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VARIATIONAL PRINCIPLES WITH SINGULARITIES IN GEODESICS, OPTICS AND DYNAMICS

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Abstract. *The calculus of variations for the action integral is formulated with a new definition of the extremality property leading to a general treatment which includes singularities of the configuration manifold and of the extremal path. Applications to the foundations of dynamics, geodesics and geometrical optics are illustrated, in an intrisic differential geometric context, on the basis of new ideas and results.*

1 INTRODUCTION

Classical dynamics may be conventionally considered to be born about 1687 with NEWTON's *Principia* and fully grown up to a well developed theory due to the contributions by EULER, LAGRANGE, LAPLACE, LEGENDRE, HAMILTON and JACOBI, during the XVIII century and the first half of the XIX century. The first attempt to provide a variational foundation to dynamics was MAUPERTUIS principle of least action. HAMILTON's action principle, inspired by earlier ideas by FERMAT and HUYGENS in optics, may be taken as the basic axiom of dynamics. FERMAT's principle in geometrical optics is the first variational statement of a general physical law. This principle provides a formidable motivation for the introduction of RIEMANN's idea of a metric tensor field varying from point to point and possibly undergoing discontinuities across singularity surfaces. It is intimately related to the concept of a geodesic and indeed may be enunciated by stating that a light-ray is a geodesic path in the euclidean space endowed with a piecewise regular metric tensor field, the optical tensor. A similar, singular situation occurs when geodesic paths are drawn on the surface of a parallelepiped, as in fastening a string around a gift-box. The calculus of variations of optical rays and dynamical trajectories may be developed in a unitary context by introducing the notion of an action one-form and of the induced action integral along a path. Optical rays and dynamical trajectories are indeed paths at which the corresponding action integral is extremal, in the sense that it has a vanishing first variation. We adopt hereafter an extended definition of extremality which is more suitable to deal with singularities and also more satisfactory from an epistemological point of view.

A theory based on a variational principle has on its side the pleasant flavour offered by an extremality property, the generality of the mathematical context, the natural way in which discontinuities may be dealt with and the direct connection with computational methodologies. The variational approach is best developed by a recourse to concepts and methods of calculus on manifolds, whose basic notions and results, due mainly to the pioneering contributions of MARIUS SOPHUS LIE, HENRI POINCARÉ and ELIE CARTAN may be found in[3,4,10,12,13,15]. In deriving the differential and jump conditions of extremality, REYNOLDS transport theorem, STOKES's formula, CARTAN's magic formula and PALAIS' formula for the exterior derivative of a differential form, are the playmates. The main contribution is a new formulation of the action principle which leads to EULER's extremality differential and jump conditions. This approach provides a general form of the LAGRANGE's law which includes NOETHER's theorem as a special case. By an affine connection in the configuration manifold, LAGRANGE's law may be expressed in terms of the covariant derivative and of the newly introduced notion of base derivative of the Lagrangian functional. HAMILTON's point of view is illustrated and generalized to the case in which the Lagrangian functional is convex and subdifferentiable but not necessarily everywhere differentiable. This situation occurs in geometrical optics and in geodesics where extremality is imposed on the lenght of the path, so that the Lagrangian is a norm in the tangent space and the Hamiltonian is the indicator of the unit ball in the dual space. Accordingly the conjugacy map from the tangent into the cotangent space is univocal but not invertible. This extension provides the suitable theoretical tool to derive the *eikonal* equation from the HAMILTON-JACOBI partial differential equation.

2 PREMISES

We summarize hereafter concepts, results and notations of calculus on manifolds referred to in the sequel. We consider a non-finite dimensional differentiable manifold $\mathbb{M}$ modeled on a linear BANACH's space E. The tangent bundle $\mathbb{TM}$ is the collection of the tangent spaces at the points of $\mathbb{M}$ and the dual cotangent bundle $\mathbb{T}^*\mathbb{M}$ is the collection of the cotangent spaces, i.e. of the linear spaces of bounded linear forms on the tangent spaces. Push-forward and its inverse, the pull-back, of scalar, vector and tensor fields due to a diffeomorphism $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ are respectively denoted by $\boldsymbol{\varphi}\!\uparrow$ and $\boldsymbol{\varphi}\!\downarrow$. The usual notation in differential geometry is $\boldsymbol{\varphi}_* = \boldsymbol{\varphi}\!\uparrow$ and $\boldsymbol{\varphi}^* = \boldsymbol{\varphi}\!\downarrow$ but then too many stars appear in the geometrical sky (duality, HODGE operator). A dot $\cdot$ denotes linear dependence on subsequent arguments and the crochet $\langle , \rangle$ denotes a duality pairing. The variational analysis performed in this paper is mainly based on the following tools of calculus on manifolds[12,13,15]. The first tool is the POINCARÉ-STOKES' formula which states that the integral of a differential $(k-1)$-form $\boldsymbol{\omega}^{k-1}$ on the boundary chain $\partial\Sigma$ of a kD submanifold Σ of $\mathbb{M}$ is equal to the integral of its exterior derivative $d\boldsymbol{\omega}^{k-1}$, a differential k-form, on Σ i.e.

$$\int_{\Sigma} d\boldsymbol{\omega}^{k-1} = \oint_{\partial\Sigma} \boldsymbol{\omega}^{k-1}\,.$$

The second tool is LIE's derivative of a vector field $\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ along a flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ with velocity $\mathbf{v} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$:

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} = \partial_{\lambda=0}\,(\boldsymbol{\varphi}_\lambda\!\downarrow\!\mathbf{w})\,,$$

which is equal to the antisymmetric LIE-bracket: $\mathcal{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v},\mathbf{w}] = -[\mathbf{w},\mathbf{v}]$ defined by: $d_{[\mathbf{v},\mathbf{w}]}f = d_{\mathbf{v}}d_{\mathbf{w}}f - d_{\mathbf{w}}d_{\mathbf{v}}f$, for any $f \in \mathrm{C}^2(\mathbb{M}\,;\mathcal{R})$.

The LIE derivative of a differential form $\boldsymbol{\omega}^k \in \mathrm{C}^1(\mathbb{M}\,;\Lambda^k(\mathbb{TM}))$ is similarly defined by $\mathcal{L}_{\mathbf{v}}\boldsymbol{\omega}^k = \partial_{\lambda=0}\,(\boldsymbol{\varphi}_\lambda\!\downarrow\!\boldsymbol{\omega}^k)$. The third tool is REYNOLDS' transport formula:

$$\int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_\Sigma \boldsymbol{\varphi}_\lambda\!\downarrow\!\boldsymbol{\omega}^k \quad \Longrightarrow \quad \partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_\Sigma \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k\,,$$

and the fourth tool is the extrusion formula

$$\partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\Sigma)} \boldsymbol{\omega}^k = \int_\Sigma (d\boldsymbol{\omega}^k)\mathbf{v} + \int_{\partial\Sigma} \boldsymbol{\omega}^k\mathbf{v}\,,$$

and the related CARTAN's magic formula (or homotopy formula): $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k = (d\boldsymbol{\omega}^k)\mathbf{v} + d(\boldsymbol{\omega}^k\mathbf{v})$ where the $(k-1)$-form $\boldsymbol{\omega}^k\mathbf{v} = \boldsymbol{\omega}^k \cdot \mathbf{v}$ is the contraction performed by taking $\mathbf{v}$ as the first argument of the form $\boldsymbol{\omega}^k$. The homotopy formula may be readily inverted to get PALAIS formula for the exterior derivative. Indeed, by LEIBNIZ rule for the LIE derivative, we have that, for any two vector fields $\mathbf{v},\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$:

$$d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{w} = (\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^1) \cdot \mathbf{w} - d(\boldsymbol{\omega}^1\mathbf{v}) \cdot \mathbf{w} = d_{\mathbf{v}}\,(\boldsymbol{\omega}^1\mathbf{w}) - \boldsymbol{\omega}^1 \cdot [\mathbf{v},\mathbf{w}] - d_{\mathbf{w}}\,(\boldsymbol{\omega}^1\mathbf{v})\,.$$

The expression at the r.h.s. of PALAIS formula fulfils the tensoriality criterion, as quoted in[11,15]. The exterior derivative of a differential one-form is thus well-defined as a differential two-form, since its value at a point depends only on the values of the argument vector fields at that point. The same algebra may be repeatedly applied to deduce PALAIS formula for the exterior derivative of a k-form.

2.1 Action principle and Euler conditions

The status of a system is described by a point belonging to a piecewise differentiable manifold $\mathbb{M}$, the *phase space*. Piecewise regularity is an assumption dictated by epistemological reasons and by the modeling of many applications where either the trajectory or the phase space or both have singularities which cannot be eliminated. Examples of singularities are provided by geodesic paths on polyhedral surfaces, by light rays propagating across interfaces between optical media with different refraction properties, and by dynamical models subject to sharp unilateral constraints. Piecewise regular geodesics, on a regular manifold, have been considered in the context of the calculus of variations by MILNOR[2]. A piecewise differentiable manifold $\mathbb{M}$ is a pair $\{\mathbb{M}\,,\mathcal{T}(\mathbb{M})\}$ made of a C^0-manifold $\mathbb{M}$ and of a *regularity patchwork* $\mathcal{T}(\mathbb{M})$ which is a finite family of disjoint C^1-submanifolds of $\mathbb{M}$ such that the union of their closures is a covering of $\mathbb{M}$.

The closure of each submanifold is called an *element* of the patchwork. The disjoint union of the boundaries of the elements, deprived of the boundary of $\mathbb{M}$, is the set of *singularity interfaces* $\mathcal{I}(\mathbb{M})$ associated with the patchwork $\mathcal{T}(\mathbb{M})$. A piecewise regular trajectory $\mathbf{\Gamma} \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{M})$, with regularity patchwork $\{\,t_0, \ldots, t_n\,\}$ in the time interval $I = \{t_0\,, t_n\}$, is a path $\mathbf{\Gamma} \in \mathrm{C}^0(I\,;\mathbb{M})$ such that $\mathbf{\Gamma}_i \in \mathrm{C}^1(\{t_{(i-1)}\,, t_i\}\,;\mathbb{M})$ where $\{t_{(i-1)}\,, t_i\} \in \mathcal{T}(I)$ with $i = 1, \ldots, n$ is a regularity time interval. The evolution of the system along a piecewise regular trajectory $\mathbf{\Gamma} \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{M})$ is assumed to be governed by a variational condition on its signed-length, evaluated according to a piecewise regular differential one-form, the *action one-form* $\boldsymbol{\omega}^1 \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\omega}}(\mathbb{M})\,;\mathbb{T}^*\mathbb{M})$, with $\mathcal{T}_{\boldsymbol{\omega}}(\mathbb{M})$ regularity patchwork. The test fields for the variational condition are vector fields with values in a subbundle $V_{\mathrm{TEST}} \subset \mathbb{T}\mathbb{M}$, dubbed the *test-subbundle*. The *action integral* is the signed-length of a path $\mathbf{\Gamma} \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{M})$ in the phase-space, evaluated according to the action one-form:

$$\int_{\mathbf{\Gamma}} \boldsymbol{\omega}^1 \,.$$

A *virtual flow* of $\mathbf{\Gamma}$ in $\mathbb{M}$ is a flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ whose velocity field at $\mathbf{\Gamma}$ belongs to the test-subbundle $V_{\mathrm{TEST}} \subset \mathbb{T}\mathbb{M}$. The restriction of the test-subbundle to $\mathbf{\Gamma}$ will be denoted $V_{\mathrm{TEST}}(\mathbf{\Gamma})$.

Axiom 1 (Action principle) *A trajectory $\mathbf{\Gamma} \subset \mathbb{M}$ of the system with action one-form $\boldsymbol{\omega}^1 \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\omega}}(\mathbb{M})\,;\mathbb{T}^*\mathbb{M})$, is a piecewise regular path $\mathbf{\Gamma} \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{M})$ such that the action integral is extremal in the sense that it fulfils the variational condition:*

$$\partial_{\lambda=0} \int_{\boldsymbol{\varphi}_\lambda(\mathbf{\Gamma})} \boldsymbol{\omega}^1 = \int_{\partial\mathbf{\Gamma}} \boldsymbol{\omega}^1 \cdot \mathbf{v} \,,$$

for any virtual flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ with virtual velocity $\mathbf{v} \in \mathrm{C}^0(\mathbf{\Gamma}\,;V_{\mathrm{TEST}}(\mathbf{\Gamma}))$.

This means that the initial rate of increase of the $\boldsymbol{\omega}^1$-length of the trajectory $\mathbf{\Gamma}$ along a test virtual flow is equal to the outward $\boldsymbol{\omega}^1$-flux of the virtual velocities at the end points. Denoting by $\mathbf{x}_1$ and $\mathbf{x}_2$ the initial and final end points of $\mathbf{\Gamma}$, we have that $\partial\mathbf{\Gamma} = \mathbf{x}_2 - \mathbf{x}_1$ (a 0-chain) and the boundary integral may be written as

$$\int_{\partial\mathbf{\Gamma}} \boldsymbol{\omega}^1 \cdot \mathbf{v} = (\boldsymbol{\omega}^1 \cdot \mathbf{v})(\mathbf{x}_2) - (\boldsymbol{\omega}^1 \cdot \mathbf{v})(\mathbf{x}_1) \,.$$

The extremality of the action integral is a problem of *calculus of variations* on a nonlinear manifold. The necessary and sufficient differential condition for a regular path to be a trajectory is called the Euler's condition. The classical result of Euler deals with regular paths and fixed end points and is formulated in coordinates. The statement introduced in the next proposition is instead concerned with non-fixed end points and piecewise regular paths, so that extremality is expressed in terms of differential and jump conditions. The variational problem is discussed in coordinate-free form.

Theorem 1 (Euler's condition with singularities) *A path* $\mathbf{\Gamma} \subset \mathbb{M}$ *is a trajectory if and only if the tangent vector field* $\mathbf{v_\Gamma} \in \mathrm{C}^1(\mathcal{T}(\mathbf{\Gamma})\,;\mathbb{T}\mathbf{\Gamma})$ *meets, in each element of a regularity patchwork* $\mathcal{T}(\mathbf{\Gamma})$ *, the differential condition*

$$d\boldsymbol{\omega}^1 \cdot \mathbf{v_\Gamma} \cdot \mathbf{v} = 0\,, \quad \forall\, \mathbf{v} \in \mathrm{C}^0(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))\,,$$

and, at the singularity interfaces $\mathcal{I}(\mathbf{\Gamma})$ *, the jump conditions*

$$[[\boldsymbol{\omega}^1 \mathbf{v}]] = 0\,, \quad \forall\, \mathbf{v} \in \mathrm{C}^0(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))\,.$$

Remark 1 *The fulfillment of the local conditions is necessary and sufficient for the fulfilment of the action principle under various equivalent boundary conditions. Indeed the equivalence*

$$(d\boldsymbol{\omega}^1 \cdot \mathbf{v})\,\mathbf{\Gamma} = 0 \quad \Longleftrightarrow \quad d\boldsymbol{\omega}^1 \cdot \mathbf{v_\Gamma} \cdot \mathbf{v} = 0\,, \quad \forall\, \mathbf{v} \in \mathrm{C}^1(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))\,,$$

still holds when the space $\mathrm{C}^1(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))$ *is substituted by any linear subspace which contains the space* $\mathrm{C}^\infty_0(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))$ *of indefinitely differentiable test vector fields vanishing in a neighbourhood of the end points. However, the assumption that the field* $\mathbf{v} \in \mathrm{C}^1(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))$ *vanishes at each endpoint of* $\mathbf{\Gamma}$ *, usually made in stating the abstract action principle on a manifold*[5]*, is too special and even unsatisfactory from the epistemological point of view (see remark 3).*

The next two corollaries of proposition 1 are due to the first author[15].

Corollary 1 (A symmetry condition) *The differential condition fulfilled by a trajectory* $\mathbf{\Gamma} \subset \mathbb{M}$ *may equivalently be written as*

$$d_{\mathbf{v_\Gamma}}(\boldsymbol{\omega}^1 \cdot \mathbf{v}) = d_{\mathbf{v}}(\boldsymbol{\omega}^1 \cdot \mathbf{v_\Gamma})\,, \quad \forall\, \mathbf{v} \in \mathrm{C}^0(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))\,,$$

where $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{T}\mathbb{M})$ *is an extension of the virtual velocity* $\mathbf{v} \in \mathrm{C}^0(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))$ *and* $\mathbf{v_\Gamma} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{T}\mathbb{M})$ *is the extension of* $\mathbf{v_\Gamma} \in \mathrm{C}^0(\mathbf{\Gamma}\,; V_{\mathrm{TEST}}(\mathbf{\Gamma}))$ *performed by pushing it along the flow* $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ *generated by* $\mathbf{v} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{T}\mathbb{M})$ *.*

Proof. The result follows from proposition 1 by a direct application of Palais formula: $d\boldsymbol{\omega}^1 \cdot \mathbf{v} \cdot \mathbf{v_\Gamma} = d_{\mathbf{v}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v_\Gamma}) - d_{\mathbf{v_\Gamma}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v}) - \boldsymbol{\omega}^1 \cdot [\mathbf{v}, \mathbf{v_\Gamma}]$. Indeed, by tensoriality of the exterior derivative, the r.h.s. is independent of the extensions of $\mathbf{v}$ and $\mathbf{v_\Gamma}$. Moreover the special extension of $\mathbf{v_\Gamma}$ implies that $[\mathbf{v}, \mathbf{v_\Gamma}] = 0$.

Corollary 2 (Abstract Noether's theorem) *If the functional* $\boldsymbol{\omega}^1 \cdot \mathbf{v_\Gamma}$ *enjoys the extremality property:* $d_{\mathbf{v}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v_\Gamma}) = 0$ *, then the functional* $\boldsymbol{\omega}^1 \cdot \mathbf{v}$ *is constant along the trajectory* $\mathbf{\Gamma} \subset \mathbb{M}$ *.*

3 CONTINUUM VS RIGID-BODY DYNAMICS

The abstract theory concerning the action principle may be applied to continuum mechanics by envisaging a suitable phase-space to describe motions. A continuous body is identified with an open, connected, reference domain $\mathbb{B} \subset \mathbb{S}$ embedded in the euclidean space $\{\mathbb{S}, \mathbf{g}\}$. A configuration $\boldsymbol{\varphi} \in C^1(\mathbb{B};\mathbb{S})$ of a continuous body $\mathbb{B} \subset \mathbb{S}$ is an injective map with the property of being a diffeomorphic transformation onto its range.

The configuration-space $\mathbb{C}$ is assumed to be a differentiable manifold modeled on a Banach space. The *velocity phase-space* is the tangent bundle $\mathbb{TC}$ and the *momentum phase-space* is the cotangent bundle $\mathbb{T}^*\mathbb{C}$. The *velocity-time phase-space* $\mathbb{TC} \times I$ is the cartesian product of the velocity-space $\mathbb{TC}$ and an open time interval I, and the *momentum-time phase-space* is $\mathbb{T}^*\mathbb{C} \times I$. These two phase-spaces are respectively adopted in the Lagrangian and the Hamiltonian descriptions of dynamics. Vectors tangent to the velocity-time phase-space $\mathbb{TC} \times I$ are in the bundle $\mathbb{TTC} \times \mathbb{T}I$ whose elements are pairs $\{\delta\mathbf{v}, \delta t\} \in \mathbb{T}_{\mathbf{v}}\mathbb{TC} \times \mathbb{T}_t I$.

Denoting by $\boldsymbol{\pi} \in C^1(\mathbb{TC};\mathbb{C})$ the projector on the base manifold, the velocity of the configuration $\boldsymbol{\pi}(\mathbf{v}) \in \mathbb{C}$, corresponding to a tangent vector $\delta\mathbf{v} \in \mathbb{T}_{\mathbf{v}}\mathbb{TC}$ is found by acting on it with the differential $d\boldsymbol{\pi}(\mathbf{v}) \in BL(\mathbb{T}_{\mathbf{v}}\mathbb{TC};\mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C})$ of the projector, to get: $d\boldsymbol{\pi}(\mathbf{v}) \cdot \delta\mathbf{v} \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C}$. A section $\mathbf{X} \in C^1(\mathbb{TC};\mathbb{TTC})$ of $\boldsymbol{\pi}_{\mathbb{TC}} \in C^1(\mathbb{TTC};\mathbb{TC})$, is such that $\boldsymbol{\pi}_{\mathbb{TC}} \circ \mathbf{X} = \mathbf{id}_{\mathbb{TC}}$. The tangent map $T\boldsymbol{\pi} \in C^1(\mathbb{TTC};\mathbb{TC})$, defined by $(T\boldsymbol{\pi} \circ \mathbf{X})(\mathbf{v}) = d\boldsymbol{\pi}(\mathbf{v}) \cdot \mathbf{X}(\mathbf{v})$, maps each vector $\mathbf{X}(\mathbf{v})$ into the velocity of the configuration $\boldsymbol{\pi}(\mathbf{v}) \in \mathbb{C}$.

3.1 Holonomic vs non-holonomic constraints

A dynamical system is said to be subject to *ideal constraints* if the admissibile velocities are imposed to belong to a vector sub-bundle $\mathcal{A}$ of $\mathbb{TC}$, that is, a bundle with base manifold $\mathbb{C}$ and fibers which are linear subspaces of the tangent spaces to $\mathbb{C}$.

The subbundle $\mathcal{A}$ is integrable if for any $\mathbf{x} \in \mathbb{C}$ there exists a (local) submanifold (the integral manifold) $\mathbb{I}_{\mathcal{A}} \subset \mathbb{C}$ thru $\mathbf{x}$ such that $\mathbb{TI}_{\mathcal{A}}$ is $\mathcal{A}$ restricted to $\mathbb{I}_{\mathcal{A}}$. If the sub-bundle $\mathcal{A}$ is integrable, the ideal constraints are said to be *holonomic*. Frobenius theorem states that integrability holds if and only if the sub-bundle $\mathcal{A}$ is involutive, in the sense that for any pair of vector fields $\mathbf{X}, \mathbf{Y} \in C^1(\mathbb{C};\mathcal{A})$ in the vector sub-bundle $\mathcal{A}$ of $\mathbb{TC}$ we have that

$$[\mathbf{X}, \mathbf{Y}] = \mathcal{L}_{\mathbf{X}}\,\mathbf{Y} \in C^1(\mathbb{C};\mathcal{A}).$$

3.2 Rigidity constraint

Two configurations $\boldsymbol{\varphi}_1 \in C^1(\mathbb{B};\mathbb{S})$ and $\boldsymbol{\varphi}_2 \in C^1(\mathbb{B};\mathbb{S})$ are metric-equivalent if $\boldsymbol{\varphi}_2{\downarrow}\mathbf{g} = \boldsymbol{\varphi}_1{\downarrow}\mathbf{g}$. Then the diffeomorphic map $\boldsymbol{\varphi}_2 \circ \boldsymbol{\varphi}_1^{-1} \in C^1(\boldsymbol{\varphi}_1(\mathbb{B});\boldsymbol{\varphi}_2(\mathbb{B}))$ is a metric-preserving (or rigid) transformation of the configuration $\boldsymbol{\varphi}_1 \in C^1(\mathbb{B};\mathbb{S})$ into the configuration $\boldsymbol{\varphi}_2 \in C^1(\mathbb{B};\mathbb{S})$. By the metric-equivalence relation so introduced, the manifold $\mathbb{C}$ is partitioned into a family of disjoint connected rigidity-classes $\mathbb{C}_R$ which are submanifolds of $\mathbb{C}$.

The elements of the tangent space $\mathbb{T}_{\boldsymbol{\varphi}}\mathbb{C}_R$ to a rigidity-class $\mathbb{C}_R$ at $\boldsymbol{\varphi} \in \mathbb{C}_R$ are the infinitesimal isometries $\mathbf{v} \in V_{\text{TEST}}$, that is, the vector fields $\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\varphi}(\mathbb{B})\,;\mathbb{S})$ fulfilling the EULER-KILLING condition[15]: $\mathcal{L}_{\mathbf{v}}\mathbf{g} = 2\,\mathbf{g}\,(\mathrm{sym}\,\nabla\mathbf{v}) = 0$. The LIE derivative of the metric tensor is defined by: $\mathcal{L}_{\mathbf{v}}\mathbf{g} := \partial_{\lambda=0}\,\boldsymbol{\chi}_\lambda{\downarrow}\mathbf{g}$ where $\boldsymbol{\chi}_\lambda \in \mathrm{C}^1(\boldsymbol{\chi}(\mathbb{B})\,;\mathbb{S})$ is the flow generated by $\mathbf{v} = \partial_{\lambda=0}\,\boldsymbol{\chi}_\lambda$ and $\boldsymbol{\chi}_\lambda{\downarrow}\mathbf{g}$ is the pull back along $\boldsymbol{\chi}_\lambda \in \mathrm{C}^1(\boldsymbol{\chi}(\mathbb{B})\,;\mathbb{S})$ of the metric tensor: $(\boldsymbol{\chi}_\lambda{\downarrow}\mathbf{g})(\mathbf{a},\mathbf{b}) = \mathbf{g}(d\boldsymbol{\chi}_\lambda(\mathbf{a}), d\boldsymbol{\chi}_\lambda(\mathbf{b}))$ for all $\mathbf{a},\mathbf{b} \in \mathbb{T}\mathbb{S}$.

4 HAMILTON'S ACTION PRINCIPLE

For a dynamical system governed by a lagrangian functional $L_t \in \mathrm{C}^1(\mathbb{T}\mathbb{C}\,;\mathcal{R})$, the classical HAMILTON's principle is stated in terms of the lagrangian one-form $\mathrm{L}(\mathbf{v},t)\,dt \in \mathbb{T}^*(\mathbb{T}\mathbb{C} \times I)$ on the velocity-time phase-space and of the corresponding action integral. Being $dt \cdot \{\dot{\mathbf{v}}_t\,, 1_t\} = dt \cdot 1_t = 1$, we have that

$$\int_{\boldsymbol{\Gamma}_I} \mathrm{L}(\mathbf{v}_t,t)\,dt = \int_I (\mathrm{L}(\mathbf{v}_t,t)\,dt \cdot \{\dot{\mathbf{v}}_t\,, 1_t\})\,dt = \int_I L_t(\mathbf{v}_t)\,dt\,,$$

where $\boldsymbol{\Gamma}_I$ is the time-parametrized phase-trajectory in the velocity-time phase-space and $\{\dot{\mathbf{v}}_t\,, 1_t\}$ is the relevant speed.

Proposition 1 (Standard Hamilton's principle) *A trajectory of a dynamical system is a time-parametrized path $\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{C})$ in the configuration manifold fulfilling the extremality condition*

$$\partial_{\lambda=0} \int_I L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t))\,dt = 0\,,$$

for any flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ in the configuration manifold whose velocity field $\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{T}\mathbb{C})$ is an infinitesimal isometry at each point of the path $\boldsymbol{\gamma}$ and vanishes at its end points.

Remark 2 *The kinetic energy of a mechanical system, and hence the Lagrangian functional $L_t \in \mathrm{C}^1(\mathbb{T}\mathbb{C}\,;\mathcal{R})$, is defined only on the trajectory of the body in the space. On the other hand, to formulate* HAMILTON*'s principle, the Lagrangian has to be evaluated on paths which are variations of the trajectory. When dealing with continuum dynamics the mass-form has to be dragged by the virtual flow and an explicit statement must be made concerning this point.*

Remark 3 *The original definition of extremality in the calculus of variations, and hence also of* HAMILTON*'s principle in dynamics, is unsatisfactory from the principle-theoretic point of view. Indeed, it is a natural requirement that a extremality property, characterizing a special class of paths, be formulated so that any piece of a special path is special too and the chain of two subsequent special paths is special too. The formulation of extremality in terms of flows vanishing at the end points of a path does not fulfill this requirement.*

4.1 The Hamiltonian phase space

The lagrangian functional may be expressed in terms of a conjugate functional by means of the FENCHEL-LEGENDRE transform[1]. If the Lagrangian $L_t \in \mathrm{C}^1(\mathbb{T}\mathcal{C}\,;\mathcal{R})$ is a convex functional everywhere subdifferentiable in its domain, the Hamiltonian $H_t \in \mathrm{C}^1(\mathbb{T}^*\mathcal{C}\,;\mathcal{R})$ is fiberwise defined as the convex functional conjugate to the Lagrangian according to the transformation rules:

$$H_t(\mathbf{v}^*) = \sup_{\mathbf{v}\in\mathbb{T}_{\boldsymbol{\pi}^*(\mathbf{v}^*)}\mathcal{C}} \{\, \langle \mathbf{v}^*, \mathbf{v} \rangle - L_t(\mathbf{v}) \,\} \quad \Longleftrightarrow \quad L_t(\mathbf{v}) = \sup_{\mathbf{v}^*\in\mathbb{T}^*_{\boldsymbol{\pi}(\mathbf{v})}\mathcal{C}} \{\, \langle \mathbf{v}^*, \mathbf{v} \rangle - H_t(\mathbf{v}^*) \,\}\,.$$

The fiber-subdifferentials $d_{\mathrm{F}} L_t(\mathbf{v}) \subset \mathbb{T}^*\mathcal{C}$ and $d_{\mathrm{F}} H_t(\mathbf{v}^*) \subset \mathbb{T}\mathcal{C}$ are the closed convex sets in which the suprema are attained, so that:

$$L_t(\mathbf{v}) + H_t(\mathbf{v}^*) = \langle \mathbf{v}^*, \mathbf{v} \rangle\,, \quad \begin{cases} \mathbf{v} \ \in d_{\mathrm{F}} H_t(\mathbf{v}^*) \subset \mathbb{T}\mathcal{C}\,, \\ \mathbf{v}^* \in d_{\mathrm{F}} L_t(\mathbf{v}) \ \ \subset \mathbb{T}^*\mathcal{C}\,. \end{cases}$$

If the convex conjugate functionals are differentiable, the conjugacy correspondence is one to one and reduces to the classical LEGENDRE transform. An important special case of a multivalued conjugacy is met when the Lagrangian $L_t \in \mathrm{C}^1(\mathbb{T}\mathcal{C}\,;\mathcal{R})$ is a norm: $L_t(\mathbf{v}) = \|\mathbf{v}\|_{\mathbf{g}}$ and the Hamiltonian $H_t \in \mathrm{C}^1(\mathbb{T}^*\mathcal{C}\,;\mathcal{R})$ is the convex indicator of the unit ball $B^1(\mathbb{T}^*\mathcal{C}, \mathbf{g}^{-1})$. Then the subdifferential relation is expressed by the normality rule: $\mathbf{v} \in \mathcal{N}_{B^1(\mathbb{T}^*\mathcal{C},\mathbf{g}^{-1})}(\mathbf{v}^*)$ and, for $\mathbf{v} \neq 0$, the projection $\mathbf{v}^* = \mathbf{P}_{B^1(\mathbb{T}^*\mathcal{C},\mathbf{g}^{-1})}(\mathbf{v})$ is well-defined[8,9]. This situation occurs in the study of geodesics and in geometrical optics.

Then, being $d\boldsymbol{\pi}^*(\mathbf{v}^*_t) \cdot \dot{\mathbf{v}}^*_t = \mathbf{v}_t$, we may define the one-form

$$\boldsymbol{\omega}^1_{H_t}(\mathbf{v}^*) := \boldsymbol{\theta}(\mathbf{v}^*) - H_t(\mathbf{v}^*)\, dt \in \mathbb{T}^*_{(\mathbf{v}^*,t)}(\mathbb{T}^*\mathcal{C} \times I)\,.$$

where $\boldsymbol{\theta}(\mathbf{v}^*) \cdot \delta\mathbf{v}^* = \langle \mathbf{v}^*, d\boldsymbol{\pi}^*(\mathbf{v}^*) \cdot \delta\mathbf{v}^* \rangle\,, \quad \forall\, \delta\mathbf{v}^* \in \mathbb{T}_{\mathbf{v}^*}\mathbb{T}^*\mathcal{C}$ so that

$$\begin{aligned} \int_I L_t(\mathbf{v}_t)\, dt &= \int_I (\langle \mathbf{v}^*_t, \mathbf{v}_t \rangle - H_t(\mathbf{v}^*_t))\, dt = \int_I (\boldsymbol{\theta}(\mathbf{v}^*_t) \cdot \dot{\mathbf{v}}^*_t - H_t(\mathbf{v}^*_t)\, dt \cdot 1_t)\, dt \\ &= \int_I \boldsymbol{\omega}^1_{H_t}(\mathbf{v}^*_t) \cdot \{\dot{\mathbf{v}}^*_t\,, 1_t\}\, dt = \int_{\boldsymbol{\Gamma}^*_I} \boldsymbol{\omega}^1_{H_t}\,. \end{aligned}$$

A flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathcal{C}\,;\mathcal{C})$ in the configuration manifold induces, in the *velocity phase-space*, a lifted phase-flow $T\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{T}\mathcal{C}\,;\mathbb{T}\mathcal{C})$ whose velocity field $\mathbf{v}_{T\boldsymbol{\varphi}} = \partial_{\lambda=0}\, T\boldsymbol{\varphi}_\lambda$ is the *phase-velocity* and a conjugate phase-flow $\boldsymbol{\psi}_\lambda \in \mathrm{C}^1(\mathbb{T}^*\mathcal{C}\,;\mathbb{T}^*\mathcal{C})$ with velocity field $\mathbf{v}_{\boldsymbol{\psi}} = \partial_{\lambda=0}\, \boldsymbol{\psi}_\lambda$. To the trajectory $\boldsymbol{\gamma} \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathcal{C})$ in the configuration manifold, there correspond a lifted phase-trajectory $\boldsymbol{\Gamma}_I \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{T}\mathcal{C} \times I)$ in the *velocity phase-space* and a conjugate phase-trajectory $\boldsymbol{\Gamma}^*_I \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{T}^*\mathcal{C} \times I)$ in the *momentum phase-space*.

The action principle for the one-form $\boldsymbol{\omega}^1_{H_t}$ is thus expressed by the variational condition:

$$\partial_{\lambda=0} \int_{\boldsymbol{\psi}_\lambda \circ \boldsymbol{\Gamma}^*_I} \boldsymbol{\omega}^1_{H_t} = \int_{\partial \boldsymbol{\Gamma}^*_I} \boldsymbol{\omega}^1_{H_t} \cdot \{ \mathbf{v}_{\boldsymbol{\psi}} \, , 0 \} \, ,$$

for any test flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ such that the velocity $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{TC})$ is an infinitesimal isometry of the trajectory $\boldsymbol{\gamma} \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{C})$. Localizing, the differential condition reads:

$$d\boldsymbol{\theta}(\mathbf{v}^*_t) \cdot \dot{\mathbf{v}}^*_t \cdot \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}^*_t) = -\langle \, dH_t(\mathbf{v}^*_t), \mathbf{v}_{\boldsymbol{\psi}}(\mathbf{v}^*_t) \, \rangle \, ,$$

Being $\boldsymbol{\omega}^1_{H_t} \cdot \{ \mathbf{v}_{\boldsymbol{\psi}} \, , 0 \} = \boldsymbol{\theta}(\mathbf{v}^*_t) \cdot \mathbf{v}_{\boldsymbol{\psi}} = \langle \, \mathbf{v}^*, d\boldsymbol{\pi}^*(\mathbf{v}^*) \cdot \mathbf{v}_{\boldsymbol{\psi}} \, \rangle = \langle \, \mathbf{v}^*, \mathbf{v}_{\boldsymbol{\varphi}} \, \rangle$, the boundary term vanishes if $\mathbf{v}_{\boldsymbol{\varphi}} = 0$ at the end points of $\boldsymbol{\gamma}$, which means that initial and final configurations are held fixed by the flow. This is the usual assumption made in formulating the action principle[10]. It can be shown[15] that, if the variational condition is formulated for any flow in the momentum-time phase-space, then the EULER extremality condition yields also the LEGENDRE transformation rule: $d\boldsymbol{\pi}^*(\mathbf{v}^*_t) \cdot \dot{\mathbf{v}}^*_t = d_{\mathrm{F}} H_t(\mathbf{v}^*_t)$.

The action principle may be rewritten in terms of the Lagrangian as:

$$\partial_{\lambda=0} \int_I L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t)) \, dt = \int_{\partial I} \langle \, d_{\mathrm{F}} \mathrm{L}_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \, \rangle \, dt \, .$$

Integrating by parts on each regularity interval in $\mathcal{T}(I)$, we have

$$\begin{aligned} \int_{\mathcal{T}(I)} \partial_{\lambda=0} \, L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t)) \, dt = & \int_{\mathcal{T}(I)} \partial_{\tau=t} \, \langle \, d_{\mathrm{F}} \mathrm{L}_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau)) \, \rangle \, dt \\ & + \int_{\mathcal{I}(I)} \langle \, [[d_{\mathrm{F}} \mathrm{L}_t(\mathbf{v}_t)]], \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \, \rangle \, dt \, . \end{aligned}$$

Theorem 2 (The law of dynamics) *A trajectory of the system is a time-parametrized piecewise regular path* $\boldsymbol{\gamma} \in \mathrm{C}^1(\mathcal{T}(I)\,;\mathbb{C})$ *in the configuration manifold* $\mathbb{C}$*, fulfilling the differential condition:*

$$\partial_{\tau=t} \, \langle \, d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau)) \, \rangle = d_{\mathbf{v}_{T\boldsymbol{\varphi}(\mathbf{v}_t)}} L_t(\mathbf{v}_t) \, ,$$

at regular points and the jump conditions $\langle \, [[d_{\mathrm{F}} L_t(\mathbf{v}_t)]], \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t)) \, \rangle = 0$*, for any test flow* $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *whose velocity* $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))$ *at the actual configuration* $\boldsymbol{\pi}(\mathbf{v}_t) \in \mathbb{C}$ *is an infinitesimal isometry.*

Remark 4 *To evaluate the expression of the law of dynamics in the form derived above, it is compelling to assign the flows* $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *at least in a neighborhood of* $\boldsymbol{\pi}(\mathbf{v}_t) \in \boldsymbol{\gamma}$ *and not just the initial velocity* $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))$ *at the actual configuration* $\boldsymbol{\pi}(\mathbf{v}_t) \in \boldsymbol{\gamma}$*. By tensoriality, the flows* $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *leading to the same value of* $\mathbf{v}_{T\boldsymbol{\varphi}}(\mathbf{v}_t) \in \mathbb{T}_{\mathbf{v}_t}\mathbb{TC}$ *are equivalent. Anyway, we shall see that this expression of the law of dynamics is equivalent to one in which virtual flows enters in the analysis only thru their virtual velocity. This fundamental result states that dynamical equilibrium depends only on the kinematical constraints pertaining to the body-placement under consideration.*

Remark 5 *In the variational expression of the law of dynamics, the test fields* $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(\boldsymbol{\gamma}\,;\mathbb{TC})$ *are infinitesimal isometries at the trajectory* $\boldsymbol{\gamma} \in \mathrm{C}^1(I\,;\mathbb{C})$. *This rigidity constraint reveals that the dynamical equilibrium at a given configuration is independent of the material properties. The evaluation of the equilibrium configuration requires in general to take into account the constitutive properties of the material and hence to get rid of the rigidity constraint. This task can be accomplished in complete generality by the method of* Lagrange *multipliers. In continuum mechanics, a field of* Lagrange *multipliers in duality with the rigidity constraint is called a stress field in the body*[14].

Remark 6 *The general expression of the law of dynamics implies, as a trivial corollary, a statement which extends to continuum dynamics* E. Noether*'s theorem*[10,12,16]. *Indeed from the law of dynamics we infer that*

$$\partial_{\lambda=0}\, L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t)) = 0 \quad \Longrightarrow \quad \partial_{\tau=t}\,\langle\, d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle = 0\,,$$

while the extension of Noether*'s theorem consists in the weaker statement:*

$$L_t(T\boldsymbol{\varphi}_\lambda(\mathbf{v}_t)) = L_t(\mathbf{v}_t) \quad \Longrightarrow \quad \partial_{\tau=t}\,\langle\, d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_\tau))\,\rangle = 0\,,$$

for all flows $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{C})$ *whose velocity field is an infinitesimal isometry of* $\boldsymbol{\gamma}$ *.*

5 DYNAMICS IN A MANIFOLD WITH A CONNECTION

Let us assume that the configuration manifold $\mathbb{C}$ be endowed with an affine connection ∇ and the associated parallel transport. We denote by $\mathbf{c}_{\tau,t}\Uparrow$ the parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1(I\,;\mathbb{C})$ from the point $\mathbf{c}(t) \in \mathbb{C}$ to the point $\mathbf{c}(\tau) \in \mathbb{C}$, setting $\mathbf{c}_{t,\tau}\Downarrow \,:=\, \mathbf{c}_{\tau,t}\Uparrow$. The *base derivative* of a functional $f \in \mathrm{C}^1(\mathbb{TC}\,;\mathcal{R})$ at $\mathbf{v} \in \mathbb{TC}$ along a vector $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v})) \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C}$ is defined by:

$$\langle\, d_{\mathrm{B}} f(\mathbf{v}), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}))\,\rangle := \partial_{\lambda=0}\, f(\boldsymbol{\varphi}_\lambda \Uparrow \mathbf{v})\,.$$

The definition is well-posed[15] since the r.h.s. depends linearly on $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v})) \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{v})}\mathbb{C}$ for any fixed $\mathbf{v} \in \mathbb{TC}$. The base derivative provides the rate of change of $f \in \mathrm{C}^1(\mathbb{TC}\,;\mathcal{R})$ when the base point $\boldsymbol{\pi}(\mathbf{v}) \in \mathbb{C}$ is dragged by the flow while the velocity $\mathbf{v} \in \mathbb{TC}$ is parallel transported along the flow. Let $\textsc{tors}(\mathbf{v},\mathbf{u}) = \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v},\mathbf{u}] \in \mathbb{TC}$ be the evaluation of the torsion of the connection ∇ on the pair $\mathbf{v},\mathbf{u} \in \mathbb{TC}$. The next statement provides the form taken by the differential law of dynamics in terms of a connection in the configuration manifold[15].

Proposition 2 (The law of dynamics in terms of a connection) *In a configuration manifold* $\mathbb{C}$ *with an affine connection* ∇ *the differential law of dynamics takes the special form*

$$\begin{aligned}\langle\, \partial_{\tau=t}\, d_{\mathrm{F}} L_\tau(\mathbf{v}_t) + \nabla_{\mathbf{v}_t}(d_{\mathrm{F}} L_t \circ \mathbf{v}_t) - d_{\mathrm{B}} L_t(\mathbf{v}_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle \\ = \langle\, d_{\mathrm{F}} L_t(\mathbf{v}_t), \textsc{tors}(\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_{\boldsymbol{\gamma}})(\boldsymbol{\pi}(\mathbf{v}_t))\,\rangle\,,\end{aligned}$$

for any virtual velocity field $\mathbf{v}_{\boldsymbol{\varphi}} = \partial_{\lambda=0}\,\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C}\,;\mathbb{TC})$ *which is an infinitesimal isometry at the configuration* $\boldsymbol{\pi}(\mathbf{v}_t)$ *.*

5.1 Hamilton's law of dynamics

HAMILTON's law is deduced from LAGRANGE's law by a translation in terms of co-vectors $\mathbf{v}^* \in \mathbb{T}^*\mathbb{C}$ by means of LEGENDRE's transform and of the next result[15], whose special case in linear spaces is referred to as DONKIN's theorem (1854) by GANTMACHER[6].

Proposition 3 (Base derivatives of Legendre transforms) *In a manifold $\mathbb{C}$ with an affine connection ∇ the following relation holds:*

$$d_{\mathrm{B}}H_t(\mathbf{v}^*) + d_{\mathrm{B}}L_t(d_{\mathrm{F}}H_t(\mathbf{v}^*)) = 0\,.$$

From propositions 2 and 3 we then get:

Proposition 4 (Hamilton's canonical equations) *If the configuration manifold $\mathbb{C}$ is endowed with an affine connection ∇, the differential law of dynamics takes the form*

$$\begin{cases}\langle\, \partial_{\tau=t}\,\mathbf{v}^*_\tau + \nabla_{\mathbf{v}_t}\mathbf{v}^* + d_{\mathrm{B}}H_t(\mathbf{v}^*_t), \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\pi}^*(\mathbf{v}^*_t))\,\rangle = \langle\, \mathbf{v}^*_t, \mathrm{TORS}(\mathbf{v}_{\boldsymbol{\varphi}}, \mathbf{v}_t)(\boldsymbol{\pi}^*(\mathbf{v}^*_t))\,\rangle\,, \\ \mathbf{v}_t = d_{\mathrm{F}}H_t(\mathbf{v}^*_t)\,. \end{cases}$$

6 HAMILTON-JACOBI EQUATION

Let us assume that thru any point in a neighbourhood of the point $\{\mathbf{x}\,, t\} \in \mathbb{C} \times I$ there is a unique trajectory starting at a fixed point $\{\mathbf{x}_0\,, t_0\} \in \mathbb{C} \times I$. Then the action integral defines an *action functional* $J \in \mathrm{C}^1(\mathbb{C} \times I\,; \mathcal{R})$ according to the relation:

$$J(\mathbf{x}, t) := \int_{\boldsymbol{\gamma}} \mathrm{L}_t(\dot{\boldsymbol{\gamma}}(t))\,dt = \int_{\boldsymbol{\Gamma}^*_I} \boldsymbol{\omega}^1_{H_t}\,,$$

and we have the following results[10,15]:

Lemma 1 (Differential of the action functional) *The differential of the action functional $J \in \mathrm{C}^1(\mathbb{C} \times I\,; \mathcal{R})$ is given by*

$$dJ(\mathbf{x}, t) = \mathbf{v}^*_t - H_t(\mathbf{v}^*_t)dt \in \mathbb{T}^*_{(\mathbf{x},t)}(\mathbb{C} \times I) \quad \Longleftrightarrow \quad \begin{cases} dJ_t(\mathbf{x}) = \mathbf{v}^*_t\,, \\ \partial_{\tau=t}\,J_\tau(\mathbf{x}) = H_t(\mathbf{v}^*_t)\,. \end{cases}$$

Theorem 3 (Hamilton-Jacobi equation) *The action functional $J \in \mathrm{C}^1(\mathbb{C} \times I\,; \mathcal{R})$ fulfils the* HAMILTON-JACOBI *equation:*

$$\partial_{\tau=t}\,J_\tau + H_t \circ dJ_t = 0\,.$$

In geometrical optics and in the theory of geodesics the action integral is the lenght of the path. Accordingly the Lagrangian is given by $L_t(\mathbf{v}) = \|\mathbf{v}\|_{\mathbf{g}}$ and the Hamiltonian $H_t \in \mathrm{C}^1(\mathbb{T}^*\mathbb{C}\,; \mathcal{R})$ is the convex indicator of the unit ball $B^1(\mathbb{T}^*\mathbb{C}, \mathbf{g}^{-1})$. Then $\partial_{\tau=t}\,J_\tau = 0$ and $\|dJ_t(\mathbf{x})\|_{\mathbf{g}^{-1}} = \|\mathbf{v}^*_t\|_{\mathbf{g}^{-1}} = 1$. The gradient $\nabla J(\mathbf{x}) := \mathbf{g}^{-1}dJ_t(\mathbf{x})$ fulfils the *eikonal* equation

$$\|\nabla J(\mathbf{x})\|_{\mathbf{g}} = 1\,.$$

This is an improvement of the treatments by CHOQUET-BRUHAT[4] and JOHN[7].

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