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SHEAR DEFORMABILITY OF THIN-WALLED BEAMS WITH ARBITRARY CROSS SECTIONS

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SUMMARY

Formulas for the computation of the shear deformability of thin-walled prismatic beams can be found in the technical literature only in the special case of symmetric cross sections. In order to fill this gap a formulation of the flexural behaviour of thin-walled beams taking into account transverse shear deflections is developed in the present paper. On this basis, the general expression of the shear centre location and the shear deformability tensor for open and closed sections of arbitrary shape are given and their properties discussed. In the case of polygonal, circular and arc-shaped cross sections explicit formulas, which can be suitably implemented for automatic computations, are provided. For the sake of completeness, the expression of the stiffness tensor for prismatic beams, previously obtained by the first two authors in a co-ordinate-free version, is reported. Finally, a numerical example is carried out and comparisons with the results given by Cowper[1] for symmetric cross sections are presented.

INTRODUCTION

In the analysis of the flexural behaviour of elastic beams it is often necessary to take account of the effect of transverse shear deformations. In fact, shear deformability can play a significant role in the evaluation of the global stiffness of a structure and in the distribution of the internal forces among its constitutive members. This is the case, for instance, of the structural skeletons of tall buildings, subjected to horizontal forces, when the presence of concrete shear walls is predominant.

As is well known,[2,3] in the Saint Venant beam theory, the transverse deflections are produced by the flexural curvature while the warping of the cross section is due to the shear strains associated with the shearing stresses. However, in the Saint Venant theory the kinematical restraints are absent while, in the technical beam theory, boundary conditions on the flexural rotations of sections, assumed perfectly rigid in their own plane, are imposed. Therefore the warping of the cross section due to the shearing strains results in an additional transverse deflection of the beam.

Recently[4,5] a satisfactory formulation of the flexural behaviour of thin-walled beams with arbitrary cross section has been given in the framework of the technical beam theory. In particular it has been shown that the principal axes of the shear deformability tensor, which relates the shearing force to the dual kinematical variable, are different, as a general rule, from the ones of the bending deformability tensor. This means that, when the cross section of the beam is not symmetric, the effects due to a shearing force cannot be considered separately in the two principal planes of inertia, as is usually assumed in the technical literature.

In the case of open thin-walled sections, a relevant contribution to illustrate this point has

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been provided by Capurso,[6] who derived the correct expression of the shear deformability tensor referred to the principal axes of the bending deformability tensor.

However, a general treatment which allows one to compute the shear factors tensor for thin-walled beams with non-symmetric cross sections is still lacking in the literature on the subject.

In this paper a simple and direct procedure is presented for the evaluation of the shear deformability tensor and for the shear centre location of open and closed thin-walled sections of arbitrary shape. In addition, a general expression of the stiffness tensor which takes account of the shear deformability is provided. As an example of different orientation between the principal directions of the bending and shear deformability tensors, explicit computations for a thin-walled L-section are carried out. A comparison with the shear factors values computed by Cowper[1] for Timoshenko beam theory in the case of symmetric cross sections is finally performed.

KINEMATIC MODEL

Let us consider a prismatic thin-walled beam whose longitudinal axis is oriented by means of a unit vector $\mathbf{k}$. By assuming an arbitrary point as origin of a reference system, each point of the beam will be identified by the corresponding position vector $\mathbf{p}$. Denoting by A the area of a cross section, the elastic centroid O will then be located by the vector

$$\mathbf{p}_O = \frac{\int_A E(\mathbf{p})\mathbf{p}\,\mathrm{d}A}{\int_A E(\mathbf{p})\,\mathrm{d}A} \tag{1}$$

where $E(\mathbf{p})$ is the Young's modulus of the generic point at the position $\mathbf{p}$. It is convenient to introduce the relative location of the points on a cross section with respect to the elastic centroid by means of the position vector $\mathbf{r}$ defined by

$$\mathbf{r} = \mathbf{p} - \mathbf{p}_O \tag{2}$$

According to the well-known Bernoulli–Navier hypothesis, it is assumed in the technical beam theory that the cross sections remain plane after deformation and perfectly rigid in their own plane. Hence the parameters defining the kinematic model of the beam are the axial displacement of the elastic centroid, the flexural and torsional rotation of the section and the transverse displacement of the shear centre.

For what concerns a pure flexural behaviour, the significant parameters are then the flexural rotation $\boldsymbol{\phi}$ of the cross section and the transverse displacement $\mathbf{u}$ of the shear centre. The corresponding parameters of deformation are the flexural curvature of the section,

$$\mathbf{c}_f = \mathbf{k} \times \boldsymbol{\phi}' \tag{3}$$

and the transverse shearing strain of the axis of the shear centres

$$\boldsymbol{\delta} = \mathbf{u}' + \mathbf{k} \times \boldsymbol{\phi} \tag{4}$$

where the prime denotes differentiation along the axis of the beam. The extensional strain of the longitudinal fibres can be evaluated from equation (3):

$$\varepsilon(\mathbf{r}) = (\mathbf{k} \times \boldsymbol{\phi}') \cdot \mathbf{r} \tag{5}$$

and results in a linear function of the position vector $\mathbf{r}$.

THE SHEAR DEFORMABILITY TENSOR

The flexural curvature $\mathbf{c}_f$ and the transverse shearing strain $\boldsymbol{\delta}$ are related to the bending moment $\mathbf{m}_f$ and the shearing force $\mathbf{t}$, respectively, by the bending and shear deformability tensors $\mathbf{C}_f$ and $\mathbf{C}_s$:

$$\mathbf{k} \times \boldsymbol{\phi}' = \mathbf{C}_f(\mathbf{k} \times \mathbf{m}_f) \tag{6}$$

$$\boldsymbol{\delta} = \mathbf{C}_s \mathbf{t} \tag{7}$$

The bending deformability tensor $\mathbf{C}_f$ of the cross section can be readily evaluated by means of the formula

$$\mathbf{C}_f = \mathbf{J}^{-1} \tag{8}$$

where the tensor

$$\mathbf{J} = \int_A E(\mathbf{r})\mathbf{r} \otimes \mathbf{r} \, \mathrm{d}A \tag{9}$$

is the second elastic area moment with respect to the elastic centroid.

The shear deformability tensor $\mathbf{C}_s$ can be evaluated by applying the internal virtual work principle, per unit length of the beam:

$$\mathbf{t}^* \cdot \boldsymbol{\delta} = \int_A \boldsymbol{\tau}^* \cdot \boldsymbol{\gamma} \, \mathrm{d}A = \int_A \frac{\boldsymbol{\tau}^* \cdot \boldsymbol{\tau}}{G(\mathbf{r})} \mathrm{d}A \tag{10}$$

where $\mathbf{t}^*$ is an arbitrary shearing force, $\boldsymbol{\tau}^*$ is a corresponding equilibrated shear stress field, $\boldsymbol{\gamma}$ is the shear strain field associated with $\boldsymbol{\tau}$ and $G(\mathbf{r})$ is the shear modulus at the point identified by the position vector $\mathbf{r}$.

Let us now write the shear stress field $\boldsymbol{\tau}(\mathbf{r})$ in the form

$$\boldsymbol{\tau}(\mathbf{r}) = G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\frac{\mathbf{t}}{A_G} \tag{11}$$

where the factor

$$A_G = \int_A G(\mathbf{r}) \, \mathrm{d}A \tag{12}$$

has been introduced to get a dimensionless tensor $\boldsymbol{\Pi}(\mathbf{r})$. Equation (10) will then be rewritten as

$$\mathbf{t}^* \cdot \boldsymbol{\delta} = \int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\frac{\mathbf{t}^*}{A_G} \cdot \boldsymbol{\Pi}(\mathbf{r})\frac{\mathbf{t}}{A_G} \mathrm{d}A = \int_A G(\mathbf{r})\boldsymbol{\Pi}^{\mathrm{T}}(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\frac{\mathbf{t}^*}{A_G} \cdot \frac{\mathbf{t}}{A_G} \mathrm{d}A \tag{13}$$

Hence, defining the shear factors tensor as

$$\boldsymbol{\chi} = \frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}^{\mathrm{T}}(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r}) \, \mathrm{d}A \tag{14}$$

and comparing equations (13) and (7), we get the expression of the shear deformability tensor:

$$\mathbf{C}_s = \frac{\boldsymbol{\chi}}{A_G} \tag{15}$$

The tensor $\boldsymbol{\chi}$, and hence $\mathbf{C}_s$, is symmetric and turns out to be positive definite if and only if $\boldsymbol{\Pi}(\mathbf{r})$ is not singular.

A relevant property concerning the eigenvalues of the shear factors tensor is provided by the following theorem whose proof is here derived as a revised version of an original idea due to Cuomo.[7]

Theorem. The eigenvalues of the shear factors tensor $\boldsymbol{\chi}$ are greater than unity

Proof. Let $\mathbf{e}$ be the unit vector associated with an eigenvector of $\boldsymbol{\chi}$. The corresponding eigenvalue χ is then given by

$$\chi = \boldsymbol{\chi}\mathbf{e}\cdot\mathbf{e} = \frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\cdot\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\,\mathrm{d}A \tag{16}$$

In order to prove the statement let us consider the well known Cauchy–Schwarz inequality

$$\left[\frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\cdot\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\,\mathrm{d}A\right]\left[\frac{1}{A_G}\int_A G(\mathbf{r})\mathbf{e}\cdot\mathbf{e}\,\mathrm{d}A\right] \geqslant \left[\frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\cdot\mathbf{e}\,\mathrm{d}A\right]^2 \tag{17}$$

Since, by definition

$$\mathbf{t} = \int_A \boldsymbol{\tau}(\mathbf{r})\,\mathrm{d}A$$

it follows that

$$\frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\,\mathrm{d}A = \mathbf{I} \tag{18}$$

where $\mathbf{I}$ denotes the identity tensor. Then

$$\frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\cdot\mathbf{e}\,\mathrm{d}A = 1$$

Noting also that

$$\frac{1}{A_G}\int_A G(\mathbf{r})\mathbf{e}\cdot\mathbf{e}\,\mathrm{d}A = 1$$

we get, from equation (17),

$$\chi = \frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\cdot\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\,\mathrm{d}A \geqslant 1 \tag{19}$$

The equality above holds iff we have in the Cauchy–Schwarz inequality

$$\boldsymbol{\Pi}(\mathbf{r})\mathbf{e} = \alpha\mathbf{e}$$

Actually, the scalar multiplier α is found to be of unit value since, by equation (18), we can write

$$\mathbf{e} = \frac{1}{A_G}\int_A G(\mathbf{r})\boldsymbol{\Pi}(\mathbf{r})\mathbf{e}\,\mathrm{d}A = \frac{1}{A_G}\int_A G(\mathbf{r})\alpha\mathbf{e}\,\mathrm{d}A = \alpha\mathbf{e}$$

Notice that $\boldsymbol{\Pi}(\mathbf{r})$ cannot be constant since otherwise the shear strain field $\boldsymbol{\gamma}(\mathbf{r})$ would be constant over the section and equal to its mean value $\mathbf{t}/A_G$—see equation (11).

Therefore the inequality in equation (19) must be strict and the proof is complete. □

SHEAR FLOW IN THIN-WALLED SECTIONS

Explicit formulas of the shear deformability tensor $\mathbf{C}_s$ can be easily derived for thin-walled beams. Let s denote the curvilinear co-ordinate along the centre line of the cross section, $q(s)$ the shear flow across an imaginary longitudinal cut taken parallel to the axis of the beam, $\delta(s)$ the thickness of the wall and $\mathbf{d}(s)$ the unit vector tangent to the centre line.

We shall employ the usual thin-walled assumption according to which the shear stress is uniform over the thickness of the wall and directed as $\mathbf{d}(s)$:

$$\boldsymbol{\tau}(s) = \frac{q(s)}{\delta(s)}\mathbf{d}(s) \tag{20}$$

while the shear stress perpendicular to the centre line is zero.

We shall first consider the thin-walled open sections and then the closed ones.

Open thin-walled sections

Assuming as origin of the curvilinear co-ordinate system a free edge of the cross section, the relation between the shearing force $\mathbf{t}$ and the shear flow $q(s)$ can be evaluated by means of the well known equilibrium equation

$$q(s) = -\mathbf{C}_f\mathbf{t}\cdot\mathbf{s}(s) = -\mathbf{C}_f\mathbf{s}(s)\cdot\mathbf{t} \tag{21}$$

where

$$\mathbf{s}(s) = \int_{A'} E(\mathbf{r})\mathbf{r}\,\mathrm{d}A \tag{22}$$

is the first elastic area moment of the part A' of the cross section to one side of the imaginary longitudinal cut. By virtue of the hypotheses made on the shear stress distribution, equation (10) becomes†

$$\mathbf{t}^*\cdot\boldsymbol{\delta} = \int_A \boldsymbol{\tau}^*\cdot\boldsymbol{\gamma}\,\mathrm{d}A = \int_c \frac{q^*q}{G\delta}\mathrm{d}s \tag{23}$$

where c is the length of the centre line of the cross section. Hence, by means of equation (21), we have

$$\mathbf{t}^*\cdot\boldsymbol{\delta} = \int_c \frac{(\mathbf{C}_f\mathbf{s}\cdot\mathbf{t}^*)(\mathbf{C}_f\mathbf{s}\cdot\mathbf{t})}{G\delta}\mathrm{d}s = \left[\int_c \frac{\mathbf{C}_f\mathbf{s}\otimes\mathbf{C}_f\mathbf{s}}{G\delta}\mathrm{d}s\right]\mathbf{t}^*\cdot\mathbf{t} = \mathbf{t}^*\cdot\mathbf{C}_f\left[\int_c \frac{\mathbf{s}\otimes\mathbf{s}}{G\delta}\mathrm{d}s\right]\mathbf{C}_f\mathbf{t} \tag{24}$$

Defining the tensor $\boldsymbol{\Gamma}$ as

$$\boldsymbol{\Gamma} = \int_c \frac{\mathbf{s}\otimes\mathbf{s}}{G\delta}\mathrm{d}s \tag{25}$$

equation (24) can be written, by the arbitrariness of $\mathbf{t}^*$:

$$\boldsymbol{\delta} = \mathbf{C}_f\boldsymbol{\Gamma}\mathbf{C}_f\mathbf{t} \tag{26}$$

† Here, and henceforth, the explicit dependence of G, S and δ on the tangential co-ordinate s will be dropped to simplify the notations

Finally, comparing equation (26) with equation (7), we get

$$\mathbf{C}_s = \mathbf{C}_f \boldsymbol{\Gamma} \mathbf{C}_f \tag{27}$$

which yields the explicit expression of the shear deformability tensor.

Closed thin-walled sections

In the case of closed sections the shear flow can be expressed as

$$q_c(s) = q(s) + q_a \tag{28}$$

where $q(s)$ is the shear flow in the open cross section obtained by the closed one by introducing a longitudinal cut and q_a is the additional constant shear flow necessary to enforce the compatibility condition which essentially results in a null relative warping of the opposite faces of the slit. The shear flow q_a is accordingly determined by the compatibility equation:

$$\oint_c \gamma \, ds = \oint_c \frac{q(s) + q_a}{G\delta} ds = 0 \tag{29}$$

so that its expression is given by

$$q_a = -\frac{\oint_c \frac{q(s)}{G\delta} ds}{\oint_c \frac{ds}{G\delta}} = \frac{\oint_c \frac{\mathbf{C}_f \mathbf{s} \cdot \mathbf{t}}{G\delta} ds}{\oint_c \frac{ds}{G\delta}} = \mathbf{C}_f \frac{\oint_c \frac{\mathbf{s}}{G\delta} ds}{\oint_c \frac{ds}{G\delta}} \cdot \mathbf{t} \tag{30}$$

Denoting by

$$\mathbf{s}_a = -\frac{\oint_c \frac{\mathbf{s}}{G\delta} ds}{\oint_c \frac{ds}{G\delta}} \tag{31}$$

equation (28) can also be written as

$$q_c = -\mathbf{C}_f \mathbf{t} \cdot (\mathbf{s} + \mathbf{s}_a) \tag{32}$$

and equation (23) becomes

$$\mathbf{t}^* \cdot \boldsymbol{\delta} = \oint_c \frac{[q^*(s) + q_a^*][q(s) + q_a]}{G\delta} ds \tag{33}$$

Let us now apply the principle of virtual work to the true shear flow $q(s) + q_a$ associated with the shearing force $\mathbf{t}$ and to an arbitrary constant shear flow q_a^* in equilibrium with a torque. Since the external virtual work is zero we get

$$\oint_c \frac{[q(s) + q_a] q_a^*}{G\delta} ds = 0 \tag{34}$$

so that equation (33) can be simplified to

$$\mathbf{t}^* \cdot \boldsymbol{\delta} = \oint_c \frac{[q(s) q^*(s) - q_a q_a^*]}{G\delta} ds \tag{35}$$

By means of equations (21) and (30), equation (35) can also be written as

$$\mathbf{t}^* \cdot \boldsymbol{\delta} = \mathbf{C}_f \left[\oint_c \frac{\mathbf{s} \otimes \mathbf{s}}{G\delta} \mathrm{d}s - \frac{\left(\oint_c \frac{\mathbf{s}}{G\delta} \mathrm{d}s\right) \otimes \left(\oint_c \frac{\mathbf{s}}{G\delta} \mathrm{d}s\right)}{\oint_c \frac{\mathrm{d}s}{G\delta}} \right] \mathbf{C}_f \mathbf{t}^* \cdot \mathbf{t} \tag{36}$$

By the arbitrariness of $\mathbf{t}^*$, we infer that the shear deformability tensor is still expressed by equation (27) but the tensor $\boldsymbol{\Gamma}$ is now given by

$$\boldsymbol{\Gamma} = \oint_c \frac{\mathbf{s} \otimes \mathbf{s}}{G\delta} \mathrm{d}s - \frac{\left(\oint_c \frac{\mathbf{s}}{G\delta} \mathrm{d}s\right) \otimes \left(\oint_c \frac{\mathbf{s}}{G\delta} \mathrm{d}s\right)}{\oint_c \frac{\mathrm{d}s}{G\delta}} = \boldsymbol{\Gamma}_a - \boldsymbol{\Gamma}_c \tag{37}$$

Properties of the deformability matrices

The symmetry of the tensor $\boldsymbol{\Gamma}$ is apparent from equations (25) and (37). Moreover, the tensors $\mathbf{C}_f$ and $\boldsymbol{\Gamma}$ turn out to be positive definite if the singular case of narrow rectangular sections is ruled out.

For closed sections the positive definiteness of $\boldsymbol{\Gamma}$ in equation (37) can be proved by means of the Cauchy–Schwarz inequality (see Appendix I). Hence the tensor $\mathbf{C}_s$ is symmetric and positive definite since, by equation (27), it turns out to be congruent to $\boldsymbol{\Gamma}$ by means of the symmetric and positive definite tensor $\mathbf{C}_f$.

It is important to notice that the principal axes of the two tensors $\mathbf{C}_f$ and $\mathbf{C}_s$ do not coincide unless the cross section has an axis of orthogonal symmetry.

SHEAR CENTRE OF THIN-WALLED SECTIONS

It is known that the shear centre may be defined as the point of the cross section of a beam through which the shearing force should be applied in order not to produce twist. Its position can then be determined by looking for the point with respect to which the moment of the shear stress distribution due to an arbitrary shearing force vanishes.

The equation which determines the position vector $\mathbf{p}_c$ of the shear centre C with respect to the elastic centroid is thus given by

$$\int_c (\mathbf{p}_c - \mathbf{r}) \times \mathbf{d} q^* \, \mathrm{d}s = \mathbf{0} \tag{38}$$

where q^* denotes an arbitrary shear flow distribution over the cross section. In terms of scalar components along the longitudinal axis of the beam, equation (38) becomes

$$\int_c \mathbf{p}_c \times \mathbf{d} \cdot \mathbf{k} q^* \, \mathrm{d}s = \int_c \mathbf{r} \times \mathbf{d} \cdot \mathbf{k} q^* \, \mathrm{d}s \tag{39}$$

or equivalently, denoting by $\mathbf{n} = \mathbf{d} \times \mathbf{k}$ the unit vector normal to the centre line,

$$\int_c \mathbf{n} \cdot \mathbf{p}_c q^* \, \mathrm{d}s = \int_c \mathbf{n} \cdot \mathbf{r} q^* \, \mathrm{d}s \tag{40}$$

Let us express the shear flow q^* in the general form—see equation (32)—

$$q^* = -\mathbf{C}_f\mathbf{t}^*\cdot(\mathbf{s} + \mathbf{s}_a) \tag{41}$$

where $\mathbf{t}^*$ is the shearing force associated with q^*; equation (41) embodies also the case of an open section by setting $\mathbf{s}_a = \mathbf{0}$.

Equation (40) can then be written as

$$\int_c [\mathbf{n}\cdot\mathbf{p}_c][\mathbf{C}_f\mathbf{t}^*\cdot(\mathbf{s} + \mathbf{s}_a)]\,ds = \int_c [\mathbf{n}\cdot\mathbf{r}](\mathbf{C}_f\mathbf{t}^*\cdot(\mathbf{s} + \mathbf{s}_a)]\,ds \tag{42}$$

or equivalently

$$\int_c [(\mathbf{s} + \mathbf{s}_a)\otimes\mathbf{n}]\mathbf{p}_c\cdot\mathbf{C}_f\mathbf{t}^*\,ds = \int_c [(\mathbf{s} + \mathbf{s}_a)\otimes\mathbf{n}]\mathbf{r}\cdot\mathbf{C}_f\mathbf{t}^*\,ds \tag{43}$$

which, by the arbitrariness of the vector $\mathbf{C}_f\mathbf{t}^*$, yields

$$\left[\oint_c \mathbf{s}\otimes\mathbf{n}\,ds + \mathbf{s}_a\otimes\oint_c \mathbf{n}\,ds\right]\mathbf{p}_c = \oint_c (\mathbf{s}\otimes\mathbf{n})\mathbf{r}\,ds + \mathbf{s}_a\oint_c \mathbf{n}\cdot\mathbf{r}\,ds \tag{44}$$

With

$$\oint_c \mathbf{n}\,ds = \mathbf{0} \tag{45}$$

equation (44) simplifies to

$$\left[\oint_c \mathbf{s}\otimes\mathbf{n}\,ds\right]\mathbf{p}_c = \oint_c (\mathbf{s}\otimes\mathbf{n})\mathbf{r}\,ds + \mathbf{s}_a\oint_c \mathbf{n}\cdot\mathbf{r}\,ds \tag{46}$$

where $\mathbf{s}_a = \mathbf{0}$ for open sections. If the middle line c of the section is not a straight line, the tensor in square brackets in the formula above is regular and equation (46) possess a unique solution.

SHEAR FLOW IN THIN-WALLED POLYGONAL SECTIONS

We shall consider open or closed polygonal cross sections having constant thickness and elastic moduli along each branch. The parametric equation of the ith branch of the polygonal centre line is given by

$$\mathbf{r} = \mathbf{r}_i(\lambda) \doteq (1-\lambda)\mathbf{r}_i + \lambda\mathbf{r}_{i+1} \quad 0 \leqslant \lambda \leqslant 1 \tag{47}$$

where $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ are the position vectors of the ith and $(i+1)$th vertex.

Denoting by n the number of branches of the polygon and by c_i and δ_i the length and the thickness of the ith branch respectively, the second elastic area moment can be expressed as

$$\mathbf{J} = \int_A E(\mathbf{r})\mathbf{r}\otimes\mathbf{r}\,dA = \sum_{i=1}^{n} A_i \int_0^1 \mathbf{r}_i(\lambda)\otimes\mathbf{r}_i(\lambda)\,d\lambda \tag{48}$$

where $A_i = E_i\delta_i c_i$ is the elastic area of the ith branch. Setting

$$\mathbf{R}_{ij} = \mathbf{r}_i\otimes\mathbf{r}_j \tag{49}$$

equation (48) can be written as

$$\mathbf{J} = \sum_{i=1}^{n} A_i \int_0^1 [(1-\lambda)^2\mathbf{R}_{ii} + \lambda^2\mathbf{R}_{(i+1)(i+1)} + \lambda(1-\lambda)(\mathbf{R}_{i(i+1)} + \mathbf{R}_{(i+1)i})]\,d\lambda \tag{50}$$

and performing the integrations we have finally

$$\mathbf{J} = \sum_{i=1}^{n} \frac{A_i}{3} [\mathbf{R}_{ii} + \operatorname{sym} \mathbf{R}_{i(i+1)} + \mathbf{R}_{(i+1)(i+1)}] \tag{51}$$

where sym denotes the symmetric part of the tensor $\mathbf{R}_{i(i+1)}$.

The flexibility tensor $\mathbf{C}_f$ can then be easily computed by inverting the positive definite tensor $\mathbf{J}$.

Shear factors tensor for thin-walled polygonal section

In order to evaluate the shear deformability tensor (27) the expression of tensor $\mathbf{\Gamma}$ is needed. In the sequel, explicit formulas will be provided for open and closed sections.

Open polygonal sections. Let us express $\mathbf{s}(s)$ as a function of the parameter λ sweeping the ith branch:

$$\mathbf{s}(s) = \mathbf{s}_{i-1} + \mathbf{s}_i(\lambda); \quad \mathbf{s}_i(\lambda) = \int_0^{\lambda} A_i \mathbf{r} \, dt \tag{52}$$

where $\mathbf{s}_{i-1}$ is the first elastic area moment of the first $i-1$ branches with respect to the elastic centroid and $\mathbf{s}_i(\lambda)$ the first elastic area moment of the portion of the ith branch with local co-ordinate values ranging from 0 to λ. Hence we obtain

$$\begin{aligned} \mathbf{\Gamma} &= \int_c \frac{\mathbf{s} \otimes \mathbf{s}}{G\delta} ds = \sum_{i=1}^{n} \int_0^1 [\mathbf{s}_{i-1} + \mathbf{s}_i(\lambda)] \otimes [\mathbf{s}_{i-1} + \mathbf{s}_i(\lambda)] \frac{c_i}{G_i \delta_i} d\lambda \\ &= \sum_{i=1}^{n} \frac{c_i}{G_i \delta_i} \left\{ \mathbf{s}_{i-1} \otimes \mathbf{s}_{i-1} + 2 \operatorname{sym} \left[\mathbf{s}_{i-1} \otimes \int_0^1 \mathbf{s}_i(\lambda) \, d\lambda \right] + \int_0^1 \mathbf{s}_i(\lambda) \otimes \mathbf{s}_i(\lambda) \, d\lambda \right\} \end{aligned} \tag{53}$$

Since

$$\int_0^1 \mathbf{s}_i(\lambda) \, d\lambda = \int_0^1 A_i \left[\int_0^{\lambda} \mathbf{r}(t) \, dt \right] d\lambda = \frac{A_i}{6} (2\mathbf{r}_i + \mathbf{r}_{i+1}) \tag{54}$$

the explicit form of the tensor $\mathbf{\Gamma}$ becomes

$$\begin{aligned} \mathbf{\Gamma} = \sum_{i=1}^{n} \frac{c_i}{G_i \delta_i} \Bigg\{ & \mathbf{s}_{i-1} \otimes \mathbf{s}_{i-1} + 2 \operatorname{sym} \left[\mathbf{s}_{i-1} \otimes \frac{A_i}{6} (2\mathbf{r}_i + \mathbf{r}_{i+1}) \right] \\ & + (A_i)^2 \left[\frac{2}{15} \mathbf{R}_{ii} + \frac{3}{40} [\mathbf{R}_{i(i+1)} + \mathbf{R}_{(i+1)i}] + \frac{1}{20} \mathbf{R}_{(i+1)(i+1)} \right] \Bigg\} \end{aligned} \tag{55}$$

Closed polygonal sections. In the case of closed sections the additional term $\mathbf{\Gamma}_c$ on the right-hand side of equation (37) has to be evaluated. By virtue of equation (55) we have

$$\oint_c \frac{\mathbf{s}(s)}{G\delta} ds = \sum_{i=1}^{n} \int_0^1 [\mathbf{s}_{i-1} + \mathbf{s}_i(\lambda)] \frac{c_i}{G_i \delta_i} d\lambda = \sum_{i=1}^{n} \frac{c_i}{G_i \delta_i} \left[\mathbf{s}_{i-1} + \frac{A_i}{6} (2\mathbf{r}_i + \mathbf{r}_{i+1}) \right] \tag{56}$$

so that

$$\mathbf{\Gamma}_c = \frac{\left\{ \sum_{i=1}^{n} \frac{c_i}{G_i \delta_i} \left[\mathbf{s}_{i-1} + \frac{A_i}{6} (2\mathbf{r}_i + \mathbf{r}_{i+1}) \right] \right\} \otimes \left\{ \sum_{i=1}^{n} \frac{c_i}{G_i \delta_i} \left[\mathbf{s}_{i-1} + \frac{A_i}{6} (2\mathbf{r}_i + \mathbf{r}_{i+1}) \right] \right\}}{\sum_{i=1}^{n} \frac{c_i}{G_i \delta_i}} \tag{57}$$

With the expressions of $\boldsymbol{\Gamma}$ and $\mathbf{C}_f$ at hand it is an easy matter to evaluate the shear flexibility tensor $\mathbf{C}_s$ by means of equation (27) and then, by means of equation (15), the shear factors tensor.

The shear centre

The tensor on the left-hand side of the linear system (46) can be evaluated by substituting the expression of the first elastic area moment given by equation (52); then, by means of equation (54), we have

$$\int_c \mathbf{s} \otimes \mathbf{n}\, ds = \sum_{i=1}^{n} \int_0^1 [\mathbf{s}_{i-1} + \mathbf{s}_i(\lambda)] \otimes \mathbf{n}_i c_i \, d\lambda = \sum_{i=1}^{n} c_i \left[\mathbf{s}_{i-1} \otimes \mathbf{n}_i + \frac{A_i}{6}(2\mathbf{r}_i + \mathbf{r}_{i+1}) \otimes \mathbf{n}_i \right] \quad (58)$$

The two terms on the right-hand side of the linear system (46) are respectively given by

$$\int_c (\mathbf{s} \otimes \mathbf{n})\mathbf{r}\, ds = \int_c \mathbf{s}(\mathbf{n} \cdot \mathbf{r})\, ds = \sum_{i=1}^{n} c_i \left[\mathbf{s}_{i-1} + \frac{A_i}{6}(2\mathbf{r}_i + \mathbf{r}_{i+1}) \right] \mathbf{n}_i \cdot \mathbf{r}_i \quad (59)$$

the product $\mathbf{n} \cdot \mathbf{r}$ being constant along each branch, and by

$$\mathbf{s}_a \oint_c \mathbf{n} \cdot \mathbf{r}\, ds = -\frac{\sum_{i=1}^{n} \frac{c_i}{G_i \delta_i} \left[\mathbf{s}_{i-1} + \frac{A_i}{6}(2\mathbf{r}_i + \mathbf{r}_{i+1}) \right]}{\sum_{i=1}^{n} \frac{c_i}{G_i \delta_i}} \sum_{i=1}^{n} c_i \mathbf{n}_i \cdot \mathbf{r}_i \quad (60)$$

which turns out to be zero for open sections ($\mathbf{s}_a = \mathbf{0}$).

SHEAR FLOW IN THIN-WALLED CIRCULAR SECTIONS

Let us consider a homogeneous thin-walled circular section of thickness δ and centre line radius r_0. The second elastic area moment is then given by

$$\mathbf{J} = \int_A E\mathbf{r} \otimes \mathbf{r}\, dA = \delta E \oint_c \mathbf{r} \otimes \mathbf{r}\, ds = \delta r_0^2 E \oint_c \mathbf{n} \otimes \mathbf{n}\, ds = \pi \delta r_0^3 E \mathbf{I} \quad (61)$$

where $\mathbf{n}$ is the outward normal to the centre line c. To extend the definition of $\mathbf{n}$ outside c, we set $\tilde{\mathbf{n}}(\mathbf{r}) = \mathbf{r}/r_0$, so that

$$\oint_c \mathbf{n} \otimes \mathbf{n}\, ds = \oint_c \tilde{\mathbf{n}} \otimes \mathbf{n}\, ds \int_\Omega \operatorname{grad} \tilde{\mathbf{n}}\, d\Omega = \frac{\mathbf{I}}{r_0} \pi r_0^2 = \pi r_0 \mathbf{I} \quad (62)$$

since $\tilde{\mathbf{n}}(\mathbf{r}) = \mathbf{n}$ on c and $\operatorname{grad} \tilde{\mathbf{n}}(\mathbf{r}) = \mathbf{I}/r_0$.

Shear factors tensor for thin-walled circular section

In order to evaluate the shear deformability tensor (27) the expression for tensor $\boldsymbol{\Gamma}$ is needed. In the sequel explicit formulas will be provided for open and closed sections.

Open circular sections. In this case the tensor $\boldsymbol{\Gamma}$ is given by

$$\boldsymbol{\Gamma} = \oint_c \frac{\mathbf{s} \otimes \mathbf{s}}{G\delta} ds = \frac{\delta r_0^2 E^2}{G} \oint_c \left[\int_0^s \delta r_0 \mathbf{n}\, d\xi \right] \otimes \left[\int_0^s \delta r_0 \mathbf{n}\, d\xi \right] ds \quad (63)$$

The integrals in square brackets can be evaluated by means of the first Frenet formula $\mathbf{n} = -r_0 \dot{\mathbf{d}}$, where the superimposed dot denotes derivation with respect to s:

$$\begin{aligned}\boldsymbol{\Gamma} &= \frac{\delta r_0^2 E^2}{G}\oint_c \left[\int_0^s r_0 \dot{\mathbf{d}}\,\mathrm{d}\xi\right] \otimes \left[\int_0^s r_0 \dot{\mathbf{d}}\,\mathrm{d}\xi\right]\mathrm{d}s = \frac{\delta r_0^4 E^2}{G}\oint_c [\mathbf{d} - \mathbf{d}_0] \otimes [\mathbf{d} - \mathbf{d}_0]\,\mathrm{d}s \\ &= \frac{\delta r_0^4 E^2}{G}\left[\oint_c \mathbf{d} \otimes \mathbf{d}\,\mathrm{d}s + \oint_c \mathbf{d}_0 \otimes \mathbf{d}_0\,\mathrm{d}s - \oint_c \mathbf{d} \otimes \mathbf{d}_0\,\mathrm{d}s - \oint_c \mathbf{d}_0 \otimes \mathbf{d}\,\mathrm{d}s\right]\end{aligned} \tag{64}$$

$\mathbf{d}_0$ being the unit vector tangent to the centre line c at the endpoints (see Figure 1). Setting $\mathbf{d} = \mathbf{Rn}$, where

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{65}$$

is the skew orthogonal tensor which rotates an arbitrary vector counterclockwise through the angle $\pi/2$, we have, by virtue of equation (62),

$$\oint_c \mathbf{d} \otimes \mathbf{d}\,\mathrm{d}s = \oint_c \mathbf{Rn} \otimes \mathbf{Rn}\,\mathrm{d}s = \mathbf{R}\left[\oint_c \mathbf{n} \otimes \dot{\mathbf{n}}\,\mathrm{d}s\right]\mathbf{R}^{\mathrm{T}} = \pi r_0 \mathbf{I} \tag{66}$$

The second integral in equation (64) is given by

$$\oint_c \mathbf{d}_0 \otimes \mathbf{d}_0\,\mathrm{d}s = 2\pi r_0 \mathbf{d}_0 \otimes \mathbf{d}_0 \tag{67}$$

while the third and the fourth integrals are zero by equation (45).

Finally, equation (64) becomes

$$\boldsymbol{\Gamma} = \frac{\pi \delta r_0^5 E^2}{G}[\mathbf{I} + 2\mathbf{d}_0 \otimes \mathbf{d}_0] \tag{68}$$

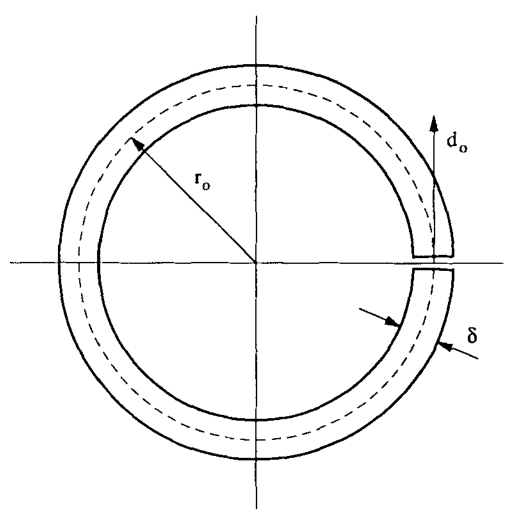

Figure 1. Thin-walled circular section

and, since $A = 2\pi r_0 \delta$, the expression for the shear factors tensor is given by

$$\chi = GA\mathbf{C}_f \boldsymbol{\Gamma} \mathbf{C}_f = 2[\mathbf{I} + 2\mathbf{d}_0 \otimes \mathbf{d}_0] \tag{69}$$

Closed circular sections. In the case of closed sections, the additional term on the right-hand side of equation (37) has to be evaluated. With

$$\begin{aligned} \oint_c \frac{\mathbf{s}}{G\delta} \mathrm{d}s &= \frac{E}{G} \oint_c \left[\int_0^s r_0 \mathbf{n} \, \mathrm{d}\xi \right] \mathrm{d}s = -\frac{r_0^2 E}{G} \oint_c \left[\int_0^s \dot{\mathbf{d}} \, \mathrm{d}\xi \right] \mathrm{d}s \\ &= -\frac{r_0^2 E}{G} \left[\oint_c \mathbf{R}\mathbf{n} \, \mathrm{d}s - \mathbf{d}_0 \oint_c \mathrm{d}s \right] = \frac{2\pi r_0^3 E}{G} \mathbf{d}_0 \end{aligned} \tag{70}$$

we have

$$\boldsymbol{\Gamma}_c = \frac{2\pi\delta r_0^5 E^2}{G} \mathbf{d}_0 \otimes \mathbf{d}_0 \tag{71}$$

Then, by virtue of equation (68), equation (37) becomes

$$\boldsymbol{\Gamma} = \frac{\pi \delta r_0^5 E^2}{G} \mathbf{I} \tag{72}$$

and

$$\chi = GA\mathbf{C}_f \boldsymbol{\Gamma} \mathbf{C}_f = 2\mathbf{I} \tag{73}$$

which is the known expression of the shear factors tensor for the closed thin circular section.

The shear centre

The tensor on the left-hand side of the linear system (46) is given by

$$\int_c \mathbf{s} \otimes \mathbf{n} \, \mathrm{d}s = \delta r_0 E \oint_c \left[\int_0^s \mathbf{n} \, \mathrm{d}\xi \right] \otimes \mathbf{n} \, \mathrm{d}s \tag{74}$$

Evaluating the integral in square brackets by means of the first Frenet formula, see equations (63) and (64), equation (74) becomes

$$\int_c \mathbf{s} \otimes \mathbf{n} \, \mathrm{d}s = -\delta r_0^2 E \left[\oint_c \mathbf{d} \otimes \mathbf{n} \, \mathrm{d}s - \mathbf{d}_0 \otimes \oint_c \mathbf{n} \, \mathrm{d}s \right] = -\delta r_0^2 E \oint_c \mathbf{d} \otimes \mathbf{n} \, \mathrm{d}s \tag{75}$$

Expressing, by the Gauss theorem, the above integral in the form

$$\oint_c \mathbf{d} \otimes \mathbf{n} \, \mathrm{d}s = \int_\Omega \operatorname{grad} \tilde{\mathbf{d}} \, \mathrm{d}\Omega \tag{76}$$

where $\tilde{\mathbf{d}} = \mathbf{R}\mathbf{r}/r_0$, equation (75) becomes

$$\int_c \mathbf{s} \otimes \mathbf{n} \, \mathrm{d}s = -\delta r_0^2 E \int_\Omega \operatorname{grad} \mathbf{d} \, \mathrm{d}\Omega = -\delta r_0^2 E \int_\Omega \frac{\mathbf{R}}{r_0} \mathrm{d}\Omega = -\delta \pi r_0^3 E \mathbf{R} \tag{77}$$

since $\operatorname{grad} \tilde{\mathbf{d}} = \mathbf{R}/r_0$.

The first term on the right-hand side of the linear system (46) can be written in the form

$$\int_c (\mathbf{s} \otimes \mathbf{n}) \mathbf{r} \, \mathrm{d}s = \int_c \mathbf{s} (\mathbf{n} \cdot \mathbf{r}) \, \mathrm{d}s = \oint_c \left[\int_0^s \delta r_0 E \mathbf{n} \, \mathrm{d}\xi \right] \mathbf{n} \cdot \mathbf{r} \, \mathrm{d}s \tag{78}$$

while the second one is zero since $\mathbf{s}_a = \mathbf{0}$. Since $\mathbf{n}\cdot\mathbf{r} = r_0$ on the centre line c and recalling equation (74) we get finally

$$\oint_c (\mathbf{s}\otimes\mathbf{n})\mathbf{r}\,\mathrm{d}s = -\delta r_0^3 E \oint_c [\mathbf{d} - \mathbf{d}_0]\,\mathrm{d}s = \delta r_0^3 E \oint_c \mathbf{d}_0\,\mathrm{d}s = 2\pi\delta r_0^4 E \mathbf{d}_0 \tag{79}$$

The position vector of the shear centre is thus given by

$$\mathbf{p}_c = -2r_0\mathbf{R}^{-1}\mathbf{d}_0 = 2r_0\mathbf{R}\mathbf{d}_0 \tag{80}$$

that is, the shear centre lies on the axis of symmetry of the open section at a distance $2r_0$ from the centre and on the opposite side of the opening.

SHEAR FLOW IN THE THIN-WALLED CIRCULAR ARC

Let us consider a homogeneous thin-walled section having the shape of an arc subtending an angle 2θ, symmetric with respect to an horizontal axis, and let r_0 be the radius of the centre line c (see Figure 2).

By making use of the first Frenet formula, the first elastic area moment with respect to the centre of the circle can be expressed as

$$\mathbf{s}_0 = E\int_A \mathbf{p}\,\mathrm{d}A = \delta E\int_c r_0\mathbf{n}\,\mathrm{d}s = -\delta r_0^2 E\int_c \dot{\mathbf{d}}\,\mathrm{d}s \tag{81}$$

where n is the normal to the centre line and $\mathbf{p} = r_0\mathbf{n}$ on c. With

$$\dot{\mathbf{d}} = \frac{\mathrm{d}\mathbf{d}}{\mathrm{d}\alpha}\dot{\alpha} = \frac{1}{r_0}\frac{\mathrm{d}\mathbf{d}}{\mathrm{d}\alpha} \tag{82}$$

where α is the angular co-ordinate sweeping the centre line of the section, equation (81) can also be written as

$$\mathbf{s}_0 = -\delta r_0^2 E\int_{-\theta}^{\theta} \frac{1}{r_0}\frac{\mathrm{d}\mathbf{d}}{\mathrm{d}\alpha} r_0\,\mathrm{d}\alpha = -\delta r_0^2 E[\mathbf{d}(\theta) - \mathbf{d}(-\theta)] \tag{83}$$

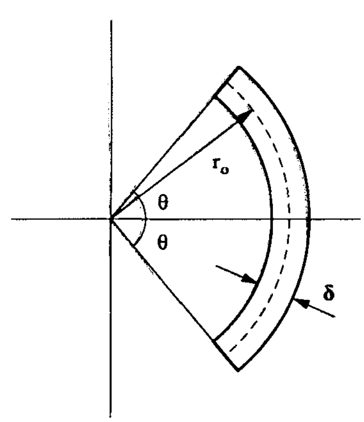

Figure 2. Thin-walled circular arc

Denoting by $\mathbf{e}_1$ the unit vector directed along the axis of symmetry of the section, equation (83) becomes

$$\mathbf{s}_0 = 2\delta r_0^2 E \sin\theta \mathbf{e}_1 \tag{84}$$

Since the elastic area of the arc is equal to $2\theta E\delta r_0$, the expression of the vector which defines the position of the centroid with respect to the origin of the reference frame is given by

$$\mathbf{p}_0 = \frac{\mathbf{s}_0}{A} = r_0 \frac{\sin\theta}{\theta} \mathbf{e}_1 \tag{85}$$

The expression of the second elastic area moment takes the form

$$\begin{aligned}\mathbf{J} &= E\int_A \mathbf{r}\otimes\mathbf{r}\,\mathrm{d}A = \delta E\int_c [\mathbf{p}-\mathbf{p}_0]\otimes[\mathbf{p}-\mathbf{p}_0]\,\mathrm{d}s \\ &= \delta E\left[\int_c \mathbf{p}\otimes\mathbf{p}\,\mathrm{d}s + \mathbf{p}_0\otimes\mathbf{p}_0\int_c \mathrm{d}s - 2\,\mathrm{sym}\left(\left(\int_c \mathbf{p}\,\mathrm{d}s\otimes\mathbf{p}_0\right)\right)\right]\end{aligned} \tag{86}$$

The first integral in the formula above can also be written in the form

$$\int_c \mathbf{p}\otimes\mathbf{p}\,\mathrm{d}s = r_0^2\int_c \mathbf{n}\otimes\mathbf{n}\,\mathrm{d}s = r_0^3\int_{-\theta}^{\theta} \mathbf{W}\mathbf{e}_1\otimes\mathbf{W}\mathbf{e}_1\,\mathrm{d}\alpha \tag{87}$$

where

$$\mathbf{W} = \begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix} \tag{88}$$

is the skew tensor which rotates an arbitrary vector through the angle α counterclockwise. After some manipulation we get

$$\int_c \mathbf{p}\otimes\mathbf{p}\,\mathrm{d}s = r_0^3\begin{bmatrix}\theta+\sin\theta\cos\theta & 0 \\ 0 & \theta-\sin\theta\cos\theta\end{bmatrix} \tag{89}$$

According to equation (85), the second integral in equation (86) becomes

$$\mathbf{p}_0\otimes\mathbf{p}_0\int_c \mathrm{d}s = 2\theta r_0\left(r_0\frac{\sin\theta}{\theta}\right)^2 \mathbf{e}_1\otimes\mathbf{e}_1 \tag{90}$$

while, by means of equations (83) and (85), the third integral can be expressed as

$$\left(\int_c \mathbf{p}\,\mathrm{d}s\right)\otimes\mathbf{p}_0 = 2r_0^3\frac{\sin^2\theta}{\theta}\mathbf{e}_1\otimes\mathbf{e}_1 \tag{91}$$

Collecting the last three terms we get finally

$$\mathbf{J} = \delta r_0^3 E\begin{bmatrix}\theta+\sin\theta\cos\theta-\dfrac{2\sin^2\theta}{\theta} & 0 \\ 0 & \theta-\sin\theta\cos\theta\end{bmatrix} \tag{92}$$

The flexibility tensor $\mathbf{C}_f$ can then be easily computed by inverting the positive definite tensor $\mathbf{J}$.

Shear factors tensor for the thin-walled arc section

In order to evaluate the shear deformability tensor (27) the expression of tensor $\mathbf{\Gamma}$ is needed:

$$\begin{aligned}\mathbf{\Gamma} &= \int_c \frac{\mathbf{s}\otimes\mathbf{s}}{G\delta}\,\mathrm{d}s = \frac{1}{G\delta}\int_c\left[\int_0^s \delta E\mathbf{r}\,\mathrm{d}\xi\right]\otimes\left[\int_0^s \delta E\mathbf{r}\,\mathrm{d}\xi\right]\mathrm{d}s \\ &= \frac{\delta r_0 E^2}{G}\int_{-\theta}^{\theta}\left[\int_{-\theta}^{\alpha} r_0[\mathbf{p}-\mathbf{p}_0]\,\mathrm{d}\gamma\right]\otimes\left[\int_{-\theta}^{\alpha} r_0[\mathbf{p}-\mathbf{p}_0]\,\mathrm{d}\gamma\right]\mathrm{d}\alpha\end{aligned} \tag{93}$$

By means of the first Frenet formula and of equation (82) we have

$$\int_{-\theta}^{\alpha}\mathbf{p}\,\mathrm{d}\gamma = -r_0^2\int_{-\theta}^{\alpha}\dot{\mathbf{d}}\,\mathrm{d}\gamma = -r_0[\mathbf{d}(\alpha)-\mathbf{d}(-\theta)] \tag{94}$$

and by equation (85)

$$\int_{-\theta}^{\alpha}\mathbf{p}_0\,\mathrm{d}\gamma = r_0\frac{\sin\theta}{\theta}\int_{-\theta}^{\alpha}\mathbf{e}_1\,\mathrm{d}\gamma = r_0\frac{\sin\theta}{\theta}(\alpha+\theta)\mathbf{e}_1 \tag{95}$$

With

$$\mathbf{d}(\alpha) = \mathbf{k}\times\mathbf{n} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \tag{96}$$

we get, after some algebra

$$\mathbf{\Gamma} = \frac{\delta r_0^5 E^2}{G}\begin{bmatrix} \theta + \frac{3}{2}\sin 2\theta + \frac{2}{3}\theta\sin^2\theta - 4\frac{\sin^2\theta}{\theta} & 0 \\ 0 & \theta - \frac{3}{2}\sin 2\theta + 2\theta\cos^2\theta \end{bmatrix} \tag{97}$$

and then, since $A = 2\delta r_0\theta$, the shear factors tensor is given by

$$\boldsymbol{\chi} = \begin{bmatrix} \dfrac{6\theta^4 + 9\theta^3\sin 2\theta + 4\theta^4\sin^2\theta - 24\theta^2\sin^2\theta}{3(\theta^2+\theta\sin\theta\cos\theta - 2\sin^2\theta)^2} & 0 \\ 0 & \dfrac{2\theta^2 - 3\theta\sin 2\theta + 4\theta^2\cos^2\theta}{(\theta-\sin\theta\cos\theta)^2} \end{bmatrix} \tag{98}$$

It can be easily verified that for $\theta = \pi$ the same solution as the one of the open circle is recovered.

The shear centre

By means of equations (85) and (94) the tensor on the left-hand side of the linear system (47) can be written in the form

$$\begin{aligned}\int_c \mathbf{s}\otimes\mathbf{n}\,\mathrm{d}s &= \int_c\left[\int_0^s \delta E\mathbf{r}\,\mathrm{d}\xi\right]\otimes\mathbf{n}\,\mathrm{d}s = \delta r_0^2 E\int_{-\theta}^{\theta}\left[\int_{-\theta}^{\alpha}(\mathbf{p}-\mathbf{p}_0)\,\mathrm{d}\gamma\right]\otimes\mathbf{n}\,\mathrm{d}\alpha \\ &= -\delta r_0^3 E\int_{-\theta}^{\theta}\left[\mathbf{d}(\alpha)\otimes\mathbf{n} - \mathbf{d}(-\theta)\otimes\mathbf{n} + \frac{\sin\theta}{\theta}(\alpha+\theta)\mathbf{e}_1\otimes\mathbf{n}\right]\mathrm{d}\alpha\end{aligned} \tag{99}$$

Performing the integrations we have finally

$$\int_c \mathbf{s} \otimes \mathbf{n}\, ds = -\delta r_0^3 E \begin{bmatrix} 0 & -\theta - \sin\theta\cos\theta + \dfrac{2\sin^2\theta}{\theta} \\ \theta - \sin\theta\cos\theta & 0 \end{bmatrix} \tag{100}$$

The right-hand side of the linear system (47) can be expressed as

$$\begin{aligned} \int_c (\mathbf{s} \otimes \mathbf{n})\mathbf{r}\, ds = \int_c (\mathbf{n}\cdot\mathbf{r})\mathbf{s}\, ds &= \int_{-\theta}^{\theta} r_0 \mathbf{n}\cdot[\mathbf{p} - \mathbf{p}_0]\left[\int_{-\theta}^{\alpha} \delta r_0 E[\mathbf{p} - \mathbf{p}_0]\, d\gamma\right] d\alpha \\ &= -\delta r_0^4 E \int_{-\theta}^{\theta} \left\{\left[1 - \frac{\sin\theta}{\theta}\cos\alpha\right]\left[[\mathbf{d}(\alpha) - \mathbf{d}(-\theta)] + \frac{\sin\theta}{\theta}(\alpha + \theta)\mathbf{e}_1\right]\right\} d\alpha \end{aligned} \tag{101}$$

and after some algebra takes the form

$$\int_c (\mathbf{s} \otimes \mathbf{n})\mathbf{r}\, ds = -\delta r_0^4 E \begin{bmatrix} 0 \\ \sin\theta - 2\theta\cos\theta + \dfrac{\sin^2\theta\cos\theta}{\theta} \end{bmatrix} \tag{102}$$

so that the position vector of the shear centre is given by

$$\mathbf{p}_c = r_0 \begin{bmatrix} \dfrac{\theta\sin\theta - 2\theta^2\cos\theta + \sin^2\theta\cos\theta}{\theta^2 - \theta\sin\theta\cos\theta} \\ 0 \end{bmatrix} \tag{103}$$

It can be easily verified that for $\theta = \pi$ the same solution as the one of the open circle is recovered and that for $\theta \to 0$, i.e. when the section reduces to a point, the vector $\mathbf{p}_c$ tends to $r_0\mathbf{e}_1$.

THE STIFFNESS TENSOR OF A BEAM ELEMENT

The flexural behaviour of beams with shear deformability can be completely described by the stiffness tensor of the beam element whose coefficients will depend upon the shear deformability tensor.

The stiffness tensor of the beam element has been already obtained by the first two authors in a completely co-ordinate-free version,[3] and it is here reported for the sake of completeness. The details of its derivation can be found in Reference 3.

Let us consider a straight beam of length l. Denoting by $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ the flexural rotations at the end sections 1 and 2 of the beam and by $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ the corresponding transverse displacements of the shear centre, the stiffness tensor can be expressed as

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{H}_{21} & \mathbf{H}_{22} \\ \mathbf{H}_{11} & \mathbf{H}_{21} & \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{H}_{12} & \mathbf{H}_{22} & \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{k} \times \boldsymbol{\phi}_1 \\ \mathbf{k} \times \boldsymbol{\phi}_2 \\ \boldsymbol{\Delta}_1 \\ \boldsymbol{\Delta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{k} \times \mathbf{b}_1 \\ \mathbf{k} \times \mathbf{b}_2 \\ \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{104}$$

where $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are the bending moments and shearing forces at the terminal sections of the beam and the unit vector $\mathbf{k}$ points from section 1 towards section 2.

Defining the adimensional tensor

$$\mathbf{A} = \frac{12\mathbf{J}\mathbf{C}_s}{l^2} \tag{105}$$

and denoting by $\mathbf{I}$ the identity tensor, the elements of the stiffness tensor are given by

$$
\begin{aligned}
\mathbf{B}_{11} &= \mathbf{B}_{22} = \frac{1}{l}(\mathbf{I}+\mathbf{A})^{-1}(4\mathbf{I}+\mathbf{A})\mathbf{J} \\
\mathbf{B}_{12} &= \mathbf{B}_{21} = \frac{1}{l}(\mathbf{I}+\mathbf{A})^{-1}(2\mathbf{I}-\mathbf{A})\mathbf{J} \\
\mathbf{H}_{11} &= \mathbf{H}_{21} = -\frac{6}{l^2}(\mathbf{I}+\mathbf{A})^{-1}\mathbf{J} \\
\mathbf{H}_{22} &= \mathbf{H}_{12} = \frac{6}{l^2}(\mathbf{I}+\mathbf{A})^{-1}\mathbf{J} \\
\mathbf{S}_{11} &= \mathbf{S}_{22} = \frac{12}{l^3}(\mathbf{I}+\mathbf{A})^{-1}\mathbf{J} \\
\mathbf{S}_{12} &= \mathbf{S}_{21} = -\frac{12}{l^3}(\mathbf{I}+\mathbf{A})^{-1}\mathbf{J}
\end{aligned}
\tag{106}
$$

As proved in Reference 3, all tensors in equation (106) turn out to be symmetric and, in addition, the diagonal elements $\mathbf{B}_{11}$, $\mathbf{B}_{22}$, $\mathbf{S}_{11}$, $\mathbf{S}_{22}$ are positive definite.

From equation (106) it is apparent that the stiffness tensor in equation (104) provides the generalization of the classical stiffness tensor for beams with shear deformability which can be found in standard textbooks and whose validity is limited to the case of symmetric cross sections, see e.g. References 8–10.

A NUMERICAL EXAMPLE

A simple application of the theory is presented here to show a relevant example in which the principal directions of the shear deformability tensor and of the bending deformability tensor turn out to be significantly different.

To this aim let us consider a thin-walled L-shaped cross section with constant thickness and having a one-half ratio between the side lengths as shown in Figure 3.

The position vectors of the vertices with respect to the origin are given by

$$
\mathbf{p}_1 = b\begin{bmatrix}1\\0\end{bmatrix},\quad \mathbf{p}_2 = \begin{bmatrix}0\\0\end{bmatrix},\quad \mathbf{p}_3 = b\begin{bmatrix}0\\2\end{bmatrix}
\tag{107}
$$

so that the elastic centroid is

$$
\mathbf{p}_0 = \frac{b}{6}\begin{bmatrix}1\\4\end{bmatrix}
\tag{108}
$$

Setting $\mathbf{r} = \mathbf{p} - \mathbf{p}_0$, we get the position of the vertices with respect to the centroid:

$$
\mathbf{r}_1 = \frac{b}{6}\begin{bmatrix}5\\-4\end{bmatrix},\quad \mathbf{r}_2 = \frac{b}{6}\begin{bmatrix}-1\\-4\end{bmatrix},\quad \mathbf{r}_3 = \frac{b}{6}\begin{bmatrix}-1\\8\end{bmatrix}
\tag{109}
$$

By applying equation (51), the second elastic area moment tensor turns out to be

$$
\mathbf{J} = \frac{Eb^3\delta}{36}\begin{bmatrix}9 & -12\\-12 & 48\end{bmatrix}
\tag{110}
$$

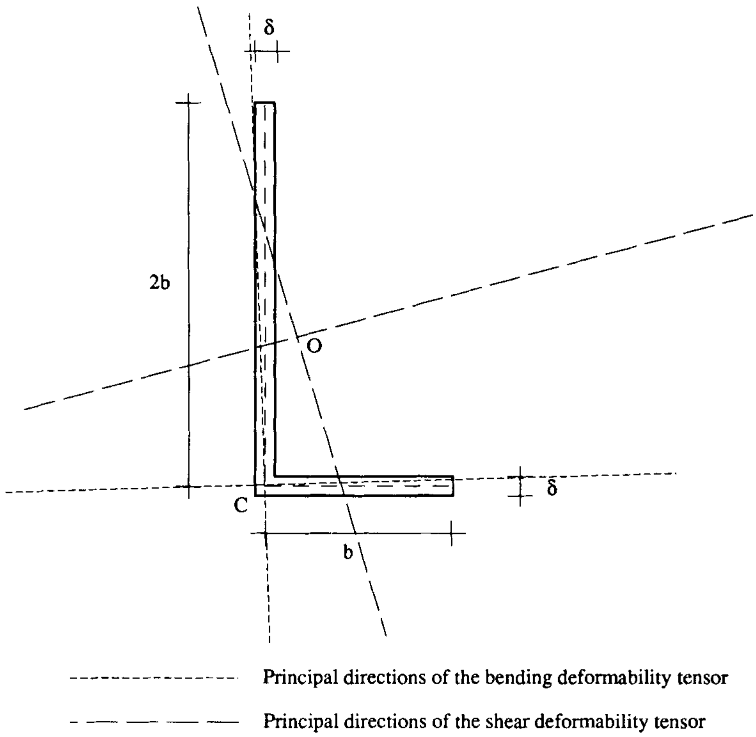

Figure 3. Thin-walled L-section

and, on the basis of the equations (55), the tensor $\mathbf{\Gamma}$ has the form

$$\mathbf{\Gamma} = \frac{E^2 b^5 \delta}{360G} \begin{bmatrix} 53 & -130 \\ -130 & 416 \end{bmatrix} \tag{111}$$

By computing the inverse of $\mathbf{J}$:

$$\mathbf{C}_{\mathrm{f}} = \frac{1}{8Eb^3\delta} \begin{bmatrix} 48 & 12 \\ 12 & 9 \end{bmatrix} \tag{112}$$

the final expression of the shear factors tensor is finally obtained, by means of equations (15) and (27):

$$\boldsymbol{\chi} = \frac{3}{40} \begin{bmatrix} 56 & 1 \\ 1 & 23 \end{bmatrix} \tag{113}$$

The eigenvalues of the tensor $\mathbf{J}$ are given by

$$\xi_1 = \frac{19 - \sqrt{233}}{24}, \quad \xi_2 = \frac{19 + \sqrt{233}}{24} \tag{114}$$

and the corresponding eigenvectors can be written as

$$\mathbf{j}_1 = \frac{-13+\sqrt{233}}{\sqrt{466-26\sqrt{233}}}\mathbf{o}_1 + \frac{8}{\sqrt{466-26\sqrt{233}}}\mathbf{o}_2$$
$$\mathbf{j}_2 = \frac{13+\sqrt{233}}{\sqrt{466+26\sqrt{233}}}\mathbf{o}_1 - \frac{8}{\sqrt{466+26\sqrt{233}}}\mathbf{o}_2 \tag{115}$$

where $\mathbf{o}_1$ and $\mathbf{o}_2$ respectively denote the unit vectors parallel to the shorter and the longer branch of the cross section.

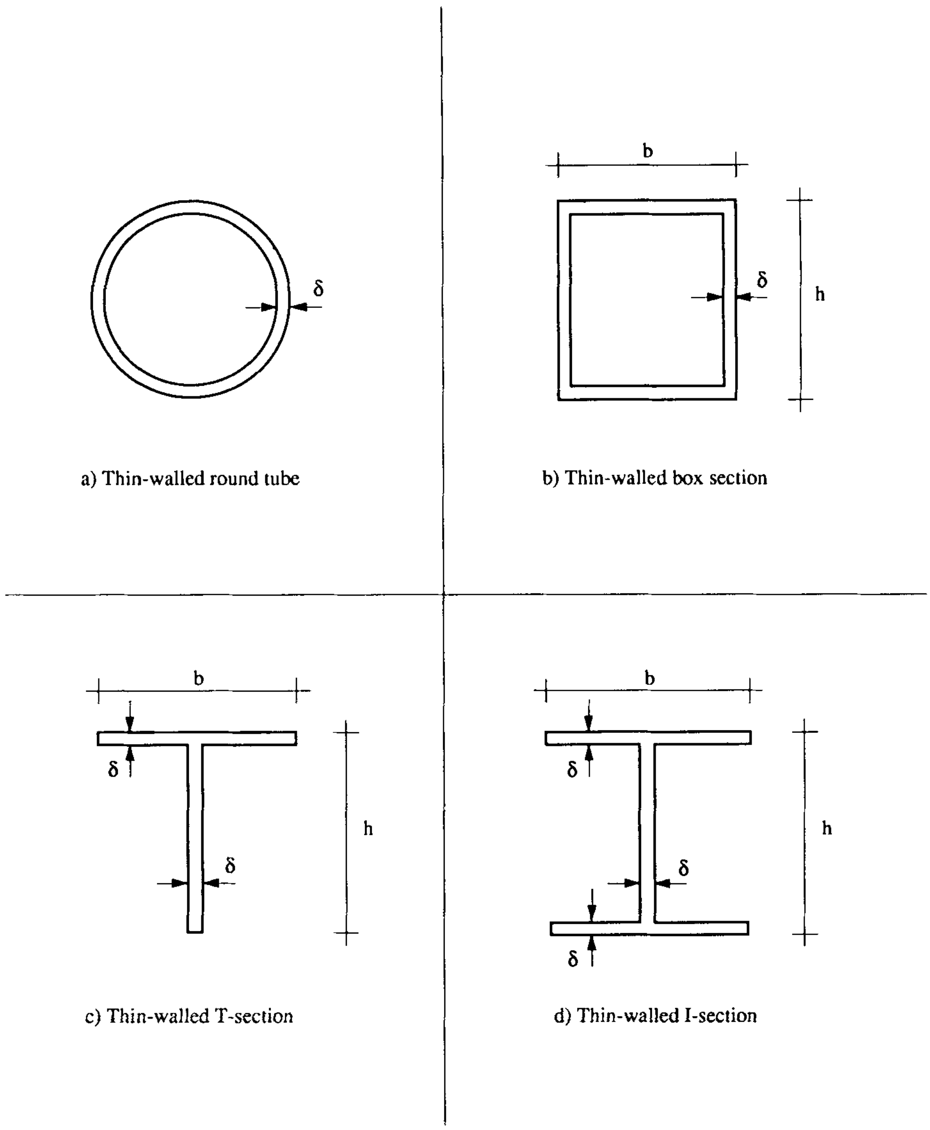

Figure 4. Thin-walled sections considered in the comparison of the shear factors values

Table I. Comparison of the computed shear factors for different values of Poisson's ratio ν

Shear factor values		Round tube	Square tube	T-section	I-section
Present theory		2·000	2·400	3·380	2·544
	$\nu = 0$	2·000	2·400	3·380	2·544
Cowper[1]	$\nu = 0{\cdot}1$	1·954	2·359	3·381	2·554
	$\nu = 0{\cdot}3$	1·885	2·296	3·383	2·569

The eigenvalues of the shear factors tensor χ are given by

$$\eta_1 = \frac{79 - \sqrt{1093}}{2}, \quad \eta_2 = \frac{79 + \sqrt{1093}}{2} \tag{116}$$

and the corresponding eigenvectors are expressed by

$$\begin{aligned}\mathbf{c}_1 &= \frac{33 + \sqrt{1093}}{\sqrt{2186 + 66\sqrt{1093}}}\mathbf{o}_1 - \frac{2}{\sqrt{2186 + 66\sqrt{1093}}}\mathbf{o}_2 \\ \mathbf{c}_2 &= \frac{33 - \sqrt{1093}}{\sqrt{2186 - 66\sqrt{1093}}}\mathbf{o}_1 - \frac{2}{\sqrt{2186 - 66\sqrt{1093}}}\mathbf{o}_2\end{aligned} \tag{117}$$

The relative rotation between the principal directions given by equations (115) and (117) is clearly depicted in Figure 3.

An automatic procedure has been implemented in a computer program to deal with thin-walled cross sections of arbitrary shape.

Four additional examples have been developed for symmetric cross sections in order to compare the values of the shear factors along the axis of symmetry with the ones obtained by Cowper[1] for Timoshenko beam theory by integrating the equations of the three-dimensional theory of elasticity.

Additional results on the shear factors values for symmetric thin-walled cross sections can be found, e.g., in References 11–13.

The shapes of the four cross sections are sketched in Figure 4 and the comparison of the results is reported in Table I. The numerical results for the three last examples refer to the special case in which $b = h$.

It has to be remarked, however, that the present method and Cowper's approach always yield the same results when the Poisson's ratio ν is zero. This occurrence can be justified by the fact that the assumed shear stress distribution satisfies the equilibrium conditions but cannot meet the compatibility requirements unless $\nu = 0$.[14]

CONCLUSIONS

It has been shown that the elastic analysis of thin-walled beams, with open and closed cross sections of arbitrary shape, can be simply and effectively achieved when transverse shear deformations need to be taken into account.

To this aim, explicit formulas have been provided for the calculation of the shear deformability tensor and shear centre location in the case of polygonal, circular and arc-shaped cross sections.

A fairly exhaustive presentation of the relevant formulas for open and closed cross sections has been developed, thus providing useful tools for practical applications.

A new co-ordinate-free expression of the stiffness tensor for beam elements which takes into account the shear deformability is provided. It generalizes the classical formulas to the case of beams with non-symmetric cross sections.

A computer code for the automatic analysis of thin-walled cross sections of arbitrary shape has been developed and some numerical examples have been carried out.

The shear deformability of thin-walled multicell cross sections, which have not been dealt with here, will be the subject of a forthcoming paper. In this more complex case, actually, the evaluation of the shear stress distribution requires the adoption of methods and procedures which are inherent to graph theory.

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APPENDIX I

Let $\mathbf{\Gamma}$ be the tensor defined by equation (37). In order to prove its positive definiteness it is sufficient to show that $\mathbf{\Gamma v}\cdot\mathbf{v} > 0$ for every constant vector $\mathbf{v}$. Since

$$\left[\oint_c \frac{\mathbf{s}\otimes\mathbf{s}}{G\delta}\,\mathrm{d}s\right]\mathbf{v}\cdot\mathbf{v} = \oint_c \frac{(\mathbf{s}\otimes\mathbf{s})\mathbf{v}\cdot\mathbf{v}}{G\delta}\,\mathrm{d}s = \oint_c \frac{(\mathbf{s}\cdot\mathbf{v})(\mathbf{s}\cdot\mathbf{v})}{G\delta}\,\mathrm{d}s \tag{118}$$

we can write

$$\mathbf{\Gamma v}\cdot\mathbf{v} = \oint_c \frac{(\mathbf{s}\cdot\mathbf{v})^2}{G\delta}\,\mathrm{d}s - \frac{\left[\left(\oint_c \frac{\mathbf{s}}{G\delta}\,\mathrm{d}s\right)\cdot\mathbf{v}\right]^2}{\oint_c \frac{\mathrm{d}s}{G\delta}} \tag{119}$$

Hence we have to show that

$$\left(\oint_c \frac{(\mathbf{s}\cdot\mathbf{v})^2}{G\delta}\,\mathrm{d}s\right)\left(\oint_c \frac{\mathrm{d}s}{G\delta}\right) > \left(\oint_c \frac{\mathbf{s}\cdot\mathbf{v}}{G\delta}\,\mathrm{d}s\right)^2 \quad \forall\,\mathbf{v}\neq\mathbf{0} \tag{120}$$

If $f(s)$ and $g(s)$ are two square integrable real functions, the Cauchy–Schwarz inequality[15] gives

$$\left|\oint_c f(s)g(s)\,\mathrm{d}s\right|^2 \leqslant \left(\oint_c f(s)^2\,\mathrm{d}s\right)\left(\oint_c g(s)^2\,\mathrm{d}s\right) \tag{121}$$

Now, with the positions

$$f(s) = \frac{\mathbf{s}\cdot\mathbf{v}}{\sqrt{G\delta}} \quad g(s) = \frac{1}{\sqrt{G\delta}} \tag{122}$$

equation (121) becomes

$$\left(\oint_c \frac{\mathbf{s}\cdot\mathbf{v}}{G\delta}\,\mathrm{d}s\right)^2 < \left(\oint_c \frac{(\mathbf{s}\cdot\mathbf{v})^2}{G\delta}\,\mathrm{d}s\right)\left(\oint_c \frac{\mathrm{d}s}{G\delta}\right) \tag{123}$$

where the equality sign has been ruled out since $\mathbf{s}\cdot\mathbf{v}$ cannot be constant unless $\mathbf{v} = \mathbf{0}$.

APPENDIX II

Notation

A	area of the beam cross section
$\mathbf{A}$	adimensional tensor—see equation (105)
A_i	elastic area of the ith branch of a polygonal section
A_G	see equation (12)
$\mathbf{B}_{11}, \mathbf{B}_{12}$	
$\mathbf{B}_{21}, \mathbf{B}_{22}$	
$\mathbf{H}_{11}, \mathbf{H}_{12}$	stiffness submatrices of the beam—see equation (106)
$\mathbf{H}_{21}, \mathbf{H}_{22}$	
$\mathbf{S}_{11}, \mathbf{S}_{12}$	
$\mathbf{S}_{21}, \mathbf{S}_{22}$	
$\mathbf{b}_1, \mathbf{b}_2$	bending moments at the end section of the beam
$\mathbf{C}_f$	bending deformability tensor of the cross section
$\mathbf{C}_s$	shear deformability tensor of the cross section
$\mathbf{c}_f$	flexural curvature of the cross section
C	shear centre
c	length of the centre line of the cross section
c_i	length of the ith branch of a polygonal section
$\mathbf{d}$	unit vector tangent to the centre line of the generic thin wall
$\mathbf{d}_0$	unit vector tangent to the centre line of the circular section at $s = 0$
$\mathbf{e}$	arbitrary unit vector
$\mathbf{e}_1$	unit vector along the axis of symmetry of an arc section
E	Young's modulus of elasticity
E_i	Young's modulus of elasticity of the ith branch of a polygonal section
$\mathbf{f}_1, \mathbf{f}_2$	shearing forces at the end section of the beam
G	shear modulus
G_i	shear modulus of the ith branch of a polygonal section
$\mathbf{I}$	identity tensor
$\mathbf{J}$	second elastic area moment tensor
$\mathbf{k}$	unit vector along the beam axis
l	length of the beam
$\mathbf{m}_f$	bending moment
$\mathbf{n}$	unit vector orthogonal to the centre line of the generic thin wall
$\mathbf{n}_i$	unit vector orthogonal to the centre line of the ith branch of a polygonal section
O	elastic centroid
$\mathbf{p}$	position vector of a generic point on a cross section with respect to an arbitrary origin
$\mathbf{p}_c$	position vector of the shear centre with respect to the elastic centroid of the cross section
$\mathbf{p}_O$	position vector of the elastic centroid with respect to an arbitrary origin
$q(s)$	shear flow at abscissa s in case of open sections
$q_c(s)$	constant shear flow at abscissa s in case of closed sections
q_a	see equation (28)
$\mathbf{R}$	see equation (65)
$\mathbf{R}_{ij}$	see equation (49)

$\mathbf{r}$	position vector of the generic point with respect to the elastic centroid of the cross section
r_0	centre line radius of circular and arc sections
$\mathbf{r}_i$	position vector of the ith vertex of a polygonal section with respect to the elastic centroid of the cross section
$\mathbf{s}(s)$	first elastic area moment vector at abscissa s with respect to the elastic centroid of the cross section
$\mathbf{s}_a$	see equation (31)
$\mathbf{s}_{i-1}$	first elastic area moment vector of the first $i-1$ branches of a polygonal section with respect to the elastic centroid of the cross section
$\mathbf{s}_i(\lambda)$	see equation (52)
s	profile co-ordinate
$\mathrm{sym}\,\mathbf{M}$	symmetric part of the tensor $\mathbf{M}$
$\mathbf{M}^T$	transpose of the tensor $\mathbf{M}$
$\mathbf{t}$	shearing force
$\mathbf{t}_1, \mathbf{t}_2$	shearing forces at the end section of the beam
$\mathbf{u}$	transverse displacement of the shear centre
$\mathbf{W}$	tensor—see equation (88)
$\boldsymbol{\chi}$	shear factors tensor of the cross section
χ	eigenvalue of the shear factors tensor of the cross section
$\boldsymbol{\delta}$	transverse shearing strain of the axis of the shear centres
δ	thickness of a generic wall of the cross section
δ_i	thickness of the ith wall of a polygonal section
$\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2$	transverse displacements of the shear centre at the end sections of the beam
ε	extensional strain of the generic longitudinal fibre of the beam
$\boldsymbol{\phi}$	flexural rotation of the cross section
$\boldsymbol{\phi}_1, \boldsymbol{\phi}_2$	flexural rotations at the end section of the beam
$\boldsymbol{\Gamma}$	tensor—see equations (25) and (37)
$\boldsymbol{\Gamma}_a, \boldsymbol{\Gamma}_c$	tensors—see equation (37)
λ	see equation (47)
$\boldsymbol{\gamma}$	shear strain vector
$\boldsymbol{\Pi}$	tensor—see equation (11)
$\boldsymbol{\tau}$	shear stress vector
θ	half-width of the angle subtending the arc section
$'$	derivative symbol along the axis of the beam
$\cdot$	scalar product
$[\dot{\ }]$	derivative symbol with respect to s
$\times$	vector product
$\otimes$	tensor product

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