Università di Napoli Federico II - DIETI
Dipartimento di Ingegneria Elettrica
e delle Tecnologie dell'Informazione

A New Look at Electro-Magnetic Induction

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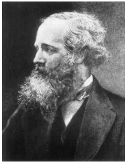
Università di Napoli Federico II, Napoli, Italia

Seminario
19 Marzo 2015

An historical sketch



James CLERK-MAXWELL (1831 - 1879)

[1]Electrodynamics from AMPÈRE to EINSTEIN (2000)

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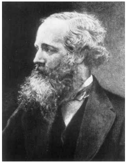

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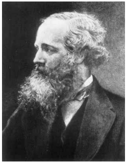

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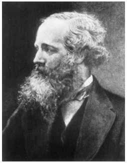

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The contribution by **J.J. Thomson** seems to have been not acknowledged and not quoted in literature until 2010 when I independently found the same expression, in intrinsic form. I have also detected a correcting factor one-half for the electric field induced on a charged body translating in a field of magnetic vortices, a factor quoted in a history book by **Darrigol** [1] and there attributed to a mistaken calculation by **J.J. Thomson**, afterwards corrected by **Hertz** and **Heavyside**.

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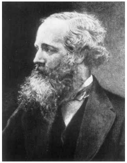

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Since the beginning of the story $2015 - 1855 = 160$ years have gone by.

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Geometry of
Space-time manifold

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Linearized Continuum Electrodynamics and Mechanics
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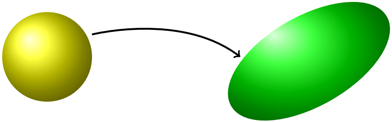
Non-Linear Continuum Electrodynamics and Mechanics
calls for Differential Geometry and Calculus on Manifolds as natural
tools for the developments of theoretical and computational models.
The role of Linear spaces is played by
tangent spaces to nonlinear manifolds.



Math1 – Tensor bundles on a manifold $\mathbf{M}$

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- Tensors $\mathbf{s}_\mathbf{x} : (\mathbf{v}_\mathbf{x}, \mathbf{v}^*_\mathbf{x}) \mapsto \mathbf{s}(\mathbf{v}_\mathbf{x}, \mathbf{v}^*_\mathbf{x})$ multilinear
- Tensorial map (2nd order)
  real-valued multilinear map $\mathbf{s}(\mathbf{v}, \mathbf{v}^*)$ that lives at points

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- Tensor fields (2nd order)
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  contravariant $\mathbf{s} : \mathbf{x} \in \mathbf{M} \mapsto \mathbf{s}(\mathbf{u}^*_{\mathbf{x}} , \mathbf{v}^*_{\mathbf{x}}) \in \mathcal{R}$
  mixed $\mathbf{s} : \mathbf{x} \in \mathbf{M} \mapsto \mathbf{s}(\mathbf{u}_{\mathbf{x}} , \mathbf{v}^*_{\mathbf{x}}) \in \mathcal{R}$

Math2 - Push forward and pull back

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Given a map $\zeta : \mathbf{M} \mapsto \mathbf{N}$ with $T\zeta : T\mathbf{M} \mapsto T\mathbf{N}$

- The pull-back of a scalar field

$$f : \mathbf{N} \mapsto \mathrm{F\scriptstyle UN}(\mathbf{N}) \quad \mapsto \quad \zeta{\downarrow}f : \mathbf{M} \mapsto \mathrm{F\scriptstyle UN}(\mathbf{M})$$

is defined by

$$(\zeta{\downarrow}f)_{\mathbf{x}} := \zeta{\downarrow}f_{\zeta(\mathbf{x})} := f_{\zeta(\mathbf{x})} \in \mathrm{F\scriptstyle UN}_{\mathbf{x}}(\mathbf{M})\,.$$

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- The push-forward of a tangent vector field

$$\mathbf{v} : \mathbf{M} \mapsto T\mathbf{M} \quad \mapsto \quad \zeta{\uparrow}\mathbf{v} : \mathbf{N} \mapsto T\mathbf{N}$$

  is defined by

$$(\zeta{\uparrow}\mathbf{v})_{\zeta(\mathbf{x})} := \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} = T_{\mathbf{x}}\zeta \cdot \mathbf{v}_{\mathbf{x}} \in T_{\zeta(\mathbf{x})}\mathbf{N}\,.$$

- Push and pull transformations of all other tensors are defined to comply with the previous ones.

Math3 – Convective and covariant derivatives

Marius Sophus Lie (1842 - 1899)

Derivatives of a tensor field $\mathbf{s} : \mathbf{M} \mapsto \mathrm{TENS}(T\mathbf{M})$
along the flow of a tangent vector field

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- Tangent vector fields and Flows

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- Lie derivative - LD (also called *convective derivative*)

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- Parallel derivative - PD (also called *covariant derivative*)

$$\boxed{\nabla_{\mathbf{v}}\,\mathbf{s} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow (\mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda})\,.}$$

Tullio Levi-Civita (1873 - 1841)

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An observer performs a double foliation of the 4D space-time manifold $\mathcal{E}$ into two complementary families of submanifolds.

- $\mathbf{Z}$ field of time-arrows tangent to 1D time-lines of isotopic events (same space location).
- $t : \mathcal{E} \mapsto \mathcal{R}$ time projection with

$$\langle\, dt, \mathbf{Z} \,\rangle = 1, \quad \text{tuning}$$

$$\mathbf{R} = dt \otimes \mathbf{Z} \quad \text{projector on time-lines}$$

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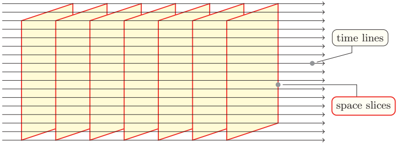

EUCLID space-time slicing.

Math5 – Differential forms

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  non-null skew-symmetric covariant tensor fields of maximal degree.

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- Skew-symmetric covariant tensors of maximal degree (equal to the manifold dimension) belong to a 1D linear space.
- Volume forms
  non-null skew-symmetric covariant tensor fields of maximal degree.
- Differential forms of degree greater than maximal vanish identically.

Math6 – Integrals of spatial volume forms

Vito VOLTERRA (1860 - 1940)

- $\mathbf{\Omega}$ compact spatial submanifold of $\mathcal{E}$
- Boundary operator $\boxed{\partial : \mathbf{\Omega} \mapsto \partial\mathbf{\Omega}}$ $\quad \dim \mathbf{\Omega} = \dim \partial\mathbf{\Omega} + 1$

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- Exterior derivative $\boxed{d : \Lambda^k(\mathbf{\Omega}) \mapsto \Lambda^{(k+1)}(\mathbf{\Omega})}$ $\quad \boxed{deg(d) = 1}$
- VOLTERRA-STOKES-KELVIN formula (d co-boundary operator)

$$\boxed{\oint_{\partial\mathbf{\Omega}} \omega = \int_{\mathbf{\Omega}} d\omega \quad \Longleftrightarrow \quad \langle \partial\mathbf{\Omega}, \omega \rangle = \langle \mathbf{\Omega}, d\omega \rangle}$$

$$\boxed{\deg(\omega) = \dim(\partial\mathbf{\Omega}) \,, \quad \deg(d\omega) = \dim(\mathbf{\Omega})}$$

Math7 – Closed and exact forms

Élie Cartan (1869 - 1951) 

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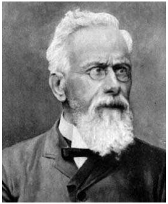
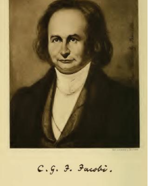
- POINCARÉ lemma:
  In a manifold contractible to a point
  (BETTI numbers vanish) closed forms are exact.



Enrico BETTI (1823 - 1892)

Math8 – Time derivative of integrals



Carl Gustav Jacob JACOBI (1840 - 1851)

$\mathbf{\Omega} \subset \mathcal{E}$ compact spatial submanifold

- JACOBI formula

  ω volume form on $\mathbf{\Omega}$, α time-lapse, $\varphi_\alpha : \mathbf{\Omega} \mapsto \mathcal{E}$ displacement

$$\boxed{\int_{\varphi_\alpha(\mathbf{\Omega})} \omega = \int_{\mathbf{\Omega}} \varphi_\alpha \!\downarrow\! \omega}$$

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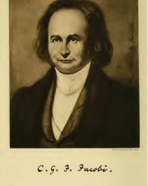

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$$\boxed{\mathcal{L}_{\mathbf{V}}\,\omega := \partial_{\alpha=0}\,(\varphi_\alpha{\downarrow}\omega)} \quad \Longrightarrow \quad \boxed{\partial_{\alpha=0} \int_{\varphi_\alpha(\mathbf{\Omega})} \omega = \int_{\mathbf{\Omega}} \mathcal{L}_{\mathbf{V}}\,\omega}$$

$$\mathbf{V} = \partial_{\alpha=0}\,\varphi_\alpha = \mathbf{v} + \mathbf{Z}\,, \qquad \mathbf{v} = \mathbf{P}\mathbf{V}$$

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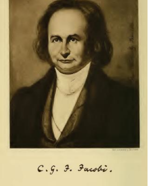

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Math9 - Extrusion and Homotopy

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Extrusion formula H.P. CARTAN (1951),

$$\boxed{\partial_{\alpha=0} \int_{\varphi_\alpha(\mathbf{\Omega})} \omega = \int_{\mathbf{\Omega}} (d\omega) \cdot \mathbf{V} + \int_{\mathbf{\Omega}} d(\omega \cdot \mathbf{V})}$$

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Recursion on the form-degree yields R.S. PALAIS formula (1954) for the exterior derivative d in terms of LIE derivatives.

$$\mathcal{L}_{\mathbf{V}}\, \omega^0 = (d\omega^0) \cdot \mathbf{V}\,,$$

$$\mathcal{L}_{\mathbf{V}}\, \omega^1 = (d\omega^1) \cdot \mathbf{V} + d(\omega^1 \cdot \mathbf{V}) = (d\omega^1) \cdot \mathbf{V} + \mathcal{L}(\omega^1 \cdot \mathbf{V})\,.$$

Math10 - Symplexes

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lenght of symplex's edges

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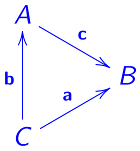
- Norm axioms



$$\|\mathbf{a}\| \geq 0\,, \quad \|\mathbf{a}\| = 0 \quad \Longrightarrow \quad \mathbf{a} = 0$$

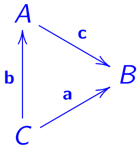
$$\|\mathbf{a}\| + \|\mathbf{b}\| \geq \|\mathbf{c}\| \quad \textbf{triangle inequality},$$

$$\|\alpha\,\mathbf{a}\| = |\alpha|\,\|\mathbf{a}\|$$

Math10 - Symplexes

lenght of symplex's edges

- Norm axioms



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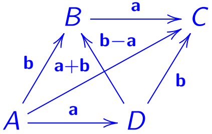
- Parallelogram rule



$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right]$$

Math11

Math11

The metric tensor

- Theorem (Fréchet – von Neumann – Jordan)

Math11

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$$\mathbf{g}(\mathbf{a}\,,\mathbf{b}) := \frac{1}{4}\left[\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right]$$

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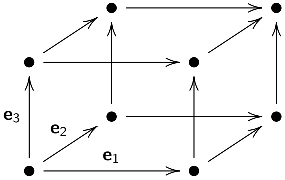
$\mathrm{VOL}\Bigg($



$\Bigg)^2 = \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$

Maurice René Fréchet (1878 - 1973)

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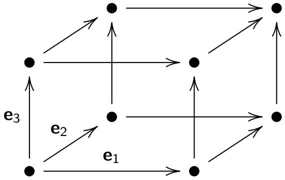

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John von Neumann (1903 - 1957)

Math11

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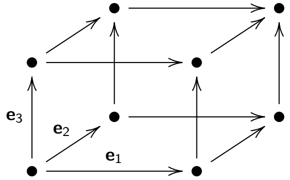

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Pascual Jordan (1902 - 1980)

Math11

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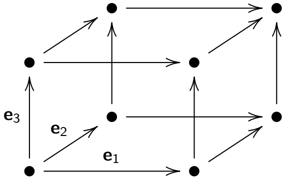

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Kosaku Yosida (1909 - 1990)

Math12

Bernhard Riemann (1826 - 1866) 

Metric tensor field: $\mathbf{g} : \mathbf{M} \mapsto \mathrm{Cov}(T\mathbf{M})$

- Riemann manifold: $(\mathbf{M}, \mathbf{g})$

Bernhard Riemann (1826 - 1866)

Metric tensor field: $\mathbf{g} : \mathbf{M} \mapsto \mathrm{Cov}(T\mathbf{M})$

- Riemann manifold: $(\mathbf{M}, \mathbf{g})$
- Fundamental theorem:
  A unique linear connection, the Levi-Civita connection, is metric and symmetric, i.e. such that
  1. $\nabla_{\mathbf{v}}\mathbf{g} = \mathbf{0}$
  2. $\nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} = [\mathbf{v}, \mathbf{u}]$

The torsion of the connection is defined by

$$\mathrm{Tors}(\mathbf{v}, \mathbf{u}) = \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v}, \mathbf{u}]$$

Math13 – EULER split formula

Math13 – Euler split formula

Leonhard Euler (1707 - 1783)

Parallel derivative of the space-time velocity field $\mathbf{V} = \mathbf{Z} + \mathbf{v}$ along the motion

$$\begin{aligned} \mathbf{a} := \nabla_{\mathbf{V}} \mathbf{V} &:= \partial_{\alpha=0}\, \varphi_\alpha \Downarrow (\mathbf{V} \circ \varphi_\alpha) \\ &= \nabla_{\mathbf{Z}} \mathbf{V} + \nabla_{\mathbf{v}} \mathbf{V} \\ &= \dot{\mathbf{v}} + \nabla_{\mathbf{v}} \mathbf{v} \end{aligned}$$

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The last expression is the celebrated EULER split formula, especially useful in problems of hydrodynamics, where it was originally conceived.
It eventually leads to the NAVIER-STOKES-ST.VENANT differential equation of motion in fluid-dynamics.

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In most treatments Euler split formula is adopted to define the so called material time derivative but the outcome is a space vector field, better to be called parallel time derivative.

Math14 – Euler's formula for the stretching

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- Stretching

$$\boldsymbol{\varepsilon}(\mathbf{v}) := \tfrac{1}{2}\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\mathrm{MAT}} = \tfrac{1}{2}\partial_{\alpha=0}\left(\boldsymbol{\varphi}_{\alpha}\!\downarrow\!\mathbf{g}_{\mathrm{MAT}}\right)$$

Math14 – Euler's formula for the stretching

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- $\mathbf{\Pi}_{\mathbf{e}} : T_{\mathbf{e}}\mathcal{S} \mapsto T_{\mathbf{e}}\mathbf{\Omega}$ projection
  $\mathbf{\Pi}^{*}_{\mathbf{e}} : T^{*}_{\mathbf{e}}\mathbf{\Omega} \mapsto T^{*}_{\mathbf{e}}\mathcal{S}$ immersion

- Euler's formula (generalized)

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \tfrac{1}{2}\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\mathrm{MAT}} = \mathbf{\Pi}^{*} \cdot \left(\tfrac{1}{2}\nabla_{\mathbf{v}}\,\mathbf{g}_{\mathrm{SPA}} + \mathrm{sym}\left(\mathbf{g}_{\mathrm{SPA}} \cdot \mathbf{L}(\mathbf{v})\right)\right) \cdot \mathbf{\Pi}$$

where $\mathbf{L} := \nabla + \mathrm{T\scriptsize{ORS}}\,.$

Math14 – Euler's formula for the stretching

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where $\mathbf{L} := \nabla + \mathrm{TORS}$.
Mixed form of the stretching tensor (standard LEVI-CIVITA connection):

$$\boxed{\tfrac{1}{2}\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\mathrm{SPA}} = \mathbf{g}_{\mathrm{SPA}}\cdot\mathrm{sym}\left(\nabla\mathbf{v}\right)}$$

since $\mathrm{TORS} = \mathbf{0}$ and $\nabla_{\mathbf{v}}\,\mathbf{g}_{\mathrm{SPA}} = \mathbf{0}$

Math15 – Differential forms vs vectors

cross product: $\mathbf{u} \times \mathbf{v} = \boldsymbol{\mu} \cdot \mathbf{u} \cdot \mathbf{v}\,,$ $\dim(\mathcal{E}_t) = 2$

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cross product: $(\mathbf{g} \cdot \mathbf{u}) \wedge (\mathbf{g} \cdot \mathbf{v}) \; = \boldsymbol{\mu} \cdot (\mathbf{u} \times \mathbf{v})\,,$ $\quad \dim(\mathcal{E}_t) = 3$

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gradient: $d\,f = \mathbf{g} \cdot \nabla f\,,$ $\dim(\mathcal{E}_t) = \textit{any}$

rotor: $d\,(\mathbf{g} \cdot \mathbf{v}) = \mathrm{rot}(\mathbf{v}) \cdot \boldsymbol{\mu}\,,$ $\dim(\mathcal{E}_t) = 2$

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divergence: $d\,(\boldsymbol{\mu} \cdot \mathbf{v}) = \mathrm{div}(\mathbf{v}) \cdot \boldsymbol{\mu}\,.$ $\dim(\mathcal{E}_t) = \textit{any}$

Math16 – Change of observer

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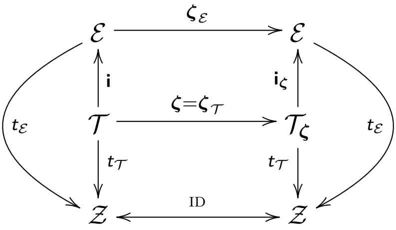
- Change of observer $\quad \zeta_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{E}\,,$ time-bundle automorphism
- Relative motion $\quad \zeta : \mathcal{T} \mapsto \mathcal{T}_\zeta\,,$ time-bundle diffeomorphism



Math16 – Change of observer

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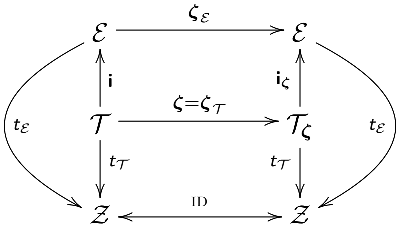
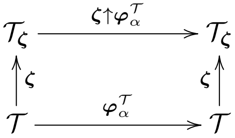


- Pushed motion



$$\Longleftrightarrow \quad (\zeta{\uparrow}\varphi_{\alpha}^{\mathcal{T}}) \circ \zeta = \zeta \circ \varphi_{\alpha}^{\mathcal{T}}\,.$$

Math17 – Time-invariance and Frame-covariance

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- Time-invariance $\quad \mathbf{s} = \varphi_\alpha \!\uparrow\! \mathbf{s}\,, \quad \varphi_\alpha : \mathcal{E} \mapsto \mathcal{E} \quad$ motion

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- Naturality of LIE derivative under diffeomorphisms

$$\boxed{\zeta\!\uparrow\!(\mathcal{L}_{\mathbf{v}}\,\mathbf{s}) = \mathcal{L}_{\zeta\uparrow\mathbf{v}}\,(\zeta\!\uparrow\!\mathbf{s})}$$

Frame-covariance of a material tensor implies frame-covariance of its time-rate.

Math18 – Frame-covariance of space-time velocity

Transformation rule

$$\mathbf{V}_{\mathcal{T}_\zeta} := \partial_{\alpha=0}\,(\zeta{\uparrow}\varphi^{\mathcal{T}}_{\alpha}) = \zeta{\uparrow}\mathbf{V}_{\mathcal{T}}\,.$$

The 4-velocity is natural with respect to frame transformations

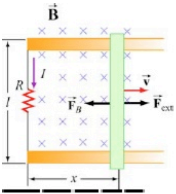
---

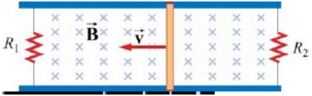
$$\zeta_{\mathcal{E}} : \begin{cases} \mathbf{x} \mapsto \mathbf{Q}(t)\cdot\mathbf{x} + \mathbf{c}(t) \\ t \mapsto t \end{cases}$$

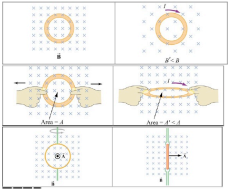
$$[T\zeta_{\mathcal{E}}]\cdot[\mathbf{V}] = \begin{bmatrix} \mathbf{Q} & (\dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}) \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}} \\ 1 \end{bmatrix}$$

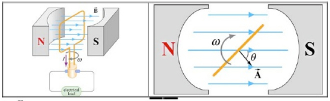
F1a – Faraday Law - examples

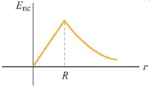
Faraday law of induction: examples



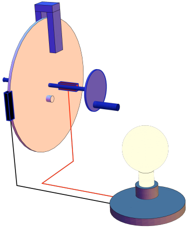


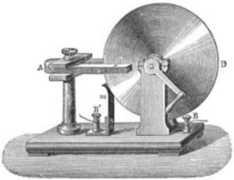


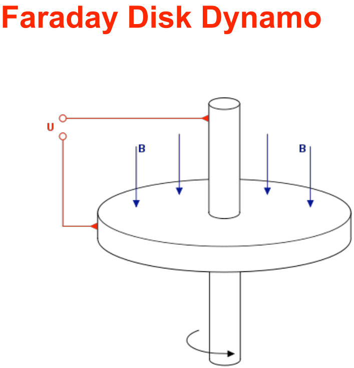




F1b – Faraday disk (1831) and flux rule







F2 – Difficulties with flux rule

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According to Feynman (1964): *as the disc rotates, the "circuit", in the sense of the place in space where the currents are, is always the same. But the part of the* **"circuit"** *in the disc is in material which is moving. Although the flux through the* **"circuit"** *is constant, there is still an EMF, as can be observed by the deflection of the galvanometer. Clearly, here is a case where the* $\mathbf{v} \times \mathbf{B}$ *force in the moving disc gives rise to an EMF which cannot be equated to a change of flux.*

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We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the rule as the combined effect of two quite separate phenomena.

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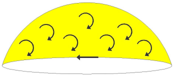
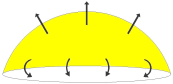
Quoting LEHNER (2010): *(The flux rule) only applies in situations when the loop during its motion or deformations maintains its material identity and is penetrated by a uniquely identifiable flux. This is neither the case for the Unipolar machine (*FARADAY *disc) nor* HERING*'s experiment. Looking back, we could have supposed this because of the spring contacts, which may have seemed minor. Brushes and sliding contacts require extra caution. In case of doubt, it is best to go back to the fundamental laws (*LORENTZ *force).*

E1 – Electromagnetic fields



inner orientation.



outer orientation.

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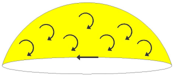

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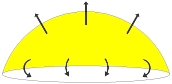

outer orientation.

$\omega^1_{\mathbf{E}} = \mathbf{g} \cdot \mathbf{E}$ electric field (inner one-form)

$\omega^2_{\mathbf{B}} = \boldsymbol{\mu} \cdot \mathbf{B}$ magnetic vortex (inner two-form)

$\omega^1_{\mathbf{A}} = \mathbf{g} \cdot \mathbf{A}$ magnetic momentum (inner one-form)

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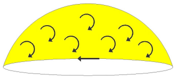
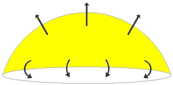


inner orientation.



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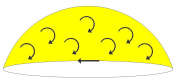
$\omega^1_{\mathbf{H}} = \mathbf{g} \cdot \mathbf{H}$ magnetic field (outer one-form)

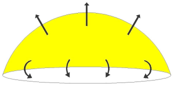
$\omega^2_{\mathbf{D}} = \boldsymbol{\mu} \cdot \mathbf{D}$ electric displacement (outer two-form)

$\omega^2_{\mathbf{J}} = \boldsymbol{\mu} \cdot \mathbf{J}$ electric current (outer two-form)

E1 – Electromagnetic fields



inner orientation.



outer orientation.

$\omega^1_{\mathbf{E}} = \mathbf{g} \cdot \mathbf{E}$ electric field (inner one-form)

$\omega^2_{\mathbf{B}} = \boldsymbol{\mu} \cdot \mathbf{B}$ magnetic vortex (inner two-form)

$\omega^1_{\mathbf{A}} = \mathbf{g} \cdot \mathbf{A}$ magnetic momentum (inner one-form)

$\omega^1_{\mathbf{H}} = \mathbf{g} \cdot \mathbf{H}$ magnetic field (outer one-form)

$\omega^2_{\mathbf{D}} = \boldsymbol{\mu} \cdot \mathbf{D}$ electric displacement (outer two-form)

$\omega^2_{\mathbf{J}} = \boldsymbol{\mu} \cdot \mathbf{J}$ electric current (outer two-form)

$$\omega^2_{\mathbf{B}} = d\omega^1_{\mathbf{A}} \quad \Longleftrightarrow \quad \mathbf{B} = \mathrm{rot}(\mathbf{A})$$

$$d\omega^2_{\mathbf{B}} = dd\omega^1_{\mathbf{A}} = \mathbf{0} \quad \Longleftrightarrow \quad \mathrm{div}(\mathbf{B}) = \mathrm{divrot}(\mathbf{A}) = 0$$

E2 – Induction law - standard

Faraday-Maxwell rule

$$-\oint_{\partial\Sigma_{\mathrm{INN}}} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\Sigma_{\mathrm{INN}})} \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\Sigma_{\mathrm{INN}}} \mathcal{L}_{\mathbf{V}}(\boldsymbol{\omega}^2_{\mathbf{B}})$$

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By Stokes formula

$$- \int_{\Sigma_{\text{INN}}} d\omega^1_{\mathbf{E}} = \int_{\Sigma_{\text{INN}}} \mathcal{L}_{\mathbf{V}}(\omega^2_{\mathbf{B}})$$

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By Stokes formula

$$- \int_{\Sigma_{\text{INN}}} d\omega^1_{\mathbf{E}} = \int_{\Sigma_{\text{INN}}} \mathcal{L}_{\mathbf{V}}(\omega^2_{\mathbf{B}})$$

Locally

$$\begin{aligned} -d\omega^1_{\mathbf{E}} &= \mathcal{L}_{\mathbf{V}}(\omega^2_{\mathbf{B}}) \\ &= \mathcal{L}_{\mathbf{Z}}(\omega^2_{\mathbf{B}}) + \mathcal{L}_{\mathbf{v}}(\omega^2_{\mathbf{B}}) \\ &= \mathcal{L}_{\mathbf{Z}}(\omega^2_{\mathbf{B}}) + (d\omega^2_{\mathbf{B}}) \cdot \mathbf{v} + d(\omega^2_{\mathbf{B}} \cdot \mathbf{v}) \end{aligned}$$

E3 – Induction law - standard

Hendrick Antoon LORENTZ (1853 - 1928)

$$
\begin{aligned}
d\omega^1_{\mathbf{E}} &= d(\mathbf{g}\cdot\mathbf{E}) = \boldsymbol{\mu}\cdot\mathrm{rot}(\mathbf{E})\,,\\
(d\omega^2_{\mathbf{B}})\cdot\mathbf{v} &= d(\boldsymbol{\mu}\cdot\mathbf{B})\cdot\mathbf{v} = \mathrm{div}(\mathbf{B})\cdot(\boldsymbol{\mu}\cdot\mathbf{v})\,,\\
d(\omega^2_{\mathbf{B}}\cdot\mathbf{v}) &= d(\boldsymbol{\mu}\cdot\mathbf{B}\cdot\mathbf{v}) = d(\mathbf{g}\cdot(\mathbf{B}\times\mathbf{v})) = \boldsymbol{\mu}\cdot(\mathrm{rot}(\mathbf{B}\times\mathbf{v}))\,.
\end{aligned}
$$

E3 – Induction law - standard

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$$\boxed{\begin{aligned} d\omega^1_{\mathbf{E}} &= d(\mathbf{g}\cdot\mathbf{E}) = \boldsymbol{\mu}\cdot\mathrm{rot}(\mathbf{E})\,,\\ (d\omega^2_{\mathbf{B}})\cdot\mathbf{v} &= d(\boldsymbol{\mu}\cdot\mathbf{B})\cdot\mathbf{v} = \mathrm{div}(\mathbf{B})\cdot(\boldsymbol{\mu}\cdot\mathbf{v})\,,\\ d(\omega^2_{\mathbf{B}}\cdot\mathbf{v}) &= d(\boldsymbol{\mu}\cdot\mathbf{B}\cdot\mathbf{v}) = d(\mathbf{g}\cdot(\mathbf{B}\times\mathbf{v})) = \boldsymbol{\mu}\cdot(\mathrm{rot}(\mathbf{B}\times\mathbf{v}))\,.\end{aligned}}$$

The differential induction law, being $\mathrm{div}(\mathbf{B}) = 0$ and $\mathcal{L}_{\mathbf{Z}}(\boldsymbol{\mu}) = \mathbf{0}$, and setting $\mathbf{B} = \mathrm{rot}(\mathbf{A})$, writes

$$\boxed{\mathrm{rot}(\mathbf{E}) = -\mathcal{L}_{\mathbf{Z}}(\mathbf{B}) + \mathrm{rot}(\mathbf{v}\times\mathbf{B}) = \mathrm{rot}(-\mathcal{L}_{\mathbf{Z}}(\mathbf{A}) + \mathbf{v}\times\mathbf{B})\,.}$$

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d\omega^1_{\mathbf{E}} &= d(\mathbf{g}\cdot\mathbf{E}) = \boldsymbol{\mu}\cdot\mathrm{rot}(\mathbf{E})\,,\\
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\end{aligned}
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$$

$-\mathcal{L}_{\mathbf{Z}}(\mathbf{A})$, transformer E.M.F. force

$\mathbf{v}\times\mathbf{B}$, motional (LORENTZ) E.M.F. force

$+\,???$, gradient of a scalar potential.

E4 – Balance principle

A new induction law is provided by a balance principle involving magnetic momentum, electric field and electrostatic potential

$$\boxed{\int_{\mathbf{\Gamma}_{\mathrm{INN}}} \boldsymbol{\omega}^1_{\mathbf{E}} + \oint_{\partial\mathbf{\Gamma}_{\mathrm{INN}}} P_{\mathbf{E}} = -\,\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\mathbf{\Gamma}_{\mathrm{INN}})} \boldsymbol{\omega}^1_{\mathbf{A}}\,.} \tag{1}$$

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Applying Lie-Reynolds transport formula, and localizing we get the differential law

$$\boxed{-\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{V}}(\boldsymbol{\omega}^1_{\mathbf{A}}) + dP_{\mathbf{E}} \,.} \tag{2}$$

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Assuming that the path $\mathbf{\Gamma}_{\text{INN}} = \partial\Sigma_{\text{INN}}$ is the boundary of an inner oriented surface Σ_{INN} undergoing a regular motion, the integral law yields the *vortex* rule (FARADAY-MAXWELL *flux rule*):

$$\boxed{-\oint_{\partial\Sigma_{\text{INN}}} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\Sigma_{\text{INN}})} \boldsymbol{\omega}^2_{\mathbf{B}} \,,} \tag{3}$$

E5 – Induction law explicated

Decomposition of space-time velocity and homotopy formula give

$$\begin{aligned} -\omega^1_{\mathbf{E}} &= \mathcal{L}_{\mathbf{V}}(\omega^1_{\mathbf{A}}) + dP_{\mathbf{E}} \\ &= \mathcal{L}_{\mathbf{Z}}(\omega^1_{\mathbf{A}}) + \mathcal{L}_{\mathbf{v}}(\omega^1_{\mathbf{A}}) + dP_{\mathbf{E}} \\ &= \mathcal{L}_{\mathbf{Z}}(\omega^1_{\mathbf{A}}) + (d\omega^1_{\mathbf{A}}) \cdot \mathbf{v} + d(\omega^1_{\mathbf{A}} \cdot \mathbf{v}) + dP_{\mathbf{E}} \end{aligned}$$

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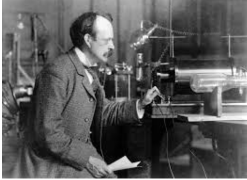
In terms of vector fields, since $\omega^1_{\mathbf{E}} = \mathbf{g} \cdot \mathbf{E}$, $\omega^1_{\mathbf{A}} = \mathbf{g} \cdot \mathbf{A}$, we have

$$\begin{aligned} \mathcal{L}_{\mathbf{Z}}(\mathbf{g} \cdot \mathbf{A}) &= \mathbf{g} \cdot \mathcal{L}_{\mathbf{Z}}(\mathbf{A}) , \quad (\mathcal{L}_{\mathbf{Z}}(\mathbf{g}) = \mathbf{0}) \\ d(\mathbf{g} \cdot \mathbf{A}) \cdot \mathbf{v} &= \boldsymbol{\mu} \cdot \mathrm{rot}(\mathbf{A}) \cdot \mathbf{v} = \mathbf{g} \cdot (\mathrm{rot}(\mathbf{A}) \times \mathbf{v}) \\ d(\mathbf{g} \cdot \mathbf{A} \cdot \mathbf{v}) &= \mathbf{g} \cdot \nabla(\mathbf{g}(\mathbf{A} , \mathbf{v})) \end{aligned}$$

E6 – J.J. Thomson force

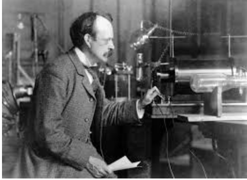


Joseph John THOMSON (1856 - 1940)

Recalling that $dP_{\mathbf{E}} = \mathbf{g} \cdot \nabla P_{\mathbf{E}}$ we get the expression

$$\boxed{\mathbf{E} = -\mathcal{L}_{\mathbf{Z}}(\mathbf{A}) + \mathbf{v} \times \mathrm{rot}(\mathbf{A}) - \nabla(\mathbf{g}(\mathbf{A}\,,\mathbf{v})) - \nabla P_{\mathbf{E}}}$$

E6 – J.J. Thomson force



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proposed by J.J. THOMSON in 1893
as explication of MAXWELL potential (1855)

$$\boxed{\Psi = \mathbf{g}(\mathbf{A}\,,\mathbf{v}) + P_{\mathbf{E}}}$$

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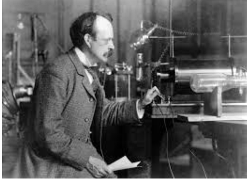

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proposed by J.J. Thomson in 1893
as explication of Maxwell potential (1855)

$$\Psi = \mathbf{g}(\mathbf{A}, \mathbf{v}) + P_{\mathbf{E}}$$

$-\mathcal{L}_{\mathbf{Z}}(\mathbf{A})$, transformer E.M.F. force
$\mathbf{v} \times \mathbf{B}$, motional E.M.F. (Lorentz force)
$-\nabla(\mathbf{g}(\mathbf{A}, \mathbf{v}))$, motional E.M.F. (J.J. Thomson force)

E7 – J.J. Thomson original

NOTES

ON

RECENT RESEARCHES IN

ELECTRICITY AND MAGNETISM

INTENDED AS A SEQUEL TO

PROFESSOR CLERK-MAXWELL'S TREATISE
ON ELECTRICITY AND MAGNETISM

BY

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Oxford
AT THE CLARENDON PRESS
1893

In the course of Maxwell's investigation of the values of X, Y, Z due to induction, the terms

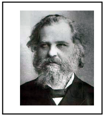
$$-\frac{d}{dx}(Fu+Gv+Hw), \quad -\frac{d}{dy}(Fu+Gv+Hw),$$
$$-\frac{d}{dz}(Fu+Gv+Hw)$$

respectively in the final expressions for X, Y, Z are included under the Ψ terms. We shall find it clearer to keep these terms separate and write the expressions for X, Y, Z as

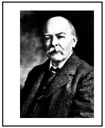
$$\left.\begin{aligned} X &= cv - bw - \frac{dF}{dt} - \frac{d}{dx}(Fu+Gv+Hw) - \frac{d\phi}{dx}, \\ Y &= aw - cu - \frac{dG}{dt} - \frac{d}{dy}(Fu+Gv+Hw) - \frac{d\phi}{dy}, \\ Z &= bu - av - \frac{dH}{dt} - \frac{d}{dz}(Fu+Gv+Hw) - \frac{d\phi}{dz}. \end{aligned}\right\} \qquad (1)$$

$$\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t}\cdot\mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.$$

E8 – Flux of electromagnetic power



Nikolay Alekseevich
UMOV (1846–1915)



John Henry
POYNTING (1852–1914)

Electric and magnetic power expended per unit volume:

$$\begin{aligned}\omega^3_{\mathrm{POWER}} &:= \omega^1_{\mathbf{E}} \wedge (\omega^2_{\mathbf{J}} + \mathcal{L}_{\mathbf{V}}(\omega^2_{\mathbf{D}})) + \omega^1_{\mathbf{H}} \wedge \mathcal{L}_{\mathbf{V}}(\omega^2_{\mathbf{B}}) \\ &= \omega^1_{\mathbf{E}} \wedge d\omega^1_{\mathbf{H}} - \omega^1_{\mathbf{H}} \wedge d\omega^1_{\mathbf{E}} \\ &= -\, d(\omega^1_{\mathbf{E}} \wedge \omega^1_{\mathbf{H}}) \quad \text{(graded derivation rule)}\end{aligned}$$

UMOV (1874)-POYNTING (1884) spatial outer two-form

$$\omega^2_{\mathrm{UMOV}} := \omega^1_{\mathbf{E}} \wedge \omega^1_{\mathbf{H}} \in \Lambda^2(\mathcal{E}) \,,$$

Balance of electromagnetic power

$$\int_{\mathbf{C}_{\mathrm{OUT}}} \omega^3_{\mathrm{POWER}} + \int_{\partial \mathbf{C}_{\mathrm{OUT}}} \omega^2_{\mathrm{UMOV}} = 0 \,.$$

E9 – Space-time forms

Harry Bateman (1882 - 1946)

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A framing $\mathbf{R} := dt \otimes \mathbf{Z}$ induces a representation formula for space-time forms $\mathbf{\Omega} \in \Lambda^k(\mathcal{E})$ in terms of time-vertical restrictions and of the time differential (extended to mobile bodies)

$$\mathbf{\Omega} = \mathbf{P}{\downarrow}\mathbf{\Omega} + dt \wedge (\mathbf{P}{\downarrow}(\mathbf{\Omega} \cdot \mathbf{V}) - (\mathbf{P}{\downarrow}\mathbf{\Omega}) \cdot \mathbf{V})\,.$$

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$$\boldsymbol{\Omega} = \mathbf{P}{\downarrow}\boldsymbol{\Omega} + dt \wedge (\mathbf{P}{\downarrow}(\boldsymbol{\Omega} \cdot \mathbf{V}) - (\mathbf{P}{\downarrow}\boldsymbol{\Omega}) \cdot \mathbf{V}) \,.$$

- The space-time Faraday two-form $\boldsymbol{\Omega}^2_{\mathbf{F}}$ is related to
  1. magnetic time-vertical space-time two form $\boldsymbol{\Omega}^2_{\mathbf{B}} := \mathbf{P}{\downarrow}\boldsymbol{\Omega}^2_{\mathbf{F}}$
  2. electric time-vertical space-time one form $\boldsymbol{\Omega}^1_{\mathbf{E}} := \mathbf{P}{\downarrow}(\boldsymbol{\Omega}^2_{\mathbf{F}} \cdot \mathbf{V})$

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- The space-time Faraday two-form $\mathbf{\Omega}^2_{\mathbf{F}}$ is related to
  1. magnetic time-vertical space-time two form $\mathbf{\Omega}^2_{\mathbf{B}} := \mathbf{P}{\downarrow}\mathbf{\Omega}^2_{\mathbf{F}}$
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by

$$\boxed{\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{\Omega}^2_{\mathbf{B}} - dt \wedge (\mathbf{\Omega}^1_{\mathbf{E}} + \mathbf{\Omega}^2_{\mathbf{B}} \cdot \mathbf{V})}$$

E10 – Space-time forms

Closeness of Faraday 2-form is equivalent to Gauss-Maxwell laws:

$$d\boldsymbol{\Omega}^2_{\mathbf{F}} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} d\boldsymbol{\omega}^2_{\mathbf{B}} = \mathbf{0}\,, \\ \mathcal{L}_{\mathbf{V}}\, \boldsymbol{\omega}^2_{\mathbf{B}} + d\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{0}\,, \end{cases}$$

and to the magnetic vortex rule (Faraday flux rule)

$$\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\Sigma_{\text{INN}})} \boldsymbol{\omega}^2_{\mathbf{B}} = - \oint_{\partial\Sigma_{\text{INN}}} \boldsymbol{\omega}^1_{\mathbf{E}}\,,$$

E11 – Space-time forms

- The space-time FARADAY 1-form $\mathbf{\Omega}^1_{\mathbf{F}}$ and the pair of
  1. magnetic space-time 1-form $\mathbf{\Omega}^1_{\mathbf{A}}$
  2. electrostatic space-time 0-form $\mathbf{\Omega}^0_{\mathbf{E}}$

are related by

$$\begin{aligned} \mathbf{\Omega}^1_{\mathbf{A}} &:= \mathbf{P}{\downarrow}\mathbf{\Omega}^1_{\mathbf{F}} && \text{magnetic time-vertical 1-form} \\ -\mathbf{\Omega}^0_{\mathbf{E}} &:= \mathbf{P}{\downarrow}(\mathbf{\Omega}^1_{\mathbf{F}} \cdot \mathbf{V}) && \text{electrostatic time-vertical 0-form} \\ \mathbf{\Omega}^1_{\mathbf{F}} &= \mathbf{\Omega}^1_{\mathbf{A}} - dt \wedge (\mathbf{\Omega}^0_{\mathbf{E}} + \mathbf{\Omega}^1_{\mathbf{A}} \cdot \mathbf{V}) && \text{FARADAY space-time 1-form} \end{aligned}$$

E12 – Space-time forms

By Poincaré lemma, closeness of Faraday 2-form ensures exactness:

$$\boxed{d\mathbf{\Omega}^2_{\mathbf{F}} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{\Omega}^2_{\mathbf{F}} = d\mathbf{\Omega}^1_{\mathbf{F}}}$$

expressed by

$$\boxed{\mathbf{\Omega}^2_{\mathbf{F}} = d\mathbf{\Omega}^1_{\mathbf{F}}} \quad \Longleftrightarrow \quad \begin{cases} \omega^2_{\mathbf{B}} &= d\omega^1_{\mathbf{A}} \,, \\ -\omega^1_{\mathbf{E}} &= \mathcal{L}_{\mathbf{V}}(\omega^1_{\mathbf{A}}) + dP_{\mathbf{E}} \,, \end{cases}$$

and by the magnetic momentum balance law

$$-\partial_{\alpha=0} \int_{\varphi_\alpha(\mathbf{\Gamma}_{\text{INN}})} \omega^1_{\mathbf{A}} = \int_{\mathbf{\Gamma}_{\text{INN}}} \omega^1_{\mathbf{E}} + \oint_{\partial\mathbf{\Gamma}_{\text{INN}}} P_{\mathbf{E}} \,,$$

E13 – Space-time matrix formulations

$$\begin{bmatrix} 0 & B_3 & -B_2 & -E_1 \\ -B_3 & 0 & B_1 & -E_2 \\ B_2 & -B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix}$$

If this matrix expression is retained also for a non-vanishing spatial velocity, the following expression is got

$$\begin{bmatrix} 0 & B_3 & -B_2 & -E_1 \\ -B_3 & 0 & B_1 & -E_2 \\ B_2 & -B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{bmatrix} = \begin{bmatrix} v_2\, B_3 - v_3\, B_2 - E_1 \\ -v_1\, B_3 + v_3\, B_1 - E_2 \\ v_1\, B_2 - v_2\, B_1 - E_3 \\ v_1\, E_1 + v_1\, E_2 + v_1\, E_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \times \mathbf{B} - \mathbf{E} \\ \\ \mathbf{g}(\mathbf{E}, \mathbf{v}) \end{bmatrix}$$

E14 – Relativistic Frame transformation - amended

Synoptic table I ($\mathbf{v} = \mathbf{0}$)	
new	**old**
$(\mathbf{E}^{\parallel}, \mathbf{E}^{\perp}) \rightarrow (\gamma\, \mathbf{E}^{\parallel}, \mathbf{E}^{\perp})$	*versus* $(\mathbf{E}^{\parallel}, \gamma\, (\mathbf{E}^{\perp} + \mathbf{w} \times \mathbf{B}))$
$(\mathbf{B}^{\parallel}, \mathbf{B}^{\perp}) \rightarrow (\mathbf{B}^{\parallel}, \gamma\, (\mathbf{B}^{\perp} - (\mathbf{w}/c^2) \times \mathbf{E}))$	*idem*
$(\mathbf{H}^{\parallel}, \mathbf{H}^{\perp}) \rightarrow (\gamma\, \mathbf{H}^{\parallel}, \mathbf{H}^{\perp})$	*versus* $(\mathbf{H}^{\parallel}, \gamma\, (\mathbf{H}^{\perp} - \mathbf{w} \times \mathbf{D}))$
$(\mathbf{D}^{\parallel}, \mathbf{D}^{\perp}) \rightarrow (\mathbf{D}^{\parallel}, \gamma\, (\mathbf{D}^{\perp} + (\mathbf{w}/c^2) \times \mathbf{H}))$	*idem*
$(\mathbf{J}^{\parallel}, \mathbf{J}^{\perp}) \rightarrow (\mathbf{J}^{\parallel}, \gamma\, \mathbf{J}^{\perp})$	*versus* $(\gamma\, (\mathbf{J}^{\parallel} - \rho\, \mathbf{w}), \mathbf{J}^{\perp})$
$\rho \rightarrow \gamma\, (\rho - \mathbf{g}(\mathbf{w}/c^2, \mathbf{J}))$	*idem*
$P_{\mathbf{E}} \rightarrow P_{\mathbf{E}}$	*versus* $\gamma\, (P_{\mathbf{E}} - \mathbf{g}(\mathbf{w}, P_{\mathbf{H}}))$
$(P_{\mathbf{H}}^{\parallel}, P_{\mathbf{H}}^{\perp}) \rightarrow (\gamma\, (P_{\mathbf{H}}^{\parallel} + (\mathbf{w}/c^2) P_{\mathbf{E}}), P_{\mathbf{H}}^{\perp})$	*idem*

E15 – Relativistic Frame transformation - amended

Synoptic table II ($\mathbf{v} \neq 0$)		
$(\mathbf{E}^{\parallel}, \mathbf{E}^{\perp})$	$\rightarrow$	$(\gamma\,(\mathbf{E}^{\parallel} - \mathbf{g}(\mathbf{v}/c\,, \mathbf{E})\,\mathbf{w}/c)\,, \mathbf{E}^{\perp})$
$(\mathbf{B}^{\parallel}, \mathbf{B}^{\perp})$	$\rightarrow$	$(\mathbf{B}^{\parallel}, \gamma\,(\mathbf{B}^{\perp} - (\mathbf{w}/c^2) \times (\mathbf{E} + \mathbf{B} \times \mathbf{v}))$
$(\mathbf{H}^{\parallel}, \mathbf{H}^{\perp})$	$\rightarrow$	$(\gamma\,\mathbf{H}^{\parallel}, \mathbf{H}^{\perp})$
$(\mathbf{D}^{\parallel}, \mathbf{D}^{\perp})$	$\rightarrow$	$(\mathbf{D}^{\parallel}, \gamma\,(\mathbf{D}^{\perp} + (\mathbf{w}/c^2) \times (\mathbf{H} - \mathbf{D} \times \mathbf{v}))$
$(\mathbf{J}^{\parallel}, \mathbf{J}^{\perp})$	$\rightarrow$	$(\mathbf{J}^{\parallel}, \gamma\,\mathbf{J}^{\perp})$
ρ	$\rightarrow$	$\gamma\,(\rho - \mathbf{g}(\mathbf{w}/c^2\,, \mathbf{J} + \rho\,\mathbf{v}))$
$P_{\mathbf{E}}$	$\rightarrow$	$P_{\mathbf{E}}$
$(P_{\mathbf{H}}^{\parallel}, P_{\mathbf{H}}^{\perp})$	$\rightarrow$	$(\gamma\,(P_{\mathbf{H}}^{\parallel} - (\mathbf{w}/c^2)(P_{\mathbf{E}} + \mathbf{g}(\mathbf{v}, P_{\mathbf{H}})))\,, P_{\mathbf{H}}^{\perp})$