Universität Innsbruck
Arbeitsbereich für Geotechnik und Tunnelbau

Geometry & Continuum Mechanics

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Short Course
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Geometric Approach to Non-Linear Continuum Mechanics

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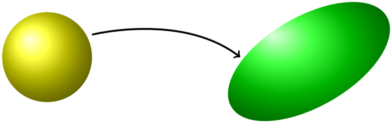
Linearized Continuum Mechanics (LCM) can be modeled by Linear Algebra (LA) and Calculus on Linear Spaces (CoLS).

Geometric Approach to Non-Linear Continuum Mechanics

Linearized Continuum Mechanics (**LCM**) can be modeled by Linear Algebra (**LA**) and Calculus on Linear Spaces (**CoLS**).

Non-Linear Continuum Mechanics (**NLCM**) calls for Differential Geometry (**DG**) and Calculus on Manifolds (**CoM**) as natural tools to develop theoretical and computational models.



Math1 - Tangent spaces

Tangent vector to a manifold:

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velocity of a curve $\mathbf{c} : [a, b] \mapsto \mathbf{M}\,, \quad \lambda \in [a, b]\,, \quad \mathbf{x} = \mathbf{c}(\lambda)$ base point

$$\mathbf{v} := \partial_{\mu=\lambda}\, \mathbf{c}(\mu) \in T_{\mathbf{x}}\mathbf{M}$$

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$$\mathbf{v}^* : T_{\mathbf{x}}\mathbf{M} \mapsto \mathcal{R} \in T^*_{\mathbf{x}}\mathbf{M} \quad \textit{linear}$$

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- A map $\zeta : \mathbf{M} \mapsto \mathbf{N}$ sends
  a curve $\mathbf{c} : [a, b] \mapsto \mathbf{M}$ into
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  a curve $\mathbf{c} : [a, b] \mapsto \mathbf{M}$ into
  a curve $\zeta \circ \mathbf{c} : [a, b] \mapsto \mathbf{N}$.
- The tangent map $T_{\mathbf{x}}\zeta : T_{\mathbf{x}}\mathbf{M} \mapsto T_{\zeta(\mathbf{x})}\mathbf{N}$
  sends a tangent vector at $\mathbf{x} \in \mathbf{M}$
  $\mathbf{v} \in T_{\mathbf{x}}(\mathbf{M}) := \partial_{\mu=\lambda}\, \mathbf{c}(\mu)$
  into a tangent vector at $\zeta(\mathbf{x}) \in \mathbf{N}$
  $T_{\mathbf{x}}\zeta \cdot \mathbf{v} \in T_{\zeta(\mathbf{x})}(\mathbf{N}) := \partial_{\mu=\lambda}\, (\zeta \circ \mathbf{c})(\mu)$

Math2 - Tangent functor

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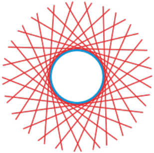
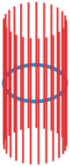
Tangent bundle



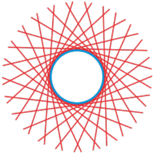


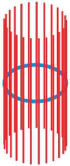
Math2 - Tangent functor

Tangent bundle

- disjoint union of tangent spaces:

$$T\mathbf{M} := \cup_{\mathbf{x}\in\mathbf{M}} T_{\mathbf{x}}\mathbf{M}$$





Math2 - Tangent functor

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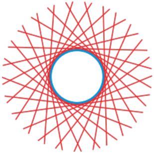

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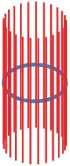

- Projection: $\boldsymbol{\tau}_{\mathbf{M}} : T\mathbf{M} \mapsto \mathbf{M}$

$$\mathbf{v} \in T_{\mathbf{x}}\mathbf{M}\,, \quad \boldsymbol{\tau}_{\mathbf{M}}(\mathbf{v}) := \mathbf{x} \quad \text{base point}$$

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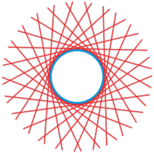

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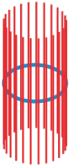

- Surjective submersion:

$T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbf{M}} : T_{\mathbf{v}}T\mathbf{M} \mapsto T_{\mathbf{x}}\mathbf{M}$ is surjective

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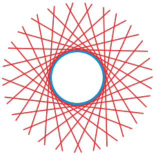

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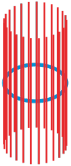

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- Tangent functor

$$\zeta : \mathbf{M} \mapsto \mathbf{N} \quad \mapsto \quad T\zeta : T\mathbf{M} \mapsto T\mathbf{N}$$

Math3 - Fiber bundles

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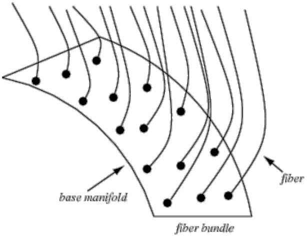
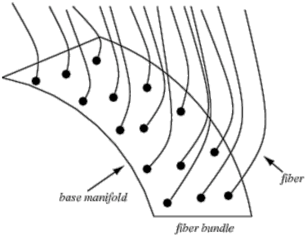
Fiber bundles



Math3 - Fiber bundles

Fiber bundles

- $\mathbf{E}, \mathbf{M}$ manifolds



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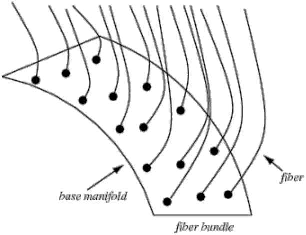

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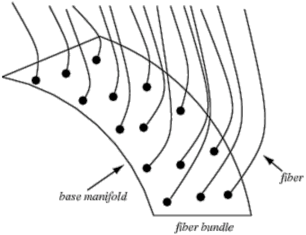

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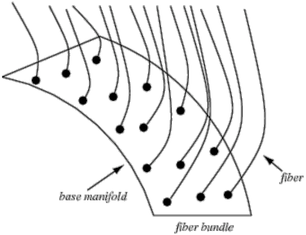

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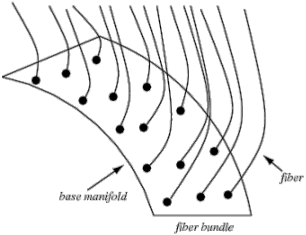
Math3 - Fiber bundles



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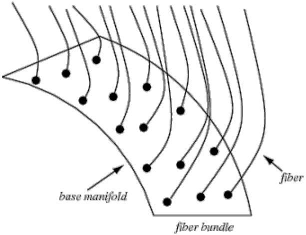

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  $\delta\mathbf{e} \in V\mathbf{E} \subset T\mathbf{E} \quad \Longrightarrow \quad T_{\mathbf{e}}\pi_{\mathbf{M},\mathbf{E}} \cdot \delta\mathbf{e} = 0$

Math4 - Fiber bundle samples

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Trivial and
non-trivial
fiber bundles

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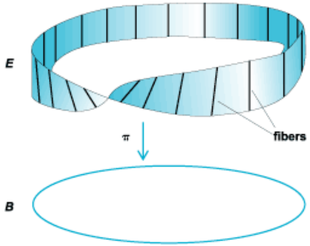

Math4 - Fiber bundle samples

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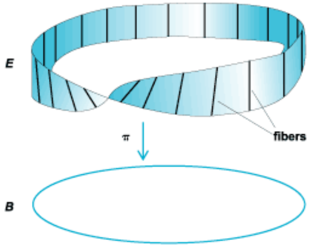
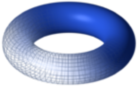
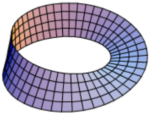
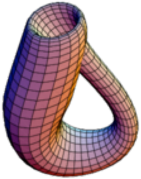




Torus



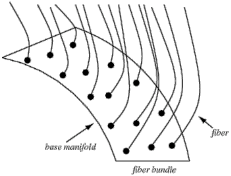
Listing-Möbius strip



Klein Bottle

Math5 - Sections

Sections of fiber bundles



Math5 - Sections

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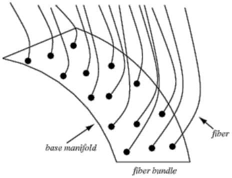

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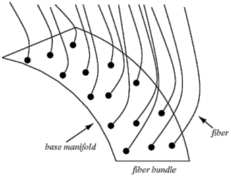

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- Fiber bundle $\pi_{\mathbf{M},\mathbf{E}} : \mathbf{E} \mapsto \mathbf{M}$
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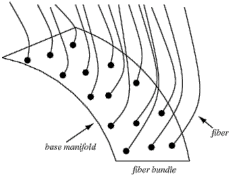

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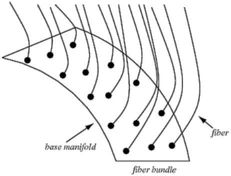

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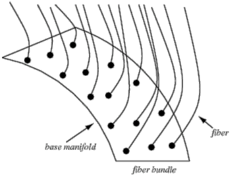
Math5 - Sections



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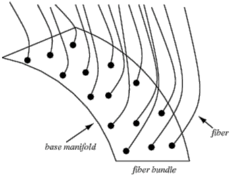
Sections of tangent and bi-tangent bundles

Math5 - Sections



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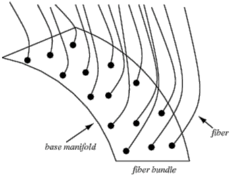
Sections of tangent and bi-tangent bundles

- Tangent vector fields:

$$\mathbf{v} : \mathbf{M} \mapsto T\mathbf{M} \ : \ \boldsymbol{\tau}_{\mathbf{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbf{M}}$$

Math5 - Sections



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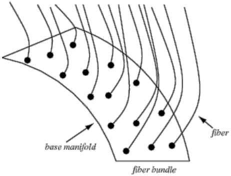
- Bi-tangent vector fields:

$$\mathbf{X} : T\mathbf{M} \mapsto TT\mathbf{M} \; : \; \boldsymbol{\tau}_{T\mathbf{M}} \circ \mathbf{X} = \mathrm{ID}_{T\mathbf{M}}$$

Math5 - Sections



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- Vertical bi-tangent vectors $\mathbf{X} \in \mathrm{Ker}\, T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbf{M}}$

Math6 - Tensor spaces

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- Covariant $\qquad \mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \mathrm{Cov}_{\mathbf{x}}(T\mathbf{M}) = L\,(T_{\mathbf{x}}\mathbf{M}^2\,;\mathcal{R}) = L\,(T_{\mathbf{x}}\mathbf{M}\,;\,T_{\mathbf{x}}^*\mathbf{M})$

Math6 - Tensor spaces

- Covariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \mathrm{Cov}_{\mathbf{x}}(T\mathbf{M}) = L\,(T_{\mathbf{x}}\mathbf{M}^2\,;\mathcal{R}) = L\,(T_{\mathbf{x}}\mathbf{M}\,;\,T_{\mathbf{x}}^*\mathbf{M})$
- Contravariant $\mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} \in \mathrm{Con}_{\mathbf{x}}(T\mathbf{M}) = L\,(T_{\mathbf{x}}^*\mathbf{M}^2\,;\mathcal{R}) = L\,(T_{\mathbf{x}}^*\mathbf{M}\,;\,T_{\mathbf{x}}\mathbf{M})$

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- Alteration rules:

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Math6 - Tensor spaces

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Tensor bundles and sections

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Math6 - Tensor spaces

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Math6 - Tensor spaces

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Tensor bundles and sections

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- with: $\boldsymbol{\tau}_{\mathbf{M}}^{\textsc{Tens}} \circ \mathbf{s}_{\mathbf{M}}^{\textsc{Tens}} = \mathrm{ID}_{\mathbf{M}}$

Math7 - Push and pull

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Given a map $\zeta : \mathbf{M} \mapsto \mathbf{N}$

- Pull-back of a scalar field

$$f : \mathbf{N} \mapsto \mathrm{F{\scriptstyle UN}}(\mathbf{N}) \quad \mapsto \quad \zeta{\downarrow}f : \mathbf{M} \mapsto \mathrm{F{\scriptstyle UN}}(\mathbf{M})$$

defined by:

$$(\zeta{\downarrow}f)_{\mathbf{x}} := \zeta{\downarrow}f_{\zeta(\mathbf{x})} := f_{\zeta(\mathbf{x})} \in \mathrm{F{\scriptstyle UN}}_{\mathbf{x}}(\mathbf{M})\,.$$

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- Push-forward of a tangent vector field

$$\mathbf{v} : \mathbf{M} \mapsto T\mathbf{M} \quad \mapsto \quad \zeta{\uparrow}\mathbf{v} : \mathbf{N} \mapsto T\mathbf{N}$$

  defined by:

$$(\zeta{\uparrow}\mathbf{v})_{\zeta(\mathbf{x})} := \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} = T_{\mathbf{x}}\zeta \cdot \mathbf{v}_{\mathbf{x}} \in T_{\zeta(\mathbf{x})}\mathbf{N}\,.$$

Math8 - Push-pull of tensor fields

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- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_\mathbf{x} \rangle = \langle T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle$$

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- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \circ T_\mathbf{x}\zeta \in \mathrm{Cov}(T\mathbf{M})_\mathbf{x}$$

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Math8 - Push-pull of tensor fields

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- Mixed tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{Mix}}_\mathbf{x} = T_\mathbf{x}\zeta \circ \mathbf{s}^{\mathrm{Mix}}_\mathbf{x} \circ T_{\zeta(\mathbf{x})}\zeta^{-1} \in \mathrm{Mix}(T\mathbf{N})_{\zeta(\mathbf{x})}$$

Math9 - Connections

Parallel transport along a curve $\mathbf{c} : [a, b] \mapsto \mathbf{M}$

Math9 - Connections

Parallel transport along a curve $\mathbf{c} : [a, b] \mapsto \mathbf{M}$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v_x} \in T_{\mathbf{x}}\mathbf{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \in T_{\mathbf{c}(\lambda)}\mathbf{M}$$

$$\mathbf{c}_{\mu,\mu} \Uparrow \mathbf{v_x} = \mathbf{v_x}$$

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$$\langle\, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}^*_\mathbf{x}, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}_\mathbf{x} \,\rangle = \mathbf{c}_{\lambda,\mu} \Uparrow \langle\, \mathbf{v}^*_\mathbf{x}, \mathbf{v}_\mathbf{x} \,\rangle$$

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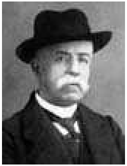

Gregorio Ricci-Curbastro (1853 - 1925)

Math9 - Connections

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- Tensor fields (by naturality)

Tullio Levi-Civita (1873 - 1941)

Math10 - Lie and parallel derivatives

Derivatives of a tensor field
$\mathbf{s} : \mathbf{M} \mapsto \mathbf{Tens}(T\mathbf{M})$
along the flow of a tangent vector field

Math10 - Lie and parallel derivatives

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- Tangent vector fields and Flows

$$\mathbf{v} : \mathbf{M} \mapsto T\mathbf{M} \qquad \mathbf{Fl}^{\mathbf{v}}_{\lambda} : \mathbf{M} \mapsto \mathbf{M}$$

$$\mathbf{v} := \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{v}}_{\lambda}$$

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- Lie derivative - LD

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{s} := \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{v}}_{\lambda}{\downarrow}(\mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

- Parallel derivative - PD

$$\nabla_{\mathbf{v}}\, \mathbf{s} := \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow (\mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda})$$

Math11

Math11

lenght of symplex's edges

lenght of symplex's edges

- Norm axioms

$$\begin{tikzcd} A \arrow[dr, "\mathbf{c}"] & \\ & B \\ C \arrow[uu, "\mathbf{b}"] \arrow[ur, "\mathbf{a}"] & \end{tikzcd}$$

$$\|\mathbf{a}\| \geq 0\,, \quad \|\mathbf{a}\| = 0 \quad \Longrightarrow \quad \mathbf{a} = 0$$
$$\|\mathbf{a}\| + \|\mathbf{b}\| \geq \|\mathbf{c}\| \quad \textbf{triangle inequality},$$
$$\|\alpha\,\mathbf{a}\| = |\alpha|\,\|\mathbf{a}\|$$

Math11

lenght of symplex's edges

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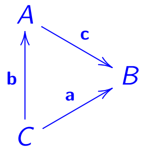

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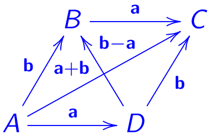
- Parallelogram rule



$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right]$$

Math12

Math12

The metric tensor

- Theorem (Fréchet – von Neumann – Jordan)

Math12

The metric tensor

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$$\mathbf{g}(\mathbf{a}\,,\mathbf{b}) := \frac{1}{4}\left[\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right]$$

Math12

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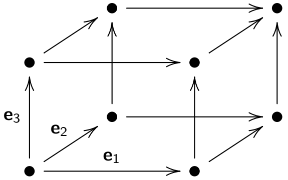
$\mathrm{VOL}\Big($



$$\Big)^2 = \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$$

Maurice René Fréchet (1878 - 1973)

Math12

The metric tensor

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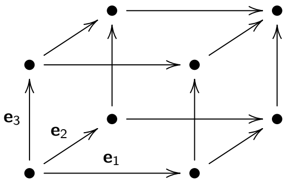
VOL (



$)^2$

$$= \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$$

John von Neumann (1903 - 1957)

Math12

The metric tensor

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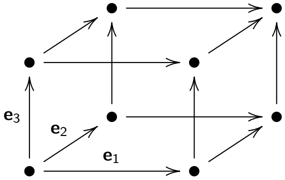

$\Big)^2 = \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$

Pascual Jordan (1902 - 1980)

Math12

The metric tensor

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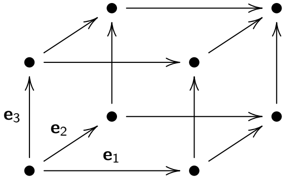

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Kosaku Yosida (1909 - 1990)

Math13

BERNHARD RIEMANN (1826 - 1866)

Math13

BERNHARD RIEMANN (1826 - 1866)

Metric tensor field: $\mathbf{g} : \mathbf{M} \mapsto \mathrm{Cov}(T\mathbf{M})$

- RIEMANN manifold: $(\mathbf{M}, \mathbf{g})$

Math13

BERNHARD RIEMANN (1826 - 1866)

Metric tensor field: $\mathbf{g} : \mathbf{M} \mapsto \mathrm{Cov}(T\mathbf{M})$

- RIEMANN manifold: $(\mathbf{M}, \mathbf{g})$
- Fundamental theorem:
  There exists a unique linear connection, the LEVI-CIVITA connection, that is metric and symmetric, i.e. such that
  1. $\nabla_{\mathbf{v}}\mathbf{g} = \mathbf{0}$
  2. $\nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} = [\mathbf{v}, \mathbf{u}]$

The torsion of the connection is defined by

$$\mathrm{TORS}(\mathbf{v}, \mathbf{u}) = \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v}, \mathbf{u}]$$

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G. Romano & R. Barretta Covariant hypo-elasticity
Eur. J. Mech. A-Solids **30** (2011) 1012–1023

G. Romano, R. Barretta, M. Diaco Geometric Continuum Mechanics
Meccanica **49** (2014) 111–133

A basic question in NLCM

- How to compare **material tensors** at corresponding points in displaced configurations of a body?

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- Devil's temptation:

In 3D bodies it might seem as natural to compare by translation the involved material vectors.
This is tacitly done in literature, when evaluating the material time-derivative of the stress tensor $\mathbf{T}$ *:*

$$\dot{\mathbf{T}}(\mathrm{p}, t) := \partial_{\tau=t}\, \mathbf{T}(\mathrm{p}, \tau) = \partial_{\tau=t}\, \boldsymbol{\varphi}_{\alpha} \Downarrow \mathbf{T}(\mathrm{p}, \tau)$$

or the material time-derivative of the director $\mathbf{n}$ *of a nematic liquid crystal:*

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These definitions are connection dependent and geometrically incorrect when considering 1D and 2D models (wires and membranes).

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- Geometric hint:

  Tangent vectors to a body placement are transformed into vectors tangent to another body placement by the tangent displacement map. This is the essence of the **COVARIANCE PARADIGM**.

Principles

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A geometrically consistent theoretical framework should be equally applicable to body models of any dimension.

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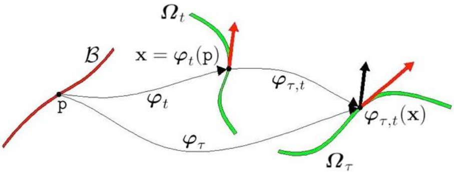

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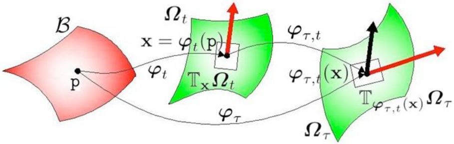

NLCM: Nonlinear Continuum Mechanics

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How to play the game
according to a full geometric approach

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- Event manifold: $\mathcal{E}$ – four dimensional RIEMANN manifold

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- Event manifold: $\mathcal{E}$ – four dimensional Riemann manifold
- Observer split into space-time: $\gamma : \mathcal{E} \mapsto \mathcal{S} \times I$

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- Event manifold: $\mathcal{E}$ – four dimensional RIEMANN manifold
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- time is absolute (Classical Mechanics)
- distance between simultaneous events $\mapsto$ space-metric
- distance between localized events $\mapsto$ time-metric

Event manifold foliation

Event manifold foliation

Each observer performs a double foliation of the 4D event manifold $\mathcal{E}$ into complementary

- 3D *space-slices* $\mathcal{S}$ of *isochronous* events (with a same corresponding time instant). $\mathbf{P}$ orthogonal projector on space slices.
- 1D *time-lines* of *isotopic* events (with a same corresponding space location). $\mathbf{Z}$ time arrow field.

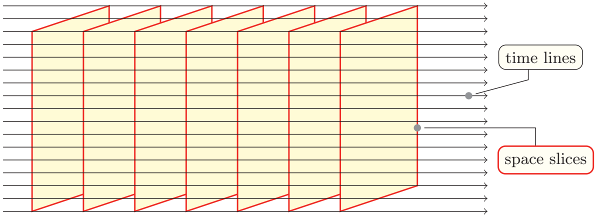

Figure : EUCLID space-time slicing.

Space-time decomposition

Space-time decomposition

Commutative diagram

$$
\begin{tikzcd}
\mathcal{T}_{\mathcal{E}} \arrow[r, "\varphi^{\mathcal{E}}_{\alpha}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{T}_{\mathcal{E}} \arrow[dd, bend left=60, "t_{\mathcal{E}}"] \\
\mathcal{T} \arrow[u, "\mathbf{i}"] \arrow[r, "\varphi^{\mathcal{T}}_{\alpha}"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[d, "t_{\mathcal{T}}"] \\
\mathcal{Z} \arrow[r, "t_{\alpha}"] & \mathcal{Z}
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{cases} \varphi^{\mathcal{E}}_{\alpha} \circ \mathbf{i} = \mathbf{i} \circ \varphi^{\mathcal{T}}_{\alpha} \,, \\ t_{\mathcal{E}} \circ \varphi^{\mathcal{E}}_{\alpha} = t_{\alpha} \circ t_{\mathcal{E}} \,, \end{cases}
$$

time translation $t_{\alpha} : \mathcal{Z} \mapsto \mathcal{Z}$ is defined by $t_{\alpha}(t) := t + \alpha \,, \quad t, \alpha \in \mathcal{Z} \,.$

Decomposition

1. a time-preserving *spatial displacement* $\varphi^{\mathcal{S}}_{\alpha} : \mathcal{E} \mapsto \mathcal{E}$,
2. a location-preserving *time step* $\varphi^{\mathcal{Z}}_{\alpha} : \mathcal{E} \mapsto \mathcal{E}$,

Commutative diagram

$$
\begin{tikzcd}
\mathcal{T}_{\mathcal{E}} \arrow[r, "\varphi^{\mathcal{S}}_{\alpha}"] \arrow[d, "\varphi^{\mathcal{Z}}_{\alpha}"'] \arrow[dr, "\varphi^{\mathcal{E}}_{\alpha}"] & \mathcal{E} \arrow[d, "\varphi^{\mathcal{Z}}_{\alpha}"] \\
\mathcal{E} \arrow[r, "\varphi^{\mathcal{S}}_{\alpha}"] & \mathcal{T}_{\mathcal{E}}
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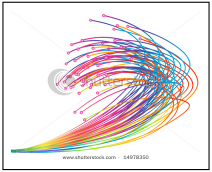
\varphi^{\mathcal{E}}_{\alpha} = \varphi^{\mathcal{S}}_{\alpha} \circ \varphi^{\mathcal{Z}}_{\alpha} = \varphi^{\mathcal{Z}}_{\alpha} \circ \varphi^{\mathcal{S}}_{\alpha} \,. \quad (1)
$$

Trajectory



- Trajectory $\mapsto$ manifold $\mathcal{T}$ with injective immersion in the event time-bundle: $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$

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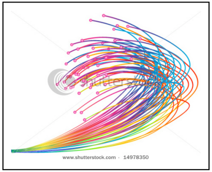

- Trajectory $\mapsto$ manifold $\mathcal{T}$ with injective immersion in the event time-bundle: $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$
- Immersed trajectory $\mathcal{T}_{\mathcal{E}} := \mathbf{i}(\mathcal{T}) \subset \mathcal{E}$ subbundle of the event time-bundle.

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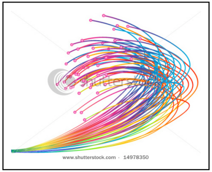

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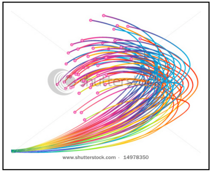

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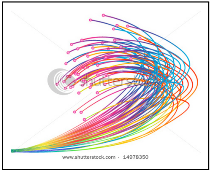

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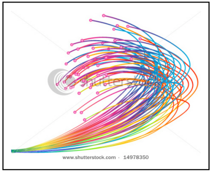

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$$\mathbf{h} \in V_{\mathbf{e}}\mathcal{T} \quad \Longleftrightarrow \quad T_{\mathbf{e}} t_{\mathcal{T}} \cdot \mathbf{h} = 0$$

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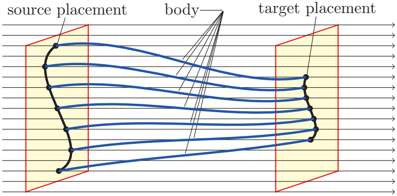
- $\mathbf{V} = \mathbf{i}{\uparrow}\mathbf{V}_{\mathcal{T}} = \mathbf{v} + \mathbf{Z}$, space and time components.

Body and particles



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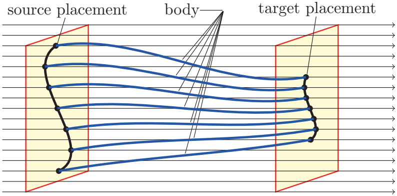

- Equivalence relation on the trajectory
  Motion related trajectory events (particle):

$$(\mathbf{e}_1\,,\mathbf{e}_2) \in \mathcal{T} \times \mathcal{T} \;:\; \mathbf{e}_2 = \varphi^{\mathcal{T}}_{t_2,t_1}(\mathbf{e}_1)\,, \quad t_i = t_{\mathcal{T}}(\mathbf{e}_i)\,, \quad i = 1,2$$

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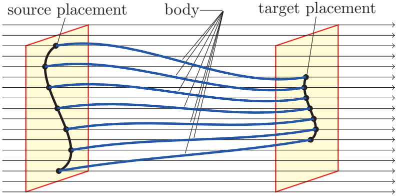

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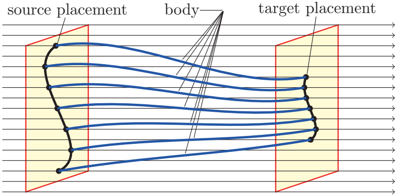
Body $=$ quotient manifold (foliation)

Body and particles



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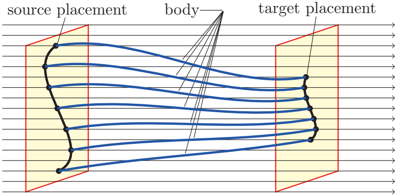

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- mass conservation

$$\int_{\boldsymbol{\Omega}_{t_1}} \mathbf{m} = \int_{\boldsymbol{\Omega}_{t_2}} \mathbf{m} \quad \Longleftrightarrow \quad \mathcal{L}_{\mathbf{V}}\mathbf{m} = \mathbf{0}$$

$\mathbf{m} : \mathcal{T} \mapsto \mathrm{VOL}(T\mathcal{T})$ mass form

Tensor fields in NLCM

Tensor fields in NLCM

Space-time fields	$\mathbf{s}_{\mathcal{E}} : \mathcal{E} \mapsto \textsc{Tens}(T\mathcal{E})$	Space-time metric tensor
Spatial fields	$\mathbf{s}_{\textsc{spa}} : \mathcal{E} \mapsto \textsc{Tens}(V\mathcal{E})$	Spatial metric tensor

Tensor fields in NLCM

Space-time fields	$\mathbf{s}_{\mathcal{E}} : \mathcal{E} \mapsto \mathrm{T{\scriptsize ENS}}(T\mathcal{E})$	Space-time metric tensor
Spatial fields	$\mathbf{s}_{\mathrm{SPA}} : \mathcal{E} \mapsto \mathrm{T{\scriptsize ENS}}(V\mathcal{E})$	Spatial metric tensor
Trajectory fields	$\mathbf{s}_{\mathcal{T}} : \mathcal{T} \mapsto \mathrm{T{\scriptsize ENS}}(T\mathcal{T})$	Trajectory-metric, trajectory speed
Material fields	$\mathbf{s}_{\mathrm{MAT}} : \mathcal{T} \mapsto \mathrm{T{\scriptsize ENS}}(V\mathcal{T})$	Material-metric, stress, stressing, stretching.

Tensor fields in NLCM

Space-time fields	$\mathbf{s}_{\mathcal{E}} : \mathcal{E} \mapsto \mathrm{TENS}(T\mathcal{E})$	Space-time metric tensor
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Material fields	$\mathbf{s}_{\mathrm{MAT}} : \mathcal{T} \mapsto \mathrm{TENS}(V\mathcal{T})$	Material-metric, stress, stressing, stretching.
Trajectory-based space-time fields	$\mathbf{s}_{\mathcal{E}} : \mathcal{T}_{\mathcal{E}} \mapsto \mathrm{TENS}(T\mathcal{E})$	Trajectory speed (immersed)
Trajectory-based spatial fields	$\mathbf{s}_{\mathrm{SPA}} : \mathcal{T}_{\mathcal{E}} \mapsto \mathrm{TENS}(V\mathcal{E})$	Virtual velocity, acceleration, force

Covariance Paradigm

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Material fields at different times along the trajectory must be compared by push along the material displacement.
Material fields on push-related trajectories must be compared by push along the relative motion.

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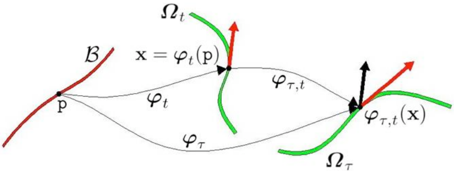
Push and parallel transport along the motion



Parallel transport does not preserve time-verticality

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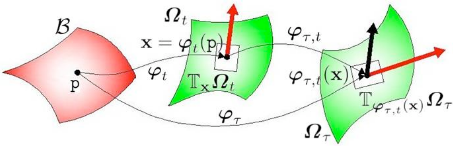

Parallel transport does not preserve time-verticality

Time derivatives $=$
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Lie Time Derivative - LTD

Marius Sophus Lie (1842 - 1899)

- Trajectory and material tensor field

$$\dot{\mathbf{s}} := \mathcal{L}_{\mathbf{V}}\,\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{V}}_{\lambda}\!\downarrow(\mathbf{s}\circ\mathbf{Fl}^{\mathbf{V}}_{\lambda})\,,$$

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Parallel Time Derivative - PTD (instead of Material Time Derivative)

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with $\mathbf{V} := \mathbf{i}{\uparrow}\mathbf{V}_{\mathcal{T}}$.

Rivers and Cogwheels

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Time derivative of the velocity field: $\mathbf{V} = \mathbf{Z} + \mathbf{v}$

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This is the celebrated EULER split formula, applicable only in special problems of hydrodynamics, where it was originally conceived.
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Notwithstanding its limitations, in most treatments of mechanics Euler split formula is improperly adopted to provide the very definition of acceleration.[1]
The result is usually named material time derivative but this is improper because the outcome is a space vector field.

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---

[1] *See e.g.*
1) C. Truesdell, *A first Course in Rational Continuum Mechanics*
Second Ed. Academic Press, New-York (1991). First Ed. (1977)
2) M.E. Gurtin, *An Introduction to Continuum Mechanics*
Academic Press, San Diego (1981)
3) J.E. Marsden & T.J.R. Hughes, *Mathematical Foundations of Elasticity*
Prentice-Hall, Redwood City, Cal. (1983)

Stretching $=$ Lie time derivative of material metric

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- $\mathbf{\Pi}_{\mathbf{e}} : T_{\mathbf{e}}\mathcal{S} \mapsto T_{\mathbf{e}}\mathbf{\Omega}$ projection
  $\mathbf{\Pi}_{\mathbf{e}}^{*} : T_{\mathbf{e}}^{*}\mathbf{\Omega} \mapsto T_{\mathbf{e}}^{*}\mathcal{S}$ immersion
- Euler's formula (generalized)

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Mixed form of the stretching tensor (standard Levi-Civita connection):

$$\boxed{\mathbf{D} := \mathbf{g}^{-1}_{\text{SPA}} \circ \frac{1}{2}\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\text{SPA}} = \operatorname{sym}(\nabla\mathbf{v})}$$

since $\text{TORS} = \mathbf{0}$ and $\nabla_{\mathbf{v}}\,\mathbf{g}_{\text{SPA}} = \mathbf{0}$

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$$\boxed{\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\sigma} = \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \!\downarrow\! \boldsymbol{\sigma})}$$

Expression in terms of LIE derivative of the immersed stress field:

$$\boxed{\mathcal{L}_{\mathbf{v}} \boldsymbol{\sigma} = \boldsymbol{\Pi} \cdot \left(\tfrac{1}{2}\, \mathcal{L}_{\mathbf{v}}\, (\mathbf{i} \!\uparrow\! \boldsymbol{\sigma}) \right) \cdot \boldsymbol{\Pi}^{*}}$$

"Objective" stress rates

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Stressing in terms of parallel derivative:

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Co-rotational stress rate tensor,
Zaremba (1903), Jaumann (1906,1911), Prager (1960):

$$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}$$

with $\dot{\mathbf{T}}$ material time derivative.

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Convective stress tensor rate,
Oldroyd (1950), Truesdell (1955), Noll (1958), Sedov (1960), Truesdell & Noll (1965):

$$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{TL}$$

Deformation gradient

The equivalence class of all material displacements whose tangent map have the common value:

$$T_{\mathbf{x}}\varphi_{\tau,t} : T_{\mathbf{x}}\mathbf{\Omega}_t \mapsto T_{\varphi_{\tau,t}(\mathbf{x})}\mathbf{\Omega}_\tau$$

- is called the *first jet* of $\varphi_{\tau,t}$ at $\mathbf{x} \in \mathbf{\Omega}_t$, in differential geometry,
- and the *relative deformation gradient* in continuum mechanics.

The chain rule between tangent maps:

$$T_{\varphi_{\tau,s}(\mathbf{x})}\varphi_{\tau,s} = T_{\varphi_{t,s}(\mathbf{x})}\varphi_{\tau,t} \circ T_{\mathbf{x}}\varphi_{t,s} \,,$$

implies the corresponding one between material deformation gradients:

$$\mathbf{F}_{\tau,s} = \mathbf{F}_{\tau,t} \circ \mathbf{F}_{t,s} \,.$$

Time rate of the deformation gradient
Standard treatment Truesdell & Noll (1965)

$$\dot{\mathbf{F}}_{t,s} = \mathbf{L}_t\, \mathbf{F}_{t,s}$$

with $\dot{\mathbf{F}}_{t,s} := \partial_{\tau=t}\, \mathbf{F}_{\tau,s}$ and $\mathbf{L}_t := \partial_{\tau=t}\, \mathbf{F}_{\tau,t}$ time derivatives ?.

$$\mathbf{L}_t(\mathbf{x}) \cdot \mathbf{h}_{\mathbf{x}} := \partial_{\tau=t}\, \mathbf{F}_{\tau,t}(\mathbf{x}) \cdot \mathbf{h}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{\Omega}_t \,, \qquad \forall\, \mathbf{h}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{\Omega}_t$$

with $\mathbf{F}_{\tau,t}(\mathbf{x}) \cdot \mathbf{h}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{\Omega}_\tau$.

Time derivatives of the deformation gradient

Time derivatives of the deformation gradient

- The LIE time derivative gives:

$$\partial_{\alpha=0} \, (T\boldsymbol{\varphi}_\alpha)^{-1} \cdot (T\boldsymbol{\varphi}_\alpha \cdot \mathbf{h}) = \partial_{\alpha=0} \, \mathbf{h} = \mathbf{0}$$

- The parallel time derivative gives:

$$\mathbf{L}(\mathbf{v}) := \partial_{\alpha=0} \, (\boldsymbol{\varphi}_\alpha \Downarrow T\boldsymbol{\varphi}_\alpha) = \nabla\mathbf{v} + \mathrm{TORS}(\mathbf{v})$$

Change of observer

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  time-bundle diffeomorphism

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "\zeta_{\mathcal{E}}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{E} \arrow[dd, bend left=60, "t_{\mathcal{E}}"] \\
\mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[r, "\zeta=\zeta_{\mathcal{T}}"] \arrow[d, "t_{\mathcal{T}}"] & \mathcal{T}_\zeta \arrow[u, "\mathbf{i}_\zeta"] \arrow[d, "t_{\mathcal{T}}"'] \\
\mathcal{Z} \arrow[r, leftrightarrow, "\mathrm{ID}"] & \mathcal{Z}
\end{tikzcd}
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\end{tikzcd}
$$

- Pushed motion

$$
\begin{tikzcd}
\mathcal{T}_{\zeta} \arrow[r, "\zeta{\uparrow}\varphi^{\mathcal{T}}_{\alpha}"] & \mathcal{T}_{\zeta} \\
\mathcal{T} \arrow[u, "\zeta"'] \arrow[r, "\varphi^{\mathcal{T}}_{\alpha}"] & \mathcal{T} \arrow[u, "\zeta"]
\end{tikzcd}
\quad \Longleftrightarrow \quad (\zeta{\uparrow}\varphi^{\mathcal{T}}_{\alpha}) \circ \zeta = \zeta \circ \varphi^{\mathcal{T}}_{\alpha}\,.
$$

Consequences of the Covariance Paradigm

Time Invariance and Frame Invariance
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- Frame Invariance $\quad \mathbf{s}_\zeta = \zeta \uparrow \mathbf{s}\,, \qquad \zeta : \mathcal{T} \mapsto \mathcal{T}_\zeta$

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Time Invariance and Frame Invariance of material fields

- Time Invariance $\quad \mathbf{s} = \varphi_\alpha \!\uparrow\! \mathbf{s}\,, \qquad \varphi_\alpha : \mathcal{E} \mapsto \mathcal{E}$
- Frame Invariance $\quad \mathbf{s}_\zeta = \zeta \!\uparrow\! \mathbf{s}\,, \qquad \zeta : \mathcal{T} \mapsto \mathcal{T}_\zeta$
- LIE time derivative along pushed motions
  Naturality of Lie derivative under diffeomorphisms

$$\mathcal{L}_{\zeta\uparrow\mathbf{v}}\,(\zeta\!\uparrow\!\mathbf{s}) = \zeta\!\uparrow\!(\mathcal{L}_{\mathbf{v}}\,\mathbf{s})$$

Frame invariance of a material tensor implies frame invariance of its time-rate.

Push of 4-velocity

Transformation rule

$$\mathbf{V}_{\mathcal{T}_\zeta} := \partial_{\alpha=0}\,(\zeta{\uparrow}\varphi^{\mathcal{T}}_\alpha) = \zeta{\uparrow}\mathbf{V}_{\mathcal{T}}\,.$$

The 4-velocity is natural with respect to frame transformations

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$$\zeta_{\mathcal{E}} : \begin{cases} \mathbf{x} \mapsto \mathbf{Q}(t)\cdot\mathbf{x} + \mathbf{c}(t) \\ t \mapsto t \end{cases}$$

$$[T\zeta_{\mathcal{E}}]\cdot[\mathbf{V}] = \begin{bmatrix} \mathbf{Q} & (\dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}) \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}} \\ 1 \end{bmatrix}$$

Straightened trajectory

Construction diffeomorphism $\boldsymbol{\xi} : \boldsymbol{\Omega}_{\mathrm{REF}} \times I \mapsto \mathcal{T}_{\mathcal{E}}$

Straightened trajectory: $\boldsymbol{\Omega}_{\mathrm{REF}} \times I$

Straightening map $\boldsymbol{\xi}^{-1} : \mathcal{T}_{\mathcal{E}} \mapsto \boldsymbol{\Omega}_{\mathrm{REF}} \times I$

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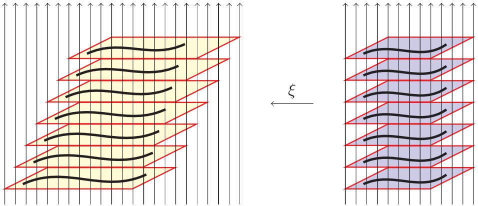

Figure : Straightening of the trajectory.

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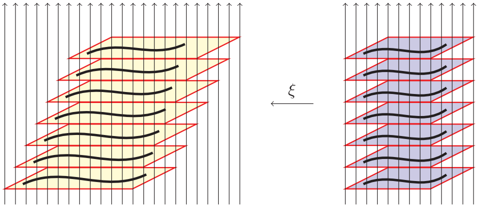

Figure : Straightening of the trajectory.

- The LIE time derivative is a partial time derivative in a straightened trajectory

$$\boldsymbol{\xi}{\downarrow}(\mathcal{L}_{\mathbf{v}}\,\mathbf{s}) = \mathcal{L}_{\mathbf{Z}}\,(\boldsymbol{\xi}{\downarrow}\mathbf{s}) = \partial_{\alpha=0}\,(\boldsymbol{\xi}{\downarrow}\mathbf{s}) \circ \mathrm{tr}_{\alpha}$$

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- Constitutive time invariance

$$\mathbf{C} = \varphi_\alpha \!\uparrow\! \mathbf{C}$$

where

$$(\varphi_\alpha \!\uparrow\! \mathbf{C})(\varphi_\alpha \!\uparrow\! \mathbf{s}) := \varphi_\alpha \!\uparrow\! (\mathbf{C}(\mathbf{s}))$$

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- **The principle of Material Frame Indifference requires that, if a set of material fields fulfills a constitutive law, then the transformed fields, when evaluated by another Euclid observer, must fulfill the same law**

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$$\boxed{\mathbf{C}_{\zeta^{\mathrm{iso}}}(\zeta^{\mathrm{iso}}\!\uparrow\mathbf{s}) = \zeta^{\mathrm{iso}}\!\uparrow(\mathbf{C}(\mathbf{s}))}$$

for any isometric relative motion $\zeta^{\mathrm{iso}} : \mathcal{T} \mapsto \mathcal{T}_{\zeta^{\mathrm{iso}}}$ induced by a change of Euclid observer $\zeta^{\mathrm{iso}}_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{E}$

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- Pure elasticity
  $\varepsilon_{\mathrm{EL}}$ elastic stretching

$$\begin{cases} \varepsilon(\mathbf{v}) & = \varepsilon_{\mathrm{EL}} \\ \varepsilon_{\mathrm{EL}} & = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \end{cases}$$

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$$\langle\, d_F \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma} \cdot \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma} \,\rangle = \text{symmetric}$$

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$$\begin{aligned} \boldsymbol{\xi}{\downarrow}\varepsilon_{\mathrm{EL}} &= d_F^2 E^*_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}}) \cdot \partial_{\alpha=0}\, (\boldsymbol{\sigma}_{\mathrm{REF}} \circ \mathrm{tr}_\alpha) \\ &= \partial_{\alpha=0}\, d_F E^*_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}}) \end{aligned}$$

- where $\boldsymbol{\sigma}_{\mathrm{REF}} = \boldsymbol{\xi}{\downarrow}\boldsymbol{\sigma}$ and $\mathrm{tr}_\alpha(\mathbf{x}, t) = (\mathbf{x}, t + \alpha)$, $\mathbf{x} \in \boldsymbol{\Omega}_{\mathrm{REF}}$.

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$$E^*_{\mathrm{REF}} := \boldsymbol{\xi}{\downarrow}E^* \quad \text{time independent}$$

Conservativeness of hyper-elasticity

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GREEN integrability of the elastic operator $\mathbf{H} = d_F^2 E^*$
as a function of the KIRCHHOFF stress tensor field
implies conservativeness:

$$\oint_I \int_{\mathbf{\Omega}_t} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\mathrm{EL}} \rangle \, \mathbf{m} \; dt = 0$$

for any cycle in the stress time-bundle,
i.e. for any stress path such that:

$$\boldsymbol{\sigma}_{t_2} = \boldsymbol{\varphi}_{t_2,t_1}\!\uparrow\!\boldsymbol{\sigma}_{t_1} \,, \quad I = [t_1, t_2]$$

Elasto-visco-plasticity

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- Constitutive law

$\varepsilon_{\mathrm{EL}}$ elastic stretching
$\varepsilon_{\mathrm{PL}}$ visco-plastic stretching

$$\begin{cases} \varepsilon(\mathbf{v}) = \varepsilon_{\mathrm{EL}} + \varepsilon_{\mathrm{PL}} & \text{stretching additivity} \\ \varepsilon_{\mathrm{EL}} = d_F^2 E^*(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} & \text{hyper-elastic law} \\ \varepsilon_{\mathrm{PL}} \in \partial_F \mathcal{F}(\boldsymbol{\sigma}) & \text{visco-plastic flow rule} \end{cases}$$

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Examples:

- The simplest rate-elastic operator is GREEN integrable and frame invariant:

$$\mathbf{H}(\mathbf{T}) := \frac{1}{2\,\mu}\,\mathbb{I} - \frac{\nu}{E}\,\mathbf{I} \otimes \mathbf{I}$$

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These results provide answers to unsolved questions posed in:

J.C. Simó & K.S. Pister, Remarks on rate constitutive equations for finite deformation problems: computational implications, *Comp. Meth. Appl. Mech. Eng.* **46** (1984) 201–215.
J. C. Simó & M. Ortiz, A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive equations, *Comp. Meth. Appl. Mech. Eng.* **49** (1985) 221–245.

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- Algorithms for numerical computations are modified to comply with the geometric theory; multiplicative decomposition of the deformation gradient is deemed geometrically inconsistent