The G-Factor Impact in NLCM

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NLCM = Non-Linear Continuum Mechanics
and
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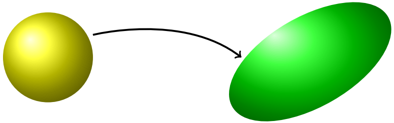
NLCM is an important source of inspiration for DG and DG is the natural tool to develop a mathematical modeling of NLCM



Prolegomena

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Hermann Weyl (1885–1955)

In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.

H. Weyl, "Invariants", Duke Mathematical Journal 5 (3): (1939) 489–502

Prolegomena

A basic question in NLCM

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- How to compare the metric and stress tensors
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- Devil's temptation:

  In 3D bodies it might seem as natural to compare by translation the traction vectors corresponding to translated normals to cutting surfaces. This is tacitly done when writing the stress time-rate as $\dot{\mathbf{T}}$ *but is geometrically untenable as may be more clearly seen by considering 1D and 2D models (wires and membranes).*

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- Hint:

  Tangent vectors to a body placement may be transformed into tangent vectors to another body placement only by means of the differential of the displacement map. This is the essence of the COVARIANCE PARADIGM.

Cable

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DIMENSIONALITY INDEPENDENCE:
A geometrically consistent framework should be equally applicable to body models of any dimension.

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Motivation for the COVARIANCE PARADIGM [1]

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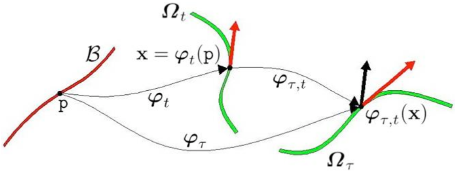

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Math1

Tangent vector to a manifold: velocity of a curve

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Cotangent vector

$$\mathbf{v}^* \in L\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathcal{R}) \in \mathbb{T}^*_{\mathbf{x}}\mathbb{M}$$

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- A map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ sends a curve $\mathbf{c} \in \mathrm{C}^1([a,b]\,;\mathbb{M})$ into a curve $\boldsymbol{\zeta} \circ \mathbf{c} \in \mathrm{C}^1([a,b]\,;\mathbb{N})$.

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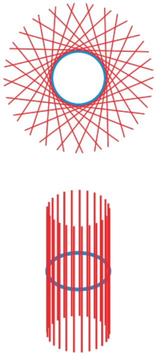
  a curve $\boldsymbol{\zeta} \circ \mathbf{c} \in \mathrm{C}^1([a, b]\,;\mathbb{N})$.
- The tangent map $T_{\mathbf{x}}\boldsymbol{\zeta} \in \mathrm{C}^0(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})$
  sends a tangent vector at $\mathbf{x} \in \mathbb{M}$
  $\mathbf{v} \in \mathbb{T}_{\mathbf{x}}(\mathbb{M}) := \partial_{\mu=\lambda}\, \mathbf{c}(\mu)$
  into a tangent vector at $\boldsymbol{\zeta}(\mathbf{x}) \in \mathbb{N}$
  $T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{v} \in \mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}(\mathbb{N}) := \partial_{\mu=\lambda}\,(\boldsymbol{\zeta} \circ \mathbf{c})(\mu)$

Math2

Math2

Tangent bundle

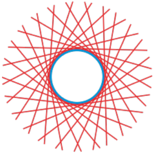


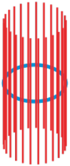
Math2

Tangent bundle

- disjoint union of tangent spaces:

$$\mathbb{TM} := \cup_{\mathbf{x}\in\mathbb{M}} \mathbb{T}_{\mathbf{x}}\mathbb{M}$$





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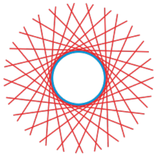

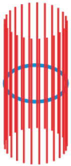

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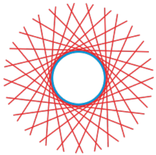

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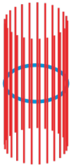

- Surjective submersion:

$T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{T}_{\mathbf{v}}\mathbb{TM}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})$ is surjective

Math2

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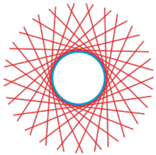

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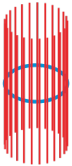

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- Tangent functor

$$\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N}) \quad \mapsto \quad T\boldsymbol{\zeta} \in \mathrm{C}^0(\mathbb{TM}\,;\mathbb{TN})$$

Math3

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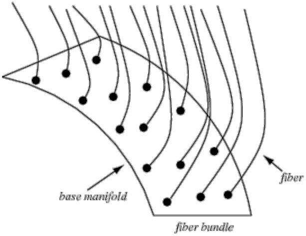
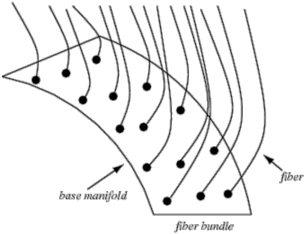
Fiber bundles



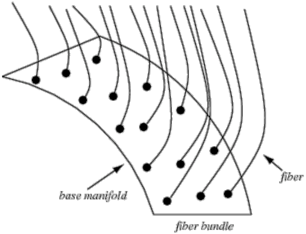
Math3

Fiber bundles

- $\mathbb{E}, \mathbb{M}$ manifolds



Math3



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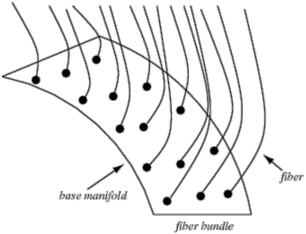
Math3



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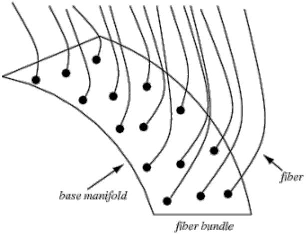

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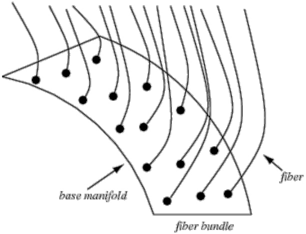

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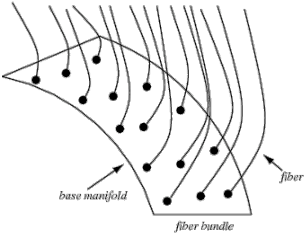
Math3



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$$\delta\mathbf{e} \in \mathbb{V}\mathrm{E} \subset \mathbb{T}\mathrm{E} \quad \Longrightarrow \quad T_{\mathbf{e}}\boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{E}} \cdot \delta\mathbf{e} = 0$$

Math4

Math4

Trivial and non-trivial fiber bundles

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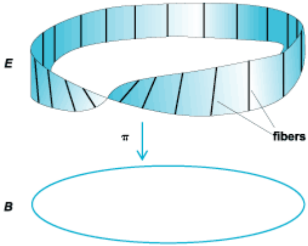
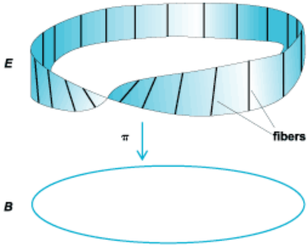
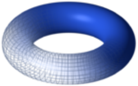


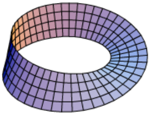
Math4

Trivial and non-trivial fiber bundles

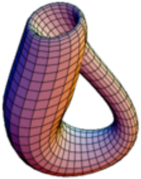




Torus



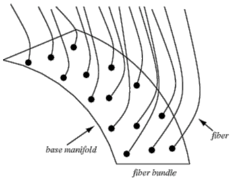
Listing-Möbius strip



Klein Bottle

Math5

Sections of fiber bundles



Math5

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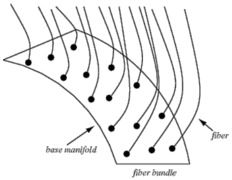

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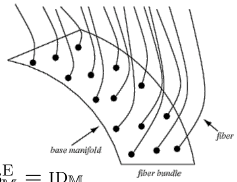

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- Fiber bundle $\boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{M})$
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Math5

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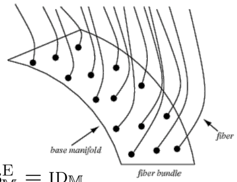

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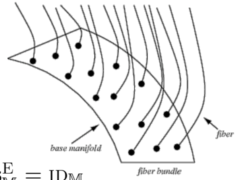

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Math5

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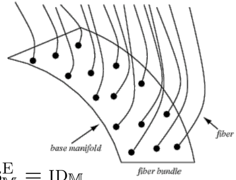

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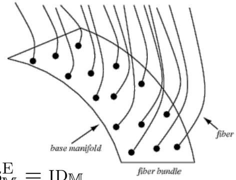
Sections of tangent and bi-tangent bundles

Math5

Sections of fiber bundles



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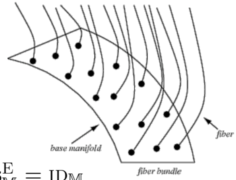
Sections of tangent and bi-tangent bundles

- Tangent vector fields:

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$

Sections of fiber bundles



- Fiber bundle $\boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{M})$
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- Tangent v.f. $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})\,, \quad \boldsymbol{\tau}_{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = \mathrm{ID}_{\mathrm{E}}$
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Sections of tangent and bi-tangent bundles

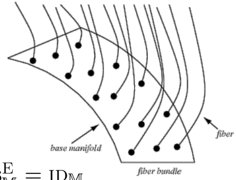
- Tangent vector fields:

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$

- Bi-tangent vector fields:

$$\mathbf{X} \in \mathrm{C}^1(\mathbb{T}\mathbb{M}\,;\mathbb{T}\mathbb{T}\mathbb{M}) \;:\; \boldsymbol{\tau}_{\mathbb{T}\mathbb{M}} \circ \mathbf{X} = \mathrm{ID}_{\mathbb{T}\mathbb{M}}$$

Math5



Sections of fiber bundles

- Fiber bundle $\boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{M})$
- Sections $\mathbf{s}_{\mathbb{M}}^{\mathrm{E}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{E})\,, \quad \boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{E}} \circ \mathbf{s}_{\mathbb{M}}^{\mathrm{E}} = \mathrm{ID}_{\mathbb{M}}$
- Tangent v.f. $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})\,, \quad \boldsymbol{\tau}_{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = \mathrm{ID}_{\mathrm{E}}$
- Vertical tangent sections $T\boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{E}} \circ \mathbf{v}_{\mathrm{E}} = 0$

Sections of tangent and bi-tangent bundles

- Tangent vector fields:
$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M}) \ : \ \boldsymbol{\tau}_{\mathbb{M}} \circ \mathbf{v} = \mathrm{ID}_{\mathbb{M}}$$
- Bi-tangent vector fields:
$$\mathbf{X} \in \mathrm{C}^1(\mathbb{T}\mathbb{M}\,;\mathbb{T}\mathbb{T}\mathbb{M}) \ : \ \boldsymbol{\tau}_{\mathbb{T}\mathbb{M}} \circ \mathbf{X} = \mathrm{ID}_{\mathbb{T}\mathbb{M}}$$
- Vertical bi-tangent vectors $\mathbf{X} \in \mathrm{Ker}\, T_{\mathbf{v}}\boldsymbol{\tau}_{\mathbb{M}}$

Math6

Tensor spaces

Math6

Tensor spaces

- Covariant
  $\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}^*_{\mathbf{x}}\mathbb{M})$

Math6

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  $\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \in \mathrm{CON}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}^*_{\mathbf{x}}\mathbb{M}^2\,;\mathcal{R}) = L(\mathbb{T}^*_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})$

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- Contravariant
  $\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \in \mathrm{CON}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M}^2 ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- Mixed
  $\mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \in \mathrm{MIX}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}_{\mathbf{x}}^*\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
- with the alteration rules:

$$\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} , \quad \mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} = \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \circ \mathbf{g}_{\mathbf{x}}^{-1} ,$$

Tensor bundles and sections

Math6

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Tensor bundles and sections

- Tensor bundle $\quad \boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{TM}) ; \mathbb{M})$

Math6

Tensor spaces

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  $\mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \in \mathrm{MIX}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} , \mathbb{T}^*_{\mathbf{x}}\mathbb{M} ; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M} ; \mathbb{T}_{\mathbf{x}}\mathbb{M})$
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$$\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} = \mathbf{g}_{\mathbf{x}} \circ \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} , \quad \mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} = \mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \circ \mathbf{g}_{\mathbf{x}}^{-1} ,$$

Tensor bundles and sections

- Tensor bundle $\boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{TM}) ; \mathbb{M})$
- Tensor field $\mathbf{s}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathbb{M} ; \mathrm{TENS}(\mathbb{TM}))$

Math6

Tensor spaces

- Covariant
  $\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}) = L\left(\mathbb{T}_{\mathbf{x}}\mathbb{M}^2\,;\mathcal{R}\right) = L\left(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}\right)$
- Contravariant
  $\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \in \mathrm{CON}_{\mathbf{x}}(\mathbb{TM}) = L\left(\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}^2\,;\mathcal{R}\right) = L\left(\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M}\right)$
- Mixed
  $\mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \in \mathrm{MIX}_{\mathbf{x}}(\mathbb{TM}) = L\left(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,,\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}\,;\mathcal{R}\right) = L\left(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M}\right)$
- with the alteration rules:

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Tensor bundles and sections

- Tensor bundle $\boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{TM})\,;\mathbb{M})$
- Tensor field $\mathbf{s}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{TENS}(\mathbb{TM}))$
- with: $\boldsymbol{\tau}_{\mathbb{M}}^{\mathrm{TENS}} \circ \mathbf{s}_{\mathbb{M}}^{\mathrm{TENS}} = \mathrm{ID}_{\mathbb{M}}$

Math7

Push and pull

Math7

Push and pull

Given a map $\zeta \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$

- Pull-back of a scalar field

$$f : \mathbb{N} \mapsto \mathrm{FUN}(\mathbb{N}) \quad \mapsto \quad \zeta\!\downarrow\! f : \mathbb{M} \mapsto \mathrm{FUN}(\mathbb{M})$$

defined by:

$$(\zeta\!\downarrow\! f)_{\mathbf{x}} := \zeta\!\downarrow\! f_{\zeta(\mathbf{x})} := f_{\zeta(\mathbf{x})} \in \mathrm{FUN}_{\mathbf{x}}(\mathbb{M})\,.$$

Push and pull

Given a map $\zeta \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$

- Pull-back of a scalar field

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- Push-forward of a tangent vector field

$$\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM}) \quad \mapsto \quad \zeta{\uparrow}\mathbf{v} : \mathbb{N} \mapsto \mathbb{TN}$$

defined by:

$$(\zeta{\uparrow}\mathbf{v})_{\zeta(\mathbf{x})} := \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} = T_{\mathbf{x}}\zeta \cdot \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N}\,.$$

Math8

Push and pull of tensor fields

Math8

Push and pull of tensor fields

- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} \rangle = \langle T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle$$

Math8

Push and pull of tensor fields

- Covectors

$$\langle\, \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}}\,\rangle = \langle\, \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_{\mathbf{x}}\,\rangle = \langle\, T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}}\,\rangle$$

- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} \circ T_{\mathbf{x}}\zeta \in \mathrm{COV}(\mathbb{TM})_{\mathbf{x}}$$

Math8

Push and pull of tensor fields

- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_\mathbf{x} \rangle = \langle T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_\mathbf{x} \rangle$$

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$$\zeta{\downarrow}\mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} \circ T_\mathbf{x}\zeta \in \mathrm{COV}(\mathbb{TM})_\mathbf{x}$$

- Contravariant tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{CON}}_\mathbf{x} = T_\mathbf{x}\zeta \circ \mathbf{s}^{\mathrm{CON}}_\mathbf{x} \circ T^*_{\zeta(\mathbf{x})}\zeta \in \mathrm{CON}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

Math8

Push and pull of tensor fields

- Covectors

$$\langle \zeta{\downarrow}\mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle = \langle \mathbf{v}^*_{\zeta(\mathbf{x})}, \zeta{\uparrow}\mathbf{v}_{\mathbf{x}} \rangle = \langle T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{v}^*_{\zeta(\mathbf{x})}, \mathbf{v}_{\mathbf{x}} \rangle$$

- Covariant tensors

$$\zeta{\downarrow}\mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} = T^*_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{COV}}_{\zeta(\mathbf{x})} \circ T_{\mathbf{x}}\zeta \in \mathrm{COV}(\mathbb{TM})_{\mathbf{x}}$$

- Contravariant tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{CON}}_{\mathbf{x}} = T_{\mathbf{x}}\zeta \circ \mathbf{s}^{\mathrm{CON}}_{\mathbf{x}} \circ T^*_{\zeta(\mathbf{x})}\zeta \in \mathrm{CON}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

- Mixed tensors

$$\zeta{\uparrow}\mathbf{s}^{\mathrm{MIX}}_{\mathbf{x}} = T_{\mathbf{x}}\zeta \circ \mathbf{s}^{\mathrm{MIX}}_{\mathbf{x}} \circ T_{\zeta(\mathbf{x})}\zeta^{-1} \in \mathrm{MIX}(\mathbb{TN})_{\zeta(\mathbf{x})}$$

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a, b] ; \mathbb{M})$

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a, b] ; \mathbb{M})$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu}\Uparrow \mathbf{v_x} \in \mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$$

$$\mathbf{c}_{\mu,\mu}\Uparrow \mathbf{v_x} = \mathbf{v_x}$$

$$\mathbf{c}_{\lambda,\mu}\Uparrow \circ\, \mathbf{c}_{\mu,\nu}\Uparrow \,=\, \mathbf{c}_{\lambda,\nu}\Uparrow$$

Math9

Parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1([a, b]\,;\mathbb{M})$

- Vector fields

$$\mathbf{x} = \mathbf{c}(\mu)\,, \quad \mathbf{v_x} \in \mathbb{T}_\mathbf{x}\mathbb{M} \quad \mapsto \quad \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \in \mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$$

$$\mathbf{c}_{\mu,\mu} \Uparrow \mathbf{v_x} = \mathbf{v_x}$$
$$\mathbf{c}_{\lambda,\mu} \Uparrow \circ\, \mathbf{c}_{\mu,\nu} \Uparrow \, = \mathbf{c}_{\lambda,\nu} \Uparrow$$

- Covector fields $\mathbf{v}^*_\mathbf{x} \in \mathbb{T}^*_\mathbf{x}\mathbb{M}$ (naturality)

$$\langle\, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v}^*_\mathbf{x}, \mathbf{c}_{\lambda,\mu} \Uparrow \mathbf{v_x} \,\rangle = \mathbf{c}_{\lambda,\mu} \Uparrow \langle\, \mathbf{v}^*_\mathbf{x}, \mathbf{v_x} \,\rangle$$

- Tensor fields (naturality)

NLCM: Nonlinear Continuum Mechanics

NLCM: Nonlinear Continuum Mechanics

How to play the game

NLCM: Nonlinear Continuum Mechanics

How to play the game

Kinematics

NLCM: Nonlinear Continuum Mechanics

How to play the game

Kinematics

- Events manifold: $\mathbb{E}$ – four dimensional

NLCM: Nonlinear Continuum Mechanics

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Kinematics

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- Observer split into space-time: $\mathcal{S} \times I$

NLCM: Nonlinear Continuum Mechanics

How to play the game

Kinematics

- Events manifold: E – four dimensional
- Observer split into space-time: $\mathcal{S} \times I$
- time is absolute (Classical Mechanics)

NLCM: Nonlinear Continuum Mechanics

How to play the game

Kinematics

- Events manifold: $\mathbb{E}$ – four dimensional
- Observer split into space-time: $\mathcal{S} \times I$
- time is absolute (Classical Mechanics)
- distance between simultaneous events $\mapsto$ metric tensor

Events manifold fibrations

Events manifold fibrations

- Time and space fibrations: $\gamma : \mathrm{E} \mapsto \mathcal{S} \times I$ (observer)

Events manifold fibrations

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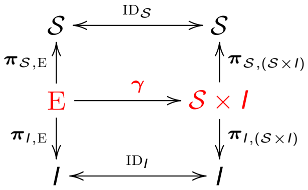

$\iff$

$$\boldsymbol{\pi}_{I,\mathrm{E}} = \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \circ \boldsymbol{\gamma}$$
$$\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} = \boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)} \circ \boldsymbol{\gamma}$$

Events manifold fibrations

- Time and space fibrations: $\gamma : \mathrm{E} \mapsto \mathcal{S} \times I$ (observer)

$$
\begin{array}{ccc}
\mathcal{S} & \overset{\mathrm{ID}_{\mathcal{S}}}{\longleftrightarrow} & \mathcal{S} \\
\uparrow \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} & & \uparrow \boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)} \\
\mathrm{E} & \overset{\gamma}{\longrightarrow} & \mathcal{S} \times I \\
\downarrow \boldsymbol{\pi}_{I,\mathrm{E}} & & \downarrow \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \\
I & \overset{\mathrm{ID}_{I}}{\longleftrightarrow} & I
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{l}
\boldsymbol{\pi}_{I,\mathrm{E}} = \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \circ \boldsymbol{\gamma} \\
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\end{array}
$$

- Time-vertical subbundle: spatial vectors

$$\mathbf{v} \in \mathbb{V}_{\mathbf{e}}\mathrm{E} \quad \Longleftrightarrow \quad T_{\mathbf{e}}\boldsymbol{\pi}_{I,\mathrm{E}} \cdot \mathbf{v} = 0$$

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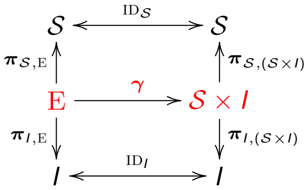

$$\iff \quad \begin{aligned} \boldsymbol{\pi}_{I,\mathrm{E}} &= \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \circ \boldsymbol{\gamma} \\ \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} &= \boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)} \circ \boldsymbol{\gamma} \end{aligned}$$

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- $\mathbf{v}_{\mathbf{e}} \in \mathbb{V}_{\mathbf{e}}\mathrm{E} \quad \iff \quad \gamma{\uparrow}\mathbf{v}_{\mathbf{e}} = (v_{\mathbf{x},t}\,, 0_t) \in \mathbb{T}_{\mathbf{x}}\mathcal{S} \times \mathbb{T}_t I$

Trajectory and evolution

Trajectory and evolution

- Trajectory: $\mathcal{T}_{\varphi} \subset \mathrm{E}$; subbundle of the events time-bundle
  ball (dim=3+1)
  membrane (dim=2+1)
  wire (dim=1+1)

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- time fibration $\mapsto$ fibers: body placements $\boldsymbol{\Omega}_t$
- vertical tangent fibration $\mapsto$ material vectors $\mathbf{v}_{\varphi}$
- Evolution operator: $\boldsymbol{\varphi}$
- Law of determinism (Chapman-Kolmogorov):

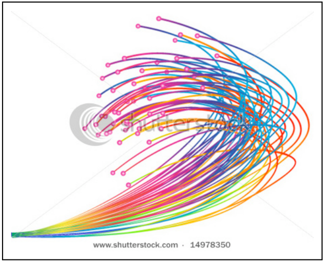
$$\boldsymbol{\varphi}_{\tau,s} = \boldsymbol{\varphi}_{\tau,t} \circ \boldsymbol{\varphi}_{t,s}$$

- Displacements: diffeomorphisms between placements

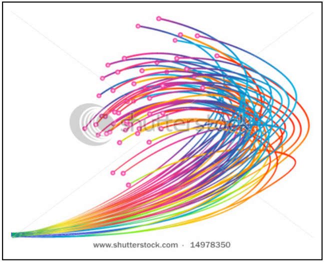
$$\boldsymbol{\varphi}_{\tau,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t \,;\boldsymbol{\Omega}_\tau)\,, \quad \tau, t \in I$$

Body and particles

Body and particles



Body and particles

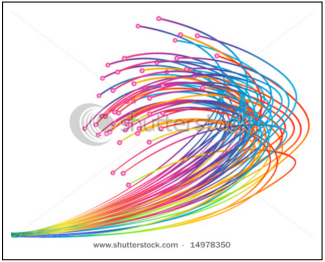


- Equivalence relation on the trajectory:

$$(\mathbf{e}_1\,,\mathbf{e}_2) \in \mathcal{T}_{\varphi} \times \mathcal{T}_{\varphi} \;:\; \mathbf{e}_2 = \varphi_{t_2,t_1}(\mathbf{e}_1)\,.$$

with $t_i = \pi_{I,\mathrm{E}}(\mathbf{e}_i)\,, \quad i = 1,2\,.$

Body and particles



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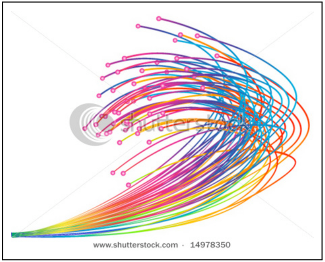
Body $=$ quotient manifold (foliation)

Body and particles



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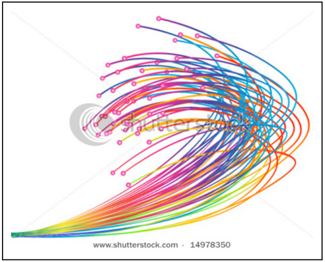
Body = quotient manifold (foliation)
Particles = equivalence classes (folia)

Body and particles



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Body = quotient manifold (foliation)
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- mass conservation

Tensor fields in NLCM

Tensor bundles

Tensor fields in NLCM

Tensor bundles

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Fields in Continuum Mechanics and Thermodynamics

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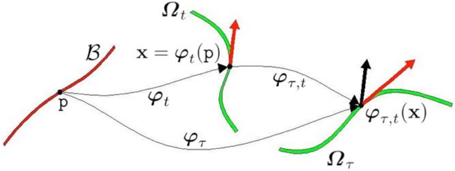

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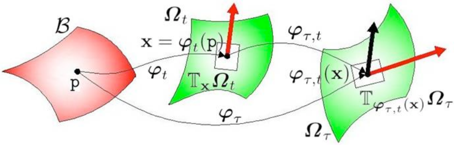

Time derivatives along the motion

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Lie time derivative - LTD (Convective time derivative - CTD)

- Material tensor field

$$\dot{\mathbf{s}}_{\boldsymbol{\varphi},t} := \mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{s}_{\boldsymbol{\varphi}} = \partial_{\tau=t}\left(\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{s}_{\boldsymbol{\varphi},\tau}\right)$$

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Material time-derivative - MTD
(Parallel time-derivative - PTD)

- Material-based spatial fields

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Beware of LEIBNIZ (Gottfried Wilhelm von, 1646 - 1716)

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MTD of the velocity field - Acceleration

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The latter is D'ALEMBERT-D. BERNOULLI formula, applicable only in special problems in hydrodynamics, where it was conceived. This eventually led to the NAVIER-STOKES-ST.VENANT differential equation of motion

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Both conditions are not fulfilled in solid mechanics, as a rule.

A sample of objective stress tensors

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Convective stress tensor rates in Truesdell & Noll (1965):

$$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{L}^{T}\mathbf{T} + \mathbf{T}\mathbf{L}$$

Co-rotational stress tensor rates in Truesdell & Noll (1965):

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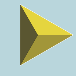
Both formulas rely on Leibniz rule and on treating the material stress tensor field as a spatial valued tensor field

Math10

Math10



lenght of symplex's edges

Math10

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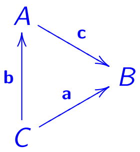
- Norm axioms

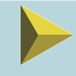


$$\|\mathbf{a}\| \geq 0\,, \quad \|\mathbf{a}\| = 0 \quad \Longrightarrow \quad \mathbf{a} = 0$$

$$\|\mathbf{a}\| + \|\mathbf{b}\| \geq \|\mathbf{c}\| \quad \textbf{triangle inequality},$$

$$\|\alpha\,\mathbf{a}\| = |\alpha|\,\|\mathbf{a}\|$$

Math10



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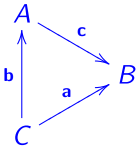

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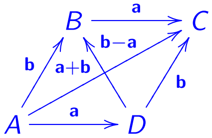
- Parallelogram rule



$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\left[\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right]$$

Math11

Math11

The metric tensor

- Theorem (Fréchet – von Neumann – Jordan)

Math11

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$$\mathbf{g}(\mathbf{a}\,,\mathbf{b}) := \frac{1}{4}\left[\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right]$$

Math11

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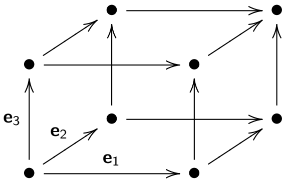

$$\Big)^2 = \det\begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1)\cdots\mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$$

Kosaku Josida (1909 - 1990)

Math11

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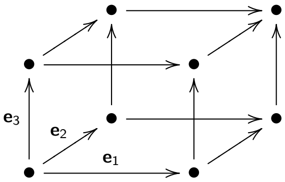

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Maurice René Fréchet (1878 - 1973)

Math11

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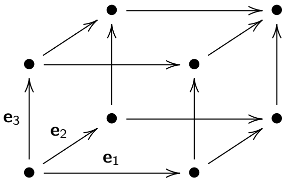

$\Big)^2 = \det \begin{bmatrix} \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_1) \cdots \mathbf{g}(\mathbf{e}_1\,,\mathbf{e}_3) \\ \cdots \quad \cdots \quad \cdots \\ \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_1) \cdots \mathbf{g}(\mathbf{e}_3\,,\mathbf{e}_3) \end{bmatrix}$

John von Neumann (1903 - 1957)

Math11

The metric tensor

- Theorem (Fréchet – von Neumann – Jordan)

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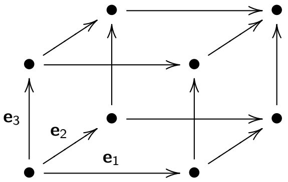
VOL(



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Pascual Jordan (1902 - 1980)

Material pull back of the spatial metric tensor

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Stretching

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Leonhard Euler (1707 - 1783) 

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Lie (convective) time derivative

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  $\mathbf{g}\circ\nabla\mathbf{v}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathrm{COV}(\mathbb{T}_{\boldsymbol{\Omega}_t}\mathcal{S}))$
- Mixed form of the stretching tensor (standard):

$$\mathbf{D}_{\boldsymbol{\varphi},t} := \mathbf{g}^{-1}_{\boldsymbol{\varphi},t}\circ\frac{1}{2}\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{g}_{\boldsymbol{\varphi}} = \mathrm{sym}\,(\nabla^{\mathrm{MAT}}\mathbf{v}_{\boldsymbol{\varphi},t})$$

Stress and stressing

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Lie (convective) time derivative

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- Stress: $\boldsymbol{\sigma}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}} ; \mathrm{CON}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}}))$
  contravariant material tensor field in duality with the stretching covariant material tensor field:
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$$\begin{aligned}\mathcal{L}_{\boldsymbol{\varphi},t}\,\mathbf{s}_{\mathrm{E}} := \partial_{\tau=t}\,(\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{s}_{\mathrm{E},\tau}) &= \partial_{\tau=t}\;\mathbf{s}_{\mathrm{E},\tau} + \partial_{\tau=t}\,\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!(\mathbf{s}_{\mathrm{E},t}\circ\boldsymbol{\varphi}_{\tau,t})\\ &= \partial_{\tau=t}\;\mathbf{s}_{\mathrm{E},\tau} + \mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi},t}}\,\mathbf{s}_{\mathrm{E},t}\end{aligned}$$

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- Expression of Lie derivative in terms of parallel derivative
  $\mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi},t}}\;\mathbf{s}_{\mathrm{E},t} = \nabla_{\mathbf{v}_{\boldsymbol{\varphi},t}}\,\mathbf{s}_{\mathrm{E},t} - 2\,\mathrm{sym}\,(\nabla\mathbf{v}_{\boldsymbol{\varphi},t} \circ \mathbf{s}_{\mathrm{E},t})$

Change of observer

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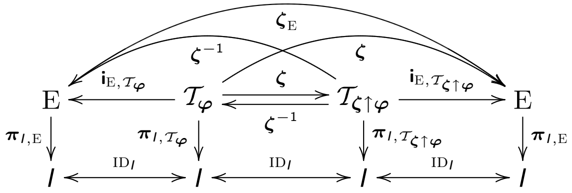
- Relative motion: $\quad \boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathcal{T}_{\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}})\,,$ diffeomorphism



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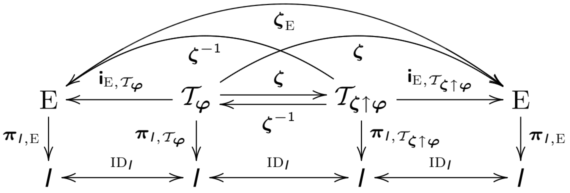
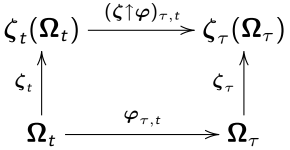


- Pushed motion:



$$\Longleftrightarrow \quad (\zeta\uparrow\varphi)_{\tau,t} = \zeta_\tau \circ \varphi_{\tau,t} \circ \zeta_t^{-1}$$

Consequences of the Covariance Paradigm

Time independence and Invariance of material fields

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- Invariance $\mathbf{s}_{\zeta\uparrow\varphi} = \zeta\!\uparrow\!\mathbf{s}_{\varphi}$
- Push of Lie time derivative to reference

$$\varphi_{t,\mathrm{REF}}\!\downarrow\!(\mathcal{L}_{\varphi,t}\,\mathbf{s}_{\varphi}) = \partial_{\tau=t}\,\varphi_{\tau,\mathrm{REF}}\!\downarrow\!\mathbf{s}_{\varphi,\tau}$$

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- Lie time derivative along pushed motions

$$\mathcal{L}_{(\zeta\uparrow\varphi),t}\,(\zeta\!\uparrow\!\mathbf{s}_{\varphi}) = \zeta_t\!\uparrow\!\mathcal{L}_{\varphi,t}\,\mathbf{s}_{\varphi}$$

Constitutive laws

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- Constitutive operator $\mathbf{H}_{\varphi}$

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A material bundle morphism whose domain and codomain are Whitney products of material tensor bundles

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- Constitutive invariance under relative motions

$$\mathbf{H}_{\zeta\uparrow\varphi} = \zeta\!\uparrow\!\mathbf{H}_{\varphi}$$

$$(\zeta\!\uparrow\!\mathbf{H}_{\varphi})(\zeta\!\uparrow\!\mathbf{s}_{\varphi}) = \zeta\!\uparrow\!(\mathbf{H}_{\varphi}(\mathbf{s}_{\varphi}))$$

Hypo-elasticity

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- Constitutive hypo-elastic law

$$\begin{cases} \dot{\varepsilon}_{\varphi} = \mathbf{e}_{\varphi} \\ \mathbf{e}_{\varphi} = \mathbf{H}_{\varphi}^{\mathrm{HYPO}}(\boldsymbol{\sigma}_{\varphi}) \cdot \dot{\boldsymbol{\sigma}}_{\varphi} \end{cases}$$

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$$\langle\, d_F \mathbf{H}_{\varphi}^{\mathrm{HYPO}}(\boldsymbol{\sigma}_{\varphi}) \cdot \delta\boldsymbol{\sigma}_{\varphi} \cdot \delta_1\boldsymbol{\sigma}_{\varphi}, \delta_2\boldsymbol{\sigma}_{\varphi} \,\rangle = \text{symmetric} \quad \Longrightarrow$$

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- GREEN integrability

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Elasticity

- Elastic constitutive operator:
  hypo-elastic constitutive operator which is integrable and time independent

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- pull-back to reference:

$$\begin{aligned} \boldsymbol{\varphi}_{t,\mathrm{REF}}\!\downarrow\mathbf{e}_{\varphi,t} &= d_F^2 E_{\mathrm{REF}}^*(\boldsymbol{\varphi}_{t,\mathrm{REF}}\!\downarrow\boldsymbol{\sigma}_{\varphi,t}) \cdot \partial_{\tau=t}\ \boldsymbol{\varphi}_{\tau,\mathrm{REF}}\!\downarrow\boldsymbol{\sigma}_{\varphi,\tau} \\ &= \partial_{\tau=t}\, d_F E_{\mathrm{REF}}^*(\boldsymbol{\varphi}_{\tau,\mathrm{REF}}\!\downarrow\boldsymbol{\sigma}_{\varphi,\tau}) \end{aligned}$$

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$$\begin{aligned} \varphi_{\tau,\mathrm{REF}} &:= \varphi_{\tau,t} \circ \varphi_{t,\mathrm{REF}} \\ E_{\mathrm{REF}}^* &:= \varphi_{t,\mathrm{REF}}\!\downarrow\! E_{\varphi,t}^* \quad \text{time independent} \end{aligned}$$

Conservativeness of hyper-elasticity

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GREEN integrability of the elastic operator $\mathbf{H}_{\boldsymbol{\varphi}}$ implies conservativeness:

$$\oint_{I} \int_{\Omega_t} \langle \boldsymbol{\sigma}_{\boldsymbol{\varphi},t}, \mathbf{e}_{\boldsymbol{\varphi},t} \rangle \, \mathbf{m}_{\boldsymbol{\varphi},t} \; dt = 0$$

for any cycle in the stress time bundle,

i.e. for any stress path $\boldsymbol{\sigma}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(I\,;\mathrm{CON}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}}))$ such that:

$$\boldsymbol{\sigma}_{\boldsymbol{\varphi},t_2} = \boldsymbol{\varphi}_{t_2,t_1}\!\uparrow\!\boldsymbol{\sigma}_{\boldsymbol{\varphi},t_1} \,, \quad I = [t_1, t_2]$$

Elasto-visco-plasticity

Elasto-visco-plasticity

- Constitutive law

$$
\begin{cases}
\dot{\varepsilon}_{\varphi} = \mathbf{e}_{\varphi} + \mathbf{p}_{\varphi} & \text{stretching additivity} \\
\mathbf{e}_{\varphi} = d_F^2 E_{\varphi}^*(\boldsymbol{\sigma}_{\varphi}) \cdot \dot{\boldsymbol{\sigma}}_{\varphi} & \text{hyper-elastic law} \\
\mathbf{p}_{\varphi} \in \partial_F \mathcal{F}_{\varphi}(\boldsymbol{\sigma}_{\varphi}) & \text{visco-plastic flow rule}
\end{cases}
$$

Reference strains

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- total strain in the time interval $I = [s, t]$:

$$\varepsilon_{\varphi,t,s} := \varphi_{t,s}\!\downarrow\!\mathbf{g}_{\varphi,t} - \mathbf{g}_{\varphi,s}$$

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Material Frame Indifference (MFI)

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Sufficient conditions

- Material fields must be frame invariant
- Constitutive operators must be frame invariant

MFI in elasto-visco-plasticity

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- the visco-plastic flow rule is frame invariant

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Consequences

- treatments of constitutive behaviors in the nonlinear range should be revised and reformulated
- algorithms for numerical computations must be modified to comply with the covariant theory; multiplicative decomposition of the deformation gradient is geometrically inconsistent