Error estimates in mixed elastostatics

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Key words: mixed elastostatics, error estimate, LBB condition.

Summary. Error estimates in mixed elastostatics is a topic of great interest in computational mechanics. An assessment of the approximation energy error is provided in terms of a parameter h which is the elements' diameter in the finite element method. A sufficient condition for the convergence in energy of the approximate solution is expressed in terms of suitable properties of the interpolating subspaces. The result contributes an alternative form of the well known LBB condition.

1 PRELIMINARY NOTIONS

The formal framework for the analysis of linear structural models makes reference to the following pairs of dual Hilbert spaces: the kinematic-force pair $\mathcal{V}$, $\mathcal{F}$ and the strain-stress pair $\mathcal{D}$, $\mathcal{S}$. The kinematic operator $\mathbf{B} \in BL(\mathcal{V},\mathcal{D})$ gives the linearized strain due to a prescribed displacement field and the dual equilibrium operator $\mathbf{B}' \in BL(\mathcal{S},\mathcal{F})$ provides the force system in equilibrium with a given stress field. Here $BL()$ stands for bounded linear map. Stress and strain spaces $\mathcal{S}$, $\mathcal{D}$ may be identified with a pivot Hilbert space $\mathcal{H}$ (square-integrable fields). The inner product in $\mathcal{H}$ is denoted by $(\!(\,\cdot\,,\,\cdot\,)\!)$ and the duality pairing in $\mathcal{F}\times\mathcal{V}$ by $\langle\cdot,\cdot\rangle$. The pair of dual kinematic and equilibrium operators defines the bilinear form $\mathbf{b}(\mathbf{v},\boldsymbol{\sigma}) := (\!(\,\boldsymbol{\sigma}\,,\,\mathbf{B}\mathbf{v}\,)\!) = \langle \mathbf{B}'\boldsymbol{\sigma},\mathbf{v}\rangle$ for any $\boldsymbol{\sigma}\in\mathcal{S}$ and $\mathbf{v}\in\mathcal{V}$. Linear boundary constraints define a linear subspace $\mathcal{L}\subset\mathcal{V}$ of conforming displacements and a Korn type inequality implies that the range of $\mathbf{B}\in BL(\mathcal{V},\mathcal{D})$ is closed, that is $\|\mathbf{B}\mathbf{u}\|_{\mathcal{H}} \;\geq\; c_{\mathbf{b}}\,\|\mathbf{u}\|_{\mathcal{L}/\mathrm{Ker}\,\mathbf{B}}$ for any conforming displacement field $\mathbf{u}\in\mathcal{L}$ and any conformity space $\mathcal{L}$. The internal elastic compliance is expressed by a continuous symmetric and positive bilinear form $\mathbf{c}(\boldsymbol{\sigma},\boldsymbol{\tau}) = (\!(\,\mathbf{C}\boldsymbol{\sigma}\,,\,\boldsymbol{\tau}\,)\!)$ on $\mathcal{S}\times\mathcal{S}$, where $\mathbf{C}\in BL(\mathcal{S},\mathcal{D})$. The elements of the kernel of $\mathbf{C}$ are the elastically ineffective stress fields. We assume the $\mathcal{H}$-ellipticity of $\mathbf{c}$, i.e. $\mathbf{c}(\boldsymbol{\sigma},\boldsymbol{\sigma}) \;\geq\; \alpha\,\|\boldsymbol{\sigma}\|^2_{\mathcal{H}/\mathrm{Ker}\,\mathbf{C}}$ for any $\boldsymbol{\sigma}\in\mathcal{S}$. The external elastic stiffness is expressed by a continuous symmetric and positive bilinear form $\mathbf{k}(\mathbf{u},\mathbf{v}) = \langle\mathbf{K}\mathbf{u},\mathbf{v}\rangle$ on $\mathcal{V}\times\mathcal{V}$, with $\mathbf{K}\in BL(\mathcal{V},\mathcal{F})$. It is assumed to fulfil the inequality $\mathbf{k}(\mathbf{u},\mathbf{u}) \;\geq\; c_{\mathbf{k}}\,\|\mathbf{u}\|^2_{\mathcal{L}/(\mathrm{Ker}\,\mathbf{K}\cap\mathrm{Ker}\,\mathbf{B})}$ for any $\mathbf{u}\in\mathrm{Ker}\,\mathbf{B}\cap\mathcal{L}$. The elements of the kernel of $\mathbf{K}$ are kinematic fields compatible with null reactions. The mixed elastostatic problem is formulated in operator form as

$$\mathbb{M})\quad \begin{cases}\mathbf{K}\mathbf{u}+\mathbf{B}'\boldsymbol{\sigma}=\mathbf{f}\\ \mathbf{B}\mathbf{u}-\mathbf{C}\boldsymbol{\sigma}=\boldsymbol{\delta}\end{cases} \quad\text{or}\quad \mathbf{S}\begin{vmatrix}\mathbf{u}\\ \boldsymbol{\sigma}\end{vmatrix} = \begin{vmatrix}\mathbf{K} & \mathbf{B}'\\ \mathbf{B} & -\mathbf{C}\end{vmatrix}\begin{vmatrix}\mathbf{u}\\ \boldsymbol{\sigma}\end{vmatrix} = \begin{vmatrix}\mathbf{f}\\ \boldsymbol{\delta}\end{vmatrix}\,,$$

where $\mathbf{S} \in BL(\mathcal{V} \times \mathcal{S}, \mathcal{F} \times \mathcal{D})$ is called the structural operator. The former equation is the equilibrium condition, in which $\mathbf{f} \in \mathcal{F}$ is the external load, $-\mathbf{K}\mathbf{u} \in \mathcal{F}$ is the reaction of the external elastic constraints, $\mathbf{B}'\boldsymbol{\sigma} \in \mathcal{F}$ is the external force in equilibrium with the stress $\boldsymbol{\sigma}$. The latter equation is the kinematic compatibility condition in which $\boldsymbol{\delta} \in \mathcal{D}$ is an imposed strain, $\mathbf{C}\boldsymbol{\sigma} \in \mathcal{D}$ is the elastic strain and $\mathbf{B}\mathbf{u} \in \mathcal{D}$ is the strain field compatible with the displacement $\mathbf{u}$. The variational form of the mixed elastostatic problem is given by

$$\mathbb{MV}) \quad \begin{cases} \mathbf{k}(\mathbf{u},\mathbf{v}) + \mathbf{b}(\mathbf{v},\boldsymbol{\sigma}) = \langle \mathbf{f}, \mathbf{v} \rangle & \mathbf{u} \in \mathcal{V}, \quad \forall\, \mathbf{v} \in \mathcal{V}, \\ \mathbf{b}(\mathbf{u},\boldsymbol{\tau}) - \mathbf{c}(\boldsymbol{\sigma},\boldsymbol{\tau}) = \langle \boldsymbol{\delta}, \boldsymbol{\tau} \rangle & \boldsymbol{\sigma} \in \mathcal{S}, \quad \forall\, \boldsymbol{\tau} \in \mathcal{S}. \end{cases}$$

Such problems have been firstly analysed in the pioneering works [1], [2], [3], [4] and in [5], [6]. A comprehensive presentation of the state of the art can be found in [7]. In the sequel conforming displacement fields will be taken into account. From a mathematical point of view the formulation of a discrete model consists in imposing that the dispacement and stress fields belong to finite dimensional linear subspaces $\mathcal{L}_h \subset \mathcal{L}$ and $\mathcal{S}_h \subset \mathcal{H}$. Active forces ℓ belong to the linear space $\mathcal{F}_h$ which is dual of $\mathcal{L}_h \subset \mathcal{L}$ in the topology induced by $\mathcal{L}$. It is worth observing that, if $\mathcal{L}$ is a normed space, all the norms induced on $\mathcal{L}_h \subset \mathcal{L}$ are equivalent [8]. The reactive forces $\mathbf{r}_h$ belong to the discrete space $\mathcal{R}_h = \mathcal{L}_h^{\perp}$. The discrete mixed elastic problem can then be defined in the following form:

$$\mathbb{MV}_h) \quad \begin{cases} \mathbf{k}(\mathbf{u}_h,\mathbf{v}_h) + \mathbf{b}(\boldsymbol{\sigma}_h,\mathbf{v}_h) = \langle \ell, \mathbf{v}_h \rangle, & \mathbf{u}_h \in \mathcal{L}_h, \quad \forall\, \mathbf{v}_h \in \mathcal{L}_h, \\ \mathbf{b}(\boldsymbol{\tau}_h,\mathbf{u}_h) - \mathbf{c}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) = (\!(\boldsymbol{\delta}, \boldsymbol{\tau}_h)\!), & \boldsymbol{\sigma}_h \in \mathcal{S}_h, \quad \forall\, \boldsymbol{\tau}_h \in \mathcal{S}_h. \end{cases}$$

This problem is well-posedness since the involved spaces are finite dimensional.

2 ERROR ESTIMATE

Let us assume that the uniqueness and well-posedness conditions of the continuum problem and the uniqueness condition of the displacement field of the discrete problem are fulfilled. We will provide an estimate of the approximation error in energy, defined by $\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathcal{H}}$. Following the treatment developed in [7], we employ the triangle inequality to conclude that

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathcal{H}} \le\ & \|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}} + \\ & + \|\mathbf{u}_h - \overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}}, \end{aligned}$$

for any $\overline{\mathbf{u}}_h \in \mathcal{L}_h$ and $\overline{\boldsymbol{\sigma}}_h \in \mathcal{S}_h$. The first step consists in increasing the term $\|\mathbf{u}_h - \overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}}$ by means the distance $\|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}}$. To this end we observe that the problems $\mathbb{MV})$ and $\mathbb{MV}_h)$ yield the following relations:

$$\mathbb{P}) \quad \begin{cases} \mathbf{k}(\mathbf{u} - \overline{\mathbf{u}}_h, \mathbf{v}_h) + \mathbf{b}(\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h, \mathbf{v}_h) = \mathbf{k}(\mathbf{u}_h - \overline{\mathbf{u}}_h, \mathbf{v}_h) + \mathbf{b}(\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}}_h, \mathbf{v}_h), \\ \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u} - \overline{\mathbf{u}}_h) - \mathbf{c}(\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) = \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h - \overline{\mathbf{u}}_h) - \mathbf{c}(\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h). \end{cases}$$

The terms at r.h.s. of $\mathbb{P})$ are continuous linear functionals on $\mathcal{L}_h$ and $\mathcal{S}_h$. The solution of the problem $\mathbb{P})$ is bounded above by the following inequality:

$$\|\mathbf{u}_h - \overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}_h - \overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}} \le m_h \left[\|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma} - \overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}} \right],$$

where m_h is a positive and bounded nonlinear function of $\|\mathbf{c}\|$, $\|\mathbf{k}\|$, $c_{\mathbf{b}h}$, $c_{\mathbf{k}h}$, α_h on bounded subsets. By triangle inequality we deduce thus that

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathcal{H}} \le (1+m_h)\,(\|\mathbf{u}-\overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}})\,,$$

for any $\overline{\mathbf{u}}_h \in \mathcal{L}_h$ and $\overline{\boldsymbol{\sigma}}_h \in \mathcal{S}_h$. Setting $c_h = 1+m_h$ we conclude that

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathcal{H}} \le c_h \Big[\inf_{\overline{\mathbf{u}}_h\in\mathcal{L}_h} \|\mathbf{u}-\overline{\mathbf{u}}_h\|_{\mathcal{L}} + \inf_{\overline{\boldsymbol{\sigma}}_h\in\mathcal{S}_h} \|\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}}\Big].$$

3 *LBB* CONDITION AND CONVERGENCE

If the constant c is independent of h, the convergence in energy of the approximate solution to the exact one is ensured if there are sufficient properties of interpolation of the discrete subspaces. In the literature a condition which guarantees such properties is referred to as Ladyzhenskaya-Babuška-Brezzi's condition (*LBB condition*, see [1], [2], [3], [4], [5], [7]). An alternative form of LBB condition is provided in the next theorem.

Theorem 1. *Let the mixed elastic problem* $\mathbb{MV})$ *be well-posed with an unique solution and the elasticity of the structure be not singular so that* $Ker\,\mathbf{C} = \{\mathbf{o}\}$ *with the kinematic operator a* Korn*'s operator. Further, let us assume that the families of the interpolating linear subspaces* $\mathcal{L}_h \subset \mathcal{L}$ *and* $\mathcal{S}_h \subset \mathcal{H}$ *meet the conditions*

$$a)\quad \mathbf{B}\,\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\,\mathbf{o}\,\}\,,$$
$$b)\quad \mathbf{B}\,\mathcal{L}_h + \mathcal{S}_h^{\perp} \quad \textit{uniformly closed in} \quad \mathcal{H}\,.$$

Then an asymptotic estimate of the approximation error can be inferred from an asymptotic estimate of the interpolation error.

Proof. Let us preliminarily observe that the condition $a)$ is equivalent to $Ker\,\mathbf{B}_h = Ker\,\mathbf{B} \cap \mathcal{L}_h$. The uniqueness of the displacement solution of the continuous problem, given that $Ker\,\mathbf{K} \cap Ker\,\mathbf{B} = \{\mathbf{o}\}$, implies $Ker\,\mathbf{K}_h \cap Ker\,\mathbf{B}_h = Ker\,\mathbf{K} \cap Ker\,\mathbf{B} \cap \mathcal{L}_h = \{\,\mathbf{o}\,\}$. Hence the uniqueness of solution of the discrete problem $\mathbb{MV}_h)$ in terms of interpolating displacement fields is met. The ellipticity condition on $Ker\,\mathbf{B}$ of the bilinear form $\mathbf{k}$

$$\mathbf{k}\,(\mathbf{u},\mathbf{u}) \ge c_{\mathbf{k}}\,\|\mathbf{u}\|^2_{\mathcal{L}/(\mathrm{Ker}\,\mathbf{K}\cap\mathrm{Ker}\,\mathbf{B})}\,, \qquad \forall\,\mathbf{u} \in Ker\,\mathbf{B}\,,$$

can be rewritten as $\mathbf{k}\,(\mathbf{u},\mathbf{u}) \ge c_{\mathbf{k}}\,\|\mathbf{u}\|^2_{\mathcal{L}}$ for any $\mathbf{u} \in Ker\,\mathbf{B}$, so that:

$$\mathbf{k}\,(\mathbf{u}_h,\mathbf{u}_h) \ge c_{\mathbf{k}}\,\|\mathbf{u}_h\|^2_{\mathcal{L}} \quad \forall\,\mathbf{u}_h \in Ker\,\mathbf{B}_h = Ker\,\mathbf{B} \cap \mathcal{L}_h\,,$$

i.e. the uniform ellipticity on $Ker\,\mathbf{B}_h$ of the bilinear form $\mathbf{k}$. The condition $b)$ is equivalent to the uniform closure of the family of subspaces $\mathrm{Im}\,\mathbf{B}_h = \mathbf{B}\mathcal{L}_h + \mathcal{S}_h^{\perp}$ which is expressed by the inequality

$$\sup_{\boldsymbol{\tau}_h\in\mathcal{S}_h} \frac{((\,\boldsymbol{\tau}_h\,,\,\mathbf{B}\mathbf{u}_h\,))}{\|\boldsymbol{\tau}_h\|_{\mathcal{H}}} \ge c_{\mathbf{b}}\,\|\mathbf{u}_h\|_{\mathcal{H}/\mathrm{Ker}\,\mathbf{B}_h} \quad \forall\,\mathbf{u}_h \in \mathcal{L}_h\,,$$

with $c_{\mathbf{b}}$ independent of h. Then the inequality above together with the problem $\mathbb{P})$ allows us to state that $\|\mathbf{u}_h-\overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}_h-\overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}} \le m\,(\|\mathbf{u}-\overline{\mathbf{u}}_h\|_{\mathcal{L}} + \|\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}})$, where m is a nonlinear function of $\|\mathbf{c}\|$, $\|\mathbf{k}\|$, $c_{\mathbf{b}}$, $c_{\mathbf{k}}$, α, and is positive and bounded on bounded subsets [8]. By the triangle inequality we deduce that

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{L}}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathcal{H}} \le (1+m)\left[\|\mathbf{u}-\overline{\mathbf{u}}_h\|_{\mathcal{L}}+\|\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}}\right],$$

for any $\overline{\mathbf{u}}_h \in \mathcal{L}_h$ and $\overline{\boldsymbol{\sigma}}_h \in \mathcal{S}_h$. Setting $c = 1+m$ we conclude that

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathcal{L}}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathcal{H}} \le c\left[\inf_{\overline{\mathbf{u}}_h\in\mathcal{L}_h}\|\mathbf{u}-\overline{\mathbf{u}}_h\|_{\mathcal{L}}+\inf_{\overline{\boldsymbol{\sigma}}_h\in\mathcal{S}_h}\|\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}}\right]. \qquad \blacksquare$$

Remark 1. Observing that $\mathbf{B}\,\mathcal{L}_h \cap \mathcal{S}_h^{\perp} = \{\,\mathbf{o}\,\}$, the uniform closure condition in $\mathcal{H}$ of the family $\mathbf{B}\,\mathcal{L}_h + \mathcal{S}_h^{\perp}$ can be expressed by $\|\boldsymbol{\Pi}\mathbf{B}\mathbf{u}_h\|_{\mathcal{H}} \ge c\,\|\mathbf{B}\mathbf{u}_h\|_{\mathcal{H}}$ for any $\mathbf{u}_h \in \mathcal{L}_h$, in which $\boldsymbol{\Pi} \in BL(\mathcal{H};\mathcal{H})$ is the orthogonal projector on $\mathcal{S}_h \subset \mathcal{H}$ [8]. Hence this condition is an alternative expression of the LBB condition. ■

Theorem 1 shows that the approximation error is bounded above by the interpolation error. The asymptotic estimate, i.e. as $h \to 0$, of the decrease rate of the interpolation error $\mathrm{Err}(h) = \inf_{\overline{\mathbf{u}}_h\in\mathcal{L}_h}\|\mathbf{u}-\overline{\mathbf{u}}_h\|_{\mathcal{L}} + \inf_{\overline{\boldsymbol{\sigma}}_h\in\mathcal{S}_h}\|\boldsymbol{\sigma}-\overline{\boldsymbol{\sigma}}_h\|_{\mathcal{H}}$ is provided by the polynomial interpolation theory that leads to the exponential formula: $\mathrm{Err}(h) \le \beta\,h^k\,(\|\mathbf{u}\|_{\mathcal{L}}+\|\boldsymbol{\sigma}\|_{\mathcal{H}})$. In a two-logaritmic scale the exponential law with exponent k transforms to the linear law with slope equal to k that is $\ln(\mathrm{Err}(h)) \le \ln(\beta\,(\|\mathbf{u}\|_{\mathcal{L}}+\|\boldsymbol{\sigma}\|_{\mathcal{H}})) + k\,\ln(h)$.

References

1. Ladyszhenskaya OA, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.
2. Babuška I, Aziz AK, *The Mathematical Foundations of the Finite Element Method with Application to Partial Differential Equations*, Part I, Academic Press, New York, 1972.
3. Babuška I, The finite element method with lagrangian multipliers, *Numer. Math.* 20, 1973, 179-192.
4. Brezzi F, On the existence, uniqueness and approximation of saddle point problems arising from lagrangian multipliers, *R.A.I.R.O. Anal. Numer.* 8, 1974, 129-151.
5. Oden JT, Problems with Constraints. In: *The Finite Element Handbook*, Kardestuncer H (Editor), ch.9.4, 2.321-2.326, McGraw-Hill, New York, 1987.
6. Roberts JE, Thomas JM, Mixed and Hybrid Methods. In: *Handbook of Numerical Analysis, Finite Element Methods*, Part I, Vol.II, Ciarlet PG, Lions JL (Editors), North-Holland, Amsterdam, 1989.
7. Brezzi F, Fortin M, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.
8. Romano G, *Theory of structural models*, Part I, Elements of Linear Analysis, University of Naples Federico II, 2000.