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The geometry of nonlinear elasticity

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Abstract Elasticity is the prototype of constitutive models in Continuum Mechanics. In the nonlinear range, the elastic model claims for a geometrically consistent physico-mathematical formulation providing also the logical premise for linearized approximations. A theoretic framework is envisaged here with the aim of contributing a conceptually clear, physically consistent, and computationally convenient formulation. A reasoning about the physics of the model, from a geometric point of view, leads to conceive constitutive relations as instantaneous incremental responses to a finite set of tensorial state variables and to their time rates along the space-time motion. Integrability of the tangent elastic compliance, existence of an elastic stress potential, and conservativeness of the elastic response, under the conservation of mass, are given a brand new treatment. Finite elastic strains have no physical interpretation in the new rate theory, and referential local placements are appealed to, just as loci for operations of linear calculus. Frame invariance is assessed with a consistent geometric treatment, and the clear distinction between the new notion and the property of isotropy is pointed out, thus overcoming the improper statement of material frame indifference. Extension of the theory to elasto-visco-plastic constitutive models is briefly addressed. Basic computational steps are described to illustrate feasibility and convenience of calculations according to the new theory of elasticity.

1 Introduction

Elasticity, as described in modern treatments, can be dated back to the foundational treatments by Cauchy [1,2] and Green [3,4] in the first half of the nineteenth century. Cauchy's treatment was concerned with the linearized small displacement theory, while finite strains were considered by Green.

We develop hereafter a conceptual and methodological revisitation of the theory of elasticity, with the aim of contributing to the resolution of difficulties in present treatments dealing with the general case in which no linearizing approximations are made. To motivate the need for this revisitation, let us address some significant points.

In most treatments of elastic and elasto-plastic mechanical models, in the presence of finite displacements, it is still affirmed that a rate formulation of elasticity clashes against the inherent lack of conservativeness of

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the constitutive model which could lead to perform a nonvanishing elastic work in closed cycles of deformation. This comment is made without a clear mathematical treatment or without even mentioning the relevant bibliographic source, most times with the more or less implicit statement that *it is well-known that ...* A quite recent instance of such unsupported claim can be found in [5].

The conviction above is based on the formulation of the hypo-elastic model due to Truesdell [6] and stems, with any evidence, from the analysis made by Bernstein in [7] and subsequently referred to in [8–11].

The issue has been first addressed and positively answered in full generality in [12]. The hypo-elastic model can be suitably reformulated, and explicit, applicable conservativeness conditions can be enunciated. Moreover, the long-lasting debate, about definitions of time rates of the stress tensor along the motion, can be given a definitive and univocal answer.

With these results at hand, the rate theory of elasticity can be displayed without conceptual and operational difficulties.

The ensuing theory is also free from the troublesome task of giving a reasonable definition of a priori local reference placements. In this respect, we observe that the introduction of reference placements can only be avoided in the context of a rate theory, where geometric objects involved in constitutive relations are tensor fields on body placements, and evaluated on the trajectory at fixed time during the motion (hereafter called *material* tensor fields).

Local reference placements are usefully considered as effective computational tools to perform mathematical operations in a linear context, either for theoretical or for numerical intents.

1. Theoretical proofs involving material tensors may be conveniently carried out by push forward to a local reference placement. There operations of linear calculus, such as differentiations or integrations, are performed, and the results pertaining to the final time instant are pulled back to the actual trajectory, at the same time instant. An instance is provided in Sect. 14 where the procedure is applied to prove the connection between hyper-elasticity and conservativeness.
2. Numerical computations in the nonlinear range are carried out by iterative algorithms based on the discretization of the evolution process into a finite number of time steps. The data pertaining to the beginning of a time step are pushed forward to a local reference placement where linear computations are carried out. The results pertaining to the end of the time step are then pulled back to the actual trajectory to provide the value of the material tensors on the final placement of the body.

The procedure sketched at item 2 is reminiscent of the one adopted in the finite element method to solve problems of Continuum Mechanics.

A peculiar feature of the rate formulation is that no constitutive meaning is attributed to notions which do not pertain to a definite time instant and therefore are not represented by material tensors.

Such is, for instance, the notion of finite elastic strain, which is obtained by integration in a time step of the push forward of the elastic stretching to a local reference placement. The result is therefore a referential notion, which cannot be pulled back to the trajectory at a definite placement, as discussed in Sect. 13.

A careful revisitation is also devoted to the comparison of constitutive relations formulated by observers in relative motion. The standard approach to this issue, essentially due to Noll [13], is in fact affected by two main difficulties.

1. The former is that in evaluating the transformation law for the deformation gradient, under a change of observer, it is assumed that the local reference placement, where the fist leg of the deformation gradient is located, is not operated on by the transformation. A way out of this inconsistency was shown by Liu in [14].
2. The latter difficulty concerns the principle of material frame indifference (MFI) and is more essential. The reason is that distinct observers provide distinct visions of the trajectory manifold, the only communication channel between these two separate points of view being provided by the relative motion between observers. As discussed in [15] and illustrated below in Sect. 16, this argument leads to the need of substituting the MFI principle, which is concerned with only one constitutive operator. The new principle of Constitutive Frame Invariance (CFI) deals instead with two constitutive operators, detected by the observers in relative motion and clarifies also the basic distinction from the notion of isotropy of material response.

Another addressed point concerns the evaluations of time rates of material tensors along the motion. This is a most debated topic in nonlinear Continuum Mechanics which is fully discussed in Sect. 18.

The geometric analysis developed in the paper has also led to commenting upon well-known topics in Continuum Mechanics, such as the Kirchhoff stress tensor and the Piola-Kirchhoff referential tensor, whose expressions are revised, respectively, in Remark 1 of Sect. 7 and in Sect. 9.

2 State of the art

In the general context of finite displacements, the state of the art presently referred to in the literature is the one contributed in *The Non-Linear Field Theories of Mechanics* (NLFTM) by Truesdell and Noll [16]. The finite strain constitutive theory of elasticity exposed in [16], Sect. 43, stipulates between the Cauchy stress tensor $\mathbf{T}$ and the deformation gradient $\mathbf{F}$ the relation

$$\mathbf{T} = \mathbb{G}(\mathbf{F}). \tag{1}$$

A rate theory is also exposed in [16], Sect. 99, based on the original proposal by Truesdell [6] under the name *hypo-elasticity*

$$\dot{\mathbf{T}} = \mathbb{H}(\mathbf{T}, \mathbf{L}(\mathbf{v})), \tag{2}$$

with $\dot{\mathbf{T}}$ material time derivative and $\mathbf{L}(\mathbf{v}) = \nabla\mathbf{v}$ velocity gradient.

The *reduction* argument adduced by Noll [17], relying on a previous work by Richter [18], is usually applied to rewrite the finite strain elastic law in a reference local placement as

$$\mathbf{S} = \mathbb{G}_{\mathrm{REF}}(\mathbf{U}), \tag{3}$$

and the hypo-elastic law as

$$\overset{\circ}{\mathbf{T}} = \mathbb{H}(\mathbf{T}) \cdot \mathbf{D}(\mathbf{v}); \tag{4}$$

$\mathbf{S}$ is the symmetric Piola-Kirchhoff referential stress,
$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ is the referential right stretch,
$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{W}(\mathbf{v})\,\mathbf{T} - \mathbf{T}\,\mathbf{W}(\mathbf{v})$ is the Jaumann co-rotational stress-rate,
$\mathbf{D}(\mathbf{v}) = \operatorname{sym}\nabla\mathbf{v}$ is the stretching,
$\mathbb{H}$ is the elastic tangent stiffness, nonlinearly dependent on the stress $\mathbf{T}$.

The *reduction* procedure is based on an appeal to the principle of *Material Frame Indifference* (MFI) enunciated by Noll in [13]. A careful analysis reveals, however, that the formal expression of the principle of MFI is affected by a geometrically improper interpretation of the relation between points of view of distinct observers [15]. The correct geometric formulation of frame invariance leads to the new principle of *Constitutive Frame Invariance* (CFI) as substitute of the MFI, with the consequence that *reduction* procedures are not feasible. These items will be discussed in Sect. 16.

As explicitly observed in [16], Sect. 80, the elasticity map $\mathbb{G}$ in Eq. (1) depends on the choice of a reference local placement. Consequently, the theory requires an assumption concerning invariance with respect to this choice. But this invariance eventually amounts to assume that the elasticity map $\mathbb{G}$ does not depend at all on the deformation gradient. Moreover, *not* discussed in [16] are the following issues.

1. The formula (1) states a relation between a tensor $\mathbf{T}$, based at an event on the trajectory, and a two point tensor $\mathbf{F}$, pertaining to a pair of events. This contradiction cannot be resolved just by imposing an invariance property, as observed above. To be more explicit about this comment, one should imagine to perform a thought experiment to evaluate the constitutive properties of an elastic material. Assuming that the stress state and its time rate are evaluated by means of statical measurements and theoretical reasonings and that nonelastic phenomena are excluded by a careful testing procedure, the dual state variable, allowed to enter in the description of the material behavior at that time, is the time rate of change of metric properties, the *stretching*. A finite strain or a deformation gradient do on the contrary refer to a start and to a target body placement. The latter is the current placement, while the former is not detectable by laboratory tests.
2. The formula (1) should better describe the change in the elastic deformation in response to a given change of stress. In this way, the geometric change of deformation can be directly evaluated as the sum of various contributions described by different inelastic constitutive responses to changes of various state variables, such as stress, temperature, electromagnetic fields, and internal structural parameters.

The assumption that the deformation gradient is a driving factor in describing the constitutive behavior of elastic materials, embodied in (1), contrasts with the physical evidence that materials do not react to isometric displacements. Noll's *reduction* argument, intended to eliminate the incongruence, is a belated remedy based on an appeal to the geometrically incorrect statement of MFI, as evidenced above. The further remedy, adduced to include plasticity and other inelastic behaviors by means of a chain decomposition of the deformation gradient into inelastic and elastic parts [19,20], was worse than the disease. Indeed, concerning intermediate local placements and ordering of parts in the chain, troubles soon began and are still persistent after some fifty years. This clear indication of inadequacy of the proposal was not effective in dissuading many valuable researchers from perseverating, and now, the poisoning remedy has risen to the role of *deus ex machina* in formulating geometrically nonlinear constitutive behaviors [21].

The reason why it was and is still commonly considered to be difficult to give up with the untenable chain decomposition of the deformation gradient is that a satisfactory rate theory of elasticity was not at hand.

The hypo-elastic law expressed by Eq. (4) is indeed affected by drawbacks concerning the following issues.

1. In formula (4), the stress-rate $\overset{\circ}{\mathbf{T}}$ suffers from a longly debated intrinsic indeterminacy which cannot be resolved without a consistent geometric treatment.
2. In order to give to hypo-elasticity the physical role of a satisfactory elastic model, applicable integrability and conservativeness conditions are required.
3. The formula (4) should rather describe the rate of change in the elastic response to a given rate of change of the stress. In fact, rates of change in the response may well be due to causes other than stress rates of change, such, for instance, as temperature rates of change, and the geometric stretching will in general also include rate inelastic responses of the material.

Items 1 and 2 were already treated in [16], Sects. 99, 100, but the indeterminacy was not resolved, being rather accepted as an unavoidable feature of rate theories. Integrability was discussed in [7] by performing a comparison between the hypo-elastic law (2) and the time derivative of the elastic law (1) leading to problematic conclusions. The same treatment of integrability was also adopted, but with a simpler exposition, in [22]. These unsuccessful investigations led researchers involved in computational issues to strive to abandon the rate elastic model [8–11].

All these difficulties can, however, be overcome by undertaking a new, geometric line of attack to the problem, as is being evolving in a recent research activity [12,15,23–30].

The leading ideas are the following.

1. New physico-geometric notions of *material* and *spatial* fields, both defined on the trajectory manifold, are introduced. Constitutive properties are described in terms of material fields. Comparisons of material tensors at different times are performed in a natural way by push along the motion[1]. Spatial fields are instead to be compared by choosing a parallel transport along the motion in the trajectory manifold.
2. The *geometric stretching* is defined in a natural manner as the covariant tensor given by one-half the Lie derivative of the material metric tensor along the space-time motion.
3. The *stress* is described by a material contravariant tensor in duality with the *geometric stretching*. The duality interaction between stress and stretching provides the mechanical power per unit mass.
4. The elastic response is expressed in rate form by defining the *elastic stretching* as a covariant tensor depending nonlinearly on the *stress* and linearly on its time rate, the *stressing*, which is the Lie derivative of the stress tensor along the space-time motion.
5. The *geometric stretching* is assumed to be the result of the (commutative) addition of various physical contributions such as *elastic stretching*, *thermal stretching*, *visco-plastic stretching*, *phase-change stretching*, *electromagnetic stretching*, *growth stretching*, etc., provided by specific models of constitutive response in function of current values of the state variables, such as stress, temperature, and internal parameters, and of the relevant time rates along the motion.

A geometrically consistent constitutive theory can then be developed with integrability, frame invariance, and computational methods fully available, with a clear physical interpretation of the involved fields and with direct experimental strategies designable for testing material properties. These capabilities will be evidenced in the present paper with explicit reference to elasticity.

For the reader's convenience, an essential background of geometric notions and properties, referred to in the text, is provided in the Appendix.

[1] The notion of *naturality* is illustrated in detail in Sect. 4.1.

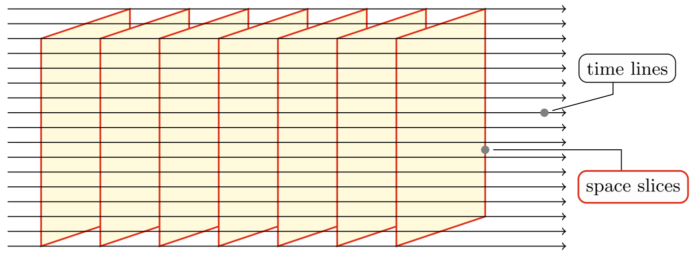


Fig. 1 Euclid space-time slicing

3 Continuum kinematics

The theory of Continuum Mechanics is best developed in the framework of a four-dimensional manifold of events $\mathbf{e} \in \mathcal{E}$.

Each observer performs a double foliation of the 4D events manifold into complementary 3D *space-slices* $\mathcal{S}$ of *isochronous* events (with a same corresponding time instant) and 1D *time-lines* $\mathcal{Z}$ of *isotopic* events (with a same corresponding spatial location).

Time lines do not intersect with one another, and each time line intersects a spatial slice just at one point. Analogously, spatial slices do not intersect with one another, and each spatial slice intersects a time line just at one point, as sketched in Fig. 1.

Each *time-line* is parametrized by time in such a way that a *time-projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{R}$ assigns the same time instant $t_{\mathcal{E}}(\mathbf{e}) \in \mathcal{R}$ to each event in a *spatial-slice*, that is

$$t_{\mathcal{E}}(\overline{\mathbf{e}}) = t_{\mathcal{E}}(\mathbf{e}), \quad \forall\, \overline{\mathbf{e}} \in \mathcal{S}. \tag{5}$$

The velocities of *time-lines* define the field of *time arrows* $\mathbf{Z} : \mathcal{E} \mapsto T\mathcal{E}$.[2]

The tangent space $T_{\mathbf{e}}\mathcal{E}$ at any event $\mathbf{e} \in \mathcal{E}$ is split into a complementary pair of a 3D time-vertical subspace $V_{\mathbf{e}}\mathcal{E}$ (tangent to a spatial slice) and a 1D time-horizontal subspace $H_{\mathbf{e}}\mathcal{E}$ (tangent to a time line) generated by a time arrow $\mathbf{Z} \in T_{\mathbf{e}}\mathcal{E}$.

The time-projection $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{R}$ and the time arrow $\mathbf{Z} \in T_{\mathbf{e}}\mathcal{E}$ can be *tuned* so that

$$\langle dt_{\mathcal{E}}, \mathbf{Z} \rangle = 1. \tag{6}$$

In the tangent bundle $T\mathcal{E}$, the *time-vertical* subbundle $V\mathcal{E}$ (*time-horizontal subbundle* $H\mathcal{E}$) is the disjoint union of all time-vertical (time-horizontal) subspaces. They are, respectively, called *space* bundle and *time* bundle.

In the familiar Euclid setting of classical Mechanics, the spatial slices and the *time-projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ are the same for all observers (universality of time).

A reference frame $\{\mathbf{e}_i\,;\, i = 0, 1, 2, 3\}$ for the event manifold is *adapted* if $\mathbf{e}_0 = \mathbf{Z}$ and $\mathbf{e}_i \in V\mathcal{E}$, $i = 1, 2, 3$.

Definition 1 (*Trajectory and motion*) The *trajectory manifold* is the geometric object investigated in Mechanics, characterized by embedding[3] $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$ into the event manifold $\mathcal{E}$ such that the image $\mathcal{T}_{\mathcal{E}} := \mathbf{i}(\mathcal{T})$ is a submanifold of $\mathcal{E}$. The *motion* along the trajectory

$$\{\boldsymbol{\varphi}_\alpha : \mathcal{T} \mapsto \mathcal{T},\ \alpha \in \mathcal{R}\} \tag{7}$$

is a simultaneity preserving one-parameter family of maps fulfilling the composition rule

$$\boldsymbol{\varphi}_\beta \circ \boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_{(\alpha+\beta)}, \tag{8}$$

for any pair of time-lapses $\alpha, \beta \in \mathcal{R}$. Each $\boldsymbol{\varphi}_\alpha : \mathcal{T} \mapsto \mathcal{T}$ is a *displacement*.

The trajectory can alternatively be considered as a manifold $\mathcal{T}$ by itself or as a submanifold $\mathcal{T}_{\mathcal{E}} = \mathbf{i}(\mathcal{T}) \subset \mathcal{E}$ of the event manifold. Then, a $(1+n)$ coordinate system is adopted on $\mathcal{T}$, while an adapted 4D space-time coordinate system in $\mathcal{E}$ is adopted on $\mathcal{T}_{\mathcal{E}}$.

The trajectory inherits from the events manifold the time projection $t_{\mathcal{T}} := t_{\mathcal{E}} \circ \mathbf{i} : \mathcal{T} \mapsto \mathcal{Z}$ which defines a time bundle whose fibers of simultaneous events are the *body placements*, as sketched in Fig. 2 for a 1D body.

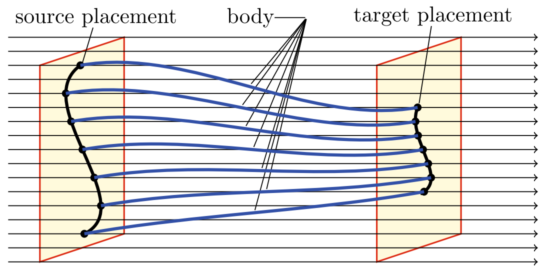


Fig. 2 Particles, body and placements

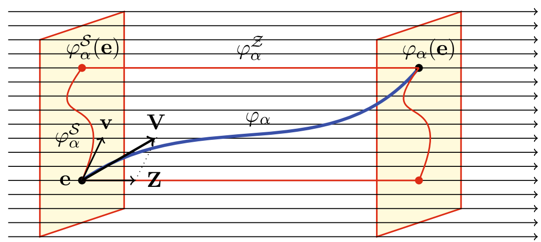


Fig. 3 Displacement decomposition

The space-time displacement $\boldsymbol{\varphi}_\alpha^{\mathcal{E}} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ and the trajectory displacement $\boldsymbol{\varphi}_\alpha : \mathcal{T} \mapsto \mathcal{T}$ are related by the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\mathcal{E}} \arrow[r, "\boldsymbol{\varphi}_\alpha^{\mathcal{E}}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{T}_{\mathcal{E}} \arrow[dd, bend left=60, "t_{\mathcal{E}}"] \\ \mathcal{T} \arrow[u, "\mathbf{i}"] \arrow[r, "\boldsymbol{\varphi}_\alpha"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[d, "t_{\mathcal{T}}"] \\ \mathcal{Z} \arrow[r, "\mathrm{TR}_\alpha"] & \mathcal{Z} \end{tikzcd} \tag{9}$$

where the time translation $\mathrm{TR}_\alpha : \mathcal{Z} \mapsto \mathcal{Z}$ is defined by

$$\mathrm{TR}_\alpha \circ t_{\mathcal{T}} := t_{\mathcal{T}} + \alpha, \quad \alpha \in \mathcal{R}. \tag{10}$$

Definition 2 (*Material particles and body manifold*) The physical notion of *material particle* corresponds in the geometric view to a time-parametrized curve of events in the trajectory, related by the motion as follows:

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathcal{T} \ : \ \mathbf{e}_2 = \boldsymbol{\varphi}_\alpha(\mathbf{e}_1). \tag{11}$$

Accordingly, we will say that a geometrical object is defined *along* (not *at*) a material particle. Events belonging to a *material particle* form a class of equivalence, and the quotient manifold so induced in the trajectory is the *body manifold*.

The notions of *material particle*, *body placement*, and *body manifold* are schematically depicted in Fig. 2.

As sketched in Fig. 3, to a space-time displacement $\boldsymbol{\varphi}_\alpha^{\mathcal{E}} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$, there corresponds a pair of maps:

1. a time-preserving *spatial displacement* $\boldsymbol{\varphi}_\alpha^{\mathcal{S}} : \mathcal{E} \mapsto \mathcal{E}$,
2. a location-preserving *time step* $\boldsymbol{\varphi}_\alpha^{\mathcal{Z}} : \mathcal{E} \mapsto \mathcal{E}$,

which fulfill the commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\boldsymbol{\varphi}_\alpha^{\mathcal{S}}"] \arrow[d, "\boldsymbol{\varphi}_\alpha^{\mathcal{Z}}"'] \arrow[dr, "\boldsymbol{\varphi}_\alpha^{\mathcal{E}}"] & \mathcal{E} \arrow[d, "\boldsymbol{\varphi}_\alpha^{\mathcal{Z}}"] \\ \mathcal{E} \arrow[r, "\boldsymbol{\varphi}_\alpha^{\mathcal{S}}"] & \mathcal{T} \end{tikzcd} \iff \boldsymbol{\varphi}_\alpha^{\mathcal{E}} = \boldsymbol{\varphi}_\alpha^{\mathcal{S}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{Z}} = \boldsymbol{\varphi}_\alpha^{\mathcal{Z}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{S}}. \tag{12}$$

[2] *Zeit* is the German word for *Time*.

[3] An embedding is an injective immersion, defined in Appendix, whose co-restriction is continuous with the inverse.

The spatial motion $\{\boldsymbol{\varphi}_\alpha^{\mathcal{S}} : \mathcal{E} \mapsto \mathcal{E},\ \alpha \in \mathcal{Z}\}$ is generated by intersecting each spatial slice with the time lines passing through the events of each material particle, as represented by thin red lines in Fig. 3.

The space-time *velocity* of the motion is defined by the derivative

$$\mathbf{V} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{E}} \in T\mathcal{T}_{\mathcal{E}} \tag{13}$$

Taking the time derivative of (12) and applying Leibniz rule, we get the decomposition into spatial and time components

$$\mathbf{V} = \mathbf{v} + \mathbf{Z}, \qquad \mathbf{v} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{S}}, \qquad \mathbf{Z} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathcal{Z}}. \tag{14}$$

Since from (9) $t_{\mathcal{E}} \circ \boldsymbol{\varphi}_\alpha^{\mathcal{E}} = \mathrm{TR}_\alpha \circ t_{\mathcal{E}}$, the space-time velocity has a unitary component on the time axis in an adapted frame. Indeed, from tuning (6), we infer that

$$\langle dt_{\mathcal{E}}, \mathbf{V}\rangle = \langle dt_{\mathcal{E}}, \mathbf{v}\rangle + \langle dt_{\mathcal{E}}, \mathbf{Z}\rangle = \langle dt_{\mathcal{E}}, \mathbf{Z}\rangle = 1. \tag{15}$$

Since the target is a general treatment of elasticity theory, the trajectory is allowed to be a $(1+n)$-dimensional submanifold of space-time with $n = 1, 2, 3$ to model, respectively, elastic wires, membranes, and balls.

This is a somewhat distinctive feature with respect to usual formulations of the theory, confined to 3D spatial models.

4 Spatial and material tensor fields

On the basis of the geometric framework set forth, new notions of spatial and material fields can be introduced in a natural way.

As a warning for the reader, we emphasize that these notions, which make no appeal to a local reference placement, do not comply with the homonymic nomenclature of usage in the literature, in the wake of the one adopted in [16].

Spatial and *material* vector fields are in fact both defined on body placements in the trajectory, the former being tangent to spatial slices, while the latter to body placements.

The space-time bundle $(T\mathcal{E})_{\mathcal{T}}$ is the restriction of the tangent bundle $T\mathcal{E}$ to vectors based on the trajectory manifold.

Definition 3 (*Spatial bundle*) The *spatial bundle* $(V\mathcal{E})_{\mathcal{T}}$ is the sub-bundle of the space-time bundle $(T\mathcal{E})_{\mathcal{T}}$ made of time-vertical tangent vectors

$$(V\mathcal{E})_{\mathcal{T}} := \{\mathbf{v}_{\mathcal{E}} \in (T\mathcal{E})_{\mathcal{T}} \quad \text{such that} \quad \langle dt_{\mathcal{E}}, \mathbf{v}_{\mathcal{E}}\rangle = 0\}. \tag{16}$$

Definition 4 (*Material bundle*) The *material bundle* $V\mathcal{T}$ is the sub-bundle of the tangent trajectory bundle $T\mathcal{T}$ made of time-vertical tangent vectors

$$V\mathcal{T} := \{\mathbf{v}_{\mathcal{T}} \in T\mathcal{T} \quad \text{such that} \quad \langle dt, \mathbf{v}_{\mathcal{T}}\rangle = 0\}, \tag{17}$$

and the immersed *material bundle* $V\mathcal{T}_{\mathcal{E}}$ is defined by

$$V\mathcal{T}_{\mathcal{E}} := \{\mathbf{v}_{\mathcal{E}} \in T\mathcal{T}_{\mathcal{E}} \quad \text{such that} \quad \langle dt_{\mathcal{E}}, \mathbf{v}_{\mathcal{E}}\rangle = 0\}. \tag{18}$$

Spatial and material tensors are multilinear maps acting, respectively, on spatial and material vectors [30].

Basic geometric notions are collected in the Appendix for useful reference. Some readily needed items are recalled hereafter[4].

A *covariant* (*contravariant*) second-order tensor is a bilinear scalar function of tangent (cotangent) vectors or equivalently a linear operator from the tangent (cotangent) to the cotangent (tangent) space.

A *mixed* tensor is a bilinear scalar function of a tangent and a cotangent vector or equivalently a linear operator from the tangent space to itself.

All tensor fields of interest in Continuum Mechanics are defined on the trajectory manifold and are therefore either *spatial* or *material* tensor fields, according to Definitions 3 and 4.

[4] In the literature, covariant, contravariant, and mixed components of tensors are still improperly considered. These notions pertain instead to the tensors themselves as scalar bilinear maps.

The only (and important) exception is the *metric tensor field* which is defined on the whole event manifold $\mathcal{E}$.

Acceleration, force, and metric are *spatial* vector, covector, and tensor fields. Stress, stressing, stretching, heat flux, temperature, and thermodynamical potentials are *material* tensor, vector, and scalar fields. Only material fields are allowed to enter in constitutive relations. These involve in fact material tensors and their time rates along the motion.

4.1 Natural comparison of material tensors

Definition 5 (*Naturality*) A notion concerning *material* tensors is said to be *natural* if it depends only on the *metric* properties of the event manifold and on the *motion*, no other arbitrary assumption (such as the choice of a parallel transport) being involved [15,30].

To perform the time derivative of a material tensor field along the motion, a transportation tool must be employed to bring the base point to the event in the trajectory corresponding to the evaluation time, prior to taking the derivative with respect to the time lapse.

In this respect, we underline that two material tensor fields $\mathbf{s}_1$ and $\mathbf{s}_2$, based at a same event $\mathbf{e} \in \mathcal{T}$ on the trajectory, are naturally compared by taking the difference between their evaluation on any pair of argument vectors based at that event.

The question, of how to compare material tensors based at distinct events along a particle on the trajectory, requires a more careful geometric examination. At its root, there is the question concerning the comparison of material vectors tangent to distinct placements of the body along a particle. We call attention to the following items.

1. The comparison requires the availability of a map apt to transform, in a linear and invertible way, a tangent vector based at an event along a particle, into another one based at the evaluation event, where subtraction can be operated upon.
2. The temptation of defining equality by parallel transport is to be rejected because an unnatural choice is involved. The same comment holds if equality is defined by invariance of cartesian components.
3. Parallel transport is moreover not feasible for lower dimensional bodies since parallel transported material vectors (tangent to a placement) will in general no more be material (tangent to the transformed placement), see Fig. 5.

The natural way to compare the values of a material tensor field along a particle consists in pulling back by the displacement map and leads to the following definition.

Definition 6 (*Material time invariance*) *Time-invariance* of a *material* tensor $\mathbf{s}_{\text{MAT}} \in \text{TENS}(V\mathcal{T})$ along the motion, means fulfilment of the pull-back relation

$$\mathbf{s}_{\text{MAT}} = \boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{s}_{\text{MAT}}. \tag{19}$$

According to this definition, the time rate is evaluated as *Lie derivative* along the motion, still a material tensor field, see Fig. 4 where two events belonging to the same particle P are considered.

Definition 7 (*Material time derivative*) The *time derivative* of a *material* tensor field $\mathbf{s}_{\text{MAT}} \in \text{TENS}(V\mathcal{T})$ is naturally provided by the Lie derivative along the motion

$$\dot{\mathbf{s}}_{\text{MAT}} := \mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\, \mathbf{s}_{\text{MAT}} := \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{s}_{\text{MAT}}). \tag{20}$$

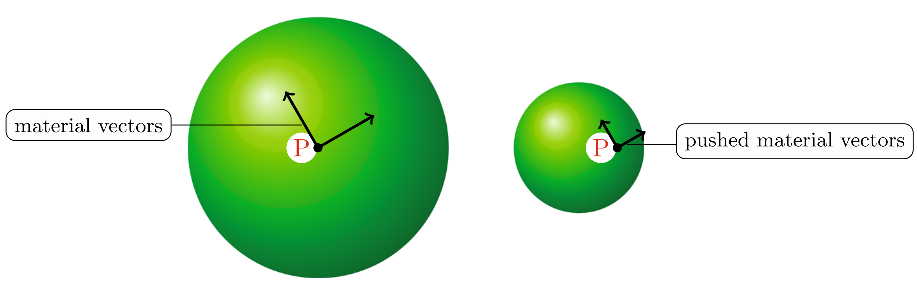


Fig. 4 Push of vectors tangent to a material surface

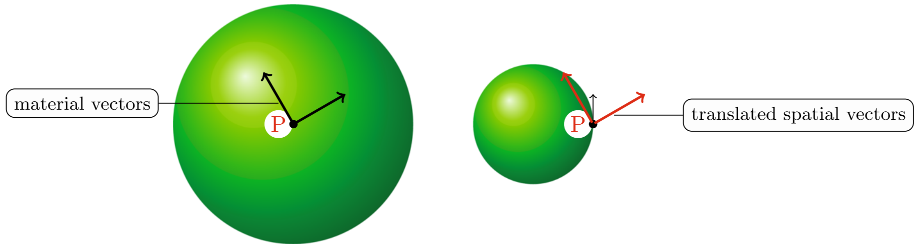


Fig. 5 Parallel transport of vectors tangent to a material surface

4.2 Comparison of spatial tensors by parallel transport

In general, the time derivative of a spatial tensor field along the motion cannot be performed by pull back along the motion, because, for lower dimensional bodies, the immersed material bundle is only a proper sub-bundle of the spatial tensor bundle, and therefore, the tangent displacement cannot operate on spatial vectors.

Accordingly, a *not natural* choice of spatial parallel transport in the event manifold $\mathcal{E}$ is needed. In the Euclid framework, the parallel transport by translation is tacitly assumed.

Anyway, even in the Euclid framework, different choices of parallel transport are possible and may be more convenient, sometimes, for instance when curvilinear coordinate systems are considered [29].

The choice of a linear parallel transport leads to the following notions.

Definition 8 (*Spatial time invariance*) *Time-invariance* of a spatial tensor $\mathbf{s}_{\text{SPA}} \in \text{TENS}((V\mathcal{E})_{\mathcal{T}})$ along the motion, means fulfilment of the transport-back relation

$$\mathbf{s}_{\text{SPA}} = \boldsymbol{\varphi}_\alpha \Downarrow \mathbf{s}_{\text{SPA}}. \tag{21}$$

Definition 9 (*Spatial time derivative*) The definition of time derivative of a *spatial* tensor field $\mathbf{s} \in \text{TENS}((V\mathcal{E})_{\mathcal{T}})$ along the motion is provided by the parallel derivative

$$\dot{\mathbf{s}}_{\text{SPA}} := \nabla_{\mathbf{V}} \mathbf{s}_{\text{SPA}} := \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha \Downarrow \mathbf{s}_{\text{SPA}}). \tag{22}$$

The derivative defined in (22) is usually split into spatial and time components by setting

$$\nabla_{\mathbf{V}} \mathbf{s}_{\text{SPA}} = \nabla_{\mathbf{v}} \mathbf{s}_{\text{SPA}} + \nabla_{\mathbf{Z}} \mathbf{s}_{\text{SPA}}. \tag{23}$$

The split form (23) of the parallel derivative (22) is commonly named *material time derivative*, but this nomenclature is not appropriate because the field resulting from (22) is *not* a material tensor, but a spatial tensor, as evidenced by the sketch in Fig. 5.

Moreover, the split (23) is in general not performable because the vectors $\mathbf{v}$ and $\mathbf{Z}$ may be transversal to the trajectory, for lower dimensional bodies.

The spatial time derivative of the tangent displacement $\mathbf{F} := T\boldsymbol{\varphi}_\alpha^{\mathcal{E}} : V\mathcal{T}_{\mathcal{E}} \mapsto V\mathcal{T}_{\mathcal{E}}$ is the spatial tensor defined, according to (22), by[5]

$$\mathbf{L}(\mathbf{v}) := \dot{\mathbf{F}} = \partial_{\alpha=0} (\boldsymbol{\varphi}_\alpha^{\mathcal{E}} \Downarrow T\boldsymbol{\varphi}_\alpha^{\mathcal{E}}). \tag{24}$$

In Lemma 1 the following general formula provides an expression in terms of the spatial velocity field:

$$\mathbf{L}(\mathbf{v}) = \nabla\mathbf{v} + \text{TORS}(\mathbf{v}). \tag{25}$$

The usual formula $\mathbf{L}(\mathbf{v}) = \nabla\mathbf{v}$ is recovered when a linear torsion-free connection, such as a translation, is adopted.

The *spatial* tensor field $\mathbf{L}(\mathbf{v})$ is *not* a *natural* notion, being dependent on the choice of a linear connection in the event manifold. Therefore, being neither *material* nor *natural*, its appearance in constitutive relations must be carefully avoided [30]. This comment contributes to deprive of physical basis treatments of geometrically nonlinear Continuum Mechanics in which the tangent displacement $\mathbf{F} := T\boldsymbol{\varphi}_\alpha^{\mathcal{E}} : V\mathcal{T}_{\mathcal{E}} \mapsto V\mathcal{T}_{\mathcal{E}}$ and its spatial time derivative $\mathbf{L}(\mathbf{v}) := \dot{\mathbf{F}} : V\mathcal{T}_{\mathcal{E}} \mapsto (V\mathcal{E})_{\mathcal{T}_{\mathcal{E}}}$ play the role of state variables. Although this inadequacy was already been pointed out in [12,15,30], such unphysical treatments are still being proposed [5].

[5] The standard notations $\mathbf{F}$ and $\dot{\mathbf{F}}$ are recalled here for direct comparison with treatments in the literature.

5 Metric tensor

Continuum Mechanics is founded on the geometric notion of a metric which provides the mathematical tool to evaluate the length of any spatial path.

Definition 10 (*Spatial metric*) A *spatial metric field* is a twice covariant tensor field on the time-vertical bundle

$$\mathbf{g}_{\mathcal{S}} \in \mathrm{C}\scriptstyle{\mathrm{OV}}\displaystyle(V\mathcal{E}) \tag{26}$$

which is symmetric and positive definite, that is

$$\begin{gathered} \mathbf{g}_{\mathcal{S}}(\mathbf{d}_1, \mathbf{d}_2) = \mathbf{g}_{\mathcal{S}}(\mathbf{d}_2, \mathbf{d}_1), \\ \begin{cases} \mathbf{g}_{\mathcal{S}}(\mathbf{d}, \mathbf{d}) \geq 0, \\ \mathbf{g}_{\mathcal{S}}(\mathbf{d}, \mathbf{d}) = 0 \quad \Longrightarrow \quad \mathbf{d} = \mathbf{0} \end{cases} \end{gathered} \tag{27}$$

for any $\mathbf{d}, \mathbf{d}_1, \mathbf{d}_2 \in V\mathcal{E}$.

In any reference system $\{\mathbf{d}_i\ ;\ i = 1, 2, 3\}$ in the spatial bundle, the spatial metric field is represented by a symmetric 3×3 Gram matrix

$$g_{ij} := [\mathbf{g}_{\mathcal{S}}(\mathbf{d}_i, \mathbf{d}_j)] \quad \Longrightarrow \quad \mathrm{G}\scriptstyle{\mathrm{RAM}}\displaystyle(\mathbf{g}_{\mathcal{S}}) = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}. \tag{28}$$

On the other hand, the elastic constitutive theory is called to provide a mathematical model of the response of a material body to a change in the state of stress. Experience tells us that a local stress change in an elastic material results in a change in the local metric properties of the body.

These are measured by the material metric as follows.

Definition 11 (*Material metric*) A *material metric field* is the twice covariant tensor field $\mathbf{g} \in \mathrm{C}\scriptstyle{\mathrm{OV}}\displaystyle(V\mathcal{T})$ on the material bundle defined by the pull-back of the spatial metric to the material bundle,

$$\mathbf{g} = \mathbf{i}\!\downarrow\!\mathbf{g}_{\mathcal{S}}, \tag{29}$$

which means that for any $\mathbf{a}, \mathbf{b} \in V\mathcal{T}$

$$\mathbf{g}(\mathbf{a}, \mathbf{b}) = \mathbf{g}_{\mathcal{S}}(T\mathbf{i} \cdot \mathbf{a}, T\mathbf{i} \cdot \mathbf{b}). \tag{30}$$

The genesis of the material metric $\mathbf{g} \in \mathrm{C}\scriptstyle{\mathrm{OV}}\displaystyle(V\mathcal{T})$ can be traced back to the measurement of the sides of a simplex in a tangent linear space to a body manifold.

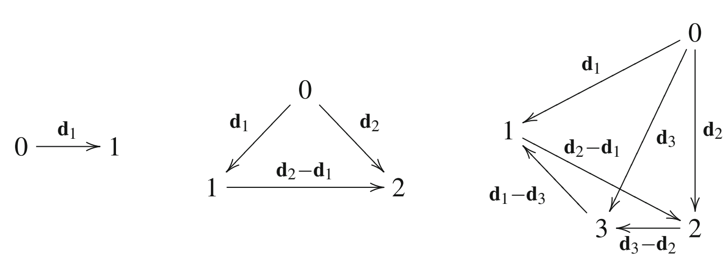
For a body dimension $n = 1, 2, 3$, the simplex is a segment, a triangle or a tetrahedron, respectively, with $C_{n+1,2} = (n+1)n/2$ sides ($C_{2,2} = 1$ $(n = 1)$, $C_{3,2} = 3$ $(n = 2)$, $C_{4,2} = 6$ $(n = 3)$), as depicted in the schemes (32).

The binomial coefficient $C_{n+1,2}$ gives also the number of components of the symmetric Gram matrix $G_{ij} := \mathbf{g}(\mathbf{d}_i, \mathbf{d}_j)$, $i, j = 1, \dots, n$ of the metric tensor with respect to the basis of material vectors $\mathbf{d}_1, \dots, \mathbf{d}_n$.

The material metric tensor is expressed in terms of the side lengths of a simplex by the formula [30]

$$\mathbf{g}(\mathbf{d}_i, \mathbf{d}_j) := \tfrac{1}{2} \left(\|\mathbf{d}_i\|^2 + \|\mathbf{d}_j\|^2 - \|\mathbf{d}_i - \mathbf{d}_j\|^2 \right), \tag{31}$$

which stays at the root of the experimental way of doing direct strain measurements, as explained in Sect.6. Bilinearity of (31) is a mathematical result due to Frechét, von Neumann, Jordan [31].



(32)

If the body is $2D$, and the reference system in the material bundle is $\{\mathbf{d}_i \in V\mathcal{T}, ; i = 1, 2\}$, the Gram matrix of the material metric field is given by

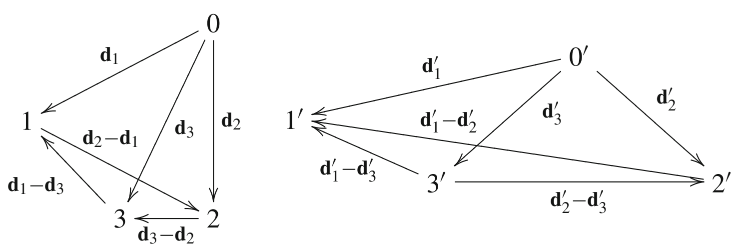
$$g_{ij} := \mathbf{g}(\mathbf{d}_i, \mathbf{d}_j) \implies \textsc{Gram}(\mathbf{g}) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}. \tag{33}$$

At each event on the trajectory, the material metric $\mathbf{g} \in \textsc{Cov}(V\mathcal{T})$ is a bilinear scalar-valued map on the tangent linear space to the body placement, equivalent to an invertible linear map from the tangent onto the cotangent space, and we may write $\mathbf{g} : V\mathcal{T} \mapsto V\mathcal{T}^*$, with inverse $\mathbf{g}^{-1} : V\mathcal{T}^* \mapsto V\mathcal{T}$ since the metric tensor is positive definite.

The material metric tensor $\mathbf{g}$ and its inverse $\mathbf{g}^{-1}$ are the standard tools to perform, by composition, alterations of material tensors, i.e., transformation from one kind (covariant, contravariant or mixed) to another one, see Appendix.

6 Geometric stretch and stretching

The notion of geometric stretch of a body is based on the comparison of the lengths of the sides of an infinitesimal simplex before and after a displacement in a time lapse α, as depicted in (34).



(34)

By definition, the vectors $\mathbf{d}'_1, \dots, \mathbf{d}'_n$ of the transformed basis in the target placement are related to the vectors $\mathbf{d}_1, \dots, \mathbf{d}_n$ of the basis in the source placement by the tangent map $T\boldsymbol{\varphi}_\alpha$ to the transformation, so that[6]

$$\mathbf{d}'_i = T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_i, \quad i = 1, \dots, n. \tag{35}$$

The length measurements permit to evaluate the Gram matrix defined by

$$\mathbf{g}(\mathbf{d}'_i, \mathbf{d}'_j) - \mathbf{g}(\mathbf{d}_i, \mathbf{d}_j) = \mathbf{g}(T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_i, T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_j) - \mathbf{g}(\mathbf{d}_i, \mathbf{d}_j). \tag{36}$$

By fibrewise linearity of the material tangent map $T\boldsymbol{\varphi}_\alpha : V\mathcal{T} \mapsto V\mathcal{T}$ and by bilinearity of the metric tensor, we infer that the notion of finite stretch is independent of the choice of a particular simplex. Indeed, introducing the pull back of the material metric tensor

$$(\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{g})(\mathbf{d}_i, \mathbf{d}_j) := \mathbf{g}(T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_i, T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_j) = \mathbf{g}((T\boldsymbol{\varphi}_\alpha)^A \cdot T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_i, \mathbf{d}_j), \tag{37}$$

the basis $\mathbf{d}_1, \dots, \mathbf{d}_n$ may be eliminated from the formula to get

$$\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{g} = \mathbf{g} \cdot (T\boldsymbol{\varphi}_\alpha)^A \cdot T\boldsymbol{\varphi}_\alpha. \tag{38}$$

The tensorial measure of the finite stretch in passing from the placement at time $t \in \mathcal{Z}$ to the one at time $\tau = t + \alpha \in \mathcal{Z}$ is thus

$$\boldsymbol{\eta}_\alpha := \tfrac{1}{2}(\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{g} - \mathbf{g}) \in \textsc{Cov}(V\mathcal{T}) \tag{39}$$

with the factor one-half introduced for convenience. The mixed alteration

$$\tfrac{1}{2}((T\boldsymbol{\varphi}_\alpha)^A \cdot T\boldsymbol{\varphi}_\alpha - \mathbf{I}) \in \textsc{Mix}(V\mathcal{T}), \tag{40}$$

is the *strain* (or *stretch*) tensor field introduced by Green [3,4].

[6] The dot $(\cdot)$ denotes that dependence on the subsequent item is linear in each tensor fiber. When redundant, it will be omitted in the sequel.

The geometric strain, measured at any point of a body of dimension $n = 1, 2, 3$ in a time lapse α, depends on $C_{n+1,2} = (n+1)n/2$ (respectively $1, 3, 6$) independent components of the symmetric Green tensor.

The material tangent displacement $T\boldsymbol{\varphi}_\alpha : V\mathcal{T} \mapsto V\mathcal{T}$, whose matrix has $n \times n$ components, does not yield by itself any tool for length measurements, and hence, the naming of *deformation gradient* and the role of driving geometric object in constitutive relations appear as unnatural.

In fact, to recover a physically consistent formulation, a reduction procedure was conceived by Noll in [17]. This procedure is confined to a 3D body (to apply polar decomposition) and is based on a formulation of the MFI principle, critically addressed in [15] and further discussed below in Sect. 16.

Definition 12 (*Geometric stretching*) The *geometric stretching* is the material tensor field defined in a natural way as time derivative of the Green strain. In geometric terms, it is one-half the Lie *derivative* of the material metric tensor field along the motion, i.e., the symmetric, twice covariant material tensor field defined by

$$\boldsymbol{\varepsilon}(\mathbf{v}) := \tfrac{1}{2}\mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\,\mathbf{g} := \partial_{\alpha=0}\,\tfrac{1}{2}(\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{g}) \in \mathrm{C{\small OV}}(V\mathcal{T}). \tag{41}$$

By resorting to the formula $\mathbf{V} = \mathbf{v} + \mathbf{Z}$ exposed in (14), a decomposition of the stretching into time and spatial contributions may be envisaged.

The exceptional geometrical property of the *spatial metric tensor field* $\mathbf{g}_\mathcal{S} \in \mathrm{C{\small OV}}(V\mathcal{E})$ is that it is defined on the whole event manifold and this is a discriminant against all other tensor fields of Continuum Mechanics which are defined only on the trajectory.

Indeed, since the material metric $\mathbf{g}$ is defined only on the trajectory, the Lie derivatives along the vector fields $\mathbf{v} \in (V\mathcal{E})_\mathcal{T}$ and $\mathbf{Z} \in (H\mathcal{E})_\mathcal{T}$, which may point out of the trajectory, must necessarily be performed on the spatial metric field $\mathbf{g}_\mathcal{S}$, defined on the whole event manifold, to get

$$\mathcal{L}_\mathbf{V}\,\mathbf{g}_\mathcal{S} = \mathcal{L}_\mathbf{v}\,\mathbf{g}_\mathcal{S} + \mathcal{L}_\mathbf{Z}\,\mathbf{g}_\mathcal{S}. \tag{42}$$

Time independence of the spatial metric $\mathbf{g}_\mathcal{S}$ in the Euclid space-time of Classical Mechanics is expressed by the conditions

$$\mathcal{L}_\mathbf{Z}\,\mathbf{g}_\mathcal{S} = \partial_{\alpha=0}\,(\boldsymbol{\varphi}^\mathcal{Z}_\alpha\!\downarrow\!\mathbf{g}_\mathcal{S}) = \partial_{\alpha=0}\,\boldsymbol{\varphi}^\mathcal{Z}_\alpha\!\Downarrow\mathbf{g}_\mathcal{S} = \nabla_\mathbf{Z}\,\mathbf{g}_\mathcal{S} = \mathbf{0}, \tag{43}$$

so that

$$\mathcal{L}_\mathbf{V}\,\mathbf{g}_\mathcal{S} = \mathcal{L}_\mathbf{v}\,\mathbf{g}_\mathcal{S}. \tag{44}$$

In order to recover a material tensor field, it is then expedient to introduce the *orthogonal projector* $\boldsymbol{\Pi} : (V\mathcal{E})_\mathcal{T} \mapsto V\mathcal{T}$ from the spatial bundle onto the material bundle, with adjoint tangent immersion $\boldsymbol{\Pi}^A = T\mathbf{i} : V\mathcal{T} \mapsto (V\mathcal{E})_\mathcal{T}$ defined by the identity

$$\mathbf{g}(\boldsymbol{\Pi}\cdot\mathbf{u},\mathbf{d}) = \mathbf{g}_\mathcal{S}(\mathbf{u},\boldsymbol{\Pi}^A\cdot\mathbf{d}), \quad \forall\,\mathbf{d}\in V\mathcal{T}, \quad \forall\,\mathbf{u}\in(V\mathcal{E})_\mathcal{T}. \tag{45}$$

We then have the following result.

Proposition 1 (Euler tensor) *The material stretching* $\boldsymbol{\varepsilon}(\mathbf{v}) \in \mathrm{C{\small OV}}(V\mathcal{T})$ *and the spatial Euler tensor* $\mathbf{D}(\mathbf{v}) \in \mathrm{M{\small IX}}((V\mathcal{E})_\mathcal{T})$*, defined by*

$$\mathbf{g}_\mathcal{S}\cdot\mathbf{D}(\mathbf{v}) = \tfrac{1}{2}\mathcal{L}_\mathbf{v}\mathbf{g}_\mathcal{S}, \tag{46}$$

are related by

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{g}\cdot\boldsymbol{\Pi}\cdot\mathbf{D}(\mathbf{v})\cdot\boldsymbol{\Pi}^A, \tag{47}$$

whereby the material tensor $\boldsymbol{\Pi}\cdot\mathbf{D}(\mathbf{v})\cdot\boldsymbol{\Pi}^A \in \mathrm{M{\small IX}}(V\mathcal{T})$ *is the mixed alteration of the stretching.*

Proof Setting $\boldsymbol{\varepsilon}_\mathbf{v} \equiv \boldsymbol{\varepsilon}(\mathbf{v})$, by naturally Lie derivatives with respect to push (20.13), from (29), (41) and (44) we get

$$\begin{aligned}\boldsymbol{\varepsilon}_\mathbf{v}(\mathbf{a},\mathbf{b}) &= \tfrac{1}{2}\mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}(\mathbf{i}\!\downarrow\!\mathbf{g}_\mathcal{S})(\mathbf{a},\mathbf{b}) = \mathbf{i}\!\downarrow\!(\tfrac{1}{2}\mathcal{L}_\mathbf{V}\,\mathbf{g}_\mathcal{S})(\mathbf{a},\mathbf{b})\\ &= \tfrac{1}{2}(\mathcal{L}_\mathbf{v}\mathbf{g}_\mathcal{S})(\boldsymbol{\Pi}^A\mathbf{a},\boldsymbol{\Pi}^A\mathbf{b}) = \mathbf{g}_\mathcal{S}(\mathbf{D}(\mathbf{v})\cdot\boldsymbol{\Pi}^A\mathbf{a},\boldsymbol{\Pi}^A\mathbf{b})\\ &= \mathbf{g}(\boldsymbol{\Pi}\cdot\mathbf{D}(\mathbf{v})\cdot\boldsymbol{\Pi}^A\mathbf{a},\mathbf{b}), \qquad \forall\,\mathbf{a},\mathbf{b}\in V\mathcal{T},\end{aligned} \tag{48}$$

which yields (47). $\square$

Next, we provide a general formula for the $\mathbf{g}_\mathcal{S}$-symmetric Euler tensor $\mathbf{D}(\mathbf{v}) \in \mathrm{M{\small IX}}((V\mathcal{E})_\mathcal{T})$ in terms of parallel derivatives according to any linear connection in the spatial bundle.

Proposition 2 (Euler formula) *The Euler tensor* $\mathbf{D}(\mathbf{v}) \in \mathrm{MIX}((V\mathcal{E})_{\mathcal{T}})$ *may be evaluated in terms of parallel derivatives by the formula*

$$\mathbf{D}(\mathbf{v}) = \tfrac{1}{2}\,\mathbf{G}(\mathbf{v}) + \tfrac{1}{2}\,(\nabla + \mathrm{TORS})(\mathbf{v}) + \tfrac{1}{2}\,(\nabla + \mathrm{TORS})(\mathbf{v})^{A}, \tag{49}$$

where $\mathbf{g}_{\mathcal{S}} \cdot \mathbf{G}(\mathbf{v}) := \nabla_{\mathbf{v}}\, \mathbf{g}_{\mathcal{S}}$. *If the connection is metric and torsion free*[7], *being* $\mathbf{G}(\mathbf{v}) = \mathbf{0}$ *and* $\mathrm{TORS}(\mathbf{v}) = \mathbf{0}$ *we get the standard expression*

$$\mathbf{D}(\mathbf{v}) = \tfrac{1}{2}\,(\nabla\mathbf{v} + (\nabla\mathbf{v})^{A}) = \mathrm{sym}\,\nabla\mathbf{v}. \tag{50}$$

Proof From (20.22.3) we have

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{g}_{\mathcal{S}} = \nabla_{\mathbf{v}}\, \mathbf{g}_{\mathcal{S}} + \mathbf{g}_{\mathcal{S}} \cdot (\nabla + \mathrm{TORS})(\mathbf{v})) + (\nabla + \mathrm{TORS})(\mathbf{v}))^{*} \cdot \mathbf{g}_{\mathcal{S}}, \tag{51}$$

which, taking into account the relation between dual and adjoint,

$$(\nabla + \mathrm{TORS})(\mathbf{v}))^{*} \cdot \mathbf{g}_{\mathcal{S}} = \mathbf{g}_{\mathcal{S}} \cdot (\nabla + \mathrm{TORS})(\mathbf{v}))^{A}, \tag{52}$$

and setting $\mathbf{g}_{\mathcal{S}} \cdot \mathbf{G}(\mathbf{v}) := \nabla_{\mathbf{v}}\, \mathbf{g}_{\mathcal{S}}$, gives the formula

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{g}_{\mathcal{S}} = \mathbf{g}_{\mathcal{S}} \cdot \big(\mathbf{G}(\mathbf{v}) + \big((\nabla + \mathrm{TORS})(\mathbf{v}) + (\nabla + \mathrm{TORS})(\mathbf{v})^{A}\big)\big). \tag{53}$$

Then, by definition (46), we get (49). □

In Continuum Mechanics, the trajectory manifold is characterized by the property of mass conservation along the motion, expressed, for any body placement $\boldsymbol{\Omega}$, by

$$\int_{\boldsymbol{\Omega}} \mathbf{m} = \int_{\varphi_{\alpha}(\boldsymbol{\Omega})} \mathbf{m} \quad \Longleftrightarrow \quad \mathcal{L}_{\mathbf{V}}\, \mathbf{m} = 0. \tag{54}$$

The material volume $\boldsymbol{\mu}$ is the exterior form (see fn. 17) induced by the material metric $\mathbf{g}$ so that the unitary n-cube has a unit volume. The mass $\mathbf{m}$ is the material exterior form defined by the proportionality $\mathbf{m} = \rho\,\boldsymbol{\mu}$, where ρ is the scalar mass density. The property (54) will be resorted to in the proof of Proposition 5 in Sect. 14.

7 Stress and stressing

The notion of stress, as dual basic ingredient of Continuum Mechanics, is introduced by appealing to the natural duality between contravariant and covariant tensors, see Appendix.

In a body manifold of dimension n let us assign

1. any vector basis $\{\mathbf{d}_i,\ i = 1, \ldots, n\}$ in $V\mathcal{T}$
2. and the dual covector basis $\{\mathbf{d}^k,\ k = 1, \ldots, n\}$ in $(V\mathcal{T})^{*}$, so that $\langle \mathbf{d}^k, \mathbf{d}_j \rangle = \delta^k_{.j}$.

The matrices associated with the *stress* $\boldsymbol{\sigma} \in \mathrm{CON}(V\mathcal{T})$ and the *stretching* $\boldsymbol{\varepsilon}_{\mathbf{v}} \equiv \boldsymbol{\varepsilon}(\mathbf{v}) \in \mathrm{COV}(V\mathcal{T})$ are

$$\boldsymbol{\sigma}^{ij} = \boldsymbol{\sigma}(\mathbf{d}^i, \mathbf{d}^j), \qquad \boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon}_{\mathbf{v}}(\mathbf{d}_i, \mathbf{d}_j), \tag{55}$$

and the duality pairing between them takes the *invariant* expression

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\mathbf{v}} \rangle = \boldsymbol{\sigma}^{ij}\, \boldsymbol{\varepsilon}_{ij}. \tag{56}$$

By the symmetry of $\boldsymbol{\varepsilon}_{\mathbf{v}} \in \mathrm{COV}(V\mathcal{T})$, the stress $\boldsymbol{\sigma} \in \mathrm{CON}(V\mathcal{T})$ may be assumed to be symmetric without loss of generality.

[7] In a Riemann manifold, the Levi-Civita connection is uniquely defined by the property of being torsion-free and metric.

Definition 13 (*Stress tensor*) The stress field $\boldsymbol{\sigma} \in \mathrm{C}\text{ON}(V\mathcal{T})$ is a twice contravariant symmetric material tensor field whose duality pairing $\langle , \rangle$ with the covariant geometric stretching $\boldsymbol{\varepsilon}(\delta\mathbf{v}) \in \mathrm{C}\text{OV}(V\mathcal{T})$ provides the mechanical power expended per unit mass

$$\int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \mathbf{m} = \langle \mathbf{f}, \delta\mathbf{v} \rangle, \tag{57}$$

where $\delta\mathbf{v} : \boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}$ is any virtual spatial velocity and $\mathbf{f}$ is the force system acting on the body placement $\boldsymbol{\Omega}$.

The symmetry of $\boldsymbol{\sigma} \in \mathrm{C}\text{ON}(V\mathcal{T})$ entails $\mathbf{g}$-symmetry of the mixed stress tensor $\mathbf{K} = \boldsymbol{\sigma} \cdot \mathbf{g} \in \mathrm{M}\text{IX}(V\mathcal{T})$, known in the literature under the name of Kirchhoff stress, as follows from the relations

$$\boldsymbol{\sigma}(\mathbf{g} \cdot \mathbf{a}, \mathbf{g} \cdot \mathbf{b}) = \mathbf{g}(\boldsymbol{\sigma} \cdot \mathbf{g} \cdot \mathbf{a}, \mathbf{b}) = \mathbf{g}(\mathbf{K} \cdot \mathbf{a}, \mathbf{b}), \quad \forall\, \mathbf{a}, \mathbf{b} \in V\mathcal{T}. \tag{58}$$

By the same argument, the $\mathbf{g}$-symmetry of the Cauchy stress tensor $\mathbf{T} \in \mathrm{M}\text{IX}(V\mathcal{T})$ follows from the definition $\mathbf{T} := \rho\, \mathbf{K}$. Then, from the equality

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}) \rangle = \langle \boldsymbol{\sigma}, \mathbf{g} \cdot \mathbf{g}^{-1} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \rangle = \langle \boldsymbol{\sigma} \cdot \mathbf{g}, \mathbf{g}^{-1} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \rangle,$$

resorting to (47), we get the relation

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}) \rangle\, \mathbf{m} = \langle \mathbf{K}, \boldsymbol{\Pi} \cdot \mathbf{D}(\mathbf{v}) \cdot \boldsymbol{\Pi}^{A} \rangle\, \mathbf{m} = \langle \mathbf{T}, \boldsymbol{\Pi} \cdot \mathbf{D}(\mathbf{v}) \cdot \boldsymbol{\Pi}^{A} \rangle\, \boldsymbol{\mu}, \tag{59}$$

where the last term provides the mechanical power per unit volume.

The mixed Cauchy stress tensor $\mathbf{T} \in \mathrm{M}\text{IX}(V\mathcal{T})$ is usually adopted in the linearized theory under small displacements, since its flux through a surface provides the force of interaction per unit area.

The contravariant Kirchhoff stress tensor $\boldsymbol{\sigma} \in \mathrm{C}\text{ON}(V\mathcal{T})$ is, however, the one suitable in the general theory, when invariance under diffeomorphic transformations and conservation of mass are involved. The relevance of this observation will become evident in Sects. 14 and 16.

The components of the contravariant Kirchhoff stress tensor $\boldsymbol{\sigma} \in \mathrm{C}\text{ON}(V\mathcal{T})$ with respect to the covector basis $\{\mathbf{d}^k\}$ are given by

$$\sigma^{ij} := \boldsymbol{\sigma}(\mathbf{d}^i, \mathbf{d}^j) \quad \Longrightarrow \quad \mathrm{G}\text{RAM}(\boldsymbol{\sigma}) = \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{21} & \sigma^{22} & \sigma^{23} \\ \sigma^{31} & \sigma^{32} & \sigma^{33} \end{bmatrix}, \tag{60}$$

and the components of the mixed Kirchhoff stress tensor $\mathbf{K} = \boldsymbol{\sigma} \cdot \mathbf{g} \in \mathrm{M}\text{IX}(V\mathcal{T})$ with respect to the vector basis $\{\mathbf{d}_k\}$

$$K^{i}_{.j} := \langle \mathbf{d}^i, \mathbf{K} \cdot \mathbf{d}_j \rangle \quad \Longrightarrow \quad \mathrm{G}\text{RAM}(\mathbf{K}) = \begin{bmatrix} K^{1}_{.1} & K^{1}_{.2} & K^{1}_{.3} \\ K^{2}_{.1} & K^{2}_{.2} & K^{2}_{.3} \\ K^{3}_{.1} & K^{3}_{.2} & K^{3}_{.3} \end{bmatrix} \tag{61}$$

setting $g_{ij} := \mathbf{g}(\mathbf{d}_i, \mathbf{d}_j)$ are related by $K^{i}_{.j} = \sigma^{ik}\, g_{kj}$.

Remark 1 The relation $\mathbf{T} = \rho\, \mathbf{K}$, between mixed Kirchhoff and Cauchy stress tensors, differs from the one adopted in the literature [32, p. 156], [33, p. 329] consisting in the relation

$$\mathbf{K} = J_\alpha\, \mathbf{T}, \quad \text{with} \quad J_\alpha := \det(T\boldsymbol{\varphi}_\alpha). \tag{62}$$

In this equality, only the term J_α depends on the time lapse α, usually omitted in standard treatments, and this fact is not admissible. Analogous objections hold true if transformations from reference placements are considered, instead of displacements along the motion.

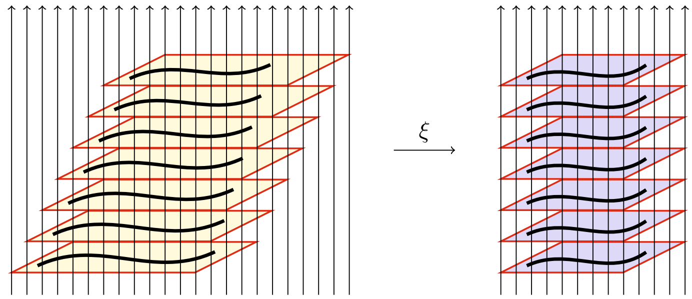


Fig. 6 Straightening of the trajectory

Definition 14 (*Stress time invariance*) A stress tensor is *time invariant* if it is dragged along the motion[8]:

$$\boldsymbol{\sigma} = \boldsymbol{\varphi}_\alpha \!\downarrow\! \boldsymbol{\sigma}, \tag{63}$$

with push forward of the contravariant stress tensor expressed by (20.6) as

$$(\boldsymbol{\varphi}_\alpha \!\downarrow\! \boldsymbol{\sigma}) = (T\boldsymbol{\varphi}_\alpha)^{-1} \cdot \boldsymbol{\sigma} \cdot (T\boldsymbol{\varphi}_\alpha)^{-*}. \tag{64}$$

The motivation for Definition 14 is twofold.

1. The temptation of defining equality by invariance of cartesian components is faulty because the unnatural choice of a parallel transport is required, as discussed below in Sect. 4.1.
2. Parallel transport is not feasible for lower dimensional bodies since parallel transported material vectors (tangent at a placement) will in general no more be material (tangent at the transformed placement), see Fig. 5.

The comparison between material tensor fields, at distinct time instants along a particle, can only be made, in a natural way, by pull back along the motion, and this leads to the following definition.

Definition 15 (*Kirchhoff stressing*) The *stressing* $\dot{\boldsymbol{\sigma}}$ is the Lie derivative of the contravariant Kirchhoff stress tensor along the motion

$$\dot{\boldsymbol{\sigma}} := \mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\boldsymbol{\sigma} := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha \!\downarrow\! \boldsymbol{\sigma}). \tag{65}$$

The time rate of the material stress tensor $\boldsymbol{\sigma} \in \mathrm{C{\scriptsize ON}}(V\mathcal{T})$ is thus evaluated in perfect analogy with the definition (41) of time rate of the material metric tensor $\mathbf{g} \in \mathrm{C{\scriptsize OV}}(V\mathcal{T})$. Anyway, two significant differences emerge.

1. The material metric is the pull back of the spatial metric (whose domain is the whole event manifold) to the trajectory manifold, while the material stress field is defined only on the trajectory.
2. In the Euclid framework, the spatial metric is *time-independent*, i.e. (43) holds, while such a notion is meaningless for the material stress which is defined only on the trajectory, so that the Lie derivative along the time motion may be not feasible.

8 Local straightened trajectory

Theoretical investigations and computational procedures are conveniently carried out by a local transformation of the trajectory $\mathcal{T}$ into a straightened one which is a tube generated by the time axis [15,30], as depicted in Fig. 6.

The cross sections of the tube are copies of a *local reference manifold* $\boldsymbol{\Omega}_{\mathrm{REF}} \subset \mathcal{R}^n$, with n dimension of the body.

The referential translation along the time axis,

$$\boldsymbol{\theta}_\alpha : \boldsymbol{\Omega}_{\mathrm{REF}} \times \mathcal{Z} \mapsto \boldsymbol{\Omega}_{\mathrm{REF}} \times \mathcal{Z}, \tag{66}$$

[8] Equality between material tensors based at distinct events is impracticable.

leaves the location in $\boldsymbol{\Omega}_{\text{REF}}$ invariant, according to the definition

$$\boldsymbol{\theta}_\alpha(\mathbf{e}_{\text{REF}}, t) := (\mathbf{e}_{\text{REF}}, t+\alpha), \quad \mathbf{e}_{\text{REF}} \in \boldsymbol{\Omega}_{\text{REF}}, \quad t, \alpha \in \mathcal{Z}, \tag{67}$$

and is related to the motion by $\boldsymbol{\theta}_\alpha = \boldsymbol{\chi}\!\uparrow\!\boldsymbol{\varphi}_\alpha$, according to the commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\boldsymbol{\varphi}_\alpha"] \arrow[d, "\boldsymbol{\chi}"'] & \mathcal{T} \arrow[d, "\boldsymbol{\chi}"] \\ \boldsymbol{\Omega}_{\text{REF}} \times \mathcal{Z} \arrow[r, "\boldsymbol{\theta}_\alpha"] & \boldsymbol{\Omega}_{\text{REF}} \times \mathcal{Z} \end{tikzcd} \quad \Longleftrightarrow \quad \boldsymbol{\chi} \circ \boldsymbol{\varphi}_\alpha = \boldsymbol{\theta}_\alpha \circ \boldsymbol{\chi}. \tag{68}$$

For any material tensor field $\mathbf{s} \in \text{TENS}(V\mathcal{T})$ on $\mathcal{T}$, setting $\mathbf{s}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\mathbf{s}$, we infer from (68) that

$$\boldsymbol{\theta}_\alpha\!\downarrow\!\mathbf{s}_{\text{REF}} = \boldsymbol{\theta}_\alpha\!\downarrow\!(\boldsymbol{\chi}\!\uparrow\!\mathbf{s}) = (\boldsymbol{\chi}^{-1} \circ \boldsymbol{\theta}_\alpha)\!\downarrow\!\mathbf{s} = (\boldsymbol{\varphi}_\alpha \circ \boldsymbol{\chi}^{-1})\!\downarrow\!\mathbf{s} = \boldsymbol{\chi}\!\uparrow\!(\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{s}). \tag{69}$$

Being $\boldsymbol{\chi}\!\uparrow\!\mathbf{V} = \partial_{\alpha=0}\,\boldsymbol{\theta}_\alpha$, by appealing to naturality of Lie derivatives with respect to push (20.13) and setting $T\boldsymbol{\theta}_\alpha$ equal to the identity mapping, we get

$$\begin{aligned} \boldsymbol{\chi}\!\uparrow\!(\mathcal{L}_{\mathbf{V}}\,\mathbf{s}) &= \mathcal{L}_{\boldsymbol{\chi}\uparrow\mathbf{V}}\,(\boldsymbol{\chi}\!\uparrow\!\mathbf{s}) = \partial_{\alpha=0}\,(\boldsymbol{\theta}_\alpha\!\downarrow\!\mathbf{s}_{\text{REF}}) \\ &= \partial_{\alpha=0}\,(T\boldsymbol{\theta}_{-\alpha} \circ \mathbf{s}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha) = \partial_{\alpha=0}\,(\mathbf{s}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha). \end{aligned} \tag{70}$$

The straightening map transforms then Lie derivatives of material tensor fields along the actual motion into Lie derivatives of the transformed tensor fields along the time translation, and these are the standard partial time derivatives.

The referential rate elastic constitutive operator is defined by the push

$$\mathbf{H}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\mathbf{H} \tag{71}$$

which, in analogy with (19), consists in the identity

$$\boldsymbol{\chi}\!\uparrow\!(\mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}) = \mathbf{H}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \cdot \dot{\boldsymbol{\sigma}}_{\text{REF}}.$$

The push forward $\boldsymbol{\chi}\!\uparrow$ preserves symmetry of covariant and contravariant tensors and fulfills the commutativity property

$$d_F \circ \boldsymbol{\chi}\!\uparrow = \boldsymbol{\chi}\!\uparrow \circ\, d_F. \tag{72}$$

Integrability of the rate elastic constitutive operator, expressed by (91), assures then integrability of the referential rate elastic constitutive operator.

From (20.6), setting $\mathbf{g}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\mathbf{g}$, we get the push forward relations

$$\begin{cases} \mathbf{g}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\mathbf{g} = (T\boldsymbol{\chi})^{-*} \cdot \mathbf{g} \cdot T\boldsymbol{\chi}^{-1}, & (73.1) \\ \boldsymbol{\sigma}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\boldsymbol{\sigma} = (T\boldsymbol{\chi}) \cdot \boldsymbol{\sigma} \cdot (T\boldsymbol{\chi})^{*}, & (73.2) \\ \boldsymbol{\sigma}_{\text{REF}} \cdot \mathbf{g}_{\text{REF}} = \boldsymbol{\chi}\!\uparrow\!(\boldsymbol{\sigma} \cdot \mathbf{g}) = T\boldsymbol{\chi} \cdot (\boldsymbol{\sigma} \cdot \mathbf{g}) \cdot (T\boldsymbol{\chi})^{-1}. & (73.3) \end{cases}$$

Then $\mathbf{K} = \boldsymbol{\sigma} \cdot \mathbf{g}$ is $\mathbf{g}$-symmetric and $\mathbf{K}_{\text{REF}} = \boldsymbol{\sigma}_{\text{REF}} \cdot \mathbf{g}_{\text{REF}}$ is $\mathbf{g}_{\text{REF}}$-symmetric[9].

[9] Symmetry of a mixed tensor is intended to hold with respect to a metric tensor. Symmetrizable mixed tensors have real eigenvalues and a basis of orthonormal eigenvectors with respect to the relevant metric.

9 A caveat on Piola–Kirchhoff tensor

In [16], Sect. 43A, it is said that *"The second Piola–Kirchhoff tensor is introduced so as to provide a representation of stress that, while transformed back to the reference configuration, is still a symmetric tensor."*

The expression of the second Piola–Kirchhoff tensor provided in [16] may be recovered by evaluating the $\mathbf{g}$-symmetric formula

$$\boldsymbol{\sigma}_{\text{REF}} \cdot \mathbf{g} = T\boldsymbol{\chi} \cdot \boldsymbol{\sigma} \cdot (T\boldsymbol{\chi})^* \cdot \mathbf{g} = T\boldsymbol{\chi} \cdot (\boldsymbol{\sigma} \cdot \mathbf{g}) \cdot (T\boldsymbol{\chi})^A, \tag{74}$$

where by (20.7) the relation $(T\boldsymbol{\chi})^* \cdot \mathbf{g} = \mathbf{g} \cdot (T\boldsymbol{\chi})^A$ has been resorted to[10].

It should be observed that the straightened trajectory and the actual trajectory are $\boldsymbol{\chi}$-isomorphic but quite distinct manifolds. Adoption of the material metric $\mathbf{g}$ to get the second Piola–Kirchhoff tensor, as mixed alteration of the contravariant pull back stress $\boldsymbol{\sigma}_{\text{REF}}$, is not legitimate when the push forward to a reference manifold is considered. Indeed, the metric $\mathbf{g}$ is defined only on the actual trajectory and *not* on the reference manifold.

In most referential treatments of nonlinear continuum mechanics, the metric tensor which symmetry refers to is not explicitly mentioned, but tacitly assumed to be the spatial metric, e.g., [33]. In the wake of [16,35,36], the mixed second Piola–Kirchhoff tensor is commonly assumed as referential stress, without awareness of the involved conceptual and operational difficulty.

The *natural* way of performing the alteration of $\boldsymbol{\sigma}_{\text{REF}}$ is by means of the push forward $\mathbf{g}_{\text{REF}}$ of the material metric $\mathbf{g}$ to the reference manifold, to get the referential stress exposed in (73.3). This result can also be directly obtained by push of the mixed Kirchhoff stress $\mathbf{K} \in \text{MIX}(V\mathcal{T})$ to the reference manifold. A collateral advantage is that the spectra of push-forward-related mixed tensors are identical.

10 Rate elasticity

The rate elastic behavior is expressed by means of a *constitutive operator* $\mathbf{C}$ which depends in a nonlinear way on the *stress* and linearly on the *stressing*.

Definition 16 (*Rate elasticity*) The rate elastic law defines the *elastic stretching* $\mathbf{el} \in \text{COV}(V\mathcal{T})$ as a symmetric covariant material tensor expressed by

$$\mathbf{el} := \mathbf{C}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}) = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, \tag{75}$$

where $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\, \boldsymbol{\sigma} \in \text{CON}(V\mathcal{T})$. The linear operator $\mathbf{H}(\boldsymbol{\sigma})$ is the *tangent elastic compliance*.

A *purely elastic* behavior is characterized by the additional condition $\boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{el}$ stating that the elastic stretching $\mathbf{el}$ is equal to the geometric stretching $\boldsymbol{\varepsilon}(\mathbf{v})$.

The modeling of an elasto-visco-plastic constitutive response will be briefly described in Sect. 15.

We remark that the alternative choice of the contravariant stress $\boldsymbol{\sigma} \in \text{CON}(V\mathcal{T})$ or of the mixed stress $\mathbf{K} = \boldsymbol{\sigma} \cdot \mathbf{g} \in \text{MIX}(V\mathcal{T})$ makes a basic difference in constitutive relations because their rates are not linearly related. Indeed

$$\mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\, \mathbf{K} = \mathcal{L}_{\mathbf{V}}(\boldsymbol{\sigma} \cdot \mathbf{g}) = (\mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\boldsymbol{\sigma}) \cdot \mathbf{g} + \boldsymbol{\sigma} \cdot \mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\mathbf{g}. \tag{76}$$

This difference disappears in the linearized theory since the assumption that $\mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\mathbf{g} = \mathbf{0}$ leads to the relation $\mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\mathbf{K} = (\mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\boldsymbol{\sigma}) \cdot \mathbf{g}$.

Definition 17 (*Constitutive time invariance*) The constitutive operator $\mathbf{C}$ is *time-invariant* along the motion, if it fulfills the relation

$$\mathbf{C} = \boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{C}, \tag{77}$$

with the pull back of the constitutive operator $\mathbf{C}$ defined by the identity

$$(\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{C})(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma}, \boldsymbol{\varphi}_\alpha\!\downarrow\!\dot{\boldsymbol{\sigma}}) = \boldsymbol{\varphi}_\alpha\!\downarrow\!(\mathbf{C}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}})). \tag{78}$$

We underline that while the stretching is a material tensor field describing the rate of distorsion of the body, the material metric by itself has no place in constitutive relations since it is just the restriction of the spatial metric to material vectors.

The new formulation of rate elasticity proposed in Definition 16 differs from the original one of hypo-elasticity recalled in (4) in two main aspects.

[10] The incorrect formula $\boldsymbol{\sigma}_{\text{REF}} \cdot \mathbf{g} = T\boldsymbol{\chi} \cdot (\boldsymbol{\sigma} \cdot \mathbf{g}) \cdot (T\boldsymbol{\chi})^*$ is reported in [34].

1. The new statement in Definition 16 provides the very definition of *elastic stretching* $\mathbf{el} \in \mathrm{Cov}(V\mathcal{T})$ in terms of the material stress tensor and of its time derivative. The geometric stretching $\boldsymbol{\varepsilon}(\mathbf{v})$, or equivalently the Euler tensor $\mathbf{D}(\mathbf{v}) \in \mathrm{Mix}((V\mathcal{E})_{\mathcal{T}})$, does not appear in the constitutive relation unless the very special assumption of *pure elasticity* is laid down.
2. Integrability of the hypo-elastic law (4) was intended as a condition apt to recover the elastic law (1) in terms of the deformation gradient [7]. This misstatement was the main cause of failure in the construction of a rate theory of elasticity. On the contrary, the issue of integrability of the *rate elastic operator* $\mathbf{H}$ appearing in (75) is intended to hold in each linear stress fibre of the bundle $\mathrm{Con}(V\mathcal{T})$ without even mentioning a reference placement, as illustrated in Sect. 11 below. The new treatment leads in a direct way to assess the basic property of conservativeness of the elastic constitutive operator with respect to stress cycles, which is the mathematical expression of the nondissipative character of elasticity.

11 Integrability

Integrability of the *rate elastic operator* $\mathbf{H}$ has two levels.

1. Cauchy integrability is the symmetry condition on the fiber derivative $d_F\mathbf{H}$ [11] of the *rate elastic operator* $\mathbf{H}$ and assures existence of a potential $\boldsymbol{\Psi}$ in the stress bundle such that
$$\mathbf{H} = d_F\boldsymbol{\Psi}. \tag{79}$$
2. Green integrability is the symmetry condition on the *rate elastic operator* $\mathbf{H}$ itself. In addition to the previous condition, it assures existence of a scalar elastic potential E in the stress fiber, such that
$$\boldsymbol{\Psi} = d_F E \quad \Longleftrightarrow \quad \mathbf{H} = d_F^2 E. \tag{80}$$

The adjoint $\mathbf{H}(\boldsymbol{\sigma})^A$ of the linear operator $\mathbf{H}(\boldsymbol{\sigma})$ is defined, for all $\delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma}$, by the identity (here the metric plays no role)
$$\langle \mathbf{H}(\boldsymbol{\sigma})^A \cdot \delta_2\boldsymbol{\sigma}, \delta_1\boldsymbol{\sigma} \rangle = \langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma} \rangle. \tag{81}$$
The basic result of constitutive elasticity theory is the following.

Proposition 3 (Integrability conditions) *Cauchy integrability of the rate elastic operator* $\mathbf{H}$ *is equivalent to the symmetry condition*
$$(d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma})^A = d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}, \tag{82}$$
on the directional fiber derivative $d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}$ *of the tangent elastic compliance* $\mathbf{H}(\boldsymbol{\sigma})$ *along any stress variation* $\delta\boldsymbol{\sigma}$. *The further symmetry condition on the tangent elastic compliance* $\mathbf{H}(\boldsymbol{\sigma})$
$$\mathbf{H}(\boldsymbol{\sigma})^A = \mathbf{H}(\boldsymbol{\sigma}), \tag{83}$$
ensures GREEN-*integrability (hyper-elasticity).*

Proof In the material bundle of contravariant tensors, see Appendix, the symmetry condition on the fiber derivative is equivalent to the vanishing, along any stress loop in a fiber of the material bundle, of the integral
$$\oint_{\ell} \langle \mathbf{H}(\ell(\lambda)) \cdot \delta\boldsymbol{\sigma}, \dot{\ell}(\lambda) \rangle \, d\lambda = 0.$$

Here, $\ell(\lambda)$ with $\lambda \in [0,1]$ is any loop in the stress fiber, so that $\ell(0) = \ell(1)$, with tangent vector $\dot{\ell}(\lambda) := \partial_{\mu=\lambda}\, \ell(\mu)$.

Vanishing of loop integrals implies that, for any fixed stress $\delta\boldsymbol{\sigma}$, there exists a scalar potential defined, up to an additive constant, by the integral
$$\boldsymbol{\Psi}_{\delta\boldsymbol{\sigma}}(\boldsymbol{\sigma}) = \int_0^1 \langle (\mathbf{H} \circ \mathbf{c})(\lambda) \cdot \delta\boldsymbol{\sigma}, \dot{\mathbf{c}}(\lambda) \rangle \, d\lambda,$$

[11] The fiber derivative is performed by holding the linear stress fiber fixed.

where $\mathbf{c}(\lambda)$ with $\lambda \in [0, 1]$ is any path between stress points $\mathbf{c}(0) = \mathbf{0}$ and $\mathbf{c}(1) = \boldsymbol{\sigma}$. Hence

$$d_F \boldsymbol{\Psi}_{\delta\boldsymbol{\sigma}}(\boldsymbol{\sigma}) = \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}.$$

Recalling the definition (81) of the adjoint operator $\mathbf{H}(\boldsymbol{\sigma})^A$, the expression of the potential may be rewritten as

$$\boldsymbol{\Psi}_{\delta\boldsymbol{\sigma}}(\boldsymbol{\sigma}) = \int_0^1 \langle \mathbf{H}(\mathbf{c}(\lambda))^A \cdot \dot{\mathbf{c}}(\lambda), \delta\boldsymbol{\sigma} \rangle \, d\lambda = \langle \boldsymbol{\Psi}(\boldsymbol{\sigma}), \delta\boldsymbol{\sigma} \rangle,$$

with the strain-valued stress-potential $\boldsymbol{\Psi}$ defined in the stress fiber by

$$\boldsymbol{\Psi}(\boldsymbol{\sigma}) := \int_0^1 \mathbf{H}(\mathbf{c}(\lambda))^A \cdot \dot{\mathbf{c}}(\lambda) \, d\lambda.$$

Being $d_F \boldsymbol{\Psi}_{\delta\boldsymbol{\sigma}}(\boldsymbol{\sigma}) = \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}$, for any fixed $\delta\boldsymbol{\sigma}$, we infer that

$$d_F \boldsymbol{\Psi} = \mathbf{H}.$$

Fulfillment of the further symmetry property

$$d_F \boldsymbol{\Psi}(\boldsymbol{\sigma}) = \mathbf{H}(\boldsymbol{\sigma}) = \mathbf{H}(\boldsymbol{\sigma})^A = (d_F \boldsymbol{\Psi}(\boldsymbol{\sigma}))^A$$

implies the vanishing, in each stress fiber, of any loop integral

$$\oint \langle \boldsymbol{\Psi}(\ell(\lambda)), \dot{\ell}(\lambda) \rangle \, d\lambda = 0.$$

This ensures existence of a scalar-valued stress-potential $E : \mathrm{C\small{ON}}(V\mathcal{T}) \mapsto \mathcal{R}$ which may be computed as

$$\begin{aligned} E(\boldsymbol{\sigma}) &= \int_0^1 (\boldsymbol{\Psi} \circ \mathbf{c})(\lambda) \cdot \dot{\mathbf{c}}(\lambda) \, d\lambda \\ &= \int_0^1 d\lambda \int_0^\lambda (\mathbf{H} \circ \mathbf{c})(\xi) \cdot \dot{\mathbf{c}}(\xi) \, d\xi, \end{aligned} \tag{84}$$

so that $\boldsymbol{\Psi} = d_F E$ and $\mathbf{H} = d_F \boldsymbol{\Psi} = d_F^2 E$. □

Remark 2 A rate elastic law similar to Eq. (75) has been considered in [37] and in a large number of contributions, for instance [38–42]. According to their treatment, integrability of the rate elastic law (75) requires that a co-rotational logarithmic stress rate is considered. For a purely elastic behavior, the integrated law results in a relation between Hencky logarithmic strain

$$\boldsymbol{\varepsilon}_H(\boldsymbol{\varphi}_\alpha) := \frac{1}{2} \ln(\mathbf{F}\mathbf{F}^A) = \frac{1}{2} \ln(T\boldsymbol{\varphi}_\alpha \cdot (T\boldsymbol{\varphi}_\alpha)^A) \tag{85}$$

and the fiber derivative of the scalar stress potential evaluated at the (standard) mixed Kirchhoff stress

$$\boldsymbol{\varepsilon}_H(\boldsymbol{\varphi}_\alpha) = d_F \boldsymbol{\Psi}(J_\alpha \mathbf{T}). \tag{86}$$

Hidden in these expressions is the assumption of a reference local placement. This is a basic distinction from our analysis, exposed in Sect. 10, where no reference manifold is resorted to and conservation of mechanical energy is implied by the Green integrability condition, see Sect. 14.

12 Elasticity

We may now give the new mathematical definition of an elastic material in the full non-linear range.

Definition 18 (*Elasticity and hyper-elasticity*) A rate elastic constitutive law, defined by

$$\mathbf{el} := \mathbf{C}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}) = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \tag{87}$$

is *elastic* (resp. *hyper-elastic*) if the constitutive operator is time-invariant according to Definition 17, i.e.,

$$\mathbf{C} = \boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{C}, \tag{88}$$

and the *rate elastic operator* $\mathbf{H}$ is Cauchy (resp. GREEN) integrable, and admits an injective Cauchy potential $\boldsymbol{\Psi}$, so that

$$\mathbf{el} = d_F \boldsymbol{\Psi}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, \qquad (\mathbf{el} = d_F^2 E(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}). \tag{89}$$

Time invariance is expressed in terms of the potentials by the condition

$$\boldsymbol{\Psi} = \boldsymbol{\varphi}_\alpha \!\uparrow\! \boldsymbol{\Psi}, \qquad (E = \boldsymbol{\varphi}_\alpha \!\uparrow\! E). \tag{90}$$

The physical idea of elasticity requires that the rate elastic law must comply with the following requirements.

1. The elastic constitutive response does not vary with time.
2. An elastic material, when subjected to any isometric displacement, does not change its state of stress.
3. An elastic material does not dissipate or generate mechanical energy in any motion which leaves invariant the state of stress.

Property 1 is expressed by the condition (88). Properties 2 and 3 are proven to hold under the integrability conditions enunciated in Proposition 3, i.e., that for all stress $\boldsymbol{\sigma}$ and for all stress variations $\delta\boldsymbol{\sigma}$ it is

$$\begin{cases} (d_F \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma})^A = d_F \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}, \\ \mathbf{H}(\boldsymbol{\sigma})^A = \mathbf{H}(\boldsymbol{\sigma}). \end{cases} \tag{91}$$

These last two results are provided below in Propositions 4 and 5, by relying on a local referential formulation. Property 3 leads to introduce the notion of levels of elastic energy attached to equivalence classes of stress states along the motion, as illustrated in Sect. 14.

13 Referential finite elasticity

Setting $\boldsymbol{\Psi}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\boldsymbol{\Psi}$, $\mathbf{H}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\mathbf{H}$, $\boldsymbol{\sigma}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\boldsymbol{\sigma}$ and $\mathbf{el}_{\text{REF}} := \boldsymbol{\chi}\!\uparrow\!\mathbf{el}$, the constitutive relation of an elastic (rate elastic, time-invariant, Cauchy integrable) material is expressed in the straightened trajectory by

$$\begin{aligned} \mathbf{el}_{\text{REF}} &= \mathbf{H}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \cdot \partial_{\alpha=0}\, \boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha \\ &= d_F \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \cdot \partial_{\alpha=0}\, \boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha \\ &= \partial_{\alpha=0}\, \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha) \end{aligned} \tag{92}$$

with the time translation $\boldsymbol{\theta}_\alpha : \boldsymbol{\Omega}_{\text{REF}} \times \mathcal{Z} \mapsto \boldsymbol{\Omega}_{\text{REF}} \times \mathcal{Z}$ defined by (67).

Integration in a time lapse α yields the expression of the finite-step referential elastic strain, given by

$$\mathbf{El}_{\text{REF}}(\alpha) := \int_0^\alpha \mathbf{el}_{\text{REF}}\, dt = \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha) - \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}). \tag{93}$$

Contrary to usual formulations of the referential elastic law [10,35,43–45], the expression above relates the referential elastic strain, accumulated in the time lapse α, to the variation of the referential elastic response to the referential stress, in the same time lapse.

Assuming invertibility of the stress-potential $\boldsymbol{\Psi}_{\text{REF}}$, the referential stress after the time lapse α can be written as

$$\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha = \boldsymbol{\Psi}_{\text{REF}}^{-1}\big((\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}})) + \mathbf{El}_{\text{REF}}(\alpha)\big). \tag{94}$$

13.1 Special referential formulations

To compare with other formulations of nonlinear elasticity, we must introduce special assumptions.

1. If the rate elastic constitutive operator $\mathbf{H}$ does not depend on the stress, the referential potential $\boldsymbol{\Psi}_{\mathrm{REF}}$ is linear in the referential stress $\boldsymbol{\sigma}_{\mathrm{REF}}$, and the referential elastic law (94) can be written as

$$\boldsymbol{\sigma}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha} - \boldsymbol{\sigma}_{\mathrm{REF}} = \boldsymbol{\Psi}_{\mathrm{REF}}^{-1}(\mathbf{El}_{\mathrm{REF}}(\alpha)). \tag{95}$$

Assuming Green integrability, so that $\boldsymbol{\Psi}_{\mathrm{REF}} = d_F E_{\mathrm{REF}}$, the scalar potential $E_{\mathrm{REF}} := \boldsymbol{\chi}\!\uparrow\! E$ is quadratic.
2. If a *pure elastic* response is considered, that is, if $\mathbf{el} = \boldsymbol{\varepsilon}(\mathbf{v})$, the referential elastic stretching is one-half the partial time derivative of the referential metric tensor

$$\mathbf{el}_{\mathrm{REF}} = \boldsymbol{\chi}\!\uparrow\!(\tfrac{1}{2}\mathcal{L}_{\mathbf{v}}\,\mathbf{g}) = \tfrac{1}{2}\partial_{\alpha=0}\,(\mathbf{g}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha}), \tag{96}$$

and the finite-step referential total strain is got by integration in time,

$$\mathbf{El}_{\mathrm{REF}}(\alpha) = \tfrac{1}{2}\int_0^{\alpha} \mathbf{g}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha}\, d\alpha = \tfrac{1}{2}\mathbf{g}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha} - \tfrac{1}{2}\mathbf{g}_{\mathrm{REF}}. \tag{97}$$

3. The finite-step referential elastic strain pertaining to *pure elasticity* may then be written as

$$\tfrac{1}{2}\mathbf{g}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha} - \tfrac{1}{2}\mathbf{g}_{\mathrm{REF}} = \boldsymbol{\Psi}_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha}) - \boldsymbol{\Psi}_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}}), \tag{98}$$

with the referential metric tensor $\mathbf{g}_{\mathrm{REF}}$ given by (73.1).

In the linear realm of the straightened trajectory, linear operations (such as time integration) are feasible, but the results can be transformed back to the actual trajectory only if they refer to a definite time instant.

As a consequence, concepts such as *finite elastic strain* or *finite plastic strain*, which are integrals over a time interval of the corresponding stretching, cannot be interpreted as material tensors on the actual trajectory and are bound to live in an arbitrarily fixed referential manifold.

On the contrary, the computational evaluation of referential stress field at the end of a time step can be transformed back to the actual trajectory to provide the corresponding actual stress field at the same instant of time.

These features, which apply also to the geometrically linearized context, where the actual trajectory is assumed to be straight, are often overlooked in theoretical and computational treatments of elasticity.

Proposition 4 (Stress invariance and isometries) *In Cauchy elastic materials any displacement which leaves the stress tensor invariant is an isometry. The converse implication, that any isometric displacement leaves the stress tensor invariant, holds under the assumption that the Cauchy elastic stress-potential $\boldsymbol{\Psi}$ is injective.*

Proof From the referential elastic relation (93) it follows that

$$\boldsymbol{\sigma}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha} = \boldsymbol{\sigma}_{\mathrm{REF}} \quad \Longrightarrow \quad \mathbf{El}_{\mathrm{REF}}(\alpha) = \mathbf{0}.$$

The result follows from (69) by pulling back (98) to the trajectory. Vice versa, if a pure elastic response is considered, by (97) isometry of the displacement implies that $\mathbf{El}_{\mathrm{REF}}(\alpha) = \mathbf{0}$, and hence (93) gives

$$\boldsymbol{\Psi}_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha}) = \boldsymbol{\Psi}_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}}).$$

Injectivity of the elastic stress-potential $\boldsymbol{\Psi}$ implies that the referential potential $\boldsymbol{\Psi}_{\mathrm{REF}} := \boldsymbol{\chi}\!\uparrow\!\boldsymbol{\Psi}$ is injective too. Equality $\boldsymbol{\sigma}_{\mathrm{REF}} \circ \boldsymbol{\theta}_{\alpha} = \boldsymbol{\sigma}_{\mathrm{REF}}$ between referential stress tensors after the time lapse $\alpha \in \mathcal{R}$ thus follows, and this entails equivalence of material stress tensor fields. □

Pushing the elastic potential by a straightening diffeomorphism, a time-independent potential is got. Potentials on straightened trajectories in diffeomorphic correspondence are related by push.

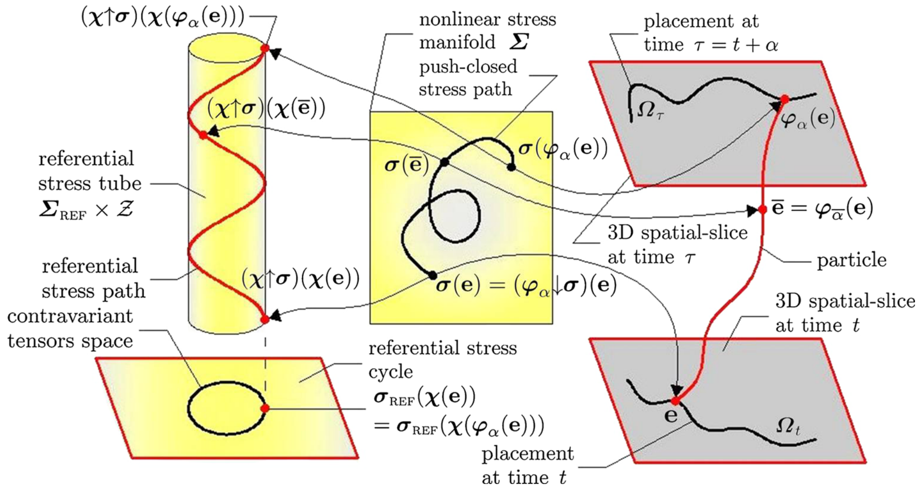


Fig. 7 Stress cycle

14 Elastic energy

We are now ready to provide an answer to the third requirement for an elastic behavior, enunciated in Sect. 12.

Definition 19 (*Elastic work*) The elastic work per unit mass, performed in a time lapse, is the integral of the elastic power per unit mass, over the particle segment $\boldsymbol{\varphi}_t$ with $t \in [0, \alpha]$, i.e.

$$\int_0^\alpha \langle \boldsymbol{\sigma}, \mathbf{el} \rangle\, \mathbf{m} \circ \boldsymbol{\varphi}_t\, dt. \tag{99}$$

Definition 20 (*Conservativeness*) The constitutive operator $\mathbf{C}$ of rate elasticity is *conservative* if the elastic work per unit mass, performed on a particle segment $\boldsymbol{\varphi}_t$ with $t \in [0, \alpha]$, vanishes when the corresponding stress path $\boldsymbol{\sigma} \circ \boldsymbol{\varphi}_t$ with $t \in [0, \alpha]$ is push-closed,

$$\int_0^\alpha \langle \boldsymbol{\sigma}, \mathbf{el} \rangle\, \mathbf{m} \circ \boldsymbol{\varphi}_t\, dt = \int_0^\alpha \langle \boldsymbol{\sigma}, \mathbf{C}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}) \rangle\, \mathbf{m} \circ \boldsymbol{\varphi}_t\, dt = 0. \tag{100}$$

Push-closedness of the stress path $\boldsymbol{\sigma} \circ \boldsymbol{\varphi}_t$, $t \in [0, \alpha]$ means that the stress takes equivalent values at the endpoints, i.e. by (63) that $\boldsymbol{\sigma} = \boldsymbol{\varphi}_\alpha \!\downarrow\! \boldsymbol{\sigma}$.

Conservativeness according to Definition 20 amounts to require that the stress path, when pulled back to a straightened trajectory, projects to a closed path of referential stress in any referential placement, as depicted in Fig. 7.

From Proposition 4, it is inferred that conservativeness according to Definition 20 implies conservativeness with respect to *no total strain* paths, that is, paths such that $\mathbf{g} = \boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{g}$. The converse implication requires the assumption of a *purely elastic* behavior with an injective Cauchy elastic potential.

The next result shows that conservation of mass and GREEN's integrability of the rate elastic operator, when expressed in terms of the Kirchhoff stress tensor, implies conservation of elastic energy.

Proposition 5 (Hyper-elasticity and conservativeness) *The constitutive operator of a rate elastic material, which is hyper-elastic, is conservative.*

Proof Conservativeness of the constitutive operator may be formulated by performing a push to a straightened trajectory by $\boldsymbol{\chi} : \mathcal{T} \mapsto \boldsymbol{\Omega}_{\mathrm{REF}} \times \mathcal{Z}$. Conservation of mass ensures that the referential mass density $\mathbf{m}_{\mathrm{REF}} := \boldsymbol{\chi} \!\uparrow\! \mathbf{m}$ is time-independent. Setting $\boldsymbol{\Psi}_{\mathrm{REF}} := \boldsymbol{\chi} \!\uparrow\! \boldsymbol{\Psi}$ and $\mathbf{el}_{\mathrm{REF}} := \boldsymbol{\chi} \!\uparrow\! \mathbf{el}$, by Cauchy integrability we have that

$$\mathbf{el}_{\mathrm{REF}} = \partial_{\alpha=0}\, \boldsymbol{\Psi}_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}} \circ \boldsymbol{\theta}_\alpha),$$

and the referential elastic power per unit mass may be written as

$$\begin{aligned}\langle \boldsymbol{\sigma}_{\text{REF}}, \mathbf{el}_{\text{REF}} \rangle &= \langle \boldsymbol{\sigma}_{\text{REF}}, \partial_{\alpha=0}\, \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha) \rangle \\ &= \partial_{\alpha=0}\, \langle \boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha, \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha) \rangle \\ &\quad - \langle \partial_{\alpha=0}\, (\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha), \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \rangle .\end{aligned}$$

By Green integrability, setting $E_{\text{REF}} = \boldsymbol{\chi}\!\uparrow\! E$ we get $\boldsymbol{\Psi}_{\text{REF}} = d_F E_{\text{REF}}$, so that

$$\langle \partial_{\alpha=0}\, \boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha, \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \rangle = \partial_{\alpha=0}\, E_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha).$$

The referential elastic power takes then the expression

$$\langle \boldsymbol{\sigma}_{\text{REF}}, \mathbf{el}_{\text{REF}} \rangle = \partial_{\alpha=0}\, (\langle \boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha, \boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha) \rangle - E_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}} \circ \boldsymbol{\theta}_\alpha)),$$

whose integral along a closed referential stress path vanishes. □

By Proposition 5, the mechanical work per unit mass, spent in performing an elastic deformation process along a particle, depends only on the equivalence gap between stress tensors in the source and in the target events.

The variation of elastic energy per unit mass, in passing from one event to another one along a particle, is evaluated by the integral

$$\int_0^\alpha \langle \boldsymbol{\sigma}, \mathbf{el} \rangle\, \mathbf{m} \circ \boldsymbol{\varphi}_t \, dt .$$

According to the new geometric theory of elasticity, in the general nonlinear context of finite displacements, the notion of elastic energy ought to be introduced as a scalar functional whose domain is the *nonlinear stress manifold* $\boldsymbol{\Sigma} \in \text{Con}(V\mathcal{T})$.

The notion of *elastic strain energy* expressed by (95) is in fact confined to a referential placement and can be considered as a material functional only in linearized approximations, where actual and straightened trajectories are identified.

Under the special assumptions that the process is *purely elastic* and that the Cauchy elastic *stress-potential* $\boldsymbol{\Psi}$ is injective, Proposition 4 entails that the elastic *stress-energy* per unit mass takes the same value at events related by an isometric displacement.

15 Elasto-visco-plasticity

Once that the rate elastic model has been properly formulated and the right conditions for time invariance (Definition 17) and conservativeness (Proposition 5) have been assessed, the rate model of *elasto-visco-plastic* constitutive behavior, which is of primary applicative interest in NLCM, can be described by the relations:

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{el} + \mathbf{pl}, & \text{stretching additivity,} \\ \mathbf{el} = d_F^2 E(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, & \text{hyper-elastic law,} \\ \mathbf{pl} \in \partial_F \mathcal{F}(\boldsymbol{\sigma}), & \text{visco-plastic flow rule.} \end{cases} \tag{101}$$

Here, $\mathcal{F} \subset \text{Fun}(V\mathcal{T})$ is the visco-plastic potential, fiberwise subdifferentiable and convex [46,47].

These constitutive relations are in fact extensions of the classical formula introduced, with reference to visco-elasticity, by James Clerk-Maxwell in [48].

Neither the *elastic stretching* $\mathbf{el} \in \mathrm{C}^1(\mathcal{T}, \text{Sym}(V\mathcal{T}))$ nor the *visco-plastic stretching* $\mathbf{pl} \in \mathrm{C}^1(\mathcal{T}, \text{Sym}(V\mathcal{T}))$ are Lie time derivatives of finite stretch fields. Elastic and plastic stretching should consistently *not* be denoted by a superimposed dot, as often made in the literature. An exception is the notation adopted in [49].

Elasto-plasticity is modeled by assuming that the stress potential is the indicator function of a convex set of admissible stresses $\mathcal{K} \subset \mathrm{SYM}^*(V\mathcal{T})$, so that $\mathcal{F}(\boldsymbol{\sigma}) = \sqcup_{\mathcal{K}}(\boldsymbol{\sigma})$ and the constitutive relations become

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{el} + \mathbf{pl}, & \text{stretching additivity,} \\ \mathbf{el} = d_F^2 E(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, & \text{elastic law,} \\ \mathbf{pl} \in \partial_F \sqcup_{\mathcal{K}} (\boldsymbol{\sigma}) = \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}), & \text{plastic flow rule.} \end{cases} \tag{102}$$

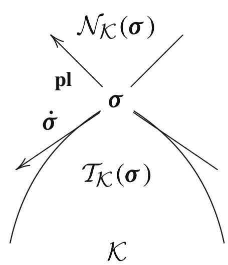
where E is the scalar stress elastic potential corresponding to Green integrability, ∂_F is the fiber-subdifferential of the convex indicator $\sqcup_{\mathcal{K}}$ of the elastic domain $\mathcal{K}$ at $\boldsymbol{\sigma} \in \mathcal{K}$, and $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})$ is the outward normal cone, with $\mathbf{pl}$ *plastic stretching* tensor. The plastic flow rule may equivalently be expressed by the fiberwise variational inequality

$$\langle \mathbf{pl}, \overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \rangle \leq 0, \quad \boldsymbol{\sigma} \in \mathcal{K}, \quad \forall \overline{\boldsymbol{\sigma}} \in \mathcal{K}. \tag{103}$$

Incremental elasto-plasticity is modeled by

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{el} + \mathbf{pl}, & \text{stretching additivity,} \\ \mathbf{el} = d_F^2 E(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, & \text{elastic law,} \\ \mathbf{pl} \in \mathcal{N}_{\mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})}(\dot{\boldsymbol{\sigma}}), & \text{rate plastic flow rule.} \end{cases} \tag{104}$$

This rate rule of plastic flow is more stringent than the plastic flow rule formulated by (102) and is equivalent to the following fiberwise complementarity condition, known as Prager's *rule*:



$$\begin{cases} \langle \mathbf{pl}, \dot{\boldsymbol{\sigma}} \rangle = 0, \\ \mathbf{pl} \in \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}), \\ \dot{\boldsymbol{\sigma}} \in \mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma}) \,. \end{cases} \tag{105}$$

Here, $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}), \mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})$ are the normal and the tangent cones to the elastic domain $\mathcal{K}$ at the stress point $\boldsymbol{\sigma} \in \mathcal{K}$, and $\mathcal{N}_{\mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})}(\dot{\boldsymbol{\sigma}})$ is the normal cone to the tangent cone $\mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})$, at the stress-rate point $\dot{\boldsymbol{\sigma}} \in \mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})$.

16 Frame invariance

A *change of frame* in the Euclid space-time is an isometric automorphism $\boldsymbol{\zeta}_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{E})$ in the event time bundle, characterized by the property

$$\boldsymbol{\zeta}_{\mathcal{E}} \!\downarrow\! \mathbf{g}_{\mathcal{S}} = \mathbf{g}_{\mathcal{S}}. \tag{106}$$

Definition 21 (*Trajectory transformation*) A *trajectory transformation* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T}_{\boldsymbol{\zeta}})$ is induced by a *change of frame* $\boldsymbol{\zeta}_{\mathcal{E}} \in \mathrm{C}^1(\mathcal{E}\,;\mathcal{E})$, as described by the commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\zeta_{\mathcal{E}}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{E} \arrow[dd, bend left=60, "t_{\mathcal{E}\zeta}"] \\ \mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[r, "\zeta"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T}_{\zeta} \arrow[u, "\mathbf{i}_{\zeta}"] \arrow[d, "t_{\mathcal{T}\zeta}"] \\ \mathcal{Z} \arrow[r, leftrightarrow, "\mathrm{ID}"] & \mathcal{Z} \end{tikzcd} \tag{107}$$

The material metric tensors $\mathbf{g} \in \mathrm{COV}(V\mathcal{T})$ and $\mathbf{g}_{\zeta} \in \mathrm{COV}(V\mathcal{T}_{\zeta})$, in the source and the target trajectory time bundle, are defined by pull back according to the relevant immersions

$$\mathbf{g} := \mathbf{i}\!\downarrow\!\mathbf{g}_{\mathcal{S}}, \qquad \mathbf{g}_{\zeta} := \mathbf{i}_{\zeta}\!\downarrow\!\mathbf{g}_{\mathcal{S}}. \tag{108}$$

From (106) and (107), it follows that $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T}_{\boldsymbol{\zeta}})$ is an *isometric isomorphism* between the two trajectory time bundles, as seen by distinct observers, according to the property

$$\boldsymbol{\zeta}{\downarrow}\mathbf{g}_{\boldsymbol{\zeta}} = \mathbf{g}. \tag{109}$$

The trajectory transformation $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T}_{\boldsymbol{\zeta}})$ relates the motion $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ and the pushed motion $\boldsymbol{\zeta}{\uparrow}\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\zeta}}\,;\mathcal{T}_{\boldsymbol{\zeta}})$ according to the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\boldsymbol{\zeta}} \arrow[r, "\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha"] & \mathcal{T}_{\boldsymbol{\zeta}} \\ \mathcal{T} \arrow[u, "\boldsymbol{\zeta}"'] \arrow[r, "\boldsymbol{\varphi}_\alpha"] & \mathcal{T} \arrow[u, "\boldsymbol{\zeta}"'] \end{tikzcd} \iff (\boldsymbol{\zeta}{\uparrow}\boldsymbol{\varphi}_\alpha)\circ\boldsymbol{\zeta} = \boldsymbol{\zeta}\circ\boldsymbol{\varphi}_\alpha. \tag{110}$$

The space-time velocity $\mathbf{V} := \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha$ is then also transformed by push,

$$\mathbf{V}_{\boldsymbol{\zeta}} := \partial_{\alpha=0}\,\boldsymbol{\zeta}{\uparrow}\boldsymbol{\varphi}_\alpha = \boldsymbol{\zeta}{\uparrow}\mathbf{V}. \tag{111}$$

Remark 3 The usual transformation law for the spatial velocity due to a relative rigid motion between observers is recovered by considering the explicit expression of an isometric frame-transformation in the Euclid space which, in terms of time-dependent rotation $\mathbf{Q}$ and translation $\mathbf{c}$, is defined by

$$\boldsymbol{\zeta}_{\mathcal{E}} : \begin{cases} \mathbf{x} \mapsto \mathbf{Q}(t)\cdot\mathbf{x} + \mathbf{c}(t) \\ t \mapsto t\,. \end{cases} \tag{112}$$

The associated Jacobi space-time block matrix is given by

$$[T\boldsymbol{\zeta}_{\mathcal{E}}] = \begin{bmatrix} \mathbf{Q} & (\dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}) \\ \mathbf{0} & 1 \end{bmatrix}. \tag{113}$$

The pushed velocity $\boldsymbol{\zeta}{\uparrow}\mathbf{V}$ has then the space-time block expression

$$[T\boldsymbol{\zeta}_{\mathcal{E}}]\cdot[\mathbf{V}] = \begin{bmatrix} \mathbf{Q} & (\dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}) \\ \mathbf{0} & 1 \end{bmatrix}\cdot\begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}} \\ 1 \end{bmatrix} \tag{114}$$

which provides the usual transformation rule for the spatial velocity.

By definition frame-transformations in the Euclid space are isometric, and hence by (109) the metric tensor is frame-invariant: $\mathbf{g}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\mathbf{g}$.

A natural axiom of Continuum Mechanics, which formalizes the physical requirement that material behavior must be independent of the special observer performing measurements, is enunciated below.

Axiom 1 (Material Frame Invariance) *All material tensors are Euclid frame invariant.*

Denoting by a subscript $()_{\boldsymbol{\zeta}}$ transformed material tensor fields, frame invariance of stress and elastic stretching is expressed by

$$\boldsymbol{\sigma}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\sigma}, \quad \mathbf{el}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\mathbf{el}. \tag{115}$$

It follows that the elastic power $\langle\boldsymbol{\sigma}, \mathbf{el}\rangle$ is frame-invariant,

$$\langle\boldsymbol{\sigma}_{\boldsymbol{\zeta}}, \mathbf{el}_{\boldsymbol{\zeta}}\rangle = \langle\boldsymbol{\zeta}{\uparrow}\boldsymbol{\sigma}, \boldsymbol{\zeta}{\uparrow}\mathbf{el}\rangle = \boldsymbol{\zeta}{\uparrow}\langle\boldsymbol{\sigma}, \mathbf{el}\rangle. \tag{116}$$

Proposition 6 (Frame invariance of time rates) *Frame-invariance of a material tensor field implies frame-invariance of its time-rate along the motion*

$$\mathbf{s}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\mathbf{s} \implies \mathcal{L}_{\mathbf{V}_{\boldsymbol{\zeta}}}\,\mathbf{s}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}(\mathcal{L}_{\mathbf{V}}\,\mathbf{s}).$$

Proof The push transformation law for the space-time velocity (111) and the naturality property of the Lie derivative with respect to push, expressed for any tensor field $\mathbf{s}$ on the trajectory, by

$$\mathcal{L}_{\boldsymbol{\zeta}\uparrow\mathbf{V}}(\boldsymbol{\zeta}{\uparrow}\mathbf{s}) = \boldsymbol{\zeta}{\uparrow}(\mathcal{L}_{\mathbf{V}}\,\mathbf{s}), \tag{117}$$

lead to assess the result. □

By (65), it follows that invariance of stress implies invariance of stressing,

$$\boldsymbol{\sigma}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma} \quad \Longrightarrow \quad \dot{\boldsymbol{\sigma}}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}\!\uparrow\!\dot{\boldsymbol{\sigma}}. \tag{118}$$

In providing a mathematical model of mechanical properties of materials, it is natural to require that a Constitutive Frame Invariance principle (CFI) be fulfilled.

Principle 1 (Constitutive Frame Invariance) *Frame invariance requires that constitutive operators described by distinct observers must fulfil the relation*

$$\mathbf{C}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{C}. \tag{119}$$

For a rate elastic material (75), the pushed constitutive operator $\boldsymbol{\zeta}\!\uparrow\!\mathbf{C}$ is defined by the property that the transformed triplet $(\boldsymbol{\zeta}\!\uparrow\!\mathbf{el}, \boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma}, \boldsymbol{\zeta}\!\uparrow\!\dot{\boldsymbol{\sigma}})$ will fulfills the transformed rate elastic law

$$\boldsymbol{\zeta}\!\uparrow\!\mathbf{el} = (\boldsymbol{\zeta}\!\uparrow\!\mathbf{C})(\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma}, \boldsymbol{\zeta}\!\uparrow\!\dot{\boldsymbol{\sigma}}), \tag{120}$$

if and only if the triplet $(\mathbf{el}, \boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}})$ fulfills the rate elastic law (75). The requirement (119) of CFI may be then expressed by the equivalence

$$\mathbf{el} = \mathbf{C}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}) \quad \Longleftrightarrow \quad \boldsymbol{\zeta}\!\uparrow\!\mathbf{el} = \mathbf{C}_{\boldsymbol{\zeta}}(\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma}, \boldsymbol{\zeta}\!\uparrow\!\dot{\boldsymbol{\sigma}}). \tag{121}$$

The physical requirement expressed by CFI is that the results of laboratory experiments performed by two observers should be comparable in a *natural* way, that is, on the basis of the knowledge of their relative motion.

If the first observer detects a constitutive operator $\mathbf{C}$, relating state variables with constitutive responses, he is able to foretell the result of the same experiment performed by the second observer, who detects a constitutive operator $\mathbf{C}_{\boldsymbol{\zeta}}$ relating the transformed state variables and the transformed constitutive responses.

Remark 4 The principle of *Material Frame Indifference* (MFI), as enunciated in [16], p. 403, and amended in [14] to take into account that both the source and the target placements are transformed by a change of observer[12], when translated in geometric notations, consists in the requirement that

$$\mathbf{C} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{C}, \tag{122}$$

where the isometry $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}})$, fulfilling the following sub-diagram of 107, describes the relative motion between observers

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\zeta"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T}_{\zeta} \arrow[d, "\mathcal{T}_{\zeta}"] \\ \mathcal{Z} \arrow[r, "\mathrm{ID}"] & \mathcal{Z} \end{tikzcd} \tag{123}$$

The constitutive maps $\mathbf{C}$ and $\boldsymbol{\zeta}\!\uparrow\!\mathbf{C}$ involved in the equality (122) refer to constitutive descriptions made by distinct observers and have therefore domains and codomains which are material tensor bundles based on distinct trajectory manifolds $\mathcal{T}$ and $\mathcal{T}_{\boldsymbol{\zeta}}$. By definition (120), the equality in (122) expressing MFI amounts to require that

$$\mathbf{el} = \mathbf{C}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}) \quad \Longleftrightarrow \quad \boldsymbol{\zeta}\!\uparrow\!\mathbf{el} = \mathbf{C}(\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma}, \boldsymbol{\zeta}\!\uparrow\!\dot{\boldsymbol{\sigma}}). \tag{124}$$

The r.h.s. of this equivalence is, however, geometrically incorrect because the constitutive operator $\mathbf{C}$ acts on material tensors based on the trajectory $\mathcal{T}$, while the argument material tensors at the r.h.s. (124) are based on the transformed trajectory $\mathcal{T}_{\boldsymbol{\zeta}}$.

In place of the incorrect equality (122) expressing the MFI, the geometrically consistent requirement (119) of the CFImust be adopted. The constitutive maps $\mathbf{C}_{\boldsymbol{\zeta}}$ and $\boldsymbol{\zeta}\!\uparrow\!\mathbf{C}$ involved in the equality (119) are in fact based on the same transformed trajectory manifold $\mathcal{T}_{\boldsymbol{\zeta}}$, and their equality may well be imposed and checked by a single observer.

[12] The same amendment was proposed by the first author in [50, Tomo I, p. 57].

Definition 22 (*Material isotropy*) The condition of *isotropy* of a constitutive operator $\mathbf{C}$ consists in the requirement that, for any simultaneity preserving isometric transformation $\boldsymbol{\xi} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$, the following equality holds:

$$\mathbf{C} = \boldsymbol{\xi}\!\uparrow\!\mathbf{C}, \tag{125}$$

a condition explicitly expressed by the equivalence

$$\mathbf{el} = \mathbf{C}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}) \quad \Longleftrightarrow \quad \boldsymbol{\xi}\!\uparrow\!\mathbf{el} = \mathbf{C}(\boldsymbol{\xi}\!\uparrow\!\boldsymbol{\sigma}, \boldsymbol{\xi}\!\uparrow\!\dot{\boldsymbol{\sigma}}). \tag{126}$$

The isometry $\boldsymbol{\xi} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ is defined by the condition $\mathbf{g} = \boldsymbol{\xi}\!\downarrow\!\mathbf{g}$ and by the commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\boldsymbol{\xi}"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T} \arrow[d, "t_{\mathcal{T}}"] \\ \mathcal{Z} \arrow[r, "\mathrm{ID}"] & \mathcal{Z} \end{tikzcd} \tag{127}$$

The physical interpretation of the condition (125)–(126) is that a single observer gets the same constitutive response by testing two mutually rotated material specimens[13].

The similarity, between the statement (122)–(124) of the MFI and the condition of isotropy (125)–(126), has been the source of confusion that led to sustain the unphysical conclusion that MFI implies isotropy of material behavior [16] fn. 1, Sect. 47, p. 140 and fn. 2, Sect. 99, p. 403.

The geometric treatment reveals instead a basic difference because $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T}_{\boldsymbol{\zeta}})$ is an isometric transformation between two distinct trajectories, induced by a change of observer according to Definition 21, and the map $\boldsymbol{\xi} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ is an isometric transformation in a single trajectory, evaluated by a single observer.

As a consequence, the statement (122)–(124) of the MFI is untenable, as observed above, while the requirement (125), (126) of isotropy is correctly formulated.

The geometric treatment reveals that the requirement of *Constitutive Frame Invariance* formulated in (119) has nothing to do with material isotropy because two distinct constitutive operators, evaluated by distinct observers, are involved.

The results exposed in Sects. 11, 12, 13, 14, 16, give to the new rate elastic law the right position of basic constitutive model for elasticity[14].

17 Computational scheme

In computational algorithms for the detection of the static evolution of an elastic structure undergoing large displacements, the equilibrium process is approximated by finite-step incremental solutions in time.

To underline the decisive computational role of a referential formulation, let us briefly describe the iterative scheme leading to the solution of an elastostatic problem in a finite time step.

The *control process* is described by a time-parametrized curve $\mathbf{c} : \mathcal{Z} \mapsto \mathrm{C}$ in a control manifold C.

The loading acting on a body at time $\tau = t + \alpha \in \mathcal{Z}$ is assumed to depend on the control point $\mathbf{c}(\tau)$ and on the displacement $\boldsymbol{\varphi}_\alpha$ from the placement $\boldsymbol{\Omega}_t$, so that we may write

$$\mathbf{f}(\tau) = \mathcal{P}(\mathbf{c}(\tau), \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}_t)) : \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}_t) \mapsto T^*\mathcal{S}.$$

An iterative trial and error procedure may be formulated as follows.

1. The starting point is an equilibrium solution at time $t \in \mathcal{Z}$ under the loading

$$\mathbf{f}(t) = \mathcal{P}(\mathbf{c}(t), \boldsymbol{\Omega}_t) : \boldsymbol{\Omega}_t \mapsto T^*\mathcal{S}$$

which, in conjunction with the stress field $\boldsymbol{\sigma}_t$, fulfills the virtual power variational equality

$$\langle \mathbf{f}, \delta\mathbf{v} \rangle_t = \int_{\boldsymbol{\Omega}_t} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m}.$$

13 The group of isometries such that (125) holds characterizes the symmetry properties of the material.

14 The stress independent isotropic rate elastic law fulfills the CFI principle [15].

2. An initial guess for the displacement $\boldsymbol{\varphi}_\alpha$ consequent to an update of the input control point from $\mathbf{c}(t)$ to $\mathbf{c}(\tau)$ may be obtained by updating the loading to $\mathbf{f} = \mathcal{P}(\mathbf{c}(\tau), \boldsymbol{\Omega}_t) : \boldsymbol{\Omega}_t \mapsto T^*\mathcal{S}$. The placement $\boldsymbol{\Omega}_t$ is assumed as reference placement. The vector field $\mathbf{u} : \boldsymbol{\Omega}_t \mapsto T\mathcal{S}$ is the incremental displacement from $\boldsymbol{\Omega}_t$, and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the corresponding stretching (linearized strain increment). The constitutive stress response is then expressed by
$$\boldsymbol{\sigma}_{\text{REF}} = \boldsymbol{\Psi}^{-1}_{\text{REF}}\big(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_t) + \boldsymbol{\varepsilon}(\mathbf{u})\big).$$
The solution of the geometrically linearized elastostatic problem
$$\begin{aligned}\langle \mathbf{f}, \delta\mathbf{v}\rangle_t &= \langle \mathcal{P}(\mathbf{c}(\tau), \boldsymbol{\Omega}_t), \delta\mathbf{v}\rangle \\ &= \int_{\boldsymbol{\Omega}_t} \langle \boldsymbol{\Psi}^{-1}_{\text{REF}}\big(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_t) + \boldsymbol{\varepsilon}(\mathbf{u})\big), \boldsymbol{\varepsilon}(\delta\mathbf{v})\rangle\, \mathbf{m}\end{aligned} \tag{128}$$
yields the spatial vector field $\mathbf{u} : \boldsymbol{\Omega}_t \mapsto T\mathcal{S}$ as first trial incremental displacement from $\boldsymbol{\Omega}_t$.
3. Setting $\boldsymbol{\varphi}_\alpha(\mathbf{e}) = \mathbf{u}(\mathbf{e}) + \mathbf{e}$ for $\mathbf{e} \in \boldsymbol{\Omega}_t$, the first trial placement of the body at time τ is evaluated by the assignment $\boldsymbol{\Omega} = \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}_t)$.
4. The control algorithm provides the loading update
$$\mathbf{f} = \mathcal{P}(\mathbf{c}(\tau), \boldsymbol{\Omega}) : \boldsymbol{\Omega} \mapsto T^*\mathcal{S},$$
and the trial referential finite-step elastic strain on $\boldsymbol{\Omega}_t$ is evaluated by
$$\mathbf{El}_{\text{REF}}(\alpha) = \tfrac{1}{2}(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g} - \mathbf{g}).$$
5. According to (94), the updated stress on $\boldsymbol{\Omega} = \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}_t)$ is given by
$$\boldsymbol{\sigma} = \boldsymbol{\varphi}_\alpha{\uparrow}\boldsymbol{\chi}{\downarrow}\big(\boldsymbol{\Psi}^{-1}_{\text{REF}}\big(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) + \mathbf{El}_{\text{REF}}(\alpha)\big)\big), \tag{129}$$
and the related elastic response at the trial placement $\boldsymbol{\Omega}$ is evaluated by the virtual power variational expression
$$\langle \mathbf{r}, \delta\mathbf{v}\rangle = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v})\rangle\, \mathbf{m}.$$
6. If the ratio, between a suitable norm of the force gap $\mathbf{f} - \mathbf{r} : \boldsymbol{\Omega} \mapsto T^*\mathcal{S}$ and the norm of the loading $\mathbf{f}$, is less than a prescribed tolerance, an approximated fixed point of the algorithm is deemed to be reached, and the iterations stop.
7. Otherwise, the force gap $\mathbf{f} - \mathbf{r}$ is applied to perform a correction of the previous guess concerning the displacement $\boldsymbol{\varphi}_\alpha$ from $\boldsymbol{\Omega}_t$. This task is accomplished by assuming the previous guess $\boldsymbol{\Omega}$ as reference placement. The constitutive stress response, to the previous stress trial $\boldsymbol{\sigma}$ and to a linearized strain increment $\boldsymbol{\varepsilon}(\mathbf{u})$, is expressed by
$$\boldsymbol{\sigma}_{\text{REF}} = \boldsymbol{\Psi}^{-1}_{\text{REF}}\big(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}) + \boldsymbol{\varepsilon}(\mathbf{u})\big),$$
and the solution of the geometrically linearized elastostatic problem
$$\langle \mathbf{f} - \mathbf{r}, \delta\mathbf{v}\rangle = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\Psi}^{-1}_{\text{REF}}\big(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}) + \boldsymbol{\varepsilon}(\mathbf{u})\big), \boldsymbol{\varepsilon}(\delta\mathbf{v})\rangle\, \mathbf{m} \tag{130}$$
yields the spatial vector field $\mathbf{u} : \boldsymbol{\Omega} \mapsto T\mathcal{S}$ as trial incremental displacement from $\boldsymbol{\Omega}$. The update of the displacement map from $\boldsymbol{\Omega}_t$ is given by the assignment $\boldsymbol{\varphi}_\alpha(\mathbf{e}) = \boldsymbol{\varphi}_\alpha(\mathbf{e}) + \mathbf{u}(\boldsymbol{\varphi}_\alpha(\mathbf{e}))$ for $\mathbf{e} \in \boldsymbol{\Omega}_t$. Setting $\boldsymbol{\Omega} = \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}_t)$, the iteration loop proceeds from item 4.

After convergence, the next time step is performed starting from the placement $\boldsymbol{\Omega}_\tau = \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}_t)$ under the force $\mathbf{f}(\tau) = \mathcal{P}(\mathbf{c}(\tau), \boldsymbol{\Omega}_\tau) : \boldsymbol{\Omega}_\tau \mapsto T^*\mathcal{S}$.

More complex constitutive behaviors can be modeled by adding to the rate elastic law suitable relations depending also on internal parameters and on their time derivatives and defining other stretching phenomena such as the ones induced by thermal variations, visco-plasticity, or phase transformation.

18 Discussion about standard stress-rate formulations

In the literature, a large number of proposals of stress rates have been made [6,51–59]. A systematic approach in terms of convective derivatives of different tensor alterations was provided by [60] and [61, p. 100].

In all these treatments, the time derivative along the motion of the stress tensor is split into the sum of partial time and spatial derivatives. This procedure, which gives a deceptive feeling of computational convenience, is geometrically incorrect.

To motivate our assertion, we observe that the split of the space-time displacement $\boldsymbol{\varphi}_{\alpha}^{\mathcal{E}} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ into a time step and spatial displacement, as in (12), (14) cannot be invoked to get a corresponding decomposition of the time derivative of material fields, as discussed in Sect. 4.1.

Neglecting for a moment this criticism, let us consider a purely formal procedure leading to the split formulae, as exposed in [60,61].

1. The Lie derivative and the parallel derivative of the stress tensor along the space-time motion are split in time and space components

$$\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\sigma} = \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\sigma} + \mathcal{L}_{\mathbf{Z}}\,\boldsymbol{\sigma}, \tag{131}$$

$$\nabla_{\mathbf{V}}\,\boldsymbol{\sigma} = \nabla_{\mathbf{v}}\,\boldsymbol{\sigma} + \nabla_{\mathbf{Z}}\,\boldsymbol{\sigma}. \tag{132}$$

2. The LIE derivative $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\sigma}$ along a spatial motion is expressed in terms of parallel (covariant) derivatives according to a Levi-Civita (metric and torsion-free) connection in the spatial slice, see (20.22.4), by the formula

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\sigma} = \nabla_{\mathbf{v}}\,\boldsymbol{\sigma} - \nabla\mathbf{v}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\nabla\mathbf{v})^{*}. \tag{133}$$

3. Taking account of the relation (20.7) between dual and adjoint operators and of the following equality valid in Euclid space-time

$$\mathcal{L}_{\mathbf{Z}}\,\boldsymbol{\sigma} = \partial_{\alpha=0}\,\boldsymbol{\varphi}_{\alpha}^{\mathcal{Z}}\!\downarrow\!\boldsymbol{\sigma} = \partial_{\alpha=0}\,\boldsymbol{\varphi}_{\alpha}^{\mathcal{Z}}\!\Downarrow\!\boldsymbol{\sigma} = \nabla_{\mathbf{Z}}\,\boldsymbol{\sigma} \tag{134}$$

   the formula (133) is finally written as

$$\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\sigma} = \nabla_{\mathbf{Z}}\,\boldsymbol{\sigma} + \nabla_{\mathbf{v}}\,\boldsymbol{\sigma} - \nabla\mathbf{v}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\nabla\mathbf{v})^{*}, \tag{135}$$

   with the mixed counterpart $\overset{\circ}{\mathbf{K}} = (\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\sigma})\cdot\mathbf{g}$ given by

$$\overset{\circ}{\mathbf{K}} = \dot{\mathbf{K}} - \nabla\mathbf{v}\cdot\mathbf{K} - \mathbf{K}\cdot(\nabla\mathbf{v})^{A}, \tag{136}$$

   where the $\dot{\mathbf{K}} := \nabla_{\mathbf{V}}\,\mathbf{K}$ is called the *material time derivative*[15].
   Assuming an isometric motion, we have that $\operatorname{sym}\nabla\mathbf{v} = \mathbf{0}$ and hence $\nabla\mathbf{v} = \mathbf{W}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^{A})$. Consequently, being $\mathbf{W}(\mathbf{v})^{A} = -\mathbf{W}(\mathbf{v})$, from (136) we get the Jaumann rate considered in (4):

$$\overset{\circ}{\mathbf{K}} = \dot{\mathbf{K}} - \mathbf{W}(\mathbf{v})\cdot\mathbf{K} + \mathbf{K}\cdot\mathbf{W}(\mathbf{v}). \tag{137}$$

The splitting (131) and (132) and the ensuing formulae (135) or (136), (137), although spread throughout the literature on nonlinear continuum mechanics, are affected by the following flaws.

1. The additive decompositions of the Lie derivative (131) and of the parallel derivative (132), along the space-time motion, have no general validity being subject to the special assumption that spatial and temporal components $\mathbf{v}$, $\mathbf{Z}$ of the velocity $\mathbf{V}$ are *not transversal* to the trajectory, see Fig. 3. Consequently, (133) and (134) cannot be applied to get (135).
2. The parallel (covariant) derivative (132) along the space-time velocity $\mathbf{V}$ requires to evaluate the backward parallel transport of the stress field along the material particle, but the resulting tensor field will not in general still belong to the material bundle but rather to the spatial bundle. As a result, the parallel derivative cannot appear in constitutive relations (4). This comment applies to the formulation of hypo-elasticity given in [16].

[15] The nomenclature is improper because the parallel derivative will, as a rule, yield a spatial tensor.

The comments above point out again the peculiar position of Euler formula expressing the mixed material stretching in terms of parallel derivatives, a formula whose general validity relies on the fact that the material metric tensor field is the restriction to the material bundle of the spatial metric tensor field which is defined on the whole event manifold. No other material tensor in Continuum Mechanics has this property, and hence, no general formula in terms of parallel derivatives can be found to express its time derivative along the motion.

These basic difficulties with the representation (135) are, however, of no concern because theoretical and numerical computations are conveniently performed in full generality by pushing stress $\boldsymbol{\sigma}$ and stressing $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{V}}\boldsymbol{\sigma}$ material tensors to a local straightened trajectory where Lie derivatives along the motion reduce to partial time derivatives, as shown in Sects. 8 and 13.

19 Conclusions

The treatment of constitutive models developed in [16], after an initial diffidence and also harsh criticism [62], went acquiring an increasing consensus till the sign that presently it is considered as the standard in the relevant literature.

Also, the treatise [61] on *Mathematical Foundations of Elasticity* is based on it, and most formulations, proposed by teams of computational biased researchers and mathematical minded scholars, have been built on this basis [8–10].

More recent contributions in elasticity and electro-elasticity, such as [37,63–72] some of which adopting a geometric formalism, are still in the wake of [16,73].

The impressive success of the theory there exposed is here contrasted by comments and results of a new physical and geometrical analysis which leads to the conclusion that the edifice of constitutive constructions in theoretical and computational mechanics should be properly revised [12,15,29,30].

The discussions and the comments on the seminal treatments in [16,35] have guided to a new approach to elasticity in the nonlinear geometric range.

Introduction of the elastic model in rate form is the theoretically motivated natural choice, directly related to experimental evidence, see, e.g. [74].

A leading principle is that only material tensors, defined on the actual body placement, can appear in constitutive relations.

A major result of the geometric approach consists in recognizing that finite elastic strains can only be defined in a local straightened trajectory as integrals over a time interval of the pull back of elastic stretching to the local straightened trajectory. The finite elastic strain is then a referential tensor field which cannot be pushed forward to a material field on a body placement, since it does not refer to a single instant of time, but to a time lapse. Accordingly, no physical significance can be attributed to it.

It is, however, emphasized that for mathematical investigations and for computational implementations, the recourse to a referential formulation is an effective tool to perform linear operations and that the notion of a referential finite elastic strain may be usefully introduced by fixing a local straightened trajectory. It may well vary from point to point of the discretized model, as in the finite element method.

A noteworthy outcome of the geometric treatment is that the elastic energy is a scalar potential depending on the stress to within the equivalence relation expressed by time invariance. This is an innovative notion with respect to usual treatments where, as an heritage of linearized approximations, the elastic energy is assumed to be a function of a strain measure.

The careful geometric investigation reveals also that well-known and diffusely adopted tools, such as the Kirchhoff stress and the second Piola–Kirchhoff referential stress, are introduced with improper geometric definitions, see Eqs. (62) and (74). Such is also the principle of *Material Frame Indifference*, universally adopted by followers of [13,16,17,73,75], see Sect. 16.

The incompleteness of the geometric framework adopted in formulations of elasticity and hypo-elasticity has been the source of difficulties in theoretical Continuum Mechanics and in its applications to engineering problems. A main negative consequence is due to the progressive discard of rate formulations by theoreticians and computationalists in favor of a finite deformation formulation based on a chain (noncommutative) decomposition of the deformation gradient into an inelastic and an elastic part.

The choice of reference and intermediate local configurations has, however, no clear physical ground and in dealing with the modeling of the effects of thermal variations, plastic dislocations, phase transformations, or tissue growth [21], the ordering of multiple contributions to the total deformation gradient is hard to be motivated.

The evidence of such intrinsic bugs in this proposal has, however, been considered ever more as an unkindness of nature to be accepted *obtorto collo* in the absence of a satisfactory rate theory of elasticity.

This vacancy has now been filled by an effective geometric treatment.

The simpler and physically sound engineering way of doing nonlinear simulations and computations in Structural Mechanics can be trodden again, but now with a solid theoretical foundation and a clear methodological framework fully available.

Computational details and numerical evidences resulting from the implementation of an algorithm will be provided in forthcoming papers.

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Appendix: Geometric background

The investigation about transformations from a given manifold $\mathbb{M}$ to another one $\mathbb{N}$ is a basic task in Continuum Mechanics. For instance, one is faced with this task when dealing with motions, changes of observer and computational schemes.

A manifold is a geometric object which generalizes the notion of a curve, surface or ball in the Euclid space. It is characterized by a family of local charts which are differentiable and invertible maps onto open sets in model linear space, say $\mathcal{R}^n$. Then n is the manifold dimension. The inverse maps provide local coordinate systems. Velocities of parametrized curves through a point $\mathbf{e} \in \mathbb{M}$ on a manifold, are tangent vectors at that point and describe the tangent linear space $T_{\mathbf{e}}\mathbb{M}$. The dual space of real-valued linear maps on $T_{\mathbf{e}}\mathbb{M}$ is denoted by $T^*_{\mathbf{e}}\mathbb{M} = (T_{\mathbf{e}}\mathbb{M})^*$ and its elements are called covectors at $\mathbf{e} \in \mathbb{M}$. To a smooth transformation $\boldsymbol{\phi} : \mathbb{M} \mapsto \mathbb{N}$ it corresponds, at each point $\mathbf{e} \in \mathbb{M}$, a linear infinitesimal transformation $T_{\mathbf{e}}\boldsymbol{\phi} : T_{\mathbf{e}}\mathbb{M} \mapsto T_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N}$ between the tangent spaces, called the *differential*, whose action on the tangent vector $\mathbf{u}_{\mathbf{e}} := \partial_{s=0}\, \mathbf{c}(s) \in T_{\mathbf{e}}\mathbb{M}$ to a curve $\mathbf{c} : \mathcal{R} \mapsto \mathbb{M}$, at the point $\mathbf{e} = \mathbf{c}(0)$, is defined by

$$T_{\mathbf{e}}\boldsymbol{\phi} \cdot \mathbf{u}_{\mathbf{e}} = \partial_{s=0}\, (\boldsymbol{\phi} \circ \mathbf{c})(s). \tag{20.1}$$

A dot $\cdot$ denotes linear dependence on subsequent arguments belonging to linear spaces. A circle $\circ$ denotes composition of maps. A chochét $\langle\, , \,\rangle$ denotes the bilinear, non-degenerate duality between pairs of dual linear spaces $(T_{\mathbf{e}}\mathbb{M}, T^*_{\mathbf{e}}\mathbb{M})$ or $(T_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N}, T^*_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N})$. The dual linear map

$$(T_{\mathbf{e}}\boldsymbol{\phi})^* : T^*_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N} \mapsto T^*_{\mathbf{e}}\mathbb{M}$$

is defined, for any $\mathbf{u}_{\mathbf{e}} \in T_{\mathbf{e}}\mathbb{M}$ and $\mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})} \in T_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N}$, by the identity

$$\langle T_{\mathbf{e}}\boldsymbol{\phi} \cdot \mathbf{u}_{\mathbf{e}}, \mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})} \rangle = \langle \mathbf{u}_{\mathbf{e}}, (T_{\mathbf{e}}\boldsymbol{\phi})^* \cdot \mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})} \rangle. \tag{20.2}$$

The *tangent* bundle $T\mathbb{M}$ and the *cotangent* bundle $T^*\mathbb{M}$ are disjoint unions respectively of the linear tangent spaces and of the dual spaces based at points of the manifold.

The global transformation between tangent bundles $T\boldsymbol{\phi} : T\mathbb{M} \mapsto T\mathbb{N}$ is called the *tangent transformation*. The operator T, acting on manifolds and on maps between them, is named the *tangent functor*.

Zeroth-order tensors are just real-valued functions. Second-order tensors at $\mathbf{e} \in \mathbb{M}$ are bilinear maps on pairs of vectors or covectors based at that point. They are named covariant, contravariant or mixed depending on whether the arguments are both vectors, both covectors or a vector and a covector. The corresponding linear tensor spaces at $\mathbf{e} \in \mathbb{M}$ are denoted by $\mathrm{FUN}(T_{\mathbf{e}}\mathbb{M})$, $\mathrm{COV}(T_{\mathbf{e}}\mathbb{M})$, $\mathrm{CON}(T_{\mathbf{e}}\mathbb{M})$, $\mathrm{MIX}(T_{\mathbf{e}}\mathbb{M})$. First-order covariant tensors are covectors, and first-order contravariant tensors are tangent vectors. Second-order tensors at $\mathbf{e} \in \mathbb{M}$ are equivalently defined as linear operators from a tangent or cotangent space to another such space at that point:

$$\begin{aligned} (\mathbf{s}_{\mathrm{COV}})_{\mathbf{e}} &: T_{\mathbf{e}}\mathbb{M} \mapsto T^*_{\mathbf{e}}\mathbb{M} \in \mathrm{COV}(T_{\mathbf{e}}\mathbb{M}), \\ (\mathbf{s}_{\mathrm{CON}})_{\mathbf{e}} &: T^*_{\mathbf{e}}\mathbb{M} \mapsto T_{\mathbf{e}}\mathbb{M} \in \mathrm{CON}(T_{\mathbf{e}}\mathbb{M}), \\ (\mathbf{s}_{\mathrm{MIX}})_{\mathbf{e}} &: T_{\mathbf{e}}\mathbb{M} \mapsto T_{\mathbf{e}}\mathbb{M} \in \mathrm{MIX}(T_{\mathbf{e}}\mathbb{M}). \end{aligned} \tag{20.3}$$

A covariant tensor $\mathbf{g}^{\mathbb{M}}_{\mathbf{e}} \in \mathrm{COV}(T_{\mathbf{e}}\mathbb{M})$ is non-degenerate if

$$\mathbf{g}^{\mathbb{M}}_{\mathbf{e}}(\mathbf{u}_{\mathbf{e}}, \mathbf{w}_{\mathbf{e}}) = 0 \quad \forall\, \mathbf{w}_{\mathbf{e}} \in T_{\mathbf{e}}\mathbb{M} \quad \Longrightarrow \quad \mathbf{u}_{\mathbf{e}} = \mathbf{0}_{\mathbf{e}}.$$

The corresponding linear operator $\mathbf{g}_{\mathbf{e}}^{\mathbb{M}} : T_{\mathbf{e}}\mathbb{M} \mapsto T_{\mathbf{e}}^{*}\mathbb{M}$ is then invertible and provides a tool to change tensorial type (alterations). The most important alterations are those which transform covariant or contravariant tensors into mixed ones and vice versa,

$$(\mathbf{s}_{\text{COV}})_{\mathbf{e}} \in \text{COV}(T_{\mathbf{e}}\mathbb{M}) \Longrightarrow (\mathbf{g}_{\mathbf{e}}^{\mathbb{M}})^{-1} \cdot (\mathbf{s}_{\text{COV}})_{\mathbf{e}} \in \text{MIX}(T_{\mathbf{e}}\mathbb{M}),$$
$$(\mathbf{s}_{\text{CON}})_{\mathbf{e}} \in \text{CON}(T_{\mathbf{e}}\mathbb{M}) \Longrightarrow (\mathbf{s}_{\text{CON}})_{\mathbf{e}} \cdot \mathbf{g}_{\mathbf{e}}^{\mathbb{M}} \in \text{MIX}(T_{\mathbf{e}}\mathbb{M}).$$

Symmetry of covariant or contravariant tensors means invariance of their values under an exchange of the two arguments. A pseudo-metric tensor is a non-degenerate covariant tensor which is symmetric, i.e.

$$\mathbf{g}_{\mathbf{e}}^{\mathbb{M}}(\mathbf{u}_{\mathbf{e}}, \mathbf{w}_{\mathbf{e}}) = \mathbf{g}_{\mathbf{e}}^{\mathbb{M}}(\mathbf{w}_{\mathbf{e}}, \mathbf{u}_{\mathbf{e}}).$$

A metric tensor $\mathbf{g}_{\mathbf{e}}^{\mathbb{M}} \in \text{COV}(T_{\mathbf{e}}\mathbb{M})$ is symmetric and positive definite, i.e. such that

$$\mathbf{u}_{\mathbf{e}} \neq \mathbf{0} \Longrightarrow \mathbf{g}_{\mathbf{e}}^{\mathbb{M}}(\mathbf{u}_{\mathbf{e}}, \mathbf{u}_{\mathbf{e}}) > 0.$$

A tensor bundle $\text{TENS}(T\mathbb{M})$ is the disjoint union of tensor fibers which are linear tensor spaces based at points of the manifold.

A bundle is characterized by a projection operator $\boldsymbol{\pi}_{\mathbb{M}} : \text{TENS}(T\mathbb{M}) \mapsto \mathbb{M}$ which assigns to each element $\mathbf{s}_{\mathbf{e}} \in \text{TENS}(T_{\mathbf{e}}\mathbb{M})$ of the bundle the corresponding base point $\mathbf{e} \in \mathbb{M}$. The fibers $\boldsymbol{\pi}_{\mathbb{M}}^{-1}(\mathbf{e})$ are the inverse images of the projection and are assumed to be related to each other by diffeomorphic transformations, so that they are all of the same dimension.

A *tensor field* is a map $\mathbf{s} : \mathbb{M} \mapsto \text{TENS}(T\mathbb{M})$ from a manifold $\mathbb{M}$ to a tensor bundle $\text{TENS}(T\mathbb{M})$ such that a point $\mathbf{e} \in \mathbb{M}$ is mapped to a tensor based at the same point, i.e. such that $\boldsymbol{\pi}_{\mathbb{M}} \circ \mathbf{s}$ is the identity map on $\mathbb{M}$. In geometrical terms it is said that a tensor field is a *section* of a tensor bundle.

A transformation $\boldsymbol{\phi} : \mathbb{M} \mapsto \mathbb{N}$ maps a curve on $\mathbb{M}$ into a curve in $\mathbb{N}$ and, under suitable assumptions, scalar, vector and covector fields from $\mathbb{M}$ onto $\boldsymbol{\phi}(\mathbb{M}) \subset \mathbb{N}$ (push forward $\uparrow$) and vice versa (pull back $\downarrow$)[16].

A synopsis is provided below. Assumptions of differentiability and invertibility of the differential are claimed whenever needed by formulae [47].

Push forward from $\mathbb{M}$ on $\boldsymbol{\phi}(\mathbb{M})$, $\boldsymbol{\phi} : \mathbb{M} \mapsto \mathbb{N}$ injective:

$$\begin{aligned}
\psi : \mathbb{M} \mapsto \mathcal{R}, &\quad (\boldsymbol{\phi}\uparrow\psi)_{\boldsymbol{\phi}(\mathbf{e})} = \psi_{\mathbf{e}},\\
\mathbf{v} : \mathbb{M} \mapsto T\mathbb{M}, &\quad (\boldsymbol{\phi}\uparrow\mathbf{v})_{\boldsymbol{\phi}(\mathbf{e})} = T_{\mathbf{e}}\boldsymbol{\phi} \cdot \mathbf{v}_{\mathbf{e}},\\
\mathbf{v}^{*} : \mathbb{M} \mapsto T^{*}\mathbb{M}, &\quad \langle \boldsymbol{\phi}\uparrow\mathbf{v}^{*}, \mathbf{w} \rangle_{\boldsymbol{\phi}(\mathbf{e})} = \langle \mathbf{v}_{\mathbf{e}}^{*}, (T_{\mathbf{e}}\boldsymbol{\phi})^{-1} \cdot \mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})} \rangle.
\end{aligned} \tag{20.4}$$

Pull back from $\boldsymbol{\phi}(\mathbb{M})$ to $\mathbb{M}$:

$$\begin{aligned}
\phi : \mathbb{N} \mapsto \mathcal{R}, &\quad (\boldsymbol{\phi}\downarrow\phi)_{\mathbf{e}} = \phi_{\boldsymbol{\phi}(\mathbf{e})},\\
\mathbf{w} : \mathbb{N} \mapsto T\mathbb{N}, &\quad (\boldsymbol{\phi}\downarrow\mathbf{w})_{\mathbf{e}} = (T_{\mathbf{e}}\boldsymbol{\phi})^{-1} \cdot \mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})},\\
\mathbf{w}^{*} : \mathbb{N} \mapsto T^{*}\mathbb{N}, &\quad \langle \boldsymbol{\phi}\downarrow\mathbf{w}^{*}, \mathbf{v} \rangle_{\mathbf{e}} = \langle \mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})}^{*}, T_{\mathbf{e}}\boldsymbol{\phi} \cdot \mathbf{v}_{\mathbf{e}} \rangle.
\end{aligned} \tag{20.5}$$

Push-pull relations for second-order covariant, contravariant and mixed tensors, are defined so that their scalar values be invariant and are given by the formulae

$$\begin{aligned}
(\boldsymbol{\phi}\downarrow\mathbf{s}_{\text{COV}})_{\mathbf{e}} &= (T_{\mathbf{e}}\boldsymbol{\phi})^{*} \cdot (\mathbf{s}_{\text{COV}})_{\boldsymbol{\phi}(\mathbf{e})} \cdot T_{\mathbf{e}}\boldsymbol{\phi} \in \text{COV}(T_{\mathbf{e}}\mathbb{M}),\\
(\boldsymbol{\phi}\uparrow\mathbf{s}_{\text{CON}})_{\boldsymbol{\phi}(\mathbf{e})} &= T_{\mathbf{e}}\boldsymbol{\phi} \cdot (\mathbf{s}_{\text{CON}})_{\mathbf{e}} \cdot (T_{\mathbf{e}}\boldsymbol{\phi})^{*} \quad \in \text{CON}(T_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N}),\\
(\boldsymbol{\phi}\uparrow\mathbf{s}_{\text{MIX}})_{\boldsymbol{\phi}(\mathbf{e})} &= T_{\mathbf{e}}\boldsymbol{\phi} \cdot (\mathbf{s}_{\text{MIX}})_{\mathbf{e}} \cdot (T_{\mathbf{e}}\boldsymbol{\phi})^{-1} \quad \in \text{MIX}(T_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N}).
\end{aligned} \tag{20.6}$$

These transformation rules play an important role in CM since the metric tensor is covariant and the dual stress tensor is contravariant. As the result of a push, a mixed tensor symmetric with respect to a metric tensor is transformed into a mixed tensor symmetric with respect to the pushed metric tensor.

[16] In differential geometry push and pull are respectively denoted by low and high asteriscs $_{*};^{*}$ [76,77]. This standard notation leads however to consider too many similar stars in the geometric sky, i.e. push, pull, duality, Hodge operator.

A *morphism* $\mathbf{\Phi}$ over $\boldsymbol{\phi}$ is made of a pair of maps $(\mathbf{\Phi}, \boldsymbol{\phi})$ between tensor bundles and their base manifolds that preserve the tensorial fibers, as expressed by the commutative diagram

$$\begin{tikzcd} \mathrm{TENS}(T\mathbb{M}) \arrow[r, "\mathbf{\Phi}"] \arrow[d, "\boldsymbol{\pi}_{\mathbb{M}}"'] & \mathrm{TENS}(T\mathbb{N}) \arrow[d, "\boldsymbol{\pi}_{\mathbb{N}}"] \\ \mathbb{M} \arrow[r, "\boldsymbol{\phi}"] & \mathbb{N} \end{tikzcd} \iff \boldsymbol{\pi}_{\mathbb{N}} \circ \mathbf{\Phi} = \boldsymbol{\phi} \circ \boldsymbol{\pi}_{\mathbb{M}}.$$

Morphisms that are invertible and differentiable with the inverse, are named *diffeomorphisms*. Important instances of diffeomorphisms are the displacements from a placement of a body to another one, changes of observer, and straightening out maps. On the other hand, differentiable maps which are *not* diffeomorphisms are, for instance the following

1. *immersions* (maps with injective tangent map)
2. *submersions* (maps with surjective tangent map)
3. *projections* (surjective submersions).

The $(\mathbf{g}^{\mathbb{M}}, \mathbf{g}^{\mathbb{N}})$-adjoint tangent map

$$(T_{\boldsymbol{\phi}(\mathbf{e})}\boldsymbol{\phi})^A \in L\,(T_{\boldsymbol{\phi}(\mathbf{e})}\boldsymbol{\phi}(\mathbb{M})\,;\, T_{\mathbf{e}}\mathbb{M})$$

is pointwise defined by

$$(T_{\boldsymbol{\phi}(\mathbf{e})}\boldsymbol{\phi})^A := (\mathbf{g}^{\mathbb{M}}_{\mathbf{e}})^{-1} \cdot (T_{\mathbf{e}}\boldsymbol{\phi})^* \cdot \mathbf{g}^{\mathbb{N}}_{\boldsymbol{\phi}(\mathbf{e})}, \tag{20.7}$$

as expressed by the commutative diagram

$$\begin{tikzcd} (T\mathbb{M})^* & (T\mathbb{N})^* \arrow[l, "(T\boldsymbol{\phi})^*"'] \\ T\mathbb{M} \arrow[u, "\mathbf{g}^{\mathbb{M}}"] & T\mathbb{N} \arrow[l, "(T\boldsymbol{\phi})^A"] \arrow[u, "\mathbf{g}^{\mathbb{N}}"'] \end{tikzcd} \iff \mathbf{g}^{\mathbb{M}} \cdot (T\boldsymbol{\phi})^A = (T\boldsymbol{\phi})^* \cdot \mathbf{g}^{\mathbb{N}}. \tag{20.8}$$

By (20.2), the relation (20.7) may be written as an identity

$$\mathbf{g}^{\mathbb{N}}_{\boldsymbol{\phi}(\mathbf{e})}(\mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})}, T_{\mathbf{e}}\boldsymbol{\phi} \cdot \mathbf{u}_{\mathbf{e}}) = \mathbf{g}^{\mathbb{M}}_{\mathbf{e}}((T_{\boldsymbol{\phi}(\mathbf{e})}\boldsymbol{\phi})^A \cdot \mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})}, \mathbf{u}_{\mathbf{e}}),$$

for any $\mathbf{u}_{\mathbf{e}} \in T_{\mathbf{e}}\mathbb{M}$ and $\mathbf{w}_{\boldsymbol{\phi}(\mathbf{e})} \in T_{\boldsymbol{\phi}(\mathbf{e})}\mathbb{N}$.

The Lie derivative of a vector field $\mathbf{h} \in \mathrm{C}^1(\mathbb{M}\,;T\mathbb{M})$ according to a vector field $\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,;T\mathbb{M})$ is defined by considering the flow $\mathbf{Fl}^{\mathbf{u}}_{\lambda}$ generated by solutions of the differential equation $\mathbf{u} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{u}}_{\lambda}$ and by differentiating the pull back along the flow

$$\mathcal{L}_{\mathbf{u}}\,\mathbf{h} := \partial_{\lambda=0}\,(\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\!\mathbf{h}) = \partial_{\lambda=0}\,T\mathbf{Fl}^{\mathbf{u}}_{-\lambda} \cdot (\mathbf{h} \circ \mathbf{Fl}^{\mathbf{u}}_{\lambda}). \tag{20.9}$$

The Lie derivative of a tensor field is defined in an analogous way, and the Lie derivative of scalar fields coincides with the directional derivative.

The commutator of tangent vector fields $\mathbf{u}, \mathbf{h} \in \mathrm{C}^1(\mathbb{M}\,;T\mathbb{M})$ is the skew-symmetric tangent-vector-valued operator defined by

$$[\mathbf{u}, \mathbf{h}]f := (\mathcal{L}_{\mathbf{u}}\mathcal{L}_{\mathbf{h}} - \mathcal{L}_{\mathbf{h}}\mathcal{L}_{\mathbf{u}})f \tag{20.10}$$

with $f \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{FUN}(T\mathbb{M}))$ a scalar field. A basic theorem concerning Lie derivatives states that $\mathcal{L}_{\mathbf{u}}\mathbf{h} = [\mathbf{u}, \mathbf{h}]$ and hence the commutator of tangent vector fields is called the Lie bracket. For any injective morphism $\boldsymbol{\phi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ the Lie bracket enjoys the push-naturality property [47]

$$\boldsymbol{\phi}\!\uparrow\!(\mathcal{L}_{\mathbf{u}}\,\mathbf{h}) = \boldsymbol{\phi}\!\uparrow\![\mathbf{u}, \mathbf{h}] = [\boldsymbol{\phi}\!\uparrow\!\mathbf{u}, \boldsymbol{\phi}\!\uparrow\!\mathbf{h}] = \mathcal{L}_{\boldsymbol{\phi}\uparrow\mathbf{u}}\,\boldsymbol{\phi}\!\uparrow\!\mathbf{h}. \tag{20.11}$$

For any tensor field $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{TENS}(T\mathbb{M}))$ the Lie derivative is defined by

$$\mathcal{L}_{\mathbf{u}}\,\mathbf{s} := \partial_{\lambda=0}\,(\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\!\mathbf{s}) = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\!(\mathbf{s} \circ \mathbf{Fl}^{\mathbf{u}}_{\lambda}), \tag{20.12}$$

and the push-naturality property can be extended to

$$\boldsymbol{\phi}\!\uparrow\!(\mathcal{L}_{\mathbf{u}}\,\mathbf{s}) = \mathcal{L}_{\boldsymbol{\phi}\uparrow\mathbf{u}}\,\boldsymbol{\phi}\!\uparrow\!\mathbf{s}. \tag{20.13}$$

By commutativity between push and composition, the Leibniz rule for the $\partial_{\lambda=0}$ derivative yields the analogous Leibniz rule for Lie derivatives of tensor fields,

$$\mathcal{L}_{\mathbf{u}}\,(\mathbf{s}_{\mathrm{CON}} \cdot \mathbf{s}_{\mathrm{COV}}) = (\mathcal{L}_{\mathbf{u}}\,\mathbf{s}_{\mathrm{CON}}) \cdot \mathbf{s}_{\mathrm{COV}} + \mathbf{s}_{\mathrm{CON}} \cdot (\mathcal{L}_{\mathbf{u}}\,\mathbf{s}_{\mathrm{COV}}). \tag{20.14}$$

Given a volume-form $\boldsymbol{\mu} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{MXF}(T\mathbb{M}))$, the associated divergence operator div is defined by the equality

$$\mathcal{L}_{\mathbf{u}}\,\boldsymbol{\mu} = \mathrm{div}(\mathbf{u})\,\boldsymbol{\mu}. \tag{20.15}$$

A noteworthy property [47] is that for any scalar field $f \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{FUN}(T\mathbb{M}))$:

$$\mathcal{L}_{\mathbf{u}}\,(f\,\boldsymbol{\mu}) = \mathcal{L}_{(f\,\mathbf{u})}\,\boldsymbol{\mu}. \tag{20.16}$$

A volume-form induces a measure defined by $\mathrm{MEAS}(\boldsymbol{\mu}) := \mathrm{SIGNUM}(\boldsymbol{\mu})\,\boldsymbol{\mu}$. The *density* associated with a scalar field $\rho : \mathcal{T} \mapsto \mathcal{R}$ and with a volume-form $\boldsymbol{\mu} \in \mathrm{MXF}(V\mathcal{T})$ is the product $\rho\,\mathrm{MEAS}(\boldsymbol{\mu})$.

A linear connection ∇ in a manifold $\mathbb{M}$ fulfills the characteristic properties of a point derivation [78, Vol. III, XVII-18],

$$\begin{aligned} \nabla_{\mathbf{w}}(\mathbf{u}_1 + \mathbf{u}_2) &= \nabla_{\mathbf{w}}\mathbf{u}_1 + \nabla_{\mathbf{w}}\mathbf{u}_2, \\ \nabla_{(\mathbf{w}_1+\mathbf{w}_2)}\mathbf{u} &= \nabla_{\mathbf{w}_1}\mathbf{u} + \nabla_{\mathbf{w}_2}\mathbf{u}, \\ \nabla_{\mathbf{w}}(f\mathbf{u}) &= f\,\nabla_{\mathbf{w}}\mathbf{u} + (\nabla_{\mathbf{w}} f)\mathbf{u}, \\ \nabla_{(f\,\mathbf{w})}\mathbf{u} &= f\,\nabla_{\mathbf{w}}\mathbf{u}, \end{aligned} \tag{20.17}$$

where $f \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{FUN}(T\mathbb{M}))$, $\mathbf{u}, \mathbf{u}_i \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ and $\mathbf{w}_i \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ for $i = 1, 2$. $\nabla_{\mathbf{w}} f$ is the standard derivative of scalar fields. In terms of parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,; \mathbb{M})$, with $\mathbf{u} = \partial_{\lambda=0}\,\mathbf{c}(\lambda)$, the derivative according to the connection is the *parallel derivative*, given by

$$\nabla_{\mathbf{u}}\mathbf{w} := \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow(\mathbf{w} \circ \mathbf{c})(\lambda). \tag{20.18}$$

Parallel transported vector fields $(\mathbf{w} \circ \mathbf{c})(\lambda) = \mathbf{c}(\lambda)\Uparrow \mathbf{w}_0$ have a null parallel derivative, since

$$\begin{aligned} \nabla_{\mathbf{u}}\mathbf{w} &:= \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow(\mathbf{w} \circ \mathbf{c})(\lambda) \\ &= \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow \mathbf{c}(\lambda)\Uparrow \mathbf{w}_0 = \partial_{\lambda=0}\,\mathbf{w}_0 = 0. \end{aligned}$$

The curvature of the connection is the *tensorial*[17] map CURV, which acting on a vector field $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,; T\mathbb{M})$ gives a tangent-vector-valued two-form $\mathrm{CURV}(\mathbf{s})$ defined by[18]

$$\mathrm{CURV}(\mathbf{s})(\mathbf{u}, \mathbf{w}) := ([\nabla_{\mathbf{u}}, \nabla_{\mathbf{w}}] - \nabla_{[\mathbf{u},\mathbf{w}]})(\mathbf{s}), \tag{20.19}$$

and the torsion TORS is the tangent-vector-valued two-form defined by

$$\mathrm{TORS}(\mathbf{u}, \mathbf{w}) := \nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} - [\mathbf{u}, \mathbf{w}]. \tag{20.20}$$

Mixed tensor fields $\mathrm{TORS}(\mathbf{u})$ and $\mathrm{CURV}(\mathbf{s}, \mathbf{u})$ are defined by the identities

$$\begin{aligned} \mathrm{TORS}(\mathbf{u}) \cdot \mathbf{w} &:= \mathrm{TORS}(\mathbf{u}, \mathbf{w}) = -\mathrm{TORS}(\mathbf{w}, \mathbf{u}), \\ \mathrm{CURV}(\mathbf{s}, \mathbf{u}) \cdot \mathbf{w} &:= \mathrm{CURV}(\mathbf{s})(\mathbf{u}, \mathbf{w}) = -\mathrm{CURV}(\mathbf{s})(\mathbf{w}, \mathbf{u}). \end{aligned} \tag{20.21}$$

A connection with vanishing torsion is named *torsion-free* or *symmetric*, and a connection with vanishing curvature is said to be *curvature-free* or *flat*.

[17] Tensoriality of a multilinear map, acting on vector fields and generating a vector field, means that point values of the image field depend only on the values of the source fields at the same point. An *exterior form*, or simply a *form*, is then a vector-valued, tensorial, alternating multilinear map.

[18] The curvature form of connection on a fiber bundle and the relevant expression in terms of parallel derivatives are treated in [47].

The expression of Lie derivatives in terms of parallel derivatives is given for vectors, covectors, covariant, contravariant and mixed tensors by

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{u} = \nabla_{\mathbf{v}}\,\mathbf{u} - \mathbf{Y}(\mathbf{v})\cdot\mathbf{u}, \tag{20.22.1}$$

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{u}^* = \nabla_{\mathbf{v}}\,\mathbf{u}^* + \mathbf{u}^*\cdot\mathbf{Y}(\mathbf{v}), \tag{20.22.2}$$

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{s}_{\mathrm{COV}} = \nabla_{\mathbf{v}}\,\mathbf{s}_{\mathrm{COV}} + \mathbf{s}_{\mathrm{COV}}\cdot\mathbf{Y}(\mathbf{v}) + \mathbf{Y}(\mathbf{v})^*\cdot\mathbf{s}_{\mathrm{COV}}, \tag{20.22.3}$$

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{s}_{\mathrm{CON}} = \nabla_{\mathbf{v}}\,\mathbf{s}_{\mathrm{CON}} - \mathbf{Y}(\mathbf{v})\cdot\mathbf{s}_{\mathrm{CON}} - \mathbf{s}_{\mathrm{CON}}\cdot\mathbf{Y}(\mathbf{v})^*, \tag{20.22.4}$$

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{s}_{\mathrm{MIX}} = \nabla_{\mathbf{v}}\,\mathbf{s}_{\mathrm{MIX}} - \mathbf{Y}(\mathbf{v})\cdot\mathbf{s}_{\mathrm{MIX}} + \mathbf{s}_{\mathrm{MIX}}\cdot\mathbf{Y}(\mathbf{v}), \tag{20.22.5}$$

where $\mathbf{Y}(\mathbf{v}) := \nabla\mathbf{v} + \mathrm{TORS}(\mathbf{v})$. For an exhaustive presentation with proofs we refer the reader to [47].

A key original result is provided by next Lemma 1, which involves non-trivial notions of differential geometry, for which reference is made to [47]. The result is resorted to in (25) of Sect. 4.1.

Lemma 1 *Let a time-parametrized family* $\boldsymbol{\varphi}_\alpha : \mathbb{M} \mapsto \mathbb{M}$ *of diffeomorphisms be acted upon by the tangent functor to give* $T\boldsymbol{\varphi}_\alpha : T\mathbb{M} \mapsto T\mathbb{M}$ *and define the velocity field* $\mathbf{v} := \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha : \mathbb{M} \mapsto T\mathbb{M}$ *and the parallel time derivative to a linear connection*

$$\mathbf{L}(\mathbf{v}) := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha \Downarrow T\boldsymbol{\varphi}_\alpha) : T\mathbb{M} \mapsto T\mathbb{M}.$$

Then the parallel time derivative of the spatial velocity field

$$\nabla\mathbf{v} := \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha \Downarrow (\mathbf{v}\circ\boldsymbol{\varphi}_\alpha) : T\mathbb{M} \mapsto T\mathbb{M},$$

and the tensor field $\mathbf{L}(\mathbf{v})$ *are related by the formula*

$$\mathbf{L}(\mathbf{v}) - \nabla\mathbf{v} = \mathrm{TORS}(\mathbf{v}). \tag{20.22}$$

Proof Let us consider a curve $\mathbf{c} \in \mathrm{C}^1([-\varepsilon, \varepsilon]\,;\mathbb{M})$ with $\partial_{\lambda=0}\,\mathbf{c}(\lambda) = \mathbf{h} \in T\mathbb{M}$. The fiberwise linear *connector* $\mathcal{K} \in \mathrm{C}^1(T^2\mathbb{M}\,;T\mathbb{M})$ is related to the parallel derivative of the velocity vector field by the relation

$$\nabla\mathbf{v}\cdot\mathbf{h} := \mathcal{K}\cdot T\mathbf{v}\cdot\mathbf{h}.$$

Denoting by $T^2\mathbb{M}$ the second tangent bundle and by $\mathrm{FLIP} : T^2\mathbb{M} \mapsto T^2\mathbb{M}$ the *canonical flip* defined by [47, 1.8.1]

$$\mathrm{FLIP}\cdot(\partial_{\alpha=0}\,\partial_{\lambda=0}\,\boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda)) = \partial_{\lambda=0}\,\partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda),$$

we get the formula

$$\begin{aligned}
\mathbf{L}(\mathbf{v})\cdot\mathbf{h} &= \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha \Downarrow (T\boldsymbol{\varphi}_\alpha\cdot\partial_{\lambda=0}\,\mathbf{c}_\lambda) = \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha \Downarrow \partial_{\lambda=0}\,\boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda)\\
&= \mathcal{K}\cdot(\partial_{\alpha=0}\,\partial_{\lambda=0}\,\boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda)) = (\mathcal{K}\circ\mathrm{FLIP})\cdot(\partial_{\lambda=0}\,\partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda))\\
&= (\mathcal{K}\circ\mathrm{FLIP})\cdot(\partial_{\lambda=0}\,\mathbf{v}(\mathbf{c}_\lambda)) = (\mathcal{K}\circ\mathrm{FLIP})\cdot(T\mathbf{v}\cdot\mathbf{h}).
\end{aligned}$$

The conclusion follows from the expression of the torsion-form of a linear connection in terms of the *connector* [47, 1.8.12],

$$\mathrm{TORS}(\mathbf{v}, \mathbf{h}) = (\mathcal{K}\circ\mathrm{FLIP} - \mathcal{K})\cdot T\mathbf{v}\cdot\mathbf{h},$$

and from the definition of the tensor field $\mathrm{TORS}(\mathbf{v})$. □

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