mathbf{S}_{\mathbb{M}}, \end{aligned}$$

since $(\mathbf{N} + \mathbf{S}\mathbf{M}_{\mathbb{M}})^T\mathbf{n} = 0$. Moreover, substituting $(\mathbf{n} \otimes \mathbf{T}) : \nabla\mathbf{n} = \mathrm{tr}(\nabla\mathbf{n}(\mathbf{n} \otimes \mathbf{T})) = 0$, we have

$$\begin{aligned} \mathbf{g}(\mathrm{Div}(\mathbf{n} \otimes \mathbf{T}), \mathbf{n}) &= \mathrm{div}((\mathbf{T} \otimes \mathbf{n})\mathbf{n}) - (\mathbf{n} \otimes \mathbf{T}) : \nabla\mathbf{n} = \mathrm{div}\mathbf{T}, \\ \mathbf{g}(\mathrm{Div}(\mathbf{n} \otimes \mathbf{T}), \mathbf{v}_{\mathbb{M}}^{||}) &= \mathbf{g}((\nabla\mathbf{n})\mathbf{T}, \mathbf{v}_{\mathbb{M}}^{||}) + (\mathrm{div}\mathbf{T})\mathbf{g}(\mathbf{n}, \mathbf{v}_{\mathbb{M}}^{||}) = \mathbf{g}(\mathbf{S}\mathbf{T}, \mathbf{v}_{\mathbb{M}}^{||}). \end{aligned}$$

Hence the differential equilibrium equation may be split into

$$\begin{aligned} -\mathrm{div}(\mathbf{N}_{\mathbb{M}} + \mathbf{S}_{\mathbb{M}}\mathbf{M}_{\mathbb{M}}) - \mathbf{S}_{\mathbb{M}}\mathbf{T}_{\mathbb{M}} &= \mathbf{p}^{||}, \\ -\mathrm{div}\,\mathrm{div}\mathbf{M}_{\mathbb{M}} + (\mathbf{N}_{\mathbb{M}} + \mathbf{S}_{\mathbb{M}}\mathbf{M}_{\mathbb{M}}) : \mathbf{S}_{\mathbb{M}} &= p_{\mathbf{n}}. \end{aligned}$$

11.2. ***Interpretation of the boundary terms.*** To get a clearer mechanical interpretation of the boundary virtual work, we split the boundary spin $\dot{\mathbf{n}}$ into the bending and the torsional components: $\dot{\mathbf{n}} = \dot{n}_b\mathbf{n}_{\partial\mathbb{M}} + \dot{n}_t\mathbf{t}_{\partial\mathbb{M}}$ with $\dot{n}_b = -\mathbf{g}_{\mathbb{M}}(\partial_{\mathbf{n}_{\partial\mathbb{M}}}\mathbf{v}_{\mathbb{M}}, \mathbf{n})$ and $\dot{n}_t = -\mathbf{g}_{\mathbb{M}}(\partial_{\mathbf{t}_{\partial\mathbb{M}}}\mathbf{v}_{\mathbb{M}}, \mathbf{n})$. Setting $\mathbf{M}_{\mathbb{M}}\mathbf{n}_{\partial\mathbb{M}} = M_b\mathbf{n}_{\partial\mathbb{M}} + M_t\mathbf{t}_{\partial\mathbb{M}}$ we have

$$\oint_{\partial\mathbb{M}} \mathbf{g}_{\mathbb{M}}(\mathbf{M}_{\mathbb{M}}\mathbf{n}_{\partial\mathbb{M}}, \dot{\mathbf{n}})\boldsymbol{\mu}_{\partial\mathbb{M}} = \oint_{\partial\mathbb{M}} (M_b\dot{n}_b + M_t\dot{n}_t)\boldsymbol{\mu}_{\partial\mathbb{M}}.$$

Then, given

$$\begin{aligned}\oint_{\partial\mathbb{M}} M_t \dot{n}_t \boldsymbol{\mu}_{\partial\mathbb{M}} &= -\oint_{\partial\mathbb{M}} \mathbf{g}(\partial_{\mathbf{t}_{\partial\mathbb{M}}} \mathbf{v}_{\mathbb{M}}, M_t \mathbf{n}) \boldsymbol{\mu}_{\partial\mathbb{M}} \\ &= -\oint_{\partial\mathbb{M}} \partial_{\mathbf{t}_{\partial\mathbb{M}}} (M_t v_{\mathbf{n}}) \boldsymbol{\mu}_{\partial\mathbb{M}} + \oint_{\partial\mathbb{M}} \mathbf{g}(\partial_{\mathbf{t}_{\partial\mathbb{M}}} (M_t \mathbf{n}), \mathbf{v}_{\mathbb{M}}) \boldsymbol{\mu}_{\partial\mathbb{M}} \\ &= \oint_{\partial\mathbb{M}} (\partial_{\mathbf{t}_{\partial\mathbb{M}}} M_t) v_{\mathbf{n}} \boldsymbol{\mu}_{\partial\mathbb{M}} + \oint_{\partial\mathbb{M}} M_t \mathbf{g}(\mathbf{S}\mathbf{t}_{\partial\mathbb{M}}, \mathbf{v}_{\mathbb{M}}) \boldsymbol{\mu}_{\partial\mathbb{M}},\end{aligned}$$

the boundary virtual work may be written as

$$\oint_{\partial\mathbb{M}} \mathbf{g}((\mathbf{N}+\mathbf{S}\mathbf{M}_{\mathbb{M}})\mathbf{n}_{\partial\mathbb{M}} + M_t \mathbf{S}\mathbf{t}_{\partial\mathbb{M}}, \mathbf{v}_{\mathbb{M}}^{||}) \boldsymbol{\mu}_{\partial\mathbb{M}} + \oint_{\partial\mathbb{M}} (T + \partial_{\mathbf{t}_{\partial\mathbb{M}}} M_t) v_{\mathbf{n}} \boldsymbol{\mu}_{\partial\mathbb{M}} + \oint_{\partial\mathbb{M}} M_b \dot{n}_b \boldsymbol{\mu}_{\partial\mathbb{M}}.$$

Hence the boundary equilibrium conditions are given by

$$\begin{aligned} &(\mathbf{N}+\mathbf{S}\mathbf{M}_{\mathbb{M}})\mathbf{n}_{\partial\mathbb{M}} + M_t \mathbf{S}\mathbf{t}_{\partial\mathbb{M}} = \mathbf{t}^{||}, && \text{dual of} \quad \mathbf{v}_{\mathbb{M}}^{||}, \\ &T + \partial_{\mathbf{t}_{\partial\mathbb{M}}} M_t = t_{\mathbf{n}}, && \text{dual of} \quad v_{\mathbf{n}}, \\ &M_b = m, && \text{dual of} \quad \dot{n}_b. \end{aligned}$$

These boundary conditions differ from the ones reported in the literature which will be referred to as the standard boundary conditions. Indeed, the bending couple M_b, per-unit-length along the boundary, is here considered as the force system dual of the bending spin

$$\dot{n}_b = \mathbf{g}_{\mathbb{M}}(\dot{\mathbf{n}}, \mathbf{n}_{\partial\mathbb{M}}) = -\mathbf{g}_{\mathbb{M}}(\partial_{\mathbf{n}_{\partial\mathbb{M}}} \mathbf{v}_{\mathbb{M}}, \mathbf{n}).$$

In the literature, the bending couple per-unit-length along the boundary has typically been assumed to be the force system dual of the derivative $\partial_{\mathbf{n}_{\partial\mathbb{M}}} v_{\mathbf{n}}$ of the virtual velocity $v_{\mathbf{n}}$, normal to the middle surface, along the outward normal to the boundary. This choice has been apparently suggested by the analogous formula for flat plates. However, when the shell is not flat, the two choices are not equivalent since

$$\dot{n}_b = -\partial_{\mathbf{n}_{\partial\mathbb{M}}} v_{\mathbf{n}} + \mathbf{g}_{\mathbb{M}}(\mathbf{S}\mathbf{n}_{\partial\mathbb{M}}, \mathbf{v}_{\mathbb{M}}^{||}).$$

In fact, the derivative $\partial_{\mathbf{n}_{\partial\mathbb{M}}} v_{\mathbf{n}}$ has not a clear kinematical meaning, while apparently $\dot{n}_b = \mathbf{g}_{\mathbb{M}}(\dot{\mathbf{n}}, \mathbf{n}_{\partial\mathbb{M}})$ is the appropriate expression for the bending rate of the normal fibers of the shell at a boundary. To highlight the connection between the new and the standard boundary equilibrium conditions we observe that

$$\begin{aligned} M_b \dot{n}_b &= M_b \mathbf{g}_{\mathbb{M}}(\dot{\mathbf{n}}, \mathbf{n}_{\partial\mathbb{M}}) = M_b \mathbf{g}_{\mathbb{M}}(\mathbf{S}\mathbf{v}_{\mathbb{M}} - \nabla v_{\mathbf{n}}, \mathbf{n}_{\partial\mathbb{M}}) \\ &= \mathbf{g}_{\mathbb{M}}(\mathbf{S}\mathbf{v}_{\mathbb{M}}, \mathbf{M}_{\mathbb{M}}\mathbf{n}_{\partial\mathbb{M}}) - \mathbf{g}_{\mathbb{M}}(\mathbf{S}\mathbf{v}_{\mathbb{M}}, M_t \mathbf{t}_{\partial\mathbb{M}}) - M_b \partial_{\mathbf{n}_{\partial\mathbb{M}}} v_{\mathbf{n}} \\ &= \mathbf{g}_{\mathbb{M}}(\mathbf{S}\mathbf{M}_{\mathbb{M}}\mathbf{n}_{\partial\mathbb{M}} - M_t \mathbf{S}\mathbf{t}_{\partial\mathbb{M}}, \mathbf{v}_{\mathbb{M}}) - M_b \partial_{\mathbf{n}_{\partial\mathbb{M}}} v_{\mathbf{n}}. \end{aligned}$$

Substituting this expression into the boundary virtual work we recover the standard boundary conditions:

$$\begin{aligned} &(\mathbf{N}+2\mathbf{S}\mathbf{M}_{\mathbb{M}})\mathbf{n}_{\partial\mathbb{M}} = \mathbf{t}^{||}, && \text{dual of} \quad \mathbf{v}_{\mathbb{M}}^{||}, \\ &T + \partial_{\mathbf{t}_{\partial\mathbb{M}}} M_t = t_{\mathbf{n}}, && \text{dual of} \quad v_{\mathbf{n}}, \\ &M_b = m^*, && \text{dual of} \quad -\partial_{\mathbf{n}_{\partial\mathbb{M}}} v_{\mathbf{n}}. \end{aligned}$$

Remark 11.1. The split form of the differential condition of equilibrium is reported in coordinate expression in [Ciarlet 2000; 2005]. The coordinate form of the standard boundary conditions can be found in [Ciarlet 2005, Theorem 4.4-4]. The unsplit differential condition and the new form of the boundary conditions are contributed here for the first time in the context of a duality approach to shell theory. The treatment of shell equilibrium, according to the so-called direct approach in the restricted theory, is reviewed in the comprehensive article by [Naghdi 1972, section 15] and in the book by [Libai and Simmonds 1998]. In this context, a differential equilibrium equation formally similar to our was provided in [Steele 1971].

12. Equilibrium in a reference placement

In the nonlinear analysis of elastic shells the equilibrium condition must be written in terms of fields defined in a reference placement since constitutive laws are expressed in terms of strain measures from a natural reference placement. Moreover, in computational mechanics the suitable form of the equilibrium condition is the variational one in terms of virtual work. Once the referential stress fields are evaluated, it is needed to recover the physically significant stress fields in the actual placement. To this end we remark that, by the consistency of the shell strain measure, proven in Section 6.1, the rigid virtual displacements can be characterized as vector fields in the kernel of the rates of the strain measure from a fixed reference placement $\mathbb{M}$ to the moving actual one $\boldsymbol{\varphi}(\mathbb{M})$. The strain measure associated with the configuration map $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M};\mathbb{S})$ is given by the pair of strain fields:

$$2\mathbf{E}_{\boldsymbol{\varphi}} := \partial\boldsymbol{\varphi}^T\partial\boldsymbol{\varphi} - \mathbf{I}_{\mathbb{M}}, \qquad \mathbf{K}_{\boldsymbol{\varphi}} := \partial\boldsymbol{\varphi}^T\mathbf{S}_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi} - \mathbf{S}_{\mathbb{M}}.$$

Let $\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S};\mathbb{S})$ be a flow with velocity $\mathbf{v} = \partial_{t=s}\mathbf{Fl}^{\mathbf{v}}_{t,s} \in \mathrm{C}^1(\mathbb{S};\mathbb{T}_{\mathbb{S}})$. The tangent membrane and curvature strain rates at $\boldsymbol{\varphi}(\mathbb{M})$, are the time derivative $\partial_{t=s}$ of the shell strains in the transformation from $\boldsymbol{\varphi}(\mathbb{M})$ to $(\mathbf{Fl}^{\mathbf{v}}_{t,s}\circ\boldsymbol{\varphi})(\mathbb{M})$ evaluated at $\boldsymbol{\varphi}(\mathbb{M})$, and are given by

$$\dot{\mathbf{E}}_{\boldsymbol{\varphi}(\mathbb{M})}(\mathbf{v}) = \operatorname{sym}\nabla\mathbf{v}, \quad \dot{\mathbf{K}}_{\boldsymbol{\varphi}(\mathbb{M})}(\mathbf{v}) = \nabla\dot{\mathbf{n}}_{\boldsymbol{\varphi}} + (\mathbf{S}_{\boldsymbol{\varphi}}\nabla\mathbf{v})^T.$$

The secant membrane and curvature strain rates at $\boldsymbol{\varphi}(\mathbb{M})$, are the time derivative $\partial_{t=s}$ of the shell strains in the transformation from $\mathbb{M}$ to $(\mathbf{Fl}^{\mathbf{v}}_{t,s}\circ\boldsymbol{\varphi})(\mathbb{M})$ evaluated at $\boldsymbol{\varphi}(\mathbb{M})$, and are given by

$$\dot{\mathbf{E}}_{\{\mathbb{M},\boldsymbol{\varphi}(\mathbb{M})\}}(\mathbf{v}) := \partial_{t=s}\mathbf{E}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}\circ\boldsymbol{\varphi}} = \operatorname{sym}(\partial\boldsymbol{\varphi}^T\partial\mathbf{v}\partial\boldsymbol{\varphi}) = \partial\boldsymbol{\varphi}^T\dot{\mathbf{E}}_{\boldsymbol{\varphi}(\mathbb{M})}(\mathbf{v})\partial\boldsymbol{\varphi},$$

$$\dot{\mathbf{K}}_{\{\mathbb{M},\boldsymbol{\varphi}(\mathbb{M})\}}(\mathbf{v}) := \partial_{t=s}\mathbf{K}_{\mathbf{Fl}^{\mathbf{v}}_{t,s}\circ\boldsymbol{\varphi}} = \partial\boldsymbol{\varphi}^T(\nabla\dot{\mathbf{n}}_{\boldsymbol{\varphi}} + (\mathbf{S}_{\boldsymbol{\varphi}}\nabla\mathbf{v})^T)\partial\boldsymbol{\varphi} = \partial\boldsymbol{\varphi}^T\dot{\mathbf{K}}_{\boldsymbol{\varphi}(\mathbb{M})}(\mathbf{v})\partial\boldsymbol{\varphi}.$$

The shell stress tensors in the reference placement are defined, in tems of the shell stress tensors in the actual placement, by the invertible relations

$$\mathbf{N}_{\boldsymbol{\varphi}(\mathbb{M})} = J_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}\mathbf{N}_{\{\mathbb{M},\boldsymbol{\varphi}(\mathbb{M})\}}\partial\boldsymbol{\varphi}^T, \qquad \mathbf{M}_{\boldsymbol{\varphi}(\mathbb{M})} = J_{\boldsymbol{\varphi}}\partial\boldsymbol{\varphi}\mathbf{M}_{\{\mathbb{M},\boldsymbol{\varphi}(\mathbb{M})\}}\partial\boldsymbol{\varphi}^T,$$

where J_{φ} is the jacobian determinant associated with the map $\varphi \in C^1(\mathbb{M}; \mathbb{S})$. Then we have the equality:

$$\int_{\varphi(\mathbb{M})} \mathbf{N}_{\varphi(\mathbb{M})} : \dot{\mathbf{E}}_{\varphi(\mathbb{M})}(\mathbf{v})\boldsymbol{\mu} + \int_{\varphi(\mathbb{M})} \mathbf{M}_{\varphi(\mathbb{M})} : \dot{\mathbf{K}}_{\varphi(\mathbb{M})}(\mathbf{v})\boldsymbol{\mu} \\ = \int_{\mathbb{M}} \mathbf{N}_{\{\mathbb{M},\varphi(\mathbb{M})\}} : \dot{\mathbf{E}}_{\{\mathbb{M},\varphi(\mathbb{M})\}}(\mathbf{v}) J_{\varphi}\boldsymbol{\mu} + \int_{\mathbb{M}} \mathbf{M}_{\{\mathbb{M},\varphi(\mathbb{M})\}} : \dot{\mathbf{K}}_{\{\mathbb{M},\varphi(\mathbb{M})\}}(\mathbf{v}) J_{\varphi}\boldsymbol{\mu},$$

for any virtual displacement $\mathbf{v} \in C^1(\varphi(\mathbb{M}); \mathbb{T}_{\mathbb{S}})$ of the actual placement of the shell.

13. Shear deformable and polar shell models

In what follows we outline the main ingredients of two shell models proposed in the literature and widely adopted for computational purposes. In the former model the shear defomability of the shell is approximately taken into account since the transversal fibers are allowed to rotate with respect to the middle surface. In the latter model, a polar structure is introduced with the aim to match the shell kinematic parameters with that of a shear deformable beam. We will not develop these models here in full detail but will illustrate some critical remarks that provide guidelines for their modification or rejection.

13.1. ***Shells with transversal shear deformation.*** In this shell model transversal shear deformation is simulated by means of rigid oriented *needles* hinged at the points of the middle surface. The middle surface has then the geometrical structure of a fiber manifold and is commonly called a Cosserat surface [Naghdi 1972]. However, such a nickname may lead to the misleading conclusion that the need for a polar structure of the ambient space is implied. The needles are described by a field of directors $\mathbf{d} \in C^1(\mathbb{M}; S^2(1))$ defined on $\mathbb{M}$ and with values on the unit sphere $S^2(1)$, which is a 2D submanifold of the euclidean space $\mathbb{S}$. Directors play a role similar to that of normal versors in the Kirchhoff–Love shell model and hence the shell kinematics describes a 2D model which is a constrained three-dimensional Cauchy continuum model. In a model with inextensible needles, the shell thickness is assumed to be small enough to ensure that for all $\mathbf{x} \in U_{\mathbb{M}}$ there exists an unique decomposition such that:

$$\mathbf{x} = \mathbf{P_d}(\mathbf{x}) + r(\mathbf{x})\mathbf{d}(\mathbf{P_d}(\mathbf{x})) = \mathbf{P_d}(\mathbf{x}) + r(\mathbf{x})\mathbf{d}(\mathbf{x}),$$

with $\mathbf{P_d} \in C^1(U_{\mathbb{M}}; \mathbb{M})$ and $\partial\mathbf{P_d}(\mathbf{x}) \in BL(\mathbb{T}_{\mathbb{S}}(\mathbf{x}); \mathbb{T}_{\mathbb{M}}(\mathbf{P_d}(\mathbf{x})))$. The surfaces which are level sets of the function $r \in C^2(U_{\mathbb{M}}; \mathfrak{R})$ do not form a foliation of the shell middle surface. Indeed $\mathbf{g}(\partial r, \mathbf{d}) = 1$ and $\mathbf{g}(\mathbf{d}, \mathbf{n})\partial r = \mathbf{n}$. Hence r is not a distance function unless $\mathbf{d} = \mathbf{n}$. The derivative $\partial\mathbf{d} \in BL(\mathbb{T}_{\mathbb{M}}; \mathbb{T}_{\mathbb{S}})$ is not necessarily symmetric, so that $\mathbf{d} \in \operatorname{Ker} \partial\mathbf{d}^T$ and $\operatorname{Im} \partial\mathbf{d} \subseteq (\operatorname{Span}\partial\mathbf{d})^{\perp}$. Taking the derivative we get: $\mathbf{h} = \partial_{\mathbf{h}}\mathbf{P_d} + r\partial\mathbf{d}\partial_{\mathbf{h}}\mathbf{P_d} + (\partial_{\mathbf{h}}r)\mathbf{d}$.

The transformation $\boldsymbol{\psi}_{\varphi,\mathbf{d}} \in C^1(U_{\mathbb{M}}; \mathbb{S})$ of the shell, induced by a diffeomorphic transformation $\varphi \in C^1(\mathbb{M}; \mathbb{S})$ of the middle surface $\mathbb{M}$ and by a field of directors $\mathbf{d}_{\varphi} \in C^1(\varphi(\mathbb{M}); \mathbb{T}_{\mathbb{S}})$, is given by

$$\boldsymbol{\psi}_{\varphi,\mathbf{d}}(\mathbf{x}) := \varphi(\mathbf{P_d}(\mathbf{x})) + r(\mathbf{x})\mathbf{d}_{\varphi}(\varphi(\mathbf{P_d}(\mathbf{x}))), \qquad \text{for all } \mathbf{x} \in U_{\mathbb{M}}.$$

with $\mathbf{d}_{\varphi} \in C^1(\varphi(\mathbb{M}); S^2(1))$ director field on $\varphi(\mathbb{M})$.

Then

$$\partial_{\mathbf{h}}\boldsymbol{\psi}_{\varphi,\mathbf{d}_{\varphi}} = (\mathbf{I} + r\partial\mathbf{d}_{\varphi})\partial\varphi\partial_{\mathbf{h}}\mathbf{P_d} + (\partial_{\mathbf{h}}r)\mathbf{d}_{\varphi} \circ \varphi,$$

and Green strain is:

$$\mathbf{G}_{\boldsymbol{\varphi},\mathbf{d}_{\boldsymbol{\varphi}}} = \partial\mathbf{P}_{\mathbf{d}}^{T}(\partial\boldsymbol{\varphi}^{T}\partial\boldsymbol{\varphi} + r\,\mathrm{sym}(\partial\boldsymbol{\varphi}^{T}\partial_{\mathbb{M}}(\mathbf{d}_{\boldsymbol{\varphi}}\circ\boldsymbol{\varphi})) + \frac{1}{2}r^{2}\partial_{\mathbb{M}}(\mathbf{d}_{\boldsymbol{\varphi}}\circ\boldsymbol{\varphi})^{T}\partial_{\mathbb{M}}(\mathbf{d}_{\boldsymbol{\varphi}}\circ\boldsymbol{\varphi}))\partial\mathbf{P}_{\mathbf{d}} + \mathrm{sym}(\partial\mathbf{P}_{\mathbf{d}}^{T}\partial\boldsymbol{\varphi}^{T}(\mathbf{d}_{\boldsymbol{\varphi}}\circ\boldsymbol{\varphi}\otimes\partial r)) - \mathbf{I}.$$

In the relevant literature, [Simo and Fox 1989; VuQuoc et al. 2000], membranal, curvature and shear strains for the shell have been defined as:

$$2\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}(\mathbf{a},\mathbf{b}) := \mathbf{g}(\partial_{\mathbf{a}}\boldsymbol{\varphi},\partial_{\mathbf{b}}\boldsymbol{\varphi}) - \mathbf{g}(\mathbf{a},\mathbf{b}),$$

$$\boldsymbol{\chi}_{\boldsymbol{\varphi},\mathbf{d}_{\boldsymbol{\varphi}}}(\mathbf{a},\mathbf{b}) := \mathbf{g}(\partial_{\mathbf{a}}(\mathbf{d}_{\boldsymbol{\varphi}}\circ\boldsymbol{\varphi}),\partial_{\mathbf{b}}\boldsymbol{\varphi}) - \mathbf{g}(\partial_{\mathbf{a}}\mathbf{d},\mathbf{b}),$$

$$\boldsymbol{\delta}_{\boldsymbol{\varphi},\mathbf{d}_{\boldsymbol{\varphi}}}(\mathbf{a}) := \mathbf{g}(\mathbf{d}_{\boldsymbol{\varphi}}\circ\boldsymbol{\varphi},\partial_{\mathbf{a}}\boldsymbol{\varphi}) - \mathbf{g}(\mathbf{d},\mathbf{a}),$$

where $\mathbf{a},\mathbf{b}\in\mathbb{T}_{\mathbb{M}}$. The membranal strain is a 2×2 symmetric tensor, while the curvature strain is a 2×2 unsymmetric tensor and the shear strain is a 2D vector. We have then $3+4+2=9$ scalar strain parameters. Despite of its wide usage, this strain measure is redundant. A more economical set of strain fields can be envisaged by substituting the unsymmetrical curvature strain $\boldsymbol{\chi}_{\boldsymbol{\varphi},\mathbf{d}}$ with its symmetric part or by retaining the same membranal and curvature strains of the Kirchhoff–Love shell model, adding to these the shear strain measure. We have then $3+3+2=8$ scalar strain parameters. In the literature no attention seems to have been payed to the issue of redundancy but the topic has been only indirectly treated by assuming that the shell stress tensor dual of the curvature strain be symmetric [Naghdi 1972; Simo and Fox 1989]. Moreover, an explicit expression of the Green strain in terms of shell strains is still lacking. A computational analysis of the shell model with transversal shear strain can be carried out by a direct recourse to the expression of the Green strain provided above thus avoiding the introduction of shell strain fields.

13.2. ***Polar Shells.*** In the polar model of a shell, a rigid triad is hinged at each point of the middle surface and arbitrary rotations are allowed for. This is called the shell model with drilling rotations [Fox and Simo 1992; Ibrahimbegovic 1994]. A placement of the polar shell is described by a placement of the middle surface $\mathbb{M}$ and by a field of rotations which simulate a rigid body kinematics along the thickness of the shell. The ambient space is then the Cosserat manifold $\mathbb{S}\times\mathrm{SO}(3)$ which is a trivial fiber bundle whose base manifold is the euclidean space $\mathbb{S}$. The nonlinear fiber manifold SO(3) is the three-dimensional compact group of rotations, the special orthogonal group. A configuration change is a map $\mathbf{u}_{t,s}:\mathbb{M}\mapsto\mathbb{S}\times\mathrm{SO}(3)$ described by a configuration change $\boldsymbol{\varphi}_{t,s}\in C^{1}(\mathbb{M};\mathbb{S})$ of the base manifold $\mathbb{M}$ and by a field of rotations $\mathbf{Q}_{t,s}\in C^{1}(\mathbb{M};\mathrm{SO}(3))$, with respect to a reference triad. Then, at each point $\mathbf{x}\in\mathbb{M}$, we have $\mathbf{u}_{t,s}(\mathbf{x})=\{\boldsymbol{\varphi}_{t,s}(\mathbf{x}),\mathbf{Q}_{t,s}(\mathbf{x})\}\in\mathbb{S}\times\mathrm{SO}(3)$. In the literature [Fox and Simo 1992; Ibrahimbegovic 1994], the strain measure is defined to be the pair

$$\begin{cases}\mathbf{Q}_{t,s}^{T}\partial_{\mathbb{M}}\boldsymbol{\varphi}_{t,s}-\mathbf{I}_{\mathbb{M}}, & \text{membrane-shear strain,}\\ \mathbf{Q}_{t,s}^{T}\partial_{\mathbb{M}}\mathbf{Q}_{t,s}, & \text{wryness strain,}\end{cases}$$

where $\partial_{\mathbb{M}}$ denoted the derivative along tangent vectors in $\mathbb{T}_{\mathbb{M}}$ and $\mathbf{I}_{\mathbb{M}}$ is the identity in $\mathbb{T}_{\mathbb{M}}$. Since the tangent spaces $\mathbb{T}_{\mathbb{M}}(\mathbf{x})$ are 2D and $\mathbf{Q}$ belongs to the three-dimensional manifold SO(3), the vanishing of the wryness is expressed by 6 scalar differential conditions. The vanishing of the membrane-shear strain is also expressed by 6 scalar differential conditions. By decomposing the membrane-shear strain into its normal and tangent components, the membrane and shear components of the strain measure are given by [Ibrahimbegovic 1994]:

$$\boldsymbol{\varepsilon}_{\boldsymbol{\varphi},\mathbf{Q}}(\mathbf{a},\mathbf{b}) := \mathbf{g}(\partial\boldsymbol{\varphi}\cdot\mathbf{a}_1, \mathbf{Q}\mathbf{a}_2) - \mathbf{g}(\mathbf{a}_1,\mathbf{a}_2), \qquad \mathbf{a},\mathbf{b}\in\mathbb{T}_{\mathbb{M}},$$
$$\boldsymbol{\delta}_{\boldsymbol{\varphi},\mathbf{Q}}(\mathbf{a}) := \mathbf{g}(\partial\boldsymbol{\varphi}\cdot\mathbf{a}, \mathbf{Q}\mathbf{n}) - \mathbf{g}(\mathbf{a},\mathbf{n}), \qquad \mathbf{a}\in\mathbb{T}_{\mathbb{M}}.$$

The wryness may be also expressed in terms of the axial vector $\boldsymbol{\Omega}\mathbf{b} := \operatorname{axial}((\partial_{\mathbf{b}}\mathbf{Q})\mathbf{Q}^T)$. An expression of the wryness as a 2×2 bilinear form was provided in [Ibrahimbegovic 1994] by setting $\boldsymbol{\chi}_{\mathbf{Q}}(\mathbf{a},\mathbf{b}) := \mathbf{g}(\mathbf{Q}\mathbf{a}, \boldsymbol{\Omega}\mathbf{b})$ but this is a misstatement since only 4 scalar components are involved, instead of the 6 needed. The polar shell model is redundant. To see this, let us assume that the membrane-shear strain vanishes, that is, $\partial\boldsymbol{\varphi}(\mathbf{x})\cdot\mathbf{a} = \mathbf{Q}(\mathbf{x})\mathbf{a}$, for all $\mathbf{a}\in\mathbb{T}_{\mathbb{M}}(\mathbf{x})$. Hence $\mathbf{n}_{\boldsymbol{\varphi}}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\mathbf{n}(\mathbf{x})$. Then the membrane strain $\boldsymbol{\varepsilon}_{\boldsymbol{\varphi}}$ vanishes and moreover it is sufficient to impose the vanishing of the curvature strain $\boldsymbol{\chi}_{\boldsymbol{\varphi}}$ to ensure that the shell transformation is rigid, so that $\partial_{\mathbb{M}}\mathbf{Q} = 0$. We then need only $6+3=9$ scalar differential expressions to get a strain measure instead of the $6+6=12$ included in the strain measure of the polar shell model. This basic criticism, which stems from integrability requirements, also applies to the Cosserat polar model of a three-dimensional continuum and to the more sophisticated polar models of micromorphic and microstretch continua proposed in the literature [Eringen 1998]. A discussion on this topic is presented in [Romano 2002]. Since the polar shell model is a constrained three-dimensional Cosserat continuum, the constitutive relations of a Cauchy continuum are not applicable and this should compel the supporters of this model to envisage new nonconventional ones.

14. Concluding remarks

The initial motivation for this investigation on shell models stemmed from the unpleasant sensation that any aesthetically minded scholar feels in reading most treatises or articles on shells, and from the headaches provoked by the hoard of indexes that have to be dealt with. Also, it is hard to keep track of the geometrical significance of the quantities involved when the notation is too disturbing. In the attempt to overcome these shortcomings, several useful facts have been discovered and the ones most deserving of attention are recalled below. Basic results of surface geometry needed in a shell theory have been presented without recourse to coordinate systems or fields of frames. As a premise, the rules to be followed in the formulation of a structural model as a constrained three-dimensional Cauchy continuum are illustrated, to ensure that the model itself is well posed and that the experimental evidences, provided in the three-dimensional context, be available for the model. The *rigidity theorem* providing the uniqueness, up to isometries, of a surface with given first and second fundamental forms, is proven as a special case of a more general result of kinematics in an euclidean space. The result is given both in finite and in rate form (Theorems 5.1 and 5.2). Explicit expressions of the finite and tangent Green strains for a Kirchhoff–Love shell, in terms of the shell kinematical parameters and of the membranal and curvature strains of the shell, are provided. These basic tools are needed to properly state the variational equilibrium condition and the constitutive laws for the shell. A new expression of the tangent curvature strain is provided and allows for a direct application of the divergence theorem to get Green's formula for the shell model. The basic differential and boundary equilibrium conditions are derived in a straight way without performing the usual split into tangent and normal components. A new set of boundary conditions is provided, since it is shown that the classical definition of the bending couple does not take properly into account the bending rotation rate of the transversal fibers. Equilibrium conditions in a reference placement are also dealt with. Two shell models, presently most popular in the computational mechanics community, are also discussed (Section 13). The former is the shell with transversal shear deformation while the latter, the polar shell with drilling rotations, has been proposed in the literature to provide rotational kinematical parameters which allow for the assembly between shell elements and beam elements, with the aim to give a tool to engineers for the analysis of shells with beam-like reinforcements. The polar shell model is then a constrained Cosserat three-dimensional continuum model. A closer look at the strain measure adopted for these shell models reveals, on the basis of the discussion in Section 5, that the set of strain parameters is redundant in both cases. This unpleasant feature forces one, by duality, to introduce more stress parameters than necessary and renders the models mechanically unsound. A suitable remedy for the former model is proposed. Unfortunately, the redundancy of the polar model, also shared by the Cosserat three-dimensional continuum model, appears to be of a troublesome resolution and should eventually force researchers to open new ways for the proposal of well posed mechanical models with a polar structure or to find better alternatives.

References

[Ciarlet 2000] P. G. Ciarlet, *Theory of Shells*, vol. 3, North-Holland, Amsterdam, 2000.

[Ciarlet 2005] P. G. Ciarlet, "An Introduction to Differential Geometry with Applications to Elasticity", Technical report, City University of Hong Kong, 2005.

[Eringen 1998] A. C. Eringen, *Microcontinuum field theories*, Springer, Berlin, 1998.

[Fox and Simo 1992] D. D. Fox and J. C. Simo, "A drill rotation formulation for geometrically exact shells", *Comput. Methods Appl. Mech. Engrg.* **98** (1992), 329–342.

[Ibrahimbegovic 1994] A. Ibrahimbegovic, "Stress resultant geometrically nonlinear shell theory with drilling rotations - Part I: A consistent formulation", *Comp. Meth. Appl. Mech. Engng.* **118** (1994), 265–284.

[Libai and Simmonds 1998] A. Libai and J. G. Simmonds, *The Nonlinear Theory of Elastic Shells*, Cambridge Univ. Press, 1998.

[Marsden and Hughes 1983] J. E. Marsden and T. J. R. Hughes, *Mathematical foundations of elasticity*, Prentice-Hall, Redwood City, CA, 1983.

[Naghdi 1972] P. M. Naghdi, *Handbuch der Physik*, vol. VIa/2, Chapter The Theory of Shells and Plates, pp. 425–640, Springer, Berlin, 1972.

[Romano 2002] G. Romano, *Scienza delle Costruzioni*, Hevelius Edizioni, 2002.

[Romano et al. 2002] G. Romano, M. Diaco, and C. Sellitto, *Trends and Applications of Mathematics to Mechanics*, Chapter Tangent Stiffness of Elastic Continua on Manifolds, pp. 155–184, Springer Verlag, 2002.

[Romano et al. 2005a] G. Romano, R. Barretta, and C. Sellitto, "On the evaluation of the elastoplastic tangent stiffness", pp. 1118–1121 in *VIII International Conference on Computational Plasticity* (Barcelona), 2005.

[Romano et al. 2005b] G. Romano, M. Diaco, and R. Barretta, "Continuum mechanics: A Differential Geometric Approach", Research report, Dept. of Structural Engineering, Univ. of Naples Federico II, 2005.

[Romano et al. 2006] G. Romano, M. Diaco, and R. Barretta, "A geometric approach to the algorithmic tangent stiffness", pp. 121–129 in *III European Conference on Computational Mechanics* (Lisbon), 2006.

[Simo and Fox 1989] J. C. Simo and D. D. Fox, "On a stress resultant geometrically exact shell model—Part I: Formulation and optimal parametrization", *Comput. Methods Appl. Mech. Engrg.* **72** (1989), 267–304.

[Steele 1971] C. R. Steele, "A geometric optics solution for the thin shell equation", *Int. J. of Engng. Sci.* **9** (1971), 681–704.

[VuQuoc et al. 2000] L. VuQuoc, H. Deng, and X. G. Tan, “Geometrically-exact sandwich shells: The static case”, *Comput. Methods Appl. Mech. Engrg.* **189** (2000), 167–203.

[Yosida 1974] K. Yosida, *Functional Analysis*, 4th ed., Springer, New York, 1974.

Received 6 Dec 2005. Accepted 12 Mar 2007.

GIOVANNI ROMANO: romano@unina.it
University of Naples Federico II, Department of Structural Engineering, via Claudio 21, 80125 Naples, Italy

CARMEN SELLITTO: c.sellitto@cira.it
Centro Italiano Ricerche Aerospaziali (CIRA), LAMC (Computational Mechanics Laboratory), via Maiorise, 81043 Capua (CE), Italy

RAFFAELE BARRETTA: rabarret@unina.it
University of Naples Federico II, Department of Structural Engineering, via Claudio 21, 80125 Naples, Italy