Università di Napoli Federico II

Dottorato di Ricerca in Ingegneria delle Costruzioni

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DIST – Dipartimento di Ingegneria STrutturale

Seminari 27-29 Giugno - 2 Luglio 2012

The Geometric Approach to Non-Linear Continuum Mechanics

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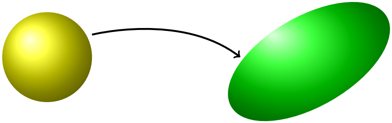
Non-Linear Continuum Mechanics (NLCM) calls instead for Differential Geometry (DG) and Calculus on Manifolds (CoM) as natural tools to develop theoretical and computational models.



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Differential Geometry provides the tools to fly higher and see what before was shadowed or completely hidden.

A basic question in NLCM

- How to compare **material tensors** at corresponding points in displaced configurations of a body?

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- Devil's temptation:

  In 3D bodies it might seem as natural to compare by translation the involved material vectors.
  This is tacitly done in literature, when evaluating the material time-derivative of the stress tensor $\mathbf{T}$ *:*

  $$\dot{\mathbf{T}}(\mathrm{p}, t) := \partial_{\tau=t}\,\mathbf{T}(\mathrm{p}, \tau)$$

  or the material time-derivative of the director $\mathbf{n}$ *of a nematic liquid crystal:*

  $$\dot{\mathbf{n}}(\mathrm{p}, t) := \partial_{\tau=t}\,\mathbf{n}(\mathrm{p}, \tau)$$

  These definitions are connection dependent and geometrically untenable when considering 1D and 2D models (wires and membranes).

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  These definitions are connection dependent and geometrically untenable when considering 1D and 2D models (wires and membranes).
- Hint: *Tangent vectors to a body placement are transformed into tangent vectors to another body placement by the tangent displacement map. This is the essence of the GEOMETRIC PARADIGM.*

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A geometrically consistent theoretical framework should be equally applicable to body models of any dimension.

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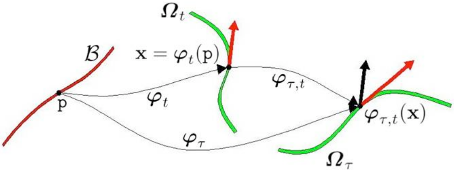

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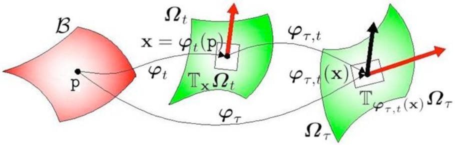

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Math1

Tangent vector to a manifold:

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velocity of a curve $\mathbf{c} \in \mathrm{C}^1([a, b] ; \mathbb{M})$, $\lambda \in [a, b]$, $\mathbf{x} = \mathbf{c}(\lambda)$ base point

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$$\mathbf{v}^* \in L\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,; \mathcal{R}) \in \mathbb{T}^*_{\mathbf{x}}\mathbb{M}$$

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- A map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ sends
  a curve $\mathbf{c} \in \mathrm{C}^1([a, b]\,;\mathbb{M})$ into
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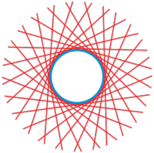
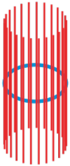
  a curve $\boldsymbol{\zeta} \circ \mathbf{c} \in \mathrm{C}^1([a,b]\,;\mathbb{N})\,.$
- The tangent map $T_{\mathbf{x}}\boldsymbol{\zeta} \in \mathrm{C}^0(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})$
  sends a tangent vector at $\mathbf{x} \in \mathbb{M}$
  $\mathbf{v} \in \mathbb{T}_{\mathbf{x}}(\mathbb{M}) := \partial_{\mu=\lambda}\, \mathbf{c}(\mu)$
  into a tangent vector at $\boldsymbol{\zeta}(\mathbf{x}) \in \mathbb{N}$
  $T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{v} \in \mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}(\mathbb{N}) := \partial_{\mu=\lambda}\,(\boldsymbol{\zeta} \circ \mathbf{c})(\mu)$

Math2

Math2

Tangent bundle



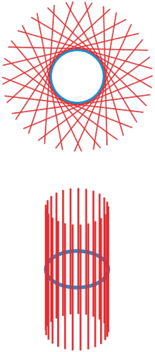


Math2

Tangent bundle

- disjoint union of tangent spaces:

$$\mathbb{T}\mathbb{M} := \cup_{\mathbf{x}\in\mathbb{M}} \mathbb{T}_{\mathbf{x}}\mathbb{M}$$



Math2

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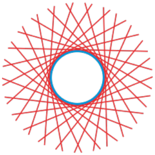
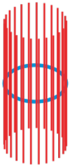


- Projection: $\boldsymbol{\tau}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{TM}\,;\mathbb{M})$

$$\mathbf{v} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}\,, \quad \boldsymbol{\tau}_{\mathbb{M}}(\mathbf{v}) := \mathbf{x} \quad \text{base point}$$



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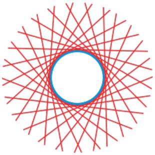

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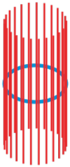
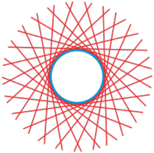


- Surjective submersion:

$T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{T}_{\mathbf{v}}\mathbb{TM}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})$ is surjective

Tangent bundle



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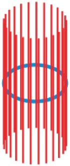

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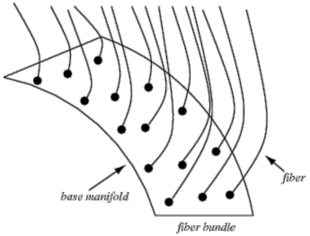
- Tangent functor

$$\zeta \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N}) \quad \mapsto \quad T\zeta \in \mathrm{C}^0(\mathbb{TM}\,;\mathbb{TN})$$

Math3

Math3

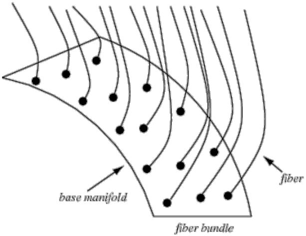
Fiber bundles



Math3

Fiber bundles

- $\mathbb{E}, \mathbb{M}$ manifolds



Math3

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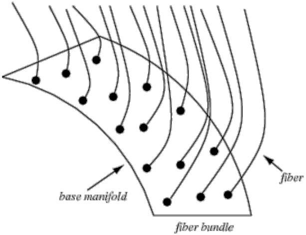

- $\mathrm{E}, \mathbb{M}$ manifolds
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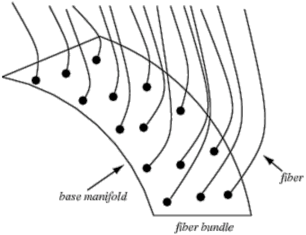
Math3



Fiber bundles

- $\mathbb{E}, \mathbb{M}$ manifolds
- Fiber bundle projection:
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- Total space: $\mathbb{E}$
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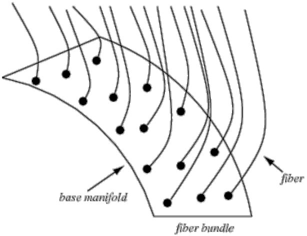
- Fiber manifold: $(\boldsymbol{\pi}_{\mathbb{M},\mathbb{E}}(\mathbf{x}))^{-1}$ based at $\mathbf{x} \in \mathbb{M}$

Math3



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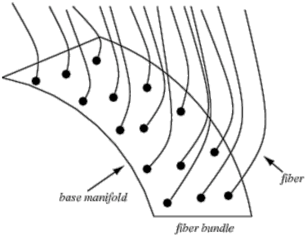
Math3



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Math3

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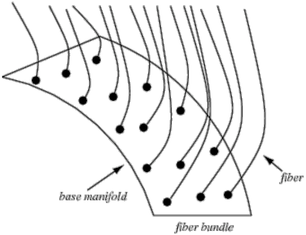

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  $\delta\mathbf{e} \in \mathbb{V}\mathrm{E} \subset \mathbb{T}\mathrm{E} \quad \Longrightarrow \quad T_{\mathbf{e}}\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \cdot \delta\mathbf{e} = 0$

Math4

Math4

Trivial and
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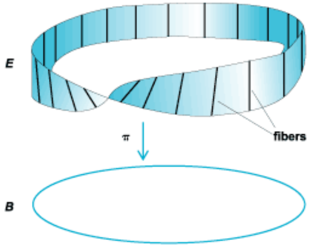
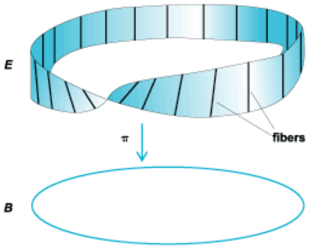
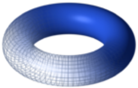


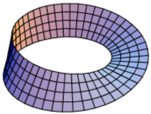
Math4

Trivial and non-trivial fiber bundles

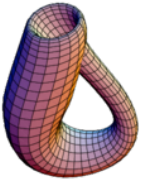




Torus



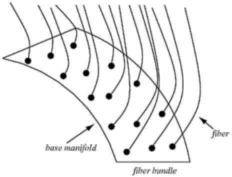
Listing-Möbius strip



Klein Bottle

Math5

Sections of fiber bundles



Math5

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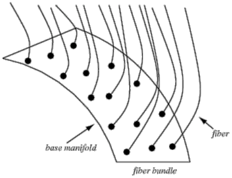

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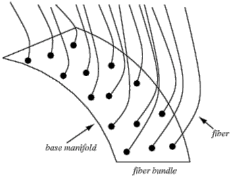

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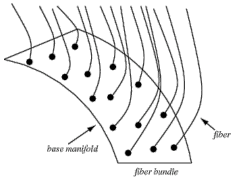
Math5



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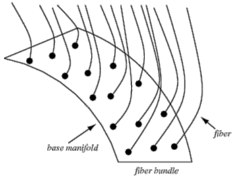

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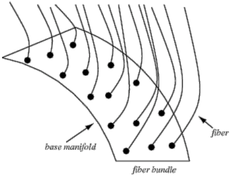
Math5



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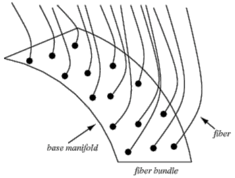
- Sections $\mathbf{s}_{\mathrm{E},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{E})\,, \quad \boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{s}_{\mathrm{E},\mathbb{M}} = \mathrm{ID}_{\mathbb{M}}$
- Tangent v.f. $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})\,, \quad \boldsymbol{\tau}_{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = \mathrm{ID}_{\mathrm{E}}$
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M},\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Sections of tangent and bi-tangent bundles

Math5



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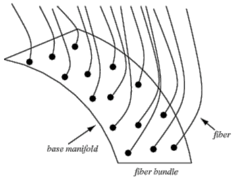
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathrm{M},\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Sections of tangent and bi-tangent bundles

- Tangent vector fields:

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$

Math5



Sections of fiber bundles

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Sections of tangent and bi-tangent bundles

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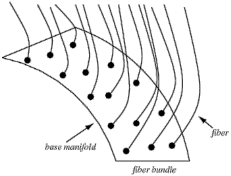
- Bi-tangent vector fields:
$$\mathbf{X} \in \mathrm{C}^1(\mathbb{T}\mathbb{M}\,;\mathbb{T}\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{T}\mathbb{M}} \circ \mathbf{X} = \mathrm{ID}_{\mathbb{T}\mathbb{M}}$$

Math5



Sections of fiber bundles

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Sections of tangent and bi-tangent bundles

- Tangent vector fields:
$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$
- Bi-tangent vector fields:
$$\mathbf{X} \in \mathrm{C}^1(\mathbb{T}\mathbb{M}\,;\mathbb{T}\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{T}\mathbb{M}} \circ \mathbf{X} = \mathrm{ID}_{\mathbb{T}\mathbb{M}}$$
- Vertical bi-tangent vectors $\mathbf{X} \in \mathrm{Ker}\, T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{M}}$

Math6

Tensor spaces

Math6

Tensor spaces

- Covariant $\quad \mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \mathrm{Cov}_{\mathbf{x}}(\mathbb{TM}) = L\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2\,;\mathcal{R}) = L\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M})$

Math6

Tensor spaces

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- Contravariant $\mathbf{s}_{\mathbf{x}}^{\textsc{Con}} \in \textsc{Con}_{\mathbf{x}}(\mathbb{TM}) = L\left(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2\,;\mathcal{R}\right) = L\left(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M}\right)$

Math6

Tensor spaces

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- Mixed $\mathbf{s}_{\mathbf{x}}^{\textsc{Mix}} \in \textsc{Mix}_{\mathbf{x}}(\mathbb{TM}) = L\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,,\mathbb{T}_{\mathbf{x}}^*\mathbb{M}\,;\mathcal{R}) = L\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})$

Math6

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- Contravariant $\mathbf{s}_{\mathbf{x}}^{\text{Con}} \in \text{Con}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed $\mathbf{s}_{\mathbf{x}}^{\text{Mix}} \in \text{Mix}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- with the alteration rules:

$$\mathbf{s}_{\mathbf{x}}^{\text{Cov}} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{s}_{\mathbf{x}}^{\text{Mix}} , \quad \mathbf{s}_{\mathbf{x}}^{\text{Con}} = \mathbf{s}_{\mathbf{x}}^{\text{Mix}} \circ \mathbf{g}_{\mathbf{x}}^{-1}$$

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Tensor bundles and sections

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\textsc{Cov}} \in \textsc{Cov}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
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Tensor bundles and sections

- Tensor bundle $\boldsymbol{\tau}_{\mathbb{M}}^{\textsc{Tens}} \in \mathrm{C}^1(\textsc{Tens}(\mathbb{TM}) ; \mathbb{M})$

Math6

Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\text{Cov}} \in \text{Cov}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}^*\mathbb{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\text{Con}} \in \text{Con}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
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Tensor bundles and sections

- Tensor bundle $\boldsymbol{\tau}_{\mathbb{M}}^{\text{Tens}} \in \text{C}^1(\text{Tens}(\mathbb{TM}) ; \mathbb{M})$
- Tensor field $\mathbf{s}_{\mathbb{M}}^{\text{Tens}} \in \text{C}^1(\mathbb{M} ; \text{Tens}(\mathbb{TM}))$

Math6

Tensor spaces

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Tensor bundles and sections

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- Tensor field $\mathbf{s}_{\mathbb{M}}^{\textsc{Tens}} \in \mathrm{C}^1(\mathbb{M}\,;\textsc{Tens}(\mathbb{TM}))$
- with: $\boldsymbol{\tau}_{\mathbb{M}}^{\textsc{Tens}} \circ \mathbf{s}_{\mathbb{M}}^{\textsc{Tens}} = \mathrm{ID}_{\mathbb{M}}$

Math7

Push and pull

Push and pull

Given a map $\zeta \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$

- Pull-back of a scalar field

$$f : \mathbb{N} \mapsto \mathrm{F\scriptsize{UN}}(\mathbb{N}) \quad \mapsto \quad \zeta{\downarrow}f : \mathbb{M} \mapsto \mathrm{F\scriptsize{UN}}(\mathbb{M})$$

defined by:

$$(\zeta{\downarrow}f)_{\mathbf{x}} := \zeta{\downarrow}f_{\zeta(\mathbf{x})} := f_{\zeta(\mathbf{x})} \in \mathrm{F\scriptsize{UN}}_{\mathbf{x}}(\mathbb{M})\,.$$

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Given a map $\zeta \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$

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- Push-forward of a tangent vector field

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM}) \quad \mapsto \quad \zeta{\uparrow}\mathbf{v} : \mathbb{N} \mapsto \mathbb{TN}$$

  defined by:

$$(\zeta{\uparrow}\mathbf{v})_{\zeta(\mathsf{x})} := \zeta{\uparrow}\mathbf{v}_{\mathsf{x}} = T_{\mathsf{x}}\zeta \cdot \mathbf{v}_{\mathsf{x}} \in \mathbb{T}_{\zeta(\mathsf{x})}\mathbb{N}\,.$$

Math8

Push and pull of tensor fields

Math8

Push and pull of tensor fields

- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_\mathbf{x} \rangle = \langle T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle$$

Math8

Push and pull of tensor fields

- Covectors

$$\left\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \right\rangle = \left\langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} \right\rangle = \left\langle\, T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \right\rangle$$

- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \circ T_{\mathbf{x}}\zeta \in \mathrm{Cov}(\mathbb{TM})_{\mathbf{x}}$$

Math8

Push and pull of tensor fields

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Push and pull of tensor fields

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- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \circ T_\mathbf{x}\zeta \in \mathrm{Cov}(\mathbb{TM})_\mathbf{x}$$

- Contravariant tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} = T_\mathbf{x}\zeta \circ \mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} \circ T^*_{\zeta(\mathbf{x})}\zeta \in \mathrm{Con}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

- Mixed tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{Mix}}_{\mathbf{x}} = T_\mathbf{x}\zeta \circ \mathbf{s}^{\mathrm{Mix}}_{\mathbf{x}} \circ T_{\zeta(\mathbf{x})}\zeta^{-1} \in \mathrm{Mix}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a, b]\,; \mathbb{M})$

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a, b]\,;\mathbb{M})$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \in \mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$$

$$\mathbf{c}_{\mu,\mu} \Uparrow \mathbf{v_x} = \mathbf{v_x}$$

$$\mathbf{c}_{\lambda,\mu} \Uparrow \circ\, \mathbf{c}_{\mu,\nu} \Uparrow \; = \mathbf{c}_{\lambda,\nu} \Uparrow$$

Math9

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- Covector fields $\mathbf{v}^*_{\mathbf{x}} \in \mathbb{T}^*_{\mathbf{x}}\mathbb{M}$ (by naturality)

$$\langle\, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}^*_{\mathbf{x}}, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}_{\mathbf{x}} \,\rangle = \mathbf{c}_{\lambda,\mu} \Uparrow \langle\, \mathbf{v}^*_{\mathbf{x}}, \mathbf{v}_{\mathbf{x}} \,\rangle$$

- Tensor fields (by naturality)

Math9

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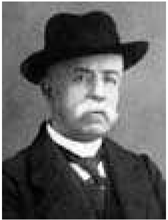

Gregorio Ricci-Curbastro (1853 - 1925)

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a,b]\,;\mathbb{M})$

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- Tensor fields (by naturality)

Tullio Levi-Civita (1873 - 1941)

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbf{Tens}(\mathbb{TM}))$

along the flow of a tangent vector field

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbf{Tens}(\mathbb{TM}))$

along the flow of a tangent vector field

- Tangent vector fields and Flows

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM}) \qquad \mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$$

$$\mathbf{v} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}$$

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbf{Tens}(\mathbb{TM}))$

along the flow of a tangent vector field

- Tangent vector fields and Flows

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM}) \qquad \mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$$

$$\mathbf{v} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}$$

- Lie derivative - LD

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{s} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}\!\downarrow(\mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

Math10

Derivatives of a tensor field

$\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbf{Tens}(\mathbb{TM}))$

along the flow of a tangent vector field

- Tangent vector fields and Flows

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- Lie derivative - LD

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{s} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}\!\downarrow(\mathbf{s}\circ\mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

- Parallel derivative - PD

$$\nabla_{\mathbf{v}}\,\mathbf{s} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Downarrow(\mathbf{s}\circ\mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

NLCM: Nonlinear Continuum Mechanics

Key contributions

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Key contributions

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2) M.E. Gurtin *An Introduction to Continuum Mechanics*
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NLCM: Nonlinear Continuum Mechanics

Key contributions

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G. Romano & R. Barretta Covariant hypo-elasticity
Eur. J. Mech. A-Solids 30 (2011) 1012–1023

G. Romano, R. Barretta, M. Diaco Basic Geometric Issues in Non-Linear Continuum Mechanics, preprint (2011).

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How to play the game
according to a full geometric approach

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- distance between simultaneous events $\mapsto$ space-metric
- distance between localized events $\mapsto$ time-metric

Math11

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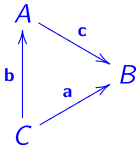
lenght of symplex's edges

Math11

lenght of symplex's edges

- Norm axioms



$$\|\mathbf{a}\| \geq 0\,, \quad \|\mathbf{a}\| = 0 \quad \Longrightarrow \quad \mathbf{a} = 0$$

$$\|\mathbf{a}\| + \|\mathbf{b}\| \geq \|\mathbf{c}\| \quad \textbf{triangle inequality},$$

$$\|\alpha\,\mathbf{a}\| = |\alpha|\,\|\mathbf{a}\|$$

Math11

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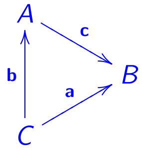

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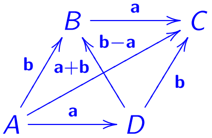
- Parallelogram rule



$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right]$$

Math12

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The metric tensor

- Theorem (Fréchet – von Neumann – Jordan)

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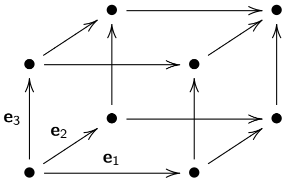
VOL (



$)^2$

$$= \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$$

Maurice René Fréchet (1878 - 1973)

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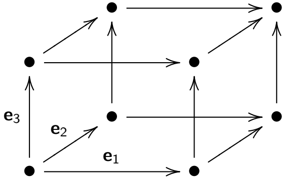

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John von Neumann (1903 - 1957)

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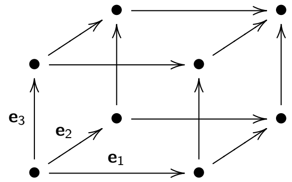

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Pascual Jordan (1902 - 1980)

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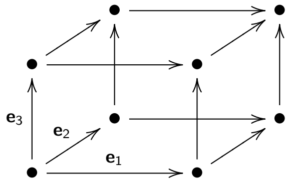

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Kosaku Yosida (1909 - 1990)

Events manifold fibrations

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- Time and space fibrations: $\gamma : \mathrm{E} \mapsto \mathcal{S} \times I$ (observer)

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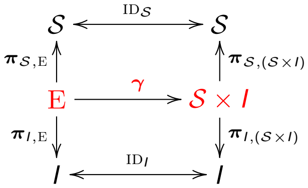

$\iff$

$$\boldsymbol{\pi}_{I,\mathrm{E}} = \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \circ \boldsymbol{\gamma}$$
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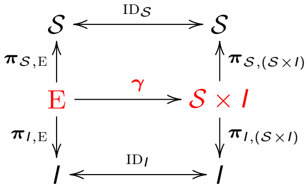

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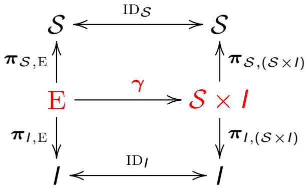

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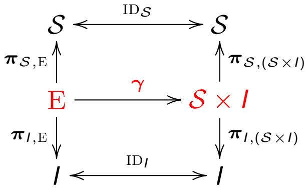

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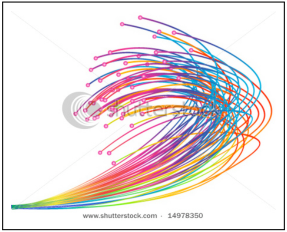
Trajectory



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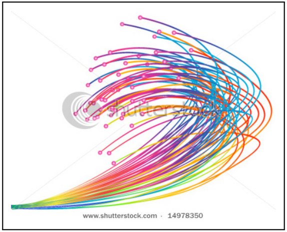

- Trajectory $\mapsto$ a manifold $\mathcal{T}_\varphi$ with injective immersion in the events time-bundle: $\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathrm{E})$

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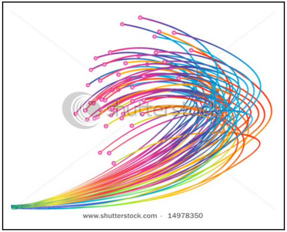

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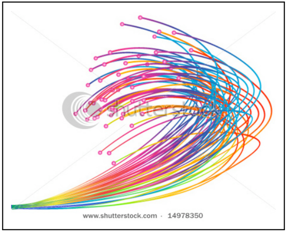

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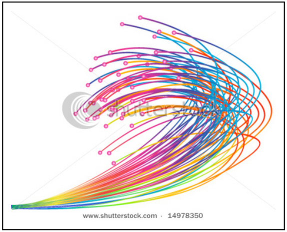
- time bundle $\mapsto$ fibers: body placements $\boldsymbol{\Omega}_t$

Trajectory



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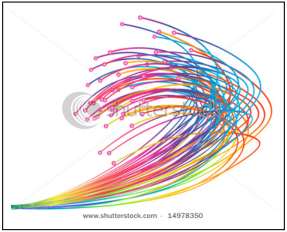

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Evolution

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- Evolution operator $\varphi^{\mathcal{T}_\varphi}$

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- Displacements: diffeomorphisms between placements

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$$\boldsymbol{\pi}_{I,\mathcal{T}_\varphi}(\boldsymbol{\varphi}^{\mathcal{T}_\varphi}_{\tau,t}(\mathbf{e}_t)) = \tau$$

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- Trajectory speed:

$$\mathbf{v}_{\mathcal{T}_\varphi}(\mathbf{e}_t) := \partial_{\tau=t}\,\varphi^{\mathcal{T}_\varphi}_{\tau,t}(\mathbf{e}_t) \quad \Longrightarrow \quad T_{\mathbf{e}}\boldsymbol{\pi}_{I,\mathcal{T}_\varphi} \cdot \mathbf{v}_{\mathcal{T}_\varphi}(\mathbf{e}_t) = 1_t$$

Body and particles

Body and particles

- Equivalence relation on the trajectory:

$$(\mathbf{e}_1\,,\mathbf{e}_2) \in \mathcal{T}_{\boldsymbol{\varphi}} \times \mathcal{T}_{\boldsymbol{\varphi}} \,:\, \mathbf{e}_2 = \boldsymbol{\varphi}^{\mathcal{T}_{\varphi}}_{t_2,t_1}(\mathbf{e}_1)\,.$$

with $t_i = \boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e}_i)\,, \qquad i = 1, 2\,.$

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Body $=$ quotient manifold (foliation)

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Body = quotient manifold (foliation)
Particles = equivalence classes (folia)

- mass conservation

$$\int_{\boldsymbol{\Omega}_{t_1}} \mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}},t_1} = \int_{\boldsymbol{\Omega}_{t_2}} \mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}},t_2} \quad \Longleftrightarrow \quad \mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}} \mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}}} = 0$$

$\mathbf{m}_{\mathcal{T}_{\boldsymbol{\varphi}}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{Vol}(\mathbb{T}\mathcal{T}_{\boldsymbol{\varphi}}))$ mass form

Tensor fields in NLCM

Tensor fields in NLCM

Space-time fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{T\small{ENS}}(\mathbb{T}\mathrm{E}))$	Space-time metric tensor
Spatial fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{T\small{ENS}}(\mathbb{V}\mathrm{E}))$	Spatial metric tensor

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Spatial fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{T{\small ENS}}(\mathbb{V}\mathrm{E}))$	Spatial metric tensor
Trajectory fields	$\mathbf{s}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathrm{T{\small ENS}}(\mathbb{T}\mathcal{T}_\varphi))$	Trajectory metric, trajectory speed
Material fields	$\mathbf{s}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathrm{T{\small ENS}}(\mathbb{V}\mathcal{T}_\varphi))$	Stress, stressing, material metric, stretching.

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Space-time fields	$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathrm{TENS}(\mathbb{T}\mathrm{E}))$	Space-time metric tensor
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Material fields	$\mathbf{s}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathrm{TENS}(\mathbb{V}\mathcal{T}_\varphi))$	Stress, stressing, material metric, stretching.
Trajectory-based space-time fields	$\mathbf{s}_{\mathrm{E},\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathrm{TENS}(\mathbb{T}\mathrm{E}))$	Trajectory speed (immersed)
Trajectory-based spatial fields	$\mathbf{s}_{\mathrm{E},\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathrm{TENS}(\mathbb{V}\mathrm{E}))$	Virtual velocity, acceleration, momentum, force

Covariance Paradigm

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Material fields at different times along the trajectory must be compared by push along the material displacement.
Material fields on push-related trajectories must be compared by push along the relative motion.

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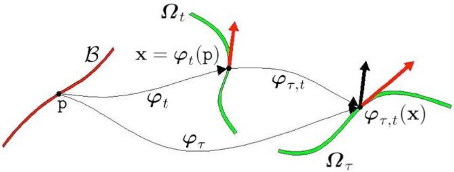
Push and parallel transport along the motion

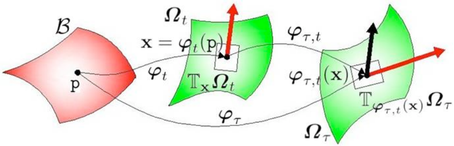


Parallel transport does not preserve time-verticality

Covariance Paradigm

Material fields at different times along the trajectory must be compared by push along the material displacement.
Material fields on push-related trajectories must be compared by push along the relative motion.

Push and parallel transport along the motion



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Lie time derivative - LTD

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Material time-derivative - MTD

- Trajectory-based space-time and spatial fields

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with $\mathbf{v}_{\mathrm{E},\mathcal{T}_{\varphi}} := \mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}} \uparrow \mathbf{v}_{\mathcal{T}_{\varphi}}$.

Rivers and Cogwheels

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[2] *See e.g.*
1) C. Truesdell, *A first Course in Rational Continuum Mechanics*
Second Ed. Academic Press, New-York (1991). First Ed. 1977
2) M.E. Gurtin, *An Introduction to Continuum Mechanics*
Academic Press, San Diego (1981)
3) J.E. Marsden & T.J.R. Hughes, *Mathematical Foundations of Elasticity*
Prentice-Hall, Redwood City, Cal. (1983)

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- Mixed form of the stretching tensor (standard):

$$\mathbf{D}_{\mathcal{T}_\varphi} := \mathbf{g}^{-1}_{\mathcal{T}_\varphi} \circ \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}} \mathbf{g}_{\mathcal{T}_\varphi} = \mathrm{sym}\,(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi})$$

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- Treatments which do not adopt a full geometric approach do not even perceive the difficulties revealed by the previous investigation.

Objective stress rate tensors

A sample of objective stress rate tensors

Co-rotational stress rate tensor, Zaremba (1903), Jaumann (1906,1911), Prager (1960):

$$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}$$

with $\dot{\mathbf{T}}$ material time derivative.

Convective stress tensor rate, Zaremba (1903), Oldroyd (1950), Truesdell (1955), Sedov (1960), Truesdell & Noll (1965):

$$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L}$$

Objective stress rate tensors

A sample of objective stress rate tensors

Co-rotational stress rate tensor, Zaremba (1903), Jaumann (1906,1911), Prager (1960):

$$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}$$

with $\dot{\mathbf{T}}$ material time derivative.

Convective stress tensor rate, Zaremba (1903), Oldroyd (1950), Truesdell (1955), Sedov (1960), Truesdell & Noll (1965):

$$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L}$$

These formulas, and similar ones in literature, rely on the application of Leibniz rule and on taking the parallel derivative of the material stress tensor field according to the trajectory connection.

The lack of regularity that may prevent to take partial time derivatives and the lack of conservation of time-verticality by parallel transport, are not taken into account.

Deformation gradient

The equivalence class of all material displacements whose tangent map have the common value:

$$T_{\mathbf{x}}\boldsymbol{\varphi}_{\tau,t} \in L\,(\mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t\,;\mathbb{T}_{\boldsymbol{\varphi}_{\tau,t}(\mathbf{x})}\boldsymbol{\Omega}_\tau\,)$$

- is called the *first jet* of $\boldsymbol{\varphi}_{\tau,t}$ at $\mathbf{x} \in \boldsymbol{\Omega}_t$ in differential geometry
- and the *relative deformation gradient* in continuum mechanics.

The chain rule between tangent maps:

$$T_{\boldsymbol{\varphi}_{\tau,s}(\mathbf{x})}\boldsymbol{\varphi}_{\tau,s} = T_{\boldsymbol{\varphi}_{t,s}(\mathbf{x})}\boldsymbol{\varphi}_{\tau,t} \circ T_{\mathbf{x}}\boldsymbol{\varphi}_{t,s}\,,$$

implies the corresponding one between material deformation gradients:

$$\mathbf{F}_{\tau,s} = \mathbf{F}_{\tau,t} \circ \mathbf{F}_{t,s}\,.$$

Time rate of deformation gradient, Truesdell & Noll (1965)

$$\dot{\mathbf{F}}_{t,s} = \mathbf{L}_t\,\mathbf{F}_{t,s}$$

with $\dot{\mathbf{F}}_{t,s} := \partial_{\tau=t}\,\mathbf{F}_{\tau,s}$ and $\mathbf{L}_t := \partial_{\tau=t}\,\mathbf{F}_{\tau,t}$ time derivatives.

$$\mathbf{L}_t(\mathbf{x})\cdot\mathbf{h}_{\mathbf{x}} := \partial_{\tau=t}\,\mathbf{F}_{\tau,t}(\mathbf{x})\cdot\mathbf{h}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t\,, \qquad \forall\,\mathbf{h}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t$$

with $\mathbf{F}_{\tau,t}(\mathbf{x})\cdot\mathbf{h}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_\tau$. The Lie time derivative gives:

$$\partial_{\tau=t}\,(T_{\mathbf{x}}\boldsymbol{\varphi}_{\tau,t})^{-1}\cdot(T_{\mathbf{x}}\boldsymbol{\varphi}_{\tau,t}\cdot\mathbf{h}_{\mathbf{x}}) = \partial_{\tau=t}\,\mathbf{h}_{\mathbf{x}} = 0$$

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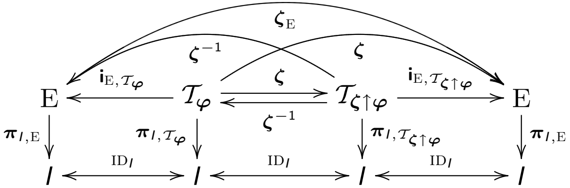

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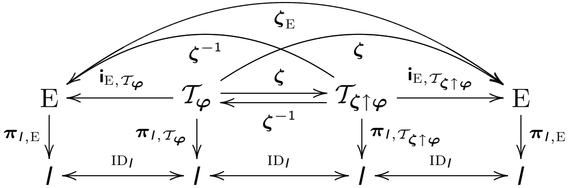
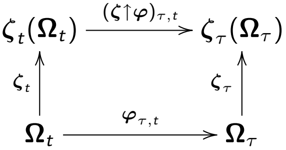


- Pushed motion



$$\iff \quad (\zeta\!\uparrow\!\varphi)_{\tau,t} = \zeta_{\tau} \circ \varphi_{\tau,t} \circ \zeta_t^{-1}$$

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Time Invariance and Frame Invariance
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Properties of Lie derivative

- Push of Lie time derivative to a fixed configuration

$$\varphi_{t,\mathrm{FIX}}\!\downarrow\!(\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\varphi}}}\,\mathbf{s}_{\mathcal{T}_{\varphi}})_t = \partial_{\tau=t}\,\varphi_{\tau,\mathrm{FIX}}\!\downarrow\!\mathbf{s}_{\mathcal{T}_{\varphi},\tau}$$

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- Lie time derivative along pushed motions

$$\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\zeta\uparrow\varphi}}}(\zeta\!\uparrow\!\mathbf{s}_{\varphi}) = \zeta\!\uparrow\!(\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\varphi}}}\,\mathbf{s}_{\mathcal{T}_{\varphi}})$$

Constitutive laws

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- Constitutive invariance under relative motions

$$\mathbf{H}_{\mathcal{T}_{\zeta\uparrow\varphi}} = \zeta\!\uparrow\!\mathbf{H}_{\mathcal{T}_\varphi}$$

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Hypo-elasticity

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- Constitutive hypo-elastic law
  $\mathbf{el}_{\mathcal{T}_\varphi}$ elastic stretching

$$\begin{cases} \dot{\varepsilon}_{\mathcal{T}_\varphi} &= \mathbf{el}_{\mathcal{T}_\varphi} \\ \mathbf{el}_{\mathcal{T}_\varphi} &= \mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}_\varphi} \end{cases}$$

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- CAUCHY integrability

$$\langle\, d_F \mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi}) \cdot \delta\boldsymbol{\sigma}_{\mathcal{T}_\varphi} \cdot \delta_1\boldsymbol{\sigma}_{\mathcal{T}_\varphi}, \delta_2\boldsymbol{\sigma}_{\mathcal{T}_\varphi} \,\rangle = \text{symmetric}$$

$$\Longrightarrow \quad \mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi}) = d_F \boldsymbol{\Phi}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi})$$

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- pull-back to reference:

$$\begin{aligned} \boldsymbol{\varphi}_{t,\mathrm{FIX}}{\downarrow}\mathbf{el}_{\mathcal{T}_\varphi,t} &= d_F^2 E^*_{\mathrm{FIX}}(\boldsymbol{\varphi}_{t,\mathrm{FIX}}{\downarrow}\boldsymbol{\sigma}_{\mathcal{T}_\varphi,t}) \cdot \partial_{\tau=t}\ \boldsymbol{\varphi}_{\tau,\mathrm{FIX}}{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau} \\ &= \partial_{\tau=t}\, d_F E^*_{\mathrm{FIX}}(\boldsymbol{\varphi}_{\tau,\mathrm{FIX}}{\downarrow}\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}) \end{aligned}$$

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$$\begin{aligned} \boldsymbol{\varphi}_{\tau,\mathrm{FIX}} &:= \boldsymbol{\varphi}_{\tau,t} \circ \boldsymbol{\varphi}_{t,\mathrm{FIX}} \\ E^*_{\mathrm{FIX}} &:= \boldsymbol{\varphi}_{t,\mathrm{FIX}}\downarrow E^*_{\mathcal{T}_\varphi,t} \quad \text{time invariant} \end{aligned}$$

Conservativeness of hyper-elasticity

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GREEN integrability of the elastic operator $\mathbf{H}_{\mathcal{T}_{\varphi}}$

as a function of the KIRCHHOFF stress tensor field

implies conservativeness:

$$\oint_{I} \int_{\Omega_t} \langle \boldsymbol{\sigma}_{\mathcal{T}_{\varphi},t}, \mathbf{el}_{\mathcal{T}_{\varphi},t} \rangle \, \mathbf{m}_{\mathcal{T}_{\varphi},t} \; dt = 0$$

for any cycle in the stress time-bundle,

i.e. for any stress path $\boldsymbol{\sigma}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(I\,; \mathrm{CON}(\mathbb{V}\mathcal{T}_{\varphi}))$

such that:

$$\boldsymbol{\sigma}_{\mathcal{T}_{\varphi},t_2} = \boldsymbol{\varphi}_{t_2,t_1}\!\uparrow\!\boldsymbol{\sigma}_{\mathcal{T}_{\varphi},t_1} \,, \quad I = [t_1, t_2]$$

Elasto-visco-plasticity

Elasto-visco-plasticity

- Constitutive law

$\mathbf{el}_{\mathcal{T}_\varphi}$ elastic stretching
$\mathbf{pl}_{\mathcal{T}_\varphi}$ visco-plastic stretching

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}_{\mathcal{T}_\varphi} = \mathbf{el}_{\mathcal{T}_\varphi} + \mathbf{pl}_{\mathcal{T}_\varphi} & \text{stretching additivity} \\ \mathbf{el}_{\mathcal{T}_\varphi} = d_F^2 E^*_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_{\mathcal{T}_\varphi}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}_\varphi} & \text{hyper-elastic law} \\ \mathbf{pl}_{\mathcal{T}_\varphi} \in \partial_F \mathcal{F}_{\mathcal{T}_\varphi}(\boldsymbol{\sigma}_\varphi) & \text{visco-plastic flow rule} \end{cases}$$

Reference strains

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- total strain in the time interval $I = [s, t]$:

$$\varepsilon_{\mathcal{T}_{\varphi},t,s} := \varphi_{t,s}{\downarrow}\mathbf{g}_{\mathcal{T}_{\varphi},t} - \mathbf{g}_{\mathcal{T}_{\varphi},s}$$

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- reference total strain:

$$\begin{aligned}
\varepsilon^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I} &:= \tfrac{1}{2}\int_I \partial_{\tau=t}\,\varphi_{\tau,\mathrm{FIX}}{\downarrow}\mathbf{g}_{\mathcal{T}_\varphi,\tau}\,dt \\
&= \tfrac{1}{2}\varphi_{t,\mathrm{FIX}}{\downarrow}\mathbf{g}_{\mathcal{T}_\varphi,t} - \tfrac{1}{2}\varphi_{s,\mathrm{FIX}}{\downarrow}\mathbf{g}_{\mathcal{T}_\varphi,s} \\
&= \tfrac{1}{2}\varphi_{s,\mathrm{FIX}}{\downarrow}\big(\varphi_{t,s}{\downarrow}\mathbf{g}_{\mathcal{T}_\varphi,t} - \mathbf{g}_{\mathcal{T}_\varphi,s}\big) = \tfrac{1}{2}\varphi_{s,\mathrm{FIX}}{\downarrow}\varepsilon_{\mathcal{T}_\varphi,t,s}
\end{aligned}$$

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- reference elastic and visco-plastic strain:

$$\mathbf{el}^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I} := \int_I \varphi_{t,\mathrm{FIX}}{\downarrow}\mathbf{el}_{\mathcal{T}_\varphi,t}\,dt\,, \qquad \mathbf{pl}^{\mathrm{FIX}}_{\mathcal{T}_\varphi,I} := \int_I \varphi_{t,\mathrm{FIX}}{\downarrow}\mathbf{pl}_{\mathcal{T}_\varphi,t}\,dt$$

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- additivity of reference strains:

$$\varepsilon^{\mathrm{FIX}}_{\mathcal{T}_\varphi, I} = \mathbf{el}^{\mathrm{FIX}}_{\mathcal{T}_\varphi, I} + \mathbf{pl}^{\mathrm{FIX}}_{\mathcal{T}_\varphi, I}$$

Material Frame Indifference (MFI)

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for any isometric relative motion $\boldsymbol{\zeta}^{\text{iso}} \in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathcal{T}_{\boldsymbol{\zeta}^{\text{iso}}\uparrow\varphi})$ **induced by a change of Euclid observer** $\boldsymbol{\zeta}^{\text{iso}}_{\mathbf{E}} \in \mathrm{C}^1(\mathbf{E}\,;\mathbf{E})$ **.**

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Equivalent condition

- Constitutive operators must be frame invariant

MFI in elasto-visco-plasticity

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- Frame invariance of the hypo-elastic operator

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Pushed operator

$$(\boldsymbol{\zeta}^{\mathrm{ISO}}\uparrow\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\varphi}}})(\boldsymbol{\zeta}^{\mathrm{ISO}}\uparrow\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}}) \cdot \boldsymbol{\zeta}^{\mathrm{ISO}}\uparrow\dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\boldsymbol{\varphi}}} = \boldsymbol{\zeta}^{\mathrm{ISO}}\uparrow(\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\varphi}}}(\boldsymbol{\sigma}_{\mathcal{T}_{\boldsymbol{\varphi}}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\boldsymbol{\varphi}}})$$

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Examples:

- the simplest hypo-elastic operator is GREEN integrable and frame invariant:

$$\mathbf{H}^{\mathrm{HYPO}}_{\mathcal{T}_{\boldsymbol{\varphi}},t}(\mathbf{T}_{\mathcal{T}_{\boldsymbol{\varphi}},t}) := \frac{1}{2\,\mu}\,\mathbb{I}_{\mathcal{T}_{\boldsymbol{\varphi}},t} - \frac{\nu}{E}\,\mathbf{I}_{\mathcal{T}_{\boldsymbol{\varphi}},t} \otimes \mathbf{I}_{\mathcal{T}_{\boldsymbol{\varphi}},t}$$

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- the visco-plastic flow rule is frame invariant

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These results provide answers to unsolved questions posed in:

J.C. Simó & K.S. Pister, Remarks on rate constitutive equations for finite deformation problems: computational implications, *Comp. Meth. Appl. Mech. Eng.* **46** (1984) 201–215.
J. C. Simó & M. Ortiz, A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive equations, *Comp. Meth. Appl. Mech. Eng.* **49** (1985) 221–245.

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- Algorithms for numerical computations must be modified to comply with the covariant theory; multiplicative decomposition of the deformation gradient should be deemed as geometrically inconsistent