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Accademia di Scienze Fisiche e Matematiche
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On time and length in special relativity

Nota del socio Giovanni Romano[1]
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Abstract - Two well-known statements in special relativity, *time dilation* (with *slower* clocks) and *longitudinal length contraction* are revisited by a geometric definition of framings and by an analysis of frame changes in the event manifold. Applied to VOIGT-LORENTZ transformations, the theory provides frame invariant notions of clock-rates and of longitudinal lengths. When time-lapses are measured by an observer along the transformed time-line detected by a moving observer, then clock-rates and time-lapses result to be increased by the relativistic factor. This relativistic effect is LARMOR *time dilation* (but with *faster* clocks). Analogously, when an observer detects the longitudinal abscissae at the end points of a small bar at rest and compares the length evaluation with the abscissae of the spatial projections, at different time instants, of the bar image in a moving frame, then the longitudinal length appears to be increased by the relativistic factor. The relativistic effect is a *longitudinal dilation* and *not* a FITZGERALD-LORENTZ *contraction*. A treatment in terms of velocities extends the analysis to nonlinear frame transformations.

Riassunto - Due ben note affermazioni della relatività speciale, dilatazione del tempo, secondo LARMOR (con orologi più lenti) e contrazione della lunghezza longitudinale secondo FITZGERALD-LORENTZ, sono rivisitate mediante una definizione geometrica dei riferimenti ed un'analisi delle relative trasformazioni di VOIGT-LORENTZ. La teoria rivela che gli effetti relativistici vanno rivisitati e modificati in dilatazione del tempo (con orologi più veloci) e dilatazione longitudinale, e *non* contrazione alla FITZGERALD-LORENTZ. La trattazione in termini di velocità estende l'analisi a trasformazioni non lineari.

[1]Department of Structures, University of Naples Federico II, via Claudio 21, 80125 - Naples, Italy. e-mail: romano@unina.it

1 PROLEGOMENA

A revisitation of well-known issues of modern physics is presented here, motivated by the emergence of improper statements concerning measurements of time and length in the context of special relativity. The basic theoretical notions are dealt with a differential geometric approach, commonly adopted in classical textbooks on relativity (Misner,Thorne and Wheeler, 1973).

The reader more oriented towards a physical point of view can however find a completely standard treatment in the concluding section, where a detailed discussion of the new outcomes of the analysis provided.

2 EVENT MANIFOLD AND OBSERVERS

Let us assume that the *event manifold* $\mathscr{E}$ is a 4-dimensional star-shaped orientable manifold without boundary.

The *tangent bundle* to $\mathscr{E}$ will be denoted by $T\mathscr{E}$ and the *exterior derivative* in the event manifold $\mathscr{E}$ will be denoted by d (doCarmo, 1994; Romano G., 2007).

A *framing* (or *observer*) consists in a criterion to assess simultaneity in the event manifold. The criterion is described in geometrical terms by a field of time-arrows $\mathbf{Z} \in \mathrm{C}^1(\mathscr{E};T\mathscr{E})$ and by a field of rank-one projectors (Whiston, 1974; Marmo and Preziosi, 2006)

$$\mathbf{R} := \omega \otimes \mathbf{Z} \tag{1}$$

with $\omega \in \Lambda^1(T\mathscr{E})$ closed one-form, $d\omega = \mathbf{0}$. Idempotency $\mathbf{R}^2 = \mathbf{R}$ is equivalent to *tuning* property $\langle \omega, \mathbf{Z} \rangle = 1$.

By POINCARÉ Lemma, $\omega = dt_{\mathscr{E}}$ with $t_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E};Z)$ projection (surjective submersion) defining the fibration of $\mathscr{E}$ over the real time-line Z.[2]

Lemma 2.1 (Space-time splitting). *The projectors* $\mathbf{R}, \mathbf{P}$

$$\mathbf{P} := \mathbf{I} - \mathbf{R}\,, \quad \mathbf{P}^2 = \mathbf{P}\,, \quad \mathbf{PR} = \mathbf{RP} = \mathbf{0}\,, \tag{2}$$

perform a splitting of tangent vectors $\mathbf{X} \in T\mathscr{E}$ *into spatial and temporal components with*

$$dt_{\mathscr{E}} \cdot \mathbf{P} = \mathbf{0}\,, \quad \mathbf{RZ} = \mathbf{Z}\,, \quad \mathrm{Ker}\, dt_{\mathscr{E}} = \mathrm{Im}\,\mathbf{P}\,. \tag{3}$$

Proof. Being $\mathbf{R}(\mathbf{X}) = (dt_{\mathscr{E}} \otimes \mathbf{Z}) \cdot \mathbf{X} = \langle dt_{\mathscr{E}}, \mathbf{X} \rangle \mathbf{Z}$, all properties are verified by a direct calculation. ■

[2] The symbol Z is taken from the German word *Zeit* for *Time*.

Under the action of a framing $\mathbf{R} := dt_{\mathscr{E}} \otimes \mathbf{Z}$ the tangent bundle $T\mathscr{E}$ splits into a WHITNEY bundle $V\mathscr{E} \times_{\mathscr{E}} H\mathscr{E}$ of time-vertical and time-horizontal tangent vectors with $V\mathscr{E} = \mathrm{Im}\,\mathbf{P}$ and $H\mathscr{E} = \mathrm{Im}\,\mathbf{R}$. The 3D fibers of $V\mathscr{E}$ are in the kernel of $dt_{\mathscr{E}} \in \Lambda^1(T\mathscr{E})$.

Both subbundles $V\mathscr{E}$ and $H\mathscr{E}$ of $T\mathscr{E}$ are integrable. To see this, we observe that FROBENIUS involutivity condition, which is expressed in terms of the LIE bracket $[\mathbf{X}, \mathbf{Y}] = \mathscr{L}_{\mathbf{X}} \mathbf{Y}$ by the implication (Romano G., 2007)

$$\mathbf{X}, \mathbf{Y} \in \mathscr{D} \quad \Longrightarrow \quad [\mathbf{X}, \mathbf{Y}] \in \mathscr{D}, \tag{4}$$

is trivially fulfilled by any 1D subbundle $\mathscr{D}$. This proves integrability of the horizontal subbundle $H\mathscr{E}$.

On the other hand, the vertical subbundle $V\mathscr{E}$ is the kernel of the closed one-form $\omega \in \Lambda^1(T\mathscr{E})$, and hence the involutivity condition may be written as

$$\omega \cdot \mathbf{X} = \mathbf{0}, \quad \omega \cdot \mathbf{Y} = \mathbf{0} \quad \Longrightarrow \quad \omega \cdot [\mathbf{X}, \mathbf{Y}] = \mathbf{0}. \tag{5}$$

By virtue of PALAIS formula

$$d\omega \cdot \mathbf{X} \cdot \mathbf{Y} = d(\omega \cdot \mathbf{Y}) \cdot \mathbf{X} - d(\omega \cdot \mathbf{X}) \cdot \mathbf{Y} - \omega \cdot [\mathbf{X}, \mathbf{Y}], \tag{6}$$

Eq. (5) may equivalently be expressed by

$$\omega \cdot \mathbf{X} = \mathbf{0}, \quad \omega \cdot \mathbf{Y} = \mathbf{0} \quad \Longrightarrow \quad d\omega \cdot \mathbf{X} \cdot \mathbf{Y} = \mathbf{0}, \tag{7}$$

and is therefore trivially implied by the closedness property $d\omega = \mathbf{0}$.

The 1D submanifolds of $\mathscr{E}$, generated as integral curves of the time-arrow field $\mathbf{Z} \in \mathrm{C}^1(\mathscr{E}; T\mathscr{E})$, are called *time-lines* while the transversal 3D submanifolds of $\mathscr{E}$ are called *space-slices*.

Events lying on a time-line are called *isotopic* since they share a common spatial position. Events lying on a space-slice are called *isochronous* (or *simultaneous*) since they share a common time instant.

By definition, time-lines are parametrized so that their velocity fields are made of time-arrows.

The integral manifolds of the vertical distribution $V\mathscr{E}$ define a *time-bundle* projection $t_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E}; Z)$ according to which a time instant $t \in Z$ is assigned to each spatial slice, so that $\mathscr{E}(t) := t_{\mathscr{E}}^{-1}(\{t\})$.

Definition 2.1 (Spatial bundle). *The spatial bundle S over Z is fibered by 3-D manifolds $S(t)$ with canonical isomorphism $\mathbf{i}_{\mathscr{E},S}(t) \in \mathrm{C}^1(S(t); \mathscr{E}(t))$ onto the 3-D submanifold $\mathscr{E}(t)$ of the 4-D space-time event manifold $\mathscr{E}$.*

For any fixed time $t \in Z$, the isomorphism $\mathbf{i}_{\mathscr{E},S}(t) \in \mathrm{C}^1(S(t);\mathscr{E}(t))$ may be acted upon by the tangent functor to provide a fibrewise defined *space-time extension*, $\mathbf{i}_{\mathscr{E},S}\!\uparrow \in \mathrm{C}^1(VS;V\mathscr{E})$ which is a global bundle isomorphism but not the tangent map of a morphism.[3]

The inverse morphism $\mathbf{i}_{\mathscr{E},S}\!\downarrow \in \mathrm{C}^1(V\mathscr{E};VS)$ is the *spatial restriction*.

Vectors in $V\mathscr{E}$, henceforth denoted by capital letters, have four space-time components in a space-time frame, while vectors in VS, denoted by small letters, have three spatial components in a space frame.

Definition 2.2 (Charts, reference systems, natural frames). *A chart is a local diffeomorphism from a subset of the event manifold $\mathscr{E}$ onto an open set of the model linear space $\mathscr{R}^4$. A reference system is the inverse map $f : \mathscr{R}^4 \mapsto \mathscr{E}$. The natural frame associated with a reference system is provided by the vector fields which are the velocity vectors $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ of the coordinate lines.*

Declaring that $\mathscr{R}^4 = \mathscr{R}_0 \times \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3 = \{\mathscr{R}_\alpha\}$ with $\alpha = 0,1,2,3$, we have that

$$\partial_\alpha \circ f := Tf \cdot 1_\alpha \quad \Longleftrightarrow \quad \partial_\alpha = f\!\uparrow\! 1_\alpha\,, \tag{8}$$

where 1_α are the vectors of the usual basis in $\mathscr{R}^4$. The cartesian projectors $\pi_\alpha : \mathscr{R}^4 \mapsto \mathscr{R}_\alpha$ define the coordinate maps $\pi_\alpha \circ f^{-1} : \mathscr{E} \mapsto \mathscr{R}_\alpha$. In the cotangent bundle, the differentials $\{d_0, d_1, d_2, d_3\}$ defined by

$$d_\alpha = T(\pi_\alpha \circ f^{-1}) = T\pi_\alpha \cdot Tf^{-1}\,, \tag{9}$$

provide the dual frame of the natural frame $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ in tangent bundle. Indeed

$$\begin{aligned} \langle d_\alpha, \partial_\beta \rangle &= T(\pi_\alpha \circ f^{-1} \circ f) \cdot 1_\beta = (T\pi_\alpha \cdot Tf^{-1} \cdot Tf) \cdot 1_\beta \\ &= T\pi_\alpha \cdot 1_\beta = \delta_{\alpha\beta}\,. \end{aligned} \tag{10}$$

Here $\delta_{\alpha\beta}$ is the KRONECKER symbol.

A change of coordinate system from $f : \mathscr{R}^4 \mapsto \mathscr{E}$ to $\hat{f} : \mathscr{R}^4 \mapsto \mathscr{E}$ is induced by a coordinate diffeomorphic transformation $\zeta_{\mathrm{num}} \in \mathrm{C}^1(\mathscr{R}^4;\mathscr{R}^4)$, according to the commutative diagram

$$\begin{array}{ccc} & \mathscr{E} & \\ {}^{f}\nearrow & & \nwarrow^{\hat{f}} \\ \mathscr{R}^4 & \xleftarrow{\;\zeta_{\mathrm{num}}\;} & \mathscr{R}^4 \end{array} \quad \Longleftrightarrow \quad \hat{f} = f \circ \zeta_{\mathrm{num}}\,. \tag{11}$$

[3] The push-pull $\uparrow$, $\downarrow$ notation is still adopted for convenience.

3 TRAJECTORY AND MOTION

The trajectory $\mathscr{T}$ is a non-linear manifold of events characterized by an injective immersion $\mathbf{i} \in \mathrm{C}^1(\mathscr{T};\mathscr{E})$ such that the immersed trajectory

$$\mathscr{T}_{\mathscr{E}} := \mathbf{i}(\mathscr{T}) \subset \mathscr{E} \tag{12}$$

is a submanifold of the event manifold.

Events in the trajectory are labeled by coordinates in that manifold whose dimensionality may in general be lower than the one of the event manifold. Events in the immersed trajectory are instead labeled by coordinates in the event manifold.

Definition 3.1 (Material bundle). *The* material bundle *is the trajectory fibration generated by the* time-bundle *projection* $t_{\mathscr{T}} = t_{\mathscr{E}} \circ \mathbf{i} \in \mathrm{C}^1(\mathscr{T};Z)$ *over the time-line* Z. *The fibres* $\mathscr{T}(t)$ *are called trajectory slices.*

The motion detected in a given framing, is a one-parameter family of automorphisms $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}_{\mathscr{E}};\mathscr{T}_{\mathscr{E}})$ of the trajectory time-bundle over the time translation $\mathrm{TR}_\alpha \in \mathrm{C}^1(Z;Z)$, defined by $\mathrm{TR}_\alpha(t) := t + \alpha$ with $t \in \mathscr{R}$ *time-instant* and $\alpha \in Z$ *time-lapse*, described by the commutative diagram

$$\begin{array}{ccc} \mathscr{T}_{\mathscr{E}} & \xrightarrow{\varphi_\alpha} & \mathscr{T}_{\mathscr{E}} \\ {\scriptstyle t_{\mathscr{E}}}\downarrow & & \downarrow{\scriptstyle t_{\mathscr{E}}} \\ Z & \xrightarrow{\mathrm{TR}_\alpha} & Z \end{array} \quad \Longleftrightarrow \quad t_{\mathscr{E}} \circ \varphi_\alpha = \mathrm{TR}_\alpha \circ t_{\mathscr{E}} \,. \tag{13}$$

Eq. (13) expresses the simultaneity preservation property of motion.

A material particle is a line (a one-dimensional manifold) whose elements are events related by the space-time motion along the trajectory. The equivalence relation

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathscr{E} \mid \exists\, \alpha \in \mathscr{R} : \mathbf{e}_2 = \varphi_\alpha(\mathbf{e}_1)\,.$$

foliates the trajectory manifold in a congruence of material lines (Romano et al., 2009a,b; Romano and Barretta, 2011, 2013a,b; Romano et al., 2014a,b,c).

The body is the disjoint union of material particles, that is the quotient manifold induced by the foliation of the trajectory manifold.

A body placement is a fibre of simultaneous trajectory-events. The placement at time $t \in I$ is then the trajectory slice $\mathscr{T}_{\mathscr{E}}(t)$.

The space-time *trajectory velocity* $\mathbf{V} \in \mathrm{C}^1(\mathscr{T}_{\mathscr{E}}; T\mathscr{T}_{\mathscr{E}})$ is the vector field defined by $\mathbf{V} := \partial_{\alpha=0}\,\varphi_\alpha$. Since motion is time-parametrized, we have that

$$\langle dt_{\mathscr{E}}, \mathbf{V} \rangle = 1\,, \quad \mathbf{V} = \mathbf{Z} + \mathbf{PV}\,, \quad \mathbf{RV} = \mathbf{Z}\,. \tag{14}$$

Spatial velocity is related to space-time velocity by $\mathbf{v} = \mathbf{i}_{\mathscr{E},S}{\downarrow}(\mathbf{PV}) \in VS$. For simplicity, we will here consider the case in which the trajectory manifold $\mathscr{T}$ is four-dimensional.

4 CHANGES OF FRAME

A *change of frame* is an automorphism $\zeta_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E};\mathscr{E})$ of the event manifold. A *trajectory transformation* $\zeta \in \mathrm{C}^1(\mathscr{T};\mathscr{T}_\zeta)$ is a diffeomorphism between trajectory manifolds, induced by a change of frame according to the commutative diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\ \zeta_{\mathscr{E}}\ } & \mathscr{E} \\ \mathbf{i}\uparrow & & \uparrow\mathbf{i}_\zeta \\ \mathscr{T} & \xrightarrow{\ \zeta\ } & \mathscr{T}_\zeta \end{array} \quad \Longleftrightarrow \quad \mathbf{i}_\zeta \circ \zeta = \zeta_{\mathscr{E}} \circ \mathbf{i}\,, \tag{15}$$

with $\mathbf{i} \in \mathrm{C}^1(\mathscr{T};\mathscr{E})$ and $\mathbf{i}_\zeta \in \mathrm{C}^1(\mathscr{T}_\zeta;\mathscr{E})$ injective immersions.

Lemma 4.1 (Pushed framings). *Under a change of frame, according to an automorphism $\zeta_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E};\mathscr{E})$, a framing $\mathbf{R} = dt_{\mathscr{E}} \otimes \mathbf{Z}$ is pushed to a framing*

$$\mathbf{R}_\zeta = dt_{\mathscr{E}_\zeta} \otimes \mathbf{Z}_\zeta\,, \tag{16}$$

with $t_{\mathscr{E}_\zeta} := \zeta_{\mathscr{E}}{\uparrow}t_{\mathscr{E}}$ and $\mathbf{R}_\zeta = \zeta_{\mathscr{E}}{\uparrow}\mathbf{R}$, $\mathbf{Z}_\zeta := \zeta_{\mathscr{E}}{\uparrow}\mathbf{Z}$, and tuning is preserved.

Proof. By definition of push of a covector field it follows that

$$\zeta_{\mathscr{E}}{\uparrow}\mathbf{R} = \zeta_{\mathscr{E}}{\uparrow}(dt_{\mathscr{E}} \otimes \mathbf{Z}) = (\zeta_{\mathscr{E}}{\uparrow}dt_{\mathscr{E}}) \otimes (\zeta_{\mathscr{E}}{\uparrow}\mathbf{Z})\,, \tag{17}$$

Setting $t_{\mathscr{E}_\zeta} := \zeta_{\mathscr{E}}{\uparrow}t_{\mathscr{E}} = t_{\mathscr{E}} \circ \zeta_{\mathscr{E}}^{-1}$, the commutativity property

$$\zeta_{\mathscr{E}}{\uparrow} \circ d = d \circ \zeta_{\mathscr{E}}{\uparrow}\,, \tag{18}$$

implies that

$$dt_{\mathscr{E}_\zeta} = d(t_{\mathscr{E}} \circ \zeta_{\mathscr{E}}^{-1}) = d(\zeta_{\mathscr{E}}{\uparrow}t_{\mathscr{E}}) = \zeta_{\mathscr{E}}{\uparrow}dt_{\mathscr{E}}\,. \tag{19}$$

The naturality property $\langle dt_{\mathscr{E}_\zeta}, \mathbf{Z}_\zeta \rangle = \zeta_{\mathscr{E}}{\uparrow}\langle dt_{\mathscr{E}}, \mathbf{Z} \rangle = 1$ assures persistence of tuning. ■

Trajectories and motions $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}\,;\mathscr{T})$ and $(\zeta{\uparrow}\varphi)_\alpha \in \mathrm{C}^1(\mathscr{T}_\zeta\,;\mathscr{T}_\zeta)$, evaluated in frames related by a trajectory transformation $\zeta \in \mathrm{C}^1(\mathscr{T}\,;\mathscr{T}_\zeta)$, are related by the commutative diagram

$$\begin{CD} \mathscr{T}_\zeta @>{(\zeta\uparrow\varphi)_\alpha}>> \mathscr{T}_\zeta \\ @A{\zeta}AA @AA{\zeta}A \\ \mathscr{T} @>{\varphi_\alpha}>> \mathscr{T} \end{CD} \qquad \Longleftrightarrow \qquad (\zeta{\uparrow}\varphi)_\alpha \circ \zeta = \zeta \circ \varphi_\alpha\,. \tag{20}$$

Definition 4.1 (Frame-invariance). *A tensor field on the trajectory manifold* $\mathbf{s} \in \mathrm{C}^1(\mathscr{T}\,;\mathrm{TENS}(T\mathscr{T}))$ *is frame-invariant under the action of a trajectory transformation* $\zeta \in \mathrm{C}^1(\mathscr{T}\,;\mathscr{T}_\zeta)$ *if it varies by push*

$$\mathbf{s}_\zeta = \zeta{\uparrow}\mathbf{s}\,. \tag{21}$$

A relation involving tensor fields is frame-invariant if it transforms by push. The pushed relation is defined by the property that its fulfilment by the involved tensor fields holds if and only if the pulled-back tensor fields do fulfill the original relation.

Lemma 4.2 (Frame-invariance of trajectory velocity). *The trajectory velocity is frame-invariant:* $\mathbf{V}_\zeta = \zeta{\uparrow}\mathbf{V}$.

Proof. Being $\mathbf{V} := \partial_{\alpha=0}\,\varphi_\alpha$ and $\mathbf{V}_\zeta := \partial_{\alpha=0}\,(\zeta_{\mathscr{E}}{\uparrow}\varphi)_\alpha$, the direct computation

$$\mathbf{V}_\zeta = \partial_{\alpha=0}\,(\zeta \circ \varphi_\alpha \circ \zeta_{\mathscr{T}}^{-1}) = T\zeta \cdot \mathbf{V} \circ \zeta_{\mathscr{T}}^{-1} = \zeta{\uparrow}\mathbf{V}\,, \tag{22}$$

gives the formula. ∎

Lemma 4.3 (Immersion and push of vector fields). *Spatial vectors according to a framing* $\mathbf{R}$*, when pushed by a change of frame, are still spatial vectors in the pushed framing* $\zeta_{\mathscr{E}}{\uparrow}\mathbf{R}$ *as expressed by the commutative diagram*

$$\begin{CD} T\mathscr{E} @>{\zeta_{\mathscr{E}}\uparrow}>> T\mathscr{E} \\ @A{\mathbf{i}\uparrow}AA @AA{\mathbf{i}_\zeta\uparrow}A \\ VS @>{\zeta_S\uparrow}>> VS_\zeta \end{CD} \qquad \Longleftrightarrow \qquad \mathbf{i}_\zeta{\uparrow} \circ \zeta_S{\uparrow} = \zeta_{\mathscr{E}}{\uparrow} \circ \mathbf{i}{\uparrow}\,. \tag{23}$$

The spatial bundle isomorphism $\zeta_S{\uparrow} \in \mathrm{C}^1(VS;VS_\zeta)$ *is induced by the space-time push* $\zeta_{\mathscr{E}}{\uparrow} \in \mathrm{C}^1(T\mathscr{E};T\mathscr{E})$ *according to a change of frame* $\zeta_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E};\mathscr{E})$*. The inverse isomorphism is* $\zeta_S{\downarrow} \in \mathrm{C}^1(VS_\zeta;VS)$.[4]

[4] The isomorphism $\zeta_S{\uparrow} \in \mathrm{C}^1(VS;VS_\zeta)$ is not the tangent map of an automorphism of the event manifold $\mathscr{E}$, unless restriction to a spatial slice is considered, see the proof of Lemma 4.4.

Proof. The push of forms is defined by invariance

$$\langle \zeta_{\mathscr{E}}\uparrow dt_{\mathscr{E}}, \zeta_{\mathscr{E}}\uparrow \mathbf{X} \rangle = \zeta_{\mathscr{E}}\uparrow \langle dt_{\mathscr{E}}, \mathbf{X} \rangle, \quad \forall \mathbf{X} \in \mathrm{C}^1(\mathscr{E}; T\mathscr{E}), \tag{24}$$

and hence $\langle dt_{\mathscr{E}}, \mathbf{X} \rangle = 0 \quad \Longrightarrow \quad \langle \zeta_{\mathscr{E}}\uparrow dt_{\mathscr{E}}, \zeta_{\mathscr{E}}\uparrow \mathbf{X} \rangle = 0$. ■

Lemma 4.4 (Simultaneity preservation). *Simultaneous events according to an initial framing* $\mathbf{R}$ *are transformed, by frame-changes in event manifold, into simultaneous events according to the pushed framing* $\mathbf{R}_\zeta = \zeta_{\mathscr{E}}\uparrow \mathbf{R}$.

Proof. The integral manifolds of a time-vertical field $\zeta_{\mathscr{E}}\uparrow \mathbf{X} \in \mathrm{C}^1(\mathscr{E}; V\mathscr{E}_\zeta)$ are space-slices got as $\zeta_{\mathscr{E}}$-images of the integral manifolds of the corresponding time-vertical field $\mathbf{X} \in \mathrm{C}^1(\mathscr{E}; V\mathscr{E})$. Then frame-changes transform simultaneous events in the initial frame into simultaneous events according to the pushed frame. It follows that the restriction of $\zeta_S\uparrow$ to a space-slice is equal to the push according to ζ_S transformation between spatial slices, induced by the $\zeta_{\mathscr{E}}$-transformation. ■

Definition 4.2 (Adapted frames). *A frame is a set of tangent vector fields which gives a basis at each point. A frame is* adapted *to a framing if one family of coordinate lines is envelop of the time-arrow field and the other three families define reference systems in the spatial slicings.*

In an adapted space-time frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ with $\mathbf{X}_0 = \mathbf{Z}$, vectors in $V\mathscr{E}$ will have a zero first component. We will denote by $\{\mathbf{X}^0, \mathbf{X}^1, \mathbf{X}^2, \mathbf{X}^3\}$ the dual frame, that is the quartet of one-forms such that

$$\langle \mathbf{X}^i, \mathbf{X}_j \rangle = \delta^i_j, \qquad i, j = 0, 1, 2, 3. \tag{25}$$

5 PUSHED NATURAL FRAMES

With reference to Def. 2.2, let us consider in the event manifold the natural frame $\partial_\alpha := Tf \cdot 1_\alpha$, with $\alpha = 0, 1, 2, 3$, associated with a coordinate system $f : \mathscr{R}^4 \mapsto \mathscr{E}$ and the dual frame d^α.

A tuned framing $\mathbf{R} = dt_{\mathscr{E}} \otimes \mathbf{Z}$ is then naturally induced by expressing the time projection as 0-th coordinate by the formula $t_{\mathscr{E}} := \pi_0 \circ f^{-1} : \mathscr{E} \mapsto \mathscr{R}_0$, and setting $\mathbf{Z} = \partial_0$. Then

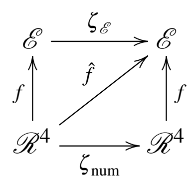
$$\begin{cases} dt_{\mathscr{E}} = d(\pi_0 \circ f^{-1}) = d\pi_0 \cdot Tf^{-1} = d_0, \\ \mathbf{Z} \circ f = \partial_0 \circ f = Tf \cdot 1_0, \end{cases} \tag{26}$$

and, with $i = 1,2,3$,

$$\begin{cases} \langle dt_{\mathscr{E}}, \mathbf{Z} \rangle = \langle d_0, \partial_0 \rangle = \langle d\pi_0 \cdot Tf^{-1}, Tf \cdot 1_0 \rangle = 1\,, \\ \langle dt_{\mathscr{E}}, \partial_i \rangle = \langle d_0, \partial_i \rangle = \langle d\pi_0 \cdot Tf^{-1}, Tf \cdot 1_i \rangle = 0\,. \end{cases} \tag{27}$$

So tuning is fulfilled and the frame vectors ∂_i are tangent to spatial slices.

A change of frame $\zeta_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E};\mathscr{E})$ is usually described by fixing a reference system $f : \mathscr{R}^4 \mapsto \mathscr{E}$ and considering the corresponding coordinate transformation $\zeta_{\text{num}} \in \mathrm{C}^1(\mathscr{R}^4;\mathscr{R}^4)$ defined by $\zeta_{\text{num}} := f^{-1} \circ \zeta_{\mathscr{E}} \circ f$. An equivalent description is got by considering another reference system $\hat{f} : \mathscr{R}^4 \mapsto \mathscr{E}$ defined by $\hat{f} := \zeta_{\mathscr{E}} \circ f$, as shown by the commutative diagram



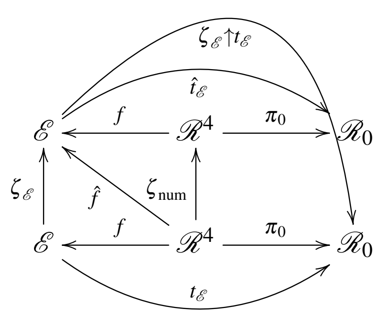
$$\Longleftrightarrow \quad \hat{f} := \zeta_{\mathscr{E}} \circ f = f \circ \zeta_{\text{num}}\,. \tag{28}$$

Under the frame change $\zeta_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E};\mathscr{E})$ induced by a reference system $f : \mathscr{R}^4 \mapsto \mathscr{E}$ and by a coordinate transformation $\zeta_{\text{num}} \in \mathrm{C}^1(\mathscr{R}^4;\mathscr{R}^4)$, the time projection in the pushed framing and its differential are given by

$$\begin{aligned} t_{\mathscr{E}_\zeta} &:= \zeta_{\mathscr{E}}{\uparrow} t_{\mathscr{E}} = t_{\mathscr{E}} \circ \zeta_{\mathscr{E}}^{-1} = \pi_0 \circ \hat{f}^{-1}\,, \\ dt_{\mathscr{E}_\zeta} &= d(\zeta_{\mathscr{E}}{\uparrow} t_{\mathscr{E}}) = \zeta_{\mathscr{E}}{\uparrow} dt_{\mathscr{E}} = dt_{\mathscr{E}} \cdot T\zeta_{\mathscr{E}}^{-1} = d\pi_0 \cdot T\hat{f}^{-1}\,. \end{aligned} \tag{29}$$

1. The pushed time projection $\zeta_{\mathscr{E}}{\uparrow} t_{\mathscr{E}} : \mathscr{E} \mapsto \mathscr{R}_0$ is then the 0-th coordinate of transformed events according to the transformed reference system $\hat{f} = f \circ \zeta_{\text{num}}$.
2. The time projection $\hat{t}_{\mathscr{E}}$ is instead the 0-th coordinate of transformed events according to the original reference system f.

The relation between these time projections is depicted by the commutative diagram



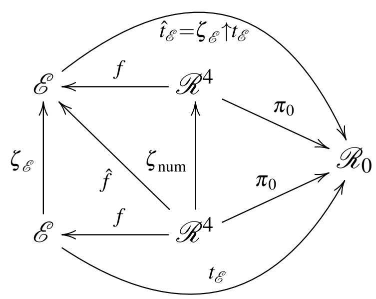
$$\Longleftrightarrow \quad \begin{aligned} &i) \quad t_{\mathscr{E}} = \pi_0 \circ f^{-1}\,, \\ &ii) \quad \zeta_{\mathscr{E}}{\uparrow} t_{\mathscr{E}} = \pi_0 \circ \hat{f}^{-1}\,, \\ &iii) \quad \hat{t}_{\mathscr{E}} \circ \zeta_{\mathscr{E}} \circ f = \pi_0 \circ \zeta_{\text{num}}\,. \end{aligned} \tag{30}$$

Then $\hat{t}_{\mathscr{E}} \circ \zeta_{\mathscr{E}} \circ f \circ \zeta_{\text{num}}^{-1} = t_{\mathscr{E}} \circ f = \pi_0$.

A peculiar property of GALILEI (or more in general EUCLID) frame changes is that the coordinate transformation $\zeta_{\text{num}} \in \mathrm{C}^1(\mathscr{R}^4; \mathscr{R}^4)$ fulfills the following *time universality property*

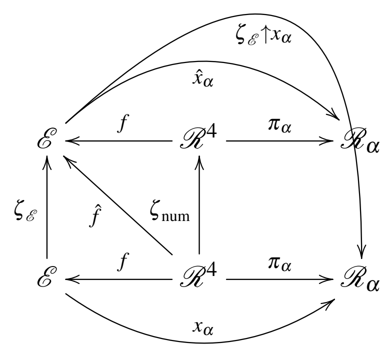
$$\pi_0 = \pi_0 \circ \zeta_{\text{num}} . \tag{31}$$

Then $\hat{t}_{\mathscr{E}} = t_{\mathscr{E}_\zeta} := \zeta_{\mathscr{E}} \uparrow t_{\mathscr{E}}$, as shown by diagram (32) below.



$$\iff \quad \begin{aligned} i)\quad & t_{\mathscr{E}} = \pi_0 \circ f^{-1} , \\ ii)\quad & \hat{t}_{\mathscr{E}} = \zeta_{\mathscr{E}} \uparrow t_{\mathscr{E}} = \pi_0 \circ \hat{f}^{-1} \end{aligned} \tag{32}$$

The diagram (30) can be extended to all coordinates as exemplified by the following diagram



(33)

6 CLOCK RATE AND TIME LAPSE

In a framing $\mathbf{R} = dt_{\mathscr{E}} \otimes \mathbf{Z}$ the physical notions of time line and of clock and clock rate are expressed by the following geometric definitions.

Definition 6.1 (Time-line path). *A time-line path* $\mathbf{p} : t \in [0, \Delta t] \to \mathscr{E}$ *is a segment of a time-line, i.e. an integral curve of the field of time-arrows, so that path velocities are time-arrows*

$$\mathbf{p}'(t) := \partial_{\mu=t}\, \mathbf{p}(\mu) = (\mathbf{Z} \circ \mathbf{p})(t) . \tag{34}$$

Definition 6.2 (Clock, Rate and Time-lapse). *The* clock *is the time differential* $dt_{\mathscr{E}}$ *and the* clock rate *is the evaluation of the time-differential along the time-arrow. In tuned framings the clock rate is unitary. The* time-lapse *is the integral of the clock rate along a time-line path* $\mathbf{p} : t \in [0, \Delta t] \to \mathscr{E}$

$$\begin{aligned}\int_{\mathbf{p}} dt_{\mathscr{E}} &:= \int_0^{\Delta t} \langle\, dt_{\mathscr{E}} \circ \mathbf{p}(t), \mathbf{p}'(t)\,\rangle\, dt \\ &= \int_0^{\Delta t} \langle\, dt_{\mathscr{E}}, \mathbf{Z}\rangle \circ \mathbf{p}(t)\, dt\,.\end{aligned} \tag{35}$$

In a tuned framing, being $\langle\, dt_{\mathscr{E}}, \mathbf{Z}\rangle = 1$, the time-lapse is equal to the time increment

$$\int_{\mathbf{p}} dt_{\mathscr{E}} = \int_0^{\Delta t} \langle\, dt_{\mathscr{E}}, \mathbf{Z}\rangle \circ \mathbf{p}(t)\, dt = \Delta t\,. \tag{36}$$

According to Lemma 4.1 tuning is preserved by a change from a given framing $\mathbf{R} = dt_{\mathscr{E}} \otimes \mathbf{Z}$ to the pushed framing $\mathbf{R}_{\zeta} = \zeta_{\mathscr{E}}{\uparrow}\mathbf{R}$ since

$$\begin{aligned}\langle\, dt_{\mathscr{E}}, \mathbf{Z}\rangle &= \langle\, \zeta_{\mathscr{E}}{\uparrow} dt_{\mathscr{E}}, \zeta_{\mathscr{E}}{\uparrow}\mathbf{Z}\rangle = \langle\, d(\zeta_{\mathscr{E}}{\uparrow} t_{\mathscr{E}}), \zeta_{\mathscr{E}}{\uparrow}\mathbf{Z}\rangle \\ &= \langle\, dt_{\mathscr{E}_{\zeta}}, \zeta_{\mathscr{E}}{\uparrow}\mathbf{Z}\rangle = 1\,.\end{aligned} \tag{37}$$

This result can be enunciated by stating that the clock rates measured in push-related framings are the same.

On the other hand, tuning may be lost under a change of frame in which the framings are not push-related.

In a pushed framing $\mathbf{R}_{\zeta} = \zeta_{\mathscr{E}}{\uparrow}\mathbf{R}$ the time-lapse evaluated along a pushed time-line path $\zeta_{\mathscr{E}} \circ \mathbf{p} : t \in [0, \Delta t] \to \mathscr{E}$ is the same as the time-lapse evaluated in the framing $\mathbf{R}$ along the corresponding time-line path $\mathbf{p} : t \in [0, \Delta t] \to \mathscr{E}$ by the formula Eq. (35). Indeed, by the commutativity property Eq. (18) and by the integral transformation formula it follows that

$$\begin{aligned}\int_{\zeta_{\mathscr{E}} \circ \mathbf{p}} dt_{\mathscr{E}_{\zeta}} &= \int_{\zeta_{\mathscr{E}} \circ \mathbf{p}} d(\zeta_{\mathscr{E}}{\uparrow} t_{\mathscr{E}}) \\ &= \int_{\zeta_{\mathscr{E}} \circ \mathbf{p}} \zeta_{\mathscr{E}}{\uparrow} dt_{\mathscr{E}} = \int_{\mathbf{p}} dt_{\mathscr{E}}\,.\end{aligned} \tag{38}$$

7 MINKOWSKI PSEUDO METRIC

A basic axiom of special relativity theory in physics is that no spatial speed can be greater than the spatial speed c of light *in vacuo*.

Denoting by $\mathbf{V} = \mathbf{v} + \mathbf{Z} \in \mathrm{C}^1(\mathscr{E}; T\mathscr{E})$ a space-time velocity vector field, with related spatial velocity field $\mathbf{v} = \mathbf{P}\mathbf{V}$, and by $\mathbf{g}_S \in \mathrm{C}^1(S; \mathrm{S}{\scriptsize \mathrm{YM}}(TS))$ the metric tensor field in the EUCLID spatial bundle, this basic axiom is expressed by the condition

$$\mathbf{g}_S(\mathbf{v},\mathbf{v}) \le c^2\,, \tag{39}$$

which defines a convex set of admissible spatial velocities in the tangent bundle TS.

The MINKOWSKI pseudo metric tensor field[5] $\mathbf{g}_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E}\,;\mathrm{SYM}(T\mathscr{E}))$ is defined in the event manifold $\mathscr{E}$ by

$$\mathbf{g}_{\mathscr{E}} := \mathbf{P}{\downarrow}\mathbf{g}_S - c^2\, dt_{\mathscr{E}} \otimes dt_{\mathscr{E}}\,. \tag{40}$$

Accordingly, the inequality Eq. (39) may be rewritten as a homogeneous condition on space-time velocities, defining the convex set

$$\mathbf{g}_{\mathscr{E}}(\mathbf{V},\mathbf{V}) \le 0\,. \tag{41}$$

The boundary of the cone generated by this convex set of space-time velocities is called the *light cone*. Recalling that $dt_{\mathscr{E}} \cdot \mathbf{PX} = 0$ for all $\mathbf{X} \in T\mathscr{E}$ and $\mathbf{PZ} = \mathbf{0}$ by definition of the projector $\mathbf{P}$ and $dt_{\mathscr{E}} \cdot \mathbf{Z} = 1$ by tuning, it may be concluded that for all $\mathbf{X},\mathbf{Y} \in T\mathscr{E}$ such that $\mathbf{RX} = \mathbf{RY} = \mathbf{Z}$ we have

$$\begin{aligned} \mathbf{g}_{\mathscr{E}}(\mathbf{X},\mathbf{Y}) &= \mathbf{g}_S(\mathbf{PX},\mathbf{PY}) - c^2\,, \\ \mathbf{g}_{\mathscr{E}}(\mathbf{PX},\mathbf{Z}) &= 0\,. \end{aligned} \tag{42}$$

In an adapted space-time frame $\{\mathbf{X}_0,\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$, with $\{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$ a $\mathbf{g}_S$-orthonormal spatial frame and $\mathbf{X}_0 = \mathbf{Z}$, the components of MINKOWSKI pseudo metric tensor field are

$$\begin{aligned} &\mathbf{g}_{\mathscr{E}}(\mathbf{X}_0,\mathbf{X}_0) = -c^2\,, \\ &\mathbf{g}_{\mathscr{E}}(\mathbf{X}_0,\mathbf{X}_i) = 0\,, \qquad i = 1,2,3\,, \\ &\mathbf{g}_{\mathscr{E}}(\mathbf{X}_i,\mathbf{X}_j) = \delta_{i,j}\,, \quad i,j = 1,2,3\,. \end{aligned} \tag{43}$$

The GRAM matrix is then

$$\begin{bmatrix} -c^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{44}$$

[5] A pseudo-metric tensor field fulfills the properties of a metric field, except positivity.

8 VOIGT-LORENTZ TRANSFORMATIONS

In the affine event manifold of special relativity a frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ adapted to a framing $\mathbf{R} = dt_{\mathscr{E}} \otimes \mathbf{Z}$ has the first vector given by $\mathbf{X}_0 = \mathbf{Z}$ and the tangent vectors $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ got by immersion of the orthonormal frame $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in the spatial slices.

Coordinates are classically denoted by $\{t, x, y, z\}$. Denoting by

$$f : \{t, x, y, z\} \in \mathscr{R}^4 \to \mathbf{e} \in \mathscr{E}\,, \tag{45}$$

a reference system for the event manifold and acting with the tangent functor, we get the diagrams

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\quad \zeta_{\mathscr{E}} \quad} & \mathscr{E} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ \mathscr{R}^4 & \xrightarrow{\zeta_{\mathrm{num}} = f^{-1} \circ \zeta_{\mathscr{E}} \circ f} & \mathscr{R}^4 \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} T\mathscr{E} & \xrightarrow{\quad T\zeta_{\mathscr{E}} \quad} & T\mathscr{E} \\ {\scriptstyle Tf}\uparrow & & \uparrow{\scriptstyle Tf} \\ T\mathscr{R}^4 & \xrightarrow{T\zeta_{\mathrm{num}} = T^{-1}f \cdot T\zeta_{\mathscr{E}} \cdot Tf} & T\mathscr{R}^4 \end{array} \tag{46}$$

where $T\zeta_{\mathrm{num}}$ is the JACOBI matrix associated with the coordinate transformation.

A VOIGT-LORENTZ frame transformation $\zeta_{\mathscr{E}}^{\mathrm{VL}} \in \mathrm{C}^1(\mathscr{E}\,;\mathscr{E})$ enjoy the characteristic property that

- spatial spherical light wave fronts evolving in a given frame still remain spherical in the same frame when the involved events are subject to such a transformation. Moreover the light propagation speed c *in vacuo* is not affected.

In the sequel these will be called VL-transformations. According to Minkowski (1908), these transformations, introduced by Lorentz (1904) and named after him by Poincaré (1906), were conceived about two decades before by Voigt (1887). The transformations were later reported by Einstein (1905) without quoting VOIGT, LORENTZ and POINCARÉ.

The expression of a linear VL-transformation and of the corresponding coordinate transformation $\zeta_{\mathrm{num}}^{\mathrm{VL}} : \mathscr{R}^4 \mapsto \mathscr{R}^4$ according to a coordinate system $f : \mathscr{R}^4 \mapsto \mathscr{E}$, is deduced as follows.

Let us start from a linear coordinate transformation [6]

$$\begin{cases} \hat{t} = \gamma t + \beta x\,, \\ \hat{x} = \sigma x + \theta t\,, \end{cases} \tag{47}$$

and define the relative translational speed w by

$$\begin{cases} \quad w = \dfrac{\partial x}{\partial t}\Big|_{\hat{x}=const} = -\dfrac{\theta}{\sigma}\,, \\ -w = \dfrac{\partial \hat{x}}{\partial \hat{t}}\Big|_{x=const} = \dfrac{\theta}{\gamma}\,. \end{cases} \tag{48}$$

Then $\theta = -w\gamma = -w\sigma$ and hence $\sigma = \gamma$. Setting $\beta = \xi\gamma$ in Eq. (47) we get

$$\begin{cases} \hat{t} = \gamma(t + \xi x)\,, \\ \hat{x} = \gamma(x - wt)\,. \end{cases} \tag{49}$$

Substituting into the condition (Poincaré, 1906)

$$\hat{x}^2 - c^2\hat{t}^2 = x^2 - c^2 t^2\,, \tag{50}$$

the l.h.s. becomes

$$\begin{aligned} \hat{x}^2 - c^2\hat{t}^2 = \gamma^2(&x^2 + w^2 t^2 - 2wxt \\ &- c^2 t^2 - c^2 x^2 \xi^2 - 2c^2 \xi x t)\,. \end{aligned} \tag{51}$$

The terms linear in x cancel if $w = -c^2\xi$ and this gives

$$\begin{cases} \hat{t} = \gamma(t - x\,(w/c^2))\,, \\ \hat{x} = \gamma(x - w\,t)\,. \end{cases} \tag{52}$$

Finally from Eq. (50) and Eq. (51), equating the coefficient of x^2 and t^2, we get the conditions

$$\begin{cases} \gamma^2(1 - w^2/c^2) = 1\,, \\ \gamma^2(c^2 - w^2) = c^2\,, \end{cases} \tag{53}$$

which are both fulfilled by setting

$$\gamma = \gamma_w := (1 - w^2/c^2)^{-\frac{1}{2}}\,. \tag{54}$$

[6] Linearity of the transformation is inessential since what in fact enters in the analysis is the tangent map of the transformation. This issue is dealt with in Sect. 9.

We may conclude that a VOIGT-LORENTZ coordinate transformation corresponding to a relative speed w in the x direction, is expressed by

$$\zeta^{\mathrm{VL}}_{\mathrm{num}} \cdot \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma_w(t - x(w/c^2)) \\ \gamma_w(x - wt) \\ y \\ z \end{bmatrix} \tag{55}$$

whose JACOBI matrix is given by

$$T\zeta^{\mathrm{VL}}_{\mathrm{num}} = \begin{bmatrix} \gamma_w & -\gamma_w\,(w/c^2) & & \\ -\gamma_w\,w & \gamma_w & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{56}$$

The inverse transformation is got by changing w into $-w$, so that γ_w is unchanged.

9 TANGENT VOIGT-LORENTZ TRANSFORMATION

The characteristic property of a nonlinear VOIGT-LORENTZ transformation $\zeta^{\mathrm{VL}}_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E};\mathscr{E})$ consists in the invariance of the MINKOWSKI pseudo-metric tensor field, expressed by the condition $\zeta^{\mathrm{VL}}_{\mathscr{E}}\!\downarrow\!\mathbf{g}_{\mathscr{E}} = \mathbf{g}_{\mathscr{E}}$ and explicitly

$$\mathbf{g}_{\mathscr{E}}(T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{V}, T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{V}) = \mathbf{g}_{\mathscr{E}}(\mathbf{V},\mathbf{V}), \quad \forall \mathbf{V} \in T\mathscr{E}. \tag{57}$$

The condition Eq. (57) assures that, if a space-time velocity field $\mathbf{V} \in T\mathscr{E}$ fulfils the basic inequality Eq. (41), the transformed velocity field $\zeta^{\mathrm{VL}}_{\mathscr{E}}\!\uparrow\!\mathbf{V} \in T\mathscr{E}$ will still fulfil the same inequality.

Let us choose as space-time frame $\{\mathbf{X}_\alpha\}$ the adapted natural frame $\partial_\alpha := Tf \cdot 1_\alpha$, with $\alpha = 0,1,2,3$, associated with the coordinate system $f : \mathscr{R}^4 \mapsto \mathscr{E}$, so that $\mathbf{X}_0 = \partial_0 = \mathbf{Z}$. The tangent functor applied to Eq. (28) gives

$$T\zeta_{\mathscr{E}} \cdot \partial_\alpha = T\zeta_{\mathscr{E}} \cdot Tf \cdot 1_\alpha = Tf \cdot T\zeta_{\mathrm{num}} \cdot 1_\alpha = T\hat{f} \cdot 1_\alpha . \tag{58}$$

Dropping the inessential vector fields $\mathbf{X}_2$ and $\mathbf{X}_3$, the components of $\mathbf{V} \in T\mathscr{E}$ and $\hat{\mathbf{V}} := \zeta^{\mathrm{VL}}_{\mathscr{E}}\!\uparrow\!\mathbf{V} \in T\mathscr{E}$ in the space-time frame $\{\mathbf{X}_\alpha\}$ are denoted by

$$\begin{cases} \mathbf{V} = v_Z\,\mathbf{Z} + v_S\,\mathbf{X}_1 , \\ \hat{\mathbf{V}} = \hat{v}_Z\,\mathbf{Z} + \hat{v}_S\,\mathbf{X}_1 . \end{cases} \tag{59}$$

with the following tangent transformation relating the velocity components

$$\begin{bmatrix} \hat{v}_Z \\ \hat{v}_S \end{bmatrix} = T\zeta^{\mathrm{VL}}_{\mathrm{num}} \begin{bmatrix} v_Z \\ v_S \end{bmatrix} = \begin{bmatrix} \gamma & \beta \\ \theta & \sigma \end{bmatrix} \cdot \begin{bmatrix} v_Z \\ v_S \end{bmatrix} \tag{60}$$

Setting $v_S = 0$, we get $\hat{v}_S = \theta\, v_Z$ and $\hat{v}_Z = \gamma v_Z$ so that the relative speed w is given by $-w = \hat{v}_S/\hat{v}_Z = \theta/\gamma$.

Setting $\hat{v}_S = 0$, we get $\theta\, v_Z + \sigma\, v_S = 0$ so that $w = v_S/v_Z = -\theta/\sigma$. Then $\sigma = \gamma$ and posing $\beta = \xi\gamma$ we write

$$\begin{bmatrix} \hat{v}_Z \\ \hat{v}_S \end{bmatrix} = \gamma \begin{bmatrix} 1 & \xi \\ -w & 1 \end{bmatrix} \cdot \begin{bmatrix} v_Z \\ v_S \end{bmatrix} \tag{61}$$

and the isometry requirement Eq. (57) amounts to the identity

$$\begin{aligned} \hat{v}_S^2 - c^2\hat{v}_Z^2 &= \gamma^2(v_S^2 + w^2 v_Z^2 - 2wv_Sv_Z \\ &\quad -c^2v_Z^2 - c^2v_S^2\xi^2 - 2c^2\xi v_Sv_Z) \\ &= v_S^2 - c^2v_Z^2\,. \end{aligned} \tag{62}$$

The terms linear in v_S cancel by setting $w = -c^2\xi$ and fulfilment of Eq. (62) implies that $\gamma = \gamma_w := (1 - w^2/c^2)^{-\frac{1}{2}}$. Hence Eq. (61) becomes

$$\begin{bmatrix} \hat{v}_Z \\ \hat{v}_S \end{bmatrix} = \gamma_w \begin{bmatrix} 1 & -w/c^2 \\ -w & 1 \end{bmatrix} \cdot \begin{bmatrix} v_Z \\ v_S \end{bmatrix} = \gamma_w \begin{bmatrix} v_Z - v_S(w/c^2) \\ v_S - v_Z w \end{bmatrix} \tag{63}$$

Since $v_Z = 1$, the ratio between space and time components of the pushed space-time velocity Eq. (63) yields the law of relativistic composition of spatial velocities

$$\frac{\hat{v}_S}{\hat{v}_Z} = \frac{v_S - w}{1 - v_S w/c^2}\,. \tag{64}$$

The transformation of the basis vectors $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ induced by the tangent transformation $T\zeta^{\mathrm{VL}}_{\mathrm{num}}$ is given by

$$\begin{bmatrix} T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{X}_0 \\ T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{X}_1 \end{bmatrix} = \gamma_w \begin{bmatrix} 1 & -w \\ -w/c^2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \end{bmatrix} \tag{65}$$

that is

$$\begin{cases} T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{X}_0 = \gamma_w\,(\mathbf{Z} - w\mathbf{X}_1)\,, \\ T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{X}_1 = \gamma_w\,(\mathbf{X}_1 - (w/c^2)\,\mathbf{Z})\,, \end{cases} \tag{66}$$

and $T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{X}_\alpha = \mathbf{X}_\alpha\,, \quad \text{for} \quad \alpha = 2,3\,.$

A tangent vector $\mathbf{X} \in T\mathscr{E}$ is called

- *Time-like* if $\mathbf{g}_{\mathscr{E}}(\mathbf{X},\mathbf{X}) < 0$,
- *Light-like* if $\mathbf{g}_{\mathscr{E}}(\mathbf{X},\mathbf{X}) = 0$,
- *Space-like* if $\mathbf{g}_{\mathscr{E}}(\mathbf{X},\mathbf{X}) > 0$.

9.1. Group property

The composition of two VOIGT-LORENTZ transformations in the same longitudinal direction, has a JACOBI matrix given by (only the upper 2×2 submatrix is considered)

$$\begin{aligned}\gamma_u\,\gamma_w \begin{bmatrix}1 & w/c^2\\ w & 1\end{bmatrix}\begin{bmatrix}1 & u/c^2\\ u & 1\end{bmatrix}\\ = \gamma_u\,\gamma_w \begin{bmatrix}1+uw/c^2 & (u+w)/c^2)\\ (u+w) & 1+uw/c^2\end{bmatrix}\end{aligned} \tag{67}$$

The relativistic composition of spatial velocities is performed, according to Eq. (64), by the function

$$\mathrm{S}(u\,,w) := \frac{u+w}{1+uw/c^2}\,, \tag{68}$$

and so, by definition Eq. (54), it follows that

$$\gamma_{\mathrm{S}(u,w)} := \Big(1-\Big(\frac{u/c+w/c}{1+uw/c^2}\Big)^2\Big)^{-1/2}. \tag{69}$$

Setting $\alpha := u/c$ and $\beta := w/c$ we may write

$$\begin{aligned}(\gamma_u\,\gamma_w)^{-2} &= (1-\alpha^2)(1-\beta^2) = 1-(\alpha^2+\beta^2)+\alpha^2\beta^2\,,\\ \gamma_{\mathrm{S}(u,w)}^{-2} &= 1-(\alpha+\beta)^2/(1+\alpha\beta)^2\\ &= (1-(\alpha^2+\beta^2)+\alpha^2\beta^2)/(1+\alpha\beta)^2\,,\\ \mathrm{S}(u\,,w) &= \frac{\alpha+\beta}{1+\alpha\beta}\,c^2\,,\end{aligned} \tag{70}$$

whence the equalities

$$\gamma_{\mathrm{S}(u,w)} = \gamma_u\,\gamma_w\,(1+uw/c^2)\,, \tag{71}$$

$$\gamma_{\mathrm{S}(u,w)}\mathrm{S}(u\,,w) = \gamma_u\,\gamma_w\,(u+w)\,, \tag{72}$$

and formula Eq. (67) may be rewritten as

$$\begin{aligned}\gamma_u\,\gamma_w \begin{bmatrix}1 & w/c^2\\ w & 1\end{bmatrix}\begin{bmatrix}1 & u/c^2\\ u & 1\end{bmatrix}\\ = \gamma_{\mathrm{S}(u,w)} \begin{bmatrix}1 & \mathrm{S}(u\,,w)/c^2)\\ \mathrm{S}(u\,,w) & 1\end{bmatrix}\end{aligned} \tag{73}$$

revealing the basic group property of VOIGT-LORENTZ transformations due to Poincaré (1906).

10 RELATIVISTIC EFFECTS

We are now able to comment in a direct manner on some relativistic effects, diffusely discussed in literature since the very beginning of the relativistic era, at the turn from the nineteenth to the twentieth century. It will be seen that usual and consolidated statements ought to be modified.

10.1. Time dilation

Definition 10.1 (World line and Proper time). *A world-line segment is a path* $\mathbf{p} : [0, \Delta\lambda] \to \mathscr{E}$ *in the events manifold such that the corresponding path-velocity field* $\mathbf{p}' : [0, \Delta\lambda] \to T\mathscr{E}$*, defined by* $\mathbf{p}'(\lambda) := \partial_{\mu=\lambda}\, \mathbf{p}(\mu)$*, is time-like. The* proper time-lapse*, associated with a world-line segment is the time-length evaluated according to* MINKOWSKI *pseudo metric tensor field, given by* [7]

$$\begin{aligned}\Delta t_{\text{PROPER}} &= \frac{1}{c} \int_{\mathbf{p}} \sqrt{\ } \circ (-\mathbf{g}_{\mathscr{E}}) \circ \text{DIAG} \\ &= \frac{1}{c} \int_0^{\Delta\lambda} (-\mathbf{g}_{\mathscr{E}}(\mathbf{p}'(\lambda), \mathbf{p}'(\lambda)))^{1/2}\, d\lambda = \frac{1}{c} \Delta s\,.\end{aligned} \tag{74}$$

The parameter s*, characterised by the property* $-\mathbf{g}_{\mathscr{E}}(\mathbf{p}'(s), \mathbf{p}'(s)) = 1$*, is the curvilinear abscissa according to* MINKOWSKI *pseudo metric.*

By the invariance property $\zeta^{\text{VL}}_{\mathscr{E}}\!\downarrow\!\mathbf{g}_{\mathscr{E}} = \mathbf{g}_{\mathscr{E}}$ it follows that the proper time is invariant under VL-transformations. Moreover if the path is a time-line, as defined by Def. 6.1, being $-\mathbf{g}_{\mathscr{E}}(\mathbf{Z}, \mathbf{Z}) = c^2$, we have that

$$\begin{aligned}\Delta t_{\text{PROPER}} &:= \frac{1}{c} \int_0^{\Delta t} (-\mathbf{g}_{\mathscr{E}}(\mathbf{p}'(t), \mathbf{p}'(t)))^{1/2}\, dt \\ &= \frac{1}{c} \int_0^{\Delta t} (-\mathbf{g}_{\mathscr{E}}(\mathbf{Z}, \mathbf{Z}))^{1/2} \circ \mathbf{p}(t)\, dt = \Delta t\,.\end{aligned} \tag{75}$$

From Eq. (36) we infer that, in a tuned framing and along a time-line, the elapsed proper time defined by Eq. (74) is equal to the time-lapse defined by Eq. (35), because both are equal to the time increment Δt.

In special relativity the change of frame induced by a VL-transformation $\zeta^{\text{VL}}_{\mathscr{E}} \in \text{C}^1(\mathscr{E}; \mathscr{E})$ is evaluated according to Eq. (66).

If the rate of the clock of an observer is evaluated along the transformed time-arrow detected by another observer in relative translational motion, the result is

$$\langle dt_{\mathscr{E}}, \zeta^{\text{VL}}_{\mathscr{E}}\!\uparrow\!\mathbf{Z} \rangle = \langle dt_{\mathscr{E}}, T\zeta^{\text{VL}}_{\mathscr{E}} \cdot \mathbf{Z} \rangle = \gamma_w \langle dt_{\mathscr{E}}, \mathbf{Z} \rangle = \gamma_w\,. \tag{76}$$

[7] The diagonal map $\text{DIAG} : T\mathscr{E} \mapsto_{\mathscr{E}} T\mathscr{E} \otimes T\mathscr{E}$ is defined by $\text{DIAG}(\mathbf{X}) := (\mathbf{X}, \mathbf{X})$.

The evaluation in Eq. (76) may be enunciated by stating that

- The time *rate* attributed to a *moving* observer, by clocks of the *rest* observer, is *faster* than the time rate measured by the *rest* observer, with amplification according to the relativistic factor.[8]

In a tuned framing $\mathbf{R} = dt_{\mathscr{E}} \otimes \mathbf{Z}$, the time-lapse evaluated by a rest clock along a time-line path $\mathbf{p} : t \in [0, \Delta t] \to \mathscr{E}$, and the time-lapse evaluated by a field of clocks of the framing $\mathbf{R}$ along the path $(\zeta_{\mathscr{E}}^{\mathrm{VL}} \circ \mathbf{p}) : t \in [0, \Delta t] \to \mathscr{E}$ pushed by a VL-transformation $\zeta_{\mathscr{E}}^{\mathrm{VL}} \in \mathrm{C}^1(\mathscr{E}; \mathscr{E})$, are related by

$$\begin{aligned} &\int_{\zeta_{\mathscr{E}}^{\mathrm{VL}} \circ \mathbf{p}} dt_{\mathscr{E}} = \int_0^{\Delta t} \langle dt_{\mathscr{E}}, T\zeta_{\mathscr{E}}^{\mathrm{VL}} \cdot \mathbf{Z} \rangle \circ \mathbf{p}\, d\lambda = \gamma_w \Delta t \\ &> \int_{\mathbf{p}} dt_{\mathscr{E}} = \int_0^{\Delta t} \langle dt_{\mathscr{E}}, \mathbf{Z} \rangle \circ \mathbf{p}\, d\lambda = \Delta t\,. \end{aligned} \tag{77}$$

In this evaluation resort was made to tuning $\langle dt_{\mathscr{E}}, \mathbf{Z} \rangle = 1$, to verticality $\langle dt_{\mathscr{E}}, \mathbf{X}_1 \rangle = 0$, and to the formulae

$$\begin{aligned} \partial_{\mu=\lambda}\, \mathbf{p}(\mu) &= (\mathbf{Z} \circ \mathbf{p})(\lambda), \\ \partial_{\mu=\lambda}\, (\zeta_{\mathscr{E}} \circ \mathbf{p})(\mu) &= (T\zeta_{\mathscr{E}} \cdot \mathbf{Z}) \circ \mathbf{p}(\lambda) \\ &= \gamma_w (\mathbf{Z} - w\mathbf{X}_1) \circ \mathbf{p}(\lambda)\,. \end{aligned} \tag{78}$$

We may conclude as follows. The *time-lapse* measured by a field of clocks of an observer along the time line of another observer in relative motion is dilated by the relativistic factor (*time dilation*) when compared with the *time-lapse* measured by a single clock along the time line of the former observer.

We remark that the difference between these time-lapses is due to the fact that the time lapse is evaluated by means of a field of clocks having, with respect to a given observer, variable spatial positions located along the time line of another observer in relative motion.[9]

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[8] This result is *not* in agreement with the statement in (Schutz, 2009, p.19) where, following (Einstein, 1905, p.10), it is said: *the clocks of* $\overline{\mathscr{O}}$ *will be measured by* $\mathscr{O}$ *to be running more slowly than those of* $\mathscr{O}$. The disagreement will be commented in Sect. 11.

[9] The fairy tale in (Schutz, 2009, 1.13) named *The twin "paradox" dissected*, may be commented upon as follows. DIANA leaves ARTEMIS and goes away for a trip at a speed $w = (24/25)c$ so that $\gamma^{-1} = 0.28$. DIANA's clock measures a time lapse of 14 when she comes back to ARTEMIS. According to Eq. (76), ARTEMIS makes the wrong evaluation that DIANA's clock is faster and estimates a time lapse of $14/.28 = 50$. The roles of DIANA and ARTEMIS can be interchanged. All this is however uninfluential and in fact DIANA and ARTEMIS will both measure on their own clocks a time lapse of 14, irrespective of whether they were or not in relative motion so that, as foreseen by the theory leading to Eq. (38), with reassurance of common sense, they still have the same age at *rendez vous*. The twins paradox is just the consequence of an incorrect conclusion.

10.2. Longitudinal length dilation

The *longitudinal-length* of a path of events is defined to be the corresponding increment of longitudinal abscissa.

Longitudinal means that abscissa develops along the direction of the relative spatial velocity between the frames related by a VL-transformation.

The *longitudinal-length* is sometimes referred to simply as the *length*, but this abuse of nomenclature may become a source of error, because *longitudinal-length* and *length* are the same only if spatial longitudinal straight paths are considered.

Let us now consider a longitudinal path of events, that is a parameterized path $\mathbf{p} : [0, \Delta\lambda] \to \mathscr{E}$ with velocity $\mathbf{p}'(\lambda) := \partial_{\mu=\lambda}\, \mathbf{p}(\mu) = \mathbf{X}_1$.

The longitudinal-length of the path in a frame adapted to a framing $\mathbf{R} := dt_{\mathscr{E}} \otimes \mathbf{Z}$, is evaluated by the integral

$$\int_{\mathbf{p}} \mathbf{X}^1 = \int_0^{\Delta\lambda} \langle\, \mathbf{X}^1, \mathbf{p}'(\lambda)\,\rangle\, d\lambda\,, \tag{79}$$

where $\mathbf{X}^1$ is the covector field of the dual frame, in duality with the vector field $\mathbf{X}_1$, as defined by Eq. (25).

The abscissa increments along a path $\mathbf{p} : [0, \Delta\lambda] \to \mathscr{E}$ and along the transformed path $\zeta_{\mathscr{E}} \circ \mathbf{p} : [0, \Delta\lambda] \to \mathscr{E}$ are the same when evaluated in push-related natural frames because, for $\alpha = 0, 1, 2, 3$, we have

$$\begin{aligned} \int_{\zeta^{\mathrm{VL}}_{\mathscr{E}} \circ \mathbf{p}} \zeta_{\mathscr{E}}\!\uparrow\!\mathbf{X}^\alpha &= \int_0^{\Delta\lambda} \langle\, \zeta_{\mathscr{E}}\!\uparrow\!\mathbf{X}^\alpha, \partial_{\mu=\lambda}\, (\zeta_{\mathscr{E}} \circ \mathbf{p})(\mu)\,\rangle\, d\lambda \\ &= \int_0^{\Delta\lambda} \langle\, \zeta_{\mathscr{E}}\!\uparrow\!\mathbf{X}^\alpha, \zeta_{\mathscr{E}}\!\uparrow\!\mathbf{p}'(\lambda)\,\rangle\, d\lambda \\ &= \int_0^{\Delta\lambda} \langle\, \mathbf{X}^\alpha, \mathbf{p}'(\lambda)\,\rangle\, d\lambda = \int_{\mathbf{p}} \mathbf{X}^\alpha\,. \end{aligned} \tag{80}$$

The velocity of the transformed path $\zeta^{\mathrm{VL}}_{\mathscr{E}} \circ \mathbf{p} : [0, \Delta\lambda] \to \mathscr{E}$ is provided by formula Eq. $(66)_2$ and the longitudinal component is given by

$$\langle\, \mathbf{X}^1, \zeta^{\mathrm{VL}}_{\mathscr{E}}\!\uparrow\!\mathbf{X}_1\,\rangle = \langle\, \mathbf{X}^1, T\zeta^{\mathrm{VL}}_{\mathscr{E}} \cdot \mathbf{X}_1\,\rangle = \gamma_w\, \langle\, \mathbf{X}^1, \mathbf{X}_1\,\rangle = \gamma_w\,. \tag{81}$$

Hence

$$\begin{aligned} &\int_{\zeta^{\mathrm{VL}}_{\mathscr{E}} \circ \mathbf{p}} \mathbf{X}^1 = \gamma_w \int_0^{\Delta\lambda} \langle\, \mathbf{X}^1, \mathbf{X}_1\,\rangle\, dx = \gamma_w\, \Delta\lambda \\ &> \int_{\mathbf{p}} \mathbf{X}^1 = \int_0^{\Delta\lambda} \langle\, \mathbf{X}^1, \mathbf{X}_1\,\rangle\, dx = \Delta\lambda\,. \end{aligned} \tag{82}$$

We may thus conclude that

- An observer who compares the *longitudinal length* of a path of *simultaneous* events with the *longitudinal length* of the pushed path of *non-simultaneous* events according to a VL-transformation, founds that the latter is increased by the relativistic factor (*longitudinal length dilation*).

11 CONCLUSIONS

Turning points outcoming from the analysis of VL-transformations may be resumed as follows.

1. The rate of a clock and the corresponding time-lapse, measured by a clock at rest, i.e. located at a fixed spatial position, appear to be increased by the relativistic factor $\gamma_w := (1 - w^2/c^2)^{-1/2}$ if the measure is performed by nearby clocks acting on the time-line of an observer moving with relative speed w (this effect is called *time dilation* because $\gamma_w \geq 1$). If the relative spatial speed w tends to the light speed c, then the dilated clock-rates and time-lapses tend to infinity.
2. The longitudinal length of a small line of simultaneous events at rest, in the direction of the relative spatial velocity $\mathbf{w}$, appears to be increased by the relativistic factor $\gamma_w := (1 - w^2/c^2)^{-1/2}$ when the longitudinal length is measured along transformed events (*length dilation*). When the relative spatial speed w tends to the light speed c, the longitudinal length tends to infinity.

As one can see, there is a perfect similarity between the two statements, as there is also between the treatments of *time dilation* and *longitudinal length dilation* effects carried out in Sect. 6 and Sect. 10.2.

The former effect of *time dilation* was first enunciated by Larmor (1897).

Our result is in accord with the findings of classical treatments of relativistic effects, such as (Lorentz, 1904), (Einstein, 1905), (Born, 1932, p.209) (Feynman, 1964, 15.4 p.154), (Landau and Lifshits, 1987, §3 p.8) and the recent one (Schutz, 2009, 1.8, p.17) for what concerns the fictitious dilation of time intervals, but with *faster* clocks instead of *slower* clocks. Indeed faster clocks measure longer time intervals and are therefore in accord with the *time dilation* effect.

In (Einstein, 1905, p.10) Sect. 4: *Physical Meaning of the Equations Obtained in Respect to Moving Rigid Bodies and Moving Clocks*, the statement concerning clock rates consists in the following affirmation (in our notations):

Between the quantities x, t *and* $\hat{t}$*, which refer to the position of the clock, we have, evidently,* $x = wt$ *and* $\hat{t} = \gamma_w(t - x(w/c^2))$*. Therefore* $\hat{t} = \gamma_w^{-1} t$*. Whence it follows that the time marked by the clock (viewed in the stationary system) is slow....*

The statement that the clock in a moving frame has a slower rate is however an incorrect conclusion.

The reason is that the relation $x = wt$ tells us that the clock is moving with relative velocity w with respect to the frame $\{t, x, y, z\}$ and hence that it stays in the fixed spatial position $\hat{x} = \gamma_w(x - wt) = 0$ in the other frame. The final relation $\hat{t} = \gamma_w^{-1} t$ gives therefore exactly the opposite answer: *the moving clock (which measures the time* t *) runs faster than the stationary clock.*

In the previous reasoning the clock located at the fixed position $\hat{x} = 0$ measures a time-lapse $\Delta\hat{t}$. According to Eq. $(84)_1$, the corresponding time-lapse $\Delta t = \gamma_w \Delta\hat{t}$ is measured by clocks located at points whose abscissae range in the interval $\Delta x = \gamma_w (w \Delta\hat{t})$.

In fact clock rates ought to be evaluated by a clock at rest, that is by a clock located at a fixed spatial position and acting on the time-arrow based at that position. Therefore the *faster* rate is in fact a fictitious rate, real rates being invariant under any change of frame, as shown by Eq. (37).

The latter of our findings is the effect of *longitudinal length dilation* which is in opposition to the relativistic *length contraction* effect, enunciated by FitzGerald (1889) and confirmed three years later by Lorentz (1892).

It is instructive to investigate about the proof of the relativistic *longitudinal length contraction* reported in (Einstein, 1905, p.10).

It consists in considering a spherical surface of radius R at rest in a frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ with orthonormal spatial subframe $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$.

In terms of the corresponding coordinates $\{t, x, y, z\}$, the space-time trajectory is described by the implicit equation

$$x^2 + y^2 + z^2 = R^2 . \tag{83}$$

Equation (83) represents, for each fixed t, a spatial spherical surface in the frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$. According to the inverse VL-transformation

$$\begin{cases} t = \gamma_w(\hat{t} + \hat{x}\,(w/c^2)) \\ x = \gamma_w(\hat{x} + w\,\hat{t}\,) \end{cases} \tag{84}$$

and $y = \hat{y}, z = \hat{z}$, the equation of the trajectory Eq. (83) in the transformed coordinates $\{\hat{t}, \hat{x}, \hat{y}, \hat{z}\}$ is given by

$$\gamma_w^2\,(\hat{x} + w\,\hat{t}\,)^2 + \hat{y}^2 + \hat{z}^2 = R^2 . \tag{85}$$

Equation (85) represents, for each fixed $\hat{t}$, a spatial ellipsoidal surface in the frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$, with longitudinal radius of length R/γ_w.

The argument in support of FITZGERALD-LORENTZ *longitudinal length contraction*, then reads (Einstein, 1905, p.10): *A rigid body which, measured in a state of rest, has the form of a sphere, therefore has in a state of motion –viewed from the stationary system– the form of an ellipsoid of revolution with the axes* $(R\sqrt{1-w^2/c^2}, R, R)$ *...omissis... the X dimension of the sphere appears shortened by the motion in the ratio* $1 : \sqrt{1-w^2/c^2}$.

This interpretation of the correspondence between the trajectories Eq. (83) and Eq. (85) is however incorrect.

To see this, let us recall that a nontrivial VL-transformation maps a body (that is a set of events laying in a spatial slice of the observer) into a set of events which appear to the same observer as non-simultaneous.

Accordingly, the observer who measures at a fixed time $\hat{t} = \hat{t}_0$ a longitudinal diameter of length $\Delta\hat{x} = \hat{x}_2 - \hat{x}_1 = 2R/\gamma_w$ between the points at the longitudinal abscissae $\hat{x}_1 = -R/\gamma_w - w\hat{t}_0$ and $\hat{x}_2 = R/\gamma_w - w\hat{t}_0$, of the spatial ellipsoidal surface Eq. (85), will evaluate a difference of longitudinal abscissae $\Delta x = x_2 - x_1 = 2R$, resulting from measurements on the transformed events. Indeed, according to Eq. (84) the longitudinal abscissae will be $x_1 = -R$ at time $t_1 = -Rw/c^2 + \hat{t}_0/\gamma_w$ and $x_2 = R$ at time $t_2 = +Rw/c^2 + \hat{t}_0/\gamma_w$. The increased longitudinal length $\Delta x = 2R$ results then from non-simultaneous measurements with a time lapse $\Delta t = t_2 - t_1 = 2Rw/c^2$. In a symmetric way, according to the direct VL-transformation

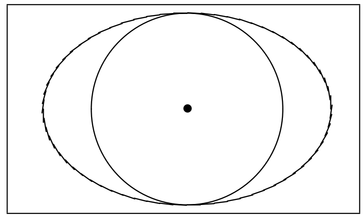
$$\begin{cases} \hat{t} = \gamma_w(t - x(w/c^2)) \\ \hat{x} = \gamma_w(x - wt) \end{cases} \tag{86}$$

and $y = \hat{y}, z = \hat{z}$, the observer who measures at a fixed time $t = t_0$ a longitudinal diameter of length $\Delta x = x_2 - x_1 = 2R$ of the spatial spherical surface Eq. (83) between the points $x_1 = -R$ and $x_2 = R$, will evaluate a difference $\Delta\hat{x} = \hat{x}_2 - \hat{x}_1 = 2R\gamma_w$ while measuring the longitudinal abscissae of the transformed events. Indeed, according to Eq. $(86)_1$, these abscissae are evaluated to be $\hat{x}_1 = \gamma_w(-R - wt_0)$ at time $\hat{t}_1 = \gamma_w(t_0 + Rw/c^2)$ and $\hat{x}_2 = \gamma_w(R - wt_0)$ at time $\hat{t}_2 = \gamma_w(t_0 - Rw/c^2)$. The increased longitudinal length $\Delta\hat{x}$ refers then to non-simultaneous measurements made in the time lapse $\Delta\hat{t} = 2\gamma_w Rw/c^2$.

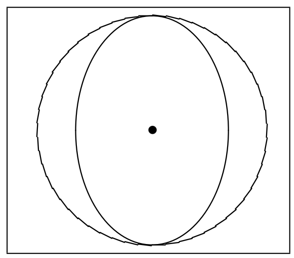
The relativistic effect of *longitudinal length dilation* is exemplified in sketches below.

In the former, a sphere of simultaneous events is evaluated to be a longitudinally dilated ellipsoidal surface resulting from the spatial projection of the transformed, non-simultaneous events.

Analogously, in the latter, the longitudinally oblated ellipsoidal surface of simultaneous events is evaluated to be a sphere resulting from the spatial projection of the transformed, non-simultaneous events.



Longitudinal length dilation of a sphere



Longitudinal length dilation of an ellipsoid

An argument, similar to the one in (Einstein, 1905, p.10), is classically exposed to evaluate the changes in the longitudinal length measurements performed by an observer on a bar at rest and on the bar image detected by another observer in relative longitudinal motion.

The evaluation, reported in classical references on relativity such as (Weyl, 1922, p.183), (Born, 1932, p.208), (Landau and Lifshits, 1987, p.11), has been reproduced till recently, see e.g. (Ferrarese and Bini, 2008, 2.11, p.50) and (Schutz, 2009, (1.11) p.18).

In these treatments, the length $L = x_2 - x_1$ of a longitudinal bar at rest (i.e. the distance of two simultaneous events located at two very near points along the longitudinal axis) is compared with the longitudinal length $\hat{L} = \hat{x}_2 - \hat{x}_1$ of the bar image, according to the VL-transformation, at a time $\hat{t}$.

Then from, Eq. (84)

$$L = x_2 - x_1 = \gamma_w(\hat{x}_2 - \hat{x}_1) = \gamma_w \hat{L}, \tag{87}$$

and the exposed conclusion is that the longitudinal length $\hat{L}$ of the transformed bar image, is smaller than the one L of the bar at rest (FITZGERALD-LORENTZ *contraction*).

This conclusion is however affected by a subtle but decisive flaw, because L cannot be considered as the length of the bar at rest inasmuch as the end point abscissae x_1 and x_2, as ruled by Eq. $(84)_1$, correspond to distinct instants of time, $t_1 = \gamma_w(\hat{t} + \hat{x}_1\,(w/c^2))$ and $t_2 = \gamma_w(\hat{t} + \hat{x}_2\,(w/c^2))$.

The correct conclusion is that the length $L = x_2 - x_1$ of the bar at rest at time t (the time is measured by a clock located at the same spatial position of the small bar) is smaller than the longitudinal length $\hat{L}$ of the bar image according to the VL-transformation, which by Eq. $(86)_2$ is given by

$$\hat{L} = \hat{x}_2 - \hat{x}_1 = \gamma_w(x_2 - x_1) = \gamma_w L. \tag{88}$$

with the abscissae $\hat{x}_1$ and $\hat{x}_2$ corresponding to two distinct time instants, $\hat{t}_1 = \gamma_w(t + x_1\,(w/c^2))$ and $\hat{t}_2 = \gamma_w(t + x_2\,(w/c^2))$.

The fact that both time and longitudinal length are dilated by a VL-transformation is also a direct consequence of the presence of the common factor γ_w in the JACOBI matrix Eq. (65).

This amplifying factor does in fact affect both the time component of the transformed time-arrow and the spatial component of transformed spatial vectors.

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