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Geometric Action Principles in Classical Dynamics

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Abstract - General principles of classical dynamics are usually developed in the framework of phase spaces, that is tangent or cotangent bundles over the control manifold. A more effective approach is proposed here by applying POINCARÉ-CARTAN theory of differential forms directly to the control manifold, so that lifting operations are completely avoided. The basic distinction between action principles and stationarity of functionals is pointed out. The EULER-LAGRANGE-HAMILTON variational theory is formulated without end constraints on the trajectory variations. A careful treatment of natural and essential conditions for the variational problem leads to a proper formulation of MAUPERTUIS action principle and to assess its equivalence to HAMILTON principle. POINCARÉ-CARTAN and HAMILTON-PONTRYAGIN hybrid principles, involving vertical variations of vector and covector fields, are addressed with an appropriate geometric approach.

Riassunto - I principi della dinamica classica sono usualmente sviluppati nel contesto di spazi delle fasi e cioè di fibrati tangenti o cotangenti sulla varietà di controllo. Si propone qui una impostazione più efficace la quale, applicando la teoria di POINCARÉ-CARTAN delle forme differenziali direttamente alla varietà di controllo, consente di evitare il sollevamento a spazi delle fasi. Una basica distinzione tra principi d'azione e stazionarietà di funzionali è posta in luce. La teoria variazionale di EULER-LAGRANGE-HAMILTON è svilup-

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pata senza imporre condizioni sui punti estremi della traiettoria. Una attenta trattazione delle condizioni naturali ed essenziali per il problema variazionale conduce ad una corretta formulazione del principio di azione di Maupertuis e ad asserirne l'equivalenza al principio di Hamilton. I principi ibridi di Poincaré-Cartan ed Hamilton-Pontryagin, con variazioni verticali di campi vettoriali e covettoriali, sono discussi in un appropriato contesto geometrico.

1 INTRODUCTION

Classical dynamics may be conventionally considered to be born about 1687 with Newton's *Principia* and grew up to a well-developed theory in the fundamental works by d'Alembert, Euler, Lagrange, Poisson, Hamilton, Jacobi, Bertrand, during the XVIII century and the first half of the XIX century (d'Alembert, 1743; Euler, 1744, 1761; Lagrange, 1788; Poisson, 1811; Hamilton, 1834, 1940; Jacobi, 1837a,b, 1884; Bertrand, 1852).

The differential geometric point of view, with the introduction of the notions of nonlinear configuration manifolds, convective derivatives along a motion and derivatives according to a parallel transport, is based on notions and methods mainly due to Sophus Lie, Henri Poincaré, Élie Cartan (Lie and Engel, 1888-1890-1893; Poincaré, 1892-1893-1899; Cartan É., 1922) and accounted, for instance, in (Klein, 1962; Godbillon, 1969; Souriau, 1970; Choquet-Bruhat, 1970; Deschamps, 1970; Spivak, 1970; Choquet-DeWitt-Dillard, 1982; Romano G., 2007).

Classical analytical mechanics deals with particle dynamics or rigid body motions. Extensions to continuous systems were treated in (Arnold, 1974; Abraham and Marsden, 1988; Marsden and Hughes, 1983; Abraham et al., 2002), by considering manifolds modeled on Banach spaces. In these treatments the formal structure of the dynamics of finite dimensional systems is however still reproduced and most results are still proposed in coordinates notation.

Our presentation makes no essential use of coordinates, with all notions and results defined and expressed in intrinsic geometrical terms. For finite dimensional control manifolds, the translation into coordinate notation is however straightforward and is explicitly reported to provide a direct comparison with existing treatments. The advantages of a geometric formulation are conceptual, since mechanical objects are properly described and retain their respective roles and rules of mutually interacting entities, and also operational, since a general formulation permits to choose the representation more suitable for the problem at hand.

We follow the common choice of taking Hamilton principle, inspired by earlier ideas by Fermat and Huygens in optics, as the basic axiom of dynamics since it has the pleasant flavour of an extremality property and, much more than this, because it leads in a natural and direct way to a general formulation of Lagrange dynamics.

In this respect we do not share the opinion in (Abraham and Marsden, 1988, Part II), *Analytical Dynamics*, section 3.8 *Variational Principles in Mechanics*, where variational principles are placed on the same ground of differential equations, with a preference for the latter in operative tasks. However, the leading position of action principles has been subsequently acknowledged in (Marsden, Patrick and Shkoller, 1998).

In fact, while it is certainly true that differential formulations are the ones more suitable for specific analytical tasks, on the other hand variational formulations, in the form of action principles, are more basic. The main motivation is that action principles just require the notions of LAGRANGE function, mass form and motion in the metric space-time.

According to the principle of geometric naturality, exposed in (Romano and Barretta, 2011, 2013; Romano et al., 2014a,b,c), the formulation of laws of dynamics must involve, as geometric objects, only the space metric in the event manifold and the motion along the trajectory. This requirement gives to integral variational principles of dynamics, a leading position in the theory. On the contrary operative differential formulations are based on the choice of a connection in the events manifold. Moreover, basic issues such as invariance under change of observer, are most readily and properly discussed in terms of action principles formulated with intrinsic treatments.

Last but not least, effective computational strategies make direct reference to variational formulations (Bailey, 1975; Riff and Baruch, 1984; Borri et al., 1985; Peters and Izadpanah, 1988; Borri et al., 1992; Borri and Bottasso, 1993). Computational issues will however not be explicitly dealt with in this contribution but a detailed geometric investigation is in progress.

Fine mathematical issues concerning calculus of variations, such as the ones discussed in (Ambrosio et al., 2000; Fonseca and Leoni, 2007), are outside the range of this presentation.

In section 2 we provide the abstract definition of an *action principle* as a variational condition for the integral of a differential form along a path which is dragged by virtual motions. No fixed end point conditions are imposed and the effect of dynamical forces is included in the formulation. A special attention is devoted to the required extension of the domain of definition of the governing differential form, an issue usually not considered in treatments akin to classical analytical mechanics.

Even in most appreciable formulations, such as (Cartan H., 1967), variations are intended to be evaluated on a functional, expressed as the integral of a differential form, along a class of curves. This interpretation leads to difficult and also critical statements concerning the topological properties of the functional space in which stationarity is to be imposed, especially if the fixed ends condition is retained (Oliva, 1798; Terra and Kobayashi, 2004a,b).

In continuum dynamics, the differential forms to be integrated on the trajectory depend on the velocity field and on the mass form, and these fields are defined only on the trajectory. In performing the variations, an extension by push along each virtual flow must then be assumed. Therefore, variational formulations in dynamics do not involve functionals to be evaluated on a class of curves.

A synthesis of notions, definitions and results of differential geometry, strictly needed in the paper, is provided in the appendix for reference and readers' convenience.

Sect. 3 presents a brief introduction to space-time kinematics and a summary of relevant definitions of mechanical objects. Sect. 4 and 5 are dedicated to the formal construction of the control manifold by means of a correspondence which is an injective immersion in the configuration manifold. Fields based in body placements are then translated into geometrico-dynamical objects in the control manifold.

HAMILTON principle and LAGRANGE action principle are enunciated and discussed in detail in Sects.6 and 7 and NOETHER theorem is deduced as a special case. MAUPERTUIS action principle is revisited in Sect.8 in the general form of a constrained action principle, in which the constraint of mechanical power balance is imposed on virtual velocities, and its equivalence to all other action principles is assessed. POINCARÉ-CARTAN action principle is introduced in Sect.9 and DONKIN theorem provides the relations to transform the canonical equation of dynamics in terms of the momentum field. Extensions to hybrid POINCARÉ-CARTAN and HAMILTON-PONTRYAGIN action principles, by the inclusion of vertical virtual variations of tangent and cotangent fields, are discussed in Sect.10.

Sect. 11 is dedicated to the formulation of action principles in terms of time integrals, and to amendment of non geometric treatments. Formulations in terms of a linear connection provide powerful theoretical and computational tools, when translated from the control manifold to the event manifold.

Comments and remarks are exposed in the final Sect.12, with a synoptic table collecting main issues and results.

2 ABSTRACT ACTION PRINCIPLE AND EULER CONDITIONS

A status of the system is described by a point in $\mathbf{M}$, the *state space*.

Definition 2.1 (Action integral). *The action integral associated with a path* Γ *in the state-space* $\mathbf{M}$ *is the signed-length of the* $1D$ *oriented compact submanifold* Γ*, evaluated according to the action* 1*-form* ω^1 *on* Γ*:*

$$\int_\Gamma \omega^1 \,. \tag{1}$$

A proper statement of the action principle requires a definition of the *virtual displacements* along which the trajectory is assumed to be varied and a definite criterion to extend the domain of definition of the action 1-form on the sheet spanned by the trajectory dragged by the virtual flow.

In formulating an action principle, virtual velocities at Γ are assumed to belong to a suitable set $\mathscr{H}_{\mathbf{M}}$ of sections of the tangent bundle $T_\Gamma\mathbf{M}$. Source terms are represented by differential forms α^1, α^2 on $T_\Gamma\mathbf{M}$.

The source 2-form α^2 is *potential* if it is defined on a neighbourhood $U(\Gamma) \subset \mathbf{M}$ of the path and is exact. This means that there exists a differential 1-form $\beta^1 : U(\Gamma) \mapsto T^*\mathbf{M}$ such that $\alpha^2 = d\beta^1$, with d exterior differentiation on $\mathbf{M}$.

Definition 2.2 (Abstract action principle). *An action principle, governed by differential* 1*-form* ω^1 *on* $\mathbf{M}$*, is a variational condition involving the rate of geometric variation of the integral of the action form along a path* $\Gamma \subset \mathbf{M}$*, due to a virtual flow, and source terms, distributed along the path (source* 2*-form* α^2*) and concentrated at singular points* $\mathrm{SING}(\Gamma)$ *(source* 1*-form* α^1*), evaluated on the virtual velocity* $\delta\mathbf{v} = \partial_{\lambda=0}\mathbf{Fl}^{\mathbf{v}}_\lambda \in \mathscr{H}_{\mathbf{M}}$ *of virtual flows* $\mathbf{Fl}^{\mathbf{v}}_\lambda : \Gamma \mapsto \mathbf{M}$

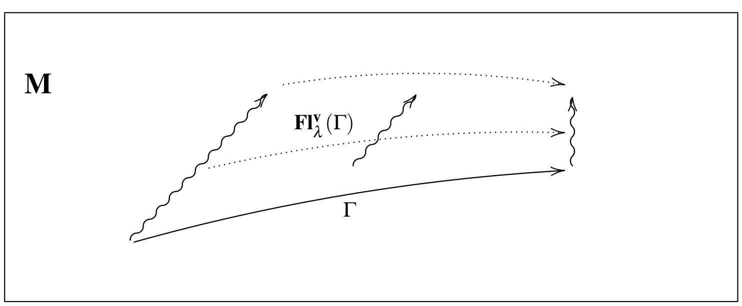
$$\boxed{\partial_{\lambda=0} \int_{\mathbf{Fl}^{\mathbf{v}}_\lambda(\Gamma)} \omega^1 - \oint_{\partial\Gamma} \omega^1 \cdot \delta\mathbf{v} = \int_\Gamma \alpha^2 \cdot \delta\mathbf{v} + \oint_{\Gamma_{\mathrm{SING}}} \alpha^1 \cdot \delta\mathbf{v}\,.} \tag{2}$$

A path Γ fulfilling the action principle is called a *trajectory* for the action form ω^1, under the effect of the sources α^1 and α^2.

Eq. (2) may be stated by saying that the rate of variation along the virtual displacement of the ω^1-integral on the oriented trajectory Γ, minus the outward boundary flux of the ω^1-virtual power (*formule de* JOSEPH BERTRAND (Cartan H., 1967, p. 132)), is equal to the virtual power performed by the source form.

Denoting by $\mathbf{x}_1$ and $\mathbf{x}_2$ the initial and final end points of Γ, it is $\partial\Gamma = \mathbf{x}_2 - \mathbf{x}_1$ (a 0-chain) and the boundary integral may be written as

$$\oint_{\partial\Gamma} \omega^1 \cdot \delta\mathbf{v} = (\omega^1 \cdot \delta\mathbf{v})(\mathbf{x}_2) - (\omega^1 \cdot \delta\mathbf{v})(\mathbf{x}_1)\,. \tag{3}$$



(4)

Definition 2.3 (Extremality principle). *The variational condition in the geometric action principle, in absence of source terms, takes the expression of an extremality condition*

$$\boxed{\partial_{\lambda=0} \int_{\mathbf{Fl}^{\mathbf{v}}_{\lambda}(\Gamma)} \omega^1 - \oint_{\partial\Gamma} \omega^1 \cdot \delta\mathbf{v} = 0\,.} \tag{5}$$

Eq. (5) *has a simple motivation when interpreting* ω^1 *as a measure of length. In changing the position of a geodesic by a virtual flow, the length tends to be invariant when boundary virtual velocity vectors fulfill the* equiprojectivity *property (vanishing of Eq.* (3)*). A familiar special instance is a straight line in* EUCLID *space.*

When the differential form to be integrated is independent of the parametrization of the integration path Γ, also the action principle, as enunciated in Eq. (2), is purely geometrical, in the sense that the variational condition results to be independent of the parametrization. This is indeed the case for FERMAT principle of least time in geometrical optics.

Time parametrization plays instead a basic role in dynamics and so the governing action principle must depend on the trajectory parametrization.

The issue is enlightened by the treatment provided in the sequel by observing explicitly that the 1-form involved in the stationarity action principle is in fact expressed by the composition of the LAGRANGE scalar-valued map with the velocity field. The governing 1-form depends then on the velocity and hence the time parametrization of the trajectory takes a central position in the theory.

The new approach to geometric dynamics reveals also that an intrinsic treatment is feasible by considering only fields in the control manifold.

Lifting to the tangent bundle can thus be bypassed, with the advantage of simplicity and reconciliation with (but also amendments to) standard treatments in coordinates. A full discussion in provided in the sequel.

The necessary and sufficient differential condition for a path to be a trajectory, a result due to Euler (1744), is provided by the next theorem.

The classical result deals with regular paths and fixed end points. The new statement considers displaceable end points and piecewise regular paths, so that extremality is expressed in terms of differential and jump conditions.

It is to be remarked that, as already evidenced in Sect. 1, the explicit appearance of the final point of the trajectory in the expression of the action principle, plays a basic role in computational implementation of dynamical problems.

Theorem 2.1 (Euler's condition). *A path* $\Gamma \subset \mathbf{M}$ *is a trajectory if and only if the tangent vector field* $\mathbf{v}_\Gamma : \Gamma \mapsto T\Gamma$ *meets, at regular points, the differential condition*

$$(d\omega^1 - \alpha^2) \cdot \delta\mathbf{v} \cdot \mathbf{v}_\Gamma = 0\,, \quad \forall\, \delta\mathbf{v} \in \mathscr{H}_\mathbf{M}\,, \tag{6}$$

and, at singularity interfaces, the jump conditions

$$[[\omega^1 \cdot \delta\mathbf{v}]] = \alpha^1\,, \quad \forall\, \delta\mathbf{v} \in \mathscr{H}_\mathbf{M}\,. \tag{7}$$

Proof. Applying the integral extrusion formula:

$$\partial_{\lambda=0} \int_{\mathbf{Fl}^{\mathbf{v}}_\lambda(\Gamma)} \omega^1 - \oint_{\partial\Gamma} \omega^1 \cdot \delta\mathbf{v} = \int_\Gamma d\omega^1 \cdot \delta\mathbf{v}\,, \tag{8}$$

the result follows upon localization. ■

Remark 2.1 (Faithful and hybrid principles). *A special nomenclature is adopted in the present paper. Variational statements conforming to the definition in Eq.* (2) *will be labeled as (*faithful*)* action principles. *In these action principles a single action* 1*-form is involved both in the line integral and in the boundary integral. Other variational statements in which two distinct* 1*-forms appear in the line integral and in the boundary integral, will be instead labeled as* hybrid action principles. *The distinction is significant because only* faithful *action principles are associated with an* EULER *condition expressed in terms of the exterior derivative of a* 1*-form, as shown by the abstract treatment leading to Eq.* (6) *of Th. 2.1.*

Remark 2.2 (Action principles vs stationarity of functionals). *As evidenced by Def. 2.2, the variational condition enunciated in the statement of an action principle, is not the stationarity condition for a functional. The basic distinction is that in an action principle the involved* 1*-form is defined only on the* 1*D trajectory manifold and is declaratively extended along each virtual flow in a natural manner. So, there is in fact no functional to be differentiated along virtual directions. Rather, in an action principle, the extensions of the* 1*-form, to be integrated on dragged trajectories, are performed according to each dragging virtual flow. The distinction is further put into evidence by the formulation expressed by Eq.* (2)*, with the elimination of the fixed-ends condition, usually included into the statement of action principles, as depicted in frame* (4)*, and with the inclusion of the boundary integral and of the effects of distributed and singular sources.*

3 KINEMATICS IN THE EVENT MANIFOLD

In the 4D manifold of events $\mathbf{e} \in \mathscr{E}$ each observer defines a *time-projection* $t_{\mathscr{E}} : \mathscr{E} \mapsto \mathscr{Z}$, that is a surjective submersion on the real line $\mathscr{Z}$ of time instants[1] and a vector field $\mathbf{Z} : \mathscr{E} \mapsto T\mathscr{E}$ of *time-arrows* in the tangent bundle $\tau_{\mathscr{E}} : T\mathscr{E} \mapsto \mathscr{E}$, fulfilling tuning $t_{\mathscr{E}}\uparrow\mathbf{Z} := 1$, as described by the commutative diagram

$$\begin{tikzcd} \mathscr{Z} \arrow[r, "1"] & T\mathscr{Z} \\ \mathscr{E} \arrow[u, "t_{\mathscr{E}}"] \arrow[r, "\mathbf{Z}"] & T\mathscr{E} \arrow[u, "dt_{\mathscr{E}}"'] \end{tikzcd} \quad \Longleftrightarrow \quad \langle dt_{\mathscr{E}}, \mathbf{Z} \rangle = 1 \circ t_{\mathscr{E}} \,. \tag{9}$$

A double foliation of the 4D events manifold $\mathscr{E}$ into complementary 3D *space-slices* $\mathscr{S}$ of *isochronous* events (with a same corresponding time instant) and 1D *time-lines* of *isotopic* events (with a same corresponding space location) is thus introduced according to FROBENIUS theorem.

The projector $dt_{\mathscr{E}} \otimes \mathbf{Z} : T\mathscr{E} \mapsto T\mathscr{E}$ splits the tangent bundle into complementary time-vertical $V\mathscr{E}$ and time-horizontal $H\mathscr{E}$ sub-bundles, with time-vertical vectors in the kernel of $dt_{\mathscr{E}} : T\mathscr{E} \mapsto T\mathscr{Z}$.

These sub-bundles are respectively called *space bundle* and *time bundle*.

In the familiar EUCLID setting of classical Mechanics, the time projection is the same for all observers (universality of time).

A reference frame $\{\mathbf{d}_i \,; i = 0,1,2,3\}$ for the event manifold is *adapted* if $\mathbf{d}_0 = \mathbf{Z}$ and $\mathbf{d}_i \in V\mathscr{E}$, $i = 1,2,3$.

Definition 3.1 (Trajectory). *The* trajectory manifold *is the geometric object investigated in Mechanics, characterized by an embedding*[2] $\mathbf{i} : \mathscr{T} \mapsto \mathscr{E}$ *into the event manifold* $\mathscr{E}$ *such that the image* $\mathscr{T}_{\mathscr{E}} := \mathbf{i}(\mathscr{T})$ *is a submanifold.*

Definition 3.2 (Motion). *The* motion *along the trajectory*

$$\{ \varphi^{\mathscr{T}}_{\alpha} : \mathscr{T} \mapsto \mathscr{T} \,,\; \alpha \in \mathscr{Z} \} \,, \tag{10}$$

is a simultaneity preserving one-parameter family of maps fulfilling the composition rule

$$\varphi^{\mathscr{T}}_{\alpha} \circ \varphi^{\mathscr{T}}_{\beta} = \varphi^{\mathscr{T}}_{(\alpha+\beta)} \,, \tag{11}$$

for any pair of time-lapses $\alpha, \beta \in \mathscr{Z}$. *Each* $\varphi^{\mathscr{T}}_{\alpha} : \mathscr{T} \mapsto \mathscr{T}$ *is a* displacement.

[1] A submersion has a surjective differential at each point. *Zeit* is the German word for *Time*.

[2] An immersion has an injective differential at each point. An embedding is an injective immersion whose co-restriction is continuous with the inverse.

The trajectory will alternatively be considered as a $(1+n)$D manifold $\mathscr{T}$ by itself or as a submanifold $\mathscr{T}_{\mathscr{E}} = \mathbf{i}(\mathscr{T}) \subset \mathscr{E}$ of the event manifold.

Then, a coordinate system is adopted on $\mathscr{T}$ while an adapted 4D space-time coordinate system in $\mathscr{E}$ is adopted on $\mathscr{T}_{\mathscr{E}}$.

The trajectory inherits from the events manifold the time projection $t_{\mathscr{T}} := t_{\mathscr{E}} \circ \mathbf{i} : \mathscr{T} \mapsto \mathscr{Z}$ which defines a time-bundle denoted by $V\mathscr{T}$ and called the *material bundle*. A fiber of simultaneous events is a *body placement*, denoted by $\Omega \subset \mathscr{T}$.

The space-time displacement $\varphi_\alpha : \mathscr{T}_{\mathscr{E}} \mapsto \mathscr{T}_{\mathscr{E}}$ and the trajectory displacement $\varphi_\alpha^{\mathscr{T}} : \mathscr{T} \mapsto \mathscr{T}$ are related by the commutative diagram

$$\begin{tikzcd}
\mathscr{T}_{\mathscr{E}} \arrow[r, "\varphi_\alpha"] \arrow[dd, bend right=60, "t_{\mathscr{E}}"'] & \mathscr{T}_{\mathscr{E}} \arrow[dd, bend left=60, "t_{\mathscr{E}}"] \\
\mathscr{T} \arrow[u, "\mathbf{i}"] \arrow[r, "\varphi_\alpha^{\mathscr{T}}"] \arrow[d, "t_{\mathscr{T}}"'] & \mathscr{T} \arrow[u, "\mathbf{i}"'] \arrow[d, "t_{\mathscr{T}}"] \\
\mathscr{Z} \arrow[r, "t_\alpha"] & \mathscr{Z}
\end{tikzcd}
\quad \iff \quad t_{\mathscr{E}} \circ \varphi_\alpha = t_\alpha \circ t_{\mathscr{E}}\,, \tag{12}$$

where the time translation $t_\alpha : \mathscr{Z} \mapsto \mathscr{Z}$ is defined by

$$t_\alpha(t) := t + \alpha\,, \quad t, \alpha \in \mathscr{Z}\,. \tag{13}$$

Definition 3.3 (Material particles and body manifold). *The physical notion of* material particle *corresponds in the geometric view to a time-parametrized curve of events in the trajectory, related by the motion as follows*

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathscr{T} \,:\, \mathbf{e}_2 = \varphi_\alpha^{\mathscr{T}}(\mathbf{e}_1)\,. \tag{14}$$

Accordingly, we will say that a geometrical object is defined along *(not* at*) a material particle. Events belonging to a* material particle *form a class of equivalence and the quotient manifold so induced in the trajectory is the* body manifold.

The space-time *velocity* of the motion is defined by the derivative

$$\mathbf{v}_{\mathscr{E}} := \partial_{\alpha=0}\, \varphi_\alpha \in T\mathscr{T}_{\mathscr{E}}\,. \tag{15}$$

Taking the time derivative of (12) we have

$$\partial_{\alpha=0}\,(t_{\mathscr{E}} \circ \varphi_\alpha) = \langle dt_{\mathscr{E}}, \mathbf{v}_{\mathscr{E}} \rangle = (\partial_{\alpha=0}\, t_\alpha) \circ t_{\mathscr{E}} = 1 \circ t_{\mathscr{E}}\,, \tag{16}$$

and comparing with Eq. (9) we get the decomposition into space and time components

$$\mathbf{v}_{\mathscr{E}} = \mathbf{v}_{\mathscr{S}} + \mathbf{Z}\,, \tag{17}$$

with $\langle dt_{\mathscr{E}}, \mathbf{v}_{\mathscr{S}} \rangle = 0$. The motions of a body is characterised by the conservation property concerning the mass, represented by a volume form $\mathbf{m} : \mathscr{T} \mapsto \mathrm{VOL}(V\mathscr{T})$ on the material bundle over the trajectory.

Definition 3.4 (Mass conservation). *Mass conservation along the motion is expressed by the pull-back and* LIE*-derivative conditions*

$$\varphi_\alpha\!\downarrow\!\mathbf{m} = \mathbf{m} \iff \mathscr{L}_{\mathbf{v}_\mathscr{E}}\,\mathbf{m} = \mathbf{0}\,, \tag{18}$$

or by the equivalent integral condition for all placements Ω

$$\int_{\varphi_\alpha(\Omega)} \mathbf{m} = \int_\Omega \varphi_\alpha\!\downarrow\!\mathbf{m} = \int_\Omega \mathbf{m}\,. \tag{19}$$

Let us set forth some basic definitions where $\mathbf{g}_{\mathrm{SPA}}$ denotes the metric tensor in the space bundle $V\mathscr{E}$.

The local LAGRANGE function per unit mass $L_\mathscr{E} : V\mathscr{E} \mapsto \mathrm{FUN}(V\mathscr{E})$ is defined for any time-vertical tangent vector $\mathbf{v}_\mathscr{S} \in V\mathscr{E}$, by

$$L_\mathscr{E}(\mathbf{v}_\mathscr{S}) := \tfrac{1}{2}\,\mathbf{g}_{\mathrm{SPA}}(\mathbf{v}_\mathscr{S}\,,\mathbf{v}_\mathscr{S}) - \Pi(\tau_\mathscr{E}(\mathbf{v}_\mathscr{S}))\,, \tag{20}$$

and is therefore the sum of two contributions:

- the local *kinetic energy* per unit mass $K_\mathscr{E} : V\mathscr{E} \mapsto \mathrm{FUN}(V\mathscr{E})$, a scalar quadratic field over the space bundle, given by

$$K_\mathscr{E}(\mathbf{v}_\mathscr{S}) := \tfrac{1}{2}\,\mathbf{g}_{\mathrm{SPA}}(\mathbf{v}_\mathscr{S}\,,\mathbf{v}_\mathscr{S}) \tag{21}$$

- and a convex *scalar potential* $\Pi : \mathscr{E} \mapsto \mathrm{FUN}(V\mathscr{E})$ defined in the whole event manifold.

Definition 3.5 (Fiber derivative of Lagrange function). *The* fiber derivative *of the function* $L_\mathscr{E} : V\mathscr{E} \mapsto \mathrm{FUN}(V\mathscr{E})$ *is the covariant tensor* $d_F L_\mathscr{E} : V\mathscr{E} \mapsto (V\mathscr{E})^*$ *pointwise defined, for any pair* $(\mathbf{v}_\mathscr{S}\,,\delta\mathbf{v}_\mathscr{S}) \in V\mathscr{E} \times_\mathscr{E} V\mathscr{E}$ *of space vectors* $\mathbf{v}_\mathscr{S}, \delta\mathbf{v}_\mathscr{S} \in V\mathscr{E}$ *having the same base point in* $\mathscr{E}$*, by*

$$\begin{aligned} \langle\, d_F L_\mathscr{E}(\mathbf{v}_\mathscr{S}), \delta\mathbf{v}_\mathscr{S}\,\rangle &:= \partial_{\lambda=0}\, L_\mathscr{E}(\mathbf{v}_\mathscr{S} + \lambda\,\delta\mathbf{v}_\mathscr{S}) \\ &= T L_\mathscr{E}(\mathbf{v}_\mathscr{S}) \cdot \mathrm{VLIFT}(\mathbf{v}_\mathscr{S}\,,\delta\mathbf{v}_\mathscr{S})\,, \end{aligned} \tag{22}$$

where the vertical lifting is given by $\mathrm{VLIFT}(\mathbf{v}_\mathscr{S}\,,\delta\mathbf{v}_\mathscr{S}) := \partial_{\lambda=0}\,(\mathbf{v}_\mathscr{S} + \lambda\,\delta\mathbf{v}_\mathscr{S})$*. In standard terms, taking the fiber derivative means that the derivative is taken while keeping fixed the base point of the argument vector.*

The fiber derivative of the LAGRANGE function is given by

$$d_F L_\mathscr{E} = d_F K_\mathscr{E} = \mathbf{g}_{\mathrm{SPA}}\,, \tag{23}$$

and provides a linear isomorphism between the dual space bundles $V\mathscr{E}$ and $(V\mathscr{E})^*$. Thus a means, to transform back and forth in a biunivocal and fiberwise linear manner between these bundles, is available.

Definition 3.6 (Parallel derivative of Lagrange function). *The* parallel derivative *of the function* $L_{\mathscr{E}} : V\mathscr{E} \mapsto \mathrm{F}_{\mathrm{UN}}(V\mathscr{E})$ *is the covariant tensor field* $\nabla L_{\mathscr{E}} : V\mathscr{E} \mapsto (V\mathscr{E})^*$ *defined by*

$$\langle \nabla L_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}}), \delta \mathbf{v}_{\mathscr{S}} \rangle := \partial_{\lambda=0}\, L_{\mathscr{E}}(\delta \varphi_{\lambda}^{\mathscr{C}})\Uparrow \mathbf{v}_{\mathscr{S}}\,, \tag{24}$$

where $(\mathbf{v}_{\mathscr{S}}, \delta \mathbf{v}_{\mathscr{S}}) \in V\mathscr{E} \times_{\mathscr{E}} V\mathscr{E}$ *and* $\delta \varphi_{\lambda}^{\mathscr{C}}\Uparrow$ *is the parallel transport along the flow* $\delta \varphi_{\lambda}^{\mathscr{C}}$ *associated with the vector field* $\delta \mathbf{v}_{\mathscr{S}}$.

In standard terms, the parallel derivative is enunciated by saying that the derivative is taken while keeping the argument vector *constant*, which means *parallel transported* along the curve chosen to change its base point. The parallel derivative depends therefore on the adoption of a linear connection, see Sect. A.

Definition 3.7 (Euler-Legendre transform). *The convex local* LAGRANGE *function* $L_{\mathscr{E}} : V\mathscr{E} \mapsto \mathrm{F}_{\mathrm{UN}}(V\mathscr{E})$ *and the convex conjugate local* HAMILTON *function* $H_{\mathscr{E}} : (V\mathscr{E})^* \mapsto \mathrm{F}_{\mathrm{UN}}((V\mathscr{E})^*)$ *on the dual bundle are related by the transform*[3]

$$\begin{cases} \mathbf{v}^*_{\mathscr{S}} = d_F L_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}})\,, \\ H_{\mathscr{E}}(\mathbf{v}^*_{\mathscr{S}}) := \langle \mathbf{v}^*_{\mathscr{S}}, \mathbf{v}_{\mathscr{S}} \rangle - L_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}})\,, \\ \mathbf{v}_{\mathscr{S}} = d_F H_{\mathscr{E}}(\mathbf{v}^*_{\mathscr{S}})\,, \end{cases} \tag{25}$$

where $\mathbf{v}_{\mathscr{S}} \in V\mathscr{E}$ *and* $\mathbf{v}^*_{\mathscr{S}} \in (V\mathscr{E})^*$. *When expressed in terms of velocities, the* HAMILTON *function yields the* energy *function* $E_{\mathscr{E}} : V\mathscr{E} \mapsto \mathrm{F}_{\mathrm{UN}}(V\mathscr{E})$, *defined by the composition*

$$E_{\mathscr{E}} = H_{\mathscr{E}} \circ d_F L_{\mathscr{E}}\,. \tag{26}$$

The action *function* $A_{\mathscr{E}} : V\mathscr{E} \mapsto \mathrm{F}_{\mathrm{UN}}(V\mathscr{E})$ *is defined by the transform Eq. (25) as*

$$A_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}}) := \langle d_F L_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}}), \mathbf{v}_{\mathscr{S}} \rangle = L_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}}) + E_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}})\,. \tag{27}$$

Definition 3.8 (Time invariance). *Invariance along the motion of a material morphism* $\phi_{\mathscr{T}} : \mathrm{T}_{\mathrm{ENS}}(V\mathscr{T}) \mapsto \mathrm{T}_{\mathrm{ENS}}(V\mathscr{T})$ *and of a space morphism* $\phi_{\mathscr{E}} : \mathrm{T}_{\mathrm{ENS}}(V\mathscr{E}) \mapsto \mathrm{T}_{\mathrm{ENS}}(V\mathscr{E})$ *are respectively defined by the conditions*

$$\begin{aligned} \phi_{\mathscr{T}}(\varphi_{\alpha}\!\uparrow\!\mathbf{s}) = \varphi_{\alpha}\!\uparrow\!(\phi_{\mathscr{T}}(\mathbf{s})) &\iff (\mathscr{L}_{\mathbf{v}_{\mathscr{T}}} \phi_{\mathscr{T}})(\mathbf{s}) = \mathbf{0}\,, \\ \phi_{\mathscr{E}}(\varphi_{\alpha}\!\Uparrow\!\mathbf{s}_{\mathscr{E}}) = \varphi_{\alpha}\!\Uparrow\!(\phi_{\mathscr{E}}(\mathbf{s}_{\mathscr{E}})) &\iff (\nabla_{\mathbf{v}_{\mathscr{E}}} \phi_{\mathscr{E}})(\mathbf{s}_{\mathscr{E}}) = \mathbf{0}\,. \end{aligned} \tag{28}$$

The invariance conditions in Eq. (28) should not be confused with the condition of time independence that will be introduced in Eq. (82).

[3] The LEGENDRE transform was first introduced by EULER, see (Arnold et al., 1988, p. 23).

4 CONFIGURATION AND CONTROL MANIFOLDS

The geometric picture of continuum dynamics takes advantage from the introduction of a possibly infinite dimensional manifold of configurations, which is a natural extension of the simplest case considered in NEWTON point-particle dynamics, where the configuration manifold is just the EUCLID space time.

Mathematical aspect of the matter are treated in (Eliasson, 1967, Th. 5.2, p. 186), and (Palais, 1968, Th. 13.6 p. 51), as discussed in (Romano et al., 2009b).

Definition 4.1 (Configuration manifold). *A* placement manifold *is a submanifold of the event manifold $\mathscr{E}$, made of isochronous events, diffeomorphic to the body placements along the trajectory. The collection of all placement manifolds is the infinite dimensional* configuration manifold $\mathscr{P}(\mathscr{E})$.

Definition 4.2 (Connection in the configuration manifold). *A linear connection $\nabla^{\mathscr{E}}$, with parallel transport $\Uparrow^{\mathscr{E}}$, in the event manifold $\mathscr{E}$ induces a linear connection and a parallel transport in the configuration manifold $\mathscr{P}(\mathscr{E})$, still denoted by the same symbols. Considering a parametrized path $\mathbf{c} : \Re \mapsto \mathscr{P}(\mathscr{E})$ and the corresponding congruence of paths in $\mathscr{E}$, the parallel transport in $\mathscr{P}(\mathscr{E})$ is performed by acting with the parallel transport along each path of the corresponding congruence in $\mathscr{E}$.*

Definition 4.3 (Control manifold). *The* control manifold $\mathscr{C}$ *is the domain of a* representation map [4]

$$\xi : \mathscr{C} \mapsto \mathscr{P}(\mathscr{E}) \,, \tag{29}$$

which is an injective immersion into the configuration manifold $\mathscr{P}(\mathscr{E})$. We will consider two kinds of controls.

1. *A* perfect control *in which the manifold $\mathscr{C}$ is infinite dimensional modeled on a suitable* BANACH *space and the* representation map *is a diffeomorphism between the control and the configuration manifolds.* [5]
2. *A* discrete control *in which the manifold $\mathscr{C}$ is finite dimensional and the* representation map *is just a (non surjective) injective immersion. Discrete controls are adopted in computational procedures. Placements in the range of the representation map are then said to be* controllable.

[4] The notion of representation map $\xi : \mathscr{C} \mapsto \mathscr{P}(\mathscr{E})$ extends to possibly infinite dimensional manifolds the map adopted for description of finite dimensional dynamical systems or for discretization of continua, in computational procedures.

[5] Perfect control of dynamical systems was investigated in (Romano et al., 2009b).

To each point $\mathbf{x} \in \mathscr{C}$ there corresponds a compact placement submanifold $\Omega = \xi(\mathbf{x}) \subset \mathscr{E}$ and hence a time instant $t_{\mathscr{E}}(\mathbf{e}) \in \mathscr{Z}$ evaluated at any $\mathbf{e} \in \Omega$. On the control manifold, a time projection $t_{\mathscr{C}} : \mathscr{C} \mapsto \mathscr{Z}$ is then defined by the composition

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "\xi"] \arrow[dr, "t_{\mathscr{C}}"'] & & \mathscr{P}(\mathscr{E}) \arrow[dl, "t_{\mathscr{E}}"] \\ & \mathscr{Z} & \end{tikzcd} \quad \Longleftrightarrow \quad t_{\mathscr{C}} := t_{\mathscr{E}} \circ \xi \,. \tag{30}$$

In the manifold $\mathscr{C}$ the *control trajectory* is a 1D path $\Gamma \subset \mathscr{C}$ whose image by $\xi : \mathscr{C} \mapsto \mathscr{P}(\mathscr{E})$ is a trajectory of controllable placements.

A motion $\varphi_\alpha^{\mathscr{C}} : \Gamma \mapsto \Gamma$ along the control trajectory Γ generates a space-time motion along the corresponding a trajectory $\mathscr{T}_{\mathscr{E}} := \xi(\Gamma) \subset \mathscr{E}$ in the event manifold, as described by the commutative diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi_\alpha^{\mathscr{C}}"] \arrow[d, "\xi"] & \Gamma \arrow[d, "\xi"'] \\ \mathscr{P}(\mathscr{T}_{\mathscr{E}}) \arrow[r, "\varphi_\alpha"] \arrow[d, "t_{\mathscr{E}}"] & \mathscr{P}(\mathscr{T}_{\mathscr{E}}) \arrow[d, "t_{\mathscr{E}}"'] \\ \mathscr{Z} \arrow[r, "t_\alpha"] \arrow[uu, bend left=60, "\gamma"] & \mathscr{Z} \arrow[uu, bend right=60, "\gamma"'] \end{tikzcd} \quad \Longrightarrow \quad \begin{cases} t_{\mathscr{E}} \circ \xi \circ \gamma = \mathrm{ID}_{\mathscr{Z}} \,, \\ \varphi_\alpha \circ \xi := \xi \circ \varphi_\alpha^{\mathscr{C}} \,. \end{cases} \tag{31}$$

The injective immersion $\gamma : \mathscr{Z} \mapsto \mathscr{C}$, with $\gamma(\mathscr{Z}) = \Gamma \subset \mathscr{C}$, provides the time-parametrization of the control trajectory.

Defining the control velocity $\mathbf{V} := \partial_{\alpha=0}\, \varphi_\alpha^{\mathscr{C}}$, from Eqs. (30) and (31) we infer that the time-component of the space-time velocity in an adaptted frame is unitary:

$$\begin{aligned} \langle dt_{\mathscr{C}}, \mathbf{V} \rangle &= \langle \xi{\downarrow} dt_{\mathscr{E}}, \mathbf{V} \rangle = \xi{\downarrow} \langle dt_{\mathscr{E}}, \xi{\uparrow}\mathbf{V} \rangle \\ &= \langle dt_{\mathscr{E}}, \mathbf{v}_{\mathscr{E}} \rangle \circ \xi = 1 \circ t_{\mathscr{C}} \,. \end{aligned} \tag{32}$$

Definition 4.4 (Lagrange bundle). *The* LAGRANGE *bundle is the time-vertical subbundle of the tangent bundle $V\mathscr{C} \subset T\mathscr{C}$ over the control manifold, that is the collection of time-vertical subspaces of tangent linear spaces attached at all points of the control manifold.*

Definition 4.5 (Hamilton bundle). *The* HAMILTON *bundle $(V\mathscr{C})^*$ is the dual subbundle of the* LAGRANGE *bundle, identifiable with the quotient bundle*

$$(T\mathscr{C})^* / (V\mathscr{C})^\circ \,. \tag{33}$$

of the cotangent bundle $(T\mathscr{C})^*$ over the subbundle $(V\mathscr{C})^\circ$ of covectors vanishing on the LAGRANGE *bundle.*

The projection maps $\tau_{\mathscr{C}} : T\mathscr{C} \mapsto \mathscr{C}$ and $\tau^*_{\mathscr{C}} : (T\mathscr{C})^* \mapsto \mathscr{C}$ bring the information about base points of tangent and cotangent vectors.

Definition 4.6 (Pontryagin bundle). *The* PONTRYAGIN *bundle $V\mathscr{C} \times_{\mathscr{C}} (V\mathscr{C})^*$ is the* WHITNEY *product of the bundles $V\mathscr{C}$ and $(V\mathscr{C})^*$, that is the collection of pairs, of time-vertical tangent subspaces and their duals, attached at the same points in the control manifold.*

The bundles $V\mathscr{C}$, $(V\mathscr{C})^*$ and $V\mathscr{C} \times_{\mathscr{C}} (V\mathscr{C})^*$ are geometric pictures of the *phase spaces* of analytical dynamics.

Definition 4.7 (Tangent mapping). *To the injective representation map $\xi : \mathscr{C} \mapsto \mathscr{P}(\mathscr{E})$ there corresponds a tangent map $T\xi : T\mathscr{C} \mapsto T(\mathscr{P}(\mathscr{E}))$ which provides an injective correspondence between vectors $\mathbf{V} \in T\mathscr{C}$ tangent to the control manifold, and space-time tangent vector fields $\mathbf{v}_{\mathscr{E}} : \Omega \mapsto T\mathscr{E}$ based at a* placement submanifold $\Omega \subset \mathscr{E}$ *and tangent to the event manifold $\mathscr{E}$, as expressed by*

$$\mathbf{v}_{\mathscr{E}} = \xi{\uparrow}\mathbf{V} \stackrel{\text{def}}{\iff} \mathbf{v}_{\mathscr{E}} \circ \xi = T\xi \cdot \mathbf{V}\,. \tag{34}$$

To time-vertical control vectors $\mathbf{v} \in V\mathscr{C}$, fulfilling condition $\langle dt_{\mathscr{C}}, \mathbf{v} \rangle = 0$, there correspond space vector fields $\mathbf{v}_{\mathscr{S}} : \Omega \mapsto V\mathscr{E}$, fulfilling condition $\langle dt_{\mathscr{E}}, \mathbf{v}_{\mathscr{S}} \rangle = 0$, according to the relation

$$\boxed{\mathbf{v}_{\mathscr{S}} = \xi{\uparrow}\mathbf{v}\,.} \tag{35}$$

Definition 4.8 (Controllable fields). *Vector fields in the image of the tangent morphism $T\xi : V\mathscr{C} \mapsto T(\mathscr{P}(\mathscr{E}))$ are said to be* controllable.

Definition 4.9 (Adapted parallel transport). *A parallel transport $\Uparrow^{\mathscr{S}}$ in the space bundle $V\mathscr{E}$, such that the transport of a controllable field along a path of controllable placements is still a controllable field, is said to be* adapted *to the subbundle $T\xi(V\mathscr{C})$ and the induced parallel transport will be denoted by $\Uparrow$.*

Definition 4.10 (Connection in the control manifold). *A linear connection $\nabla^{\mathscr{S}}$ in the space bundle $V\mathscr{E}$, with adapted parallel transport $\Uparrow^{\mathscr{S}}$, induces a linear connection ∇, with parallel transport $\Uparrow$, in the control manifold $\mathscr{C}$, as described by the commutative diagram*

$$\begin{array}{ccc} V\mathscr{C} & \xrightarrow{\quad\Uparrow\quad} & V\mathscr{C} \\ {\scriptstyle T\xi}\big\downarrow & & \big\downarrow{\scriptstyle T\xi} \\ T\xi(V\mathscr{C}) & \xrightarrow{\quad\Uparrow^{\mathscr{S}}\quad} & T\xi(V\mathscr{C}) \end{array} \iff T\xi \circ \Uparrow = \Uparrow^{\mathscr{S}} \circ T\xi\,. \tag{36}$$

By fiberwise injectivity of the tangent map $T\xi : V\mathscr{C} \mapsto T(\mathscr{P}(\mathscr{E}))$, the parallel transport $\Uparrow$ is uniquely defined by the diagram Eq. (36) and the associated connection ∇ is well-defined by the relation

$$\xi{\uparrow}(\nabla_{\delta\mathbf{v}}\mathbf{v}) = \nabla^{\mathscr{S}}_{\xi\uparrow\delta\mathbf{v}}(\xi{\uparrow}\mathbf{v})\,. \tag{37}$$

It is readily verified that symmetry of the connection $\nabla^{\mathscr{S}}$ implies symmetry of the connection ∇.

Indeed, being $\xi{\uparrow}[\mathbf{v}, \delta\mathbf{v}] = [\xi{\uparrow}\mathbf{v}, (\xi{\uparrow}\delta\mathbf{v})]$ by push naturality of LIE-brackets Eq. (**??**), the torsion forms of the two connections are related by

$$\begin{aligned}\xi{\uparrow}(\mathbf{T}(\mathbf{v}, \delta\mathbf{v})) &= \xi{\uparrow}(\nabla_{\mathbf{v}}\delta\mathbf{v} - \nabla_{\delta\mathbf{v}}\mathbf{v} - [\mathbf{v}, \delta\mathbf{v}]) \\ &= \nabla^{\mathscr{S}}_{\xi\uparrow\mathbf{v}}(\xi{\uparrow}\delta\mathbf{v}) - \nabla^{\mathscr{S}}_{\xi\uparrow\delta\mathbf{v}}\xi{\uparrow}\mathbf{v} - [\xi{\uparrow}\mathbf{v}, (\xi{\uparrow}\delta\mathbf{v})] \\ &= \mathbf{T}^{\mathscr{S}}(\mathbf{v}_{\mathscr{S}}, \delta\mathbf{v}_{\mathscr{S}}))\,.\end{aligned} \tag{38}$$

The torsion operator $\mathbf{T}(\mathbf{v})$ is defined by the identity $\mathbf{T}(\mathbf{v}) \cdot \delta\mathbf{v} = \mathbf{T}(\mathbf{v}, \delta\mathbf{v})$.

Remark 4.1 (A noteworthy example). *The control manifold adopted in the implementation of the* F.E.M. *(Finite Element Method) in structural analysis is naturally endowed with a connection induced by the spatial connection in the* EUCLID *space. The adapted connection is defined by performing the parallel transport of the space vectors based at the nodes of the discretizing space mesh and then interpolating by the shape functions to get the transported space vector fields.*

5 DYNAMICS IN THE CONTROL MANIFOLD

As we have seen, the trajectory manifold $\mathscr{T}_{\mathscr{E}}$ is naturally sliced into a family of non intersecting body placements, transversal to material lines. Integration of a material volume form μ, over a compact trajectory segment $\mathscr{T}_{\mathscr{E}}$ corresponding to a compact time interval I, can thus be performed by a space-time split based on FUBINI's theorem. Setting

$$\mu_I(\alpha) := \int_{\varphi_\alpha(\Omega)} \mu \in \Lambda^1(I)\,, \qquad \mu_{\mathscr{C}} = \gamma{\uparrow}\mu_I \in \Lambda^1(\Omega)\,, \tag{39}$$

with γ trajectory path defined by Eq. (31), we have that

$$\int_{\mathscr{T}_{\mathscr{E}}} dt_{\mathscr{E}} \wedge \mu = \int_I d\alpha \int_{\varphi_\alpha(\Omega)} \mu = \int_I \mu_I(\alpha)\, d\alpha = \int_\Gamma \gamma{\uparrow}\mu_I = \int_\Gamma \mu_{\mathscr{C}}\,, \tag{40}$$

where Ω is a compact placement manifold.

Definition 5.1 (Lagrange functional). *In the* LAGRANGE *bundle* $(V\mathscr{C})_\Gamma$ *restricted to the control trajectory, the convex* LAGRANGE *functional* $L : (V\mathscr{C})_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})_\Gamma)$ *is defined in terms of the convex local function* $L_{\mathscr{E}} : (V\mathscr{E})_{\mathscr{T}_{\mathscr{E}}} \mapsto \mathrm{FUN}((V\mathscr{E})_{\mathscr{T}_{\mathscr{E}}})$ *in the space bundle* $(V\mathscr{E})_{\mathscr{T}_{\mathscr{E}}}$*, by setting*

$$L(\mathbf{v}) := \int_\Omega L_{\mathscr{E}}(\xi\!\uparrow\!\mathbf{v})\,\mathbf{m}\,. \tag{41}$$

The definition is well-posed since the mass $\mathbf{m}$ is a material property of any placement in the trajectory. Variational principles require however to evaluate the LAGRANGE functional outside the trajectory, on placement submanifolds generated by virtual flows as images of trajectory placements. This evaluation is performed by natural extension of mass, by push along virtual flows, see Remark 6.2.

Definition 5.2 (Kinetic energy). *The* Kinetic energy $K : (V\mathscr{C})_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})_\Gamma)$ *is the quadratic functional defined in the* LAGRANGE *bundle* $(V\mathscr{C})_\Gamma$ *by*

$$K(\mathbf{v}) := \int_\Omega K_{\mathscr{E}}(\xi\!\uparrow\!\mathbf{v})\,\mathbf{m}\,. \tag{42}$$

Definition 5.3 (Action functional). *The* Action *functional* $A : (V\mathscr{C})_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})_\Gamma)$ *is defined by*

$$A(\mathbf{v}) := \int_\Omega A_{\mathscr{E}}(\xi\!\uparrow\!\mathbf{v})\,\mathbf{m}\,. \tag{43}$$

Definition 5.4 (Energy functional). *The* Energy *functional* $E : (V\mathscr{C})_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})_\Gamma)$ *is defined, in accord with Eq. (27), by*

$$E(\mathbf{v}) := \int_\Omega E_{\mathscr{E}}(\xi\!\uparrow\!\mathbf{v})\,\mathbf{m} = A(\mathbf{v}) - L(\mathbf{v})\,. \tag{44}$$

Definition 5.5 (Duality pairing). *A duality pairing between space vector fields* $\delta\mathbf{v}_{\mathscr{S}} : \Omega \mapsto V\mathscr{E}$ *and covector fields* $\mathbf{p}_S : \Omega \mapsto (V\mathscr{E})^*$ *is given by*

$$\langle\, \mathbf{p}_S, \delta\mathbf{v}_{\mathscr{S}} \,\rangle_\Omega := \int_\Omega \langle\, \mathbf{p}_S, \delta\mathbf{v}_{\mathscr{S}} \,\rangle\,\mathbf{m}\,. \tag{45}$$

A direct evaluation shows that the fiber derivative $d_F L : (V\mathscr{C})_\Gamma \mapsto (V\mathscr{C})^*_\Gamma$ of the LAGRANGE functional $L : (V\mathscr{C})_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})_\Gamma)$, and its counterpart in the event manifold defined by Eq. (22), are related by

$$\boxed{\langle d_F L(\mathbf{v}), \delta\mathbf{v} \rangle = \int_\Omega \langle d_F L_{\mathscr{E}}(\xi\!\uparrow\!\mathbf{v}), \xi\!\uparrow\!\delta\mathbf{v} \rangle\,\mathbf{m}\,,} \tag{46}$$

which, in terms of the duality pairing Eq.(45), can be rewritten as

$$\begin{cases} \mathbf{p} = d_F L(\mathbf{v})\,, \\ \mathbf{v}_{\mathscr{S}} = \xi{\uparrow}\mathbf{v}\,, \\ \mathbf{p}_S = d_F L_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}}) \\ \delta\mathbf{v}_{\mathscr{S}} = \xi{\uparrow}\delta\mathbf{v}\,, \end{cases} \quad \Longrightarrow \quad \langle \mathbf{p}, \delta\mathbf{v} \rangle = \langle \mathbf{p}_S, \delta\mathbf{v}_{\mathscr{S}} \rangle_{\Omega}\,. \tag{47}$$

Proposition 5.1 (Euler-Legendre isomorphism in control manifold). *The positive definiteness of the fiber derivative* $d_F L_{\mathscr{E}} = \mathbf{g}_{\mathrm{SPA}} : (V\mathscr{E})_{\mathscr{T}_{\mathscr{E}}} \mapsto (V\mathscr{E})^*$ *implies positive definiteness of the fiber derivative* $d_F L : (V\mathscr{C})_\Gamma \mapsto (V\mathscr{C})^*_\Gamma$*, that is*

$$\langle d_F L_{\mathscr{E}}(\mathbf{v}_{\mathscr{S}}), \mathbf{v}_{\mathscr{S}} \rangle > 0\,, \quad \forall\, \mathbf{v}_{\mathscr{S}} \in V\mathscr{E} - \{\mathbf{0}\}\,, \tag{48}$$

implies that

$$\langle d_F L(\mathbf{v}), \mathbf{v} \rangle > 0\,, \quad \forall\, \mathbf{v} \in V\mathscr{C} - \{\mathbf{0}\}\,. \tag{49}$$

Proof. Positivity of the mass measure and Eq. (48) imply Eq. (49). Invertibility follows from injectivity, for finite dimensional control manifolds. ∎

The result in Prop. 5.1 assures that the EULER-LEGENDRE transform based on the LAGRANGE functional $L : (V\mathscr{C})_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})_\Gamma)$ results in a smooth one-to-one correspondence, between the LAGRANGE bundle $\tau_{\mathscr{C}} : (V\mathscr{C})_\Gamma \mapsto \mathscr{C}$ and the HAMILTON bundle $\tau^*_{\mathscr{C}} : (V\mathscr{C})^*_\Gamma \mapsto \mathscr{C}$.

Definition 5.6 (Hamilton functional). *The conjugate to the convex* LAGRANGE *functional* $L : (V\mathscr{C})_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})_\Gamma)$*, according to* EULER-LEGENDRE *transform, is the convex* HAMILTON *functional* $H : (V\mathscr{C})^*_\Gamma \mapsto \mathrm{FUN}((V\mathscr{C})^*_\Gamma)$ *defined by*

$$\begin{cases} \mathbf{p} = d_F L(\mathbf{v})\,, \\ H(\mathbf{p}) := \langle \mathbf{p}, \mathbf{v} \rangle - L(\mathbf{v})\,, \\ \mathbf{v} = d_F H(\mathbf{p})\,. \end{cases} \tag{50}$$

Let us now introduce the injective map $\mathbf{F} : (V\mathscr{C})^*_\Gamma \mapsto T^*(\mathscr{P}(\mathscr{E}))$ which, due to invertibility of the fiber derivative $d_F L : (V\mathscr{C})_\Gamma \mapsto (V\mathscr{C})^*_\Gamma$, is well-defined by the commutative diagram

$$\begin{array}{ccc} (V\mathscr{C})^*_\Gamma & \xrightarrow{\ \mathbf{F}\ } & T^*(\mathscr{P}(\mathscr{E})) \\ {\scriptstyle d_F L}\big\uparrow & & \big\uparrow{\scriptstyle d_F L_{\mathscr{E}}} \\ (V\mathscr{C})_\Gamma & \xrightarrow{\ \xi\uparrow\ } & T(\mathscr{P}(\mathscr{E})) \end{array} \quad \Longleftrightarrow \quad \mathbf{F} \circ d_F L = d_F L_{\mathscr{E}} \circ \xi{\uparrow}\,. \tag{51}$$

Proposition 5.2. *The* HAMILTON *functional* $H : (V\mathscr{C})^*_\Gamma \mapsto \text{FUN}((V\mathscr{C})^*_\Gamma)$*, defined according to the* EULER-LEGENDRE *transform Eq.* (50)*, is related to the local* HAMILTON *function* $H_\mathscr{E} : (V\mathscr{E})^*_{\mathscr{T}_\mathscr{E}} \mapsto \text{FUN}((V\mathscr{E})^*_{\mathscr{T}_\mathscr{E}})$ *by the integral*

$$H(\mathbf{p}) = \int_\Omega H_\mathscr{E}(\mathbf{F}(\mathbf{p}))\,\mathbf{m}\,. \tag{52}$$

Proof. Setting $\mathbf{p} = d_F L(\mathbf{v})$ in Eq. (52), and $\mathbf{v}_\mathscr{S} := \xi{\uparrow}\mathbf{v}$, being

$$\mathbf{F}(\mathbf{p}) = d_F L_\mathscr{E}(\mathbf{v}_\mathscr{S})\,, \tag{53}$$

from Eq. (47) and Eq. (51) we infer that

$$\langle\,\mathbf{p},\mathbf{v}\,\rangle = \langle\,\mathbf{F}(\mathbf{p}),\mathbf{v}_\mathscr{S}\,\rangle_\Omega\,. \tag{54}$$

Then the evaluation

$$\begin{aligned} H(\mathbf{p}) &= \int_\Omega H_\mathscr{E}(\mathbf{F}(\mathbf{p}))\,\mathbf{m} = \int_\Omega \Big(\langle\,\mathbf{F}(\mathbf{p}),\mathbf{v}_\mathscr{S}\,\rangle - L_\mathscr{E}(\mathbf{v}_\mathscr{S})\Big)\,\mathbf{m} \\ &= \langle\,\mathbf{p},\mathbf{v}\,\rangle - L(\mathbf{v})\,, \end{aligned} \tag{55}$$

yields Eq. (50). ∎

Definition 5.7 (Vector and covector fields). *We will denote by* $\mathscr{H}_\mathscr{C}$ *a space of smooth time-vertical vector fields tangent to the control manifold* $\mathscr{C}$ *and by* $\mathscr{H}^*_\mathscr{C}$ *the dual space.*

The correspondence in Eq. (50) induces a diffeomorphism $\theta_L : \mathscr{H}_\mathscr{C} \mapsto \mathscr{H}^*_\mathscr{C}$ between the corresponding manifold of sections, with inverse $\theta_H : \mathscr{H}^*_\mathscr{C} \mapsto \mathscr{H}_\mathscr{C}$.

The EULER-LEGENDRE transform between the fields in $\mathscr{H}_\mathscr{C}$ and in $\mathscr{H}^*_\mathscr{C}$ is then expressed by

$$\begin{cases} \mathbf{p} = \theta_L(\mathbf{v})\,, \\ H(\mathbf{p}) + L(\mathbf{v}) = \langle\,\mathbf{p},\mathbf{v}\,\rangle\,, \\ \mathbf{v} = \theta_H(\mathbf{p})\,. \end{cases} \tag{56}$$

The action and coaction functionals are given by

$$\begin{aligned} A(\mathbf{v}) &:= \langle\,\theta_L(\mathbf{v}),\mathbf{v}\,\rangle\,, \\ B(\mathbf{p}) &:= \langle\,\theta_H(\mathbf{p}),\mathbf{p}\,\rangle\,. \end{aligned} \tag{57}$$

To avoid overburden of notations we write $\theta_L(\mathbf{v}) = (d_F L)\circ\mathbf{v}$ also as $\theta_L\circ\mathbf{v}$, the meaning being clear from the context, and similarly for other fields.

Definition 5.8 (Kinetic momentum). *The* kinetic momentum *associated with a control velocity field* $\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$ *is the* 1*-form* $\theta_L(\mathbf{v}) \in \mathscr{H}_{\mathscr{C}}^*$.

Definition 5.9 (External force). [6] *The* 1*-form* $\mathbf{f}_{\text{EXT}}(\mathbf{b}, \mathbf{t}) \in \mathscr{H}_{\mathscr{C}}^*$*, describing the* external force *acting on a controllable body placement* Ω*, is the representation of the contributing surficial and volumetric space forces forms, according to the formula*

$$\langle \mathbf{f}_{\text{EXT}}(\mathbf{b}, \mathbf{t}), \delta\mathbf{v} \rangle := \int_{\Omega} \langle \mathbf{b}, \xi \uparrow \delta\mathbf{v} \rangle\, \mu + \int_{\partial\Omega} \langle \mathbf{t}, \xi \uparrow \delta\mathbf{v} \rangle\, \partial\mu\,. \tag{58}$$

Definition 5.10 (Internal force). *The* 1*-form* $\mathbf{f}_{\text{INT}}(\sigma) \in \mathscr{H}_{\mathscr{C}}^*$ *is the representation of the* internal force *acting on a controllable body placement* Ω*, by the formula*

$$\langle \mathbf{f}_{\text{INT}}(\sigma), \delta\mathbf{v} \rangle := \int_{\Omega} \langle \sigma, \varepsilon(\xi \uparrow \delta\mathbf{v}) \rangle\, \mathbf{m}\,. \tag{59}$$

Clearly $\langle \mathbf{f}_{\text{INT}}(\sigma), \delta\mathbf{v} \rangle = 0$ for all $\xi \uparrow \delta\mathbf{v} \in \mathrm{Ker}(\varepsilon)$.

Definition 5.11 (Dynamical force). *The* 1*-form* $\mathbf{f}_{\text{DYN}} \in \mathscr{H}_{\mathscr{C}}^*$*, is the representation of the* dynamical force *acting on a controllable body placement* Ω*. It is defined by the difference between external and internal forces, as expressed, at regular points, by the formula*

$$\mathbf{f}_{\text{DYN}} := \mathbf{f}_{\text{EXT}}(\mathbf{b}, \mathbf{t}) - \mathbf{f}_{\text{INT}}(\sigma)\,. \tag{60}$$

Impulsive forces at singular points, collectively denoted by Γ_{SING}*, are represented by* 1*-forms* $\mathbf{f}_{\text{SING}} \in \mathscr{H}_{\mathscr{C}}^*$.

6 HAMILTON PRINCIPLE

Let us now provide a geometrical formulation of classical action principles of dynamics. In so doing, we drop the standard, but needlessly restrictive, assumption that variations of a trajectory segment must leave the end points fixed. By this more general approach jump conditions at singular interfaces are directly provided by the variational condition. Moreover, a satisfactory formulation is thus given from the epistemological viewpoint, as explicated in Remark 6.3. On the other hand, we add the explicit statement about the way extensions of the involved geometrical objects are performed along virtual flows. Usual treatments are substantially silent in this respect.

[6] Force systems are often ignored in variational treatments of dynamics and sometimes improperly defined as morphisms $\mathbf{f}_{\text{EXT}} : V\mathscr{C} \mapsto (V\mathscr{C})^*$ from the LAGRANGE to the HAMILTON bundle, see e.g. (Terra and Kobayashi, 2004a; Yoshimura and Marsden, 2006), a definition that violates GALILEI principle of relativity.

Definition 6.1 (Synchronous virtual variations). *Synchronous virtual variations are performed when the virtual flow* $\delta\varphi_\lambda^{\mathscr{C}} : \Gamma \mapsto \mathscr{C}$ *preserves the time projection* $t_{\mathscr{C}} : \mathscr{C} \mapsto \mathscr{Z}$, *that is*

$$\boxed{t_{\mathscr{C}} = \delta\varphi_\lambda^{\mathscr{C}}\!\downarrow\! t_{\mathscr{C}} = t_{\mathscr{C}} \circ \delta\varphi_\lambda^{\mathscr{C}}\,,} \tag{61}$$

as depicted in frame (70)*. Virtual velocities are then time vertical, since*

$$\langle dt_{\mathscr{C}}, \delta\mathbf{v} \rangle = \langle dt_{\mathscr{C}}, \partial_{\lambda=0}\, \delta\varphi_\lambda^{\mathscr{C}} \rangle = \partial_{\lambda=0}\, (t_{\mathscr{C}} \circ \delta\varphi_\lambda^{\mathscr{C}}) = 0\,. \tag{62}$$

Consequently, due to commutativity, time differentials fulfil the conditions

$$dt_{\mathscr{C}} = d(\delta\varphi_\lambda^{\mathscr{C}}\!\downarrow\! t_{\mathscr{C}}) = \delta\varphi_\lambda^{\mathscr{C}}\!\downarrow\! dt_{\mathscr{C}} \iff \mathscr{L}_{\delta\mathbf{v}} dt_{\mathscr{C}} = d(\mathscr{L}_{\delta\mathbf{v}} t_{\mathscr{C}}) = \mathbf{0}\,, \tag{63}$$

so that

$$\langle dt_{\mathscr{C}}, \delta\varphi_\lambda^{\mathscr{C}}\!\uparrow\!\mathbf{V} \rangle = \langle \delta\varphi_\lambda^{\mathscr{C}}\!\uparrow\! dt_{\mathscr{C}}, \delta\varphi_\lambda^{\mathscr{C}}\!\uparrow\!\mathbf{V} \rangle = \delta\varphi_\lambda^{\mathscr{C}}\!\uparrow\!\langle dt_{\mathscr{C}}, \mathbf{V} \rangle = 1 \circ \delta\varphi_\lambda^{\mathscr{C}}\,. \tag{64}$$

Remark 6.1 (Natural extension of the velocity field). *The space-time velocity field* $\mathbf{V} : \Gamma \mapsto T\mathscr{C}$ *along the control trajectory is extended, in a natural way, along a virtual flow by considering the velocity of the pushed motion*

$$\partial_{\alpha=0}\, (\delta\varphi_\lambda^{\mathscr{C}} \circ \varphi_\alpha^{\mathscr{C}}) = T\delta\varphi_\lambda^{\mathscr{C}} \cdot \mathbf{V} = (\delta\varphi_\lambda^{\mathscr{C}}\!\uparrow\!\mathbf{V}) \circ \delta\varphi_\lambda^{\mathscr{C}}\,. \tag{65}$$

The extended velocity is thus pushed along the virtual flow

$$\boxed{\mathbf{V} = \delta\varphi_\lambda^{\mathscr{C}}\!\uparrow\!\mathbf{V} \overset{\text{def}}{\iff} \mathbf{V} \circ \delta\varphi_\lambda^{\mathscr{C}} = T\delta\varphi_\lambda^{\mathscr{C}} \cdot \mathbf{V}\,,} \tag{66}$$

so that $[\mathbf{V}, \delta\mathbf{v}] = \mathscr{L}_{\delta\mathbf{v}}\mathbf{V} = \mathbf{0}$.

Remark 6.2 (Natural extension by virtual mass conservation). *In evaluating the* LAGRANGE *functional by Eq.* (41) *over body a placement transformed by a virtual flow, the mass* $\mathbf{m}$*, which is defined only on the trajectory, is also assumed to be extended, in a natural way, by push along the virtual flow. This procedure has the physical interpretation of virtual mass conservation and is tacitly assumed in analytical dynamics.*

Proposition 6.1 (Hamilton action principle). *The motion along the trajectory* Γ *in the control manifold* $\mathscr{C}$*, is characterized by the variational condition*[7]

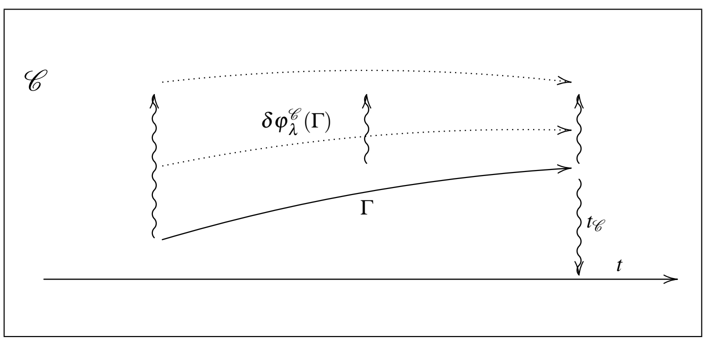
$$\boxed{\begin{aligned} \partial_{\lambda=0} \int_{\delta\varphi^{\mathscr{C}}_{\lambda}(\Gamma)} L(\mathbf{v})\, dt_{\mathscr{C}} - \oint_{\partial\Gamma} \langle\, \theta_L(\mathbf{v}), \delta\mathbf{v} \rangle \\ = \int_{\Gamma} (dt_{\mathscr{C}} \wedge \mathbf{f}_{\mathrm{DYN}}) \cdot \delta\mathbf{v} - \oint_{\Gamma_{\mathrm{SING}}} \langle \mathbf{f}_{\mathrm{SING}}, \delta\mathbf{v} \rangle\,, \end{aligned}} \tag{67}$$

for all synchronous virtual flows with velocity $\delta\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$*. At regular points, the variational condition Eq.* (67)*, expressed in terms of a linear connection* ∇*, is equivalent to* EULER-LAGRANGE-POINCARÉ *differential equation*

$$\boxed{\langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}) - \nabla L(\mathbf{v}) + \theta_L(\mathbf{v}) \cdot \mathbf{T}(\mathbf{V}), \delta\mathbf{v} \rangle = \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \rangle\,,} \tag{68}$$

which is tensorial in the synchronous virtual velocity $\delta\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$*. At singular points the motion is governed by the jump conditions*

$$[[\langle\, \theta_L(\mathbf{v}), \delta\mathbf{v} \rangle]] = \langle \mathbf{f}_{\mathrm{SING}}, \delta\mathbf{v} \rangle\,. \tag{69}$$

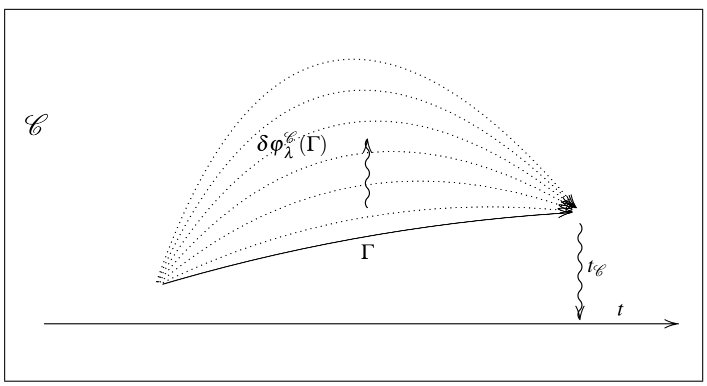


(70)

Proof. Equivalence between Eqs. (67) and (68) will be proven in Prop. 7.1. ■

[7] As reported in (Arnold et al., 1988, 2.1, p.10), FELIX KLEIN (1926) observed that: *It is astonishing that in Lagrange's work this statement may be read only between the lines. This explains the strange situation that this relation - mainly through Jacobi's influence - is generally known in Germany, and therefore also in France, as Hamilton's principle. In England no one understands this expression; there this equation is known under the correct if intuitive name of principle of stationary action.*

Remark 6.3. *The boundary integral appearing in the expression of the action principle Eq.* (67) *is usually eliminated by imposing that virtual velocities must vanish at trajectory endpoints, see e.g. (Abraham and Marsden, 1988, Sect. 3.8). This needless constraint on test fields, depicted in frame* (71)*, has however unpleasant consequences and is therefore advisable that it be eliminated. A trouble concerns the qualification of the action principle as characteristic property of the dynamical trajectory. In this respect, a natural requirement is that two subsequent trajectory segments should be chained into a resultant trajectory segment. This chain property is not fulfilled by the constrained formulation, while it is clearly met by the unconstrained formulation. Another significant advantage is that jump conditions at singular points are directly provided by the action principle. The same procedure, applied to* FERMAT *principle of least time in optics, leads the differential equation of optical geodesic and to* SNELL *interface jump conditions (Romano G., 2007).*



(71)

Remark 6.4 (Coordinates and natural frames). *In a finite dimensional control manifold, with* $\dim \mathscr{C} = m+1$*, let us consider a coordinate system in the slices of isochronous placements*

$$\phi = \{ q_i \,, i = 1, \dots m \} : \Re^m \mapsto \mathscr{C} \,, \tag{72}$$

Denoting by $\mathbf{a}_i$ *and* $\mathbf{a}^k$*, with* $i,k = 1, \dots m$*, the usual frame and the dual coframe in* $\Re^m$*, the natural frame (*repère naturel*) and coframe associated with the coordinate system are given by*

$$\begin{cases} \partial_i := \phi\!\uparrow\!\mathbf{a}_i \,, \\ dq^k := \phi\!\uparrow\!\mathbf{a}^k \,, \\ \langle dq^k, \partial_i \rangle = \phi\!\uparrow\!\langle \mathbf{a}^k, \mathbf{a}_i \rangle = \delta^k_{.i} \,. \end{cases} \tag{73}$$

Remark 6.5. *The differential law of motion Eq.* (68) *may be expressed in coordinates and in terms of the connection induced by the natural frame associated with the coordinate system. The relevant torsion form vanishes identically. Indeed, being* $\nabla_{\partial_i}\partial_j = \mathbf{0}$ *by assumption and* $[\partial_i, \partial_j] = \mathbf{0}$ *by commutativity of coordinate flows, we have that*

$$\mathbf{T}(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i - [\partial_i, \partial_j] = \mathbf{0}\,. \tag{74}$$

The EULER-LAGRANGE *law then writes*

$$\langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}) - \nabla L(\mathbf{v}), \delta\mathbf{v} \rangle = \langle \mathbf{f}_{\text{DYN}}, \delta\mathbf{v} \rangle\,, \tag{75}$$

which in coordinates takes the familiar expression

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) - \frac{dL}{dq} = Q\,, \tag{76}$$

where $\dot{q}$, $dL/d\dot{q}$, dL/dq *and* Q *are the numerical vectors of components respectively of* $\mathbf{v}$, $d_F L(\mathbf{v}) = \theta_L \circ \mathbf{v}$, $\nabla L(\mathbf{v})$ *and* $\mathbf{f}_{\text{DYN}}$.

Remark 6.6. *A special case of the differential law of motion Eq.* (68) *is described in coordinates in (Arnold et al., 1988, 2.4, p. 13), reproducing the original treatment in (Poincaré, 1901). There, the term* quasi-velocities *is adopted for the components of the space velocity in a* repère mobile *(Cartan É., 1937)* $\{\mathbf{d}_1, \ldots, \mathbf{d}_m\}$, *that is a set of* m *smooth fields which form a basis at each point. If the parallel transport of a vector is defined by the property that its components in the* repère mobile *are constant, then the parallel derivatives of the basis vector fields, according to the path independent induced connection* ∇, *vanish identically. By tensoriality of the torsion form* $\mathbf{T}(\mathbf{v}, \delta\mathbf{v})$ *the arguments can be extended by parallel transport. Consequently the torsion form can be computed at each point as coincident with the negative of the* LIE*-bracket of the vector fields generated by the extension. The components* $c_{ij}^{\cdot\cdot k}$ *of the brackets of basis vectors in the* repère mobile, *are the* structure constants *defined by*

$$[\mathbf{d}_i, \mathbf{d}_j] = c_{ij}^{\cdot\cdot k}\, \mathbf{d}_k\,. \tag{77}$$

The POINCARÉ *law takes then the expression*

$$\langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}) - \nabla L(\mathbf{v}), \delta\mathbf{v} \rangle + \theta_L(\mathbf{v}) \cdot [\mathbf{v}, \delta\mathbf{v}] = \langle \mathbf{f}_{\text{DYN}}, \delta\mathbf{v} \rangle\,, \tag{78}$$

and in coordinates

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right)_k - \left(\frac{dL}{dq}\right)_k + c_{ik}^{\cdot\cdot j}\, \dot{q}^i \left(\frac{dL}{d\dot{q}}\right)_j = Q_k\,. \tag{79}$$

In a repère naturel *all structure constants vanish and the law of motion in the control manifold takes the standard form in coordinates Eqs.* (75)*,* (76)*, as given in (Lagrange, 1788).*

Singular forces $\mathbf{f}_{\text{SING}}$ and jump conditions will be neglected in the sequel, to simplify the presentation. In classical mechanics a direct consequence of EULER-LAGRANGE-POINCARÉ condition, Eq. (68) in Prop. 6.1, is known as EMMY NOETHER theorem, see (Noether, 1918; Arnold, 1974, Sect. 20, p. 88).

Corollary 6.1 (E. Noether). *For synchronous virtual flows with* $\delta\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$*, the following implication holds*

$$\boxed{\begin{aligned}&\nabla_{\delta\mathbf{v}} L(\mathbf{v}) = \mathbf{0} \implies \\ &\langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}) - \nabla L(\mathbf{v}) + \theta_L(\mathbf{v}) \cdot \mathbf{T}(\mathbf{V}), \delta\mathbf{v} \rangle = \langle \mathbf{f}_{\text{DYN}}, \delta\mathbf{v} \rangle .\end{aligned}} \tag{80}$$

Statements in literature refer to the special case in which the connection is symmetric ($\mathbf{T} = \mathbf{0}$) *and dynamical forces vanish* ($\mathbf{f}_{\text{DYN}} = \mathbf{0}$)*, so that* NOETHER *theorem writes*

$$\boxed{\nabla_{\delta\mathbf{v}} L(\mathbf{v}) = 0 \implies \langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}), \delta\mathbf{v} \rangle = 0 .} \tag{81}$$

Proposition 6.2 (Balance of mechanical power). *Under the assumption that the* LAGRANGE *functional is time independent, that is*

$$\nabla_{\mathbf{Z}} L(\mathbf{v}) = 0 \iff \nabla_{\mathbf{V}} L(\mathbf{v}) = \nabla_{\mathbf{v}} L(\mathbf{v}) , \tag{82}$$

the motion fulfills the mechanical power balance

$$\boxed{\nabla_{\mathbf{V}}(E \circ \mathbf{v}) = \langle \mathbf{f}_{\text{DYN}}, \mathbf{v} \rangle ,} \tag{83}$$

stating that the time rate of the energy functional along the motion is equal to the power expended by dynamical forces.

Proof. Setting $\delta\mathbf{v} = \mathbf{v}$ in Eq. (68) and noting that $\mathbf{T}(\mathbf{V}, \mathbf{v}) = \mathbf{T}(\mathbf{v}, \mathbf{v}) = \mathbf{0}$, the proof follows by the relations

$$\begin{aligned}&\langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}), \mathbf{v} \rangle - \nabla_{\mathbf{v}} L(\mathbf{v}) \\ &= \langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}), \mathbf{v} \rangle - \nabla_{\mathbf{V}} L(\mathbf{v}) + \nabla_{\mathbf{Z}} L(\mathbf{v}) \\ &= \nabla_{\mathbf{V}} \langle \theta_L \circ \mathbf{v}, \mathbf{v} \rangle - \langle \theta_L \circ \mathbf{v}, \nabla_{\mathbf{V}} \mathbf{v} \rangle - \nabla_{\mathbf{V}} L(\mathbf{v}) + \nabla_{\mathbf{Z}} L(\mathbf{v}) \\ &= \nabla_{\mathbf{V}} \langle \theta_L \circ \mathbf{v}, \mathbf{v} \rangle - \nabla_{\mathbf{V}}(L \circ \mathbf{v}) + \nabla_{\mathbf{Z}} L(\mathbf{v}) \\ &= \nabla_{\mathbf{V}}(E \circ \mathbf{v}) + \nabla_{\mathbf{Z}} L(\mathbf{v}) = \langle \mathbf{f}_{\text{DYN}}, \mathbf{v} \rangle ,\end{aligned} \tag{84}$$

where splitting Eq. (A.5) and definition Eq. (44) have been resorted to. ■

7 LAGRANGE PRINCIPLE

The formal structure of an action principle, as defined by Eq. (2), is not respected in HAMILTON principle Eq. (67), since a pair of 1-forms on $\mathscr{C}$ are there involved:

- the 1-form $(L \circ \mathbf{v})\, dt_{\mathscr{C}}$ in the integral over the trajectory,
- the 1-form $\theta_L \circ \mathbf{v}$ in the integral over the trajectory boundary.

According to the nomenclature in Rem. 2.1, HAMILTON principle is then an *hybrid* action principle. An equivalent *faithful* action principle will however be enunciated in Prop. 7.1 by introducing of the following item.[8]

Definition 7.1 (Lagrange action form). *The* LAGRANGE *action* 1*-form on the trajectory manifold* $\omega_L^1 \circ \mathbf{v} : \Gamma \mapsto (T\mathscr{C})^*_\Gamma$, *is induced by the map* $\omega_L^1 : (T\mathscr{C})_\Gamma \mapsto (T\mathscr{C})^*_\Gamma$ *defined by*

$$\boxed{\omega_L^1(\mathbf{v}) := \theta_L(\mathbf{v}) - E(\mathbf{v})\, dt_{\mathscr{C}}\,.} \tag{85}$$

Lemma 7.1 (Equality between path integrals). *On a time parameterised path* Γ *we have the equalities*

$$\int_\Gamma L(\mathbf{v})\, dt_{\mathscr{C}} = \int_\Gamma \omega_L^1(\mathbf{v})\,. \tag{86}$$

$$\oint_{\partial\Gamma} \langle\, \theta_L(\mathbf{v}), \delta\mathbf{v} \,\rangle = \oint_{\partial\Gamma} \langle\, \omega_L^1(\mathbf{v}), \delta\mathbf{v} \,\rangle\,. \tag{87}$$

Moreover, by synchronicity of virtual flows, Eq. (86) *holds also on dragged paths*

$$\int_{\delta\varphi_\lambda^{\mathscr{C}}(\Gamma)} L(\mathbf{v})\, dt_{\mathscr{C}} = \int_{\delta\varphi_\lambda^{\mathscr{C}}(\Gamma)} \omega_L^1(\mathbf{v})\,. \tag{88}$$

Proof. By EULER-LEGENDRE transform, the evaluation $\langle\, dt_{\mathscr{C}}, \mathbf{V} \,\rangle = 1$ gives

$$\langle\, \omega_L^1(\mathbf{v}), \mathbf{V} \,\rangle = \langle\, \theta_L(\mathbf{v}), \mathbf{V} \,\rangle - (E(\mathbf{v}))\langle\, dt_{\mathscr{C}}, \mathbf{V} \,\rangle = L(\mathbf{v}) = \langle\, L(\mathbf{v})\, dt_{\mathscr{C}}, \mathbf{V} \,\rangle\,, \tag{89}$$

and the evaluation $\langle\, dt_{\mathscr{C}}, \delta\mathbf{v} \,\rangle = 0$ yields

$$\langle\, \omega_L^1(\mathbf{v}), \delta\mathbf{v} \,\rangle = \langle\, \theta_L(\mathbf{v}), \delta\mathbf{v} \,\rangle - (E(\mathbf{v}))\langle\, dt_{\mathscr{C}}, \delta\mathbf{v} \,\rangle = \langle\, \theta_L(\mathbf{v}), \delta\mathbf{v} \,\rangle\,. \tag{90}$$

This proves Eq. (86) and Eq. (87). Eq. (88) holds since $\langle\, dt_{\mathscr{C}}, \delta\varphi_\lambda^{\mathscr{C}}\!\uparrow\!\mathbf{V} \,\rangle = 1$ due to synchronicity Eq. (64). ■

[8] In (Arnold, 1974)) is said: *the form* θ_L *seems here to appear out of thin air. In the following paragraph we will see how the idea of using this form arose form optics*. A formal motivation is suggested by Lemma 7.1.

Proposition 7.1 (Lagrange action principle). *The motion along the trajectory Γ in the control manifold $\mathscr{C}$, is characterized by the extremality condition*

$$\boxed{\partial_{\lambda=0} \int_{\delta\varphi_\lambda^{\mathscr{C}}(\Gamma)} \omega_L^1(\mathbf{v}) - \oint_{\partial\Gamma} \langle \omega_L^1(\mathbf{v}), \delta\mathbf{v} \rangle = \int_\Gamma (dt_{\mathscr{C}} \wedge \mathbf{f}_{\mathrm{DYN}}) \cdot \delta\mathbf{v} \,,} \tag{91}$$

for all synchronous virtual flows with velocity $\delta\mathbf{v} = \partial_{\lambda=0}\, \delta\varphi_\lambda^{\mathscr{C}} \in \mathscr{H}_{\mathscr{C}}$. The corresponding EULER *differential condition writes*

$$\boxed{\langle d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} + d(E \circ \mathbf{v}), \delta\mathbf{v} \rangle = \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \rangle \,,} \tag{92}$$

which, in terms of a linear connection ∇, is expressed by

$$\boxed{\langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}) - \nabla L(\mathbf{v}) + \theta_L(\mathbf{v}) \cdot \mathbf{T}(\mathbf{V}), \delta\mathbf{v} \rangle = \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \rangle \,.} \tag{93}$$

The whole expression at the l.h.s. of LAGRANGE *law of dynamics Eq. (93) is independent of the choice of a linear connection.*

Proof. Equivalence between action principles in Eq. (91) and Eq. (67) follows directly from the equalities Eq. (88) and (87). The proof of Eq. (93) is carried out as follows. The extrusion formula Eq. (8), applied to Eq. (91), gives

$$\int_\Gamma d(\omega_L^1 \circ \mathbf{v}) \cdot \delta\mathbf{v} = \int_\Gamma (dt_{\mathscr{C}} \wedge \mathbf{f}_{\mathrm{DYN}}) \cdot \delta\mathbf{v} \,. \tag{94}$$

and hence, by Eq. (96), the EULER differential condition

$$d(\omega_L^1 \circ \mathbf{v}) \cdot \delta\mathbf{v} \cdot \mathbf{V} = d\Big((\theta_L \circ \mathbf{v}) - (E \circ \mathbf{v})\, dt_{\mathscr{C}}\Big) \cdot \delta\mathbf{v} \cdot \mathbf{V} = -\langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \rangle \,, \tag{95}$$

Being $\langle dt_{\mathscr{C}}, \mathbf{V} \rangle = 1$ and $\langle dt_{\mathscr{C}}, \delta\mathbf{v} \rangle = 0$, we have that

$$\begin{aligned} d\Big((E \circ \mathbf{v})\, dt_{\mathscr{C}}\Big) \cdot \delta\mathbf{v} \cdot \mathbf{V} &= \Big(d(E \circ \mathbf{v}) \wedge dt_{\mathscr{C}}\Big) \cdot \delta\mathbf{v} \cdot \mathbf{V} = d(E \circ \mathbf{v}) \cdot \delta\mathbf{v} \,, \\ (dt_{\mathscr{C}} \wedge \mathbf{f}_{\mathrm{DYN}}) \cdot \delta\mathbf{v} \cdot \mathbf{V} &= (dt_{\mathscr{C}} \cdot \delta\mathbf{v})(\mathbf{f}_{\mathrm{DYN}} \cdot \mathbf{V}) - (dt_{\mathscr{C}} \cdot \mathbf{V})(\mathbf{f}_{\mathrm{DYN}} \cdot \delta\mathbf{v}) \\ &= -\langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \rangle \,. \end{aligned} \tag{96}$$

Moreover, from Lemma A.1 and the formula in Eq. (A.1), being

$$\begin{aligned} \langle \theta_L(\mathbf{v}), \mathbf{V} \rangle &= \langle \theta_L(\mathbf{v}), \mathbf{v} \rangle \,, \\ \langle \theta_L(\mathbf{v}), \nabla_{\delta\mathbf{v}} \mathbf{V} \rangle &= \langle \theta_L(\mathbf{v}), \nabla_{\delta\mathbf{v}} \mathbf{v} \rangle \,, \end{aligned} \tag{97}$$

we get

$$
\begin{cases}
d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} \cdot \delta\mathbf{v} = 2\,\text{skew}\,(\nabla(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} \cdot \delta\mathbf{v}) \\
\qquad\qquad\qquad\qquad + \langle\, \theta_L(\mathbf{v}), \mathbf{T}(\mathbf{V}\,, \delta\mathbf{v})\,\rangle \\
\qquad\qquad\qquad\qquad = \langle \nabla_{\mathbf{V}}(\theta_L \circ \mathbf{v}), \delta\mathbf{v} \rangle - \langle \nabla_{\delta\mathbf{v}}(\theta_L \circ \mathbf{v}), \mathbf{V} \rangle \\
\qquad\qquad\qquad\qquad + \langle\, \theta_L(\mathbf{v}), \mathbf{T}(\mathbf{V}\,, \delta\mathbf{v})\,\rangle\,, \\
\langle \nabla_{\delta\mathbf{v}}(\theta_L \circ \mathbf{v}), \mathbf{V} \rangle = \nabla_{\delta\mathbf{v}} \langle\, \theta_L \circ \mathbf{v}, \mathbf{V} \rangle - \langle\, \theta_L(\mathbf{v}), \nabla_{\delta\mathbf{v}}\mathbf{V} \rangle \\
\qquad\qquad\qquad\qquad = \nabla_{\delta\mathbf{v}}(A \circ \mathbf{v}) - \langle\, \theta_L(\mathbf{v}), \nabla_{\delta\mathbf{v}}\mathbf{V} \rangle\,, \\
\nabla_{\delta\mathbf{v}}(A \circ \mathbf{v}) = \nabla_{\delta\mathbf{v}} A(\mathbf{v}) + \langle d_F A(\mathbf{v}), \nabla_{\delta\mathbf{v}}\mathbf{v} \rangle\,, \\
d(E \circ \mathbf{v}) \cdot \delta\mathbf{v} = \nabla_{\delta\mathbf{v}}(E \circ \mathbf{v}) = \nabla_{\delta\mathbf{v}} E(\mathbf{v}) + \langle d_F E(\mathbf{v}), \nabla_{\delta\mathbf{v}}\mathbf{v} \rangle\,.
\end{cases}
\tag{98}
$$

The EULER-LEGENDRE transform gives

$$
\theta_L(\mathbf{v}) = d_F A(\mathbf{v}) - d_F E(\mathbf{v})\,, \qquad \nabla_{\delta\mathbf{v}} A - \nabla_{\delta\mathbf{v}} E = \nabla_{\delta\mathbf{v}} L\,, \tag{99}
$$

and Eq. (93) follows. ■

8 MAUPERTUIS PRINCIPLE

Attribution of the least action principle to MAUPERTUIS was at the centre of an ugly dispute with KÖNIG, who sustained that the principle was first enunciated by LEIBNIZ in a letter to HERMANN in 1707, some 37 years before. The original of the letter was however never found and its existence was even questioned by supporters of MAUPERTUIS.

The principle enunciated in (Maupertuis, 1744) was also formulated in more precise terms and in the same year in (Euler, 1744). An undiscussed parental attribution is still lacking. Therefore the principle could also be properly named LEIBNIZ-EULER-MAUPERTUIS least action principle, although the names of LAGRANGE and JACOBI should be added with full credit to the list. Following the tradition we will simply refer to it as the MAUPERTUIS least action principle.

The least action principle is also referred to in literature as stationarity principle for the *reduced action*, to underline that the 1-form $\theta_L \circ \mathbf{v}$ is got from the 1-form $\omega^1_L \circ \mathbf{v} := \theta_L \circ \mathbf{v} - (E \circ \mathbf{v})\, dt_{\mathscr{C}}$ appearing in LAGRANGE action principle Prop. 7.1, by dropping the term $(E \circ \mathbf{v})\, dt_{\mathscr{C}}$ involving the energy (Arnold, 1974).

A proper formulation of the principle has always been reported as a challenging task in literature.

Difficulties in providing a clear statement were reported in (Jacobi, 1837a,b, 1884) and thence repeated in literature till recently, see (Goldstein, 1950; Godbillon, 1971; Arnold, 1974; Landau and Lifšits, 1976; Abraham and Marsden, 1988). In (Arnold, 1974, p. 246), it is said:

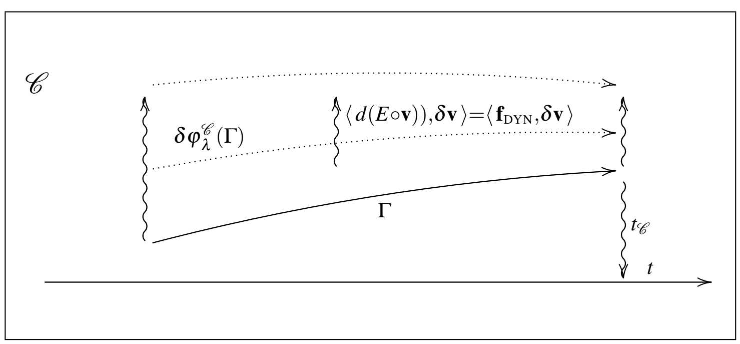
*"In almost all textbooks, even the best, this principle is presented so that it is impossible to understand" (*C. JACOBI*, Lectures on Dynamics). I do not choose to break with tradition. A very interesting "proof" of* MAUPERTUIS *principle is in Section 44 of the mechanics textbook of* LANDAU *and* LIFŠITS *(Mechanics, Oxford, Pergamon 1960).*

In (Abraham and Marsden, 1988, footnote on p. 249), it is written: *We thank* M. SPIVAK *for helping us to formulate this theorem correctly. The authors, like many others (we were happy to learn), were confused by the standard textbook statements. For instance the misterious variation "Δ" in* GOLDSTEIN *[1950, p. 228] corresponds to our enlargement of the variables by $c \to (\tau, c)$.*

We provide here a quite general formulation in which the power balance constraint is imposed only on test fields, as shown in the diagram Eq. (100).

Standard statements in literature are affected by an misformulation which results in the *variational crime*[9] described below:

- The natural condition, concerning balance of mechanical power along the motion, is imposed as an essential condition on admissible paths and on virtual velocities. On the contrary, balance of virtual mechanical power is an essential condition to be imposed on virtual velocities, while balance of mechanical power along the motion is a natural outcome of the variational principle and holds only under the special assumption of time independence of the LAGRANGE functional.



(100)

[9] We imitate here the title of an interesting chapter in a nice book on the finite element method (Strang and Fix, 1988).

Generality is achieved by the new formulation in Prop. 8.1 under two respects:

1. The end point of the trajectory segment are left free to vary, according to the assumed virtual flow in the control manifold.
2. The original constraint of energy conservation, in which both motions and virtual flows are imposed to evolve while leaving the energy functional constant, is replaced by the sole constraint that virtual velocities must fulfils balance of virtual power, with no constraints concerning the motion.

The new form of MAUPERTUIS least action principle, enunciated in Prop. 8.1 and sketched in diagram Eq. (100), is named MAUPERTUIS extremality principle in agreement with Def. 2.3, and is shown to be equivalent to HAMILTON action principle of Prop. 6.1, by performing a comparison of the ensuing EULER differential conditions.

Proposition 8.1 (Maupertuis extremality principle). *The trajectory* Γ *is characterized by an extremality condition for the path integral of the kinetic momentum*

$$\boxed{\partial_{\lambda=0} \int_{\delta\varphi_\lambda^{\mathscr{C}}(\Gamma)} \theta_L(\mathbf{v}) = \oint_{\partial\Gamma} \langle \theta_L(\mathbf{v}), \delta\mathbf{v} \rangle \,,} \tag{101}$$

in the class of synchronous virtual flows whose velocity $\delta\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$ *fulfils the constraint of virtual power balance*

$$\boxed{\langle d(E \circ \mathbf{v}), \delta\mathbf{v} \rangle = \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \rangle \,.} \tag{102}$$

By extrusion formula Eq. (8), the variational condition Eq. (101) is equivalent to the differential condition

$$\boxed{d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} \cdot \delta\mathbf{v} = 0 \,,} \tag{103}$$

for all time-vertical virtual velocity fields $\delta\mathbf{v} : \Gamma \mapsto V\mathscr{C}$ *fulfilling the virtual power balance Eq. (102).*

Proof. For notational convenience, we introduce the covector $\mathbf{f}_{\mathrm{EQ}} : (T\mathscr{C})_\Gamma \mapsto \Re$ defined by $\mathbf{f}_{\mathrm{EQ}} := \mathbf{f}_{\mathrm{DYN}} - d(E \circ \mathbf{v})$, with dual $\mathbf{f}^*_{\mathrm{EQ}} : \Re \mapsto (T\mathscr{C})^*_\Gamma$. The differential law Eq. (103), with the linear constraint Eq. (102) on the virtual velocities, may then be expressed by the conditions

$$\begin{aligned} & d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} \in (\mathrm{Ker}\, \mathbf{f}_{\mathrm{EQ}})^{\circ} \\ \Longleftrightarrow\ & d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} \in \mathrm{Im}\, \mathbf{f}^{*}_{\mathrm{EQ}} \\ \Longleftrightarrow\ & d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} = \mathbf{f}^{*}_{\mathrm{EQ}}(\lambda) \\ \Longleftrightarrow\ & d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} \cdot \delta\mathbf{v} = \langle \mathbf{f}^{*}_{\mathrm{EQ}}(\lambda), \delta\mathbf{v} \rangle = \lambda \, \langle \mathbf{f}_{\mathrm{EQ}}, \delta\mathbf{v} \rangle \,. \end{aligned} \tag{104}$$

For a given space velocity $\mathbf{v} \in V\mathscr{E}$, the solutions of Eq. (103) describe a 1D manifold with the space-time velocity parameterized by

$$\mathbf{V} = \mathbf{v} + (p/p_0)\,\mathbf{Z}\,, \tag{105}$$

with p_0 normalizing value to be fixed so that the time component of the space-time velocity is unitary. The function $\hat{p} : \Re \mapsto \Re$ is defined by imposing equality in last of Eqs. (104):

$$d(\theta_L \circ \mathbf{v}) \cdot (\mathbf{v} + (\hat{p}(\lambda)/p_0)\,\mathbf{Z}) \cdot \delta\mathbf{v} = \lambda \, \langle \mathbf{f}_{\mathrm{DYN}} - d(E \circ \mathbf{v}), \delta\mathbf{v} \rangle \,. \tag{106}$$

Setting $p_0 = \hat{p}(1)$, the solution of the least action principle Prop. 8.1, corresponding to $\lambda = 1$, is then

$$\mathbf{V} = \mathbf{v} + (\hat{p}(1)/p_0)\,\mathbf{Z} = \mathbf{v} + \mathbf{Z}\,. \tag{107}$$

Hence the space-time velocity in Eq. (107) is also the solution of EULER-LAGRANGE Eq. (93). The converse statement, that the solution of Eq. (93)

$$d(\theta_L \circ \mathbf{v}) \cdot \mathbf{V} \cdot \delta\mathbf{v} = (\mathbf{f}_{\mathrm{DYN}} - d(E \circ \mathbf{v})) \cdot \delta\mathbf{v}\,, \tag{108}$$

is also solution of Eqs. Eq. (102), (103), is trivial. ■

Remark 8.1. *The solution of the stationarity principle Eq.* (91) *is also solution of the action principle characterized by the extremality condition in Eq.* (101) *under the integral constraint*

$$\boxed{\int_{\Gamma} \langle d(E \circ \mathbf{v}), \delta\mathbf{v} \rangle \, dt_{\mathscr{C}} = \int_{\Gamma} \langle \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \rangle \, dt_{\mathscr{C}}\,.} \tag{109}$$

In turn the solution the extremality condition with the integral constraint Eq. (109) *is also solution of the least action principle Eq.* (101) *with the pointwise constraint Eq.* (102)*. The equivalence proved in Prop. 8.1 closes the path of implications so that all these formulations are in fact equivalent one another.*

MAUPERTUIS principle was discussed in (Romano et al., 2009a) where a geometric treatment based on the lifting to the LAGRANGE bundle $V\mathscr{C}$ was provided.

The next lemma yields the basic relations for the formulation of HAMILTON canonical law of dynamics in terms of a linear connection ∇. In (Gantmacher, 1970) the analogous result in coordinates is referred to as DONKIN's theorem (Donkin, 1854).

Lemma 9.1 (Donkin theorem). *For any linear connection ∇ and associated parallel transport $\Uparrow$, parallel derivatives of* LAGRANGE *and* HAMILTON *functionals fulfil the relations*

$$\begin{cases} \nabla_{\delta \mathbf{v}} H + \nabla_{\delta \mathbf{v}} L \circ \theta_H = 0 \,, \\ \nabla_{\delta \mathbf{v}} L + \nabla_{\delta \mathbf{v}} H \circ \theta_L = 0 \,. \end{cases} \tag{110}$$

Proof. Recalling that, by definition, the parallel transport of a covector fulfils the invariance property

$$\langle \delta \varphi_\lambda^{\mathscr{C}} \Uparrow \mathbf{p}, \delta \varphi_\lambda^{\mathscr{C}} \Uparrow \mathbf{v} \rangle = \langle \mathbf{p}, \mathbf{v} \rangle \,, \tag{111}$$

we infer that

$$\begin{aligned} \nabla_{\delta \mathbf{v}} H(\mathbf{p}) &:= \partial_{\lambda=0} \, H(\delta \varphi_\lambda^{\mathscr{C}} \Uparrow \mathbf{p}) \,, \\ \nabla_{\delta \mathbf{v}} L(\mathbf{v}) &:= \partial_{\lambda=0} \, L(\delta \varphi_\lambda^{\mathscr{C}} \Uparrow \mathbf{v}) \,, \\ \nabla_{\delta \mathbf{v}} \langle \mathbf{p}, \mathbf{v} \rangle &= \partial_{\lambda=0} \, \langle \delta \varphi_\lambda^{\mathscr{C}} \Uparrow \mathbf{p}, \delta \varphi_\lambda^{\mathscr{C}} \Uparrow \mathbf{v} \rangle = \partial_{\lambda=0} \, \langle \mathbf{p}, \mathbf{v} \rangle = 0 \,, \end{aligned} \tag{112}$$

and the result follows from EULER-LEGENDRE transform Eq. (56). ■

Proposition 9.1 (Hamilton law of dynamics). *The* EULER-LAGRANGE-POINCARÉ *law of motion is equivalent to* HAMILTON *law expressed, in terms of kinetic momentum, by*

$$\boxed{\begin{cases} \mathbf{p} = \theta_L(\mathbf{v}) \,, \\ \langle \nabla_{\mathbf{V}} \mathbf{p} + \nabla H(\mathbf{p}) + \mathbf{p} \cdot \mathbf{T}(\mathbf{V}), \delta \mathbf{v} \rangle = \langle \mathbf{f}_{\text{DYN}}, \delta \mathbf{v} \rangle \,, \end{cases}} \tag{113}$$

for all synchronous $\delta \mathbf{v} \in \mathscr{H}_{\mathscr{C}}$, *with the jump condition* $[[\langle \mathbf{p}, \delta \mathbf{v} \rangle]] = \langle \mathbf{f}_{\text{SING}}, \delta \mathbf{v} \rangle$ *at singular points.*

Proof. Applying EULER-LEGENDRE transform Eq. (56) and DONKIN relation Eq. (110) to LAGRANGE law Eq. (93), we get HAMILTON law Eq. (113). ■

Proposition 9.2 (Time rate of Hamilton functional). *Let us assume that the* HAMILTON *functional is time independent that is*

$$\nabla_{\mathbf{Z}} H(\mathbf{p}) = 0 \iff \nabla_{\mathbf{V}} H(\mathbf{p}) = \nabla_{\mathbf{v}} H(\mathbf{p}) \,. \tag{114}$$

Then its rate along the motion is equal to the power expended by dynamical forces

$$\boxed{\nabla_{\mathbf{V}} (H \circ \mathbf{p}) = \langle \mathbf{f}_{\mathrm{DYN}}, \mathbf{v} \rangle \,.} \tag{115}$$

Proof. Setting $\delta\mathbf{v} = \mathbf{v}$ in Eq. (113), noting that

$$\mathbf{T}(\mathbf{V}, \mathbf{v}) = \mathbf{T}(\mathbf{v}, \mathbf{v}) = \mathbf{0} \tag{116}$$

and recalling that $\mathbf{v} = d_F H(\mathbf{p})$, the result follows from the relations

$$\begin{aligned} \langle \nabla_{\mathbf{V}} \mathbf{p}, \mathbf{v} \rangle + \langle \nabla H(\mathbf{p}), \mathbf{v} \rangle &= d_F H(\mathbf{p}) \cdot \nabla_{\mathbf{V}} \mathbf{p} + \nabla_{\mathbf{V}} H(\mathbf{p}) - \nabla_{\mathbf{Z}} H(\mathbf{p}) \\ &= \nabla_{\mathbf{V}} (H \circ \mathbf{p}) - \nabla_{\mathbf{Z}} H(\mathbf{p}) = \langle \mathbf{f}_{\mathrm{DYN}}, \mathbf{v} \rangle \,, \end{aligned} \tag{117}$$

where Eqs. (A.5) and (56) have been resorted to. ■

HAMILTON differential law of dynamics Eq. (113) can be derived from EULER condition of an action principle governed by a 1-form. Proof is omitted for brevity.

Definition 9.1 (Poincaré-Cartan action form). *The* POINCARÉ-CARTAN *action* 1*-form* $\omega^1_H \circ \mathbf{p} : \mathscr{C} \mapsto (V\mathscr{C})^*_\Gamma$ *on the control manifold is induced by the map* $\omega^1_H : (V\mathscr{C})^*_\Gamma \mapsto (V\mathscr{C})^*_\Gamma$ *defined by*

$$\boxed{\omega^1_H(\mathbf{p}) := \mathbf{p} - H(\mathbf{p})\, dt_{\mathscr{C}} \,.} \tag{118}$$

The LAGRANGE map $\omega^1_L : (V\mathscr{C})_\Gamma \mapsto (V\mathscr{C})^*_\Gamma$ is related to the POINCARÉ-CARTAN map $\omega^1_H : (V\mathscr{C})^*_\Gamma \mapsto (V\mathscr{C})^*_\Gamma$ by the LEGENDRE diffeomorphism $\theta_L : (V\mathscr{C})_\Gamma \mapsto (V\mathscr{C})^*_\Gamma$ with inverse $\theta_H : (V\mathscr{C})^*_\Gamma \mapsto (V\mathscr{C})_\Gamma$.

Indeed, being $E = H \circ \theta_L$, we have the commutative diagram

$$\begin{tikzcd} & (T\mathscr{C})^* & \\ T\mathscr{C} \arrow[ur, "\omega^1_L"] \arrow[rr, shift left, "\theta_L"] & & (T\mathscr{C})^* \arrow[ul, "\omega^1_H"'] \arrow[ll, shift left, "\theta_H"] \end{tikzcd} \quad \iff \quad \begin{cases} \omega^1_L = \omega^1_H \circ \theta_L \,, \\ \omega^1_H = \omega^1_L \circ \theta_H \,. \end{cases} \tag{119}$$

Proposition 9.3 (Poincaré-Cartan action principle). *The motion along the trajectory* Γ *in the control manifold* $\mathscr{C}$*, is characterized by the action principle governed by the* POINCARÉ-CARTAN *action form*

$$\boxed{\partial_{\lambda=0} \int_{\delta\varphi_\lambda^{\mathscr{C}}(\Gamma)} \omega_H^1 \circ \mathbf{p} - \oint_{\partial\Gamma} \langle\, \omega_H^1 \circ \mathbf{p}, \delta\mathbf{v} \,\rangle = \int_\Gamma (dt_{\mathscr{C}} \wedge \mathbf{f}_{\mathrm{DYN}}) \cdot \delta\mathbf{v}\,,} \tag{120}$$

where $\mathbf{p} = \theta_L(\mathbf{v}) \in \mathscr{H}_{\mathscr{C}}^*$ *is the momentum field and the variational condition holds for all synchronous* $\delta\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$*. The localization of Eq.* (120)*, resorting to Eq.* (118)*, yields the differential condition*

$$\boxed{\langle\, d(\omega_H^1 \circ \mathbf{p}) \cdot \mathbf{V}, \delta\mathbf{v} \,\rangle = \langle\, d\mathbf{p} \cdot \mathbf{V} - d(H \circ \mathbf{p}), \delta\mathbf{v} \,\rangle = \langle\, \mathbf{f}_{\mathrm{DYN}}, \delta\mathbf{v} \,\rangle\,,} \tag{121}$$

that will be referred to as the EULER-HAMILTON *law. Expressed in terms of a linear connection, Eq.* (121) *gives* HAMILTON *law Eq.* (113) *since from Eq.* (A.1) *and Eq.* $(\mathrm{A.5})_2$ *we infer that*

$$\begin{aligned} d(\omega_H^1 \circ \mathbf{p}) \cdot \mathbf{V} \cdot \delta\mathbf{v} = \langle\, \nabla_{\mathbf{V}} \mathbf{p}, \delta\mathbf{v} \,\rangle + \nabla_{\delta\mathbf{v}} H(\mathbf{p}) + \langle\, \mathbf{p}, \mathbf{T}(\mathbf{V}, \delta\mathbf{v}) \,\rangle \\ + \langle\, \theta_H(\mathbf{p}) - \mathbf{v}, \nabla_{\delta\mathbf{v}} \mathbf{p} \,\rangle\,, \end{aligned} \tag{122}$$

with the last term vanishing due to the assumption that $\mathbf{v} = \theta_H(\mathbf{p})$.

According to the nomenclature introduced in Rem. 2.1, both the LAGRANGE action principle of Prop. 7.1 and POINCARÉ-CARTAN action principle of Prop. 9.3, are *faithful* action principles, and such is also MAUPERTUIS least action principle Eq. (101). The HAMILTON principle enunciated by Eq. (67) is instead a *hybrid* variational principle.

In a finite dimensional control manifold, with $\dim \mathscr{C} = m+1$, let us consider a coordinate system $\phi : \mathscr{Z} \times \Re^m \mapsto \mathscr{C}$ adapted to the time-fibration, composed by the time parameterization map $\gamma : \mathscr{Z} \mapsto \mathscr{C}$ of the control trajectory line $\Gamma \subset \mathscr{C}$ (0-th coordinate) and by a system of coordinates in the slices of isochronous placements

$$\phi = \{\, q_i\,, i = 1, \dots m \,\} : \Re^m \mapsto \mathscr{C}\,. \tag{123}$$

We may then set the definitions

$$\begin{cases} \dot{q} = \dfrac{dp}{dt} := \phi{\downarrow}\mathbf{v}\,, \quad \delta q := \phi{\downarrow}\delta\mathbf{v}\,, \quad p := \phi{\downarrow}\mathbf{p}\,, \\ \delta p := \phi{\downarrow}(\nabla_{\delta\mathbf{v}}\, \mathbf{p})\,, \\ \delta\dot{q} := \phi{\downarrow}(\nabla_{\delta\mathbf{v}}\, \mathbf{v})\,, \quad (\delta q)^{\cdot} := \phi{\downarrow}(\nabla_{\mathbf{V}}\, \delta\mathbf{v})\,. \end{cases} \tag{124}$$

By extension Eq. (66) we have that $[\mathbf{V}, \delta\mathbf{v}] = \mathbf{0}$ and hence

$$\mathbf{T}(\mathbf{V}, \delta\mathbf{v}) = \nabla_{\delta\mathbf{v}}\mathbf{V} - \nabla_{\mathbf{V}}\delta\mathbf{v} - [\mathbf{V}, \delta\mathbf{v}] = \nabla_{\delta\mathbf{v}}\mathbf{V} - \nabla_{\mathbf{V}}\delta\mathbf{v}\,. \tag{125}$$

Adopting a symmetric connection ($\mathbf{T} = \mathbf{0}$), $\delta\dot{q}$ will be equal to $(\delta q)^{\cdot}$ since

$$\delta\dot{q} = \phi{\downarrow}(\nabla_{\delta\mathbf{v}}\mathbf{v}) = \phi{\downarrow}(\nabla_{\delta\mathbf{v}}\mathbf{V}) = \phi{\downarrow}(\nabla_{\mathbf{V}}\delta\mathbf{v}) = (\delta q)^{\cdot}. \tag{126}$$

The time-vertical 1-form $\mathbf{p} \in (V\mathscr{C})^*_\Gamma$ is then expressed, in the coordinates introduced in Remark 6.4, as a linear combination of the covectors of the dual frame:

$$\mathbf{p} = p\,dq := \sum_{k=1,n} p_k\,dq^k\,, \tag{127}$$

and its exterior derivative is given by [10]

$$d\mathbf{p} = dp \wedge dq := \sum_{k=1,n} dp_k \wedge dq^k\,. \tag{128}$$

The POINCARÉ-CARTAN action 1-form of Eq. (118) is then written as

$$\omega^1_H \circ \mathbf{p} := \mathbf{p} - H(\mathbf{p})\,dt_{\mathscr{C}} = p\,dq - H(q\,,p)\,dt_{\mathscr{C}}\,, \tag{129}$$

and HAMILTON law Eq. (113) takes the standard expression

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{dH}{dp} & \iff \qquad \mathbf{v} = d_F H(\mathbf{p})\,, \\[2mm] -\dfrac{dp}{dt} = \dfrac{dH}{dq} - Q & \iff \quad -\nabla_{\mathbf{V}}\mathbf{p} = \nabla H(\mathbf{p}) - \mathbf{f}_{\mathrm{DYN}}\,. \end{cases} \tag{130}$$

10 VARIATIONAL PRINCIPLES WITH VERTICAL VARIATIONS

The present status of proposed formulations is the following.

1. In (Gantmacher, 1970, 3.17, p. 96) a *second form of* HAMILTON *principle* in *phase space* is enunciated without proof. Analogous statements are the *principle of least action in phase space* in (Arnold, 1974, 9.45 C, p. 244) and the HAMILTON principle in *phase space* in (Marsden and Ratiu, 1998, 8.1.6, p. 224) and (Yoshimura and Marsden, 2006, 3.9). In these statements, the POINCARÉ-CARTAN action principle Eq. (120) of the present paper, is modified by dropping the fiberwise relation $\mathbf{v} = d_F H(\mathbf{p})$ so that the covector $\mathbf{p} \in (V\mathscr{C})^*_\Gamma$ is free to vary in the relevant time-vertical cotangent fiber, that is with the base point kept fixed.

[10] In literature the expressions in components Eq. (127) and Eq. (128) are improperly attributed to the canonical 1-form on the cotangent bundle and to its exterior derivative.

2. In (Yoshimura and Marsden, 2006, 3.1) a HAMILTON-PONTRYAGIN principle in *phase space* is also enunciated by introducing a pair of tangent-cotangent vectors $(\mathbf{u}, \mathbf{p}) \in V\mathscr{C} \times_\Gamma (V\mathscr{C})^*$ with $\mathbf{u} \in (V\mathscr{C})_\Gamma$ related to the time-vertical velocity field $\mathbf{v} \in (V\mathscr{C})_\Gamma$ by a fiberwise LAGRANGE constraint in which the covector field $\mathbf{p} \in (V\mathscr{C})^*_\Gamma$ plays the role of controller.

In Prop. 10.1 it will be shown that POINCARÉ-CARTAN action principle of Prop. 9.3 can be extended, by changing the 1-form to be integrated, to an equivalent hybrid principle. Therein the covector field $\mathbf{p} \in \mathscr{H}^*_\mathscr{C}$ is left free to vary in each cotangent fiber (that is at fixed base points) and the momentum relation $\mathbf{p} = \theta_L \circ \mathbf{v}$ is recovered as a natural condition stemming from the variational principle. In Prop. 10.2 the geometric formulation of the HAMILTON-PONTRYAGIN principle considered in (Yoshimura and Marsden, 2006) is provided.

To design the modifications to be brought to POINCARÉ-CARTAN action 1-form, let us preliminarily derive two integral equalities along the trajectory. The property $\langle dt_\mathscr{C}, \mathbf{V} \rangle = 1$ leads, for any $\mathbf{p} \in (V\mathscr{C})^*_\Gamma$, to the equalities

$$\langle \mathbf{p}, \mathbf{V} \rangle = \langle \mathbf{p}, \mathbf{v} \rangle = \langle \mathbf{p}, \mathbf{v} \rangle \langle dt_\mathscr{C}, \mathbf{V} \rangle = \langle \langle \mathbf{p}, \mathbf{v} \rangle dt_\mathscr{C}, \mathbf{V} \rangle . \tag{131}$$

Being $\langle dt_\mathscr{C}, \delta\mathbf{v} \rangle = 0$, the definition $\omega^1_H \circ \mathbf{p} := \mathbf{p} - H(\mathbf{p})\, dt_\mathscr{C}$ gives

$$\begin{cases} (\omega^1_H \circ \mathbf{p}) \cdot \mathbf{V} = \langle \mathbf{p}, \mathbf{v} \rangle - H(\mathbf{p}) = \Big(\langle \mathbf{p}, \mathbf{v} \rangle - H(\mathbf{p}) \Big)\, dt_\mathscr{C} \cdot \mathbf{V} , \\ (\omega^1_H \circ \mathbf{p}) \cdot \delta\mathbf{v} = \langle \mathbf{p}, \delta\mathbf{v} \rangle . \end{cases} \tag{132}$$

From Eq. (131) and Eq. $(132)_1$ the following equalities are inferred

$$\int_\Gamma \mathbf{p} = \int_\Gamma \langle \mathbf{p}, \mathbf{v} \rangle\, dt_\mathscr{C} , \tag{133}$$

and

$$\int_\Gamma \omega^1_H \circ \mathbf{p} = \int_\Gamma \Big(\langle \mathbf{p}, \mathbf{v} \rangle - H(\mathbf{p}) \Big)\, dt_\mathscr{C} . \tag{134}$$

The 1-forms ω^1_{PC} and ω^1_{HP} that will be adopted in POINCARÉ-CARTAN and HAMILTON-PONTRYAGIN hybrid principles, with $(\mathbf{u}, \mathbf{p}) \in (V\mathscr{C})_\Gamma \times_\Gamma (V\mathscr{C})^*_\Gamma$, are given by

$$\boxed{\begin{aligned} &1) \quad \omega^1_{\mathrm{PC}} \circ (\mathbf{p}, \mathbf{v}) := \Big(\langle \mathbf{p}, \mathbf{v} \rangle - H(\mathbf{p}) \Big)\, dt_\mathscr{C} , \quad \forall \mathbf{p} \in (V\mathscr{C})^*_\Gamma , \\ &2) \quad \omega^1_{\mathrm{HP}} \circ (\mathbf{u}, \mathbf{p}, \mathbf{v}) := \Big(\langle \mathbf{p}, \mathbf{v} - \mathbf{u} \rangle + (L(\mathbf{u})) \Big)\, dt_\mathscr{C} . \end{aligned}} \tag{135}$$

Setting $\mathbf{u} = \mathbf{v}$, Eq. $(135)_2$ yields the 1-form $(L \circ \mathbf{v})\, dt_\mathscr{C}$ of HAMILTON principle, Prop. 6.1. Let us illustrate in detail the geometric treatment of the variational principles corresponding to the 1-forms ω^1_{PC} and ω^1_{HP}, omitting proofs for brevity.

The former principle is well-known but the standard formulation provided in literature does not comply with the geometric features of the variational problem whose essential ingredient is the variation of the trajectory, according to dragging virtual flows in the control manifold. The issue will be accurately commented in Sect. 11. Two distinct kinds of virtual flows are considered.

1. Virtual flows in the control manifold $\mathscr{C}$, that drag the trajectory $\Gamma \subset \mathscr{C}$, and hence the feet of all tensor fields based on it. These flows are generated by vector fields of synchronous virtual velocities $\delta\mathbf{v} \in (V\mathscr{C})_\Gamma$ in the LAGRANGE bundle over the control manifold.
2. Virtual flows in the HAMILTON bundle $(V\mathscr{C})^*_\Gamma$, that drag the covectors $\mathbf{p} \in (V\mathscr{C})^*_\Gamma$, while keeping the base points fixed. These flows are generated by vector fields $\delta\mathbf{X} \in T_\mathbf{p}(V\mathscr{C})^*_\Gamma$ tangent to HAMILTON bundle and vertical according to the cotangent bundle projection, i.e. such that

$$T\tau^*_\mathscr{C} \cdot \delta\mathbf{X} = \mathbf{0} \in T\mathscr{C}\,. \tag{136}$$

Vectors fulfilling Eq. (136) are said to be *vertical* and are univocally obtainable as vertical liftings[11] of covector fields, so that we may set

$$\delta\mathbf{X} = \mathrm{VLIFT}(\mathbf{p}, \delta\mathbf{p})\,, \qquad \mathbf{p}, \delta\mathbf{p} \in (V\mathscr{C})^*_\Gamma\,, \tag{137}$$

where $\mathrm{VLIFT}(\mathbf{p}, \delta\mathbf{p}) = \partial_{\lambda=0}\,(\mathbf{p} + \lambda\,\delta\mathbf{p}) = \partial_{\lambda=0}\,(\lambda\,\delta\mathbf{p}) = \delta\mathbf{p}$.

Adopting the usual identification between parallel vectors tangent to a linear space at distinct points, we will set $\delta\mathbf{X} = \mathrm{VLIFT}(\mathbf{p}, \delta\mathbf{p}) \equiv \delta\mathbf{p}$.

The statement of Prop. 10.1 puts into evidence the manner how an intrinsic formulation is feasible without performing the lifting of the trajectory to the cotangent phase space, and how boundary terms are to be taken into account.

Proposition 10.1 (Poincaré-Cartan hybrid principle). *The motion along the trajectory* Γ *in the control manifold* $\mathscr{C}$, *is characterized by the hybrid action principle*

$$\boxed{\begin{aligned}\partial_{\mu=0}\,\partial_{\lambda=0} \int_{\delta\varphi^{\mathscr{C}}_\lambda(\Gamma)} \omega^1_{\mathrm{PC}} \circ (\mathbf{Fl}^{\delta\mathbf{p}}_\mu(\mathbf{p})\,, \mathbf{v}) - \oint_{\partial\Gamma} \langle\, \omega^1_H \circ \mathbf{p}, \delta\mathbf{v} \rangle \\ = \int_\Gamma (dt_\mathscr{C} \wedge \mathbf{f}_{\mathrm{DYN}}) \cdot \delta\mathbf{v}\,,\end{aligned}} \tag{138}$$

for all synchronous virtual velocity fields $\delta\mathbf{v} \in \mathscr{H}_\mathscr{C}$ *and dual virtual covector fields* $\delta\mathbf{p} \in \mathscr{H}^*_\mathscr{C}$.

[11] The meaning here is that the relevant virtual variations are performed while keeping fixed the base point on the trajectory.

The latter principle was considered in (Yoshimura and Marsden, 2006, 3.1), but without the boundary term.

In our treatment, the variations induced in the tangent and cotangent fields by the virtual displacement of the trajectory are properly taken into account and no boundary (end points) condition is assumed. The local conditions are coincident with the ones in (Yoshimura and Marsden, 2006) because the additional terms appearing in our result are killed by the auxiliary conditions generated by vertical variations. As usual, the identification $\mathrm{VLIFT}(\mathbf{u}\,,\delta\mathbf{u}) \equiv \delta\mathbf{u}$ is made.

Proposition 10.2 (Hamilton-Pontryagin hybrid principle). *The motion along the trajectory Γ in the control manifold $\mathscr{C}$, is characterized by the hybrid action principle*

$$\boxed{\begin{aligned}&\partial_{\theta=0}\,\partial_{\mu=0}\,\partial_{\lambda=0}\int_{\delta\varphi^{\mathscr{C}}_{\lambda}(\Gamma)}\omega^{1}_{\mathrm{HP}}\circ(\mathbf{Fl}^{\delta\mathbf{u}}_{\theta}(\mathbf{u}),\mathbf{Fl}^{\delta\mathbf{p}}_{\mu}(\mathbf{p}),\mathbf{v})\\ &\qquad\qquad -\oint_{\partial\Gamma}\langle\theta_{L}\circ\mathbf{v},\delta\mathbf{v}\rangle=\int_{\Gamma}(dt_{\mathscr{C}}\wedge\mathbf{f}_{\mathrm{DYN}})\cdot\delta\mathbf{v}\,,\end{aligned}} \tag{139}$$

for all synchronous virtual velocities $\delta\mathbf{v}\in\mathscr{H}_{\mathscr{C}}$, and vector fields $\delta\mathbf{u}\in\mathscr{H}_{\mathscr{C}}$ and covector fields $\delta\mathbf{p}\in\mathscr{H}^{*}_{\mathscr{C}}$.

11 STANDARD FORMULATIONS

It is instructive to reproduce explicitly the standard treatment of POINCARÉ-CARTAN variational principle, in order to perform a direct comparison with the intrinsic formulation provided in Eq. (138) of Prop. 10.1.

The standard expression of the principle, as reported in (Arnold, 1974, 9.45 C, p. 244), but with the addition of the boundary integral, is the following

$$\begin{aligned}\delta\int_{t_1}^{t_2}(p\,\dot{q}-H)\,dt-\oint_{t_1}^{t_2}(p\,\delta q)&=\delta\int_{t_1}^{t_2}(p\,\dot{q}-H)\,dt-\int_{t_1}^{t_2}(p\,\delta q)^{\cdot}\,dt\\ &=\int_{t_1}^{t_2}\Big(\dot{q}\,\delta p-\dot{p}\,\delta q-\frac{\partial H}{\partial p}\delta p-\frac{\partial H}{\partial q}\delta q\Big)\,dt\\ &=\int_{t_1}^{t_2}\Big((\dot{q}-\frac{\partial H}{\partial p})\,\delta p-(\dot{p}+\frac{\partial H}{\partial q})\,\delta q\Big)\,dt=0\,,\end{aligned} \tag{140}$$

where resort was made to the commutation property $\delta\dot{q}=(\delta q)^{\cdot}$ and to the product rule $(p\,\delta q)^{\cdot}=p\,\delta\dot{q}+\dot{p}\,\delta q$, with the *variations* δp and δq assumed to be independent of one another, with δq vanishing at the end points t_1,t_2.

The EULER conditions of the variational principle Eq. (140) provide HAMILTON laws of dynamics, in the form corresponding to a standard connection by translation

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad -\dot{p} = \frac{\partial H}{\partial q}. \tag{141}$$

Simple as it is, the previous argument leads in fact to the correct equations of the motion. But this is really an interesting example of how a geometrically incorrect analysis may lead to correct results.

The issue is delicate and the difficulty is subtle and deserves a careful examination. The basic difference, with respect to our treatment in Prop. 9.3, is that the geometric formulation takes naturally into account the rate of variation of the covector field $\mathbf{p} \in \mathscr{H}_{\mathscr{C}}^*$ due to a virtual velocity of the trajectory.

The rate of variation of a covector field $\mathbf{p} \in \mathscr{H}_{\mathscr{C}}^*$ is composed of a vertical variation (a free variation in each fiber at fixed base point in $\mathscr{C}$) and of a non-vertical variation $\nabla_{\delta\mathbf{v}}\mathbf{p}$ linearly dependent on the virtual velocity $\delta\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$, as observed in Sect. 10. Only the former vertical rate of variation is considered in standard treatments, see e.g. (Gantmacher, 1970; Arnold, 1974; Yoshimura and Marsden, 2006).

The subtle point is that evaluation of the latter, non-vertical, variation is in fact not essential for correctness of the result, since the relevant term in EULER condition vanishes by imposing that the covector field is equal to kinetic momentum. This equality is either assumed as an essential condition (in Prop. 9.3) or derived as a natural condition (in Prop. 10.1). Consequently, although the non-vertical variation $\nabla_{\delta\mathbf{v}}\mathbf{p}$ and the virtual velocity $\delta\mathbf{v} \in \mathscr{H}_{\mathscr{C}}$ are *not* independent, since the former depends linearly on the latter, the result arrived at by ignoring this dependence is, after all, the correct one.

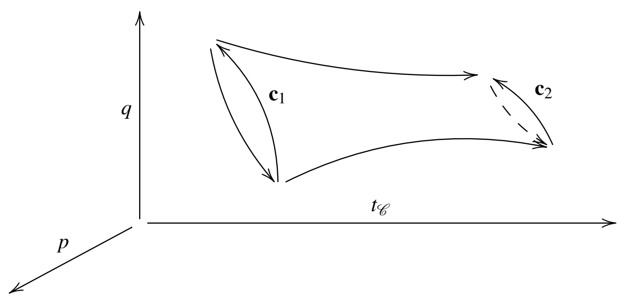
The improper procedure of ignoring non-vertical variation was, as a matter of fact, adopted in all treatments of variational dynamics, probably due to the adoption in Eq. (140) of the δ as a variation operator acting in an unspecified manner on the time integral, with the trajectory not appearing explicitly.

Ambiguity of the variation symbol δ helps in forgetting that the trajectory, which is the manifold on which the involved dynamical objects (velocity and kinetic momentum) are based, is dragged by the virtual flow and therefore the common foot of these objects has to follow the moving ground (trajectory) where they are based on. Anyway, also in purportedly geometrical treatments, the possibly misleading δ notation is still adopted, together with formulations in coordinates.

Remark 11.1. *We emphasise that no operational meaning was given in the present treatment to the symbol δ which is here adopted just as a prefix apt to denote* test fields *in variational statements. This is a distinction from the common but* *inadvisable usage of attributing to* δ *the meaning of an often not explicitly defined variation operator.*

The schematic diagram Eq. (142) reproduces classical pictures such as (Gantmacher, 1970, 3.17, p. 96, fig. 32), and (Arnold, 1974, 9.44 C, p. 236, fig. 182; 9.44 D, p. 238, fig. 183).

These representations are geometrically unsuitable, because the fibered structure of *phase space* is not taken into proper account.



(142)

The resulting scheme, widely diffused in treatments of analytical mechanics, is eventually misleading since the variables $t_{\mathscr{C}}, q, p$ are depicted as they were just cartesian coordinates and not coordinates in a frame adapted to the nonlinear HAMILTON bundle.

In a consistent geometric analysis, coordinate variations cannot be considered as mutually independent. In the picture Eq. (142) the cotangent fibers based at distinct points of the control space are merged into a unique representative linear subspace, an identification which is not admissible since the corresponding covector fields refer to distinct placements of the body.

12 CONCLUSIONS AND RESULTS

Application of geometry to dynamics is a classical subject of investigations and has a quite long history with many brilliant contributions, well-known to scholars involved in the matter. We do not even try to provide a necessarily incomplete list of valuable contributions, but the references in the citations could help in the task. This notwithstanding, the topic is still out of the target of most scholars, even mathematically trained ones, mainly due to a discouraging complexity of notions and notations. The present contribution was focused on the task of illustrating feasibility of a proper geometric treatment of dynamics in the control manifold by resorting to simple but powerful notions and methods. The recurse to more sophisticated geometric constructions involving the second tangent bundle or the tangent to the cotangent bundle is avoided. The theory is thus freed from unneeded complications, rendering the treatment readily addressable to a larger audience of non specialists in differential geometry. Variational principles are under the spotlight and classical treatments are revisited by basic tools of modern differential geometry, with the aim of dropping unnecessary special assumptions, while retaining or also recovering hypotheses which are basic for a proper geometrical analysis. The outcome is a new formulation of basic results, with an innovative analysis of the connections between various variational statements, and with a special care in elucidating hidden difficulties.

Main results achieved by the investigation are listed below and main topics are summarized in the synoptic table, with the *hybrid* labelling explicated in Rem. 2.1. The exterior forms included in curly braces are respectively the bulk and the boundary forms of the relevant hybrid action principle.

Synoptic table		
Tangent bundle $T\mathscr{C}$		**action forms**
HAMILTON	hybrid principle	$\begin{cases} (L\circ\mathbf{v})\,dt_{\mathscr{C}}\,, \\ \theta_L\circ\mathbf{v}\,. \end{cases}$
LAGRANGE action principle		$\omega^1_L\circ\mathbf{v}=\theta_L\circ\mathbf{v}-(E\circ\mathbf{v})\,dt_{\mathscr{C}}$
MAUPERTUIS least action principle		$\theta_L\circ\mathbf{v}\,,$ with the constraint $\langle d(E\circ\mathbf{v})),\delta\mathbf{v}\rangle=\langle\mathbf{f}_{\mathrm{DYN}},\delta\mathbf{v}\rangle\,.$
Cotangent bundle $(T\mathscr{C})^*$		**action forms / loop integrals**
POINCARÉ-CARTAN action principle		$\omega^1_H\circ\mathbf{p}=\mathbf{p}-(H\circ\mathbf{p})\,dt_{\mathscr{C}}\,,$ with $\mathbf{p}=\theta_L\circ\mathbf{v}\,.$
POINCARÉ-CARTAN	hybrid principle	$\begin{cases} (\langle\mathbf{p},\mathbf{v}\rangle-(H\circ\mathbf{p}))\,dt_{\mathscr{C}}\,, \\ \mathbf{p}-(H\circ\mathbf{p})\,dt_{\mathscr{C}}\,. \end{cases}$
Whitney bundle $T\mathscr{C}\times_{\mathscr{C}}(T\mathscr{C})^*$		**action form**
HAMILTON-PONTRYAGIN	hybrid principle	$\begin{cases} (\langle\mathbf{p},\mathbf{v}-\mathbf{u}\rangle+(L\circ\mathbf{u}))\,dt_{\mathscr{C}}\,, \\ \theta_L\circ\mathbf{v}\,. \end{cases}$

All variational principles listed therein are shown, after suitable and even drastic revisitation, to be equivalent statements of the law of dynamics.

- The primary target is to show that a geometric treatment of dynamics can be performed in the natural context of control manifold. This result avoids the recourse to more sophisticated geometric constructions based on considering tangent or cotangent manifolds over the control manifold and the relevant (bi-)tangent bundles. A substantial step towards popularizing geometric methods in dynamics, presently still confined to small nuclei of specialists, is thus made.

- A first issue is an appropriate presentation of what an action principle is defined to be. There are basic differences with variational statements dealing with stationarity of a functional. Action principles consider variations of the integral of an exterior form over a manifold dragged by a virtual flow, the stationarity manifold being called the trajectory. The involved exterior form is well-defined only on the trajectory and, to perform the variations, extensions along virtual flows must be explicitly declared. There is no functional for a stationarity condition to be imposed.

- Another issue is the elimination of needlessly restrictive, and even epistemologically incorrect, constraints on the variations, classically assumed to have null velocities at the trajectory end points. The adoption of unconstrained variations leads to deduce both differential and jump conditions from the action principle and plays a significant role in computational schemes.

- It is emphasised that two main procedures are available to perform comparisons and derivatives. The former procedure is naturally induced by the motion, or by the virtual flow, and consists in a pull-back operation and in the related LIE-differentiation. The latter is induced by the choice of a connection and of the related parallel transport and derivative. The specific problem at hand may suggest the convenient choice of a connection. The usual one in EUCLID spaces is parallel transport by translation, but even in EUCLID spaces other connections may be more suitable. An important instance is given by the adapted connection generated by the control.

- A general formula is contributed to evaluate exterior derivatives of differential forms in terms of parallel derivatives. The formula is a convenient tool to provide an operative expression of EULER differential condition. Arbitrary linear connections are considered and the role of the torsion form is underlined.

- HAMILTON action principle is shown to be equivalent to a constrained action principle in which virtual velocities fulfil a virtual mechanical power balance. This action principle extends and corrects standard but inappropriate definitions of MAUPERTUIS least action principle.
- POINCARÉ-CARTAN action principle for kinetic momentum is formulated in the control manifold and is shown to be equivalent to HAMILTON law of dynamics.
- A discussion, about variational principles in phase spaces as formulated in literature, witnesses the role played by geometrical notions in dynamics. These hybrid principles are reformulated in the natural context of the control manifold, with the introduction of vertical variations (i.e. variations at fixed base point) as appropriate geometric notions.

In conclusion the hope is that this innovative presentation of the geometric approach to dynamics will result into a wider acceptance of the powerful and conceptually clear framework provided by basic differential geometry.

A AUXILIARY RESULTS

The next results, contributed in (Romano G., 2007), provides the expression of the exterior derivative of a 1-form in terms of a linear connection and two split formulae that have been resorted to in the treatment of action principles.

Proposition A.1 (Exterior derivative in terms of a connection). *The exterior derivative* $d\,\omega^1$ *of a* 1*-form* $\omega^1 \in \Lambda^1(T\mathbf{M})$ *is expressed in terms of a linear connection* ∇ *by the formula*

$$\boxed{d\omega^1 = \nabla\omega^1 - (\nabla\omega^1)^A + \omega^1 \cdot \mathbf{T}\,,} \tag{A.1}$$

where the 2*-forms at the r.h.s. are defined by*

$$\begin{aligned}(\nabla\omega^1)\cdot\mathbf{a}\cdot\mathbf{b} &= (\nabla_{\mathbf{a}}\,\omega^1)\cdot\mathbf{b}\,,\\ (\nabla\omega^1)^A\cdot\mathbf{a}\cdot\mathbf{b} &= (\nabla_{\mathbf{b}}\,\omega^1)\cdot\mathbf{a}\,,\\ (\omega^1\cdot\mathbf{T})\cdot\mathbf{a}\cdot\mathbf{b} &= \omega^1\cdot\mathbf{T}(\mathbf{a},\mathbf{b})\,,\quad \forall\,\mathbf{a},\mathbf{b}\in T\mathbf{M}\,.\end{aligned} \tag{A.2}$$

Let $(\phi\,,\mathrm{ID}_{\mathbf{M}})$ be a smooth non-linear morphism, between the tensor bundles $\mathrm{TENS}_1(T\mathbf{M})$ and $\mathrm{TENS}_2(T\mathbf{M})$, described by the commutative diagram

$$\begin{tikzcd} \mathrm{TENS}_1(T\mathbf{M}) \arrow[r, "\phi"] \arrow[d, "\pi_{\mathrm{TENS}_1}"'] & \mathrm{TENS}_2(T\mathbf{M}) \arrow[d, "\pi_{\mathrm{TENS}_2}"] \\ \mathbf{M} \arrow[r, "\mathrm{ID}_{\mathbf{M}}"] & \mathbf{M} \end{tikzcd} \iff \pi_{\mathrm{TENS}_2}\circ\phi = \pi_{\mathrm{TENS}_1}\,. \tag{A.3}$$

Lemma A.1 (Differential split formulae). *Let the tensor field*

$$\phi \circ \mathbf{s} : \mathbf{M} \mapsto \mathrm{TENS}_2(T\mathbf{M}) \,, \tag{A.4}$$

be the composition of the morphism $\phi : \mathrm{TENS}_1(T\mathbf{M}) \mapsto \mathrm{TENS}_2(T\mathbf{M})$ *with a tensor field* $\mathbf{s} : \mathbf{M} \mapsto \mathrm{TENS}_1(T\mathbf{M})$. *The* LIE *and parallel derivatives along the flow* $\varphi_\alpha := \mathbf{Fl}^{\mathbf{v}}_\alpha$ *of a vector field* $\mathbf{v} : \mathbf{M} \mapsto T\mathbf{M}$, *may then be expressed by the split formulae*

$$\boxed{\begin{aligned} &\mathscr{L}_{\mathbf{v}}(\phi \circ \mathbf{s}) = (\mathscr{L}_{\mathbf{v}}\phi)(\mathbf{s}) + d_F\phi(\mathbf{s}) \cdot \mathscr{L}_{\mathbf{v}}\mathbf{s} \,, \\ &\nabla_{\mathbf{v}}(\phi \circ \mathbf{s}) = (\nabla_{\mathbf{v}}\phi)(\mathbf{s}) + d_F\phi(\mathbf{s}) \cdot \nabla_{\mathbf{v}}\mathbf{s} \,. \end{aligned}} \tag{A.5}$$

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