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On natural derivatives and the curvature formula in fiber bundles

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Abstract - Natural derivatives of a section of a fiber bundle are defined as tangent vector fields on the image of the section. A local extension to vector fields in the tangent bundle leads to a direct proof of the formula expressing the curvature of a connection in terms of vertical derivatives. The result is based on the tensoriality property of the horizontal lifting and extends to nonlinear connections on fiber bundles a well-known formula for linear connections on vector bundles.

Riassunto - Le derivate naturali di una sezione di una varietà fibrata sono definite come campi vettoriali tangenti all'immagine della sezione. Un'estensione locale a campi vettoriali nel fibrato tangente consente una dimostrazione diretta della formula che esprime la curvatura di una connessione in termini di derivate verticali. Il risultato è basato sul ricorso alla proprietà di tensorialità del sollevamento orizzontale ed estende a connessioni non lineari su varietà fibrate una nota formula relativa a connessioni lineari su fibrati vettoriali.

1 INTRODUCTION

The notion of connection on a fiber bundle was introduced by Charles Ehresmann (1950) and investigated in (Libermann, 1969, 1973, 1982).

Standard references on the topic are the article by Kobayashi (1957) and the text by Kobayashi and Nomizu (1963).

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The analysis developed in this paper makes a more direct reference to the treatment of the matter presented in (Kolar, Michor and Slovak, 1993).

Let us recall some well-known facts. In the tangent bundle to a fiber bundle, the vertical distribution is naturally defined as the subbubdle of vectors tangent to the total manifold, with a null projection on the base manifold.

The vertical distribution is always integrable and the leaves of the induced foliation are the fibers themselves.

The general definition of a connection as a (regular) field of projectors on the vertical subspaces of the tangent spaces to a fiber bundle, splits each tangent space into two complementary subspaces, the vertical and the horizontal ones. This leads naturally to the question about integrability of the horizontal distribution.

The involutivity condition provided by FROBENIUS theorem leads to the definition of the curvature as obstruction against integrability of the horizontal distribution, see e.g. (Kolar et al., 1993).

In this context a new result, stated in Theorem 4.1 below, builds a direct bridge between the expression of the curvature in terms of horizontal lifts, which is the one naturally stemming from FROBENIUS involutivity condition, and the expression of the curvature in terms of covariant derivatives, more suitable for applications.

The new result extends the well-known expression of the curvature for linear connections on vector or principal bundles, to general connections on fiber bundles.

The proof is based on the novel definition of natural derivatives, and on an extension, of the natural derivative of a section, to a vector field in the tangent bundle. A direct, powerful tensoriality argument leads to assess equality between the expressions of curvature in terms of horizontal lifts and in terms of vertical derivatives.

The analysis moves along the same line of thought as for instance the one declared in (Mangiarotti and Modugno, 1984), by trying to avoid unnecessary recourse to additional geometric structures.

In this respect the assumptions and the result of our Theorem 4.1 should be compared with the ones in (Kobayashi and Nomizu, 1963) Chapter III Theorem 5.1, in (Choquet-Bruhat, DeWitt-Morette, Dillard-Bleick, 1989) Chapter V-bis Section A.5, in (Mangiarotti and Sardanashvily, 2000) Chapter 2 Section 2.4, and in (Michor, 2007) Corollary 19.16, dealing with the curvature of linear connections on vector bundles.

2 CONNECTION ON A FIBER BUNDLE

The preliminary notions and definitions exposed in this section are detailedly illustrated in (Saunders, 1989), (Lang, 1995), (Romano, G., 2007).

A circle $\circ$ denotes a chain composition and a dot $\cdot$ (sometimes omitted) denotes a fiberwise $\Re$-linear chain composition.

Let us consider two differentiable manifolds $\mathbf{M}, \mathbf{N}$, with the relevant tangent bundles with projections $\tau_{\mathbf{M}} \in \mathrm{C}^1(T\mathbf{M};\mathbf{M})$ and $\tau_{\mathbf{N}} \in \mathrm{C}^1(T\mathbf{N};\mathbf{N})$.

The two manifolds are said to be related by a morphism $\phi \in \mathrm{C}^1(\mathbf{M};\mathbf{N})$, if the following commutative diagram holds

$$\begin{array}{ccc} T\mathbf{M} & \xrightarrow{T\phi} & T\mathbf{N} \\ {\scriptstyle \tau_{\mathbf{M}}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathbf{N}}} \\ \mathbf{M} & \xrightarrow{\phi} & \mathbf{N} \end{array} \iff \tau_{\mathbf{N}} \circ T\phi = \phi \circ \tau_{\mathbf{M}}\,, \tag{1}$$

where T is the tangent functor.

A vector field $\mathbf{X} \in \mathrm{C}^1(\varphi(\mathbf{M});T\mathbf{N})$ is φ-related to a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbf{M};T\mathbf{M})$ if

$$\mathbf{X} \circ \varphi = T\varphi \cdot \mathbf{u}\,. \tag{2}$$

For a diffeomorphism $\varphi \in \mathrm{C}^1(\mathbf{M};\mathbf{N})$ the push and pull operations are then denoted by $\mathbf{X} = \varphi{\uparrow}\mathbf{u}$ and $\mathbf{u} = \varphi{\downarrow}\mathbf{X}$. The usual notation is $\varphi{\uparrow} = \varphi_*$ and $\varphi{\downarrow} = \varphi^*$ but then too many stars do appear in the geometrical sky (push, duality, HODGE star).

A fiber bundle is a surjective submersion $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$ with $\mathbf{E}$ the total manifold and $\mathbf{M}$ the base manifold, i.e. $\mathbf{im}(\mathbf{p}) = \mathbf{M}$ and $\mathbf{im}(T\mathbf{p}(\mathbf{e})) = T_{\mathbf{p}(\mathbf{e})}\mathbf{M}$ for all $\mathbf{e} \in \mathbf{E}$. The vertical distribution is the subbundle $V\mathbf{E} := \mathbf{ker}(T\mathbf{p})$ of the tangent bundle $T\mathbf{E}$.

A section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$ is a morphism such that $\mathbf{p} \circ \mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{M})$ is the identity. The fiber at $\mathbf{x} \in \mathbf{M}$ is the set $\mathbf{E}_{\mathbf{x}} := \mathbf{p}^{-1}(\mathbf{x})$ which is assumed to be isomorphic to a standard fiber manifold.

Definition 2.1 (Connection). *A connection* $\mathrm{P_V} \in \Lambda^1(\mathbf{E};T\mathbf{E})$ *in a fiber bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$ *is an idempotent vector-valued one-form, which is pointwise a projector on vertical subspaces:*

$$\mathrm{P_V} \circ \mathrm{P_V} = \mathrm{P_V}\,, \tag{3}$$

with $\mathbf{im}(\mathrm{P_V}(\mathbf{e})) = \mathbf{ker}(T\mathbf{p}(\mathbf{e}))$. *Horizontal vectors are the ones in the kernel* $\mathbf{ker}(\mathrm{P_V}(\mathbf{e}))$ *of the connection. The projector on the horizontal distribution* $H\mathbf{E}$ *is denoted by* $\mathrm{P_H} = \mathbf{id}_{T\mathbf{E}} - \mathrm{P_V}$, *so that* $\mathrm{P_H} \circ \mathrm{P_H} = \mathrm{P_H}$ *and* $\mathrm{P_H} \circ \mathrm{P_V} = \mathrm{P_V} \circ \mathrm{P_H} = 0$.

The pull-back bundle of a fiber bundle $\mathbf{p} \in \mathrm{C}^1(T\mathbf{E};\mathbf{E})$ by a section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$ is the fiber bundle $\mathbf{s}{\downarrow}\mathbf{p} \in \mathrm{C}^1(\mathbf{s}{\downarrow}T\mathbf{E};\mathbf{M})$ whose fiber at $\mathbf{x} \in \mathbf{M}$ is the tangent space $T_{\mathbf{s}(\mathbf{x})}\mathbf{E}$ of $\mathbf{p} \in \mathrm{C}^1(T\mathbf{E};\mathbf{E})$.

The tangent to a section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$ of a fiber bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$ along a vector field $\mathbf{v} \in \mathrm{C}^0(\mathbf{M};T\mathbf{M})$ is a section $T\mathbf{s} \cdot \mathbf{v} \in \mathrm{C}^1(\mathbf{M};\mathbf{s}{\downarrow}T\mathbf{E})$ of the pull-back bundle $\mathbf{s}{\downarrow}\mathbf{p} = \mathrm{C}^1(\mathbf{s}{\downarrow}T\mathbf{E};\mathbf{M})$.

Definition 2.2 (Natural derivative). *In a fiber bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$*, for any section* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$*, the natural derivative along a vector field* $\mathbf{v} \in \mathrm{C}^0(\mathbf{M};T\mathbf{M})$ *is the vector field* $T_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbf{M});T\mathbf{E})$ *in the tangent bundle* $\tau_{\mathbf{E}} \in \mathrm{C}^1(T\mathbf{E};\mathbf{E})$ *defined by*

$$T_{\mathbf{v}} \cdot \mathbf{s} := T\mathbf{s} \cdot \mathbf{v} \in \mathrm{C}^1(\mathbf{M};T\mathbf{E})\,. \tag{4}$$

For any $\mathbf{x} \in \mathbf{M}$ we have that $T_{\mathbf{v}}(\mathbf{s}_{\mathbf{x}}) = T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} \in T_{\mathbf{s}_{\mathbf{x}}}\mathbf{E}$. The natural derivative $T_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbf{M});T\mathbf{E})$ and the vector field $\mathbf{v} \in \mathrm{C}^1(\mathbf{M};T\mathbf{M})$, and the relevant flows are $\mathbf{p}$-related according to the commutative diagrams

$$\begin{tikzcd} \mathbf{s}(\mathbf{M}) \subset \mathbf{E} \arrow[r, "\mathbf{Fl}^{T_{\mathbf{v}}}_{\lambda}"] \arrow[d, "\mathbf{p}"'] & \mathbf{E} \arrow[d, "\mathbf{p}"] \\ \mathbf{M} \arrow[r, "\mathbf{Fl}^{\mathbf{v}}_{\lambda}"] & T\mathbf{M} \end{tikzcd} \iff \mathbf{p} \circ \mathbf{Fl}^{T_{\mathbf{v}}}_{\lambda} = \mathbf{Fl}^{\mathbf{v}}_{\lambda} \circ \mathbf{p}\,, \tag{5}$$

$$\begin{tikzcd} \mathbf{s}(\mathbf{M}) \subset \mathbf{E} \arrow[r, "T_{\mathbf{v}}"] \arrow[d, "\mathbf{p}"'] & T\mathbf{E} \arrow[d, "T\mathbf{p}"] \\ \mathbf{M} \arrow[r, "\mathbf{v}"] & T\mathbf{M} \end{tikzcd} \iff T\mathbf{p} \cdot T_{\mathbf{v}} = \mathbf{v} \circ \mathbf{p}\,. \tag{6}$$

It is apparent that the natural derivative is tensorial in $\mathbf{v} \in \mathrm{C}^0(\mathbf{M};T\mathbf{M})$ since the differential $T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s} \in T_{\mathbf{s}(\mathbf{x})}\mathbf{E}$ is an $\Re$-linear function of the vector $\mathbf{v}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{M}$. The next statement enunciates the well known property of naturality of the LIE bracket with respect to relatedness, (see e.g. (Kolar et al., 1993) Lemma 3.10 or (Romano, G., 2007) Lemma 1.3.4).

Lemma 2.1 (Morphism-related vector fields and Lie brackets). *If the vector fields* $\mathbf{X},\mathbf{Y} \in \mathrm{C}^1(\mathbf{N};T\mathbf{N})$ *and* $\mathbf{u},\mathbf{v} \in \mathrm{C}^1(\mathbf{M};T\mathbf{M})$ *are related by a morphism* $\varphi \in \mathrm{C}^1(\mathbf{M};\mathbf{N})$*, then also their* LIE *brackets are* φ*-related:*

$$\left.\begin{aligned} \mathbf{X} \circ \varphi = T\varphi \cdot \mathbf{u} \\ \mathbf{Y} \circ \varphi = T\varphi \cdot \mathbf{v} \end{aligned}\right\} \implies [\mathbf{X},\mathbf{Y}] \circ \varphi = T\varphi \cdot [\mathbf{u},\mathbf{v}]\,. \tag{7}$$

Setting $T_{\mathbf{v}} \circ \varphi := T\varphi \cdot \mathbf{v}$ *for any morphism* $\varphi \in \mathrm{C}^1(\mathbf{M};\mathbf{N})$*, we have that* $T\varphi \cdot [\mathbf{u},\mathbf{v}] = T_{[\mathbf{u},\mathbf{v}]} \circ \varphi$ *and the result may be stated as*

$$[T_{\mathbf{u}}, T_{\mathbf{v}}] = T_{[\mathbf{u},\mathbf{v}]}\,. \tag{8}$$

Tensoriality is a crucial property of a multilinear scalar or vector valued map, meaning that it *lives at points* (Spivak, 1979), i.e. that its point-values depend only on the values of the argument fields at that point.

A standard tensoriality criterion for multilinear forms on $\mathbf{M}$ is provided by $C^\infty(\mathbf{M};\Re)$-linearity (see (Kolar et al., 1993) Lemma 7.3 or (Lang, 1995) Lemma 2.3 of Ch. VIII).

Although not needed in evaluating the LIE bracket $[T_{\mathbf{u}}, T_{\mathbf{v}}]$ on $\mathbf{s}(\mathbf{M})$, for the developments illustrated in Theorem 4.1 it is essential to extend the domain of the natural derivatives $T_{\mathbf{u}}, T_{\mathbf{v}} \in C^1(\mathbf{s}(\mathbf{M}); T\mathbf{E})$ outside the range $\mathbf{s}(\mathbf{M}) \subset \mathbf{E}$ of the section $\mathbf{s} \in C^1(\mathbf{M};\mathbf{E})$, so that they can be considered as (local) tangent vector fields $T_{\mathbf{u}}, T_{\mathbf{v}} \in C^1(\mathbf{E}; T\mathbf{E})$ with the further property of being projectable. This task can be accomplished by the following construction.

Lemma 2.2 (Extension by foliation). *The natural derivative $T_{\mathbf{v}} \in C^1(\mathbf{s}(\mathbf{M}); T\mathbf{E})$ of a section $\mathbf{s} \in C^1(\mathbf{M};\mathbf{E})$ in a fiber bundle $\mathbf{p} \in C^1(\mathbf{E};\mathbf{M})$, along a vector field $\mathbf{v} \in C^0(\mathbf{M}; T\mathbf{M})$, can be extended, in the bundle $\tau_{\mathbf{E}} \in C^1(T\mathbf{E};\mathbf{E})$, to a (local) tangent vector field $T_{\mathbf{v}} \in C^1(\mathbf{E}; T\mathbf{E})$ which projects on the vector field $\mathbf{v} \in C^0(\mathbf{M}; T\mathbf{M})$, i.e. we have that, locally in $\mathbf{E}$:*

$$\begin{cases} \tau_{\mathbf{E}} \circ T_{\mathbf{v}} = \mathbf{id}_{\mathbf{E}}\,, \\ T\mathbf{p} \cdot T_{\mathbf{v}} = \mathbf{v} \circ \mathbf{p}\,. \end{cases} \tag{9}$$

Proof. The extension may be performed by considering a (local) foliation of the total manifold $\mathbf{E}$, whose leaves are transversal to the fibers and include the folium $\mathbf{s}(\mathbf{M})$. The existence of at least a local foliation with these characteristics can be inferred by acting with a local bundle chart, which maps (locally) the image of the section into the trivial bundle image of the chart, and, subsequently, with a local chart which maps (locally) the fibers in their linear model space.

The foliation is performed by translation in the linear image of the fibers and the resulting leaves are mapped back to get the leaves in the total manifold. It is thus possible to define the map $\sigma \in C^1(\mathbf{E}; C^1(\mathbf{M};\mathbf{E}))$ which to each $\mathbf{e} \in \mathbf{E}$ associates the (local) section $\sigma_{\mathbf{e}} \in C^1(\mathbf{M};\mathbf{E})$ by

$$\sigma_{\mathbf{e}}(\mathbf{x}) := \Sigma_{\mathbf{e}} \cap \mathbf{E}_{\mathbf{x}}\,, \quad \forall \mathbf{e} \in \mathbf{E}\,, \tag{10}$$

whose range is the leaf $\Sigma_{\mathbf{e}}$ through $\mathbf{e} \in \mathbf{E}$.

The extension of $T_{\mathbf{v}}$ is (locally) defined by $T_{\mathbf{v}}(\mathbf{e}) := T_{\mathbf{p(e)}}\sigma_{\mathbf{e}} \cdot \mathbf{v}_{\mathbf{p(e)}}$ and gives a vector field since $\tau_{\mathbf{E}}(T_{\mathbf{p(e)}}\sigma_{\mathbf{e}} \cdot \mathbf{v}_{\mathbf{p(e)}}) = \mathbf{e}$ for all $\mathbf{e} \in \mathbf{E}$.

Moreover this extension projects on $\mathbf{v} \in C^0(\mathbf{M}; T\mathbf{M})$ since

$$\begin{aligned} T_{\mathbf{p(e)}}\mathbf{p} \cdot T_{\mathbf{v}}(\mathbf{e}) &= T_{\mathbf{p(e)}}\mathbf{p} \cdot T_{\mathbf{p(e)}}\sigma_{\mathbf{e}} \cdot \mathbf{v}_{\mathbf{p(e)}} \\ &= T_{\mathbf{p(e)}}(\mathbf{p} \circ \sigma_{\mathbf{e}}) \cdot \mathbf{v}_{\mathbf{p(e)}} = \mathbf{v}_{\mathbf{p(e)}}\,. \end{aligned} \tag{11}$$

Being $\sigma_{\mathbf{e}}(\mathbf{p}(\mathbf{e})) = \mathbf{e}$ the extension $T_{\mathbf{v}}(\mathbf{e}) := T_{\mathbf{p(e)}}\sigma_{\mathbf{e}} \cdot \mathbf{v}_{\mathbf{p(e)}}$ may be written as $(T_{\mathbf{v}} \circ \sigma_{\mathbf{e}})(\mathbf{p}(\mathbf{e})) = (T\sigma_{\mathbf{e}} \cdot \mathbf{v})(\mathbf{p}(\mathbf{e}))$ which, by surjectivity of $\mathbf{p}$, means that (locally)

$$T_{\mathbf{v}} \circ \sigma_{\mathbf{e}} = T\sigma_{\mathbf{e}} \cdot \mathbf{v}\,, \quad \forall \mathbf{x} \in \mathbf{M}\,. \tag{12}$$

If $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{E}$ are such that $\Sigma_{\mathbf{e}_1} = \Sigma_{\mathbf{e}_2}$, then $\sigma_{\mathbf{e}_1} = \sigma_{\mathbf{e}_2}$. If $\mathbf{e} \in \mathbf{s}(\mathbf{M})$, the section $\sigma_{\mathbf{e}} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$ is in fact coincident with $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$. ■

Definition 2.3 (Horizontal lift). *In a bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$ the horizontal lift $\mathbf{H} \in \mathrm{C}^1(\mathbf{E} \times_{\mathbf{M}} T\mathbf{M}; T\mathbf{E})$ is a right inverse of $(\tau_{\mathbf{E}}, T\mathbf{p}) \in \mathrm{C}^1(T\mathbf{E}; \mathbf{E} \times_{\mathbf{M}} T\mathbf{M})$ such that the map $\mathbf{H}_{\mathbf{s}_\mathbf{x}} \in \mathrm{C}^1(T\mathbf{M}; T\mathbf{E})$, defined by $\mathbf{H}_{\mathbf{s}_\mathbf{x}}(\mathbf{v}_\mathbf{x}) = \mathbf{H}(\mathbf{s}_\mathbf{x}, \mathbf{v}_\mathbf{x})$ for all $\mathbf{v}_\mathbf{x} \in T_\mathbf{x}\mathbf{M}$, is a linear homomorphism from the tangent bundle $\tau_{\mathbf{M}} \in \mathrm{C}^1(T\mathbf{M};\mathbf{M})$ to the tangent bundle $\tau_{\mathbf{E}} \in \mathrm{C}^1(T\mathbf{E};\mathbf{E})$, i.e:*

$$\begin{cases} (\tau_{\mathbf{E}}, T\mathbf{p}) \circ \mathbf{H} = \mathbf{id}_{\mathbf{E} \times_{\mathbf{M}} T\mathbf{M}}, \\ \mathbf{H}_{\mathbf{s}_\mathbf{x}}(\alpha\, \mathbf{u}_\mathbf{x} + \beta\, \mathbf{v}_\mathbf{x}) = \alpha\, \mathbf{H}_{\mathbf{s}_\mathbf{x}}(\mathbf{u}_\mathbf{x}) + \beta\, \mathbf{H}_{\mathbf{s}_\mathbf{x}}(\mathbf{v}_\mathbf{x}) \in T_{\mathbf{s}_\mathbf{x}}\mathbf{E}, \end{cases} \tag{13}$$

with $\mathbf{s}_\mathbf{x} \in \mathbf{E}_\mathbf{x}$ and $\mathbf{u}_\mathbf{x}, \mathbf{v}_\mathbf{x} \in T_\mathbf{x}\mathbf{M}$ and $\alpha, \beta \in \Re$.

Lemma 2.3 (Horizontal lifts and horizontal projectors). *A horizontal projector $\mathrm{P_H} \in \mathrm{C}^1(T\mathbf{E}; T\mathbf{E})$, induces a horizontal lift defined by*

$$\mathbf{H}(\mathbf{s}_\mathbf{x}, \mathbf{v}_\mathbf{x}) := \mathrm{P_H} \cdot T_\mathbf{x}\mathbf{s} \cdot \mathbf{v}_\mathbf{x} \in H_{\mathbf{s}_\mathbf{x}}\mathbb{B}, \quad \forall \mathbf{s}_\mathbf{x} \in \mathbf{E}_\mathbf{x}, \quad \mathbf{v}_\mathbf{x} \in T_\mathbf{x}\mathbf{M}, \tag{14}$$

where $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$ is an arbitrary section extension of $\mathbf{s}_\mathbf{x} \in \mathbf{E}_\mathbf{x}$. Vice versa, a horizontal lift $\mathbf{H} \in \mathrm{C}^1(\mathbf{E} \times_{\mathbf{M}} T\mathbf{M}; T\mathbf{E})$ induces a horizontal projector given by

$$\mathrm{P_H} := \mathbf{H} \circ (\tau_{\mathbf{E}}, T\mathbf{p}). \tag{15}$$

Proof. Eq. (14) yields a horizontal lift since:

$$((\tau_{\mathbf{E}}, T\mathbf{p}) \circ \mathbf{H})(\mathbf{s}_\mathbf{x}, \mathbf{v}_\mathbf{x}) = (\tau_{\mathbf{E}}, T\mathbf{p}) \cdot \mathrm{P_H} \cdot T_\mathbf{x}\mathbf{s} \cdot \mathbf{v}_\mathbf{x} = (\mathbf{s}_\mathbf{x}, \mathbf{v}_\mathbf{x}). \tag{16}$$

The homomorphism $\mathrm{P_H} := \mathbf{H} \circ (\tau_{\mathbf{E}}, T\mathbf{p})$ is idempotent due to the equality

$$\begin{aligned} \mathrm{P_H} \circ \mathrm{P_H} &= \mathbf{H} \circ (\tau_{\mathbf{E}}, T\mathbf{p}) \circ \mathbf{H} \circ (\tau_{\mathbf{E}}, T\mathbf{p}) \\ &= \mathbf{H} \circ \mathbf{id}_{\mathbf{E} \times_{\mathbf{M}} T\mathbf{M}} \circ (\tau_{\mathbf{E}}, T\mathbf{p}) = \mathrm{P_H}, \end{aligned} \tag{17}$$

and horizontal due to the identity

$$\begin{aligned} ((\tau_{\mathbf{E}}, T\mathbf{p}) \circ \mathrm{P_H})(\mathbf{X}) &= ((\tau_{\mathbf{E}}, T\mathbf{p}) \circ \mathbf{H} \circ (\tau_{\mathbf{E}}, T\mathbf{p}))(\mathbf{X}) \\ &= (\tau_{\mathbf{E}}(\mathbf{X}), T\mathbf{p}(\mathbf{X})). \end{aligned} \tag{18}$$

Eq. (15) yields then a horizontal projector. ■

Definition 2.4 (Vertical derivative). *The vertical derivative is the vertical component of the natural derivative:*

$$\overline{\nabla}_{\mathbf{v}}\mathbf{s} := \mathrm{P_V} \cdot T_{\mathbf{v}} \cdot \mathbf{s} \in \mathrm{C}^1(\mathbf{M}; V\mathbf{E})\,. \tag{19}$$

Setting $\mathbf{Hs} = \mathrm{P_H} \circ T\mathbf{s}$ and $\overline{\nabla}\mathbf{s} = \mathrm{P_V} \circ T\mathbf{s}$, it is $T\mathbf{s} = \overline{\nabla}\mathbf{s} + \mathbf{Hs} \in \mathrm{C}^1(T\mathbf{M}; T\mathbf{E})$ and $T_{\mathbf{v}} = \overline{\nabla}_{\mathbf{v}} + \mathbf{H_v} \in \mathrm{C}^1(\mathbf{s}(\mathbf{M}); T\mathbf{E})$ with $\overline{\nabla}_{\mathbf{v}} = \mathrm{P_V} \circ T_{\mathbf{v}}$ and $\mathbf{H_v} = \mathrm{P_H} \circ T_{\mathbf{v}}$.

Lemma 2.4 (Projectability). *The horizontal lift* $\mathbf{H_v} \in \mathrm{C}^1(\mathbf{s}(\mathbf{M}); H\mathbf{E})$ *is* $\mathbf{p}$*-related to the vector field* $\mathbf{v} \in \mathrm{C}^1(\mathbf{M}; T\mathbf{M})$ *according to the commutative diagram*

$$\begin{tikzcd} \mathbf{s}(\mathbf{M}) \subset \mathbf{E} \arrow[r, "\mathbf{H_v}"] \arrow[d, "\mathbf{p}"'] & T\mathbf{E} \arrow[d, "T\mathbf{p}"] \\ \mathbf{M} \arrow[r, "\mathbf{v}"] & T\mathbf{M} \end{tikzcd} \iff T\mathbf{p} \cdot \mathbf{H_v} = \mathbf{v} \circ \mathbf{p}\,. \tag{20}$$

Proof. Being, by definition $T\mathbf{p} \circ \overline{\nabla}_{\mathbf{v}} = 0$, from the decomposition $T_{\mathbf{v}} = \overline{\nabla}\mathbf{v} + \mathbf{H_v} \in \mathrm{C}^1(\mathbf{E}; T\mathbf{E})$ it follows that

$$T\mathbf{p} \circ T_{\mathbf{v}} = T\mathbf{p} \cdot \overline{\nabla}_{\mathbf{v}} + T\mathbf{p} \cdot \mathbf{H_v} = T\mathbf{p} \cdot \mathbf{H_v}\,. \tag{21}$$

The $\mathbf{p}$-relatedness of $\mathbf{H_v}$ and $\mathbf{v}$ is inferred from that of $T_{\mathbf{v}}$ in Eq. (8). ■

Naturality of LIE brackets with respect to relatedness and Lemma 2.4 give:

$$\begin{aligned} T\mathbf{p} \cdot [\mathbf{H_u}, \mathbf{H_v}] &= [T\mathbf{p} \cdot \mathbf{H_u}, T\mathbf{p} \cdot \mathbf{H_v}] \\ &= [\mathbf{u} \circ \mathbf{p}, \mathbf{v} \circ \mathbf{p}] = [\mathbf{u}, \mathbf{v}] \circ \mathbf{p} \in \mathrm{C}^1(\mathbf{E}; T\mathbf{M})\,. \end{aligned} \tag{22}$$

Lemma 2.5 (Injectivity). *The horizontal lift* $\mathbf{Hs} \in \mathrm{C}^1(T\mathbf{M}; H\mathbf{E})$*, along a cross section* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$ *of a fiber bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}; \mathbf{M})$*, is a fiberwise injective homomorphism, i.e.* $\mathbf{H_x s} \in BL(T_{\mathbf{x}}\mathbf{M}; H_{\mathbf{s(x)}}\mathbf{E})$ *is an injective linear map at each* $\mathbf{x} \in \mathbf{M}$.

Proof. We must prove that $\mathbf{ker}(\mathbf{H_x s}) = \{0\}$. We first investigate the linear differential $T_{\mathbf{x}}\mathbf{s} \in BL(T_{\mathbf{x}}\mathbf{M}; T_{\mathbf{s(x)}}\mathbf{E})$. By the characteristic property of a section, $\mathbf{p} \circ \mathbf{s} = \mathbf{id_M}$ it is: $T_{\mathbf{s(x)}}\mathbf{p} \cdot T_{\mathbf{x}}\mathbf{s} \cdot \mathbf{v_x} = T_{\mathbf{x}}(\mathbf{p} \circ \mathbf{s}) \cdot \mathbf{v_x} = \mathbf{v_x}$ for all $\mathbf{v_x} \in T_{\mathbf{x}}\mathbf{M}$. It follows that $\mathbf{ker}(T_{\mathbf{x}}\mathbf{s}) = \{0\}$ and $\mathbf{im}(T_{\mathbf{x}}\mathbf{s}) \cap \mathbf{ker}(T_{\mathbf{s(x)}}\mathbf{p}) = \{0\}$. The injectivity of $T_{\mathbf{x}}\mathbf{s}$ implies that: $\dim \mathbf{im}(T_{\mathbf{x}}\mathbf{s}) = \dim T_{\mathbf{x}}\mathbf{M}$. Being $T_{\mathbf{x}}\mathbf{s} = \overline{\nabla}_{\mathbf{x}}\mathbf{s} + \mathbf{H_x s}$ with $\mathbf{im}(\overline{\nabla}_{\mathbf{x}}\mathbf{s}) \subseteq \mathbf{ker}(T_{\mathbf{s(x)}}\mathbf{p})$, we have that $T_{\mathbf{s(x)}}\mathbf{p} \cdot \mathbf{H_x s} \cdot \mathbf{v_x} = T_{\mathbf{s(x)}}\mathbf{p} \cdot T_{\mathbf{x}}\mathbf{s} \cdot \mathbf{v_x} = \mathbf{v_x}$ for all $\mathbf{v_x} \in T_{\mathbf{x}}\mathbf{M}$. It follows that $\mathbf{ker}(\mathbf{H_x s}) = \{0\}$ and $\mathbf{im}(\mathbf{H_x s}) \cap \mathbf{ker}(T_{\mathbf{s(x)}}\mathbf{p}) = \{0\}$ with $\dim \mathbf{im}(\mathbf{H_x s}) = \dim T_{\mathbf{x}}\mathbf{M}$. ■

Theorem 2.1 (Homomorphism). *The horizontal lift* $\mathbf{Hs} \in \mathrm{C}^1(T\mathbf{M}; H\mathbf{E})$ *along a section* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$ *of a fiber bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}; \mathbf{M})$ *is a vector bundle homomorphism between the bundle* $\tau_{\mathbf{M}} \in \mathrm{C}^1(T\mathbf{M}; \mathbf{M})$ *and the pull-back bundle* $\mathbf{s}{\downarrow}\tau_{\mathbf{E}} \in \mathrm{C}^1(\mathbf{s}{\downarrow}H\mathbf{E}; \mathbf{M})$ *which is fiberwise invertible and tensorial in* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$.

Proof. Let $\dim \mathbf{M} = \dim T_{\mathbf{x}}\mathbf{M} = m$ and $\dim \mathbb{F} = f$ where $\mathbb{F}$ is the typical fiber. Then $\dim \mathbf{E} = \dim T_{\mathbf{s}(\mathbf{x})}\mathbf{E} = m + f$. So that $\dim V_{\mathbf{s}(\mathbf{x})}\mathbf{E} = f$ and $\dim H_{\mathbf{s}(\mathbf{x})}\mathbf{E} = m$. By reasons of dimensions the injectivity of $\mathbf{H_x s} \in BL(T_{\mathbf{x}}\mathbf{M}; H_{\mathbf{s}(\mathbf{x})}\mathbf{E})$ implies then its surjectivity. Moreover let $\bar{\mathbf{s}} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$ be another section such that $\bar{\mathbf{s}}(\mathbf{x}) = \mathbf{s}(\mathbf{x})$. Then, for any $\mathbf{v_x} \in T_{\mathbf{x}}\mathbf{M}$, being $T_{\mathbf{v_x}}\mathbf{s}, T_{\mathbf{v_x}}\bar{\mathbf{s}} \in T_{\mathbf{s}(\mathbf{x})}\mathbf{E}$, we have that $T\mathbf{p} \cdot (T_{\mathbf{v_x}}\mathbf{s} - T_{\mathbf{v_x}}\bar{\mathbf{s}}) = 0$ and hence that $\mathbf{H}_{\mathbf{v_x}}\mathbf{s} = \mathrm{P_H} \cdot T_{\mathbf{v_x}}\mathbf{s} = \mathrm{P_H} \cdot T_{\mathbf{v_x}}\bar{\mathbf{s}} = \mathbf{H}_{\mathbf{v_x}}\bar{\mathbf{s}} \in BL(T_{\mathbf{x}}\mathbf{M}; H_{\mathbf{s}(\mathbf{x})}\mathbf{E})$. To a tangent vector $\mathbf{v_x} \in T_{\mathbf{x}}\mathbf{M}$ there corresponds a horizontal vector $\mathbf{H}_{\mathbf{v_x}}\mathbf{s} \in H_{\mathbf{s}(\mathbf{x})}\mathbf{E}$ which depends only on the value of $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$ at $\mathbf{x} \in \mathbf{M}$. ■

3 CURVATURE OF A CONNECTION

The vertical distribution of a fiber bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}; \mathbf{M})$ is integrable and the leaves of the induced foliation are the fibers of the bundle.

By FROBENIUS theorem, see e.g. (Kolar et al., 1993), (Lang, 1995), integrability of a distribution is equivalent to involutivity, i.e. to closeness of the distribution under LIE bracket operation.

Since vertical vectors are, by definition, related to null-sections, from Eq. (7) we infer that also their LIE brackets are related to null-sections, a result in accord with the fact that the vertical distribution is integrable, a property equivalent to vanishing of the vector-valued lifted *cocurvature* form:

$$\overline{\mathbf{R^c}}(\mathbf{X}, \mathbf{Y}) := -\mathrm{P_H} \cdot [\widehat{\mathrm{P_V}\mathbf{X}}, \widehat{\mathrm{P_V}\mathbf{Y}}] = \mathbf{0}, \tag{23}$$

for any $\mathbf{X}, \mathbf{Y} \in T\mathbf{E}$. Here $(\widehat{\mathrm{P_V}\mathbf{X}}, \widehat{\mathrm{P_V}\mathbf{Y}}) \in \mathrm{C}^1(\mathbf{E}; T\mathbf{E})$ is any pair of vector fields extension of the vectors $\mathrm{P_V}\mathbf{X}, \mathrm{P_V}\mathbf{Y} \in T\mathbf{E}$, since tensoriality follows from the $\mathrm{C}^\infty(\mathbf{E}; \mathfrak{R})$-linearity of the cocurvature form.

The involutivity condition $[\widehat{\mathrm{P_H}\mathbf{X}}, \widehat{\mathrm{P_H}\mathbf{Y}}] \in \mathrm{C}^1(\mathbf{E}; H\mathbf{E})$, to be imposed for the integrability of the horizontal distribution, is equivalently expressed by the vanishing of the lifted *curvature* form defined by (Kolar et al., 1993)

$$\overline{\mathbf{R}}(\mathbf{X}, \mathbf{Y}) := -\mathrm{P_V} \cdot [\widehat{\mathrm{P_H}\mathbf{X}}, \widehat{\mathrm{P_H}\mathbf{Y}}], \quad \forall \mathbf{X}, \mathbf{Y} \in T\mathbf{E}. \tag{24}$$

Tensoriality follows from the $\mathrm{C}^\infty(\mathbf{E}; \mathfrak{R})$-linearity of the curvature form, as shown below. Let us denote by $\Lambda^k(\mathbf{M}; T\mathbf{M})$ the space of tangent-valued k-forms on a manifold $\mathbf{M}$.

Proposition 3.1 (Tensoriality of the curvature). *The curvature of a connection* $\mathrm{P_V} \in \Lambda^1(\mathbf{E}; T\mathbf{E})$ *in a fiber bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}; \mathbf{M})$ *is a vertical-vector valued, horizontal* 2*-form* $\overline{\mathbf{R}} \in \Lambda^2(\mathbf{E}; V\mathbf{E})$*, that is a* 2*-form vanishing on vertical vectors and taking values in the vertical distribution.*

Proof. A direct verification of the tensoriality, based on $\mathrm{C}^\infty(\mathbf{E}; \Re)$-linearity, yields the result:

$$
\begin{aligned}
-\overline{\mathbf{R}}(\mathbf{X}, f\mathbf{Y}) &:= \mathrm{P_V} \cdot [\widehat{\mathrm{P_H}\mathbf{X}}, \widehat{\mathrm{P_H}\mathbf{Y}}] \\
&= f\,\mathrm{P_V} \cdot [\widehat{\mathrm{P_H}\mathbf{X}}, \widehat{\mathrm{P_H}\mathbf{Y}}] + (\mathscr{L}_{\mathrm{P_H}\mathbf{X}} f)\,(\mathrm{P_V} \cdot \mathrm{P_H})(\mathbf{Y}) \\
&= -f\,\overline{\mathbf{R}}(\mathbf{X}, \mathbf{Y}), \quad \forall f \in \mathrm{C}^1(\mathbf{E}; \Re),
\end{aligned}
\tag{25}
$$

since $\mathrm{P_V} \cdot \mathrm{P_H} = 0$. Similarly $\overline{\mathbf{R}}(f\mathbf{X}, \mathbf{Y}) = f\,\overline{\mathbf{R}}(\mathbf{X}, \mathbf{Y})$. ■

Theorem 3.1. *For any given section* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$*, the curvature of a connection* $\mathrm{P_V} \in \Lambda^1(\mathbf{E}; V\mathbf{E})$ *is expressed by a* 2*-form* $\mathbf{R_s} \in \Lambda^2(\mathbf{M}; \mathbf{s}{\downarrow}V\mathbf{E})$ *with values in the pull-back of the vertical distribution by the section* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$*, defined in terms of horizontal lifts by*

$$
\begin{aligned}
\mathbf{R}(\mathbf{s})(\mathbf{u}, \mathbf{v}) &:= \overline{\mathbf{R}}(\mathbf{H_u}, \mathbf{H_v}) \cdot \mathbf{s} \\
&= (\mathbf{H}_{[\mathbf{u},\mathbf{v}]} - [\mathbf{H_u}, \mathbf{H_v}]) \cdot \mathbf{s}, \quad \forall \mathbf{u}, \mathbf{v} \in \Lambda^0(\mathbf{M}; T\mathbf{M}),
\end{aligned}
\tag{26}
$$

The 2*-form* $\mathbf{R}(\mathbf{s}) \in \Lambda^2(\mathbf{M}; \mathbf{s}{\downarrow}V\mathbf{E})$ *is tensorial in* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$*.*

Proof. We rely on the properties of tensoriality and horizontality of the curvature two-form $\mathbf{R} \in \Lambda^2(\mathbf{E}; V\mathbf{E})$ stated in Proposition 3.1 and on the tensorial isomorphism of the horizontal lifts stated in Theorem 2.1. Accordingly, the point value of the lifted curvature $\overline{\mathbf{R}}(\mathbf{X}, \mathbf{Y}) = -\mathrm{P_V} \circ [\widehat{\mathrm{P_H}\mathbf{X}}, \widehat{\mathrm{P_H}\mathbf{Y}}]$ at $\mathbf{b} \in \mathbf{E_x}$ depends only on the vectors $\mathrm{P_H}\mathbf{X_b}, \mathrm{P_H}\mathbf{Y_b} \in T_\mathbf{b}\mathbf{E}$.

Moreover, by Theorem 2.1, given any section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$ such that $\mathbf{s_x} = \mathbf{b}$, there exists a uniquely determined pair of vectors $\mathbf{u_x}, \mathbf{v_x} \in T_\mathbf{x}\mathbf{M}$, such that

$$
\mathbf{H}_{\mathbf{u_x}}\mathbf{s} = (\mathrm{P_H}\mathbf{X})(\mathbf{s_x}), \quad \mathbf{H}_{\mathbf{v_x}}\mathbf{s} = (\mathrm{P_H}\mathbf{Y})(\mathbf{s_x}) \tag{27}
$$

and the pair $\mathbf{u_x}, \mathbf{v_x} \in T_\mathbf{x}\mathbf{M}$ does not depend on the choice of the section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$ such that $\mathbf{s_x} = \mathbf{b}$.

Then the curvature two-form $\overline{\mathbf{R}} \in \Lambda^2(\mathbf{E}; V\mathbf{E})$, evaluated on pairs of horizontal lifts, defines the field $\mathbf{R}(\mathbf{s})(\mathbf{u}, \mathbf{v}) := -\mathrm{P_V} \circ [\mathbf{H_u}, \mathbf{H_v}] \circ \mathbf{s} \in \mathrm{C}^1(\mathbf{M}; V\mathbf{E})$ for any pair of vector fields $\mathbf{u}, \mathbf{v} \in \mathrm{C}^0(\mathbf{M}; T\mathbf{M})$ on the tangent bundle and any section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M}; \mathbf{E})$ of the fiber bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}; \mathbf{M})$.

By tensoriality, for any section $\mathbf{s} \in C^1(\mathbf{M};\mathbf{E})$ the field $\mathbf{R}(\mathbf{s}) \in \Lambda^2(\mathbf{M};V\mathbf{E})$ is a vector-valued two-form on $\mathbf{M}$ with values in $\mathbf{s}{\downarrow}V\mathbf{E}$ and for any pair $\mathbf{u},\mathbf{v} \in C^0(\mathbf{M};T\mathbf{M})$ the field $\mathbf{R}(\mathbf{u},\mathbf{v}) \in \Lambda^1(\mathbf{M};\mathbf{s}{\downarrow}V\mathbf{E})$ is a vertical-valued vector field along $\mathbf{s} \in C^1(\mathbf{M};\mathbf{E})$.

Moreover, by Lemma 2.4, horizontal lifts are projectable and we have the relations

$$\left.\begin{aligned} T\mathbf{p}\cdot[\mathbf{H_u},\mathbf{H_v}] &= [\mathbf{u},\mathbf{v}]\circ\mathbf{p} \\ T\mathbf{p}\cdot\mathbf{H}_{[\mathbf{u},\mathbf{v}]} &= [\mathbf{u},\mathbf{v}]\circ\mathbf{p} \end{aligned}\right\} \implies T\mathbf{p}\cdot([\mathbf{H_u},\mathbf{H_v}]-\mathbf{H}_{[\mathbf{u},\mathbf{v}]}) = 0\,. \tag{28}$$

Then $\mathbf{H}_{[\mathbf{u},\mathbf{v}]}$ is the horizontal component of $[\mathbf{H_u},\mathbf{H_v}]$ and we get the equality: $[\mathbf{H_u},\mathbf{H_v}]-\mathbf{H}_{[\mathbf{u},\mathbf{v}]} = \mathrm{P_V}\cdot[\mathbf{H_u},\mathbf{H_v}] \iff \mathbf{H}_{[\mathbf{u},\mathbf{v}]} = \mathrm{P_H}\cdot[\mathbf{H_u},\mathbf{H_v}]$. ■

4 VERTICAL DERIVATIVE

Lemma 4.1 (Vertical derivative as Lie derivative). *In a fiber bundle $\mathbf{p} \in C^1(\mathbf{E};\mathbf{M})$ with a connection, the vertical derivative is given by the generalized* LIE *derivative*

$$\overline{\nabla}_{\mathbf{v}}\mathbf{s} = \mathscr{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}_{\lambda}^{(\mathbf{H_v},\mathbf{v})}{\downarrow}\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}_{-\lambda}^{\mathbf{H_v}}\cdot\mathbf{s}\circ\mathbf{Fl}_{\lambda}^{\mathbf{v}}\,. \tag{29}$$

Proof. By LEIBNIZ rule $\mathscr{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s} = T\mathbf{s}\cdot\mathbf{v}-\mathbf{H_v}\mathbf{s} = T_{\mathbf{v}}\mathbf{s}-\mathbf{H_v}\mathbf{s}$. Then, being $\mathscr{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s} \in C^1(\mathbf{M};V\mathbf{E})$ and $\mathbf{H_v}\mathbf{s} \in C^1(\mathbf{M};H\mathbf{E})$, by uniqueness of the vertical-horizontal split, we get that $\overline{\nabla}_{\mathbf{v}}\mathbf{s} := \mathrm{P_V}\cdot T_{\mathbf{v}}\mathbf{s} = \mathscr{L}_{(\mathbf{H_v},\mathbf{v})}\mathbf{s}$. ■

Definition 4.1 (Parallel transport). *Let $\mathbf{p} \in C^1(\mathbf{E};\mathbf{M})$ be a fiber bundle with a connection. The parallel transport $\mathbf{Fl}_{\lambda}^{\mathbf{v}}{\Uparrow}\mathbf{s} \in C^1(\mathbf{M};\mathbf{E})$ of a section $\mathbf{s} \in C^1(\mathbf{M};\mathbf{E})$ along the flow $\mathbf{Fl}_{\lambda}^{\mathbf{v}} \in C^1(\mathbf{M};\mathbf{M})$ is defined by*

$$\mathbf{Fl}_{\lambda}^{\mathbf{v}}{\Uparrow}\mathbf{s} := \mathbf{Fl}_{\lambda}^{\mathbf{H_v}}\circ\mathbf{s} = (\mathbf{Fl}_{\lambda}^{(\mathbf{H_v},\mathbf{v})}{\uparrow}\mathbf{s})\circ\mathbf{Fl}_{\lambda}^{\mathbf{v}}\,, \tag{30}$$

so that $\mathbf{p}\circ\mathbf{Fl}_{\lambda}^{\mathbf{v}}{\Uparrow}\mathbf{s} = \mathbf{p}\circ\mathbf{Fl}_{\lambda}^{\mathbf{H_v}}\circ\mathbf{s} = \mathbf{Fl}_{\lambda}^{\mathbf{v}}\circ\mathbf{p}\circ\mathbf{s} = \mathbf{Fl}_{\lambda}^{\mathbf{v}}$.

From the definition of parallel transport and Lemma 4.1 we infer that the vertical derivative and the horizontal lift are given by

$$\begin{aligned} \overline{\nabla}_{\mathbf{v}}\mathbf{s} &= \partial_{\lambda=0}\,\mathbf{Fl}_{-\lambda}^{\mathbf{H_v}}\cdot\mathbf{s}\circ\mathbf{Fl}_{\lambda}^{\mathbf{v}} = \partial_{\lambda=0}\,\mathbf{Fl}_{-\lambda}^{\mathbf{v}}{\Uparrow}\mathbf{s}\circ\mathbf{Fl}_{\lambda}^{\mathbf{v}}\,, \\ \mathbf{H_v}\mathbf{s} &= \partial_{\lambda=0}\,\mathbf{Fl}_{\lambda}^{\mathbf{H_v}}\cdot\mathbf{s} = \partial_{\lambda=0}\,\mathbf{Fl}_{\lambda}^{\mathbf{v}}{\Uparrow}\mathbf{s}\,. \end{aligned} \tag{31}$$

Since the horizontal lift $\mathbf{H_v}$ is defined pointwise in $\mathbf{M}$, the parallel transport along a curve in $\mathbf{M}$ of a section defined only on that curve is meaningful and so is for the vertical derivative.

Definition 4.2 (Geodesic). *A curve $c \in \mathrm{C}^1(I;\mathbf{M})$ in a manifold with a connection is a geodesic if the velocity field $\mathbf{v} \in \mathrm{C}^1(I;T\mathbf{M})$ of the curve fulfils the condition*

$$(\overline{\nabla}_{\mathbf{v}}\mathbf{v})(\mathbf{x}) := \partial_{\mu=\lambda}\, c_{\mu,\lambda}\Downarrow \mathbf{v}(\mathbf{x}) = 0\,. \tag{32}$$

Here $\overline{\nabla}$ is the vertical derivative, the velocity at $\mathbf{x} = c(\lambda)$ is given by $(\mathbf{v}\circ c)(\lambda) := \partial_{\mu=\lambda}\, c_\mu$ and $c_{\mu,\lambda}\Downarrow = c_{\lambda,\mu}\Uparrow$ is the parallel transport from c_μ to c_λ along the curve.

Geodesic curves are related to the notion of *spray*, introduced in (Ambrose, Palais and Singer, 1960), see also (Dieudonne, 1969; Lang, 1995).

Definition 4.3 (Spray). *A section $\mathbf{X} \in \mathrm{C}^1(T\mathbf{M};TT\mathbf{M})$ of the tangent bundle $\tau_{T\mathbf{M}} \in \mathrm{C}^1(TT\mathbf{M};T\mathbf{M})$ is called a spray if it is also a section of the bundle $T\tau_{\mathbf{M}} \in \mathrm{C}^1(TT\mathbf{M};T\mathbf{M})$, that is if*

$$T\tau_{\mathbf{M}}\cdot\mathbf{X} = \tau_{T\mathbf{M}}\circ\mathbf{X} = \mathbf{id}_{T\mathbf{M}}\,. \tag{33}$$

Lemma 4.2 (Geodesics and sprays). *Let $\mathbf{S} \in \mathrm{C}^1(T\mathbf{M};TT\mathbf{M})$ be a spray and $\mathbf{v}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{M}$ a tangent vector. Then, for any connection compatible with the spray, i.e. such that $\mathbf{H}(\mathbf{v},\mathbf{v}) = \mathbf{S}(\mathbf{v})$, the base curve below the flow line of the spray through the vector $\mathbf{v}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{M}$, is a geodesic curve through the base point $\mathbf{x} \in \mathbf{M}$.*

Proof. Let $\lambda \mapsto \mathbf{Fl}^{\mathbf{S}}_{\lambda}(\mathbf{v}_{\mathbf{x}})$ be the flow line of the spray through the vector $\mathbf{v}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{M}$. The projected curve on the base manifold $\lambda \mapsto (\tau_{\mathbf{M}}\circ\mathbf{Fl}^{\mathbf{S}}_{\lambda})(\mathbf{v}_{\mathbf{x}})$, has velocity field $\mathbf{v}_{\mathbf{S}} \in \mathrm{C}^1(I;T\mathbf{M})$ given by

$$\begin{aligned}(\mathbf{v}_{\mathbf{S}}\circ c)(\lambda) &:= \partial_{\mu=\lambda}\,(\tau_{\mathbf{M}}\circ\mathbf{Fl}^{\mathbf{S}}_{\mu})(\mathbf{v}_{\mathbf{x}})\\ &= T\tau_{\mathbf{M}}\cdot\mathbf{S}(\mathbf{Fl}^{\mathbf{S}}_{\lambda}(\mathbf{v}_{\mathbf{x}})) = \pi_{T\mathbf{M}}(\mathbf{S}(\mathbf{Fl}^{\mathbf{S}}_{\lambda}(\mathbf{v}_{\mathbf{x}}))) = \mathbf{Fl}^{\mathbf{S}}_{\lambda}(\mathbf{v}_{\mathbf{x}})\,,\end{aligned} \tag{34}$$

Being $\mathbf{H}(\mathbf{v},\mathbf{v}) = \mathbf{S}(\mathbf{v})$, the formula for the time-covariant derivative yields:

$$\overline{\nabla}_{\mathbf{v}_{\mathbf{S}}}\mathbf{v}_{\mathbf{S}} = \partial_{\mu=0}\,(\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}_{\mathbf{S}}}}_{-\mu}\circ\mathbf{Fl}^{\mathbf{S}}_{\lambda+\mu})(\mathbf{v}_{\mathbf{x}}) = \mathbf{S}(\mathbf{v}_{\mathbf{S}}) - \mathbf{H}(\mathbf{v}_{\mathbf{S}},\mathbf{v}_{\mathbf{S}}) = 0\,. \tag{35}$$

Hence the base curve is a geodesic. ■

A similar proof shows that the base curve through $\mathbf{v}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbf{M}$ below the differential of the flow line of a spray is the velocity field of a geodesic, in any connection compatible with the spray, and that the velocity field of the base points of the line is a JACOBI field (Michor, 1997).

The next original result is the main contribution of this paper. It provides, in the general context of fiber bundles, the expression of the curvature in terms of vertical derivatives.

Theorem 4.1 (Curvature and vertical derivatives). *Given a section* $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$ *of a fiber bundle* $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$ *and any pair of vector fields* $\mathbf{u},\mathbf{v} \in \mathrm{C}^1(\mathbf{M};T\mathbf{M})$*, the following identity holds on* $\mathbf{s}(\mathbf{M}) \subset \mathbf{E}$*:*

$$[\overline{\nabla}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}}] - \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + [\mathbf{H}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}}] - \mathbf{H}_{[\mathbf{u},\mathbf{v}]} = \mathbf{0}\,. \tag{36}$$

Accordingly, the vertical-valued curvature two-form $\overline{\mathbf{R}}_{\mathbf{x}}(\mathbf{s})(\mathbf{u},\mathbf{v}) \in V_{\mathbf{s}(\mathbf{x})}\mathbf{E}$ *is given by*

$$\overline{\mathbf{R}}(\mathbf{s})(\mathbf{u},\mathbf{v}) = [\overline{\nabla}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}}](\mathbf{s}) - \overline{\nabla}_{[\mathbf{u},\mathbf{v}]}(\mathbf{s})\,. \tag{37}$$

Proof. By Lemma 2.1 we know that on $\mathbf{s}(\mathbf{M}) \subset \mathbf{E}$ it is $[T_{\mathbf{u}}, T_{\mathbf{v}}] = T_{[\mathbf{u},\mathbf{v}]}$. By performing an extension of the natural derivatives, e.g. by the foliation method envisaged in Lemma 2.2, the vertical derivatives of a section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$ are consequently extended to (local) vector fields $\overline{\nabla}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{E};V\mathbf{E})$. Then, splitting into vertical and horizontal components

$$T_{\mathbf{u}} = \overline{\nabla}_{\mathbf{u}} + \mathbf{H}_{\mathbf{u}}\,, \qquad T_{\mathbf{v}} = \overline{\nabla}_{\mathbf{v}} + \mathbf{H}_{\mathbf{v}}\,, \qquad T_{[\mathbf{u},\mathbf{v}]} = \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + \mathbf{H}_{[\mathbf{u},\mathbf{v}]}\,, \tag{38}$$

by bilinearity of the LIE bracket we get

$$\begin{aligned} [\overline{\nabla}_{\mathbf{u}} + \mathbf{H}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}} + \mathbf{H}_{\mathbf{v}}] &= [\overline{\nabla}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}}] + [\mathbf{H}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}}] + [\overline{\nabla}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}}] + [\mathbf{H}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}}] \\ &= \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + \mathbf{H}_{[\mathbf{u},\mathbf{v}]}\,, \end{aligned} \tag{39}$$

which, being $[\mathbf{H}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}}] - \mathbf{H}_{[\mathbf{u},\mathbf{v}]} = \mathrm{P_V} \cdot [\mathbf{H}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}}]$, can be written as

$$[\overline{\nabla}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}}] - \overline{\nabla}_{[\mathbf{u},\mathbf{v}]} + \mathrm{P_V} \cdot [\mathbf{H}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}}] = [\mathbf{H}_{\mathbf{v}}, \overline{\nabla}_{\mathbf{u}}] + [\overline{\nabla}_{\mathbf{v}}, \mathbf{H}_{\mathbf{u}}]\,. \tag{40}$$

Tensoriality of the curvature $\mathrm{P_V} \cdot [\mathbf{H}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}}]$, as a function of the horizontal lifts $\mathbf{H}_{\mathbf{u}}$ and $\mathbf{H}_{\mathbf{v}}$, has the following implication. Let the local vector fields $\mathscr{F}^{\mathbf{x}}_{\mathbf{u}}, \mathscr{F}^{\mathbf{x}}_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{E};T\mathbf{E})$ be generated by dragging the vectors $\mathbf{H}_{\mathbf{u}_{\mathbf{x}}}, \mathbf{H}_{\mathbf{v}_{\mathbf{x}}} \in T_{\mathbf{s}_{\mathbf{x}}}\mathbf{E}$ along the flows of the (local) vector fields $\overline{\nabla}_{\mathbf{u}}, \overline{\nabla}_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{E};T\mathbf{E})$ extensions of the vertical derivatives, according to the definitions

$$\begin{aligned} \mathscr{F}^{\mathbf{x}}_{\mathbf{u}} \circ \mathbf{Fl}^{\overline{\nabla}_{\mathbf{v}}}_{\lambda} &:= T\mathbf{Fl}^{\overline{\nabla}_{\mathbf{v}}}_{\lambda} \circ \mathbf{H}_{\mathbf{u}_{\mathbf{x}}}\,, \\ \mathscr{F}^{\mathbf{x}}_{\mathbf{v}} \circ \mathbf{Fl}^{\overline{\nabla}_{\mathbf{u}}}_{\lambda} &:= T\mathbf{Fl}^{\overline{\nabla}_{\mathbf{u}}}_{\lambda} \circ \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\,. \end{aligned} \tag{41}$$

By tensoriality, in evaluating the r.h.s. of Eq.(40) at a point $\mathbf{s}(\mathbf{x}) \in \mathbf{E}$, the horizontal lifts $\mathbf{H}_{\mathbf{u}}, \mathbf{H}_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{E};T\mathbf{E})$ can be substituted by the vector fields $\mathscr{F}^{\mathbf{x}}_{\mathbf{u}}, \mathscr{F}^{\mathbf{x}}_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{E};T\mathbf{E})$. Then, by definition in Eq. (41):

$$[\mathscr{F}^{\mathbf{x}}_{\mathbf{v}}, \overline{\nabla}_{\mathbf{u}}]_{\mathbf{x}} = 0\,, \qquad [\overline{\nabla}_{\mathbf{v}}, \mathscr{F}^{\mathbf{x}}_{\mathbf{u}}]_{\mathbf{x}} = 0\,, \tag{42}$$

so that

$$[\mathbf{H}_{\mathbf{v}}, \overline{\nabla}_{\mathbf{u}}]_{\mathbf{x}} + [\overline{\nabla}_{\mathbf{v}}, \mathbf{H}_{\mathbf{u}}]_{\mathbf{x}} = [\mathscr{F}^{\mathbf{x}}_{\mathbf{v}}, \overline{\nabla}_{\mathbf{u}}]_{\mathbf{x}} + [\overline{\nabla}_{\mathbf{v}}, \mathscr{F}^{\mathbf{x}}_{\mathbf{u}}]_{\mathbf{x}} = 0\,. \tag{43}$$

The result holds for any extension of the natural derivatives and the formula for the curvature is independent of the extension, since, by tensoriality, it depends only on the values of the vertical derivatives at $\mathbf{s}(\mathbf{x})$. ■

5 CONNECTION ON A VECTOR BUNDLE

For the sake of completeness, let us resume the peculiar properties of linear connections on a vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}\,;\mathbf{M})$ to infer the relevant special expression of the curvature form.

Definition 5.1 (Linear connection). *In a vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}\,;\mathbf{M})$ a connection is linear if the pair made of the horizontal lift $\mathbf{H}_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{E}\,;H\mathbf{E})$ and of the vector field $\mathbf{v} \in \mathrm{C}^1(\mathbf{M}\,;T\mathbf{M})$ is a linear vector bundle homomorphism from the vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E}\,;\mathbf{M})$ to the vector bundle $T\mathbf{p} \in \mathrm{C}^1(T\mathbf{E}\,;T\mathbf{M})$. This means that, given two sections $\mathbf{s}_1, \mathbf{s}_2 \in \mathrm{C}^1(\mathbf{M}\,;\mathbf{E})$, the following property of $\mathbf{p}$-$T\mathbf{p}$-linearity holds:*

$$\begin{cases} \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}(\mathbf{s}_1 +_{\mathbf{p}} \mathbf{s}_2) = \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_1 +_{T\mathbf{p}} \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_2\,, \\ \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}(\alpha \cdot_{\mathbf{p}} \mathbf{s}) = \alpha \cdot_{T\mathbf{p}} \mathbf{H}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}\,, \quad \forall\, \alpha \in \Re\,. \end{cases} \tag{44}$$

Since the natural derivative is $\mathbf{p}$-$T\mathbf{p}$-linear

$$\begin{cases} T_{\mathbf{v}_{\mathbf{x}}}(\mathbf{s}_1 +_{\mathbf{p}} \mathbf{s}_2) = T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_1 +_{T\mathbf{p}} T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_2\,, \\ T_{\mathbf{v}_{\mathbf{x}}}(\alpha \cdot_{\mathbf{p}} \mathbf{s}) = \alpha \cdot_{T\mathbf{p}} T_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}\,, \quad \forall\, \alpha \in \Re\,, \end{cases} \tag{45}$$

the $\mathbf{p}$-$T\mathbf{p}$-linearity of the horizontal lift $\mathbf{H}_{\mathbf{v}_{\mathbf{x}}}$ is equivalent to $\mathbf{p}$-$T\mathbf{p}$-linearity of the vertical derivative $\overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}$:

$$\begin{cases} \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}(\mathbf{s}_1 +_{\mathbf{p}} \mathbf{s}_2) = \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_1 +_{T\mathbf{p}} \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}_2\,, \\ \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}(\alpha \cdot_{\mathbf{p}} \mathbf{s}) = \alpha \cdot_{T\mathbf{p}} \overline{\nabla}_{\mathbf{v}_{\mathbf{x}}}\mathbf{s}\,, \quad \forall\, \alpha \in \Re\,. \end{cases} \tag{46}$$

The distinguishing feature of a linear vertical bundle, with respect to a general fiber bundle, is that by means of the linear isomorphism provided by the vertical lifting, the vertical derivative $\overline{\nabla}_{\mathbf{v}}\mathbf{s} \in \mathrm{C}^1(\mathbf{M}\,;V\mathbf{E})$ can be identified with a section $\nabla_{\mathbf{v}}\mathbf{s} \in \mathrm{C}^1(\mathbf{M}\,;\mathbf{E})$ of the vector bundle. The result stated below in proposition 5.1 makes appeal to this identification and is a basic property of the covariant derivative in a linear connection (see e.g. (Kobayashi, 1957), (Kobayashi and Nomizu, 1963)).

Proposition 5.1 (Leibniz rule for the covariant derivative). *In a vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$ endowed with a linear connection, the covariant derivative $\nabla_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{s}(\mathbf{M});\mathbf{E})$ fulfils* LEIBNIZ *rule*

$$\nabla_{\mathbf{v}}(f\,\mathbf{s}) = (\nabla_{\mathbf{v}} f)\,\mathbf{s} + f\,(\nabla_{\mathbf{v}}\mathbf{s})\,. \tag{47}$$

In a vector bundle $\mathbf{p} \in \mathrm{C}^1(\mathbf{E};\mathbf{M})$ the iterated and the second covariant derivatives according to a given connection are meaningful. Hence, for any section $\mathbf{s} \in \mathrm{C}^1(\mathbf{M};\mathbf{E})$, the curvature form may be written as

$$\begin{aligned}\mathbf{R}(\mathbf{s})(\mathbf{u},\mathbf{v}) &= (\nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u},\mathbf{v}]})\,\mathbf{s} \\ &= (\nabla^2_{\mathbf{uv}} - \nabla^2_{\mathbf{vu}} + \nabla_{T(\mathbf{u},\mathbf{v})})\,\mathbf{s}\,,\end{aligned} \tag{48}$$

in terms of the second covariant derivative $\nabla^2_{\mathbf{uv}} := \nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\nabla_{\mathbf{u}}\mathbf{v}}$ and of the torsion form $T(\mathbf{u},\mathbf{v}) := \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{[\mathbf{u},\mathbf{v}]}$ which are both tensor fields. Tensoriality may be proven by relying on LEIBNIZ rule Eq. (47) to verify $\mathrm{C}^\infty(\mathbf{E};\Re)$-linearity.

Remark 5.1. *In the curvature formula provided by Theorem 4.1:*

$$\mathbf{R}(\mathbf{s})(\mathbf{u},\mathbf{v}) = [\overline{\nabla}_{\mathbf{u}},\overline{\nabla}_{\mathbf{v}}](\mathbf{s}) - \overline{\nabla}_{[\mathbf{u},\mathbf{v}]}(\mathbf{s})\,, \tag{49}$$

the term $[\overline{\nabla}_{\mathbf{u}},\overline{\nabla}_{\mathbf{v}}](\mathbf{s})$ cannot be written as $(\overline{\nabla}_{\mathbf{u}}\overline{\nabla}_{\mathbf{v}} - \overline{\nabla}_{\mathbf{v}}\overline{\nabla}_{\mathbf{u}})\,\mathbf{s}$. Indeed, since $\overline{\nabla}_{\mathbf{u}},\overline{\nabla}_{\mathbf{v}} \in \mathrm{C}^1(\mathbf{E};V\mathbf{E})$, the compositions $\overline{\nabla}_{\mathbf{u}}\overline{\nabla}_{\mathbf{v}}$ and $\overline{\nabla}_{\mathbf{v}}\overline{\nabla}_{\mathbf{u}}$, are not defined, unless the bundle is a vector bundle so that, by relying on the vertical lifting, the identification $V\mathbf{E} \simeq \mathbf{E}$ and the substitution $\overline{\nabla} \to \nabla$ can be made.

6 CONCLUSIONS

Connections on fiber bundles and their torsion and curvature forms are of primary importance in many basic issues of mathematical physics, as witnessed by a vast number of contributions in literature (see e.g. (Mangiarotti and Sardanashvily, 2000)). The topic has been revisited here with the aim of providing a direct proof of the relation between the integrability condition provided by FROBENIUS theorem and the expression of the curvature field in terms of vertical derivatives. This result, in the general form provided here, appears to be new, since classical treatments deal with linear connections on vector bundles. The contributed proof does not require additional geometric structure on the fiber bundle, being based on the notion of natural derivative of a section, on a suitable extension, by foliation, to a vector field in the tangent bundle, and on a simple but powerful tensoriality argument.

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