XX Congresso AIMETA

Bologna 12-15 Settembre 2011

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Università di Napoli Federico II, Italia

Conferenza Generale

del 13 Settembre 2011

On the Geometric Approach to Non-Linear Continuum Mechanics

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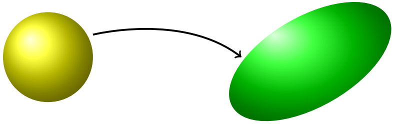
Non-Linear Continuum Mechanics (NLCM) calls instead for Differential Geometry (DG) and Calculus on Manifolds (CoM) as natural tools to develop theoretical and computational models.



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In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.

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Differential Geometry provides the tools to fly higher and see what before was shadowed or completely hidden.

A basic question in NLCM

- How to compare **material tensors** at corresponding points in displaced configurations of a body?

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- Devil's temptation:

In 3D bodies it might seem as natural to compare by translation the involved material vectors.
This is tacitly done in literature, when evaluating the material time-derivative of the stress tensor $\mathbf{T}$ *:*

$$\dot{\mathbf{T}}(\mathrm{p}, t) := \partial_{\tau=t}\, \mathbf{T}(\mathrm{p}, \tau)$$

or the material time-derivative of the director $\mathbf{n}$ *of a nematic liquid crystal:*

$$\dot{\mathbf{n}}(\mathrm{p}, t) := \partial_{\tau=t}\, \mathbf{n}(\mathrm{p}, \tau)$$

These definitions are connection dependent and geometrically untenable when considering 1D and 2D models (wires and membranes).

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  These definitions are connection dependent and geometrically untenable when considering 1D and 2D models (wires and membranes).
- Hint: *Tangent vectors to a body placement are transformed into tangent vectors to another body placement by the tangent displacement map. This is the essence of the COVARIANCE PARADIGM.*

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A geometrically consistent theoretical framework should be equally applicable to body models of any dimension.

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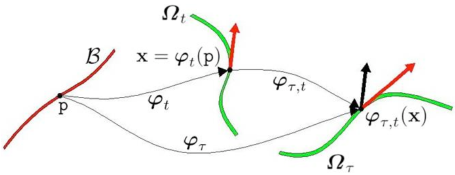

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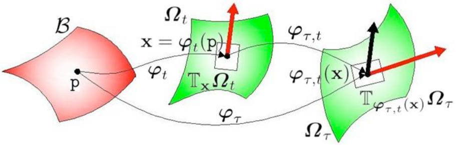

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Math1

Tangent vector to a manifold:

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velocity of a curve $\mathbf{c} \in \mathrm{C}^1([a, b]\,; \mathbb{M})\,, \quad \lambda \in [a, b]\,, \quad \mathbf{x} = \mathbf{c}(\lambda)$ base point

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- A map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{N})$ sends
  a curve $\mathbf{c} \in \mathrm{C}^1([a, b]\,; \mathbb{M})$ into
  a curve $\boldsymbol{\zeta} \circ \mathbf{c} \in \mathrm{C}^1([a, b]\,; \mathbb{N})$.

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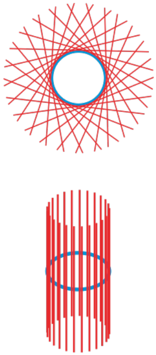
  a curve $\boldsymbol{\zeta} \circ \mathbf{c} \in \mathrm{C}^1([a,b]\,;\mathbb{N})$.
- The tangent map $T_{\mathbf{x}}\boldsymbol{\zeta} \in \mathrm{C}^0(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})$
  sends a tangent vector at $\mathbf{x} \in \mathbb{M}$
  $\mathbf{v} \in \mathbb{T}_{\mathbf{x}}(\mathbb{M}) := \partial_{\mu=\lambda}\, \mathbf{c}(\mu)$
  into a tangent vector at $\boldsymbol{\zeta}(\mathbf{x}) \in \mathbb{N}$
  $T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{v} \in \mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}(\mathbb{N}) := \partial_{\mu=\lambda}\,(\boldsymbol{\zeta} \circ \mathbf{c})(\mu)$

Math2

Math2

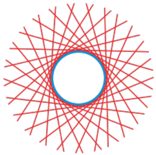
Tangent bundle

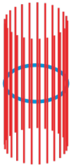


Tangent bundle

- disjoint union of tangent spaces:

$$\mathbb{TM} := \cup_{\mathbf{x}\in\mathbb{M}} \mathbb{T}_{\mathbf{x}}\mathbb{M}$$





Math2

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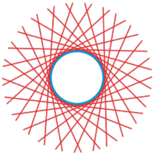
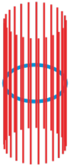


- Projection: $\boldsymbol{\tau}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{T}\mathbb{M}\,;\mathbb{M})$

$$\mathbf{v} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}\,, \quad \boldsymbol{\tau}_{\mathbb{M}}(\mathbf{v}) := \mathbf{x} \quad \text{base point}$$



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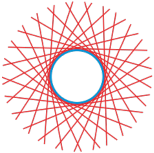

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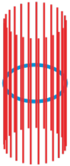

- Surjective submersion:

$T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{T}_{\mathbf{v}}\mathbb{TM}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})$ is surjective

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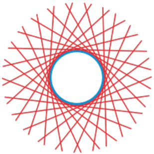

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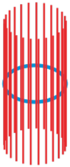

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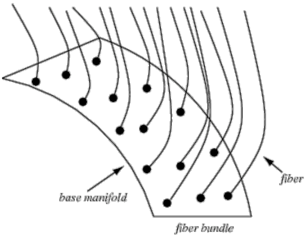
- Tangent functor

$$\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N}) \quad \mapsto \quad T\boldsymbol{\zeta} \in \mathrm{C}^0(\mathbb{TM}\,;\mathbb{TN})$$

Math3

Math3

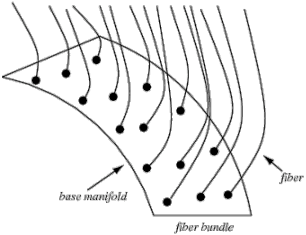
Fiber bundles



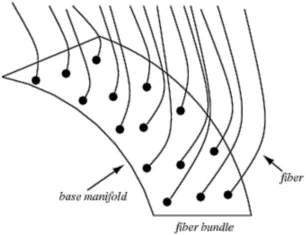
Math3

Fiber bundles

- $\mathrm{E}, \mathbb{M}$ manifolds



Math3



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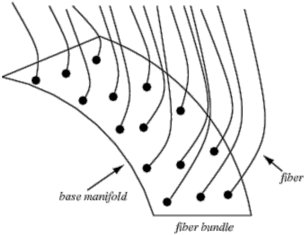
Math3



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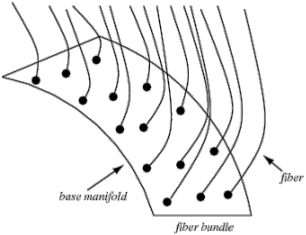
Math3



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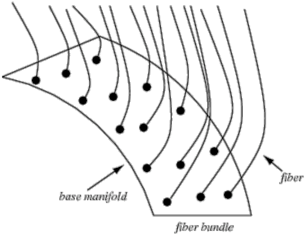
Math3



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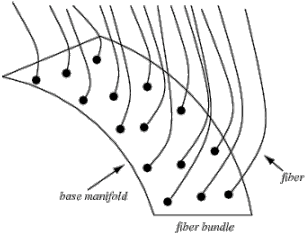
- Vertical tangent subbundle $T\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \in \mathrm{C}^0(\mathbb{V}\mathrm{E}\,;\mathbb{T}\mathbb{M})$

Math3



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- Vertical tangent subbundle $T\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \in \mathrm{C}^0(\mathbb{VE}\,;\mathbb{TM})$ with:
  $\delta\mathbf{e} \in \mathbb{VE} \subset \mathbb{TE} \quad \Longrightarrow \quad T_{\mathbf{e}}\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \cdot \delta\mathbf{e} = 0$

Math4

Math4

Trivial and non-trivial fiber bundles

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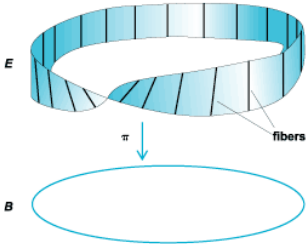

Math4

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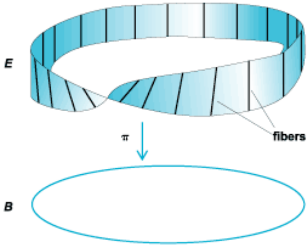
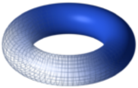
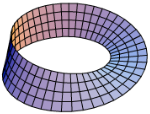
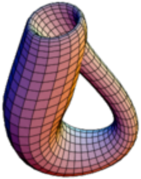


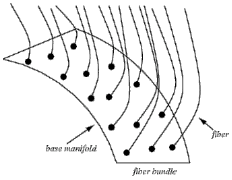

Torus


Listing-Möbius strip


Klein Bottle

Math5

Sections of fiber bundles



Math5

Sections of fiber bundles

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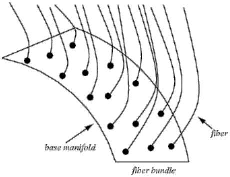
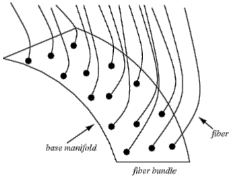


Math5



Sections of fiber bundles

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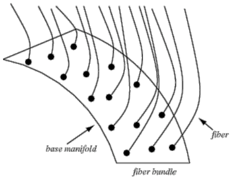
- Sections $\mathbf{s}_{\mathrm{E},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{E})\,, \quad \boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{s}_{\mathrm{E},\mathbb{M}} = \mathrm{ID}_{\mathbb{M}}$

Math5



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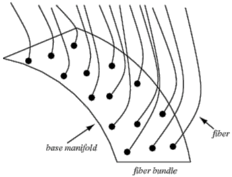

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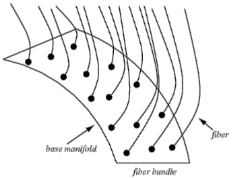
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Math5



Sections of fiber bundles

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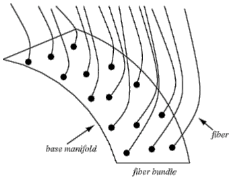
- Sections $\mathbf{s}_{\mathrm{E},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{E})\,,\quad \boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{s}_{\mathrm{E},\mathbb{M}} = \mathrm{ID}_{\mathbb{M}}$
- Tangent v.f. $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})\,,\quad \boldsymbol{\tau}_{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = \mathrm{ID}_{\mathrm{E}}$
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Sections of tangent and bi-tangent bundles

Math5



Sections of fiber bundles

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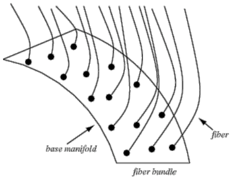
- Tangent v.f. $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})\,, \quad \boldsymbol{\tau}_{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = \mathrm{ID}_{\mathrm{E}}$
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Sections of tangent and bi-tangent bundles

- Tangent vector fields:

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$

Math5



Sections of fiber bundles

- Fiber bundle $\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{M})$
- Sections $\mathbf{s}_{\mathrm{E},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{E})\,, \quad \boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{s}_{\mathrm{E},\mathbb{M}} = \mathrm{ID}_{\mathbb{M}}$
- Tangent v.f. $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})\,, \quad \boldsymbol{\tau}_{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = \mathrm{ID}_{\mathrm{E}}$
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Sections of tangent and bi-tangent bundles

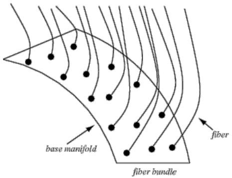
- Tangent vector fields:

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$

- Bi-tangent vector fields:

$$\mathbf{X} \in \mathrm{C}^1(\mathbb{T}\mathbb{M}\,;\mathbb{T}\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{T}\mathbb{M}} \circ \mathbf{X} = \mathrm{ID}_{\mathbb{T}\mathbb{M}}$$

Math5



Sections of fiber bundles

- Fiber bundle $\boldsymbol{\pi}_{\mathbb{M},\mathbb{E}} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{M})$
- Sections $\mathbf{s}_{\mathbb{E},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{E})\,, \quad \boldsymbol{\pi}_{\mathbb{M},\mathbb{E}} \circ \mathbf{s}_{\mathbb{E},\mathbb{M}} = \mathrm{ID}_{\mathbb{M}}$
- Tangent v.f. $\mathbf{v}_{\mathbb{E}} \in \mathrm{C}^1(\mathbb{E}\,;\mathbb{TE})\,, \quad \boldsymbol{\tau}_{\mathbb{E}} \circ \mathbf{v}_{\mathbb{E}} = \mathrm{ID}_{\mathbb{E}}$
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M},\mathbb{E}} \circ \mathbf{v}_{\mathbb{E}} = 0$

Sections of tangent and bi-tangent bundles

- Tangent vector fields:
$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM}) \;:\; \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$
- Bi-tangent vector fields:
$$\mathbf{X} \in \mathrm{C}^1(\mathbb{TM}\,;\mathbb{TTM}) \;:\; \boldsymbol{\tau}_{\mathbb{TM}} \circ \mathbf{X} = \mathrm{ID}_{\mathbb{TM}}$$
- Vertical bi-tangent vectors $\mathbf{X} \in \operatorname{Ker} T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{M}}$

Math6

Tensor spaces

Math6

Tensor spaces

- Covariant $\quad \mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^{*}\mathbb{M})$

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^{*}\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \in \mathrm{CON}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\text{Cov}} \in \text{Cov}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\text{Con}} \in \text{Con}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed $\mathbf{s}_{\mathbf{x}}^{\text{Mix}} \in \text{Mix}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \mathrm{Cov}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} \in \mathrm{Con}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed $\mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} \in \mathrm{Mix}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- with the alteration rules:

$$\mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} , \quad \mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} = \mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} \circ \mathbf{g}_{\mathbf{x}}^{-1}$$

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \mathrm{Cov}_{\mathbf{x}}(\mathbb{T}\mathbb{M}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} \in \mathrm{Con}_{\mathbf{x}}(\mathbb{T}\mathbb{M}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed $\mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} \in \mathrm{Mix}_{\mathbf{x}}(\mathbb{T}\mathbb{M}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
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Tensor bundles and sections

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \in \mathrm{CON}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed $\mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \in \mathrm{MIX}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- with the alteration rules:

$$\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} , \quad \mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} = \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \circ \mathbf{g}_{\mathbf{x}}^{-1}$$

Tensor bundles and sections

- Tensor bundle $\boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{TM}) ; \mathbb{M})$

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \mathrm{Cov}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} \in \mathrm{Con}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed $\mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} \in \mathrm{Mix}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- with the alteration rules:

$$\mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} , \quad \mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} = \mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} \circ \mathbf{g}_{\mathbf{x}}^{-1}$$

Tensor bundles and sections

- Tensor bundle $\boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{Tens}} \in \mathrm{C}^1(\mathrm{Tens}(\mathbb{TM}) ; \mathbb{M})$
- Tensor field $\mathbf{s}_{\mathbb{M}}^{\mathrm{Tens}} \in \mathrm{C}^1(\mathbb{M} ; \mathrm{Tens}(\mathbb{TM}))$

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \in \mathrm{CON}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed $\mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \in \mathrm{MIX}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- with the alteration rules:

$$\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} , \quad \mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} = \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \circ \mathbf{g}_{\mathbf{x}}^{-1}$$

Tensor bundles and sections

- Tensor bundle $\boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{TM}) ; \mathbb{M})$
- Tensor field $\mathbf{s}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathbb{M} ; \mathrm{TENS}(\mathbb{TM}))$
- with: $\boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{TENS}} \circ \mathbf{s}_{\mathbb{M}}^{\mathrm{TENS}} = \mathrm{ID}_{\mathbb{M}}$

Math7

Push and pull

Push and pull

Given a map $\zeta \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$

- Pull-back of a scalar field

$$f : \mathbb{N} \mapsto \mathrm{FUN}(\mathbb{N}) \quad \mapsto \quad \zeta{\downarrow}f : \mathbb{M} \mapsto \mathrm{FUN}(\mathbb{M})$$

defined by:

$$(\zeta{\downarrow}f)_{\mathbf{x}} := \zeta{\downarrow}f_{\zeta(\mathbf{x})} := f_{\zeta(\mathbf{x})} \in \mathrm{FUN}_{\mathbf{x}}(\mathbb{M})\,.$$

Math7

Push and pull

Given a map $\zeta \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$

- Pull-back of a scalar field

$$f : \mathbb{N} \mapsto \mathrm{F{\scriptsize UN}}(\mathbb{N}) \quad \mapsto \quad \zeta{\downarrow}f : \mathbb{M} \mapsto \mathrm{F{\scriptsize UN}}(\mathbb{M})$$

defined by:

$$(\zeta{\downarrow}f)_{\mathbf{x}} := \zeta{\downarrow}f_{\zeta(\mathbf{x})} := f_{\zeta(\mathbf{x})} \in \mathrm{F{\scriptsize UN}}_{\mathbf{x}}(\mathbb{M})\,.$$

- Push-forward of a tangent vector field

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM}) \quad \mapsto \quad \zeta{\uparrow}\mathbf{v} : \mathbb{N} \mapsto \mathbb{TN}$$

defined by:

$$(\zeta{\uparrow}\mathbf{v})_{\zeta(\mathbf{x})} := \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} = T_{\mathbf{x}}\zeta \cdot \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N}\,.$$

Math8

Push and pull of tensor fields

Math8

Push and pull of tensor fields

- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} \rangle = \langle T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle$$

Math8

Push and pull of tensor fields

- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} \rangle = \langle\, T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle$$

- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \circ T_{\mathbf{x}}\zeta \in \mathrm{Cov}(\mathbb{TM})_{\mathbf{x}}$$

Math8

Push and pull of tensor fields

- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_\mathbf{x} \rangle = \langle\, T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle$$

- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \circ T_\mathbf{x}\zeta \in \mathrm{Cov}(\mathbb{TM})_\mathbf{x}$$

- Contravariant tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{Con}}_\mathbf{x} = T_\mathbf{x}\zeta \circ \mathbf{s}^{\mathrm{Con}}_\mathbf{x} \circ T^*_{\zeta(\mathbf{x})}\zeta \in \mathrm{Con}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

Math8

Push and pull of tensor fields

- Covectors

$$\langle\, \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \,\rangle = \langle\, \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_\mathbf{x} \,\rangle = \langle\, T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \,\rangle$$

- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} \circ T_\mathbf{x}\zeta \in \mathrm{COV}(\mathbb{TM})_\mathbf{x}$$

- Contravariant tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{CON}}_\mathbf{x} = T_\mathbf{x}\zeta \circ \mathbf{s}^{\mathrm{CON}}_\mathbf{x} \circ T^*_{\zeta(\mathbf{x})}\zeta \in \mathrm{CON}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

- Mixed tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{MIX}}_\mathbf{x} = T_\mathbf{x}\zeta \circ \mathbf{s}^{\mathrm{MIX}}_\mathbf{x} \circ T_{\zeta(\mathbf{x})}\zeta^{-1} \in \mathrm{MIX}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a, b] ; \mathbb{M})$

Math9

Parallel transport along a curve $\mathbf{c} \in C^1([a, b]\,; \mathbb{M})$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \in \mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$$

$$\mathbf{c}_{\mu,\mu} \Uparrow \mathbf{v_x} = \mathbf{v_x}$$

$$\mathbf{c}_{\lambda,\mu} \Uparrow \circ\, \mathbf{c}_{\mu,\nu} \Uparrow\, = \mathbf{c}_{\lambda,\nu} \Uparrow$$

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a, b] \,; \mathbb{M})$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$$

$$\mathbf{c}_{\mu,\mu} \Uparrow \mathbf{v}_{\mathbf{x}} = \mathbf{v}_{\mathbf{x}}$$

$$\mathbf{c}_{\lambda,\mu} \Uparrow \circ\, \mathbf{c}_{\mu,\nu} \Uparrow \, = \mathbf{c}_{\lambda,\nu} \Uparrow$$

- Covector fields $\mathbf{v}^*_{\mathbf{x}} \in \mathbb{T}^*_{\mathbf{x}}\mathbb{M}$ (by naturality)

$$\langle\, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}^*_{\mathbf{x}}, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}_{\mathbf{x}} \,\rangle = \mathbf{c}_{\lambda,\mu} \Uparrow \langle\, \mathbf{v}^*_{\mathbf{x}}, \mathbf{v}_{\mathbf{x}} \,\rangle$$

- Tensor fields (by naturality)

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a,b]\,;\mathbb{M})$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \in \mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$$

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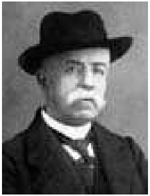
- Tensor fields (by naturality)



Gregorio Ricci-Curbastro (1853 - 1925)

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a,b]\,;\mathbb{M})$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \in \mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$$

$$\mathbf{c}_{\mu,\mu} \Uparrow \mathbf{v_x} = \mathbf{v_x}$$

$$\mathbf{c}_{\lambda,\mu} \Uparrow \circ\, \mathbf{c}_{\mu,\nu} \Uparrow \, = \mathbf{c}_{\lambda,\nu} \Uparrow$$

- Covector fields $\mathbf{v}^*_{\mathbf{x}} \in \mathbb{T}^*_{\mathbf{x}}\mathbb{M}$ (by naturality)

$$\langle\, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}^*_{\mathbf{x}}, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \,\rangle = \mathbf{c}_{\lambda,\mu} \Uparrow \langle\, \mathbf{v}^*_{\mathbf{x}}, \mathbf{v_x} \,\rangle$$

- Tensor fields (by naturality)

Tullio Levi-Civita (1873 - 1941)

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; \mathbf{Tens}(\mathbb{T}\mathbb{M}))$

along the flow of a tangent vector field

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; \mathbf{Tens}(\mathbb{TM}))$

along the flow of a tangent vector field

- Tangent vector fields and Flows

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM}) \qquad \mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{M})$$

$$\mathbf{v} := \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{v}}_{\lambda}$$

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; \mathbf{Tens}(\mathbb{TM}))$

along the flow of a tangent vector field

- Tangent vector fields and Flows

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM}) \qquad \mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{M})$$

$$\mathbf{v} := \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{v}}_{\lambda}$$

- Lie derivative - LD

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{s} := \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{v}}_{\lambda}\!\downarrow (\mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbf{Tens}(\mathbb{TM}))$

along the flow of a tangent vector field

- Tangent vector fields and Flows

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM}) \qquad \mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$$
$$\mathbf{v} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}$$

- Lie derivative - LD

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{s} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}\!\downarrow(\mathbf{s}\circ\mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

- Parallel derivative - PD

$$\nabla_{\mathbf{v}}\,\mathbf{s} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Downarrow(\mathbf{s}\circ\mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

NLCM: Nonlinear Continuum Mechanics

Key contributions

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C. Truesdell & W. Noll *The non-linear field theories of mechanics*
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2) M.E. Gurtin *An Introduction to Continuum Mechanics*
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NLCM: Nonlinear Continuum Mechanics

Key contributions

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G. Romano & R. Barretta Covariant hypo-elasticity
Eur. J. Mech. A-Solids 30 (2011) 1012–1023

G. Romano, R. Barretta, M. Diaco Basic Geometric Issues in Non-Linear Continuum Mechanics, preprint (2011).

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How to play the game
according to a full geometric approach

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Kinematics

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- Events manifold: $\mathbb{E}$ – four dimensional RIEMANN manifold

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Kinematics

- Events manifold: E – four dimensional RIEMANN manifold
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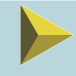
- time is absolute (Classical Mechanics)
- distance between simultaneous events $\mapsto$ space-metric
- distance between localized events $\mapsto$ time-metric

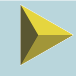
Math11

Math11



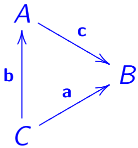
lenght of symplex's edges

Math11



lenght of symplex's edges

- Norm axioms



$$\|\mathbf{a}\| \geq 0\,, \quad \|\mathbf{a}\| = 0 \quad \Longrightarrow \quad \mathbf{a} = 0$$
$$\|\mathbf{a}\| + \|\mathbf{b}\| \geq \|\mathbf{c}\| \quad \textbf{triangle inequality},$$
$$\|\alpha\,\mathbf{a}\| = |\alpha|\,\|\mathbf{a}\|$$

lenght of symplex's edges

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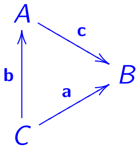

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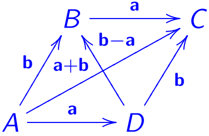
- Parallelogram rule



$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\,\big[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\big]$$

Math12

Math12

The metric tensor

- Theorem (Fréchet – von Neumann – Jordan)

Math12

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$$\mathbf{g}(\mathbf{a}\,,\mathbf{b}) := \frac{1}{4}\left[\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right]$$

Math12

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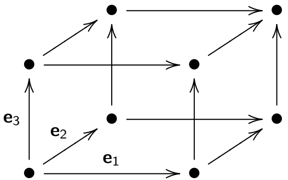
$\mathrm{VOL}\Big($



$\Big)^2 = \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1) \cdots \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1) \cdots \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$

Maurice René Fréchet (1878 - 1973)

Math12

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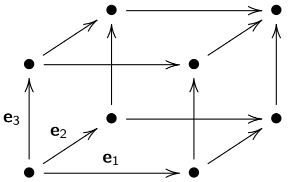

John von Neumann (1903 - 1957)

Math12

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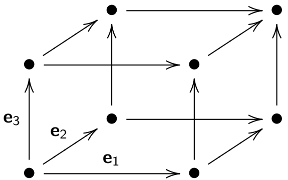

$)^2 = \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$

Pascual Jordan (1902 - 1980)

Math12

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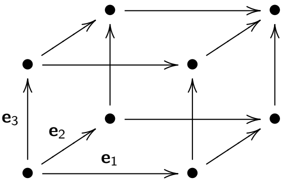
VOL (



$)^2$

$$= \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$$

Kosaku Yosida (1909 - 1990)

Events manifold fibrations

Events manifold fibrations

- Time and space fibrations: $\gamma : \mathrm{E} \mapsto \mathcal{S} \times I$ (observer)

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$$
\begin{array}{ccc}
\mathcal{S} & \xleftrightarrow{\ \mathrm{ID}_{\mathcal{S}}\ } & \mathcal{S} \\
\uparrow {\scriptstyle \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}} & & \uparrow {\scriptstyle \boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)}} \\
\mathrm{E} & \xrightarrow{\ \gamma\ } & \mathcal{S} \times I \\
\downarrow {\scriptstyle \boldsymbol{\pi}_{I,\mathrm{E}}} & & \downarrow {\scriptstyle \boldsymbol{\pi}_{I,(\mathcal{S}\times I)}} \\
I & \xleftrightarrow{\ \mathrm{ID}_{I}\ } & I
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{l}
\boldsymbol{\pi}_{I,\mathrm{E}} = \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \circ \boldsymbol{\gamma} \\
\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} = \boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)} \circ \boldsymbol{\gamma}
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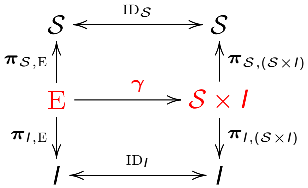

$\Longleftrightarrow$

$$\begin{aligned} \boldsymbol{\pi}_{I,\mathrm{E}} &= \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \circ \boldsymbol{\gamma} \\ \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} &= \boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)} \circ \boldsymbol{\gamma} \end{aligned}$$

- Space-time metric: $\mathbf{g}_{\mathrm{E}} := \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}{\downarrow}\mathbf{g}_{\mathcal{S}} + \boldsymbol{\pi}_{I,\mathrm{E}}{\downarrow}\mathbf{g}_{I}$

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$$

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- Time-vertical subbundle: spatial vectors

$$\mathbf{v} \in \mathbb{V}_{\mathbf{e}}\mathrm{E} \quad \iff \quad T_{\mathbf{e}}\boldsymbol{\pi}_{I,\mathrm{E}} \cdot \mathbf{v} = 0$$

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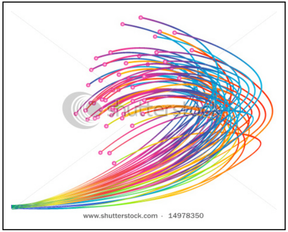
- $\mathbf{v}_{\mathbf{e}} \in \mathbb{V}_{\mathbf{e}}\mathrm{E} \quad \Longleftrightarrow \quad \gamma\uparrow\mathbf{v}_{\mathbf{e}} = (v_{\mathbf{x},t}\,, 0_t) \in \mathbb{T}_{\mathbf{x}}\mathcal{S} \times \mathbb{T}_t I$

Trajectory



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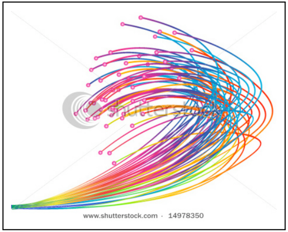
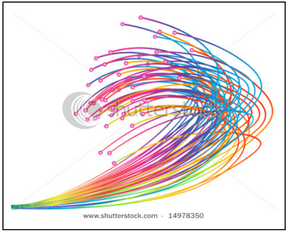


- Trajectory $\mapsto$ a manifold $\mathcal{T}_\varphi$ with injective immersion in the events time-bundle: $\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathrm{E})$

Trajectory



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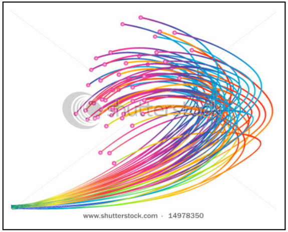
Trajectory



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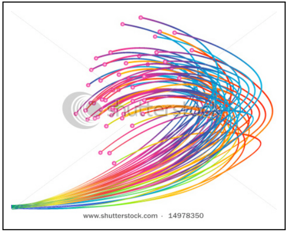
- time bundle $\mapsto$ fibers: body placements $\boldsymbol{\Omega}_t$

Trajectory



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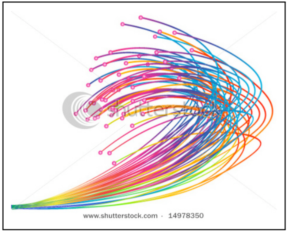
- not a space bundle $\mapsto$ fibers: irregular subsets of the observation time interval I

Trajectory



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- Time-vertical subbundle: material vectors

$$\mathbf{v} \in \mathbb{V}_{\mathbf{e}}\mathcal{T}_\varphi \quad \Longleftrightarrow \quad T_{\mathbf{e}}\boldsymbol{\pi}_{I,\mathcal{T}_\varphi} \cdot \mathbf{v} = 0$$

Evolution

Evolution

- Evolution operator $\varphi^{\mathcal{T}_{\varphi}}$

Evolution

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- Displacements: diffeomorphisms between placements

$$\varphi_{\tau,t}^{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathbf{\Omega}_t\,;\mathbf{\Omega}_\tau)\,, \quad \tau, t \in I$$

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$$\varphi^{\mathcal{T}_\varphi}_{\tau,t} \in \mathrm{C}^1(\mathbf{\Omega}_t\,;\mathbf{\Omega}_\tau)\,, \quad \tau, t \in I$$

- Law of determinism (Chapman-Kolmogorov):

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- Simultaneity of events is preserved:

$$\boldsymbol{\pi}_{I,\mathcal{T}_{\varphi}}(\boldsymbol{\varphi}^{\mathcal{T}_{\varphi}}_{\tau,t}(\mathbf{e}_t)) = \tau$$

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- Trajectory speed:

$$\mathbf{v}_{\mathcal{T}_\varphi}(\mathbf{e}_t) := \partial_{\tau=t}\,\varphi^{\mathcal{T}_\varphi}_{\tau,t}(\mathbf{e}_t) \quad \Longrightarrow \quad T_{\mathbf{e}}\boldsymbol{\pi}_{I,\mathcal{T}_\varphi} \cdot \mathbf{v}_{\mathcal{T}_\varphi}(\mathbf{e}_t) = 1_t$$

Body and particles

Body and particles

- Equivalence relation on the trajectory:

$$(\mathbf{e}_1\,,\mathbf{e}_2) \in \mathcal{T}_{\boldsymbol{\varphi}} \times \mathcal{T}_{\boldsymbol{\varphi}} \;:\; \mathbf{e}_2 = \boldsymbol{\varphi}^{\mathcal{T}_{\varphi}}_{t_2,t_1}(\mathbf{e}_1)\,.$$

with $t_i = \boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e}_i)\,, \quad i = 1,2\,.$

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Body $=$ quotient manifold (foliation)

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with $t_i = \boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e}_i)\,, \quad i = 1, 2\,.$

Body $=$ quotient manifold (foliation)
Particles $=$ equivalence classes (folia)

Body and particles

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with $t_i=\boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e}_i)\,,\qquad i=1,2\,.$

Body = quotient manifold (foliation)
Particles = equivalence classes (folia)

- mass conservation

$$\int_{\boldsymbol{\Omega}_{t_1}}\mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}},t_1}=\int_{\boldsymbol{\Omega}_{t_2}}\mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}},t_2}\quad\Longleftrightarrow\quad\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}}\mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}}}=0$$

$\mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}}}\in\mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{V{\small OL}}(\mathbb{T}\mathcal{T}_{\boldsymbol{\varphi}}))$ mass form

Tensor fields in NLCM

Tensor fields in NLCM

Space-time fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,; \mathrm{TENS}(\mathbb{T}\mathrm{E}))$	Space-time metric tensor
Spatial fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,; \mathrm{TENS}(\mathbb{V}\mathrm{E}))$	Spatial metric tensor

Tensor fields in NLCM

Space-time fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{Tens}(\mathbb{T}\mathrm{E}))$	Space-time metric tensor
Spatial fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{Tens}(\mathbb{V}\mathrm{E}))$	Spatial metric tensor
Trajectory fields	$\mathbf{s}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\mathrm{Tens}(\mathbb{T}\mathcal{T}_{\varphi}))$	Trajectory metric, trajectory speed
Material fields	$\mathbf{s}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\mathrm{Tens}(\mathbb{V}\mathcal{T}_{\varphi}))$	Stress, stressing, material metric, stretching.

Tensor fields in NLCM

Space-time fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{Tens}(\mathbb{T}\mathrm{E}))$	Space-time metric tensor
Spatial fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{Tens}(\mathbb{V}\mathrm{E}))$	Spatial metric tensor
Trajectory fields	$\mathbf{s}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\mathrm{Tens}(\mathbb{T}\mathcal{T}_{\varphi}))$	Trajectory metric, trajectory speed
Material fields	$\mathbf{s}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\mathrm{Tens}(\mathbb{V}\mathcal{T}_{\varphi}))$	Stress, stressing, material metric, stretching.
Trajectory-based space-time fields	$\mathbf{s}_{\mathrm{E},\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\mathrm{Tens}(\mathbb{T}\mathrm{E}))$	Trajectory speed (immersed)
Trajectory-based spatial fields	$\mathbf{s}_{\mathrm{E},\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\mathrm{Tens}(\mathbb{V}\mathrm{E}))$	Virtual velocity, acceleration, momentum, force

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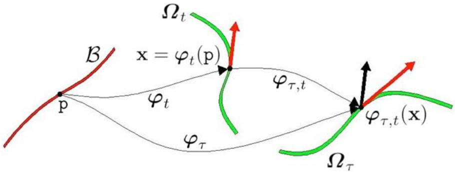

Parallel transport does not preserve time-verticality

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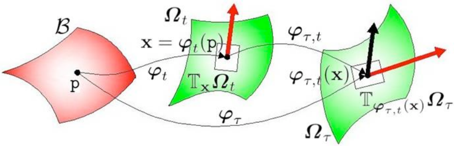

Parallel transport does not preserve time-verticality

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Lie time derivative - LTD

- Trajectory and material tensor field

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Material time-derivative - MTD

- Trajectory-based space-time and spatial fields

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with $\mathbf{v}_{\mathrm{E},\mathcal{T}_{\varphi}} := \mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}} \!\uparrow\! \mathbf{v}_{\mathcal{T}_{\varphi}}$.

Rivers and Cogwheels

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Gottfried Wilhelm von LEIBNIZ (1646 - 1716)

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Both conditions are not fulfilled in solid mechanics, in general.

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[2] *See e.g.*
1) C. Truesdell, *A first Course in Rational Continuum Mechanics*
Second Ed. Academic Press, New-York (1991). First Ed. 1977
2) M.E. Gurtin, *An Introduction to Continuum Mechanics*
Academic Press, San Diego (1981)
3) J.E. Marsden & T.J.R. Hughes, *Mathematical Foundations of Elasticity*
Prentice-Hall, Redwood City, Cal. (1983)

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- Mixed form of the stretching tensor (standard):

$$\mathbf{D}_{\mathcal{T}_\varphi} := \mathbf{g}^{-1}_{\mathcal{T}_\varphi} \circ \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}} \mathbf{g}_{\mathcal{T}_\varphi} = \mathrm{sym}\,(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi})$$

Stress and stressing

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is not performable on the time-vertical subbundle of material tensor fields because the parallel derivative $\nabla^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}}$ on the trajectory does not preserve time-verticality.

NLCM
Prolegomena
A basic question
Basic
Tangent spaces
Tangent functor
Fiber bundles
Trivial and non-trivial fiber bundles
Sections
Tensor bundle and sections
Push and pull
Push and pull of tensor fields
Parallel transport
Derivatives
Key contributions
Kinematics
Metric measurements
Metric theory
Events manifold fibrations
Trajectory
Evolution

Stress and stressing

- Stress: $\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{Con}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}}))$ in duality with the
- Stretching: $\dot{\boldsymbol{\varepsilon}}_{\mathcal{T}_{\boldsymbol{\varphi}}} := \frac{1}{2}\dot{\mathbf{g}}_{\mathcal{T}_{\boldsymbol{\varphi}}} = \frac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}}\mathbf{g}_{\mathcal{T}_{\boldsymbol{\varphi}}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{Cov}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}}))$
- Stressing: Lie time derivative

  $$\dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\boldsymbol{\varphi}},t} := (\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}}\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}})_t = \partial_{\tau=t}\,(\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}},\tau})$$

  The expression in terms of parallel derivative:

  $$\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}}\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}} = \nabla^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}}\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}} - \mathrm{sym}\,(\nabla^{\mathcal{T}_{\boldsymbol{\varphi}}}\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}\circ\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}})$$

  is not performable on the time-vertical subbundle of material tensor fields because the parallel derivative $\nabla^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}}$ on the trajectory does not preserve time-verticality.
- Treatments which do not adopt a full geometric approach do not even perceive the difficulties revealed by the previous investigation.

Objective stress rate tensors

A sample of objective stress rate tensors

Co-rotational stress rate tensor, ZAREMBA (1903), JAUMANN (1906,1911), PRAGER (1960):

$$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}$$

with $\dot{\mathbf{T}}$ material time derivative.

Convective stress tensor rate, ZAREMBA (1903), OLDROYD (1950), TRUESDELL (1955), SEDOV (1960), TRUESDELL & NOLL (1965):

$$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L}$$

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These formulas, and similar ones in literature, rely on the application of Leibniz rule and on taking the parallel derivative of the material stress tensor field according to the trajectory connection.

The lack of regularity that may prevent to take partial time derivatives and the lack of conservation of time-verticality by parallel transport, are not taken into account.

Deformation gradient

The equivalence class of all material displacements whose tangent map have the common value:

$$T_{\mathbf{x}}\boldsymbol{\varphi}_{\tau,t} \in L\,(\mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t\,;\mathbb{T}_{\boldsymbol{\varphi}_{\tau,t}(\mathbf{x})}\boldsymbol{\Omega}_\tau\,)$$

- is called the *first jet* of $\boldsymbol{\varphi}_{\tau,t}$ at $\mathbf{x} \in \boldsymbol{\Omega}_t$ in differential geometry
- and the *relative deformation gradient* in continuum mechanics.

The chain rule between tangent maps:

$$T_{\boldsymbol{\varphi}_{\tau,s}(\mathbf{x})}\boldsymbol{\varphi}_{\tau,s} = T_{\boldsymbol{\varphi}_{t,s}(\mathbf{x})}\boldsymbol{\varphi}_{\tau,t} \circ T_{\mathbf{x}}\boldsymbol{\varphi}_{t,s}\,,$$

implies the corresponding one between material deformation gradients:

$$\mathbf{F}_{\tau,s} = \mathbf{F}_{\tau,t} \circ \mathbf{F}_{t,s}\,.$$

Time rate of deformation gradient, TRUESDELL & NOLL (1965)

$$\dot{\mathbf{F}}_{t,s} = \mathbf{L}_t\,\mathbf{F}_{t,s}$$

with $\dot{\mathbf{F}}_{t,s} := \partial_{\tau=t}\,\mathbf{F}_{\tau,s}$ and $\mathbf{L}_t := \partial_{\tau=t}\,\mathbf{F}_{\tau,t}$ time derivatives.

$$\mathbf{L}_t(\mathbf{x}) \cdot \mathbf{h}_{\mathbf{x}} := \partial_{\tau=t}\,\mathbf{F}_{\tau,t}(\mathbf{x}) \cdot \mathbf{h}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t\,, \qquad \forall\,\mathbf{h}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t$$

with $\mathbf{F}_{\tau,t}(\mathbf{x}) \cdot \mathbf{h}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_\tau$. The LIE time derivative gives:

$$\partial_{\tau=t}\,(T_{\mathbf{x}}\boldsymbol{\varphi}_{\tau,t})^{-1} \cdot (T_{\mathbf{x}}\boldsymbol{\varphi}_{\tau,t} \cdot \mathbf{h}_{\mathbf{x}}) = \partial_{\tau=t}\,\mathbf{h}_{\mathbf{x}} = 0$$

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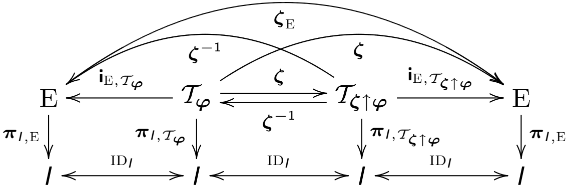
  time-bundle diffeomorphism



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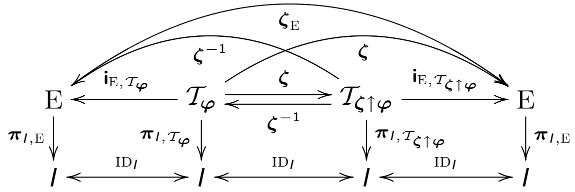
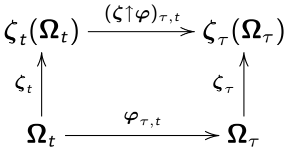


- Pushed motion



$$\Longleftrightarrow \quad (\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi})_{\tau,t} = \boldsymbol{\zeta}_\tau \circ \boldsymbol{\varphi}_{\tau,t} \circ \boldsymbol{\zeta}_t^{-1}$$

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Properties of Lie derivative

- Push of Lie time derivative to a fixed configuration

$$\varphi_{t,\mathrm{FIX}}\!\downarrow\!(\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\varphi}}}\,\mathbf{s}_{\mathcal{T}_{\varphi}})_t = \partial_{\tau=t}\,\varphi_{\tau,\mathrm{FIX}}\!\downarrow\!\mathbf{s}_{\mathcal{T}_{\varphi},\tau}$$

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- Lie time derivative along pushed motions

$$\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\zeta\uparrow\varphi}}}(\zeta\!\uparrow\!\mathbf{s}_{\varphi}) = \zeta\!\uparrow\!(\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\varphi}}}\,\mathbf{s}_{\mathcal{T}_{\varphi}})$$

Constitutive laws

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- Constitutive time invariance

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- Constitutive invariance under relative motions

$$\mathbf{H}_{\mathcal{T}_{\zeta\uparrow\varphi}} = \zeta\!\uparrow\!\mathbf{H}_{\mathcal{T}_\varphi}$$

$$(\zeta\!\uparrow\!\mathbf{H}_{\mathcal{T}_\varphi})(\zeta\!\uparrow\!\mathbf{s}_{\mathcal{T}_\varphi}) = \zeta\!\uparrow\!(\mathbf{H}_{\mathcal{T}_\varphi}(\mathbf{s}_{\mathcal{T}_\varphi}))$$

Hypo-elasticity

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- Constitutive hypo-elastic law
  $\mathbf{el}_{\mathcal{T}_{\varphi}}$ elastic stretching

$$\begin{cases} \dot{\varepsilon}_{\mathcal{T}_{\varphi}} = \mathbf{el}_{\mathcal{T}_{\varphi}} \\ \mathbf{el}_{\mathcal{T}_{\varphi}} = \mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\varphi}}(\boldsymbol{\sigma}_{\mathcal{T}_{\varphi}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\varphi}} \end{cases}$$

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- Cauchy integrability

$$\langle\, d_F \mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi}) \cdot \delta\boldsymbol{\sigma}_{\mathcal{T}_\varphi} \cdot \delta_1\boldsymbol{\sigma}_{\mathcal{T}_\varphi}, \delta_2\boldsymbol{\sigma}_{\mathcal{T}_\varphi} \,\rangle = \text{symmetric}$$

$$\Longrightarrow \quad \mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi}) = d_F \boldsymbol{\Phi}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi})$$

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- GREEN integrability

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- pull-back to reference:

$$\begin{aligned} \boldsymbol{\varphi}_{t,\mathrm{FIX}}\!\downarrow\!\mathbf{el}_{\mathcal{T}_\varphi,t} &= d_F^2 E^*_{\mathrm{FIX}}(\boldsymbol{\varphi}_{t,\mathrm{FIX}}\!\downarrow\!\boldsymbol{\sigma}_{\mathcal{T}_\varphi,t}) \cdot \partial_{\tau=t}\ \boldsymbol{\varphi}_{\tau,\mathrm{FIX}}\!\downarrow\!\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau} \\ &= \partial_{\tau=t}\, d_F E^*_{\mathrm{FIX}}(\boldsymbol{\varphi}_{\tau,\mathrm{FIX}}\!\downarrow\!\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}) \end{aligned}$$

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$$\begin{aligned} \boldsymbol{\varphi}_{\tau,\mathrm{FIX}} &:= \boldsymbol{\varphi}_{\tau,t} \circ \boldsymbol{\varphi}_{t,\mathrm{FIX}} \\ E^*_{\mathrm{FIX}} &:= \boldsymbol{\varphi}_{t,\mathrm{FIX}}\!\downarrow\! E^*_{\mathcal{T}_\varphi,t} \quad \text{time invariant} \end{aligned}$$

Conservativeness of hyper-elasticity

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GREEN integrability of the elastic operator $\mathbf{H}_{\mathcal{T}_{\varphi}}$
as a function of the KIRCHHOFF stress tensor field
implies conservativeness:

$$\oint_I \int_{\Omega_t} \langle \boldsymbol{\sigma}_{\mathcal{T}_{\varphi},t}, \mathbf{el}_{\mathcal{T}_{\varphi},t} \rangle \mathbf{m}_{\mathcal{T}_{\varphi},t} \, dt = 0$$

for any cycle in the stress time-bundle,
i.e. for any stress path $\boldsymbol{\sigma}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(I\,; \mathrm{CON}(\mathbb{V}\mathcal{T}_{\varphi}))$
such that:

$$\boldsymbol{\sigma}_{\mathcal{T}_{\varphi},t_2} = \boldsymbol{\varphi}_{t_2,t_1}\!\uparrow\!\boldsymbol{\sigma}_{\mathcal{T}_{\varphi},t_1}\,, \quad I = [t_1, t_2]$$

Elasto-visco-plasticity

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- Constitutive law

$\mathbf{el}_{\mathcal{T}_\varphi}$ elastic stretching
$\mathbf{pl}_{\mathcal{T}_\varphi}$ visco-plastic stretching

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}_{\mathcal{T}_\varphi} = \mathbf{el}_{\mathcal{T}_\varphi} + \mathbf{pl}_{\mathcal{T}_\varphi} & \text{stretching additivity} \\ \mathbf{el}_{\mathcal{T}_\varphi} = d_F^2 E^*_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}_\varphi} & \text{hyper-elastic law} \\ \mathbf{pl}_{\mathcal{T}_\varphi} \in \partial_F \mathcal{F}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_\varphi) & \text{visco-plastic flow rule} \end{cases}$$

Reference strains

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- total strain in the time interval $I = [s, t]$:

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- reference elastic and visco-plastic strain:

$$\mathbf{el}^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I} := \int_I \varphi_{t,\mathrm{FIX}}\!\downarrow\!\mathbf{el}_{\mathcal{T}_\varphi,t}\,dt\,, \qquad \mathbf{pl}^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I} := \int_I \varphi_{t,\mathrm{FIX}}\!\downarrow\!\mathbf{pl}_{\mathcal{T}_\varphi,t}\,dt$$

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- additivity of reference strains:

$$\varepsilon^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I} = \mathbf{el}^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I} + \mathbf{pl}^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I}$$

Material Frame Indifference (MFI)

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- **Any constitutive law must conform to the principle of MFI which requires that material fields, fulfilling the law, will still fulfill it when evaluated by another Euclid observer**

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for any isometric relative motion $\boldsymbol{\zeta}^{\text{iso}} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathcal{T}_{\boldsymbol{\zeta}^{\text{iso}}\uparrow\varphi})$ **induced by a change of Euclid observer** $\boldsymbol{\zeta}^{\text{iso}}_{\mathbf{E}} \in \mathrm{C}^1(\mathbf{E}\,;\mathbf{E})$**.**

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Equivalent condition

- Constitutive operators must be frame invariant

MFI in elasto-visco-plasticity

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- Frame invariance of the hypo-elastic operator

$$\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\zeta}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}}} = \boldsymbol{\zeta}^{\mathrm{ISO}}\!\uparrow\!\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\varphi}}}$$

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Pushed operator

$$(\boldsymbol{\zeta}^{\mathrm{ISO}}\!\uparrow\!\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\varphi}}})(\boldsymbol{\zeta}^{\mathrm{ISO}}\!\uparrow\!\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}}) \cdot \boldsymbol{\zeta}^{\mathrm{ISO}}\!\uparrow\!\dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\boldsymbol{\varphi}}} = \boldsymbol{\zeta}^{\mathrm{ISO}}\!\uparrow\!(\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\varphi}}}(\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\boldsymbol{\varphi}}})$$

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Examples:

- the simplest hypo-elastic operator is GREEN integrable and frame invariant:

$$\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\varphi}},t}(\mathbf{T}_{\mathcal{T}_{\boldsymbol{\varphi}},t}) := \frac{1}{2\,\mu}\,\mathbb{I}_{\mathcal{T}_{\boldsymbol{\varphi}},t} - \frac{\nu}{E}\,\mathbf{I}_{\mathcal{T}_{\boldsymbol{\varphi}},t} \otimes \mathbf{I}_{\mathcal{T}_{\boldsymbol{\varphi}},t}$$

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Pushed operator

$$(\boldsymbol{\zeta}^{\rm ISO}\!\uparrow\!\mathbf{H}^{\rm HYPO}_{\mathcal{T}_{\boldsymbol{\varphi}}})(\boldsymbol{\zeta}^{\rm ISO}\!\uparrow\!\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}}) \cdot \boldsymbol{\zeta}^{\rm ISO}\!\uparrow\!\dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\boldsymbol{\varphi}}} = \boldsymbol{\zeta}^{\rm ISO}\!\uparrow\!(\mathbf{H}^{\rm HYPO}_{\mathcal{T}_{\boldsymbol{\varphi}}}(\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\boldsymbol{\varphi}}})$$

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These results provide answers to unsolved questions posed in:

J.C. Simó & K.S. Pister, Remarks on rate constitutive equations for finite deformation problems: computational implications, *Comp. Meth. Appl. Mech. Eng.* **46** (1984) 201–215.
J. C. Simó & M. Ortiz, A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive equations, *Comp. Meth. Appl. Mech. Eng.* **49** (1985) 221–245.

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- Algorithms for numerical computations must be modified to comply with the covariant theory; multiplicative decomposition of the deformation gradient should be deemed as geometrically inconsistent