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Geometria della Meccanica del Continuo

innovazioni metodologiche e computazionali.

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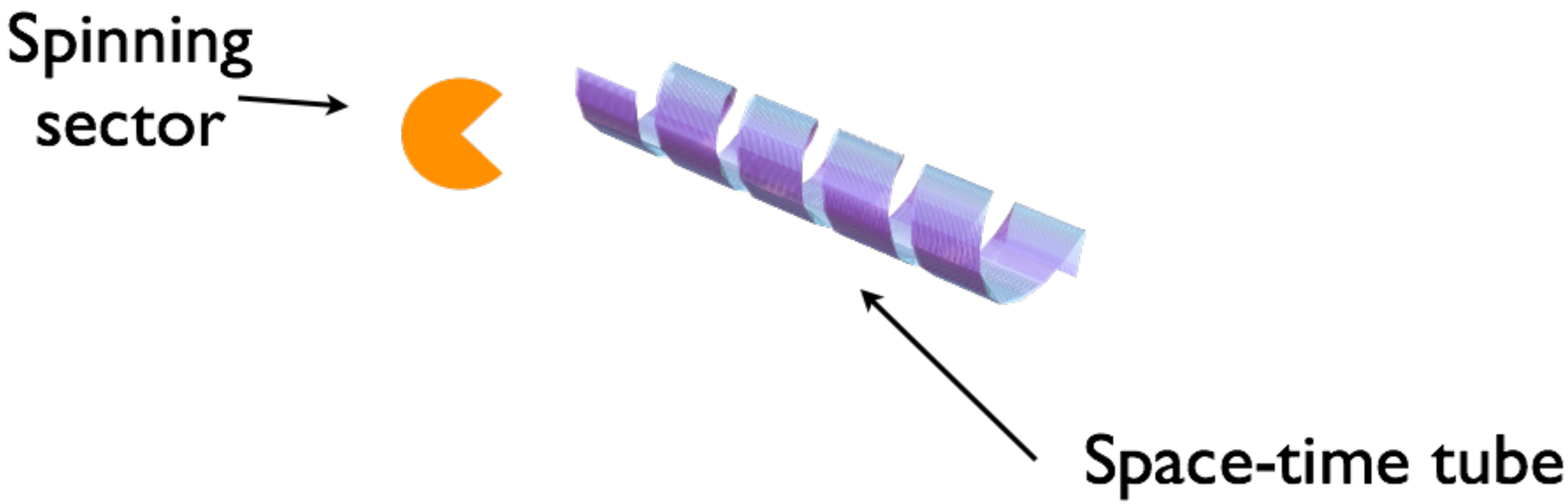
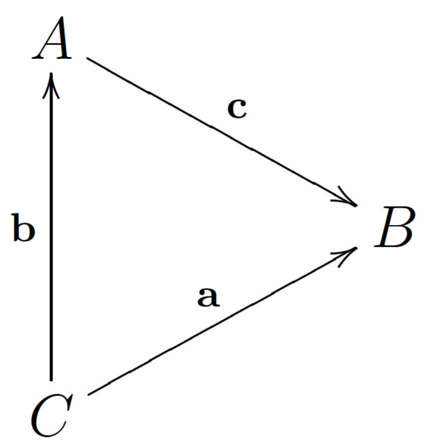
The Covariance Paradigm

in the mechanics of material behavior



Metric Tensor

normed linear space: $\mathbf{c} = \mathbf{a} - \mathbf{b}\,,$ **distance** $d(\mathbf{a}\,,\mathbf{b}) := \|\mathbf{a} - \mathbf{b}\|\,.$

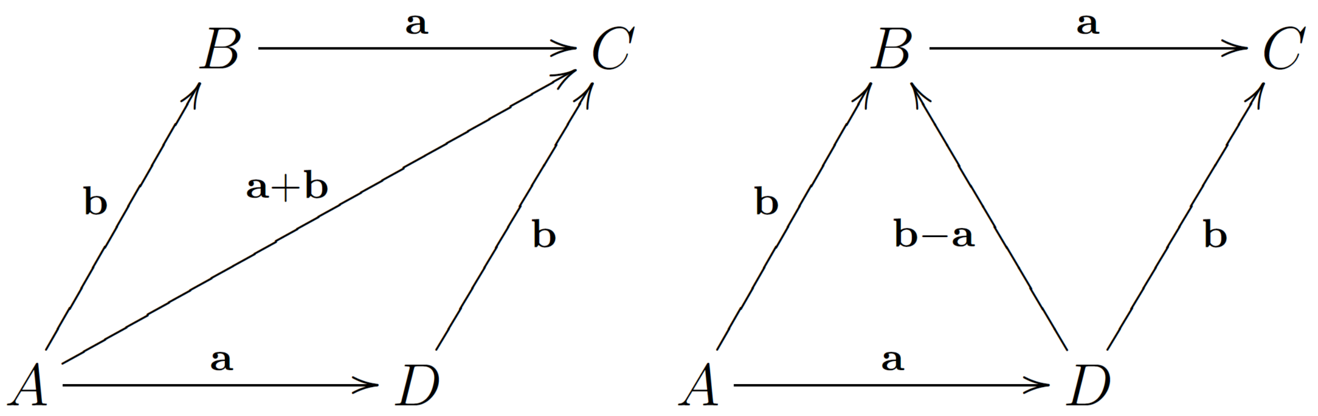


$$\|\mathbf{a}\| \geq 0 \quad \|\mathbf{a}\|^2 = 0 \quad \Longrightarrow \quad \mathbf{a} = 0\,,$$

$\|\mathbf{a}\| + \|\mathbf{b}\| \geq \|\mathbf{c}\|$ **triangle inequality**

$$\|\alpha\,\mathbf{a}\| = |\alpha|\;\|\alpha\|\,.$$

parallelogram law: $\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right]$



metric tensor

$$\mathbf{g}(\mathbf{a}\,,\mathbf{b}) := \tfrac{1}{4}\left[\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right]$$

$$= \tfrac{1}{2}\left[\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2\right]$$

$$= \tfrac{1}{2}\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2]\,.$$

with $\mathbf{g}$ bilinear, $\mathbf{g}(\mathbf{a}\,,\mathbf{b}) = \mathbf{g}(\mathbf{b}\,,\mathbf{a})$ and $\mathbf{g}(\mathbf{a}\,,\mathbf{a}) = \|\mathbf{a}\|^2\,.$

Manifolds and Fibre Bundles

A manifold $\mathbb{M}$ is the generalization of the notion of a curve or a surface in the Euclidean space. It is a set of points which can be put in a piecewise one-to-one correspondence with a linear space (the model space) with smooth transitions.

A fibre bundle $(\mathrm{BUN}(\mathbb{M}), \boldsymbol{\pi}, \mathbb{M})$ is a construction envisaged to provide a geometrical description of many basic items in mechanics and physical sciences.

It may be naïvely described as a base manifold with a fibre-manifold attached at each of its points. Each fibre is a diffeomorphic image of a given manifold called the typical fibre.

The surjective map $\boldsymbol{\pi}$, which associates with, to each point of the bundle, the base point of the relevant fibre, is called the projection.

Continuum kinematics

Differentiable geometric structures:

- The ambient space $\mathcal{S}$, in which motions take place, a finite dimensional Riemann manifold without boundary, endowed with a metric tensor field $\mathbf{g}$;
- The material body $\mathcal{B}$, a finite dimensional manifold with boundary, of dimension less than or equal to the one of the ambient space;
- The observation time interval I an open, connected subset of a line;
- The configurations manifold $\mathbb{C}$, an infinite dimensional manifold of maps which are C^1-diffeomorphisms of the body manifold onto submanifolds of the ambient space manifold.

Motions and displacements

A *motion* is described by a map $\hat{\varphi} \in C^1(\mathcal{B} \times I \,; \mathcal{S})$ from the manifold $\mathcal{B} \times I$ of *material events* into the ambient space manifold $(\mathcal{S}\,, \hat{\mathbf{g}})$.

To a motion there correspond at each time $t \in I$ a *material configuration* map $\boldsymbol{\varphi}_t \in C^1(\mathcal{B} \,; \boldsymbol{\Omega}_t)$ which is a diffeomorphisms of the body manifold $\mathcal{B}$ onto the placement manifold $\boldsymbol{\Omega}_t$.

The *material displacement* from a source placement $\boldsymbol{\Omega}_t = \boldsymbol{\varphi}_t(\mathcal{B})$ to the target placement $\boldsymbol{\Omega}_\tau = \boldsymbol{\varphi}_\tau(\mathcal{B})$, is the diffeomorphism

$$\boldsymbol{\varphi}_{\tau,t} := \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \in C^1(\boldsymbol{\Omega}_t \,; \boldsymbol{\Omega}_\tau) \,,$$

providing the position in $\boldsymbol{\Omega}_\tau$ at time $\tau \in I$ of the particle which occupies the given position in $\boldsymbol{\Omega}_t$ at time $t \in I$.

The inclusion map

To emphasize the distinction between material fields and spatial fields, it is expedient to consider the inclusion map:

$$\mathbf{i}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathcal{S})$$

We denote by:

- $\hat{\varphi}_t = \mathbf{i}_{\varphi,t} \circ \boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\mathbf{i}_{\varphi,t}(\boldsymbol{\Omega}_t))$ → the spatial configuration map,
- $\hat{\varphi}_{\tau,t} \in \mathrm{C}^1(\mathbf{i}_{\varphi,t}(\boldsymbol{\Omega}_t)\,;\mathbf{i}_{\varphi,\tau}(\boldsymbol{\Omega}_\tau))$ → the spatial displacement map,
- $\boldsymbol{\varphi}_{\tau,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\boldsymbol{\Omega}_\tau)$ → the material displacement map.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\hat{\varphi}_{\tau,t}} & \mathcal{S} \\ \uparrow{\scriptstyle \mathbf{i}_{\varphi,t}} & & \uparrow{\scriptstyle \mathbf{i}_{\varphi,\tau}} \\ \boldsymbol{\Omega}_t & \xrightarrow[\boldsymbol{\varphi}_{\tau,t}]{} & \boldsymbol{\Omega}_\tau \end{array} \quad \Longleftrightarrow \quad \hat{\varphi}_{\tau,t} \circ \mathbf{i}_{\varphi,t} := \mathbf{i}_{\varphi,\tau} \circ \boldsymbol{\varphi}_{\tau,t}\,.$$

$$\begin{array}{ccc} \mathbb{T}\mathcal{S} & \xrightarrow{T\hat{\varphi}_{\tau,t}} & \mathbb{T}\mathcal{S} \\ \uparrow{\scriptstyle T\mathbf{i}_{\varphi,t}} & & \uparrow{\scriptstyle T\mathbf{i}_{\varphi,\tau}} \\ \mathbb{T}\boldsymbol{\Omega}_t & \xrightarrow[T\boldsymbol{\varphi}_{\tau,t}]{} & \mathbb{T}\boldsymbol{\Omega}_\tau \end{array} \quad \Longleftrightarrow \quad T\hat{\varphi}_{\tau,t} \circ T\mathbf{i}_{\varphi,t} = T\mathbf{i}_{\varphi,\tau} \circ T\boldsymbol{\varphi}_{\tau,t}\,.$$

At a point $\mathbf{x} \in \mathbf{\Omega}_t$, the linear space of 0th order material tensors (scalars) is denoted by $\mathrm{FUN}_{\mathbf{x}}(\mathbf{\Omega}_t)$, the dual spaces of tangent and cotangent material vectors by $\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t$ and $\mathbb{T}^*_{\mathbf{x}}\mathbf{\Omega}_t$.

Covariant, contravariant and mixed 2nd order material tensors belong to linear spaces of scalar-valued bilinear maps (or linear operators):

$$\begin{aligned}
\mathrm{COV}_{\mathbf{x}}(\mathbf{\Omega}_t) &= L\,(\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t\,,\,\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t\,;\mathcal{R}) = L\,(\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t\,;\mathbb{T}^*_{\mathbf{x}}\mathbf{\Omega}_t)\,,\\
\mathrm{CON}_{\mathbf{x}}(\mathbf{\Omega}_t) &= L\,(\mathbb{T}^*_{\mathbf{x}}\mathbf{\Omega}_t\,,\,\mathbb{T}^*_{\mathbf{x}}\mathbf{\Omega}_t\,;\mathcal{R}) = L\,(\mathbb{T}^*_{\mathbf{x}}\mathbf{\Omega}_t\,;\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t)\,,\\
\mathrm{MIX}_{\mathbf{x}}(\mathbf{\Omega}_t) &= L\,(\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t\,,\,\mathbb{T}^*_{\mathbf{x}}\mathbf{\Omega}_t\,;\mathcal{R}) = L\,(\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t\,;\mathbb{T}_{\mathbf{x}}\mathbf{\Omega}_t)\,.
\end{aligned}$$

A generic material tensor space is denoted by $\mathrm{TENS}_{\mathbf{x}}(\mathbf{\Omega}_t)$.

At a given fixed time $t \in I$, a map $\boldsymbol{\zeta}^{\mathrm{SP}}_t \in \mathrm{C}^1(\mathbf{\Omega}_t\,;\mathcal{S})$, with the co-restriction $\boldsymbol{\zeta}_t \in \mathrm{C}^1(\mathbf{\Omega}_t\,;\boldsymbol{\zeta}^{\mathrm{SP}}_t(\mathbf{\Omega}_t))$ a diffeomorphism, will be called a *geometric displacement* to contrast its physical interpretation, of displacement at fixed time, in comparison with the one of a *material displacement* $\boldsymbol{\varphi}_{\tau,t} := \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \in \mathrm{C}^1(\mathbf{\Omega}_t\,;\mathbf{\Omega}_\tau)$ along the motion. This distinction will become significant in the discussion about time-independence and invariance in Section 5.

The push of a material scalar $f_{\varphi,t}(\mathbf{x}) \in \mathrm{FUN}_{\mathbf{x}}\boldsymbol{\Omega}_t$, along a *geometric displacement* $\boldsymbol{\zeta}_t^{\mathrm{SP}} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathcal{S})$, is a change of its base point:

$$(\boldsymbol{\zeta}_t\!\uparrow\! f_{\varphi,t})(\boldsymbol{\zeta}_t(\mathbf{x})) = f_{\varphi,t}(\mathbf{x})\,.$$

The push of a tangent material vector $\mathbf{v}_{\varphi,t}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t$ is the evaluation of the *tangent geometric displacement* $T_{\mathbf{x}}\boldsymbol{\zeta}_t \in L\,(\mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t\,;\mathbb{T}_{\boldsymbol{\zeta}_t(\mathbf{x})}\boldsymbol{\zeta}_t(\boldsymbol{\Omega}_t))$, by the formula:

$$\boldsymbol{\zeta}_t\!\uparrow\!(\mathbf{v}_{\varphi,t}(\mathbf{x})) := T_{\mathbf{x}}\boldsymbol{\zeta}_t \cdot \mathbf{v}_{\varphi,t}(\mathbf{x})\,,$$

and the push of a material cotangent vector $\mathbf{v}^*_{\varphi,t}(\mathbf{x}) \in \mathbb{T}^*_{\mathbf{x}}\boldsymbol{\Omega}_t$ is defined by invariance:

$$\langle\, \boldsymbol{\zeta}_t\!\uparrow\!\mathbf{v}^*_{\varphi,t}, \boldsymbol{\zeta}_t\!\uparrow\!\mathbf{v}_{\varphi,t}\,\rangle = \boldsymbol{\zeta}_t\!\uparrow\!\langle\, \mathbf{v}^*_{\varphi,t}, \mathbf{v}_{\varphi,t}\,\rangle\,.$$

The push of a tensor is also defined by invariance. For a twice-covariant material tensor field $\mathbf{s}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$, the push is explicitly defined, for any pair of material tangent vector fields $\mathbf{a}_{\varphi,t}, \mathbf{b}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathbb{T}\boldsymbol{\Omega}_t)$, by:

$$(\boldsymbol{\zeta}_t\!\uparrow\!\mathbf{s}_{\varphi,t})(\boldsymbol{\zeta}_t\!\uparrow\!\mathbf{a}_{\varphi,t}, \boldsymbol{\zeta}_t\!\uparrow\!\mathbf{b}_{\varphi,t}) := \boldsymbol{\zeta}_t\!\uparrow\!(\mathbf{s}_{\varphi,t}(\mathbf{a}_{\varphi,t}, \mathbf{b}_{\varphi,t}))\,.$$

Introducing the co-tangent map $T^*_{\boldsymbol{\zeta}_t(\mathrm{x})}\boldsymbol{\zeta}_t \in L\left(\mathbb{T}^*_{\boldsymbol{\zeta}_t(\mathrm{x})}\boldsymbol{\zeta}_t(\boldsymbol{\Omega}_t)\,;\mathbb{T}^*_{\mathrm{x}}\boldsymbol{\Omega}_t\right)$ such that:

$$\langle \mathbf{a}_{\boldsymbol{\zeta}_t(\mathrm{x})}, T_{\mathrm{x}}\boldsymbol{\zeta}_t \cdot \mathbf{b}_{\mathrm{x}} \rangle = \langle T^*_{\boldsymbol{\zeta}_t(\mathrm{x})}\boldsymbol{\zeta}_t \cdot \mathbf{a}_{\boldsymbol{\zeta}_t(\mathrm{x})}, \mathbf{b}_{\mathrm{x}} \rangle \,,$$

for every $\mathbf{b}_{\mathrm{x}} \in \mathbb{T}_{\mathrm{x}}\boldsymbol{\Omega}_t$ and $\mathbf{a}_{\boldsymbol{\zeta}_t(\mathrm{x})} \in \mathbb{T}^*_{\boldsymbol{\zeta}_t(\mathrm{x})}\boldsymbol{\zeta}_t(\boldsymbol{\Omega}_t)$. With the abridged notation $\mathrm{s}_{\mathrm{x}} := \mathrm{s}_{\boldsymbol{\varphi},t}(\mathrm{x})$, the pushes of covariant, contravariant and mixed material tensors are given by:

$$\begin{aligned}
\boldsymbol{\zeta}_t\!\uparrow\!\mathrm{s}^{\mathrm{COV}}_{\mathrm{x}} &= T^*_{\mathrm{x}}\boldsymbol{\zeta}_t^{-1} \circ \mathrm{s}^{\mathrm{COV}}_{\mathrm{x}} \circ T_{\boldsymbol{\zeta}_t(\mathrm{x})}\boldsymbol{\zeta}_t^{-1} \,, \\
\boldsymbol{\zeta}_t\!\uparrow\!\mathrm{s}^{\mathrm{CON}}_{\mathrm{x}} &= T_{\mathrm{x}}\boldsymbol{\zeta}_t \circ \mathrm{s}^{\mathrm{CON}}_{\mathrm{x}} \circ T^*_{\boldsymbol{\zeta}_t(\mathrm{x})}\boldsymbol{\zeta}_t \,, \\
\boldsymbol{\zeta}_t\!\uparrow\!\mathrm{s}^{\mathrm{MIX}}_{\mathrm{x}} &= T_{\mathrm{x}}\boldsymbol{\zeta}_t \circ \mathrm{s}^{\mathrm{MIX}}_{\mathrm{x}} \circ T_{\boldsymbol{\zeta}_t(\mathrm{x})}\boldsymbol{\zeta}_t^{-1} \,.
\end{aligned}$$

The *material metric field* $\mathbf{g}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$ at time $t \in I$ is induced in the configuration $\boldsymbol{\varphi}_t : \mathcal{B} \mapsto \boldsymbol{\Omega}_t$ by the spatial metric field

Alteration of tensors is defined by the relations:

$$\mathrm{s}^{\mathrm{MIX}}_{\mathrm{x}} = \mathbf{g}^{-1}_{\mathrm{x}} \circ \mathrm{s}^{\mathrm{COV}}_{\mathrm{x}} = \mathrm{s}^{\mathrm{CON}}_{\mathrm{x}} \circ \mathbf{g}_{\mathrm{x}} \,,$$

which, in components form, correspond to lowering and rising of indexes.

The adjoint $T^A_{\boldsymbol{\zeta}_t(\mathbf{x})}\boldsymbol{\zeta}_t \in L(\mathbb{T}_{\boldsymbol{\zeta}_t(\mathbf{x})}\boldsymbol{\zeta}_t(\boldsymbol{\Omega}_t)\,;\mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t)$ of the tangent map is defined by $T^A_{\boldsymbol{\zeta}_t(\mathbf{x})}\boldsymbol{\zeta}_t := \mathbf{g}^{-1}_{\mathbf{x}} \circ T^*_{\boldsymbol{\zeta}_t(\mathbf{x})}\boldsymbol{\zeta}_t \circ \mathbf{g}_{\boldsymbol{\zeta}_t(\mathbf{x})}$. Then:

$$(\boldsymbol{\zeta}_t{\downarrow}\mathbf{g}_{\boldsymbol{\zeta}_t(\mathbf{x})})^{\text{MIX}}_{\mathbf{x}} = T^A_{\boldsymbol{\zeta}_t(\mathbf{x})}\boldsymbol{\zeta}_t \circ T_{\mathbf{x}}\boldsymbol{\zeta}_t \,,$$

a formula that is referred to in evaluating the mixed form of the stretch. The pull is the push along the inverse diffeomorphism: $\boldsymbol{\zeta}_t{\downarrow} = \boldsymbol{\zeta}_t^{-1}{\uparrow}$. All these definitions extend directly to the push along a material displacement.

The *convective time-derivative* at time $t \in I$ of a *material tensor field* $\mathbf{s}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{TENS}(\boldsymbol{\Omega}_t))$, along $\boldsymbol{\varphi}^{\text{SP}} : \mathcal{B} \times I \mapsto \mathcal{S}$, is defined by:

$$\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{s}_{\boldsymbol{\varphi}} := \partial_{\tau=t}\,\boldsymbol{\varphi}_{\tau,t}{\downarrow}\mathbf{s}_{\boldsymbol{\varphi},\tau} \,.$$

The pulled-back tensors $(\boldsymbol{\varphi}_{\tau,t}{\downarrow}\mathbf{s}_{\boldsymbol{\varphi},\tau})(\mathbf{x})$ belong, for all $\tau \in I$, to the same linear tensor space $\mathrm{TENS}_{\mathbf{x}}(\boldsymbol{\Omega}_t)$ so that the derivative $\partial_{\tau=t}$ makes sense. A simple but quite important property is that the pull-back of a convective time-derivative is equal to the time-derivative of the pull-back:

$$\boldsymbol{\varphi}_{t,s}{\downarrow}(\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{s}_{\boldsymbol{\varphi}}) = \partial_{\tau=t}\,\boldsymbol{\varphi}_{\tau,s}{\downarrow}\mathbf{s}_{\boldsymbol{\varphi},\tau} \,.$$

The parallel transport of a spatial tensor along a curve $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,;\mathcal{S})$ in space is deduced, from the definition of parallel transport of a tangent spatial vector, by invariance, for a twice-covariant tensor, according to the formula:

$$(\mathbf{c}_\lambda \Uparrow \mathbf{s})(\mathbf{c}_\lambda \Uparrow \mathbf{a}\,,\mathbf{c}_\lambda \Uparrow \mathbf{b}) := \mathbf{s}(\mathbf{a}\,,\mathbf{b}) \circ \mathbf{c}_\lambda\,, \quad \lambda \in \mathcal{R}\,, \quad \mathbf{a},\mathbf{b} \in \mathbb{T}\mathcal{S}\,,$$

and similarly for other spatial tensors.

The *parallel* (or *covariant*) derivative along a curve $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,;\mathcal{S})$ of a *spatial tensor field* $\mathrm{s} \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{TENS}(\mathcal{S}))$, is accordingly defined by:

$$\nabla_{\dot{\mathbf{c}}_\lambda}\,\mathbf{s} := \partial_{\lambda=0}\,\mathbf{c}_\lambda \Downarrow (\mathbf{s} \circ \mathbf{c}_\lambda)\,,$$

where $\dot{\mathbf{c}}_\lambda := \partial_{\lambda=0}\,\mathbf{c}_\lambda$ is the parametrization velocity of the curve at $\lambda = 0$ and $\mathbf{c}_\lambda \Downarrow$ denotes the parallel transport from $\mathbf{c}(\lambda)$ to $\mathbf{c}(0)$. If the curve is time-parametrized the definition of *parallel time-derivative* is got.

The *parallel time-derivative* along the motion: $\nabla_{\boldsymbol{\varphi},t}\,\mathbf{s}^{\mathrm{SP}}_{\boldsymbol{\varphi}} := \partial_{\tau=t}\,\boldsymbol{\varphi}^{\mathrm{SP}}_{\tau,t} \Downarrow \mathbf{s}^{\mathrm{SP}}_{\boldsymbol{\varphi},\tau}$ of a spatial-valued material tensor field $\mathbf{s}^{\mathrm{SP}}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{TENS}(\mathcal{S}))$ is also well-defined.

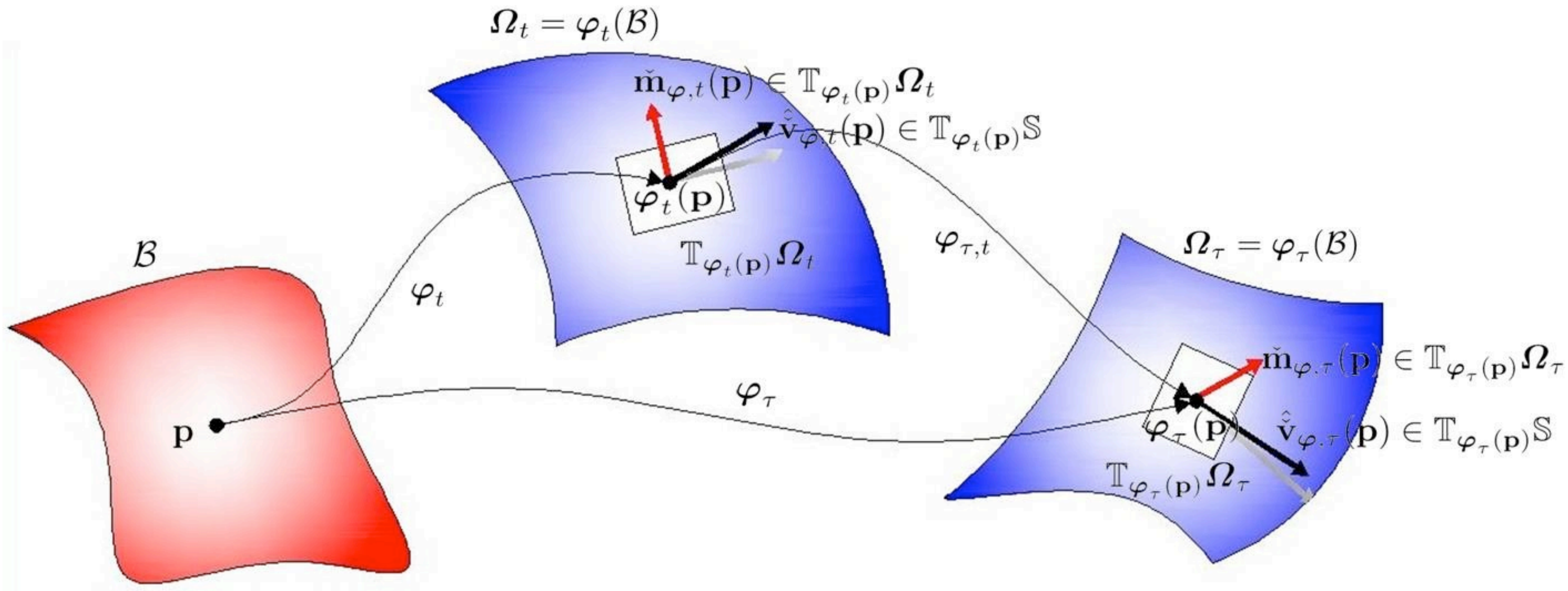
Warning

As a rule, neither the convective time-derivative nor the parallel time-derivative along the motion may be evaluated by resorting to LEIBNIZ rule to get a split into the sum of partial time and space derivatives. The celebrated D'ALEMBERT-EULER formula for the acceleration:

$$\mathbf{a}_{\boldsymbol{\varphi},t} := \partial_{\tau=t}\, \boldsymbol{\varphi}^{\mathrm{SP}}_{\tau,t}\Downarrow \mathbf{v}_{\boldsymbol{\varphi},\tau} = \partial_{\tau=t}\, \mathbf{v}_{\boldsymbol{\varphi},\tau} + \nabla_{\mathbf{v}_{\boldsymbol{\varphi},t}}\, \mathbf{v}_{\boldsymbol{\varphi},t} \,,$$

is in fact applicable only in investigations about continuous flows of a fluid in a region of space, as in problems of hydrodynamics, where it was originarily conceived. A well-known application is to the formulation of the NAVIER-STOKES-ST.VENANT equations of fluid-dynamics.

Material and spatial-valued material vector fields on a membrane



In most presentations of continuum mechanics, material fields and spatial-valued material fields are not distinguished.

The basic distinction is usually hidden by the context, in which a 3-D dimensional body manifold is considered embedded in a 3-D ambient space manifold.

Continuum Mechanics is a field theory aimed to describe the evolution of a material body in the physical ambient space.

A treatment, in the spirit of differential geometry on manifolds, induces to underline the need for a careful distinction between the various kinds of fields involved in the analysis:

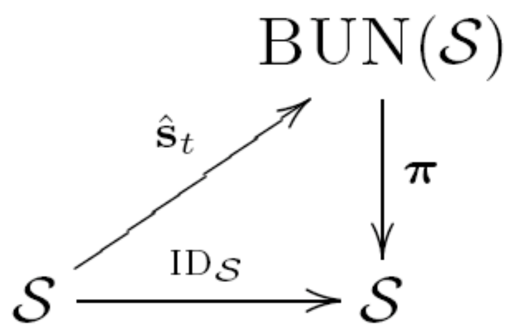
- Spatial fields;
- Material fields;
- Spatial-valued material fields;
- Material pull-back of spatial fields.
- Spatial descriptions of material fields. (D. Bernoulli – J. d'Alembert)

Spatial fields

- Spatial tensors are multilinear maps over a tangent space to the space manifold.

- Spatial fields are defined at each point of the ambient space manifold and at any time. Their values are spatial tensors based at that point, independently of whether there is a body particle crossing it or not.

A spatial field is a section $\hat{\mathbf{s}}_t \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{BUN}(\mathcal{S}))$ of the tensor bundle $(\mathrm{BUN}(\mathcal{S}), \boldsymbol{\pi}, \mathcal{S})$:



$$\Longleftrightarrow \qquad \boldsymbol{\pi} \circ \hat{\mathbf{s}}_t = \mathrm{ID}_{\mathcal{S}}$$

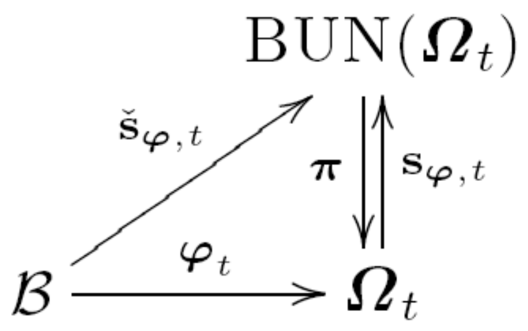
The twice covariant metric tensor field $\hat{\mathbf{g}}$ is a spatial field.

Material fields

- Material tensors are multilinear maps that operate, at each time instant, over a tangent space at a point of the body's placement along the motion.

- Material fields are defined, at each time instant, at particles of the body manifold and their values are material tensors based at the particle location evolving in the motion.

A material field at time $t \in I$ is a section $\check{\mathrm{s}}_{\varphi,t} \in \mathrm{C}^1(\mathcal{B}\,;\mathrm{BUN}(\boldsymbol{\Omega}_t))$ of the bundle $(\mathrm{BUN}(\boldsymbol{\Omega}_t), \boldsymbol{\pi}, \boldsymbol{\Omega}_t)$ along the motion:



$\iff$

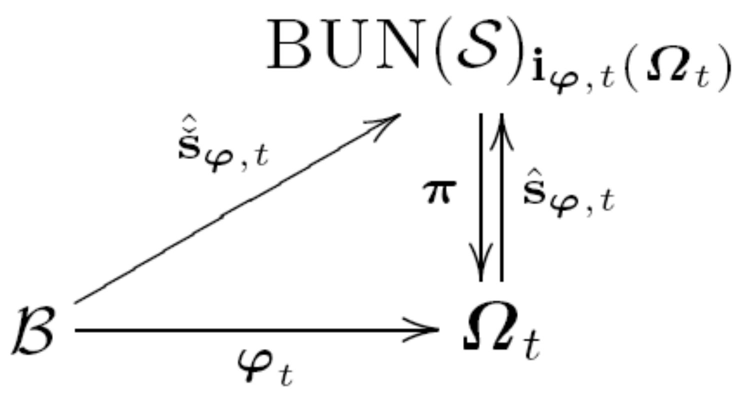
$$\boldsymbol{\pi} \circ \check{\mathrm{s}}_{\varphi,t} = \boldsymbol{\varphi}_t \,,$$
$$\mathrm{s}_{\varphi,t} = \check{\mathrm{s}}_{\varphi,t} \circ \boldsymbol{\varphi}_t^{-1} \,,$$
$$\boldsymbol{\pi} \circ \mathrm{s}_{\varphi,t} = \mathrm{ID}_{\,\boldsymbol{\Omega}_t} \,.$$

Most fields of interest in continuum mechanics are material fields, for instance, stretch, stretching, stress, stressing, temperature, heat flow, entropy, thermodynamical potentials.

Spatial-valued material fields

- Spatial-valued material fields are defined, at any instant of time, at particles of the body manifold, their values being spatial tensors based at the particle location evolving in the motion.

A spatial-valued material field is a section $\hat{\hat{\mathbf{s}}}_{\varphi,t} \in \mathrm{C}^1(\mathcal{B}\,;\mathrm{BUN}(\mathcal{S}))$ of the bundle $(\mathrm{BUN}(\mathcal{S}),\boldsymbol{\pi},\mathcal{S})$ along $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_t)$:



$$\iff \quad \begin{aligned} \boldsymbol{\pi} \circ \hat{\hat{\mathbf{s}}}_{\varphi,t} &= \boldsymbol{\varphi}_t \,, \\ \hat{\mathbf{s}}_{\varphi,t} &= \hat{\hat{\mathbf{s}}}_{\varphi,t} \circ \boldsymbol{\varphi}_t^{-1} \,, \\ \boldsymbol{\pi} \circ \hat{\mathbf{s}}_{\varphi,t} &= \mathrm{ID}_{\,\boldsymbol{\Omega}_t} \,, \end{aligned}$$

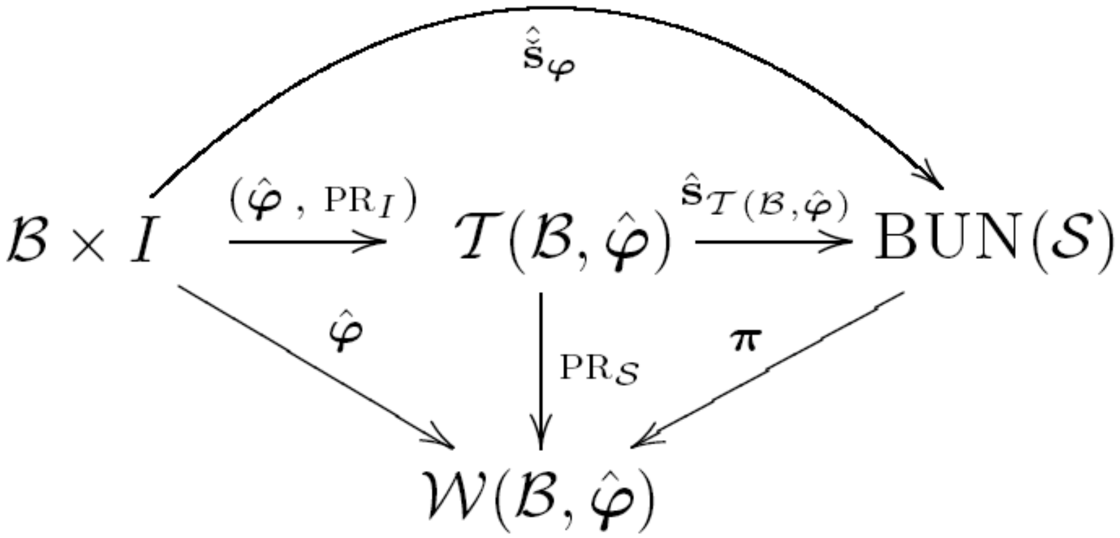
In Continuum Dynamics:
velocity, force and kinetic momentum are spatial-valued material fields.

Spatial descriptions of material fields

A material field $\hat{\hat{\mathrm{s}}}_{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,; \mathrm{BUN}(\mathcal{S}))$ admits a spatial description:

$\hat{\mathrm{s}}_{\mathcal{T}(\mathcal{B},\hat{\varphi})} \in \mathrm{C}^1(\mathcal{T}(\mathcal{B},\hat{\varphi}), \mathrm{BUN}(\mathcal{S}))\,,$

defined in the trajectory manifold $\mathcal{T}(\mathcal{B},\hat{\varphi})$ according the diagram:



$$\Longrightarrow \quad \hat{\mathrm{s}}_{\mathcal{T}(\mathcal{B},\hat{\varphi})} := \hat{\hat{\mathrm{s}}}_{\varphi} \circ (\varphi\,, \mathrm{PR}_I)^{-1}$$

- $\mathrm{PR}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{S} \times I\,; \mathcal{S})$ and $\mathrm{PR}_I \in \mathrm{C}^1(\mathcal{S} \times I\,; I)$ → cartesian projectors,
- $\hat{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,; \mathcal{S})$ → motion of the body in the ambient space,
- $\mathcal{W}(\mathcal{B},\hat{\varphi}) := \hat{\varphi}(\mathcal{B} \times I) \subset \mathcal{S}$ → spatial trajectory manifold,
- $(\hat{\varphi}\,, \mathrm{PR}_I) \in \mathrm{C}^1(\mathcal{B} \times I\,; \mathcal{S} \times I)$ → events map,
- $\mathcal{T}(\mathcal{B},\hat{\varphi}) := (\hat{\varphi}\,, \mathrm{PR}_I)(\mathcal{B} \times I)$ → trajectory manifold.

Material pull-back

- Spatial metric tensor $\mathbf{g} \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{COV}(\mathcal{S}))$

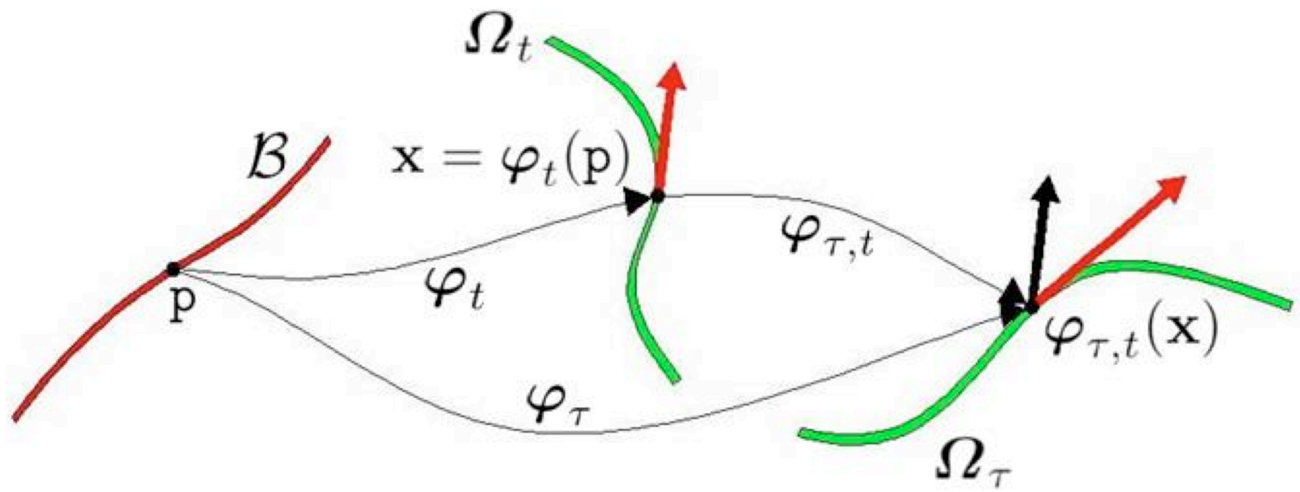
- Material metric tensor $\mathbf{g}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$

$$\mathbf{g}_{\varphi,t}(\mathbf{a}_{\varphi,t},\mathbf{b}_{\varphi,t}) := \mathbf{g}(\mathbf{i}_{\varphi,t}\!\uparrow\!\mathbf{a}_{\varphi,t},\mathbf{i}_{\varphi,t}\!\uparrow\!\mathbf{b}_{\varphi,t}) \circ \mathbf{i}_{\varphi,t}$$

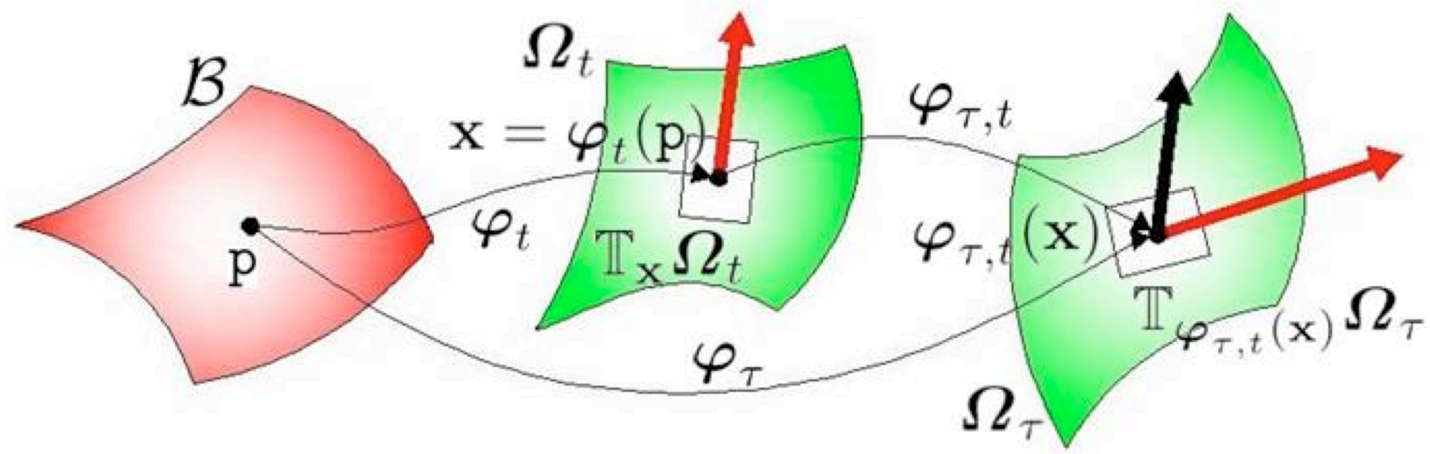
$$\mathbf{g}_{\varphi,t} = \mathbf{i}_{\varphi,t}\!\downarrow\!\mathbf{g} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\mathbb{T}\boldsymbol{\Omega}_t))$$

Bodies

- 1D: wire



- 2D: membrane



Geometric tools for comparing:

- Spatial tangent vectors at different points of the space manifold.

- Material tangent vectors at the same particle for different placements of a body.

Parallel transport along a curve:
a transformation, in the ambient space manifold, which takes a tangent vector to this manifold, that is a velocity of a curve in the space manifold, into another such tangent vector.

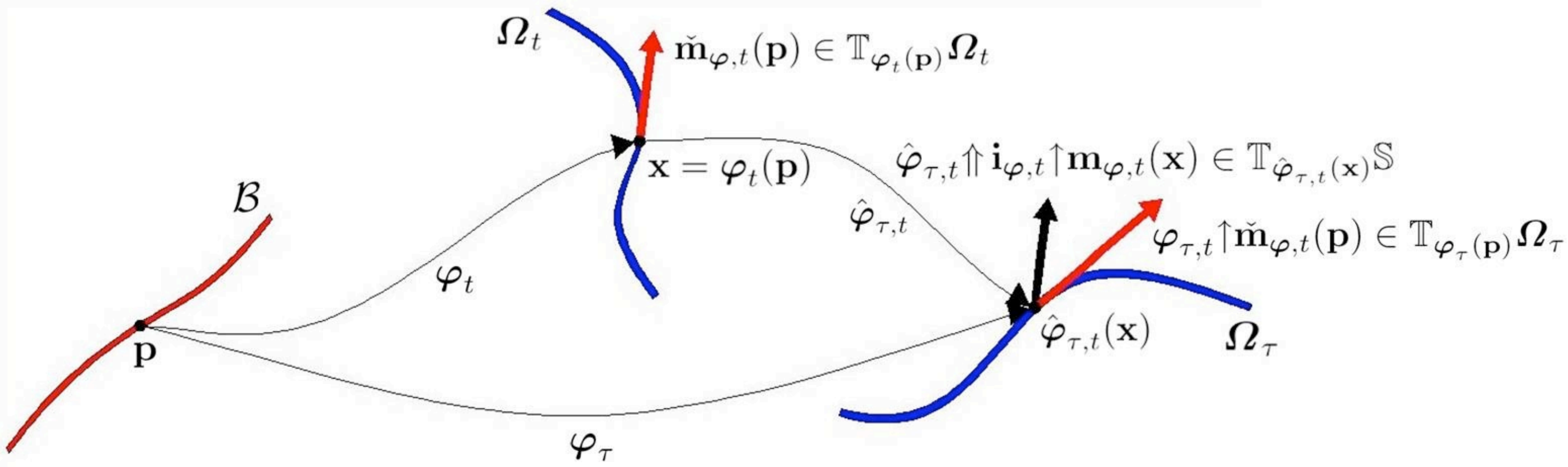
Push transformation by the material displacement map:
a transformation which takes a material tangent vector, that is a velocity of a curve in a body placement, into a material tangent vector in a displaced placement.
It is the suitable transformation tool to compare material tensors based at different configurations.

A material tangent vector is NOT in the domain of the parallel transport along a path.

Even if a material vector in a 3-D body is improperly identified with its spatial immersion, the parallel transported vector will depend on the chosen connection, a dependence which should be taken into account in the description of material behavior.

In a lower dimensional body, the image by a parallel transport along a path will, in general, no more be the spatial immersion of a material vector (see below).

Push of a material vector tangent to a wire and parallel transport of its spatial immersion



Parallel transport in space is applicable only to spatial tangent vectors.
Material tangent vectors can be only transformed by push along a diffeomorphic map (material displacement) as sketched in the figure above.

COVARIANCE PARADIGM

Material fields, pertaining to the same material body at different configurations, must be compared according to the transformation by push along the material displacement diffeomorphism.

Time-rate of a material tensor

The time rate of variation of a material tensor field

$$\mathbf{s}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{BUN}(\boldsymbol{\Omega}_t))$$

is the convective time-derivative along the motion $\hat{\varphi} \in \mathrm{C}^1(\mathcal{B} \times I\,;\mathcal{S})$:

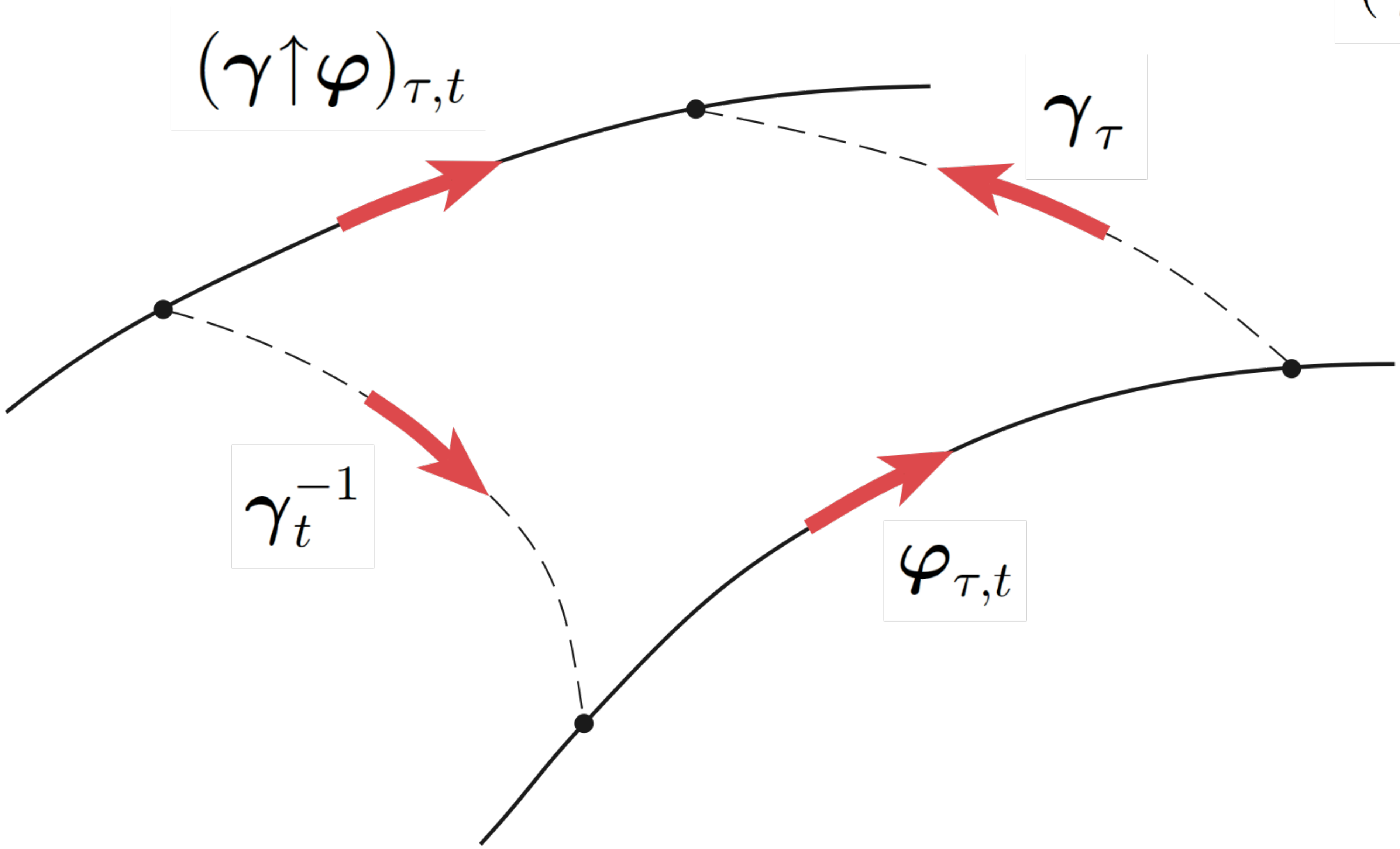
$$\boxed{\mathcal{L}_{\varphi,t}\,\mathbf{s}_{\varphi} := \partial_{\tau=t}\,\boldsymbol{\varphi}_{\tau,t}{\downarrow}\mathbf{s}_{\varphi,\tau}}$$

Relative motions

Push by a time-dependent diffeomorphism

$$\varphi_{\tau,t} := \varphi_\tau \circ \varphi_t^{-1}$$

$$(\gamma \uparrow \varphi)_{\tau,t} = \gamma_\tau \circ \varphi_{\tau,t} \circ \gamma_t^{-1}$$



Covariance and Relative motions

A pushed physical law still relates
involved fields and motions
when they are pushed forward according to a relative motion.

Group invariance

Material fields are group invariant if they are dragged by relative motions of a group:

$$\mathbf{s}_{\gamma\uparrow\varphi,t} = \gamma_t \uparrow \mathbf{s}_{\varphi,t} \,, \quad \forall\, t \in I$$

Naturality of the convective time-derivative:

$$\mathcal{L}_{(\gamma\uparrow\varphi),t}\,(\gamma\uparrow\mathbf{s}_{\varphi}) = \gamma_t\uparrow\mathcal{L}_{\varphi,t}\,\mathbf{s}_{\varphi}$$

Convective time-derivatives of group invariant fields are invariant too:

$$\mathbf{s}_{\gamma\uparrow\varphi,t} = \gamma_t\uparrow\mathbf{s}_{\varphi,t} \quad\Longrightarrow\quad \mathcal{L}_{(\gamma\uparrow\varphi),t}\,\mathbf{s}_{\gamma\uparrow\varphi,t} = \gamma_t\uparrow\mathcal{L}_{\varphi,t}\,\mathbf{s}_{\varphi}$$

A physical law, involving group invariant fields and their convective time-derivatives, is group invariant too.

Continuum Mechanics

To any pair of configurations $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_t)$ and $\boldsymbol{\varphi}_\tau \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_\tau)$
there corresponds a George Green strain (STRETCH) tensor field:

$$\tfrac{1}{2}(\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{g}_{\boldsymbol{\varphi},\tau} - \mathbf{g}_{\boldsymbol{\varphi},t}) \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}(\boldsymbol{\Omega}_t))$$

The strain rate (STRETCHING) is the material tensor field defined by:

$$\boldsymbol{\varepsilon}_{\boldsymbol{\varphi},t} := \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\varphi}} = \partial_{\tau=t}\ \tfrac{1}{2}(\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{g}_{\boldsymbol{\varphi}_\tau} - \mathbf{g}_{\boldsymbol{\varphi},t}) \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}(\boldsymbol{\Omega}_t))$$

The STRESS field $\boldsymbol{\sigma}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}^*(\boldsymbol{\Omega}_t))$ is a section of the bundle $\mathrm{SYM}^*(\boldsymbol{\Omega}_t)$ of symmetric contravariant material tensor fields, defined by duality with the stretching:

$$\langle \boldsymbol{\sigma}_{\varphi,t}, \boldsymbol{\varepsilon}_{\varphi,t} \rangle := J_1(\boldsymbol{\sigma}_{\varphi,t} \circ \boldsymbol{\varepsilon}_{\varphi,t}) \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{FUN}(\boldsymbol{\Omega}_t))$$

Material field of linear invariants of the mixed tensor field:

$$\boldsymbol{\sigma}_{\varphi,t} \circ \boldsymbol{\varepsilon}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{MIX}(\boldsymbol{\Omega}_t))$$

In Continuum Mechanics, material stress fields, whose duality pairing with the covariant stretching provide the virtual power per unit volume in the actual configuration, are contravariant Cauchy stress fields.

The ones providing the virtual power per unit mass, or per unit reference volume, are contravariant Kirchhoff stress fields. The mixed form of the covariant stretching tensor is provided by the symmetric part of the material covariant derivative of the velocity field, an outcome of Euler's formula.

Their mixed forms are Cauchy true stress and Kirchhoff true stress fields, respectively, the adjective true stemming from the fact that the boundary flux of mixed stress tensor fields provides the boundary tractions field.

The stress rate (STRESSING) is the material tensor field defined by:

$$\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi} := \partial_{\tau=t}\ \boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\boldsymbol{\sigma}_{\varphi,\tau} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{SYM}^*(\boldsymbol{\Omega}_t))\,.$$

The other tool provided by Differential Geometry for spatial-valued material fields is the covariant time-derivative along a motion $\hat{\varphi} \in \mathrm{C}^1(\mathcal{B}\times I\,;\mathcal{S})$:

$$\nabla_{\varphi,t}\,\hat{\mathbf{s}}_{\varphi} := \partial_{\tau=t}\ \hat{\varphi}_{\tau,t}\!\Downarrow\!\hat{\mathbf{s}}_{\varphi,\tau}$$

$\hat{\varphi}_{\tau,t}\!\Downarrow$ $\longrightarrow$ parallel transport from the spatial placement $\mathbf{i}_{\varphi,\tau}(\boldsymbol{\Omega}_{\tau})$
to the spatial placement $\mathbf{i}_{\varphi,t}(\boldsymbol{\Omega}_t)$.

In Continuum Dynamics, acceleration is the covariant time-derivative of the velocity, along the motion.

Remark The convective time-derivative of a material field is well-defined, while the covariant time-derivative is defined only for spatial-valued material tensor fields.

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SOME STRESS RATES PROPOSED IN LITERATURE

Truesdell rate of Cauchy stress $\longrightarrow$ $\overset{\circ}{\boldsymbol{\sigma}} = \phi_{t*}(J^{-1}\dot{\boldsymbol{S}})$

This stress rate is the Piola transformation of the material time derivative of the symmetric Piola-Kirchhoff stress $\boldsymbol{S} = J\boldsymbol{\phi}_t^*(\boldsymbol{\sigma})$

For contravariant components it has the form $\rightarrow$ $\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{l}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{t}} + \boldsymbol{\sigma}\,\mathrm{tr}(\boldsymbol{d})$

$\boldsymbol{l} = \nabla\boldsymbol{v}$ $\rightarrow$ spatial velocity gradient tesor.

Zaremba-Jaumann rate of Cauchy stress $\longrightarrow$ $\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\boldsymbol{\sigma}$

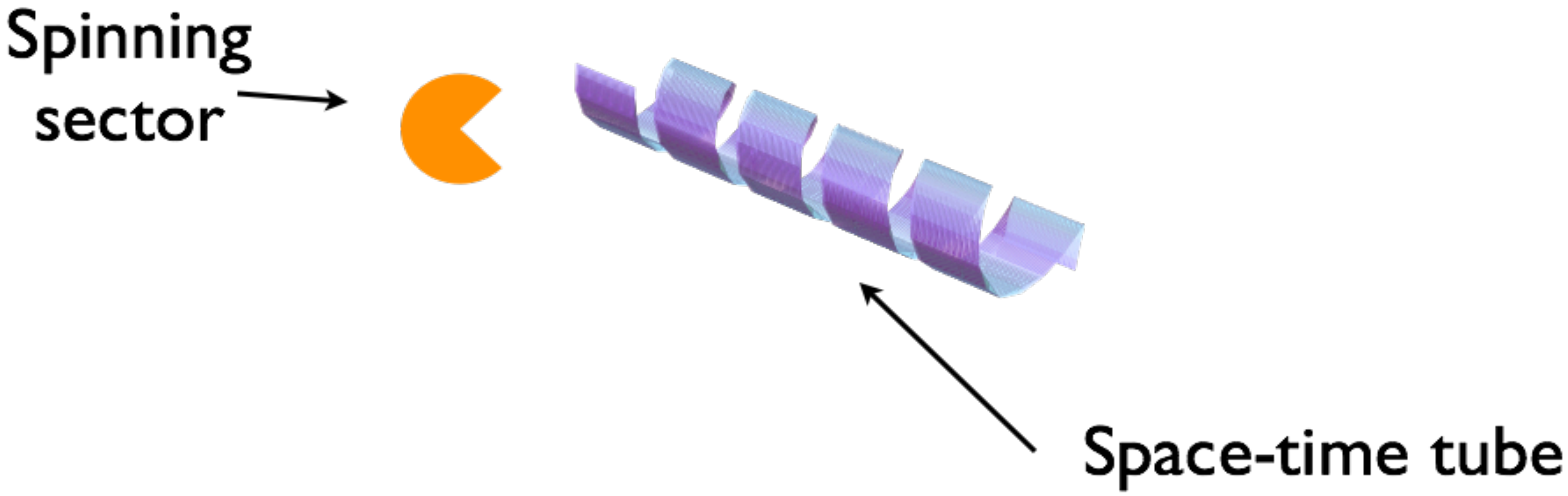
$\boldsymbol{\omega}$ $\rightarrow$ spin rate or vorticity tensor.

Zaremba-Jaumann rate of Kirchhoff stress $\longrightarrow$ $\overset{\triangledown}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\boldsymbol{\tau}$

Green-Naghdi rate of Cauchy stress $\longrightarrow$ $\hat{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\boldsymbol{\Omega} - \boldsymbol{\Omega}\cdot\boldsymbol{\sigma}$

$\boldsymbol{\Omega}$ $\rightarrow$ rate of rotation tensor related to the rotation tensor $\boldsymbol{R}$ by $\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\cdot\boldsymbol{R}^{\mathrm{t}}$.

The spatial description of a spatial-valued material field is highly irregular in time, with the exception of very special instances.



The stressing is the convective time-derivative of the material stress tensor and the stretching is the convective time-derivative of the material metric tensor along the motion. If the covariant time-derivative would be adopted to evaluate the rate of change of the metric tensor, a vanishing derivative would be got, since the standard Euclid connection is metric preserving. So, why try to use the covariant time-derivative for the stress rate?

Comments

The remedies to the lack of objectivity adopted in literature have been eventually ineffective, because the primary cause of ill-posedness was neither detected nor avoided, in the absence of a working covariance paradigm.

In this respect it is to be underlined that, although for three-dimensional bodies covariant time-derivatives of spatial immersions of material tensor fields may be performable, this tool should be treated at most as a special computational mean and not as a basic definition.

The evaluation according the Leibniz rule is subject to stringent regularity requirements and, in addition, covariant differentations are forbidden by the geometry of continuum mechanics for lower dimensional bodies (such as wires or membranes).

Synopsis

- Push transformation and convective time-differentiation of a material tensor field along a motion in the space manifold are allowed.
- Parallel transport and covariant time-differentiation of a material tensor field along a curve in the space manifold are forbidden operations for lower dimensional bodies.
- Parallel transport and covariant time-differentiation of a spatial-valued material tensor field along a curve in the space manifold are allowed.
- Push transformation and convective time-differentiation of a spatial-valued material tensor field along a motion in the space manifold are allowed.

- The split according to Leibniz rule, of the convective time-differentiation of a material tensor field along a motion in the space manifold, is not performable.

- The split according to Leibniz rule, of the convective or covariant time-differentiation of a spatial-valued material tensor field along a motion in the space manifolf, is not performable.

- The split according to Leibniz rule, of the convective or covariant time-differentiation of the spatial description of a spatial-valued material tensor field along a motion in the space manifold, is performable only under stringent regularity assumptions which are admissible in many modelings proper to fluid dynamics but are not likely to be fulfilled in solid mechanics.

Time invariance - Frame indifference - Isotropy

TIME INVARIANCE: (according to covariance)

PUSH BY MATERIAL DISPLACEMENT ALONG THE MOTION.

The response to the pushed cause acting on the pushed specimen provides the pushed effect.

MATERIAL FRAME INDIFFERENCE:

PUSH BY ISOMETRIC DISPLACEMENT AT FROZEN TIME.

The response to a rotated cause acting on a rotated specimen provides the rotated effect.

ISOTROPY:

The response to a rotated cause acting on an unrotated specimen provides the rotated effect.

Change of Observer

A *change of observer* is a time-dependent family of diffeomorphic maps $\boldsymbol{\gamma}_t^{\mathrm{OBS}} \in \mathrm{C}^1(\mathcal{S}\,;\mathcal{S})$ of the ambient space onto itself. It induces a relative motion from any given motion.

A *change of Euclid observer* requires that the change of observer $\boldsymbol{\gamma}_t^{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{S}\,;\mathcal{S})$ be an isometry: $\mathbf{g}_{\boldsymbol{\gamma}_t^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t} = \boldsymbol{\gamma}_t^{\mathrm{ISO}}\uparrow\mathbf{g}_{\boldsymbol{\varphi},t}$. Invariance under change of Euclid observer is called *frame-indifference*. The material metric tensor is frame-indifferent by definition.

A basic physical assumption is that the stress tensor is frame-indifferent, that is, invariant under a change of Euclid observer. Of course, in general, the stress tensor will not be time-independent, that is invariant under a material displacement of the body along the motion, even if the material displacement $\boldsymbol{\varphi}_{\tau,t} := \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\boldsymbol{\Omega}_\tau)$ is isometric, i.e. the material metric tensor is time-independent: $\mathbf{g}_{\boldsymbol{\varphi},\tau} = \boldsymbol{\varphi}_{\tau,t}\uparrow\mathbf{g}_{\boldsymbol{\varphi},t}$. Time-independence of the stress tensor under isometric material displacements holds however for elastic materials.

Hypo-elasticity

A hypo-elastic response of a body $\mathcal{B}$ in motion $\boldsymbol{\varphi}^{\rm SP} : \mathcal{B} \times I \mapsto \mathcal{S}$ is expressed, at each time $t \in I$, by assuming that the stretching at the configuration $\boldsymbol{\varphi}_t : \mathcal{B} \mapsto \boldsymbol{\Omega}_t$ is a function of stress and stressing.

Hypo-elastic law

The hypo-elastic response at time $t \in I$ is governed by a stress-dependent constitutive linear operator $\mathbf{H}_{\boldsymbol{\varphi},t}$ which provides the stretching $\frac{1}{2}\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\varphi}}$ corresponding to the stressing $\mathcal{L}_{\boldsymbol{\varphi},t}\,\boldsymbol{\sigma}_{\boldsymbol{\varphi}}$:

$$\dot{\varepsilon}_{\boldsymbol{\varphi},t} := \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\varphi}} = \mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \cdot \mathcal{L}_{\boldsymbol{\varphi},t}\,\boldsymbol{\sigma}_{\boldsymbol{\varphi}}\,.$$

The operator $\mathbf{H}_{\boldsymbol{\varphi},t}$ is defined on the linear space $\mathrm{CON}(\boldsymbol{\Omega}_t)$ and takes values in the linear space $L\,(\mathrm{CON}(\boldsymbol{\Omega}_t)\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$ whose elements are linear operators between the domain space $\mathrm{CON}(\boldsymbol{\Omega}_t)$ and its dual $\mathrm{COV}(\boldsymbol{\Omega}_t)$. At each $\boldsymbol{\sigma}_{\boldsymbol{\varphi},t} \in \mathrm{CON}(\boldsymbol{\Omega}_t)$ the tangent compliance $\mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t})$ is assumed to be an invertible linear operator.

In (Truesdell and Noll, 1965) the symbol $\overset{\triangle}{\mathbf{T}}$ has been proposed for the mixed form of the *stressing*, there called *convected stress rate*. In their notation, the hypo-elastic law is written in components as:

$$\mathbf{D}^{i.}_{.j} = \mathbf{H}^{i..s}_{.jr.}(\mathbf{T}^{p.}_{.q})\ \overset{\triangle}{\mathbf{T}}{}^{r.}_{.s}\ .$$

Time-independent hypo-elasticity

A hypo-elastic constitutive operator is time-independent in a time-interval I if the instantaneous operators at any pair of time instants $\tau, t \in I$ are related by push along the material displacement:

$$\mathbf{H}_{\boldsymbol{\varphi},\tau} = \boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{H}_{\boldsymbol{\varphi},t} \,,$$

the pushed operator, being defined by:

$$\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!(\mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \cdot \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\sigma}_{\boldsymbol{\varphi}}) = (\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{H}_{\boldsymbol{\varphi},t})(\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \cdot \boldsymbol{\varphi}_{\tau,t}\!\uparrow\!(\mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\sigma}_{\boldsymbol{\varphi}}) \,.$$

This means that time-invariant material tensor fields, fulfilling the constitutive relation at time $t \in I$, are still related by the law at time $\tau \in I$. To endow the mathematical definition of hypo-elastic law with a physical meaning apt to describe a material behavior, it is compelling to show independence of the change of EUCLID observer of the motion, which is the meaning of *material frame-indifference* (M.F.I.).

The physical assumption is that the stress is frame-indifferent, i.e.:

$$\boldsymbol{\sigma}_{\boldsymbol{\gamma}_t^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t} = \gamma_t^{\mathrm{ISO}}\!\uparrow\!\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}\,,$$

for any change of EUCLID observer. Frame-indifference of the material metric tensor and Proposition 5.1 assure that stressing and stretching are frame-indifferent too, i.e.

$$\mathcal{L}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t}\,\boldsymbol{\sigma}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}} = \gamma_t^{\mathrm{ISO}}\!\uparrow\!\mathcal{L}_{\boldsymbol{\varphi},t}\,\boldsymbol{\sigma}_{\boldsymbol{\varphi}}\,, \quad \mathcal{L}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}} = \gamma_t^{\mathrm{ISO}}\!\uparrow\!\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\varphi}}\,,$$

so that:

$$\begin{aligned}\gamma_t^{\mathrm{ISO}}\!\uparrow\!\tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\varphi}} = \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}} &= \mathbf{H}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t})\cdot\mathcal{L}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t}\,\boldsymbol{\sigma}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}}\\ &= \mathbf{H}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t}(\gamma_t^{\mathrm{ISO}}\!\uparrow\!\boldsymbol{\sigma}_{\boldsymbol{\varphi},t})\cdot\gamma_t^{\mathrm{ISO}}\!\uparrow\!\mathcal{L}_{\boldsymbol{\varphi},t}\,\boldsymbol{\sigma}_{\boldsymbol{\varphi}}\,.\end{aligned}$$

Hence M.F.I. holds, being expressed by the following condition of invariance under change of EUCLID observer on the hypo-elastic constitutive operator:

$$\mathbf{H}_{\boldsymbol{\gamma}^{\mathrm{ISO}}\uparrow\boldsymbol{\varphi},t} = \gamma_t^{\mathrm{ISO}}\!\uparrow\!\mathbf{H}_{\boldsymbol{\varphi},t}\,.$$

Non covariant stress rates

In literature the following expression is exposed for the *convected stress rate* (Truesdell and Noll, 1965, formula 36.20, p. 97):

$$\overset{\vartriangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{T}\mathbf{L}\,.$$

According to our notation, this formula, when written for the contravariant stress field, should be:

$$\overset{\vartriangle}{\boldsymbol{\sigma}}_{\boldsymbol{\varphi},t} = \dot{\boldsymbol{\sigma}}_{\boldsymbol{\varphi},t} - 2\,\mathrm{sym}\,(\nabla\mathbf{v}_{\boldsymbol{\varphi},t} \circ \boldsymbol{\sigma}_{\boldsymbol{\varphi},t})\,,$$

with ∇ covariant derivative, for instance the one induced by a co-ordinate system and the time-derivative along the motion defined by:

$$\dot{\boldsymbol{\sigma}}_{\boldsymbol{\varphi},t} := \partial_{\tau=t}\,\boldsymbol{\varphi}^{\mathrm{SP}}_{\tau,t}\Downarrow(\boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau} \circ \boldsymbol{\varphi}^{\mathrm{SP}}_{\tau,t})\,.$$

Here the particle is held fixed and the comparison of material stress tensors at two instants along the motion is performed by parallel transport, an operation in contrast with the prescription of the covariance paradigm. Most often, by considering the *spatial description* of the stress field, defined on the body's trajectory:

$$\mathcal{T}(\mathcal{B},\boldsymbol{\varphi}) := \{\,(\mathbf{x}^{\mathrm{SP}}, t) \mid \mathbf{x}^{\mathrm{SP}} = \boldsymbol{\varphi}^{\mathrm{SP}}(\mathrm{p}\,,t)\,, \mathrm{p} \in \mathcal{B},\; t \in I\,\}\,,$$

by: $\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\boldsymbol{\varphi})}(\mathbf{x}^{\mathrm{SP}}, t) := \boldsymbol{\sigma}_{\boldsymbol{\varphi}}(\mathrm{p}, t)\,, \quad \mathbf{x}^{\mathrm{SP}} = \boldsymbol{\varphi}(\mathrm{p}, t)\,,$ the time-derivative along the motion is split into the sum of partial time and spatial derivatives:

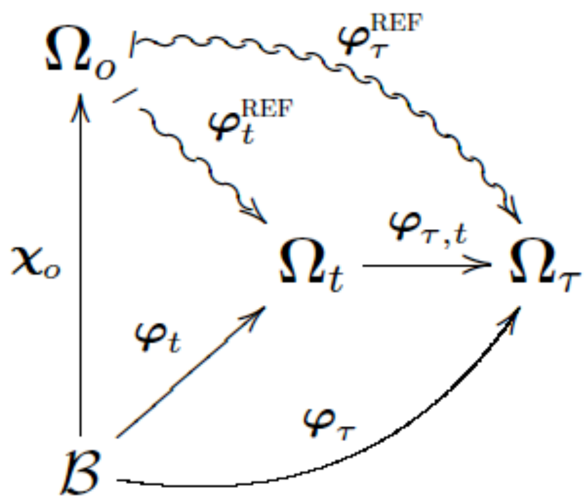
$$\dot{\boldsymbol{\sigma}}_{\mathcal{T}(\mathcal{B},\boldsymbol{\varphi}),t} = \partial_{\tau=t}\,\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\boldsymbol{\varphi}),\tau} + \nabla_{\mathbf{v}_{\boldsymbol{\varphi},t}}\,\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\boldsymbol{\varphi}),t}\,.$$

EVALUATION of the STRESS FIELD

In most computational algorithms, a basic issue concerns the evaluation of the stress field along the motion in terms of the hypo-elastic tangent stiffness at a time $t \in I$:

$$\mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\sigma}_{\boldsymbol{\varphi}} = (\mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}))^{-1} \cdot \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\varphi},t}\, \mathbf{g}_{\boldsymbol{\varphi}} \, .$$

The computation is conveniently carried out in terms of a reference configuration $\boldsymbol{\chi}_o : \mathcal{B} \mapsto \boldsymbol{\Omega}_o$. A schematic view of the relations between the body, a reference configuration and the material configurations along the motion, is provided by the diagram below:



$$\Longleftrightarrow \quad \begin{cases} \boldsymbol{\varphi}_t = \boldsymbol{\varphi}_t^{\mathrm{REF}} \circ \boldsymbol{\chi}_o \\ \boldsymbol{\varphi}_{\tau,t} = \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \\ \boldsymbol{\varphi}_{\tau,t} = \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \circ (\boldsymbol{\varphi}_t^{\mathrm{REF}})^{-1} = \boldsymbol{\varphi}_{\tau,t}^{\mathrm{REF}} \end{cases}$$

By pulling back to a reference configuration $\boldsymbol{\chi}_o : \mathcal{B} \mapsto \boldsymbol{\Omega}_o$ the hypo-elastic law writes:

$$\partial_{\tau=t}\, \tfrac{1}{2}\boldsymbol{\varphi}_{\tau}^{\mathrm{REF}}\!\downarrow\! \tfrac{1}{2}\mathbf{g}_{\boldsymbol{\varphi},\tau} = (\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! \mathbf{H}_{\boldsymbol{\varphi},t})(\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \cdot \partial_{\tau=t}\, \boldsymbol{\varphi}_{\tau}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}\,.$$

Its inverse is given by:

$$\partial_{\tau=t}\, \boldsymbol{\varphi}_{\tau}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau} = ((\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! \mathbf{H}_{\boldsymbol{\varphi},t})(\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},t}))^{-1} \cdot \partial_{\tau=t}\, \tfrac{1}{2}\boldsymbol{\varphi}_{\tau}^{\mathrm{REF}}\!\downarrow\! \tfrac{1}{2}\mathbf{g}_{\boldsymbol{\varphi},\tau}\,.$$

so that:

$$\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! (\mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t}))^{-1} = ((\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! \mathbf{H}_{\boldsymbol{\varphi},t})(\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},t}))^{-1}\,.$$

To evaluate the referential stress increment in a time interval $[s,t]$, the following integral equation should be solved:

$$\boldsymbol{\varphi}_{t}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},t} - \boldsymbol{\varphi}_{s}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},s} = \int_{s}^{t} ((\boldsymbol{\varphi}_{\theta}^{\mathrm{REF}}\!\downarrow\! \mathbf{H}_{\boldsymbol{\varphi},\theta})(\boldsymbol{\varphi}_{\theta}^{\mathrm{REF}}\!\downarrow\! \boldsymbol{\sigma}_{\boldsymbol{\varphi},\theta}))^{-1} \cdot (\tfrac{1}{2}\partial_{\tau=\theta}\, \boldsymbol{\varphi}_{\tau}^{\mathrm{REF}}\!\downarrow\! \mathbf{g}_{\boldsymbol{\varphi},\tau})\, d\theta\,.$$

The strategy adopted by Pinsky et al. (1983), for the numerical integration of the rate constitutive equation, should be mentioned as an iterative algorithm for the solution of the discretized integral equation.

SIMPLEST HYPO-ELASTIC MODEL

The simplest hypo-elastic model, corresponding to the rate form of the standard linear isotropic elasticity model adopted in the small displacements range, has been most widely adopted in computational mechanics, see e.g. (Key and Krieg, 1982). The model was investigated in (Simó and Pister, 1984; Sansour and Bednarczyk, 1993) who, by adopting the incorrect integrability conditions provided in (Bernstein, 1960), found that this hypo-elastic material is not hyper-elastic. On the contrary, on the basis of the covariant theory and of the correct integrability conditions provided above, it will be shown that the simplest hypo-elastic model is indeed hyper-elastic. Denoting the mixed forms of stretching and stressing at time $t \in I$ by:

$$\mathbf{D}_{\boldsymbol{\varphi},t} := \mathbf{g}_{\boldsymbol{\varphi},t}^{-1} \circ \tfrac{1}{2}(\mathcal{L}_{\boldsymbol{\varphi},t}\, \mathbf{g}_{\boldsymbol{\varphi}})\,, \quad \overset{\triangle}{\mathbf{T}}_{\boldsymbol{\varphi},t} := \tfrac{1}{2}(\mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\sigma}_{\boldsymbol{\varphi}}) \circ \mathbf{g}_{\boldsymbol{\varphi},t}\,,$$

the simplest hypo-elastic model is described by the linear, isotropic rate law:

$$\mathbf{D}_{\boldsymbol{\varphi},t} = \frac{1}{2\,\mu}\, \overset{\triangle}{\mathbf{T}}_{\boldsymbol{\varphi},t} - \frac{\nu}{E} J_1(\overset{\triangle}{\mathbf{T}}_{\boldsymbol{\varphi},t})\, \mathbf{I}_{\boldsymbol{\varphi},t} = \mathbf{H}^{\mathrm{MIX}}_{\boldsymbol{\varphi},t}(\mathbf{T}_{\boldsymbol{\varphi},t}) \cdot \overset{\triangle}{\mathbf{T}}_{\boldsymbol{\varphi},t}\,,$$

so that the hypo-elastic constitutive operator is given by:

$$\mathbf{H}^{\mathrm{MIX}}_{\boldsymbol{\varphi},t}(\mathbf{T}_{\boldsymbol{\varphi},t}) := \frac{1}{2\,\mu}\, \mathbb{I}_{\boldsymbol{\varphi},t} - \frac{\nu}{E}\, \mathbf{I}_{\boldsymbol{\varphi},t} \otimes \mathbf{I}_{\boldsymbol{\varphi},t}\,,$$

with E EULER (or YOUNG) modulus, ν POISSON ratio and $\mu = \frac{E}{2\,(1+\nu)}$ LAMÉ shear modulus. Here $\mathbf{I}_{\boldsymbol{\varphi},t}(\mathbf{x}) \in \mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t)$ is the identity tensor, $\otimes$ is the tensor product in the inner product tensor space $\mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t)$ and $\mathbb{I}_{\boldsymbol{\varphi},t}(\mathbf{x}) \in L\,(\mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t)\,; \mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t))$ is the identity operator.

Time-independency of the simplest hypo-elastic constitutive operator, is expressed by the equality:

$$\frac{1}{2\,\mu}\,\mathbb{I}_{\boldsymbol{\varphi},\tau} - \frac{\nu}{E}\,\mathbf{I}_{\boldsymbol{\varphi},\tau} \otimes \mathbf{I}_{\boldsymbol{\varphi},\tau} = \boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\Big(\frac{1}{2\,\mu}\,\mathbb{I}_{\boldsymbol{\varphi},t} - \frac{\nu}{E}\,\mathbf{I}_{\boldsymbol{\varphi},t} \otimes \mathbf{I}_{\boldsymbol{\varphi},t}\Big)\,,$$

which is inferred from the formulas: $\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbb{I}_{\boldsymbol{\varphi},t} = \mathbb{I}_{\boldsymbol{\varphi},\tau}$, $\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{I}_{\boldsymbol{\varphi},t} = \mathbf{I}_{\boldsymbol{\varphi},\tau}$ and:

$$\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!(\mathbf{I}_{\boldsymbol{\varphi},t} \otimes \mathbf{I}_{\boldsymbol{\varphi},t}) = (\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{I}_{\boldsymbol{\varphi},t}) \otimes (\boldsymbol{\varphi}_{\tau,t}\!\uparrow\!\mathbf{I}_{\boldsymbol{\varphi},t})\,.$$

Let us now assume that $\boldsymbol{\varphi}_t : \mathcal{B} \mapsto \boldsymbol{\Omega}_t$ is a natural, stress-free reference configuration. The hyper-elastic law may then be written in terms of the mixed GREEN's strain tensor, as:

$$\mathbf{E}_{\boldsymbol{\varphi}_{\tau,t}} = \boldsymbol{\Phi}^{\mathrm{MIX}}_{\boldsymbol{\varphi},t}(\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\boldsymbol{\varphi},\tau}) = \frac{1}{2\,\mu}\,\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\boldsymbol{\varphi},\tau} - \frac{\nu}{E} J_1(\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\boldsymbol{\varphi},\tau})\,\mathbf{I}_{\boldsymbol{\varphi},t}\,,$$

or, in inverse form:

$$\frac{1}{2\,\mu}\,\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\boldsymbol{\varphi},\tau} = \mathbf{E}_{\boldsymbol{\varphi}_{\tau,t}} + \frac{\nu}{1-2\,\nu}\,J_1(\mathbf{E}_{\boldsymbol{\varphi}_{\tau,t}})\,\mathbf{I}_{\boldsymbol{\varphi},t}\,.$$

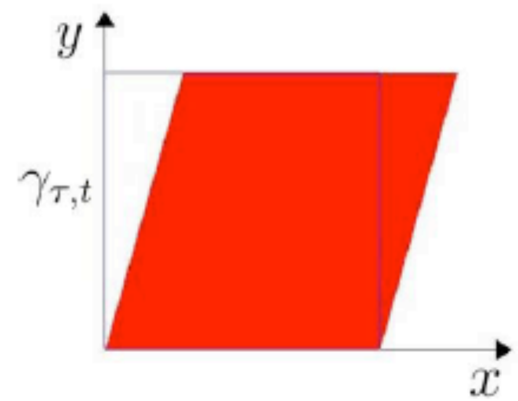
The CAUCHY true stress $\mathbf{T}_{\boldsymbol{\varphi},\tau} \in \mathrm{C}^1(\boldsymbol{\Omega}_\tau\,;\mathrm{MIX}(\boldsymbol{\Omega}_\tau))$ is recovered from the reference one $\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\boldsymbol{\varphi},\tau} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{MIX}(\boldsymbol{\Omega}_t))$ by push forward:

$$\mathbf{T}_{\boldsymbol{\varphi},\tau} = T\boldsymbol{\varphi}_{\tau,t} \circ (\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\boldsymbol{\varphi},\tau}) \circ T\boldsymbol{\varphi}_{\tau,t}^{-1}\,.$$

Simple shear

Let us consider a unit cube as a natural stress-free configuration of a body and a cartesian reference system. A simple shear, see fig. 3, is described by a material displacement whose expression in the reference system, setting $\gamma_{\tau,t} := \gamma\,(\tau - t)$, is given by:

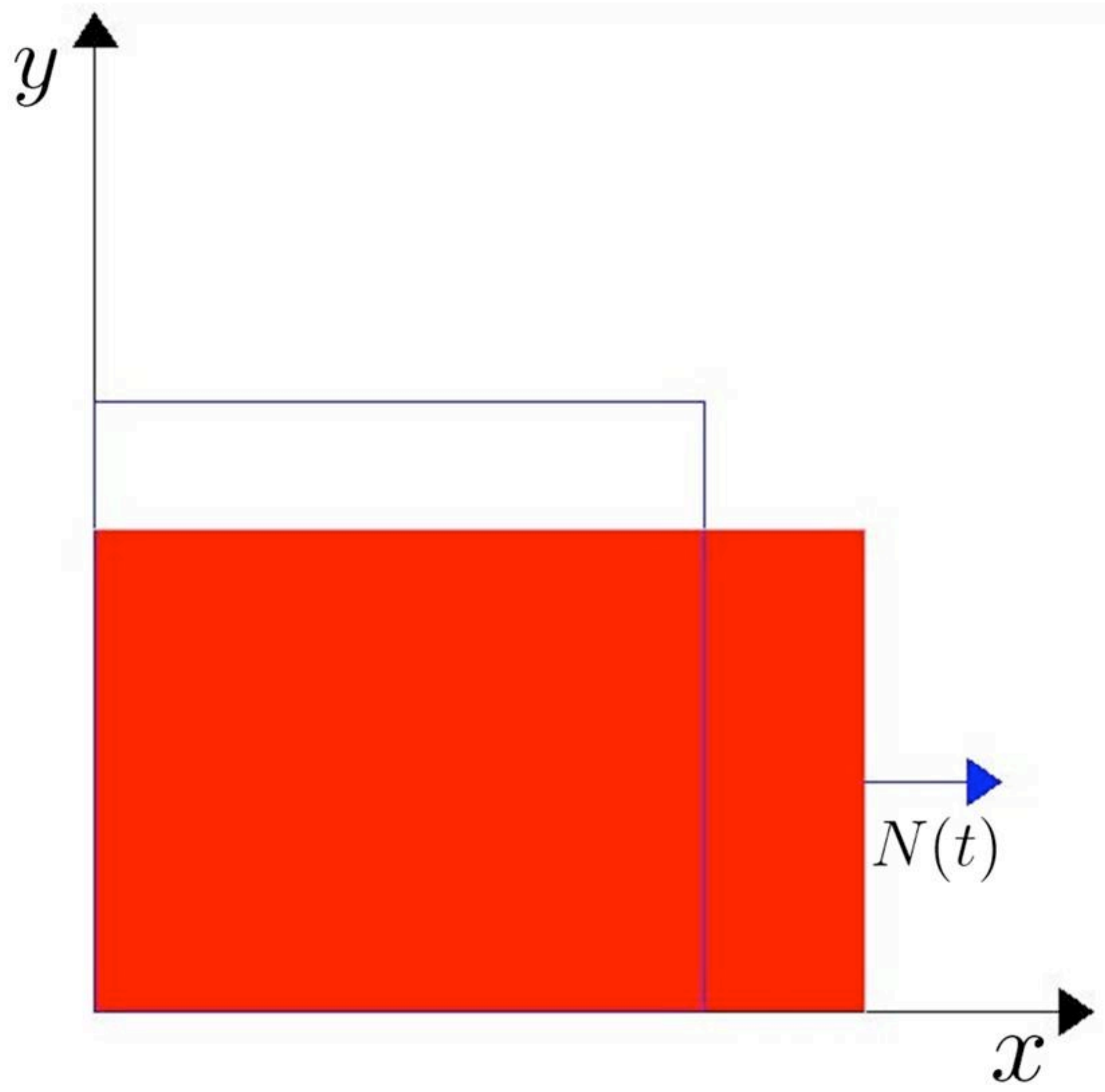
$$\boldsymbol{\varphi}_{\tau,t}(x, y, z) = (x + \gamma_{\tau,t}\, y)\, \mathbf{e}_1 + y\, \mathbf{e}_2 + z\, \mathbf{e}_3\,.$$



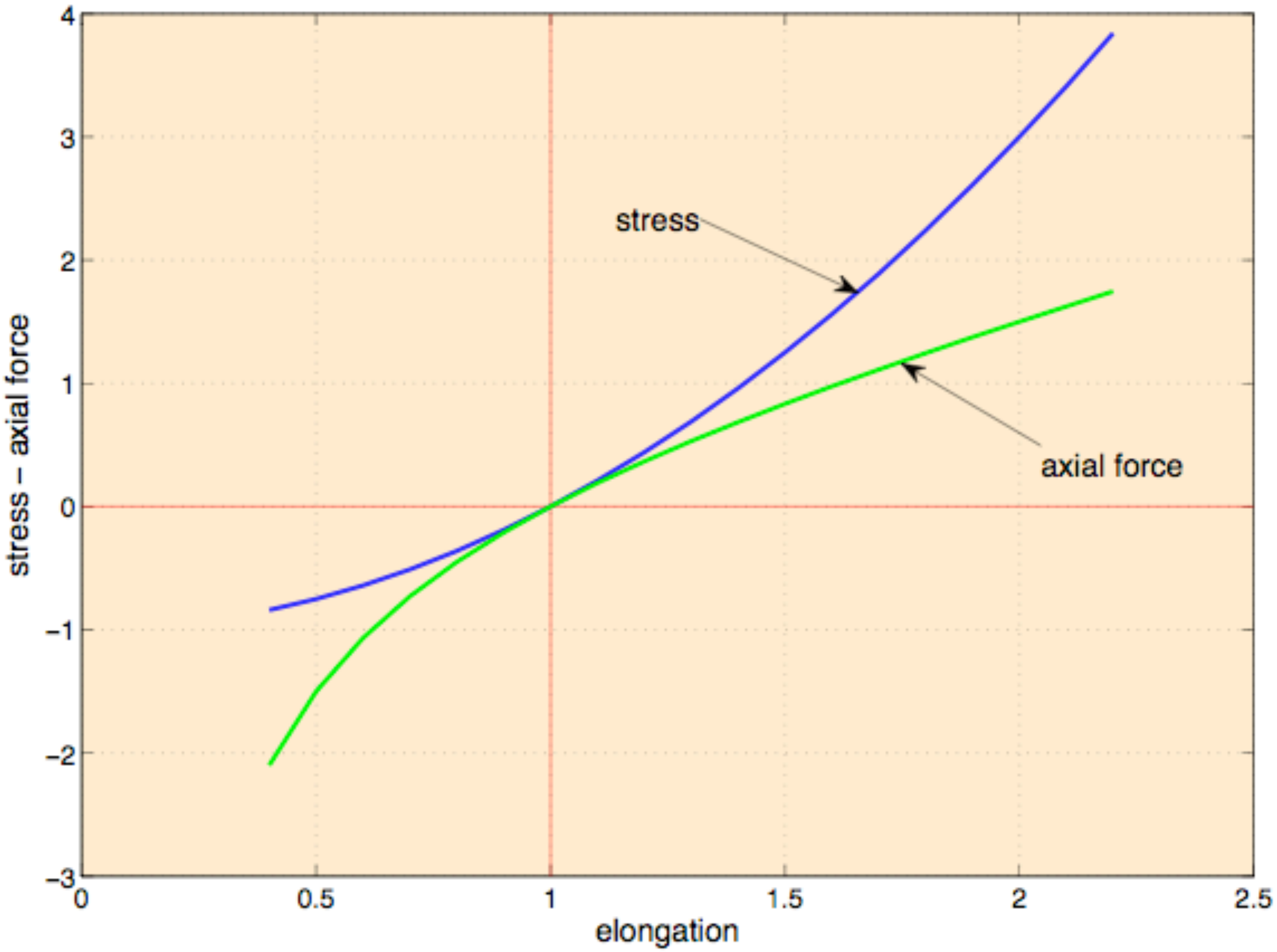
Homogeneous extension

A homogeneous extension is got by a one parameter displacement of a unitary cube:

$$\varphi_t(x,y,z) = \alpha\, t\, x\, \mathbf{e_x} + \beta\, t\, y\, \mathbf{e_y} + \ z\, \mathbf{e_z}$$



Assuming $\nu = 0$ and $\beta\, t = (\alpha\, t)^{-1}$, which corresponds to a vanishing Poisson effect and to an isochoric displacement, the normal stress $\mathbf{T}_{11}(t)$ and the resultant axial force $N(t) = A(t)\, \mathbf{T}_{11}(t) = \mu\, (\alpha\, t - 1/(\alpha\, t))$, where $A(t) = 1/(\alpha\, t)$ is the transversal area, are plotted in



Concluding remarks

- The property of covariance is formulated as variance by push instead of invariance under push.

- The principle of material frame indifference is accordingly correctly reformulated and shown to be trivially satisfied by any (covariant) rate material response.

- Spurious results, such as that material frame indifference should imply isotropy of the hypo-elastic response and of plastic yield functions, are eliminated. Accordingly, treatments devoted to recover a description of anisotropic behaviors of elastic and plastic responses should be reconsidered.

- Homogeneity and isotropy of the material are properly defined and shown to be consistent with the covariant transformation of the material response at different configurations.

- Formulations in terms of different alterations of the relevant tensors and push to other configurations may be interchanged without affecting the result, thus restoring a sound physical basis to the constitutive theory.

- The integration needed for the evaluation of the stress may be performed on the time dependent pull-backs of the stressing to a fixed reference configuration, the result being got by a subsequent push-forward to the actual configuration, in a way independent of the chosen configuration.

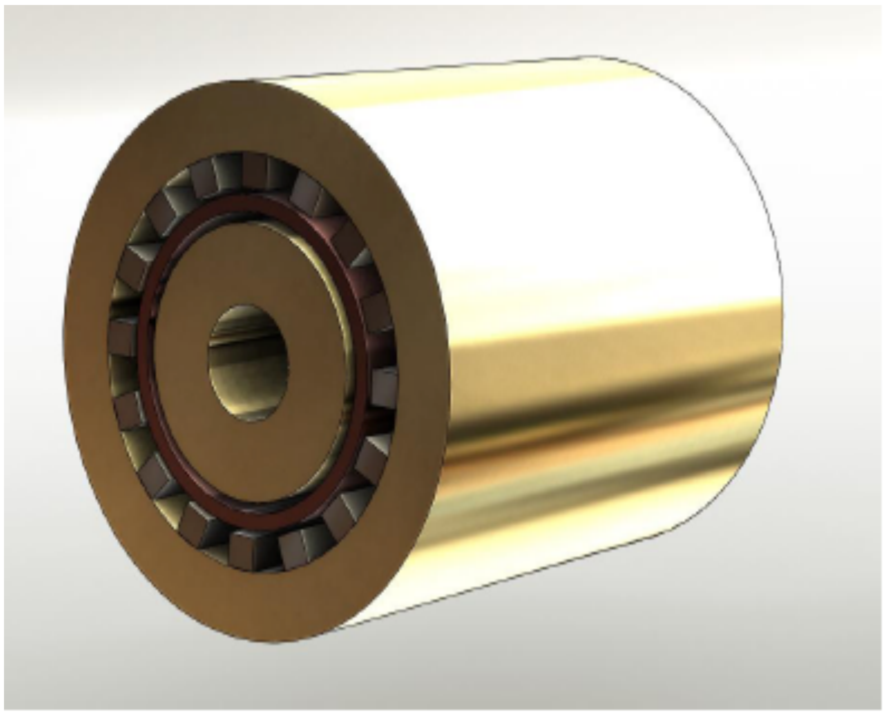
- The integrability conditions of the hypo-elastic behaviour may be checked at any fixed reference configuration and the relevant potentials may be readily computed, still in a way independent of the chosen reference configuration.

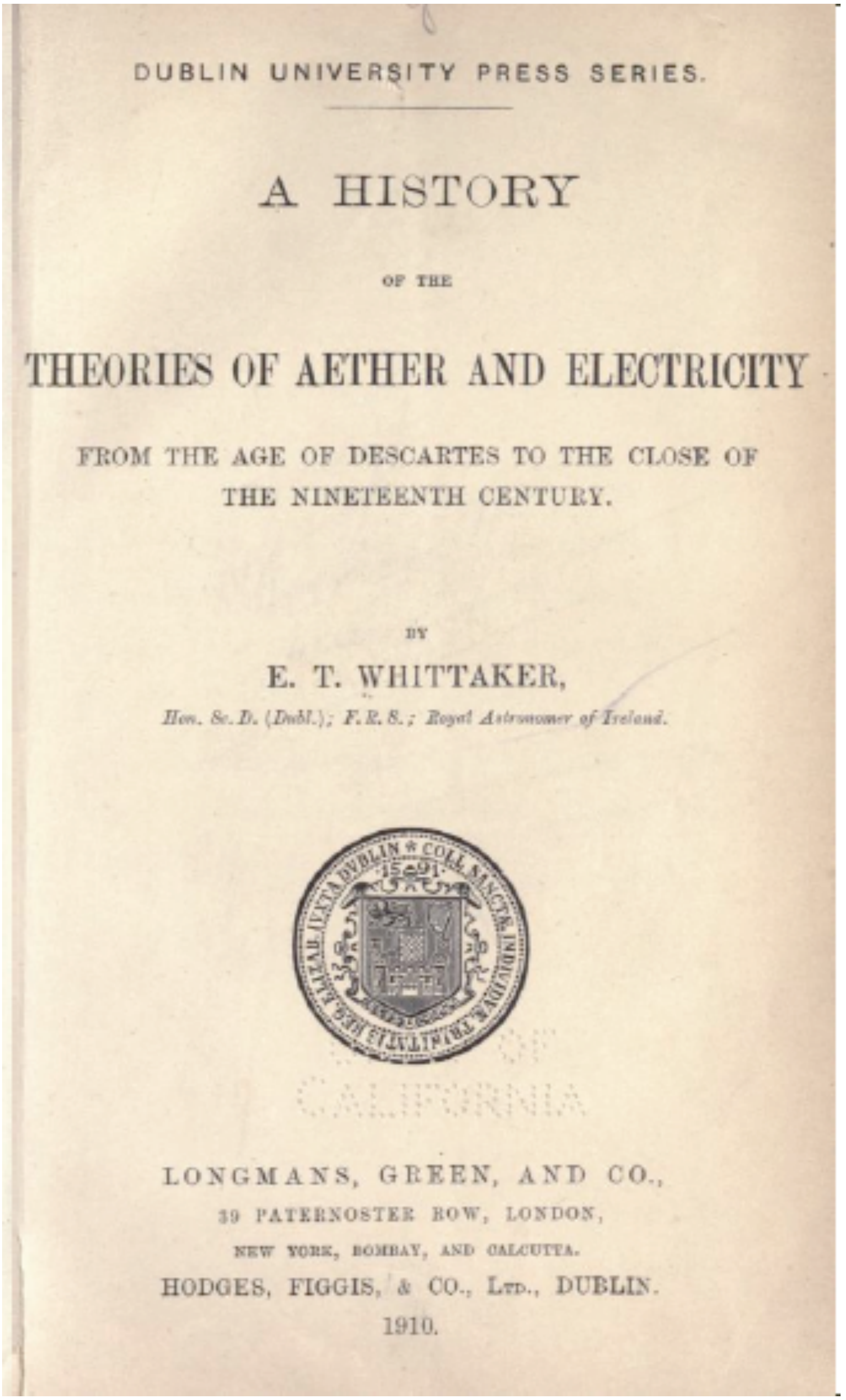
By these basic implications of the covariance paradigm most existing theoretical and computational approaches should be reviewed.

PART II

The Covariance Paradigm

in electromagnetic induction





DUBLIN UNIVERSITY PRESS SERIES.

A HISTORY

OF THE

THEORIES OF AETHER AND ELECTRICITY

FROM THE AGE OF DESCARTES TO THE CLOSE OF THE NINETEENTH CENTURY.

BY

E. T. WHITTAKER,

Hon. Sc.D. (Dubl.); F.R.S.; Royal Astronomer of Ireland.

LONGMANS, GREEN, AND CO.,

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HODGES, FIGGIS, & CO., LTD., DUBLIN.

1910.

Edmund Whittaker

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Classical Formulation of the laws of electromagnetism as introduced in most modern textbooks

Hans Christian Ørsted (1820)



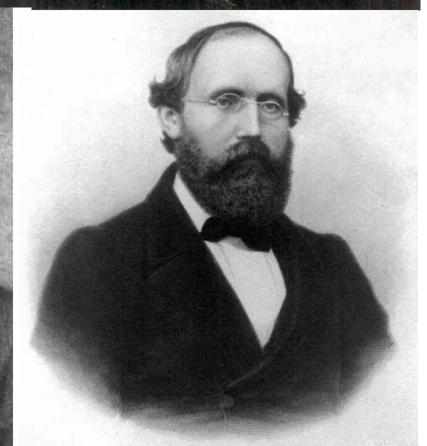
$$\oint_{\partial \boldsymbol{\Sigma}_t} \mathbf{g} \cdot \mathbf{H} = \int_{\boldsymbol{\Sigma}_t} \boldsymbol{\mu} \cdot (\partial_{\tau=t} \mathbf{D}_\tau + \mathbf{J}_\mathbf{E})$$ Maxwell(1861)-Ampère(1826)

$$\oint_{\partial \boldsymbol{\Sigma}_t} \mathbf{g} \cdot \mathbf{E} = - \int_{\boldsymbol{\Sigma}_t} \boldsymbol{\mu} \cdot (\partial_{\tau=t} \mathbf{B}_\tau)$$ Henry-Faraday(1831)

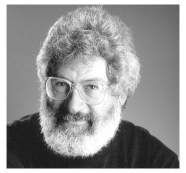
$$\oint_{\partial \boldsymbol{\Omega}} \boldsymbol{\mu} \cdot \mathbf{D} = \int_{\boldsymbol{\Omega}} \rho_\mathbf{E} \, \boldsymbol{\mu}$$ Gauss(1835)

$$\oint_{\partial \boldsymbol{\Omega}} \boldsymbol{\mu} \cdot \mathbf{B} = 0$$ Gauss(1831)



Bernhard Riemann (1858)



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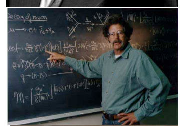

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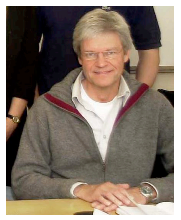


Thidé, B., 2010. Electromagnetic Field Theory (Second ed.). http://www.plasma.uu.se/CED/Book.

9.3 Electromagnetism

Classical electromagnetism is governed by Maxwell's field equations. The form of these equations depends on the physical units chosen, and changing these units introduces factors like 4π, $c =$ the speed of light, $\epsilon_0 =$ the dielectric constant and $\mu_0 =$ the magnetic permeability. The discussion in this section assumes that ϵ_0, μ_0 are constant; the choice of units is such that the equations take the simplest form; thus $c = \epsilon_0 = \mu_0 = 1$ and factors 4π disappear. We also do not consider Maxwell's equations in a material, where one has to distinguish **E** from **D**, and **B** from **H**.

Let **E**, **B**, and **J** be time dependent C^1-vector fields on $\mathbb{R}^3$ and $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ a scalar. These are said to satisfy ***Maxwell's equations*** with ***charge density*** ρ and ***current density*** **J** when the following hold:

$$\operatorname{div} \mathbf{E} = \rho \quad (\textit{Gauss's law}) \tag{9.3.1}$$

$$\operatorname{div} \mathbf{B} = 0 \quad (\textit{no magnetic sources}) \tag{9.3.2}$$

$$\operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \quad (\textit{Faraday's law of induction}) \tag{9.3.3}$$

$$\operatorname{curl} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (\textit{Ampère's law}) \tag{9.3.4}$$

E is called the ***electric field*** and **B** the ***magnetic field***.

By Stokes' theorem, equation (9.3.3) is equivalent to

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{ds} = \int_S (\operatorname{curl} \mathbf{E}) \cdot \mathbf{n}\, dS = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n}\, dS \tag{9.3.7}$$

for any closed loop ∂S bounding a surface S. The quantity $\int_{\partial S} \mathbf{E} \cdot \mathbf{ds}$ is called the ***voltage*** around ∂S. Thus, Faraday's law of induction equation (9.3.3), says that *the voltage around a loop equals the negative of the rate of change of the magnetic flux through the loop.*

Abraham, R., Marsden, J.E., Ratiu, T., 1988. Manifolds, Tensor Analysis, and Applications, second ed. (third ed. 2002) Springer Verlag, New York.

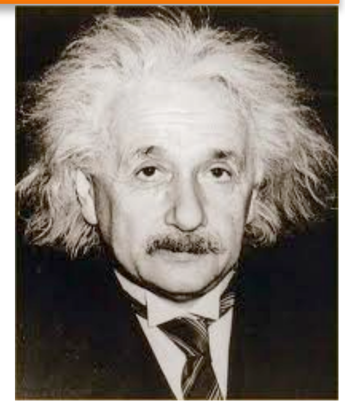
TABLE 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law

*This is also referred to as Gauss's law for magnetic fields.

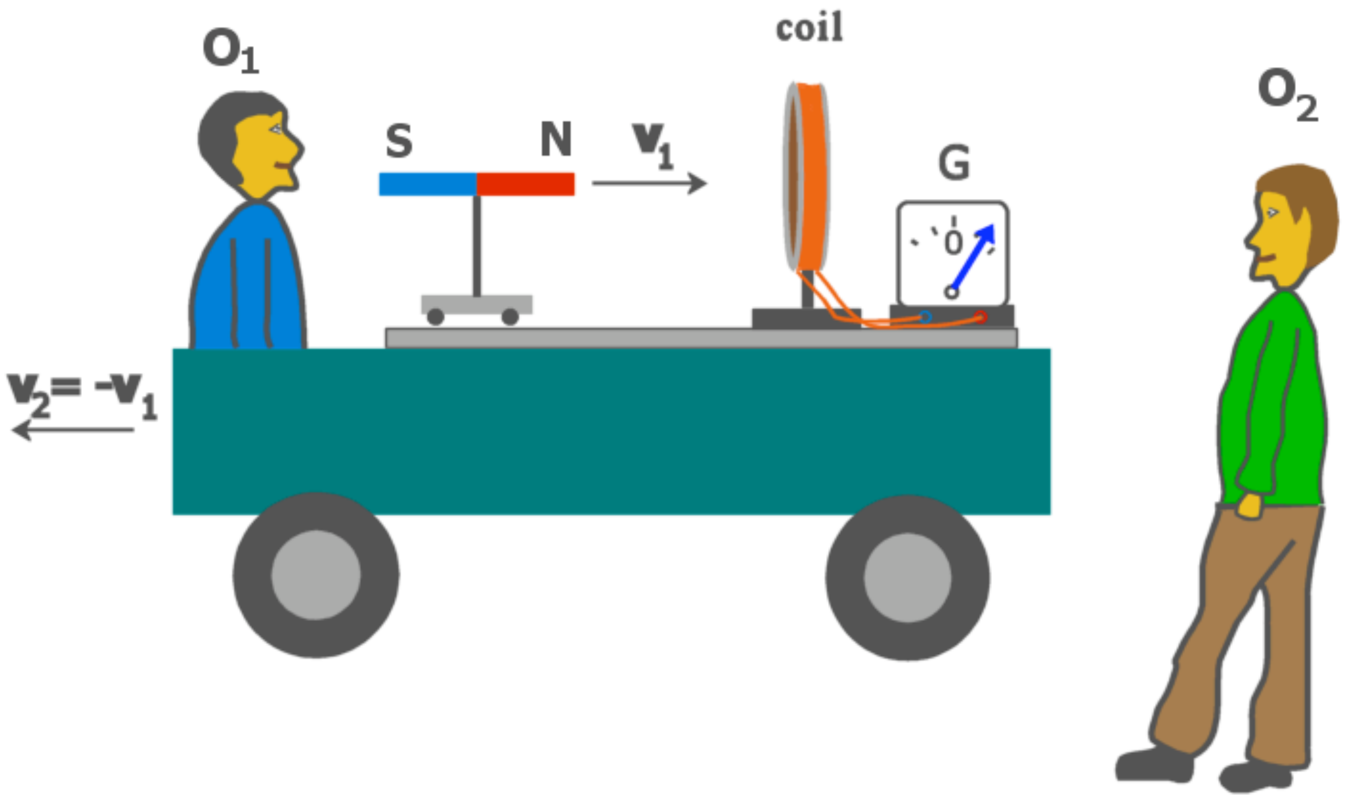
Sadiku, M.N.O., 2010. Elements of Electromagnetics (Fifth ed.). Oxford University Press. USA. ISBN-13: 9780195387759

Electrodynamics



Moving truck

Faraday law



Lorentz force

According to the most formulation of electrodynamics, the man in **blue sweater** explains the turning of the galvanometer needle by the **Faraday law** of magnetic induction, while the **green fellow** explains the same phenomenon by resorting to the **Lorentz force**.

Albert Einstein 1905

ON THE ELECTRODYNAMICS OF MOVING BODIES

By A. EINSTEIN

June 30, 1905

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Jackson, D.J. Classical Electrodynamics (1999)

When Einstein began to think about these matters there existed several possibilities:

1. The Maxwell equations were incorrect.
   The proper theory of electromagnetism was invariant under Galilean transformations.

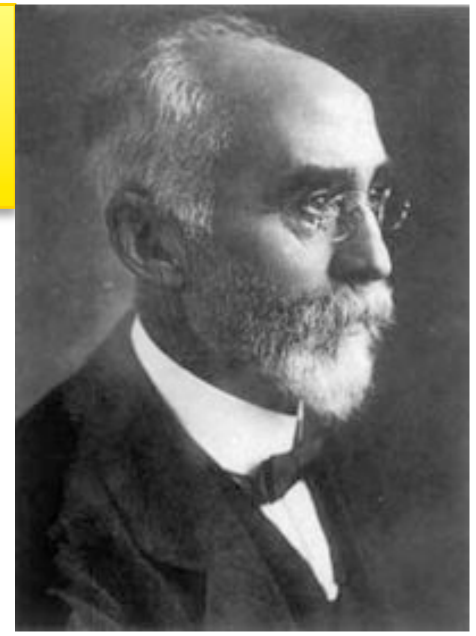
2. Galilean relativity applied to classical mechanics, but electromagnetism had a preferred reference frame, the frame in which the luminiferous ether was at rest.

3. There existed a relativity principle for classical mechanics and electromagnetism, but it was not Galilean relativity. This would imply that the laws of mechanics were in need of modification.

The first possibility was hardly viable.

Lorentz force (1892)

Richard Phillips Feynman



Hendrik Antoon Lorentz

FEYNMAN:

We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication.

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathrm{rot}\, \mathbf{E}_t = -\partial_{\tau=t}\, \mathbf{B}_\tau$$

When we said that the magnetic force on a charge was proportional to its velocity, you may have wondered: "What velocity? With respect to which reference frame?" It is,in fact, clear from the definition of B given at the beginning of this chapter that what this vector is will depend on what we choose as a reference frame for our specification of the velocity of charges. But we have said nothing about which is the proper frame for specifying the magnetic field.

Benjamin Crowell, 2010.

Electricity and Magnetism

Book 4 in the Light and Matter series

Experiments show that the magnetic force on a moving charged particle has a magnitude given by

$$|F| = q|v||B| \sin \theta ,$$

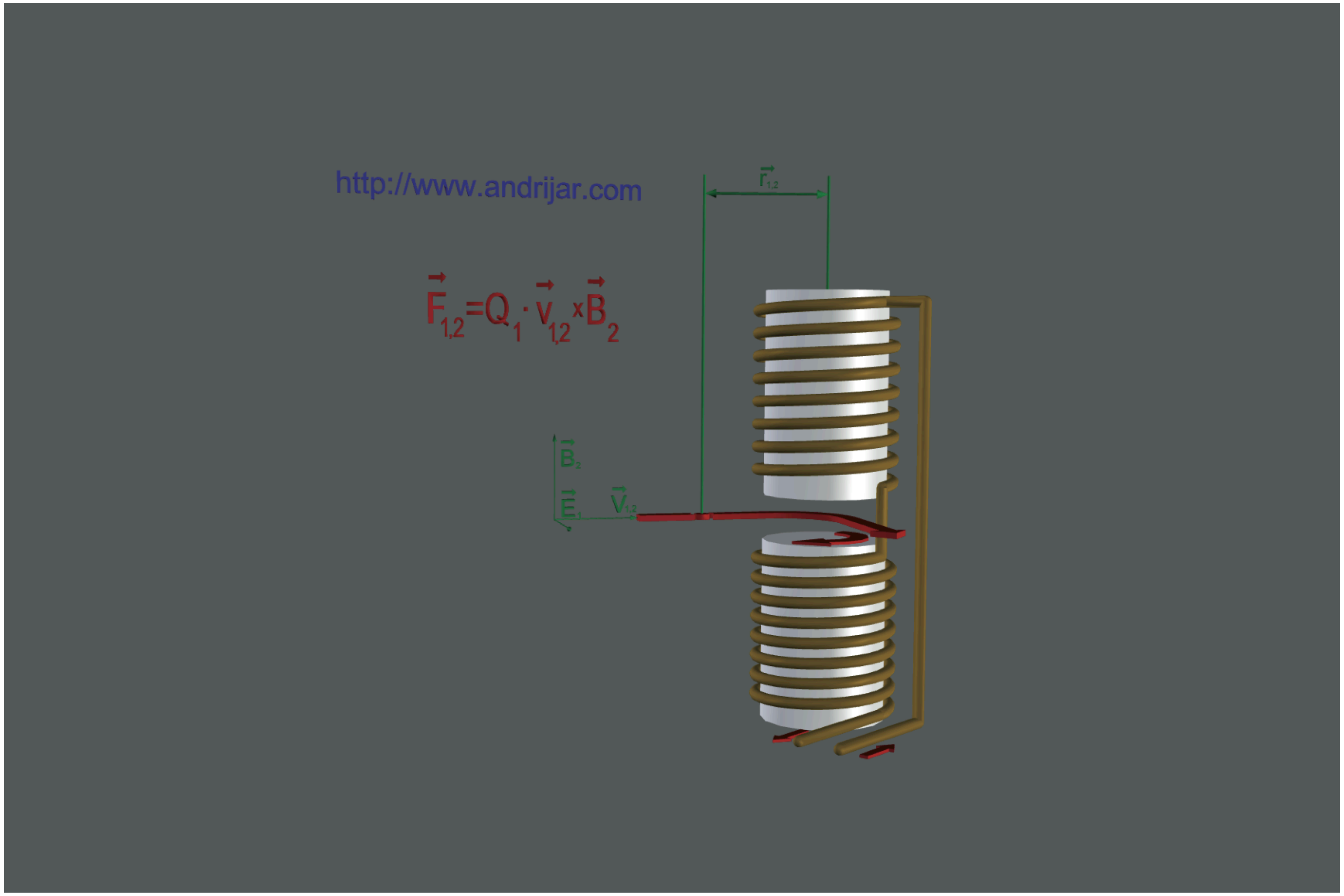
where

v is the velocity vector of the particle, and

θ is the angle between the v and B vectors.

Unlike electric and gravitational forces, magnetic forces do not lie along the same line as the field vector.

Lorentz force



In the introduction and survey of (Jackson, 1999, p.3) it is said: *Also essential for consideration of charged particle motion is the Lorentz force equation,* $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ *, which gives the force acting on a point charge* q *in the presence of electromagnetic fields.* In dealing with FARADAY's law of induction, in (Jackson, 1999, p.210) it is further said: *It is important to note, however, that the electric field* $\mathbf{E}'$ *is the electric field at* $d\boldsymbol{l}$ (an infinitesimal piece of circuit) *in the coordinate system or medium in which* $d\boldsymbol{l}$ *is at rest, since is that field that causes current to flow if a circuit is actually present.* And a little bit later (Jackson, 1999, p.211) the following formula is claimed: $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ where $\mathbf{E}$ is *the electric field in the laboratory* and $\mathbf{E}'$ is *the electric field at* $d\boldsymbol{l}$ *in its rest frame of coordinates.*

In (Sadiku, 2010, chapter 9.5) it is said that: *it is worthwhile to mention other equations that go hand in hand with Maxwell's equations. The* LORENTZ *force equation* $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ *is associated with Maxwell's equations. Also the equation of continuity is implicit in Maxwell's equations.* No mention is made of the way the observer measuring the velocity is to be selected, in writing the LORENTZ force equation.

Lorentz Force

A force on moving charges particles in the presence of magnetic **B** and electric fields **E**. The electromagnetic field tensor in MKS is given by

$$E'_x = \gamma(E_x - vB_y) \quad (1)$$

$$B'_x = \gamma\left(B_x + \frac{v}{c^2}E_y\right) \quad (2)$$

$$E'_y = \gamma(E_y + vB_x) \quad (3)$$

$$B'_y = \gamma\left(B_y - \frac{v}{c^2}E_x\right) \quad (4)$$

$$E'_z = E_z \quad (5)$$

$$B'_z = B_z, \quad (6)$$

so the Lorenz force is

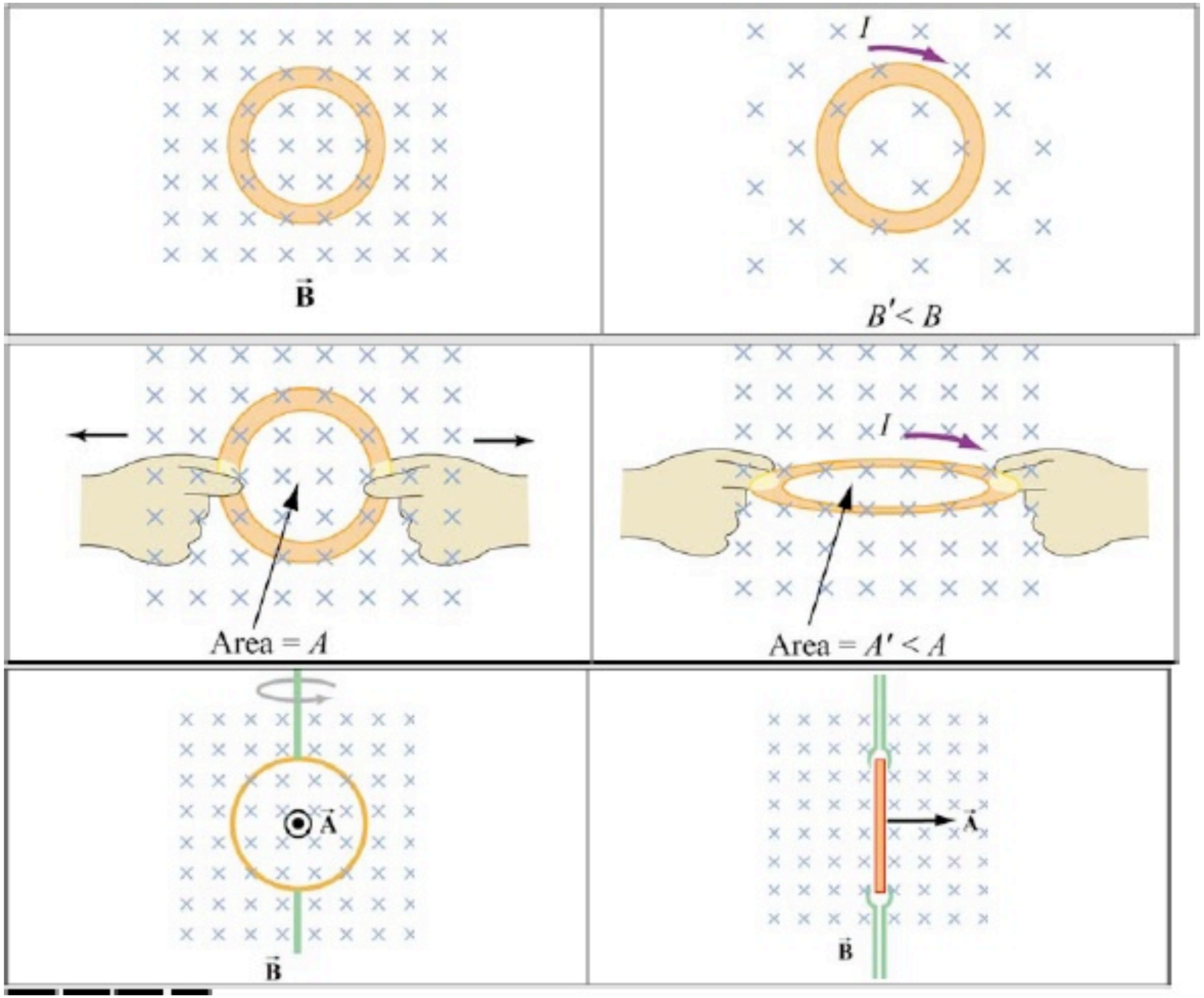
$$\mathbf{F} \equiv q\mathbf{E}' = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (7)$$

$$= q\left[-\nabla\phi - \frac{d\mathbf{A}}{dt} + \nabla(\mathbf{A}\cdot\mathbf{v})\right], \quad (8)$$

where q is the charge on a particle, **v** is its velocity, ϕ is the electric potential, and **A** is the magnetic vector potential.

SEE ALSO: Canonical Momentum, Electric Force, Magnetic Field, Magnetic Force

Faraday law of induction: examples



Geometric Formulation of the laws of electromagnetism

Faraday Law for a moving body

$$d\boldsymbol{\omega}^2_{\mathbf{B}} = 0 \qquad \text{Gauss}(1831)$$

$$-\oint_{\partial\boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\boldsymbol{\Sigma}_t)} \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{B}}$$

Maxwell-Ampère Law for a moving body

$$d\boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\rho}_{\mathbf{E}} \qquad \text{Gauss}(1835)$$

$$\oint_{\partial\boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{H}} = \partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\boldsymbol{\Sigma}_t)} \boldsymbol{\omega}^2_{\mathbf{D}} + \int_{\boldsymbol{\Sigma}_t} \boldsymbol{\omega}^2_{\mathbf{J}_{\mathbf{E}}} = \int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}_{\mathbf{E}}}$$

Romano, G.: *The laws of Electromagnetism for moving bodies and related questions (2010)*

Well-posedness of Faraday law

For any control-window $\mathbf{C}_t$:

$$\int_{\partial \mathbf{C}_t} \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\mathbf{C}_t} d(\mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}}) = \int_{\mathbf{C}_t} \mathcal{L}_{\varphi,t}\, (d\boldsymbol{\omega}^2_{\mathbf{B}}) = 0\,.$$

Well-posedness of Maxwell-Ampère law

$$\oint_{\partial \mathbf{C}_t} (\mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}}) = \int_{\mathbf{C}_t} d(\mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}}) = \int_{\mathbf{C}_t} (\mathcal{L}_{\varphi,t}\, d\boldsymbol{\omega}^2_{\mathbf{D}} + d\boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}}) = 0\,,$$

equivalent to the property of *electric charge conservation*:

$$\mathcal{L}_{\varphi,t}\, \boldsymbol{\rho}_{\mathbf{E}} + d\boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0\,,$$

or in the equivalent integral form:

$$\partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\mathbf{C}_t)} \boldsymbol{\rho}_{\mathbf{E}} + \oint_{\partial \mathbf{C}_t} \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0\,.$$

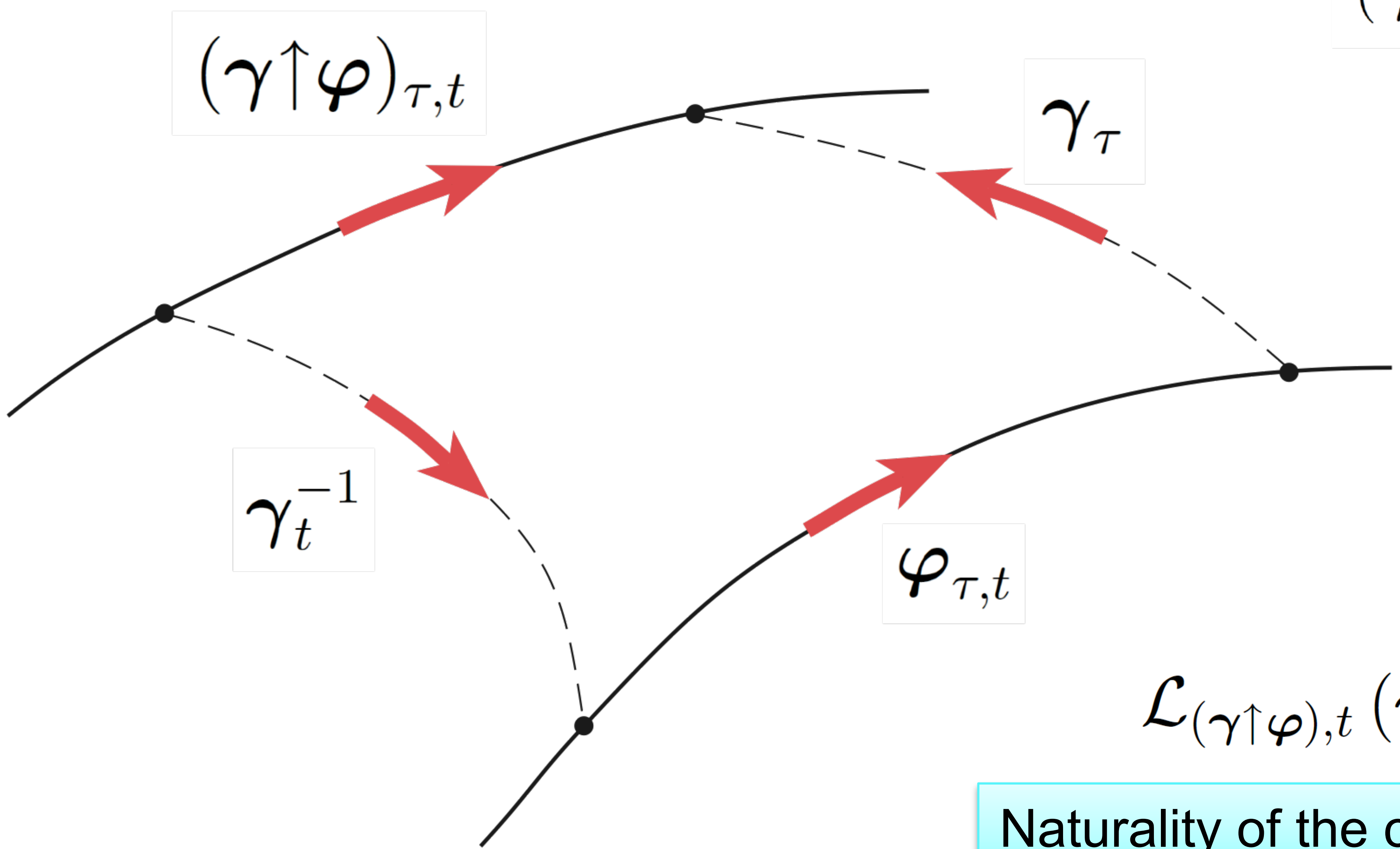
Romano, G.: *The laws of Electromagnetism for moving bodies and related questions (2010)*

Relative motions

Push by a time-dependent diffeomorphism

$$\varphi_{\tau,t} := \varphi_\tau \circ \varphi_t^{-1}$$

$$(\gamma\uparrow\varphi)_{\tau,t} = \gamma_\tau \circ \varphi_{\tau,t} \circ \gamma_t^{-1}$$



$$\mathcal{L}_{(\gamma\uparrow\varphi),t}\,(\gamma\uparrow\boldsymbol{\alpha}_\varphi) = \gamma_t\uparrow\mathcal{L}_{\varphi,t}\,\boldsymbol{\alpha}_\varphi$$

Naturality of the convective time-derivative

Covariance of electromagnetic induction laws

$$-\oint_{\partial\mathbf{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\mathbf{\Sigma}_t)} \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\mathbf{\Sigma}_t} \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{B}}$$

$$\oint_{\partial\mathbf{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{H}} = \partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\mathbf{\Sigma}_t)} \boldsymbol{\omega}^2_{\mathbf{D}} + \int_{\mathbf{\Sigma}_t} \boldsymbol{\omega}^2_{\mathbf{J}_{\mathbf{E}}} = \int_{\mathbf{\Sigma}_t} \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}_{\mathbf{E}}}$$

$$d\boldsymbol{\omega}^1_{\mathbf{E}} = -\mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{B}}$$ Faraday law

$$d\boldsymbol{\omega}^1_{\mathbf{H}} = \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}_{\mathbf{E}}}$$ Maxwell-Ampère law

Naturality of the convective time-derivative
with respect to push:

$$\mathcal{L}_{(\hat{\gamma}\uparrow\hat{\varphi}),t}\,(\hat{\gamma}\uparrow\boldsymbol{\alpha}_{\varphi}) = \hat{\gamma}_t\uparrow\mathcal{L}_{\hat{\varphi},t}\,\boldsymbol{\alpha}_{\varphi}$$

and the commutation property
between push and exterior derivative:

$$\boldsymbol{\gamma}_t\uparrow(d\boldsymbol{\alpha}_{\varphi,t}) = d(\boldsymbol{\gamma}_t\uparrow\boldsymbol{\alpha}_{\varphi,t})$$

imply covariance of the induction laws.

Lack of covariance of the standard induction laws is due to
lack of naturality of the partial time-derivative with respect to push:

$$\partial_{\tau=t}\,(\hat{\gamma}\uparrow\hat{\boldsymbol{\alpha}}_{\varphi})_{\tau} = \hat{\gamma}_t\uparrow(\partial_{\tau=t}\,\hat{\boldsymbol{\alpha}}_{\varphi,\tau}) - \mathcal{L}_{\mathbf{v}_{\hat{\gamma}},t}\,(\hat{\gamma}\uparrow\hat{\boldsymbol{\alpha}})_{\varphi}$$

Covariance of the charge conservation law

$$\partial_{\tau=t} \int_{\mathbf{\Omega}_t} \boldsymbol{\rho}_{\mathbf{E}} + \oint_{\partial\mathbf{\Omega}_t} \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0 \quad \Longleftrightarrow \quad \partial_{\tau=t}\, \boldsymbol{\rho}_{\mathbf{E},\tau} + d\boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0$$

old

$$\partial_{\tau=t} \int_{\boldsymbol{\varphi}_{\tau,t}(\mathbf{\Omega}_t)} \boldsymbol{\rho}_{\mathbf{E}} + \oint_{\partial\mathbf{\Omega}_t} \boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0 \quad \Longleftrightarrow \quad \mathcal{L}_{\boldsymbol{\varphi},t}\, \boldsymbol{\rho}_{\mathbf{E}} + d\boldsymbol{\omega}^2_{\mathbf{J}_\mathbf{E}} = 0$$

new

Naturality of the convective time-derivative with respect to push:

$$\mathcal{L}_{(\hat{\boldsymbol{\gamma}}\uparrow\hat{\boldsymbol{\varphi}}),t}\,(\hat{\boldsymbol{\gamma}}\uparrow\boldsymbol{\alpha}_{\boldsymbol{\varphi}}) = \hat{\boldsymbol{\gamma}}_t\uparrow\mathcal{L}_{\hat{\boldsymbol{\varphi}},t}\,\boldsymbol{\alpha}_{\boldsymbol{\varphi}}$$

and the commutation property between push and exterior derivative:

$$\boldsymbol{\gamma}_t\uparrow(d\boldsymbol{\alpha}_{\boldsymbol{\varphi},t}) = d(\boldsymbol{\gamma}_t\uparrow\boldsymbol{\alpha}_{\boldsymbol{\varphi},t})$$

imply covariance of the charge conservation law for moving bodies.

Covariance of electromagnetic induction laws under relative motion

$$\oint_{\partial \boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{E}} = \oint_{\partial \hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \hat{\gamma}_t \!\uparrow\! \boldsymbol{\omega}^1_{\mathbf{E}}$$

$$\int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \hat{\gamma}_t \!\uparrow\! \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \mathcal{L}_{(\hat{\gamma}\uparrow\hat{\varphi}),t}\, (\hat{\gamma} \!\uparrow\! \boldsymbol{\omega}^2_{\mathbf{B}})\,.$$

Invariance of electric field and magnetic flux

$$\gamma_t \!\uparrow\! \boldsymbol{\omega}^1_{\mathbf{E},t} = \boldsymbol{\omega}^1_{\mathbf{E},t}$$

$$\gamma_t \!\uparrow\! \boldsymbol{\omega}^2_{\mathbf{B},t} = \boldsymbol{\omega}^2_{\mathbf{B},t}$$

implies invariance of Faraday law

$$\oint_{\partial \boldsymbol{\Sigma}_t} \boldsymbol{\omega}^1_{\mathbf{E}} = -\int_{\boldsymbol{\Sigma}_t} \mathcal{L}_{\varphi,t}\, \boldsymbol{\omega}^2_{\mathbf{B}} \quad \Longleftrightarrow$$

$$\oint_{\partial \hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \boldsymbol{\omega}^1_{\mathbf{E}} = -\int_{\hat{\gamma}_t(\boldsymbol{\Sigma}_t)} \mathcal{L}_{(\hat{\gamma}\uparrow\hat{\varphi}),t}\, \boldsymbol{\omega}^2_{\mathbf{B}}\,.$$

Romano, G.: *The laws of Electromagnetism for moving bodies and related questions (2010)*

Differential formulation of Faraday law

$$d\boldsymbol{\omega}^1_{\mathbf{E}} = -\mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^2_{\mathbf{B}}$$

$$\text{rot}\,\mathbf{E}_t = -\partial_{\tau=t}\,\mathbf{B}_\tau + \text{rot}\,(\mathbf{v}_{\varphi,t}\times\mathbf{B}_t)$$

$\boldsymbol{\omega}^2_{\mathbf{B}} = d\boldsymbol{\omega}^1_{\mathbf{F}}$ Faraday potential

$d\boldsymbol{\omega}^2_{\mathbf{B}} = 0$ Gauss law

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^1_{\mathbf{F}} + dV_{\mathbf{E},t}\,,}$$

$$\mathcal{L}_{\varphi,t}\,d\,\boldsymbol{\omega}^1_{\mathbf{F}} = d\,\mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^1_{\mathbf{F}}$$

$$\mathcal{L}_{\varphi,t}\,\boldsymbol{\omega}^1_{\mathbf{F}} = \partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} + \mathcal{L}_{\mathbf{v}_{\varphi,t}}\,\boldsymbol{\omega}^1_{\mathbf{F},t} = \partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} + d(\boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}) + (d\boldsymbol{\omega}^1_{\mathbf{F},t})\cdot\mathbf{v}_{\varphi,t}\,.$$

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t}\cdot\mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.}$$

$$\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - \boldsymbol{\omega}^2_{\mathbf{B},t}\cdot\mathbf{v}_{\varphi,t} + dU_{\mathbf{E},t}$$

$$\mathbf{E}_t = -\partial_{\tau=t}\,\mathbf{F}_\tau + \mathbf{v}_{\varphi,t}\times\mathbf{B}_t + \nabla U_{\mathbf{E},t}$$

$$U_{\mathbf{E},t} = V_{\mathbf{E},t} - \boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}$$

Lorentz force

Maxwell, J.C., 1861. On Physical Lines of Force. The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science Fourth series, Part I, II, III, IV.

Electric field in a body in translational motion across a region of spatially uniform magnetic flux.

Lemma 8.1 (Linear Faraday potential). *In the* Euclid *space with the standard connection, the linear field*

$$\boldsymbol{\omega}^1_{\mathbf{F},t} := \tfrac{1}{2}\boldsymbol{\mu} \cdot \mathbf{B}_t \cdot \mathbf{r} = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{r}\,,$$

where $\mathbf{r}(\mathbf{x}) := \mathbf{x}$, *provides a* Faraday *potential for the spatially constant magnetic flux, viz.* $d\boldsymbol{\omega}^1_{\mathbf{F},t} = \boldsymbol{\omega}^2_{\mathbf{B},t}$.

Proposition 8.1 (Electric field in a translating body). *A body in translational motion across a region of spatially uniform magnetic flux experiences an electric field given by:*

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.}$$

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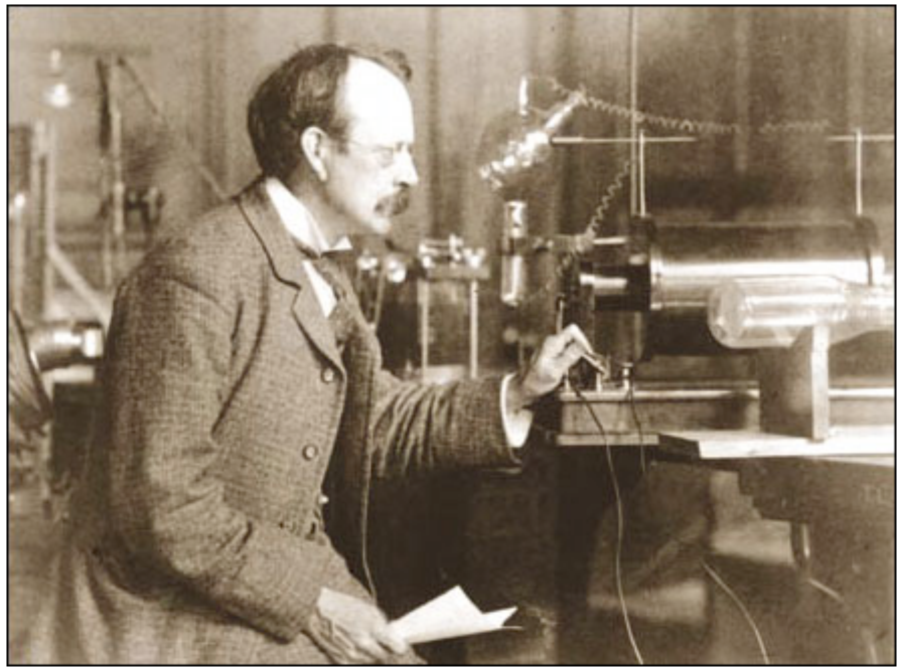

Joseph John Thomson

Cavendish Professors

* James Clerk Maxwell (1871 – 1879)
* Lord Rayleigh (1879 – 1884)
* J.J. Thomson (1884 – 1919)
* Lord Rutherford (1919 – 1937)
* William Lawrence Bragg (1938 – 1953)
* Nevill Mott (1954 – 1971)
* Brian Pippard (1971 – 1984)
* Sam Edwards (1984 – 1995)
* Richard Friend (1995 –)

J.J. Thomson was the first to apply the concept of fields to determine the electromagnetic forces on an object in terms of its properties and of external fields.
Interested in determining the electromagnetic behavior of the charged particles in cathode rays, J.J. Thomson published a paper in **1881** wherein he gave the force on the particles due to an external magnetic field as

$$\frac{1}{2} q \mathbf{v} \times \mathbf{B}.$$

J.J. Thomson was able to arrive at the correct basic form of the formula, but, because of some miscalculations and an incomplete description of the displacement current, included an incorrect numerical coefficient in front of the formula.
It was Oliver Heaviside, who had invented the modern vector notation and applied them to Maxwell's field equations, that was able to correctly derive in 1885 and 1889 the correct form of the magnetic force on a charged particle [9]. Finally, in 1892, Hendrik Antoon Lorentz derived the modern day form of the formula for the electromagnetic force.

Darrigol, O. (2000). Electrodynamics from Ampère to Einstein. Oxford University Press. ISBN 0-198-50593-0

NOTES

ON

RECENT RESEARCHES IN

ELECTRICITY AND MAGNETISM

INTENDED AS A SEQUEL TO

PROFESSOR CLERK-MAXWELL'S TREATISE
ON ELECTRICITY AND MAGNETISM

BY

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Oxford
AT THE CLARENDON PRESS
1893

In the course of Maxwell's investigation of the values of X, Y, Z due to induction, the terms

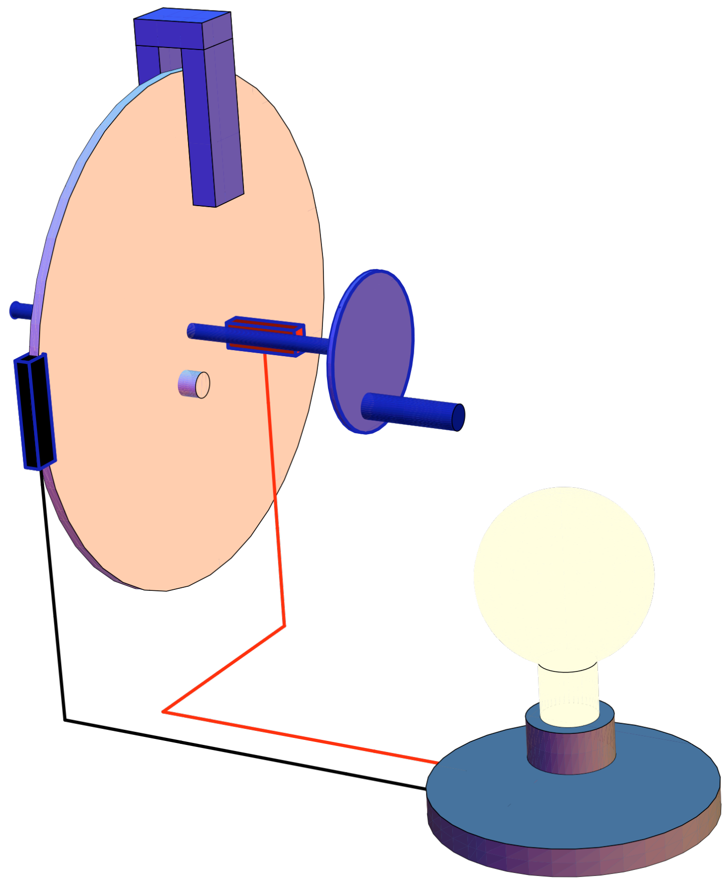
$$-\frac{d}{dx}(Fu+Gv+Hw), \quad -\frac{d}{dy}(Fu+Gv+Hw),$$
$$-\frac{d}{dz}(Fu+Gv+Hw)$$

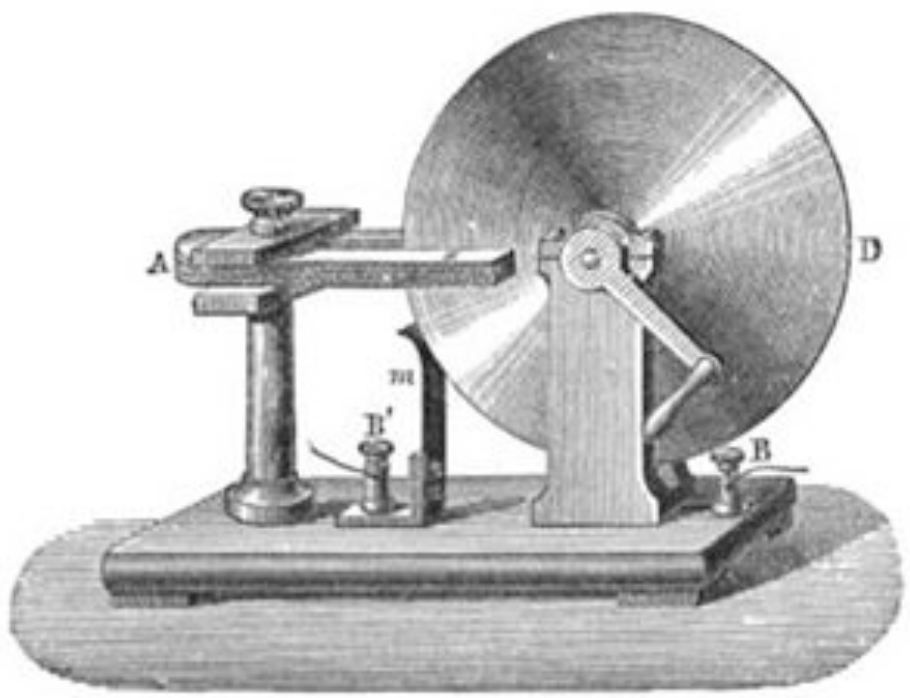
respectively in the final expressions for X, Y, Z are included under the Ψ terms. We shall find it clearer to keep these terms separate and write the expressions for X, Y, Z as

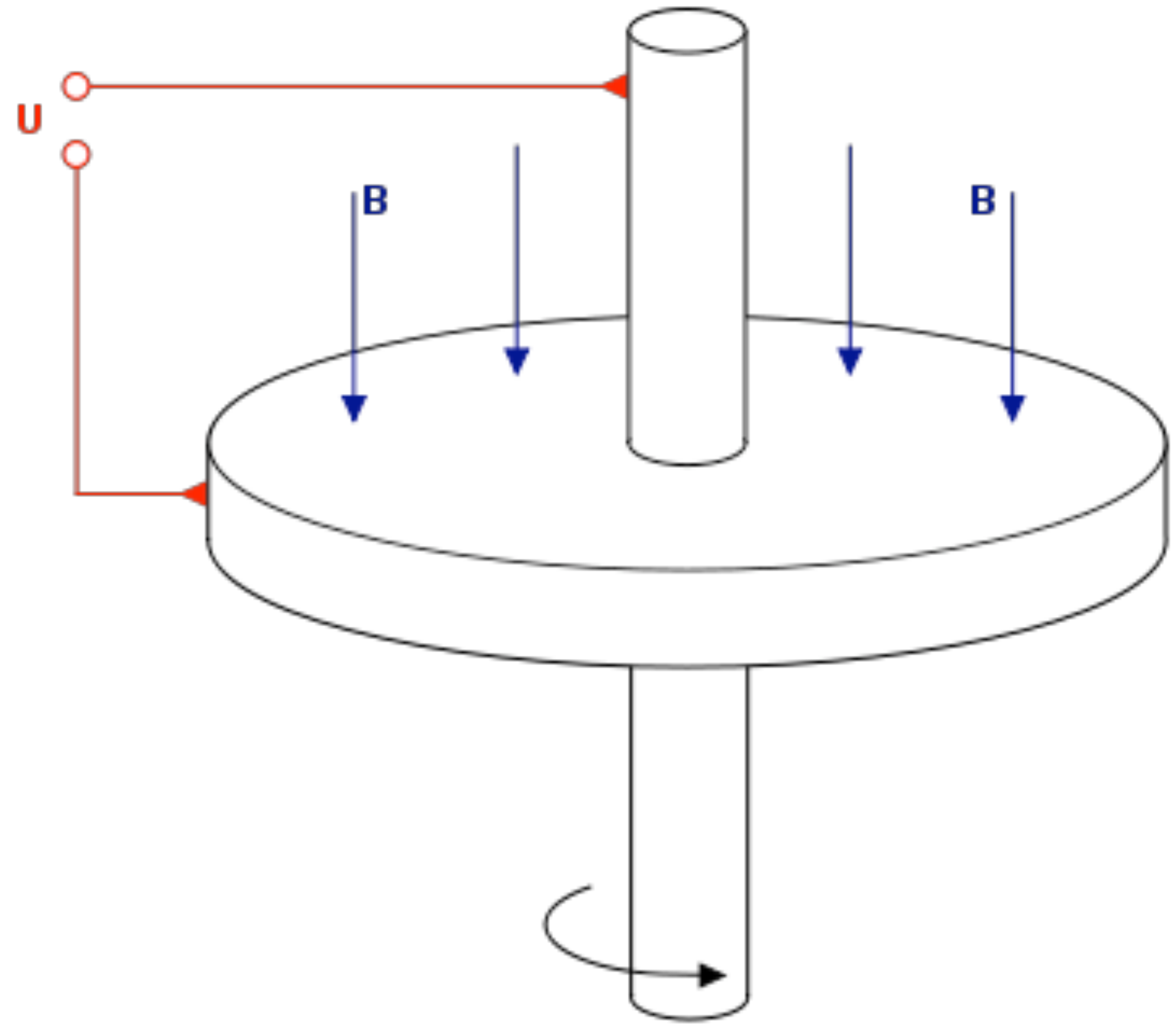
$$\left.\begin{aligned} X &= cv - bw - \frac{dF}{dt} - \frac{d}{dx}(Fu+Gv+Hw) - \frac{d\phi}{dx}, \\ Y &= aw - cu - \frac{dG}{dt} - \frac{d}{dy}(Fu+Gv+Hw) - \frac{d\phi}{dy}, \\ Z &= bu - av - \frac{dH}{dt} - \frac{d}{dz}(Fu+Gv+Hw) - \frac{d\phi}{dz}. \end{aligned}\right\} \qquad (1)$$

$$\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\,\boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t}\cdot\mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t}\cdot\mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.$$

Faraday Disk Dynamo







In the FARADAY experiments described in Section 8.1, the spatial magnetic induction flux is time-independent, so that the GALILEI observer sitting on the support of the disk axis will measure a time-independent FARADAY potential, so that: $\partial_{\tau=t}\, \boldsymbol{\omega}^1_{\mathbf{F},\tau} = 0$ and a velocity field of the spinning disk given by:

$$\mathbf{v}_{\varphi,t}(\mathbf{x}) = \boldsymbol{\Omega}_t \cdot \mathbf{r}(\mathbf{x})$$

with $\mathbf{r}(\mathbf{x}) := \mathbf{x}$ and $\mathbf{x}$ a radius vector with origin at the disk axis. Then $\nabla \mathbf{v}_{\varphi,t} = \boldsymbol{\Omega}_t$. Assuming that the magnetic flux $\boldsymbol{\omega}^2_{\mathbf{B},t}$ is spatially constant in the disk, i.e. $\nabla\, \boldsymbol{\omega}^2_{\mathbf{B},t} = 0$, in terms of the potential $\boldsymbol{\omega}^1_{\mathbf{F},t} = \frac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{r}$, we have:

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\varphi,t}}\, \boldsymbol{\omega}^1_{\mathbf{F},t} &= \nabla_{\mathbf{v}_{\varphi,t}} \boldsymbol{\omega}^1_{\mathbf{F},t} + \boldsymbol{\omega}^1_{\mathbf{F},t} \circ \nabla \mathbf{v}_{\varphi,t} \\ &= \nabla_{\mathbf{v}_{\varphi,t}}\, \boldsymbol{\omega}^1_{\mathbf{F},t} + \tfrac{1}{2}(\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{r}) \circ \boldsymbol{\Omega}_t\,, \end{aligned}$$

with the covariant derivative of the magnetic potential given by:

$$2\, \nabla_{\mathbf{v}_{\varphi,t}}\, \boldsymbol{\omega}^1_{\mathbf{F},t} = \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} + \nabla_{\mathbf{v}_{\varphi,t}} \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{r}\,.$$

For an arbitrary vector field $\mathbf{h}$ in the disk plane, we have that:

$$\begin{aligned} 2\, \langle \mathcal{L}_{\mathbf{v}_{\varphi,t}}\, \boldsymbol{\omega}^1_{\mathbf{F},t}, \mathbf{h} \rangle &= 2\, \langle \nabla_{\mathbf{v}_{\varphi,t}}\, \boldsymbol{\omega}^1_{\mathbf{F},t}, \mathbf{h} \rangle + \boldsymbol{\omega}^2_{\mathbf{B},t}(\mathbf{r}, \boldsymbol{\Omega}_t \cdot \mathbf{h}) \\ &= \boldsymbol{\omega}^2_{\mathbf{B},t}(\boldsymbol{\Omega}_t \cdot \mathbf{r}, \mathbf{h}) + \langle \nabla_{\mathbf{v}_{\varphi,t}} \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{r}, \mathbf{h} \rangle + \boldsymbol{\omega}^2_{\mathbf{B},t}(\mathbf{r}, \boldsymbol{\Omega}_t \cdot \mathbf{h}) = 0\,, \end{aligned}$$

being $\nabla\, \boldsymbol{\omega}^2_{\mathbf{B},t} = 0$ by assumption and

$$\boldsymbol{\omega}^2_{\mathbf{B},t}(\boldsymbol{\Omega}_t \cdot \mathbf{r}, \mathbf{h}) = \boldsymbol{\omega}^2_{\mathbf{B},t}(\boldsymbol{\Omega}_t \cdot (\boldsymbol{\Omega}_t \cdot \mathbf{r}), \boldsymbol{\Omega}_t \cdot \mathbf{h}) = -\boldsymbol{\omega}^2_{\mathbf{B},t}(\mathbf{r}, \boldsymbol{\Omega}_t \cdot \mathbf{h})\,.$$

The analysis reveals that the magnetically induced electric vector field in the disk vanishes identically if the magnetic flux in the disk is spatially uniform. However, to compute the electromotive force in the circuit we should take into account the discontinuity points of the velocity at the axis and at the rib brush contacts, which provide concentrated contributions to the *emf* whose sum is equal to:

$$\begin{aligned} & -\boldsymbol{\omega}^1_{\mathbf{F},t}(\mathbf{x}_1) \cdot (\boldsymbol{\Omega}_t \cdot \mathbf{x}_1) + \boldsymbol{\omega}^1_{\mathbf{F},t}(\mathbf{x}_2) \cdot (\boldsymbol{\Omega}_t \cdot \mathbf{x}_2) \\ &= -\tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{x}_1 \cdot (\boldsymbol{\Omega}_t \cdot \mathbf{x}_1) + \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{x}_2 \cdot (\boldsymbol{\Omega}_t \cdot \mathbf{x}_2)\,. \end{aligned}$$

The global *emf* is thus again coincident with the one evaluated by the integral flux formula of FARADAY in which the spinning velocity of the disk radius closing the circuit is taken into account.

$$\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\, \boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t} \cdot \mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.$$

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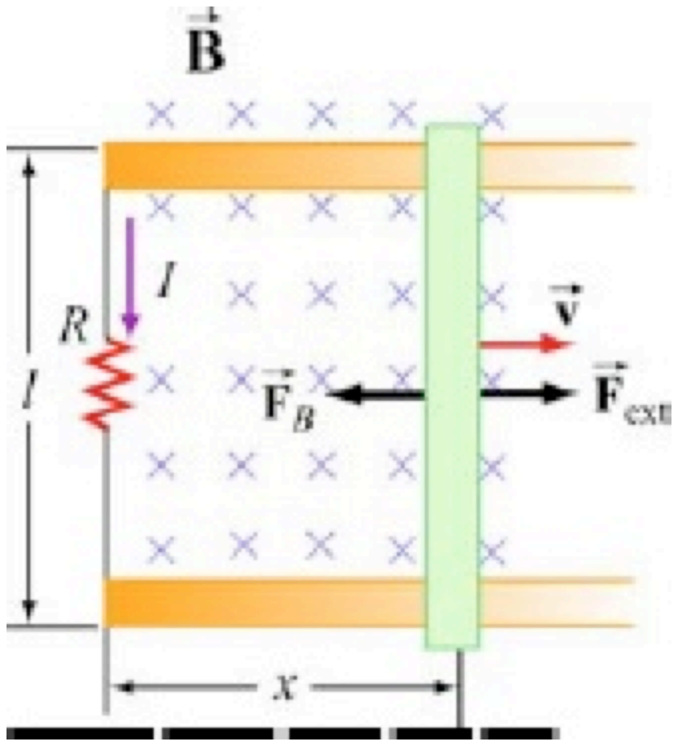
Faraday law of induction: examples



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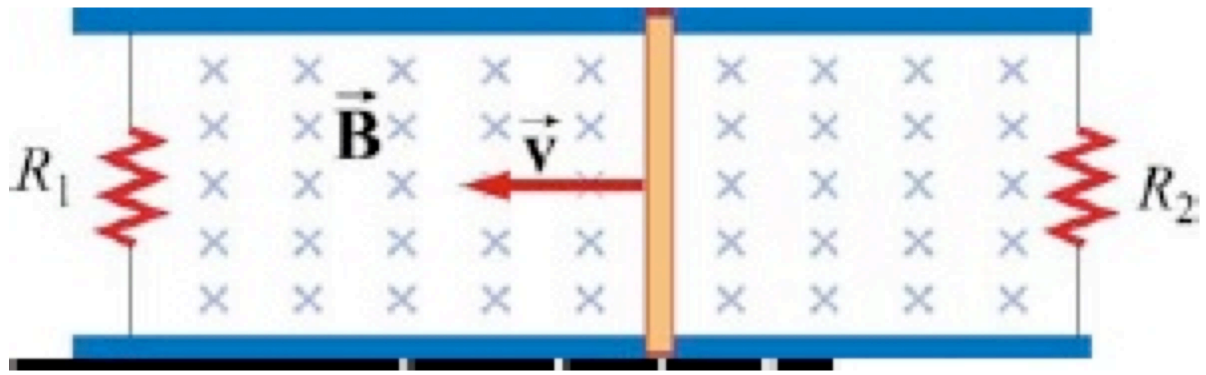

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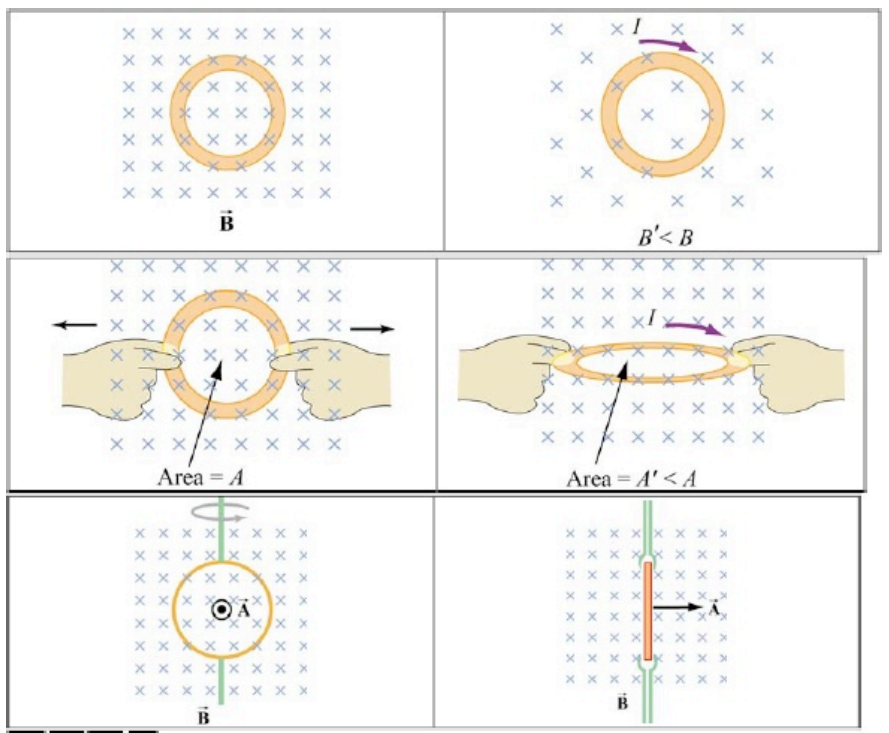

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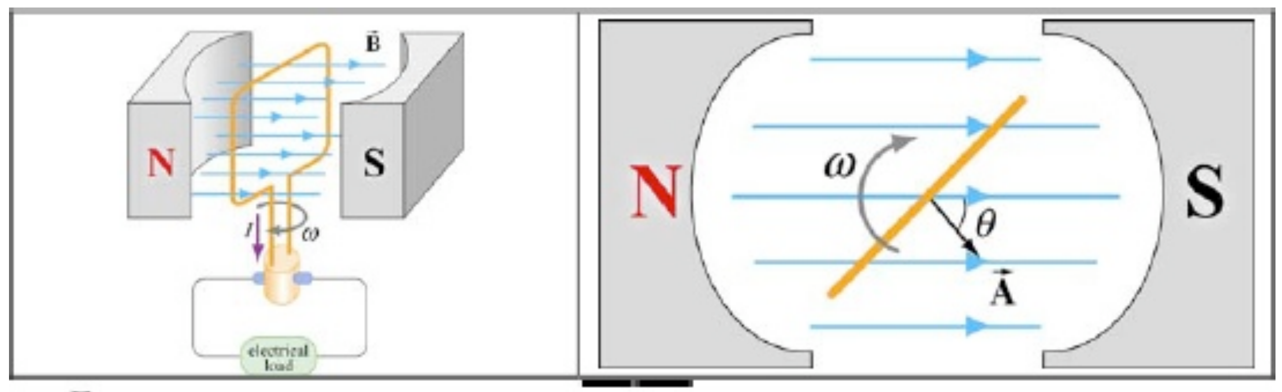

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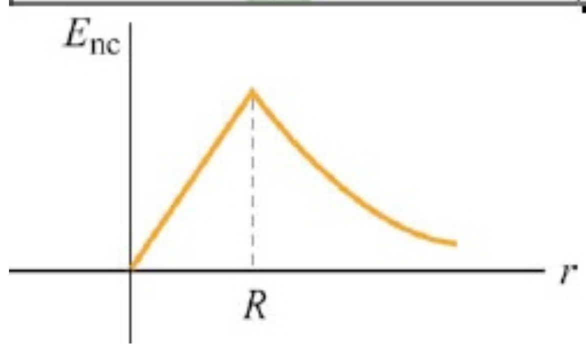

Let us consider the problem concerning the electromotive force *(emf)* generated in a conductive bar sliding on two fixed parallel rails under a transverse magnetic field which is spatially uniform and time-independent. An observer sitting on the rails measures a time independent FARADAY potential field and may thus evaluate the *emf* due to the electric field distributed along the bar is found by integration along the line from $\mathbf{x}_1$ to $\mathbf{x}_2$:

$$\boldsymbol{\omega}^1_{\mathbf{E},t} \cdot \mathbf{l} = -\tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}\,.$$

On the other hand, by the integral formula of FARADAY, the total *emf* in a circuit closed by another fixed bar is evaluated to be:

$$\oint \boldsymbol{\omega}^1_{\mathbf{E}} = -\oint \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} = -\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}\,.$$

So one-half of the total *emf* is lost as a result of the evaluation of the contribution provided by the electric field distributed along the bar. To resolve this puzzling result we have to consider that, in this example, the velocity field is no more uniform in space. Moreover, being uniform in the bar and vanishing in the rails, it presents two points of discontinuities at the sliding contacts. Then, the observer sitting on the rails measures the distributed electric field in the bar, as evaluated before, plus two impulses of *emf* concentrated at the sliding contacts, whose sum is given by

$$-(\boldsymbol{\omega}^1_{\mathbf{F},t}(\mathbf{x}_1) - \boldsymbol{\omega}^1_{\mathbf{F},t}(\mathbf{x}_2)) \cdot \mathbf{v}_{\varphi,t} = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{l} \cdot \mathbf{v}_{\varphi,t} = -\tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}$$

where $\mathbf{x}_1, \mathbf{x}_2$ are the positions of the sliding contacts and $\mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1$. Indeed the velocity jumps, in going from 1 to 2, are $\mathbf{v}_{\varphi,t}$ and $-\mathbf{v}_{\varphi,t}$ respectively. Thus, the two impulses of *emf* concentrated at the sliding contacts provide just the lost one-half of the total *emf* in the translating bar and in the sliding contacts, which therefore amounts to $-\boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} \cdot \mathbf{l}$ and is equal to the one previously computed in one stroke by the integral flux rule of FARADAY.

$$\boxed{\boldsymbol{\omega}^1_{\mathbf{E},t} = -\partial_{\tau=t}\, \boldsymbol{\omega}^1_{\mathbf{F},\tau} - d(\boldsymbol{\omega}^1_{\mathbf{F},t} \cdot \mathbf{v}_{\varphi,t}) - \boldsymbol{\omega}^2_{\mathbf{B},t} \cdot \mathbf{v}_{\varphi,t} + dV_{\mathbf{E},t}\,.}$$

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Concluding remarks

- By covariance of electromagnetic induction laws,
  Galilei invariance of the involved fields and fluxes implies
  Galilei invariance of the laws.

- This result contradicts most treatment of Electrodynamics and
  provides the correction of physically untenable statements.

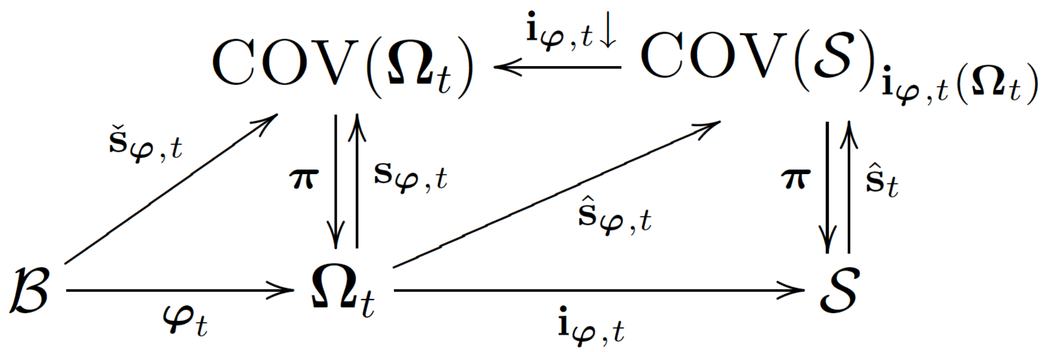
- The electrodynamic motivation for special relativity
  should be properly revised.

- Feynman's beautiful generalization stems from a proper formulation
  of the original Faraday flux principle.

- The Lorentz force law should be eliminated from physics textbooks and
  treated as a direct consequence of Faraday law, in special situations.

Material inductions of covariant spatial fields

A covariant spatial tensor field $\hat{\mathbf{s}}_t \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{COV}(\mathcal{S}))$ *at time* $t \in I$ *induces, at the configuration* $\boldsymbol{\varphi}_t \in \mathrm{C}^1(\mathcal{B}\,;\boldsymbol{\Omega}_t)$ *, a spatial-valued material field* $\hat{\mathbf{s}}_{\boldsymbol{\varphi},t} = \hat{\mathbf{s}}_t \circ \mathbf{i}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t, \mathrm{COV}(\mathcal{S}))$ *and, by co-restriction, the material fields* $\mathbf{s}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\mathcal{B}\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$ *and* $\check{\mathbf{s}}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$ *, according to the commutative diagram:*



$$\Longleftrightarrow \quad \begin{cases} \boldsymbol{\pi} \circ \hat{\mathbf{s}}_t = \mathrm{ID}_{\mathcal{S}}\,, \\ \hat{\mathbf{s}}_t \circ \mathbf{i}_{\boldsymbol{\varphi},t} = \hat{\mathbf{s}}_{\boldsymbol{\varphi},t}\,, \\ \mathbf{i}_{\boldsymbol{\varphi},t}{\downarrow}\hat{\mathbf{s}}_t = \mathbf{s}_{\boldsymbol{\varphi},t}\,, \\ \boldsymbol{\pi} \circ \mathbf{s}_{\boldsymbol{\varphi},t} = \mathrm{ID}_{\boldsymbol{\Omega}_t}\,. \end{cases}$$

The bundle $\mathrm{COV}(\mathcal{S})_{\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t)}$ *denotes the restriction of* $\mathrm{COV}(\mathcal{S})$ *to the base* $\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t) \subset \mathcal{S}$ *. The pull-back* $\mathbf{i}_{\boldsymbol{\varphi},t}{\downarrow} \in \mathrm{C}^1(\mathrm{COV}(\mathcal{S})_{\mathbf{i}_{\boldsymbol{\varphi},t}(\boldsymbol{\Omega}_t)}\,;\mathrm{COV}(\boldsymbol{\Omega}_t))$ *between covariant tensor bundles is defined in terms of the inclusion map* $\mathbf{i}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathcal{S})$ *and of the push-forward* $\mathbf{i}_{\boldsymbol{\varphi},t}{\uparrow} \in \mathrm{C}^1(\mathbb{T}\boldsymbol{\Omega}_t\,;\mathbb{T}\mathcal{S})$ *between tangent bundles, by:*

$$\mathbf{s}_{\boldsymbol{\varphi},t}(\mathbf{a}_{\boldsymbol{\varphi},t}, \mathbf{b}_{\boldsymbol{\varphi},t}) := \hat{\mathbf{s}}_{\boldsymbol{\varphi},t}(\mathbf{i}_{\boldsymbol{\varphi},t}{\uparrow}\mathbf{a}_{\boldsymbol{\varphi},t}, \mathbf{i}_{\boldsymbol{\varphi},t}{\uparrow}\mathbf{b}_{\boldsymbol{\varphi},t})\,,$$

for all $\mathbf{a}_{\boldsymbol{\varphi},t}, \mathbf{b}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathbb{T}\boldsymbol{\Omega}_t)$.

Two paradigmatic examples of the role of differential geometry in classical physics

Rate laws of mechanical material behavior **and**

Electromagnetic induction