Acta Mech 224, 211–230 (2013)
DOI 10.1007/s00707-012-0746-5

Giovanni Romano · Raffaele Barretta

On Euler's stretching formula in continuum mechanics

Received: 23 January 2012 / Revised: 7 September 2012 / Published online: 2 November 2012

Abstract The Lie time-derivative of the material metric tensor field along the motion is the proper mathematical definition of the physical notion of strain rate or stretching. Its expression, as symmetric part of the velocity gradient in Euclid space, is provided by a celebrated formula conceived by the genius of Leonhard Euler around the middle of the eighteenth century and since then reproduced in articles, books and treatises on continuum mechanics. We present here a formulation, in the proper geometric context of the four-dimensional space-time manifold endowed with an arbitrary linear connection and referring to a material body of arbitrary dimensionality. The expression involves the material time-derivative of the metric field and torsion-form and gradient of the velocity field, according to the connection induced on the trajectory. As an application, the expressions of the GRAM matrix of the stretching in natural and in normalized (or engineering) reference systems induced by orthogonal polar coordinates are provided.

1 Introduction

At the very core of continuum mechanics (CM), there is a formula envisaged by Leonhard Euler [1–3] in the second middle of 1700, see [4] Sect. 82. This celebrated result provides the proper geometric description of strain rate, or stretching (rate of stretch), defined as the symmetric part of the velocity material gradient. Vanishing of the stretching field is a non-redundant condition which is necessary and sufficient to ensure that the body is undergoing a rigid act of motion. An extension of this formula was performed, more than a century later, by Wilhelm Killing [5] who considered the more general context of a Riemann manifold endowed with the Levi-Civita connection, see for example [6,7], and infinitesimal isometries as vector fields characterized by the vanishing of the Lie time-derivative of the metric tensor along their flow. In this more general context, the stretching, when expressed as a mixed tensor, is given by the symmetric part of the material gradient of the velocity taken according to the parallel derivative induced in the trajectory by the Levi-Civita connection of the Riemann ambient space manifold [8]. The original Euler's formula has been reproduced in all treatments of CM dealing with three-dimensional bodies in motion in the three-dimensional Euclid space, see for example [4,9]. The notion of stretching plays a fundamental role in CM. Indeed, when evaluated along a virtual motion, it provides an implicit representation of the linear subspace of rigid virtual velocities, by the condition that the stretching tensor field vanishes identically in a body placement. This representation is basic for the introduction of the notion of a stress tensor field as a consequence of the variational condition of equilibrium [10]. On the

G. Romano (✉) · R. Barretta
Department of Structural Engineering, University of Naples Federico II, via Claudio 21, 80125 Naples, Italy
E-mail: romano@unina.it
Tel.: +39-081-7683729
Fax: +39-081-7683332

R. Barretta
E-mail: rabarret@unina.it

other hand, when evaluated along the real motion, the stretching provides the measure of rate of change of the material metric which enters in constitutive relations together with the stress and the stressing (rate of time change of the stress) [8].

The formulation of a general expression for the stretching, pertaining to bodies of any dimension in motion in a Riemann ambient space manifold endowed with an arbitrary linear connection, is certainly worthwhile both from the theoretical and from the applications point of view. This formulation may for instance be resorted in evaluating the stretching of a membrane moving on a curved surface or in finding the expression of the stretching of a three-dimensional body in a mobile reference system, such as the ones introduced by normalizing reference systems naturally associated with curvilinear coordinates.

To perform a general treatment, the first step is to put the topic in a proper geometric perspective. To this end, the kinematics is described in terms of the four-dimensional events manifold and of the immersed trajectory manifold where the motion and the related evolution operator are defined. A space-time description is provided by observers, with time playing an absolute role, in conformity with classical mechanics. A basic point is the classification of tangent vector fields and relevant tensor fields as spatial, material and trajectory-based spatial fields, a basic discriminant first pointed out in [8]. In CM, the stretching of a body is a material tensor field. It is the Lie time-derivative, along the motion, of the material metric tensor, provided by pull-back of the spatial metric tensor field according to the trajectory immersion map and by subsequent restriction to time-vertical tangent vectors. The second step in the treatment consists in providing a general formula for the Lie derivative of a spatial tensor field in terms of parallel derivatives and in investigating about the properties of the trajectory connection induced by a space-time connection. Since an arbitrary linear connection is considered, the expression of the Lie derivative along the motion involves the parallel derivative of the metric tensor field and the torsion of the velocity field. This general expression is a new contribution since the standard Killing formula refers to a metric preserving and symmetric (i.e. torsion-free) linear connection. On the basis of these results, the generalized Euler-Killing formula is got by expressing the Lie derivative of the space-time metric tensor field, restricted to the trajectory manifold, in terms of parallel derivatives, by performing a pull-back to the trajectory manifold and a subsequent restriction to the time-vertical tangent bundle. This procedures does not apply to material tensors of general kind. The issue deserves a special attention because difficulties faced in formulating rate constitutive relations in CM, and ensuing debates about different proposals of time-rates for the material stress tensor field, were due to attempts to get the time-rates out of performing parallel time-derivatives of the stress tensor field along the motion, an operation which does not comply with the rule dictated by the covariance paradigm, asserting that material tensor fields can be compared only by push according to a diffeomorphic transformation. Accordingly, the stress time-rate is defined as Lie derivative of the stress tensor along the motion. This rule may be got as the conclusion of a physico-geometrical reasoning aimed to fulfill the requirements of independence of trajectory dimensionality and of naturality of the notion of material time-rates, thus avoiding the arbitrariness intrinsic in the concept of connection [8]. Analogous considerations lead to define the heat flow time-rate as a Lie derivative along the motion [11], a conclusion shared by the treatments of thermal convection performed in [12–15]. The importance of assuming Lie derivatives as time-rates of material tensors is witnessed also by recent papers on thermal convection and wave propagation phenomena [16–20]. In this framework, the Euler-Killing formula occupies a peculiar position in CM, both from the physical point of view and from a purely differential-geometric perspective. As an application of the extended formula, we provide the expressions of the GRAM matrix of the stretching, relevant to the connections, respectively, induced by a natural reference system and by a mobile normalized (or engineering) reference system associated with orthogonal curvilinear coordinates.

2 Geometric background

Basic notions of tensor bundles, push-pull transformations and Lie and parallel derivatives are recalled in the following.

2.1 Tensor bundles

At a point $\mathbf{x} \in \mathbb{M}$ of a manifold $\mathbb{M}$, the linear space of 0th order tensors (scalars) is denoted by $\mathrm{FUN}_{\mathbf{x}}(\mathbb{TM})$, the dual spaces of tangent and cotangent vectors, by $\mathbb{T}_{\mathbf{x}}\mathbb{M}$ and $\mathbb{T}^*_{\mathbf{x}}\mathbb{M}$. Covariant, contravariant and mixed second-order tensors belong to linear spaces of scalar-valued bilinear maps (or linear operators) as listed:

$$\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}, \mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}),$$
$$\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \in \mathrm{CON}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}, \mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}; \mathbb{T}_{\mathbf{x}}\mathbb{M}),$$
$$\mathbf{s}_{\mathbf{x}}^{\mathrm{MIX}} \in \mathrm{MIX}_{\mathbf{x}}(\mathbb{TM}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}, \mathbb{T}_{\mathbf{x}}^{*}\mathbb{M}; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathbb{T}_{\mathbf{x}}\mathbb{M}).$$

The linear spaces of covariant and contravariant tensors are in separating duality by the pairing

$$\langle \mathbf{s}_{\mathbf{x}}^{\mathrm{CON}}, \mathbf{s}_{\mathbf{x}}^{\mathrm{COV}} \rangle := J_1(\mathbf{s}_{\mathbf{x}}^{\mathrm{CON}} \circ (\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}})^A),$$

where J_1 denotes the linear invariant, and the adjoint tensor $(\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}})^A$ is defined by the identity

$$(\mathbf{s}_{\mathbf{x}}^{\mathrm{COV}})^A(\mathbf{a}, \mathbf{b}) := \mathbf{s}_{\mathbf{x}}^{\mathrm{COV}}(\mathbf{b}, \mathbf{a}), \qquad \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}.$$

The generic tensor space is denoted by $\mathrm{TENS}_{\mathbf{x}}(\mathbb{TM})$. Spaces of symmetric covariant and contravariant tensors are denoted by $\mathrm{SYM}_{\mathbf{x}}(\mathbb{TM})$, $\mathrm{SYM}_{\mathbf{x}}^{*}(\mathbb{TM})$. Scalar-valued k-linear, alternating maps are called k-covectors. Volumes are non-vanishing k-covectors of maximal order ($k = \dim \mathbb{M}$), and the corresponding linear space is denoted by $\mathrm{VOL}_{\mathbf{x}}(\mathbb{TM})$.

Definition 1 (*Fibrations and Fiber Bundles*) Given two manifolds $\mathbb{N}$ and $\mathbb{M}$, a fibration is a triplet $(\mathbb{N}, \boldsymbol{\pi}_{\mathbb{M},\mathbb{N}}, \mathbb{M})$ where the projection map $\boldsymbol{\pi}_{\mathbb{M},\mathbb{N}} \in \mathrm{C}^1(\mathbb{N}; \mathbb{M})$ is a surjective submersion (the tangent linear map of $\boldsymbol{\pi}_{\mathbb{M},\mathbb{N}}$ at each point is surjective) onto the base manifold $\mathbb{M}$. The inverse images of the projection map are called the fibers. A fiber bundle is a fibration in which the fibers are all diffeomorphic to a given one, the typical fiber. A section $\mathbf{s}_{\mathbb{N},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}; \mathbb{N})$ is a map such that $\boldsymbol{\pi}_{\mathbb{M},\mathbb{N}} \circ \mathbf{s}_{\mathbb{N},\mathbb{M}} = \mathrm{ID}_{\mathbb{M}}$ [21].

Sections of tangent (tensor) bundles are called vector (tensor) fields. Sections whose values are k-covectors are called *k-forms*. A *volume form* is a non-vanishing form of maximal order ($k = \dim \mathbb{M}$). A *bundle morphism* $\mathbf{f} \in \mathrm{C}^1(\mathbb{N}_1; \mathbb{N}_2)$, between fiber bundles $(\mathbb{N}_1, \boldsymbol{\pi}_{\mathbb{M}_1,\mathbb{N}_1}, \mathbb{M}_1)$ and $(\mathbb{N}_2, \boldsymbol{\pi}_{\mathbb{M}_2,\mathbb{N}_2}, \mathbb{M}_2)$, is a map which respects the fibers, that is, such that, for all $\mathbf{x}, \mathbf{y} \in \mathbb{N}_1$:

$$\boldsymbol{\pi}_{\mathbb{M}_1,\mathbb{N}_1}(\mathbf{x}) = \boldsymbol{\pi}_{\mathbb{M}_1,\mathbb{N}_1}(\mathbf{y}) \quad \Longrightarrow \quad \boldsymbol{\pi}_{\mathbb{M}_2,\mathbb{N}_2}(\mathbf{f}(\mathbf{x})) = \boldsymbol{\pi}_{\mathbb{M}_2,\mathbb{N}_2}(\mathbf{f}(\mathbf{y})).$$

A diffeomorphism is a bundle morphism which is invertible with a pointwise invertible tangent map. The fibers of the tangent bundle $\boldsymbol{\pi}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{TM}; \mathbb{M})$ are tangent spaces, and the fibers of the cotangent bundle $\boldsymbol{\pi}_{\mathbb{M}}^{*} \in \mathrm{C}^1(\mathbb{T}^{*}\mathbb{M}; \mathbb{M})$ are the dual linear spaces. A tensor bundle, that is, a fiber bundle whose fibers are linear tensor spaces, will be denoted by $\boldsymbol{\pi}_{\mathbb{M}}^{\mathrm{TENS}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{TM}); \mathbb{M})$.

2.2 Push-pull transformations

The pull-back of a scalar field $f : \boldsymbol{\zeta}(\mathbb{M}) \mapsto \mathrm{FUN}(\mathbb{T}(\boldsymbol{\zeta}(\mathbb{M})))$, along a map $\boldsymbol{\zeta} \in \mathrm{C}^0(\mathbb{M}; \mathbb{N})$, is the scalar field $\boldsymbol{\zeta}\!\downarrow\! f : \mathbb{M} \mapsto \mathrm{FUN}(\mathbb{TM})$ pointwise defined by the following:

$$(\boldsymbol{\zeta}\!\downarrow\! f)_{\mathbf{x}} := \boldsymbol{\zeta}\!\downarrow\! f_{\boldsymbol{\zeta}(\mathbf{x})} := f_{\boldsymbol{\zeta}(\mathbf{x})} \in \mathrm{FUN}_{\mathbf{x}}(\mathbb{TM}).$$

The push-forward of a tangent vector field $\mathbf{v} : \mathbb{M} \mapsto \mathbb{TM}$, along a differentiable map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}; \mathbb{N})$, is the vector field $\boldsymbol{\zeta}\!\uparrow\!\mathbf{v} : \boldsymbol{\zeta}(\mathbb{M}) \mapsto \mathbb{T}(\boldsymbol{\zeta}(\mathbb{M}))$ pointwise defined by the action of the tangent map $T_{\mathbf{x}}\boldsymbol{\zeta} \in L(\mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})$, according to the formula

$$(\boldsymbol{\zeta}\!\uparrow\!\mathbf{v})_{\boldsymbol{\zeta}(\mathbf{x})} := \boldsymbol{\zeta}\!\uparrow\!\mathbf{v}_{\mathbf{x}} := T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N}.$$

The pull-back of a cotangent vector field $\mathbf{v}^{*} : \boldsymbol{\zeta}(\mathbb{M}) \mapsto \mathbb{T}^{*}(\boldsymbol{\zeta}(\mathbb{M}))$, along a differentiable map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}; \mathbb{N})$, is the cotangent field $\boldsymbol{\zeta}\!\downarrow\!\mathbf{v}^{*} : \mathbb{M} \mapsto \mathbb{T}^{*}\mathbb{M}$, with $(\boldsymbol{\zeta}\!\downarrow\!\mathbf{v}^{*})_{\mathbf{x}} = \boldsymbol{\zeta}\!\downarrow\!\mathbf{v}^{*}_{\boldsymbol{\zeta}(\mathbf{x})}$ defined by the invariance

$$\langle \boldsymbol{\zeta}\!\downarrow\!\mathbf{v}^{*}, \mathbf{v} \rangle_{\mathbf{x}} = \boldsymbol{\zeta}\!\downarrow\!\langle \mathbf{v}^{*}, \boldsymbol{\zeta}\!\uparrow\!\mathbf{v} \rangle_{\boldsymbol{\zeta}(\mathbf{x})}.$$

If the restriction of the map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}; \mathbb{N})$ to its codomain is a diffeomorphism, pull-back and push-forward are inverse operations. The pull-back of tensor fields is defined by naturality. For a twice-covariant tensor field $\mathbf{s} : \mathbb{M} \mapsto \mathrm{COV}(\mathbb{TN})$, the pull is explicitly defined, for any pair of tangent vector fields $\mathbf{u}, \mathbf{w} : \mathbb{M} \mapsto \mathbb{TM}$, by the following:

$$(\boldsymbol{\zeta}\!\downarrow\!\mathbf{s})(\mathbf{u}, \mathbf{w}) := \boldsymbol{\zeta}\!\downarrow\!(\mathbf{s}(\boldsymbol{\zeta}\!\uparrow\!\mathbf{u}, \boldsymbol{\zeta}\!\uparrow\!\mathbf{w})).$$

The co-tangent map

$$T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} := (T_{\mathbf{x}}\boldsymbol{\zeta})^* \in L\,(\mathbb{T}^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}(\mathbb{M})\,;\,\mathbb{T}^*_{\mathbf{x}}\mathbb{M}),$$

is such that, for every $\mathbf{w}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ and $\mathbf{s}_{\mathrm{COV}\,\boldsymbol{\zeta}(\mathbf{x})} \in \mathbb{T}^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}(\mathbb{M})$:

$$\langle \mathbf{s}_{\mathrm{COV}\,\boldsymbol{\zeta}(\mathbf{x})}, T_{\mathbf{x}}\boldsymbol{\zeta}\cdot\mathbf{w}_{\mathbf{x}}\rangle = \langle T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}\cdot\mathbf{s}_{\mathrm{COV}\,\boldsymbol{\zeta}(\mathbf{x})}, \mathbf{w}_{\mathbf{x}}\rangle.$$

The inverse tangent map is defined by $T^{-1}_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} := (T_{\mathbf{x}}\boldsymbol{\zeta})^{-1} \in L\,(\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}(\mathbb{M})\,;\,\mathbb{T}_{\mathbf{x}}\mathbb{M})$.

The push-pull relations for covariant, contravariant and mixed tensors, along a diffeomorphism $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\,\boldsymbol{\zeta}(\mathbb{M}) \subset \mathbb{N})$, are given by the following:

$$\begin{aligned}
\boldsymbol{\zeta}\!\downarrow\!\mathbf{s}^{\mathrm{COV}}_{\boldsymbol{\zeta}(\mathbf{x})} &= T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}\circ\mathbf{s}^{\mathrm{COV}}_{\boldsymbol{\zeta}(\mathbf{x})}\circ T_{\mathbf{x}}\boldsymbol{\zeta} \ \in \mathrm{COV}_{\mathbf{x}}(\mathbb{TM}),\\
\boldsymbol{\zeta}\!\uparrow\!\mathbf{s}^{\mathrm{CON}}_{\mathbf{x}} &= T_{\mathbf{x}}\boldsymbol{\zeta}\circ\mathbf{s}^{\mathrm{CON}}_{\mathbf{x}}\circ T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \ \in \mathrm{CON}_{\boldsymbol{\zeta}(\mathbf{x})}(\mathbb{TN}),\\
\boldsymbol{\zeta}\!\uparrow\!\mathbf{s}^{\mathrm{MIX}}_{\mathbf{x}} &= T_{\mathbf{x}}\boldsymbol{\zeta}\circ\mathbf{s}^{\mathrm{MIX}}_{\mathbf{x}}\circ T^{-1}_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \ \in \mathrm{MIX}_{\boldsymbol{\zeta}(\mathbf{x})}(\mathbb{TN}).
\end{aligned}$$

Push-forward of contravariant tensors is well defined for any differentiable morphism $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\,\boldsymbol{\zeta}(\mathbb{M}) \subset \mathbb{N})$. Pull-back of covariant tensor fields requires that the morphism is injective. Push (or pull) transformations of other fields require that this map is a diffeomorphism.

2.3 Flows, Lie derivatives and parallel derivatives

Given a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\,\mathbb{TM})$, the flow $\mathbf{Fl}^{\mathbf{v}}_\lambda \in \mathrm{C}^1(\mathcal{R}\,;\,\mathbb{M})$ is defined by the ODEs:

$$\mathbf{v}(\mathbf{x}) = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_\lambda(\mathbf{x}), \quad \mathbf{x} \in \mathbb{M}.$$

The Lie-derivative of a vector field $\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,;\,\mathbb{TM})$ along the flow of the vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\,\mathbb{TM})$ is then the vector field defined by the following:

$$\mathcal{L}_{\mathbf{v}}\,\mathbf{u} := \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_\lambda\!\downarrow\left(\mathbf{u}\circ\mathbf{Fl}^{\mathbf{v}}_\lambda\right) \in \mathrm{C}^1(\mathbb{M}\,;\,\mathbb{TM}).$$

The derivation yields a vector since all vectors $(\mathbf{Fl}^{\mathbf{v}}_\lambda\!\downarrow\!\mathbf{u})_{\mathbf{x}} := \mathbf{Fl}^{\mathbf{v}}_\lambda\!\downarrow\!\mathbf{u}(\mathbf{Fl}^{\mathbf{v}}_\lambda(\mathbf{x}))$ belong to the linear space $\mathbb{T}_{\mathbf{x}}\mathbb{M}$. The Lie derivatives of tensor fields are analogously defined in terms of the appropriate pull-back. A simple property of the Lie derivative is that, for any diffeomorphism $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\,\mathbb{N})$:

$$\boldsymbol{\zeta}\!\downarrow\!(\mathcal{L}_{\mathbf{v}}\,\mathbf{u}) = \boldsymbol{\zeta}\!\downarrow\left(\partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}}_\lambda\!\downarrow\!\mathbf{u}\right) = \partial_{\lambda=0}\,\boldsymbol{\zeta}\!\downarrow\left(\mathbf{Fl}^{\mathbf{v}}_\lambda\!\downarrow\!\mathbf{u}\right).$$

The vanishing of the Lie derivative $\mathcal{L}_{\mathbf{v}}\,\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,;\,\mathbb{TM})$ is equivalent to the commutation property of the relevant flows: $\mathbf{Fl}^{\mathbf{v}}_\lambda\circ\mathbf{Fl}^{\mathbf{u}}_\mu = \mathbf{Fl}^{\mathbf{u}}_\mu\circ\mathbf{Fl}^{\mathbf{v}}_\lambda$.

A linear connection on a manifold $\mathbb{M}$ is expressed by a derivation ∇, which we call the *parallel derivation* fulfilling the properties:

$$\begin{aligned}
\nabla_{\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2}\mathbf{u} &= \alpha_1\nabla_{\mathbf{v}_1}\mathbf{u} + \alpha_2\nabla_{\mathbf{v}_2}\mathbf{u},\\
\nabla_{\mathbf{v}}(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2) &= \alpha_1\nabla_{\mathbf{v}}\mathbf{u}_1 + \alpha_2\nabla_{\mathbf{v}}\mathbf{u}_2,\\
\nabla_{\mathbf{v}}(f\mathbf{u}) &= f\,\nabla_{\mathbf{v}}\mathbf{u} + (\nabla_{\mathbf{v}}f)\mathbf{u}.
\end{aligned}$$

The *parallel transport* induced by a linear connection along a curve $\mathbf{c} \in \mathrm{C}^1(\Lambda\,;\,\mathbb{M})$, with Λ a real interval and $\mathbf{v}(\lambda) = \partial_{\mu=\lambda}\,\mathbf{c}(\mu)$ velocity of the curve at $\lambda \in \Lambda$, is a solution of the differential equation

$$(\nabla_{\mathbf{v}}\mathbf{u})(\mathbf{c}(\lambda)) = \partial_{\mu=\lambda}\,\mathbf{c}_{\lambda,\mu}\!\Uparrow\mathbf{u}(\mathbf{c}(\mu)).$$

The linear map $\mathbf{c}_{\lambda,\mu}\!\Uparrow \in L\,(\mathbb{T}_{\mathbf{c}(\mu)}\mathbb{M}\,;\,\mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M})$ transforms vectors based on $\mathbf{c}(\mu) \in \mathbb{M}$ into vectors based on $\mathbf{c}(\lambda) \in \mathbb{M}$, so that $\mathbf{c}_{\mu,\mu}\!\Uparrow = \mathrm{ID}_{\mathbb{TM}}$ and $\mathbf{c}_{\lambda,\mu}\!\Uparrow\circ\mathbf{c}_{\mu,\nu}\!\Uparrow = \mathbf{c}_{\lambda,\nu}\!\Uparrow$ and the inverse transport is defined by $\mathbf{c}_{\mu,\lambda}\!\Downarrow := \mathbf{c}_{\lambda,\mu}\!\Uparrow$. The derivation yields a vector in $\mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$ since all vectors $\mathbf{c}_{\lambda,\mu}\!\Uparrow\mathbf{u}(\mathbf{c}(\mu))$ belong to the same linear space $\mathbb{T}_{\mathbf{c}(\lambda)}\mathbb{M}$. The parallel transport of tensor fields is defined by invariance and the Leibniz rule holds. For instance, given a linear connection ∇ on a manifold $\mathbb{M}$, for a covariant tensor field $\mathbf{s}^{\mathrm{Cov}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{Cov}(\mathbb{TM}))$ and any $\mathbf{a}, \mathbf{b} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$, we have [7]:

$$\nabla_{\mathbf{v}}\, \mathbf{s}^{\mathrm{Cov}}(\mathbf{a}, \mathbf{b}) = \nabla_{\mathbf{v}}(\mathbf{s}^{\mathrm{Cov}}(\mathbf{a}, \mathbf{b})) - \mathbf{s}^{\mathrm{Cov}}(\nabla_{\mathbf{v}}\mathbf{a}, \mathbf{b}) - \mathbf{s}^{\mathrm{Cov}}(\mathbf{a}, \nabla_{\mathbf{v}}\mathbf{b}).$$

The Lie bracket is defined as the gap of symmetry of the iterated parallel derivative [6]: $[\mathbf{v}, \mathbf{u}]\, f = \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\, f - \nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\, f$, for any scalar function $f \in \mathrm{C}^2(\mathbb{M}\,;\mathcal{R})$. It fulfills the identity $[\mathbf{v}, \mathbf{u}] = \mathcal{L}_{\mathbf{v}}\mathbf{u}$ and the push naturality property $[\boldsymbol{\zeta}\!\uparrow\!\mathbf{v}, \boldsymbol{\zeta}\!\uparrow\!\mathbf{u}] = \boldsymbol{\zeta}\!\uparrow\![\mathbf{v}, \mathbf{u}]$ where $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$. The torsion of a linear connection is the tangent vector-valued two-form defined by the following:

$$\mathrm{Tors}(\mathbf{v}(\mathbf{x}), \mathbf{u}(\mathbf{x})) := \nabla_{\mathbf{v}(\mathbf{x})}\mathbf{u} - \nabla_{\mathbf{u}(\mathbf{x})}\mathbf{v} - [\mathbf{v}, \mathbf{u}](\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\mathbb{M},$$

where $\mathbf{u}, \mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ are tangent vector fields generated by arbitrary extensions of the tangent vectors $\mathbf{u}(\mathbf{x}), \mathbf{v}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$. The definition is well posed because the torsion field is tensorial, that is, its value is independent of the extensions. The tensor field $\mathrm{Tors}(\mathbf{v}) \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{Mix}(\mathbb{M}))$ is defined by the identity:

$$\mathrm{Tors}(\mathbf{v}) \cdot \mathbf{u} = \mathrm{Tors}(\mathbf{v}, \mathbf{u}), \quad \forall\, \mathbf{u} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}.$$

3 Lie derivatives in terms of parallel derivatives

Proposition 1 *Let $\mathbb{M}$ be a manifold and ∇ a linear connection in $\mathbb{M}$. The Lie derivative of a tensor field $\mathbf{s}_{\mathrm{Cov}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{Cov}(\mathbb{TM}))$ along the flow $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ of a tangent vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ is given by the following:*

$$\begin{aligned}\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} &= \nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} + \mathbf{s}_{\mathrm{Cov}} \circ \nabla\mathbf{v} + (\nabla\mathbf{v})^* \circ \mathbf{s}_{\mathrm{Cov}} \\ &\quad + \mathbf{s}_{\mathrm{Cov}} \circ \mathrm{Tors}(\mathbf{v}) + (\mathrm{Tors}(\mathbf{v}))^* \circ \mathbf{s}_{\mathrm{Cov}}.\end{aligned}$$

If $\mathbf{s}_{\mathrm{Cov}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{Sym}(\mathbb{M}))$, the formula specializes into the following:

$$\tfrac{1}{2}(\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}}) = \tfrac{1}{2}(\nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}}) + \mathrm{Sym}\,(\mathbf{s}_{\mathrm{Cov}} \circ \nabla\mathbf{v}) + \mathrm{Sym}\,(\mathbf{s}_{\mathrm{Cov}} \circ \mathrm{Tors}(\mathbf{v})).$$

Proof. Applying the Leibniz rule to the Lie derivative and to the covariant derivative, we have that, for any $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$:

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) &= \mathcal{L}_{\mathbf{v}}\, (\mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \mathbf{w})) - \mathbf{s}_{\mathrm{Cov}}\, (\mathcal{L}_{\mathbf{v}}\mathbf{u}, \mathbf{w}) - \mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \mathcal{L}_{\mathbf{v}}\mathbf{w}), \\ (\nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) &= \nabla_{\mathbf{v}}\, (\mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \mathbf{w})) - \mathbf{s}_{\mathrm{Cov}}\, (\nabla_{\mathbf{v}}\mathbf{u}, \mathbf{w}) - \mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \nabla_{\mathbf{v}}\mathbf{w}).\end{aligned}$$

The Lie derivative and the covariant derivative of a scalar field coincide, so that $\mathcal{L}_{\mathbf{v}}\, (\mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \mathbf{w})) = \nabla_{\mathbf{v}}\, (\mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \mathbf{w}))$ and hence:

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) &= (\nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) + \mathbf{s}_{\mathrm{Cov}}\, (\nabla_{\mathbf{v}}\mathbf{u}, \mathbf{w}) + \mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \nabla_{\mathbf{v}}\mathbf{w}) \\ &\quad - \mathbf{s}_{\mathrm{Cov}}\, (\mathcal{L}_{\mathbf{v}}\mathbf{u}, \mathbf{w}) - \mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \mathcal{L}_{\mathbf{v}}\mathbf{w}).\end{aligned}$$

Moreover, since $\mathrm{Tors}(\mathbf{v}, \mathbf{u}) := \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v}, \mathbf{u}]$ we may write

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) &= (\nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) + \mathbf{s}_{\mathrm{Cov}}\, (\mathrm{Tors}(\mathbf{v}, \mathbf{u}), \mathbf{w}) + \mathbf{s}_{\mathrm{Cov}}\, (\nabla_{\mathbf{u}}\mathbf{v}, \mathbf{w}) \\ &\quad + \mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \mathrm{Tors}(\mathbf{v}, \mathbf{w})) + \mathbf{s}_{\mathrm{Cov}}\, (\mathbf{u}, \nabla_{\mathbf{w}}\mathbf{v}),\end{aligned}$$

which, by definition of the tensor field $\mathrm{Tors}(\mathbf{v}) \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{Mix}(\mathbb{TM}))$, gives the result. □

An analogous proof leads to the next formula, which provides the expression of the Lie derivative of a volume form in terms of covariant derivatives.

Proposition 2 *Let $\mathbb{M}$ be a manifold, and ∇, a linear connection in $\mathbb{M}$. The Lie derivative of a volume form $\boldsymbol{\mu} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{Vol}(\mathbb{TM}))$ along the flow $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ of a tangent vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ is given by the following:*

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\mu} = \nabla_{\mathbf{v}}\, \boldsymbol{\mu} + \mathrm{Cycle}(\boldsymbol{\mu} \circ \mathrm{Tors}(\mathbf{v}) + \boldsymbol{\mu} \circ \nabla\mathbf{v}),$$

where the operator Cycle *evaluates the sum of the values of the volume form over cyclic permutations of the argument vectors.*

Proof. Making explicit reference to a three-form and to a triplet of vector fields $\mathbf{a}, \mathbf{b}, \mathbf{c} \in C^1(\mathbb{M}\,; \mathbb{TM})$, we have that:

$$
\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\mu})(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\nabla_{\mathbf{v}}\, \boldsymbol{\mu})(\mathbf{a}, \mathbf{b}, \mathbf{c}) &+ \boldsymbol{\mu}(\mathrm{TORS}(\mathbf{v}, \mathbf{a}), \mathbf{b}, \mathbf{c}) + \boldsymbol{\mu}(\nabla_{\mathbf{a}}\mathbf{v}, \mathbf{b}, \mathbf{c}) \\
&+ \boldsymbol{\mu}(\mathbf{a}, \mathrm{TORS}(\mathbf{v}, \mathbf{b}), \mathbf{c}) + \boldsymbol{\mu}(\nabla_{\mathbf{b}}\mathbf{v}, \mathbf{c}, \mathbf{a}) \\
&+ \boldsymbol{\mu}(\mathrm{TORS}(\mathbf{v}, \mathbf{c}), \mathbf{a}, \mathbf{b}) + \boldsymbol{\mu}(\nabla_{\mathbf{c}}\mathbf{v}, \mathbf{a}, \mathbf{b}),
\end{aligned}
$$

and the result follows. □

The operator CYCLE is related to the linear invariant J_1 of a mixed tensor as follows:

$$\mathrm{CYCLE}(\boldsymbol{\mu} \circ \mathbf{s}^{\mathrm{MIX}}) \;=\; J_1(\mathbf{s}^{\mathrm{MIX}})\, \boldsymbol{\mu},$$

so that we can write the relations:

$$
\begin{aligned}
\mathrm{CYCLE}(\boldsymbol{\mu} \circ \nabla\mathbf{v}) &= J_1(\nabla\mathbf{v})\, \boldsymbol{\mu}, \\
\mathrm{CYCLE}(\boldsymbol{\mu} \circ \mathrm{TORS}(\mathbf{v})) &= J_1(\mathrm{TORS}(\mathbf{v}))\, \boldsymbol{\mu}.
\end{aligned}
$$

By tensoriality of $\nabla_{\mathbf{v}}\, \boldsymbol{\mu}$, we may define the invariant $J_o(\mathbf{v})$ by the identity:

$$\nabla_{\mathbf{v}}\, \boldsymbol{\mu} = J_o(\mathbf{v})\, \boldsymbol{\mu},$$

and the formula may be written as follows:

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\mu} = (J_o(\mathbf{v}) + J_1((\mathrm{TORS} + \nabla)(\mathbf{v})))\, \boldsymbol{\mu}.$$

By tensoriality of $\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\mu}$ and definition of divergence $\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\mu} = (\operatorname{div} \mathbf{v})\, \boldsymbol{\mu}$,

$$\operatorname{div} \mathbf{v} = J_o(\mathbf{v}) + J_1((\mathrm{TORS} + \nabla)(\mathbf{v})).$$

In terms of the volume form $\boldsymbol{\mu}_{\mathbf{g}} \in C^1(\mathbb{M}\,; \mathrm{VOL}(\mathbb{TM}))$ induced by a metric tensor field $\mathbf{g} \in C^1(\mathbb{M}\,; \mathrm{SYM}(\mathbb{TM}))$ and of the Levi-Civita connection, being $\nabla\mathbf{g} = 0$ and $\mathrm{TORS} = 0$, the implication $\nabla\mathbf{g} = 0 \Longrightarrow \nabla\boldsymbol{\mu}_{\mathbf{g}} = 0$ leads to the standard formula:

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\mu}_{\mathbf{g}} = J_1(\nabla\mathbf{v})\, \boldsymbol{\mu}_{\mathbf{g}},$$

which, by definition of divergence, may be written as follows: $\operatorname{div} \mathbf{v} = J_1(\nabla\mathbf{v})$.

4 Kinematics

The description of continuum kinematics is conveniently performed by considering a four-dimensional affine space-time, the events manifold E, and its representation by an observer. The space manifold is a Riemann manifold $(\mathcal{S}, \mathbf{g}_{\mathcal{S}})$, that is, a manifold endowed with a smooth field of metric, viz. twice-covariant symmetric and positive definite, tensors $\mathbf{g}_{\mathcal{S}} \in C^1(\mathcal{S}\,; \mathrm{COV}(\mathbb{T}\mathcal{S}))$.

Definition 2 (*Events manifold; Observers*) The events manifold E of classical mechanics is an affine trivial bundle diffeomorphic to the cartesian product $\mathcal{S} \times I$ between the affine space manifold $\mathcal{S}$ and the affine time instants interval I. The point of view of an observer consists in the choice of a diffeomorphism $\boldsymbol{\gamma} \in C^1(\mathrm{E}\,; \mathcal{S} \times I)$ so that the events manifold E is fibrated by the projections:

$$
\begin{cases}
\boldsymbol{\pi}_{I,\mathrm{E}} = \boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \circ \boldsymbol{\gamma}, \\
\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} = \boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)} \circ \boldsymbol{\gamma},
\end{cases}
$$

with $\boldsymbol{\pi}_{I,(\mathcal{S}\times I)} \in C^1(\mathcal{S} \times I\,; I)$, $\boldsymbol{\pi}_{\mathcal{S},(\mathcal{S}\times I)} \in C^1(\mathcal{S} \times I\,; \mathcal{S})$ Cartesian projections.

An observer defines the time-fibration $\boldsymbol{\pi}_{I,\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;I)$ and the space-fibration $\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathcal{S})$ as affine, surjective submersions, that is, surjective maps with surjective linear tangent map at each point. In classical mechanics, the time-fibration $\boldsymbol{\pi}_{I,\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;I)$ is assumed to be independent of the observer and hence defining an absolute time. On the contrary, the space-fibration $\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;I)$ is observer dependent.

In the events time-bundle $\boldsymbol{\pi}_{I,\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;I)$, the manifold fiber E_t, based on a time $t \in I$, collects *simultaneous events* and is diffeomorphic to the typical fiber $\mathcal{S}$, the space manifold, whose elements are spatial points $\mathbf{x} \in \mathcal{S}$. In the events space-bundle $\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathcal{S})$, the manifold fiber $\mathrm{E}_{\mathbf{x}}$, based on a position $\mathbf{x} \in \mathcal{S}$, collects *localized events* and is diffeomorphic to the typical fiber I, the time interval, whose elements are time instants $t \in I$.

Tensors bundles, constructed over the vertical tangent bundle to the events time-bundle, that is, bundles made of tensors constructed at fixed time, play a basic role in CM, and hence, we preliminarily recall the relevant notions. Tangent vectors $\mathbf{v} \in \mathbb{T}\mathrm{E}$ are velocities of curves $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,;\mathrm{E})$ drawn in the events manifold E.

Definition 3 (*Time-vertical tangent bundle*) A time-vertical tangent vector $\mathbf{v} \in \mathbb{T}_{\mathbf{e}}\mathrm{E}$ is tangent to the fiber E_t based on $t = \boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e})$, that is, $\mathbf{v} \in \mathbb{T}_{\mathbf{e}}\mathrm{E}_t$, and is characterized by the property that the velocity of the base curve vanishes:

$$T_{\mathbf{e}}\boldsymbol{\pi}_{I,\mathrm{E}} \cdot \mathbf{v} = 0.$$

Time-vertical vectors are the elements of the time-vertical subbundle $\mathbb{V}\mathrm{E} \subset \mathbb{T}\mathrm{E}$ of the tangent fibration $T\boldsymbol{\pi}_{I,\mathrm{E}} \in \mathrm{C}^0(\mathbb{T}\mathrm{E}\,;\mathbb{T}I)$. The fiber in $\mathbb{V}\mathrm{E}$ based on $\mathbf{e} \in \mathrm{E}$ is: $\mathbb{V}_{\mathbf{e}}\mathrm{E} = \mathbb{T}_{\mathbf{e}}\mathrm{E}_t$.

Definition 4 (*Induced connection in the events manifold*) A connection $\nabla^{\mathcal{S}}$ in the ambient space manifold and the usual connection ∇^I in the affine time line induce naturally a connection ∇^{E} in the events manifold E, by defining the space and time components of the parallel derivative $\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathrm{E}}}\mathbf{u}_{\mathrm{E}}$ of a vector field $\mathbf{u}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})$, along a vector field $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})$, as follows:

$$\begin{cases} \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\!\uparrow\left(\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathrm{E}}}\mathbf{u}_{\mathrm{E}}\right) := \partial_{\lambda=0}\,\left(\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \circ \mathbf{Fl}^{\mathbf{v}_{\mathrm{E}}}_{\lambda}\right) \Downarrow^{\mathcal{S}} \left(\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\!\uparrow \circ\, \mathbf{u}_{\mathrm{E}} \circ \mathbf{Fl}^{\mathbf{v}_{\mathrm{E}}}_{\lambda}\right), \\ \boldsymbol{\pi}_{I,\mathrm{E}}\!\uparrow\left(\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathrm{E}}}\mathbf{u}_{\mathrm{E}}\right) := \partial_{\lambda=0}\,\left(\boldsymbol{\pi}_{I,\mathrm{E}} \circ \mathbf{Fl}^{\mathbf{v}_{\mathrm{E}}}_{\lambda}\right) \Downarrow^{I} \left(\boldsymbol{\pi}_{I,\mathrm{E}}\!\uparrow \circ\, \mathbf{u}_{\mathrm{E}} \circ \mathbf{Fl}^{\mathbf{v}_{\mathrm{E}}}_{\lambda}\right). \end{cases}$$

Definition 4 is motivated by the fact that, as a rule, the push $\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\!\uparrow\mathbf{w}_{\mathrm{E}} \in \mathrm{C}^1(\mathcal{S}\,;\mathbb{T}\mathcal{S})$ of a vector field $\mathbf{w}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})$ is not well defined on $\mathcal{S}$, since the projection $\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathcal{S})$ is not injective, while the push of a vector $\mathbf{w}_{\mathrm{E}}(\mathbf{e}) \in \mathbb{T}_{\mathbf{e}}\mathrm{E}$ is well defined by the following:

$$\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\!\uparrow(\mathbf{w}_{\mathrm{E}}(\mathbf{e})) := T_{\mathbf{e}}\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \cdot \mathbf{w}_{\mathrm{E}}(\mathbf{e}) \in \mathbb{T}_{\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}(\mathbf{e})}\mathcal{S}.$$

Lemma 1 (Time-verticality of the parallel derivative) *The parallel derivative* $\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathrm{E}}}\mathbf{u}_{\mathrm{E}}$ *of a vector field* $\mathbf{u}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})$*, along a vector field* $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})$*, is time-vertical if either the time component of the vector field* $\mathbf{u}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathbb{T}\mathrm{E})$ *is constant in time or the vector* $\mathbf{v}_{\mathrm{E}}$ *is time-vertical.*

Proof. Each one of the conditions implies that $\boldsymbol{\pi}_{I,\mathrm{E}}\!\uparrow(\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathrm{E}}}\mathbf{u}_{\mathrm{E}}) = 0$. □

4.1 Trajectory and body motion

In introducing the basic kinematic notions, we follow an approach which simulates physical experience.

Definition 5 (*Trajectory*) The trajectory of a body is a manifold $\mathcal{T}_{\boldsymbol{\varphi}}$ with injective immersion $\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{E})$ (the tangent linear map $T_{\mathbf{e}}\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathbb{T}_{\mathbf{e}}\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathbb{T}_{\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}}(\mathbf{e})}\mathrm{E})$ of $\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}}$ at each point $\mathbf{e} \in \mathcal{T}_{\boldsymbol{\varphi}}$ is injective) in the events manifold and such that the image $\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}}(\mathcal{T}_{\boldsymbol{\varphi}}) \subset \mathrm{E}$ is a submanifold.

Whenever feasible, we will write $\mathbf{e} \in \mathrm{E}$ instead of $\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}}(\mathbf{e}) \in \mathrm{E}$ for $\mathbf{e} \in \mathcal{T}_{\boldsymbol{\varphi}}$.

Definition 6 (*Trajectory fibrations*) The trajectory is assumed to acquire the structure of a time-bundle $\boldsymbol{\pi}_{I,\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;I)$, whose time-fibers are compact, connected submanifolds, by inheriting the projection of the events time-bundle $\boldsymbol{\pi}_{I,\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;I)$, according to the relation

$$\boldsymbol{\pi}_{I,\mathcal{T}_{\varphi}} = \boldsymbol{\pi}_{I,\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}}.$$

The time-fiber $\mathcal{T}_{\boldsymbol{\varphi}}(t) := \boldsymbol{\pi}^{-1}_{I,\mathcal{T}_{\boldsymbol{\varphi}}}(t)$ at time $t \in I$ is seen by an observer as $\boldsymbol{\gamma}(\mathcal{T}_{\boldsymbol{\varphi}}(t)) = (\boldsymbol{\Omega}_t, t)$, where $\boldsymbol{\Omega}_t \subset \mathcal{S}$ is the *body placement*. A space-fibration of the trajectory $\boldsymbol{\pi}_{\mathcal{S},\mathcal{T}_{\boldsymbol{\varphi}}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathcal{S})$ is induced by the projection of the events space-bundle $\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\mathcal{S})$, according to the relation:

$$\boldsymbol{\pi}_{\mathcal{S},\mathcal{T}_{\boldsymbol{\varphi}}} = \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\varphi}}}.$$

Lemma 2 *The injective immersion* $\mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\varphi}}} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{E})$ *is a homomorphism of vertical tangent bundles.*

Proof. Indeed, being: $T\boldsymbol{\pi}_{I,\mathcal{T}_{\boldsymbol{\varphi}}} = T\boldsymbol{\pi}_{I,\mathrm{E}} \circ T\mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\varphi}}}$, it follows that time-verticality is preserved:

$$T\boldsymbol{\pi}_{I,\mathrm{E}} \cdot \mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}} = 0 \quad \Longrightarrow \quad T\boldsymbol{\pi}_{I,\mathrm{E}} \cdot (T\mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\varphi}}} \cdot \mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}) = 0,$$

and similarly for space-verticality. □

Definition 7 (*Motion and evolution operator*) A *motion* in the trajectory is a one-parameter family of automorphisms of the trajectory manifold $\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_s \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathcal{T}_{\boldsymbol{\varphi}})$ over the time shift $\mathrm{SH}_s \in \mathrm{C}^1(I\,;I)$ defined by $\mathrm{SH}_s(t) := t + s$ for all $s, t \in I$, as described by the commutative diagram:

$$\begin{array}{ccc} \mathcal{T}_{\boldsymbol{\varphi}} & \xrightarrow{\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_s} & \mathcal{T}_{\boldsymbol{\varphi}} \\ {\scriptstyle \boldsymbol{\pi}_{I,\mathcal{T}_{\boldsymbol{\varphi}}}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\pi}_{I,\mathcal{T}_{\boldsymbol{\varphi}}}} \\ I & \xleftrightarrow{\mathrm{SH}_s} & I \end{array} \quad \Longleftrightarrow \quad \boldsymbol{\pi}_{I,\mathcal{T}_{\boldsymbol{\varphi}}} \circ \boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_s = \mathrm{SH}_s \circ \boldsymbol{\pi}_{I,\mathcal{T}_{\boldsymbol{\varphi}}}.$$

The motion of a body is conveniently described by an evolution operator $\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}$ which assigns to any pair of time instants $\tau, t \in I$, the corresponding material displacement diffeomorphism $\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau,t} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}(t)\,;\mathcal{T}_{\boldsymbol{\varphi}}(\tau))$ defined by the following:

$$\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau,t}(\mathbf{e}) := \boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau-t}(\mathbf{e}),$$

for $\mathbf{e} \in \mathcal{T}_{\boldsymbol{\varphi}}(t)$. The evolution fulfills the Chapman–Kolmogorov law of *determinism* [6,7]:

$$\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau,s} = \boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau,t} \circ \boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{t,s},$$

for all $\tau, t, s \in I$, and $\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{t,\tau} = (\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau,t})^{-1}$.

The immersed evolution in the events manifold is given by $\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t} = \mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\varphi}}} \circ \boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau,t}$. The displacement $\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}$ is measured by an observer at $\boldsymbol{\gamma}(\mathbf{e}) = (\mathbf{x}, t) \in \mathcal{S} \times I$ as a pair:

$$(\boldsymbol{\varphi}^{\mathcal{S}}_{\tau,t}(\mathbf{e}), \boldsymbol{\varphi}^{I}_{\tau,t}(\mathbf{e})) := \boldsymbol{\gamma}(\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}(\mathbf{e})),$$

with $\boldsymbol{\varphi}^{I}_{\tau,t}(\mathbf{e}) = \tau - t$.

Definition 8 (*Body and particles*) The fibers of the trajectory time-bundle, made of simultaneous events, are called *body placements*. The trajectory is foliated by the evolution operator into one-dimensional submanifolds which are equivalence classes for the relation:

$$(\mathbf{e}_1, \mathbf{e}_2) \in \mathcal{T}_{\boldsymbol{\varphi}} \times \mathcal{T}_{\boldsymbol{\varphi}} \ : \ \mathbf{e}_2 = \boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{t_2,t_1}(\mathbf{e}_1),$$

with $t_i = \boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e}_i)$, $i = 1, 2$.

The determinism law ensures reflexivity, symmetry and transitivity. The trajectory foliation is called the *body*, and each folium, made of trajectory events related by the evolution operator, is called a *material particle*.

Definition 9 (*Trajectory speed*) The *trajectory speed* of a particle of the body at the event $\mathbf{e} \in \mathcal{T}_{\boldsymbol{\varphi}}$, with $t = \boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e})$, is the time-derivative of the *motion* $\boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\cdot,t}(\mathbf{e}) \in \mathrm{C}^1(I\,;\mathcal{T}_{\boldsymbol{\varphi}})$ defined by the following:

$$\mathbf{v}_{\mathcal{T}_{\boldsymbol{\varphi}}}(\mathbf{e}) := \partial_{\tau=t}\, \boldsymbol{\varphi}^{\mathcal{T}_{\boldsymbol{\varphi}}}_{\tau,t}(\mathbf{e}) \in \mathbb{T}_{\mathbf{e}}\mathcal{T}_{\boldsymbol{\varphi}}.$$

Being $\boldsymbol{\pi}_{I,\mathcal{T}_\varphi}(\boldsymbol{\varphi}^{\mathcal{T}_\varphi}_{\tau,t}(\mathbf{e})) = \tau$ for all $\tau \in I$, the trajectory speeds fulfill the condition

$$\partial_{\tau=t}\, \boldsymbol{\pi}_{I,\mathcal{T}_\varphi}(\boldsymbol{\varphi}^{\mathcal{T}_\varphi}_{\tau,t}(\mathbf{e})) = T_{\mathbf{e}}\boldsymbol{\pi}_{I,\mathcal{T}_\varphi} \cdot \mathbf{v}_{\mathcal{T}_\varphi}(\mathbf{e}) = 1_t,$$

which defines an affine subbundle $\boldsymbol{\tau}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathbb{A}_1\mathcal{T}_{\boldsymbol{\varphi}}\,; \mathcal{T}_{\boldsymbol{\varphi}})$ of the trajectory tangent bundle $\boldsymbol{\tau}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathbb{T}\mathcal{T}_{\boldsymbol{\varphi}}\,; \mathcal{T}_{\boldsymbol{\varphi}})$. Being $\mathbb{A}_0 = \mathbb{V}$, the model linear bundle is the vertical tangent time-bundle $\boldsymbol{\tau}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}}\,; \mathcal{T}_{\boldsymbol{\varphi}})$ over the trajectory manifold.

The immersion of the trajectory speed in the events manifold will be denoted by $\mathbf{v}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{v}_{\mathcal{T}_\varphi}$. An observer measures the trajectory speed as a pair $\boldsymbol{\gamma}\!\uparrow\!\mathbf{v}_{\mathcal{T}_\varphi} = (\mathbf{v}^{\mathcal{S}}_{\boldsymbol{\varphi}}, 1)$ which at time $t \in I$ is made of a spatial field component $\mathbf{v}^{\mathcal{S}}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,; \mathbb{T}\mathcal{S})$ and of a unitary time component $1_t \in \mathbb{T}_t I$.

5 Spatial and material fields

Definition 10 (*Spatial fields*) A spatial tensor field is a section

$$\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,; \mathrm{TENS}(\mathbb{V}\mathrm{E}))$$

of the tensor bundle $\boldsymbol{\pi}^{\mathrm{TENS}}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{V}\mathrm{E})\,; \mathrm{E})$ constructed over the vertical tangent bundle to the events time-bundle $\boldsymbol{\pi}_{I,\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,; I)$.

The only spatial field of interest in CM is the *spatial metric tensor field* on the events manifold $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,; \mathrm{COV}(\mathbb{V}\mathrm{E}))$, which is induced by the metric tensor field $\mathbf{g}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{S}\,; \mathrm{COV}(\mathbb{T}\mathcal{S}))$ of the Riemann space manifold by the pull-back

$$\mathbf{g}_{\mathrm{E}} = \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\!\downarrow\!\mathbf{g}_{\mathcal{S}},$$

equivalent to $\mathbf{g}_{\mathrm{E}}(\mathbf{a},\mathbf{b}) := \mathbf{g}_{\mathcal{S}}(T\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\cdot\mathbf{a}, T\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\cdot\mathbf{b})$ for all time-vertical tangent vector fields $\mathbf{a},\mathbf{b} \in \mathrm{C}^1(\mathrm{E}\,; \mathbb{V}\mathrm{E})$. The positive definiteness of the *metric tensor field* $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,; \mathrm{COV}(\mathbb{V}\mathrm{E}))$ follows by the positive definiteness of the *metric field* $\mathbf{g}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{S}\,; \mathrm{COV}(\mathbb{T}\mathcal{S}))$ and by the injectivity of the tangent map $T_{\mathbf{e}}\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in L(\mathbb{V}_{\mathbf{e}}\mathrm{E}\,; \mathbb{T}_{\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}(\mathbf{e})}\mathcal{S})$.

Definition 11 (*Material fields*) A material tensor field is a section

$$\mathbf{s}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,; \mathrm{TENS}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}}))$$

of the tensor bundle $\boldsymbol{\pi}^{\mathrm{TENS}}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}})\,; \mathcal{T}_{\boldsymbol{\varphi}})$ constructed over the vertical tangent bundle to the trajectory time-bundle $\boldsymbol{\pi}_{I,\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,; I)$.

The observer view of a material vector $\mathbf{u}_{\mathcal{T}_\varphi}(\mathbf{e}) \in \mathbb{V}_{\mathbf{e}}\mathcal{T}_{\boldsymbol{\varphi}}$, with $\boldsymbol{\gamma}(\mathbf{e}) = (\mathbf{x}, t)$ and $\mathbf{x} \in \boldsymbol{\Omega}_t$, is the pair $\boldsymbol{\gamma}\!\uparrow\!\mathbf{u}_{\mathcal{T}_\varphi}(\mathbf{e}) = (\mathbf{u}_{\boldsymbol{\varphi},t}(\mathbf{x}), 0)$ which is identified with $\mathbf{u}_{\boldsymbol{\varphi},t}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t$. To a material covector $\mathbf{s}_{\mathcal{T}_\varphi}(\mathbf{e}) \in \mathbb{V}^*_{\mathbf{e}}\mathcal{T}_{\boldsymbol{\varphi}}$, there corresponds the covector $\mathbf{s}_{\boldsymbol{\varphi},t}(\mathbf{x}) := \boldsymbol{\gamma}\!\uparrow\!\mathbf{s}_{\mathcal{T}_\varphi}(\mathbf{e}) \in \mathbb{T}^*_{\mathbf{x}}\boldsymbol{\Omega}_t$ such that

$$\langle \mathbf{s}_{\boldsymbol{\varphi},t}, \mathbf{u}_{\boldsymbol{\varphi},t} \rangle_{\mathbf{x}} = \langle \boldsymbol{\gamma}\!\uparrow\!\mathbf{s}_{\mathcal{T}_\varphi}, \boldsymbol{\gamma}\!\uparrow\!\mathbf{u}_{\mathcal{T}_\varphi} \rangle_{\boldsymbol{\gamma}(\mathbf{e})},$$

and similarly for any tensor field. Fields of primary interest in CM are *material fields*. For instance, stretch, stretching, stress, stressing, temperature, mass, entropy and thermodynamical potentials are such.

Definition 12 (*Trajectory-based space-time and spatial fields*) To a space-time tensor field $\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}(\mathcal{T}_{\boldsymbol{\varphi}})\,; \mathrm{TENS}(\mathbb{T}\mathrm{E}))$, whose domain includes the immersed trajectory manifold, there corresponds to a trajectory-based space-time field $\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,; \mathrm{TENS}(\mathbb{T}\mathrm{E}))$ which is the section of the pull-back bundle $\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\boldsymbol{\pi}^{\mathrm{TENS}}_{\mathrm{E}}$, defined by the following:

$$\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{e}) := (\mathbf{s}_{\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi})(\mathbf{e}) \in \mathrm{TENS}(\mathbb{T}_{\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}(\mathbf{e})}\mathrm{E}), \quad \mathbf{e} \in \mathcal{T}_{\boldsymbol{\varphi}}.$$

The same procedure applies to a spatial field $\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}(\mathcal{T}_{\boldsymbol{\varphi}})\,; \mathrm{TENS}(\mathbb{V}\mathrm{E}))$ to get a trajectory-based spatial field $\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,; \mathrm{TENS}(\mathbb{V}\mathrm{E}))$.

The observer view of a spatial vector $\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi} \in \mathbb{V}_{\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}(\mathbf{e})}\mathrm{E}$, with $\boldsymbol{\gamma}(\mathbf{e}) = (\mathbf{x}, t)$ and $\mathbf{x} \in \boldsymbol{\Omega}_t$, is the pair $\boldsymbol{\gamma}\uparrow\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{e}) = (\mathbf{u}^{\mathcal{S}}_{\varphi,t}(\mathbf{x}), 0)$ which is identified with $\mathbf{u}^{\mathcal{S}}_{\varphi,t}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\mathcal{S}$. To a material covector $\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{e}) \in \mathbb{V}^*_{\mathbf{e}}\mathrm{E}$, there corresponds to the covector $\mathbf{s}^{\mathcal{S}}_{\varphi,t}(\mathbf{x}) := \boldsymbol{\gamma}\uparrow\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{e}) \in \mathbb{T}^*_{\mathbf{x}}\mathcal{S}$ such that

$$\langle \mathbf{s}^{\mathcal{S}}_{\varphi,t}, \mathbf{u}^{\mathcal{S}}_{\varphi,t} \rangle_{\mathbf{x}} = \langle \boldsymbol{\gamma}\uparrow\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}, \boldsymbol{\gamma}\uparrow\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi} \rangle_{\boldsymbol{\gamma}(\mathbf{e})},$$

and similarly for any tensor field. Basic fields in continuum dynamics are *trajectory-based spatial vector or covector fields*, namely virtual velocity, acceleration, force fields.

In CM, an essential role is played by pull-back of a spatial covariant tensor field to a material covariant tensor field. This construction works only for covariant tensors because the map $\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathrm{E})$ is injective but not surjective, in general, and is possible because the tangent map $T\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathbb{T}\mathcal{T}_\varphi\,; \mathbb{T}\mathrm{E})$, being fiberwise injective, preserves the time-vertical fibration, that is, $T\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}(\mathbb{V}\mathcal{T}_\varphi) \subset \mathbb{V}\mathrm{E}$.

Definition 13 (*Trajectory metric tensor field*) The space-time metric tensor field $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}, \mathrm{Cov}(\mathbb{T}\mathrm{E}))$ induces a trajectory metric field $\mathbf{g}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi, \mathrm{Cov}(\mathbb{T}\mathcal{T}_\varphi))$ by pull-back to the trajectory according to the injective immersion $\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathrm{E})$:

$$\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_{\mathcal{T}_\varphi}, \mathbf{b}_{\mathcal{T}_\varphi}) := \mathbf{g}_{\mathrm{E}}(T\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \cdot \mathbf{a}_{\mathcal{T}_\varphi}, T\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \cdot \mathbf{b}_{\mathcal{T}_\varphi}),$$

for all $\mathbf{a}_{\mathcal{T}_\varphi}, \mathbf{b}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathbb{T}\mathcal{T}_\varphi)$, which is expressed by the pull-back:

$$\mathbf{g}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathcal{T}_\varphi, \mathrm{Cov}(\mathbb{T}\mathcal{T}_\varphi)).$$

Material metric tensors are evidently symmetric and positive definite, due to the positive definiteness of the metric tensor field $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}, \mathrm{Cov}(\mathbb{T}\mathrm{E}))$ and to the injectivity of the tangent map $T_{\mathbf{e}}\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} \in L(\mathbb{T}_{\mathbf{e}}\mathcal{T}_\varphi, \mathbb{T}_{\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}(\mathbf{e})}\mathrm{E})$.

The introduction of a material metric is usually improperly skipped in the literature, when dealing with 3-D bodies, by considering the spatial metric *tout court* as the variable appearing in constitutive relations, see for example [22]. The role of the spatial immersion is however essential for a proper geometrical treatment and becomes evident in analyzing lower dimensional bodies, such as membranes and wires.

6 Time-derivatives of tensor fields

6.1 Convective time-derivatives along the motion

Definition 14 (*Convective time-derivative*) Given a material tensor field $\mathbf{s}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathrm{Tens}(\mathbb{V}\mathcal{T}_\varphi))$ and the field of trajectory speeds $\mathbf{v}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathbb{A}_1\mathcal{T}_\varphi)$, the material tensor field defined by the following:

$$\dot{\mathbf{s}}_{\mathcal{T}_\varphi} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{s}_{\mathcal{T}_\varphi} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda}\!\downarrow(\mathbf{s}_{\mathcal{T}_\varphi} \circ \mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda}),$$

is the Lie derivative[1] (or convective derivative) along the motion.

The Leibniz rule holds, as shown by the relation:

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\langle \mathbf{s}_{\mathcal{T}_\varphi}, \mathbf{u}_{\mathcal{T}_\varphi} \rangle &= \partial_{\lambda=0}\,\langle \mathbf{s}_{\mathcal{T}_\varphi} \circ \mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda}, \mathbf{u}_{\mathcal{T}_\varphi} \circ \mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda} \rangle \\
&= \partial_{\lambda=0}\left\langle \mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda}\!\downarrow\left(\mathbf{s}_{\mathcal{T}_\varphi} \circ \mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda}\right), \mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda}\!\downarrow\left(\mathbf{u}_{\mathcal{T}_\varphi} \circ \mathbf{Fl}^{\mathbf{v}_{\mathcal{T}_\varphi}}_{\lambda}\right)\right\rangle \\
&= \langle \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{s}_{\mathcal{T}_\varphi}, \mathbf{u}_{\mathcal{T}_\varphi} \rangle + \langle \mathbf{s}_{\mathcal{T}_\varphi}, \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi} \rangle.
\end{aligned}$$

Then, the Lie time-derivative of a material tensor field $\mathbf{s}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathrm{Tens}(\mathbb{V}\mathcal{T}_\varphi))$ is well defined as a material tensor field by the formula (hereafter, written for a covector field):

$$\langle \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{s}_{\mathcal{T}_\varphi}, \mathbf{u}_{\mathcal{T}_\varphi} \rangle := \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\langle \mathbf{s}_{\mathcal{T}_\varphi}, \mathbf{u}_{\mathcal{T}_\varphi} \rangle - \langle \mathbf{s}_{\mathcal{T}_\varphi}, \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi} \rangle,$$

since verticality of the vector field $\mathbf{u}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathbb{V}\mathcal{T}_\varphi)$ implies verticality of the Lie derivative $\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi}$.

[1] It is named after the Norwegian geometer Marius Sophus Lie [23]

6.2 Parallel time-derivatives along the motion

Vectors tangent to the ambient space manifold may be transported along a curve in a parallel way dictated by the chosen connection, and the parallel transport of spatial tensor fields is defined by invariance. In CM, only trajectory-based tensor fields enter into the theory, and hence, parallel derivatives only of trajectory-based spatial fields along the motion are considered, according to the following definition.

Definition 15 (*Parallel time-derivative*) The *parallel time-derivative* of a trajectory-based space-time field, according to a given connection ∇^{E}, is the trajectory-based space-time field $\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathcal{T}_{\varphi}}} \mathbf{s}^{\mathrm{E}}_{\mathcal{T}_{\varphi}}$ defined, for any $\mathbf{e} \in \mathcal{T}_{\varphi}$ with $t = \boldsymbol{\pi}_{I,\mathrm{E}}(\mathbf{e})$, as parallel derivative along the trajectory speed $\mathbf{v}_{\mathcal{T}_{\varphi}}(\mathbf{e}) \in \mathbb{A}_{1\mathbf{e}}\mathcal{T}_{\varphi}$:

$$\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathcal{T}_{\varphi}}} \mathbf{s}^{\mathrm{E}}_{\mathcal{T}_{\varphi}} := \partial_{\lambda=0}\, \mathbf{Fl}_{\lambda}^{\mathbf{v}^{\mathrm{E}}_{\mathcal{T}_{\varphi}}} \Downarrow^{\mathrm{E}} \left(\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_{\varphi}} \circ \mathbf{Fl}_{\lambda}^{\mathbf{v}_{\mathcal{T}_{\varphi}}} \right) \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\, \mathrm{T{\scriptsize ENS}}(\mathbb{V}\mathrm{E})).$$

Definition 16 (*Pushed connection*) To a connection ∇^{E} in the events manifold and an automorphism $\boldsymbol{\zeta}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}\,;\,\mathrm{E})$, there corresponds to a pushed connection $\boldsymbol{\zeta}_{\mathrm{E}}\!\uparrow\!\nabla^{\mathrm{E}}$ defined by the following:

$$(\boldsymbol{\zeta}_{\mathrm{E}}\!\uparrow\!\nabla^{\mathrm{E}})_{\boldsymbol{\zeta}_{\mathrm{E}}\uparrow\mathbf{v}_{\mathrm{E}}}\, \boldsymbol{\zeta}_{\mathrm{E}}\!\uparrow\!\mathbf{u}_{\mathrm{E}} := \boldsymbol{\zeta}_{\mathrm{E}}\!\uparrow \left(\nabla^{\mathrm{E}}_{\mathbf{v}_{\mathrm{E}}} \mathbf{u}_{\mathrm{E}} \right).$$

By naturality of Lie brackets with respect to push, the torsion of the pushed connection is the push of the torsion.

Remark 1 This construction by push performed in Proposition 16 is not feasible to induce a connection in the immersed trajectory manifold. The reason is that the parallel derivative $(\nabla^{\mathrm{E}} \mathbf{u}^{\mathrm{E}}_{\mathcal{T}_{\varphi}} \cdot \mathbf{v}^{\mathrm{E}}_{\mathcal{T}_{\varphi}})(\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}}(\mathbf{e}))$ will in general fail to belong to the immersion $T\mathbf{i}_{\mathrm{E},\mathcal{T}_{\varphi}}(\mathbb{T}_{\mathbf{e}}\mathcal{T}_{\varphi})$ of the tangent space $\mathbb{T}_{\mathbf{e}}\mathcal{T}_{\varphi}$. The right construction of the connection induced on the trajectory by the immersion pull-back will be given in Definition 18 as an extension of the familiar notion of Levi-Civita connection induced on a submanifold of a Riemann manifold, see for example [24].

6.3 Coordinate induced connections

Two reference systems can be associated with a coordinate system in a Riemann manifold. The former is the natural one whose basis vectors at any point are the velocities of coordinate lines. The latter is obtained from the former by normalizing the basis vectors.

Correspondingly, two linear connections can be introduced by defining path-independent parallel transports in which the components of tangent vectors are left invariant. In the two reference systems, they will be, respectively, called the natural and the normalized connection. In both connections, the parallel derivatives of the reference vector fields vanish identically. In the natural connection, the Lie bracket of basis vector fields vanishes identically, due to the commutation of the corresponding flows, and hence, the torsion vanishes identically too [7]. In the normalized connection, the torsion is equal to the opposite of the Lie bracket. This last property leads to formulate the Poincaré equations as a special case of the equations of dynamics expressed in terms of an arbitrary linear connection [25,26].

7 Stretching

The notion of material metric tensor field on the trajectory, introduced in Definition 13 as pull-back of the spatial metric field according to the immersion map, provides the tool to make metric measurements on a body. The Lie derivative along the motion measures the rate of variation of the body's metric properties. We may then give the following definition.

Definition 17 (*Stretching field along the motion*) The stretching field along the motion is defined by the Lie derivative:

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\varphi}}} \mathbf{g}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\, \mathrm{S{\scriptsize YM}}(\mathbb{V}\mathcal{T}_{\varphi})).$$

It is a section of the material bundle $\boldsymbol{\pi}^{\mathrm{SYM}}_{\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathrm{S{\scriptsize YM}}(\mathbb{V}\mathcal{T}_{\varphi})\,;\,\mathcal{T}_{\varphi})$ of covariant symmetric tensors over the trajectory time-bundle $\boldsymbol{\pi}_{I,\mathcal{T}_{\varphi}} \in \mathrm{C}^1(\mathcal{T}_{\varphi}\,;\, I)$. The factor $\frac{1}{2}$ is introduced to get a convenient physical interpretation of the dual stress tensor [7].

Remark 2 The trajectory and the evolution considered in Definition 17 of stretching may be substituted by any virtual trajectory and virtual evolution defined in the spatial fiber at time $t \in I$. The velocity of a virtual evolution at time $t \in I$ is a tangent vector field $\delta\mathbf{v}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t\,;\mathbb{T}_{\boldsymbol{\Omega}_t}\mathcal{S})$, called a virtual velocity.

The expression of the mixed form of the stretching tensor field is a celebrated result due to Euler [1,3]. An extended version, valid for any linear connection in a Riemann ambient space manifold, and for bodies of any dimensionality, is a new result contributed in Proposition 3 below.

In Remark 1, it was underlined that, in the events manifold, the parallel derivative of a vector field tangent to the immersed trajectory, in the direction of a vector tangent to the immersed trajectory, does in general fail to be still tangent to the immersed trajectory. To induce a parallel derivative on the trajectory manifold, a projection is then needed and can be performed by the metric tensor defined on the events manifold, as illustrated hereafter.

Definition 18 (*Trajectory connection*) The linear connection $\nabla^{\mathcal{T}_\varphi}$ induced in the trajectory manifold, by a connection ∇^{E} in the events manifold, is introduced by considering vector fields $\mathbf{u}_{\mathcal{T}_\varphi}, \mathbf{h}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathbb{T}\mathcal{T}_{\boldsymbol{\varphi}})$ tangent to the trajectory and their immersions $\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{u}_{\mathcal{T}_\varphi}$, $\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{h}_{\mathcal{T}_\varphi}$ in the events manifold and defining the parallel derivative $\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi}$ by the pull-back

$$\mathbf{g}_{\mathcal{T}_\varphi}\circ\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\left(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}\circ\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi}\right),$$

equivalent to the orthogonal projection

$$\mathbf{g}_{\mathcal{T}_\varphi}(\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi},\mathbf{w}_{\mathcal{T}_\varphi}) := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\left(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{w}^{\mathrm{E}}_{\mathcal{T}_\varphi})\right),$$

for all $\mathbf{w}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathbb{T}\mathcal{T}_{\boldsymbol{\varphi}})$, being $\mathbf{g}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}$ and $\mathbf{w}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{w}_{\mathcal{T}_\varphi}$.

Lemma 3 (Minimum property) *The linear connection $\nabla^{\mathcal{T}_\varphi}$ induced in the trajectory manifold by the connection ∇^{E} in the events manifold, is characterized by the minimum distance property:*

$$\|\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi} - \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\left(\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi}\right)\|_{\mathbf{g}_{\mathrm{E}}} = \min_{\mathbf{w}_{\mathcal{T}_\varphi}\in\mathbb{T}\mathcal{T}_\varphi}\|\left(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi} - \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{w}_{\mathcal{T}_\varphi}\right)\|_{\mathbf{g}_{\mathrm{E}}}.$$

Proof. The minimum distance property is expressed by the variational condition

$$\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{u}^{\mathrm{E}}_{\mathcal{T}_\varphi} - \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\left(\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{u}_{\mathcal{T}_\varphi}\right),\mathbf{w}^{\mathrm{E}}_{\mathcal{T}_\varphi}) = 0,$$

which is the orthogonal projection property in Definition 18 of trajectory connection. □

The next two properties of the trajectory connection will be resorted to in the proof of Euler's stretching formula.

Lemma 4 (Parallel derivative of metric field) *The parallel derivative of the metric, according to a linear connection ∇^{E} in the events manifold, and the parallel derivative of the induced metric in the trajectory manifold according to the induced linear connection $\nabla^{\mathcal{T}_\varphi}$ are related by pull-back according to the immersion map*

$$\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{g}_{\mathcal{T}_\varphi} = \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\left(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}\right),$$

where $\mathbf{g}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}$ and $\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{h}_{\mathcal{T}_\varphi}$, with $\mathbf{h}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathbb{T}\mathcal{T}_{\boldsymbol{\varphi}})$.

Proof. Given $\mathbf{h}_{\mathcal{T}_\varphi}, \mathbf{a}_{\mathcal{T}_\varphi}, \mathbf{b}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathbb{T}\mathcal{T}_{\boldsymbol{\varphi}})$, let us set $\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{a}_{\mathcal{T}_\varphi}$, $\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{b}_{\mathcal{T}_\varphi}$ and $\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{h}_{\mathcal{T}_\varphi}$. Then, the definition of pull-back, the Leibniz rule for the parallel derivative of a covariant tensor, and the symmetry and non-degeneracy of the metric tensor, imply that

$$\begin{aligned}
(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}))(\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}) &= \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\Big((\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi})(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi})\Big)\\
&= \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\Big(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}))) - \mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}))\\
&\quad -\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}))\Big).
\end{aligned}$$

Pulling back each term we get:

$$\begin{aligned}\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}))) &= \nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi})))\\ &= \nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi})),\\ \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi})) &= \mathbf{g}_{\mathcal{T}_\varphi}(\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}),\end{aligned}$$

with the former equality following from the definition of parallel derivative of a scalar field and the latter ensuing from the Definition 18 of trajectory connection. Therefore,

$$\begin{aligned}\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!(\nabla^{\mathrm{E}}_{\mathbf{h}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi})(\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}) = {}& \nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}))\\ &-\mathbf{g}_{\mathcal{T}_\varphi}(\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi})-\mathbf{g}_{\mathcal{T}_\varphi}(\nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{b}_{\mathcal{T}_\varphi},\mathbf{a}_{\mathcal{T}_\varphi})\\ ={}& \nabla^{\mathcal{T}_\varphi}_{\mathbf{h}_{\mathcal{T}_\varphi}}\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}),\end{aligned}$$

which is the result. □

Lemma 5 (Torsion of the trajectory connection) *The torsion of the connection in the events manifold and the torsion of the induced trajectory connection are related by the pull-back property:*

$$\mathbf{g}_{\mathcal{T}_\varphi}\circ \mathrm{T{\small ORS}}^{\mathcal{T}_\varphi}(\mathbf{a}_{\mathcal{T}_\varphi}) := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\left(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}\circ \mathrm{T{\small ORS}}^{\mathrm{E}}(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi})\right).$$

Proof. With the same notations adopted in the proof of Lemma 4, the definition of *torsion* of a linear connection gives

$$\mathrm{T{\small ORS}}^{\mathrm{E}}(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi})\cdot\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi} = \nabla^{\mathrm{E}}_{\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi} - \nabla^{\mathrm{E}}_{\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi} - [\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}],$$

for all $\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}\in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathbb{T}\mathcal{T}_\varphi)$. The pull-back property in the statement follows then from Definition 18 of trajectory connection and from the relation

$$\begin{aligned}\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}([\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}],\mathbf{w}^{\mathrm{E}}_{\mathcal{T}_\varphi}) &= \mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\![\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}],\mathbf{w}_{\mathcal{T}_\varphi})\\ &= \mathbf{g}_{\mathcal{T}_\varphi}([\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}],\mathbf{w}_{\mathcal{T}_\varphi}),\end{aligned}$$

due to the naturality of Lie brackets with respect to push [7]. □

Commutativity between immersion pull-back and Lie-derivative along the motion for covariant tensor fields is shown by the next result.

Lemma 6 (Trajectory pull-back of Lie time-derivatives) *The pull-back of the Lie time-derivative of a space-time field* $\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}\in \mathrm{C}^1(\mathrm{E}\,;\mathrm{C{\small OV}}(\mathbb{T}\mathrm{E}))$ *on the trajectory is equal to the Lie time-derivative of its immersion pull-back to the trajectory:*

$$\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\left(\mathcal{L}_{\mathbf{v}^{\mathrm{E}}_{\mathcal{T}_\varphi}}\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}\right) = \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\left(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}\right).$$

Proof. The relation: $\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}\circ\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi} = \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\circ\boldsymbol{\varphi}^{\mathcal{T}_\varphi}_{\tau,t}$ implies commutativity between immersion pull-back and pull-back along the motion of a space-time covariant tensor field over the trajectory. Indeed, setting $\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{a}_{\mathcal{T}_\varphi}$, $\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{b}_{\mathcal{T}_\varphi}$, we have that

$$\begin{aligned}(\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}\!\downarrow\!\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi})(\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi}) &= \mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}\!\uparrow\!\mathbf{a}^{\mathrm{E}}_{\mathcal{T}_\varphi},\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}\!\uparrow\!\mathbf{b}^{\mathrm{E}}_{\mathcal{T}_\varphi})\\ &= \mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}\!\uparrow\!\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{a}_{\mathcal{T}_\varphi},\boldsymbol{\varphi}^{\mathrm{E}}_{\tau,t}\!\uparrow\!\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\mathbf{a}_{\mathcal{T}_\varphi})\\ &= \mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\boldsymbol{\varphi}^{\mathcal{T}_\varphi}_{\tau,t}\!\uparrow\!\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\uparrow\!\boldsymbol{\varphi}^{\mathcal{T}_\varphi}_{\tau,t}\!\uparrow\!\mathbf{b}_{\mathcal{T}_\varphi})\\ &= \boldsymbol{\varphi}^{\mathcal{T}_\varphi}_{\tau,t}\!\downarrow\left(\mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\mathbf{s}^{\mathrm{E}}_{\mathcal{T}_\varphi}\right)(\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}),\end{aligned}$$

for all $\mathbf{a}_{\mathcal{T}_\varphi},\mathbf{b}_{\mathcal{T}_\varphi}\in \mathrm{C}^1(\mathcal{T}_\varphi\,;\mathbb{T}\mathcal{T}_\varphi)$. Taking $\partial_{\tau=t}$ gives the result. □

Proposition 3 (Euler's stretching formula) *The material stretching tensor field is expressed, in terms of the trajectory connection induced by a linear connection in the events manifold, by the formula:*

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}} \mathbf{g}_{\mathcal{T}_\varphi} = \tfrac{1}{2}\nabla^{\mathcal{T}_\varphi}_{\mathbf{v}_{\mathcal{T}_\varphi}} \mathbf{g}_{\mathcal{T}_\varphi} + \mathrm{SYM}\,(\mathbf{g}_{\mathcal{T}_\varphi} \circ \mathrm{TORS}^{\mathcal{T}_\varphi})(\mathbf{v}_{\mathcal{T}_\varphi}) + \mathrm{SYM}\,(\mathbf{g}_{\mathcal{T}_\varphi} \circ \nabla^{\mathcal{T}_\varphi})(\mathbf{v}_{\mathcal{T}_\varphi}),$$

where all terms on the r.h.s. are tensorial in $\mathbf{v}_{\mathcal{T}_\varphi}$*, with the exception of the third one which is a differential operator.*

Proof. By applying the formula in Proposition 1 to the metric tensor field $\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}, \mathrm{COV}(\mathbb{V}\mathrm{E}))$ on the immersed trajectory, we may write:

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{v}^{\mathrm{E}}_{\mathcal{T}_\varphi}} \mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi} = \tfrac{1}{2}\nabla^{\mathrm{E}}_{\mathbf{v}^{\mathrm{E}}_{\mathcal{T}_\varphi}} \mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi} + \mathrm{SYM}\left(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi} \circ \mathrm{TORS}^{\mathrm{E}}\right)(\mathbf{v}^{\mathrm{E}}_{\mathcal{T}_\varphi}) + \mathrm{SYM}\left(\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi} \circ \nabla^{\mathrm{E}}\right)(\mathbf{v}^{\mathrm{E}}_{\mathcal{T}_\varphi}).$$

Then, pulling back from the events manifold to the trajectory manifold, taking into account the commutativity in Lemma 6 and the definition $\mathbf{g}_{\mathcal{T}_\varphi} := \mathbf{i}_{\mathrm{E},\mathcal{T}_\varphi}\!\downarrow\!\mathbf{g}^{\mathrm{E}}_{\mathcal{T}_\varphi}$, recalling Definition 13 of material metric tensor field, Definition 18 of trajectory connection and Lemmas 4, 5, we get the result. □

Let us now assume that the ambient space connection ∇ in $(\mathcal{S}, \mathbf{g})$ is Levi-Civita, that is, metric preserving and torsion-free, that is, $\nabla\mathbf{g} = 0$ and $\mathrm{TORS}(\mathbf{a_x}, \mathbf{b_x}) = 0 \in \mathbb{T}_{\mathbf{x}}\mathcal{S}$ for all $\mathbf{a_x}, \mathbf{b_x} \in \mathbb{T}_{\mathbf{x}}\mathcal{S}$.

This is the case for the usual Euclid connection by translation. The induced connection in the events manifold is also metric and torsion-free and such is the material connection. The generalized Euler's stretching formula of Proposition 3 reduces then to the classical formula, extended to include lower dimensional continua:

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}} \mathbf{g}_{\mathcal{T}_\varphi} &= \mathrm{SYM}\,(\mathbf{g}_{\mathcal{T}_\varphi} \circ \nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi})\\ &= \mathbf{g}_{\mathcal{T}_\varphi} \circ \mathrm{SYM}\,(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi}) \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{SYM}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}})),\end{aligned}$$

where $\mathrm{SYM}\,(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi}) := \tfrac{1}{2}(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi} + (\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi})^A) \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\varphi}}\,;\mathrm{MIX}(\mathbb{V}\mathcal{T}_{\boldsymbol{\varphi}}))$, and $(\cdot)^A$ denotes the adjoint according to the material metric $\mathbf{g}_{\mathcal{T}_\varphi}$. The mixed form of the stretching tensor is equal to the $\mathbf{g}_{\mathcal{T}_\varphi}$-symmetric part of the *trajectory parallel derivative* of the velocity field:

$$\mathbf{D}_{\mathcal{T}_\varphi} := \mathbf{g}^{-1}_{\mathcal{T}_\varphi} \circ \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}} \mathbf{g}_{\mathcal{T}_\varphi} = \mathrm{SYM}\,(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi}).$$

The symbol **D** was first adopted by Truesdell and Noll in [27] to denote the symmetric part of the velocity gradient for a 3D body. In our treatment, notation and results are revisited to distinguish material tensor fields from spatial tensor fields and to extend the analysis to lower dimensional bodies.

Proposition 4 (Euler's formula for volumetric stretching) *The material* volumetric stretching *tensor field is expressed, in terms of the trajectory connection induced by a linear connection in the events manifold, by the formula*

$$\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}} \boldsymbol{\mu}_{\mathcal{T}_\varphi} = \nabla^{\mathcal{T}_\varphi}_{\mathbf{v}_{\mathcal{T}_\varphi}} \boldsymbol{\mu}_{\mathcal{T}_\varphi} + \mathrm{CYCLE}(\boldsymbol{\mu}_{\mathcal{T}_\varphi} \circ \mathrm{TORS}^{\mathcal{T}_\varphi}(\mathbf{v}_{\mathcal{T}_\varphi}) + \boldsymbol{\mu}_{\mathcal{T}_\varphi} \circ \nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi}),$$

where $\boldsymbol{\mu}_{\mathcal{T}_\varphi}$ *is the volume form associated with the metric tensor field* $\mathbf{g}_{\mathcal{T}_\varphi}$.

Proof. From the analysis developed in Sect. 3, we recall that the invariant $J_o(\mathbf{v}_{\mathcal{T}_\varphi})$ is defined by the identity $\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}} \boldsymbol{\mu}_{\mathcal{T}_\varphi} = J_o(\mathbf{v}_{\mathcal{T}_\varphi})\, \boldsymbol{\mu}_{\mathcal{T}_\varphi}$ so that the following formula holds:

$$\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}} \boldsymbol{\mu}_{\mathcal{T}_\varphi} = \nabla^{\mathcal{T}_\varphi}_{\mathbf{v}_{\mathcal{T}_\varphi}} \boldsymbol{\mu}_{\mathcal{T}_\varphi} + J_1(\mathrm{TORS}^{\mathcal{T}_\varphi}(\mathbf{v}_{\mathcal{T}_\varphi}))\, \boldsymbol{\mu}_{\mathcal{T}_\varphi} + J_1(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi})\, \boldsymbol{\mu}_{\mathcal{T}_\varphi}.$$

□

All the terms at the r.h.s. are tensorial and such the term at the l.h.s. which is proportional to the volume form $\boldsymbol{\mu}_{\mathcal{T}_\varphi}$ with a proportionality factor defining the divergence of a tangent vector field:

$$\mathrm{div}^{\,\mathcal{T}_\varphi}(\mathbf{v}_{\mathcal{T}_\varphi}) = J_o(\mathbf{v}_{\mathcal{T}_\varphi}) + J_1(\mathrm{TORS}^{\mathcal{T}_\varphi}(\mathbf{v}_{\mathcal{T}_\varphi})) + J_1(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi}).$$

We remark that in this formula for the divergence, all terms on the r.h.s. are tensorial in $\mathbf{v}_{\mathcal{T}_\varphi}$, with the exception of the third one which is a differential operator. The previous formula provides the general expression of the divergence of a tangent vector field in terms of a linear connection. For a metric and torsion-free connection and a volume form induced by the metric, it will be $\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}} \boldsymbol{\mu}_{\mathcal{T}_\varphi} = 0$, and the formula for the divergence specializes into the usual one:

$$\mathrm{div}^{\,\mathcal{T}_\varphi}(\mathbf{v}_{\mathcal{T}_\varphi}) = J_1(\nabla^{\mathcal{T}_\varphi}\mathbf{v}_{\mathcal{T}_\varphi}).$$

8 Coordinate reference systems

A so-called *engineering* reference system is a mobile reference system in which the basis of tangent vectors at each point is got by normalizing the velocity vectors of a given orthogonal curvilinear coordinate system. To evaluate the covariant derivatives as partial derivatives of coordinates, it is expedient to define the parallel transport by declaring parallel-transported tangent vectors those with the same components in the mobile reference system. This parallel transport is path independent, and hence, a tangent vector can be extended to any other base point in a neighborhood of the original base point by performing the parallel transport along any path joining them. The torsion of the connection, evaluated on a pair of vectors, reads:

$$\mathrm{TORS}^{\mathcal{T}_\varphi}(\mathbf{a}_{\mathcal{T}_\varphi}(\mathbf{e}), \mathbf{b}_{\mathcal{T}_\varphi}(\mathbf{e})) = \nabla^{\mathcal{T}_\varphi}_{\mathbf{a}_{\mathcal{T}_\varphi}(\mathbf{e})}\mathbf{b}_{\mathcal{T}_\varphi} - \nabla^{\mathcal{T}_\varphi}_{\mathbf{b}_{\mathcal{T}_\varphi}(\mathbf{e})}\mathbf{a}_{\mathcal{T}_\varphi} - [\mathbf{a}_{\mathcal{T}_\varphi}, \mathbf{b}_{\mathcal{T}_\varphi}](\mathbf{e}),$$

with the tangent vector fields $\mathbf{a}_{\mathcal{T}_\varphi}, \mathbf{b}_{\mathcal{T}_\varphi} \in \mathrm{C}^1(\mathcal{T}_\varphi\,; \mathbb{T}\mathcal{T}_\varphi)$ built by path-independent parallel transport along coordinate lines. Accordingly, $\nabla^{\mathcal{T}_\varphi}\mathbf{a}_{\mathcal{T}_\varphi} = 0$ and $\nabla^{\mathcal{T}_\varphi}\mathbf{b}_{\mathcal{T}_\varphi} = 0$ so that:

$$\mathrm{TORS}^{\mathcal{T}_\varphi}(\mathbf{a}_{\mathcal{T}_\varphi}(\mathbf{e}), \mathbf{b}_{\mathcal{T}_\varphi}(\mathbf{e})) = -[\mathbf{a}_{\mathcal{T}_\varphi}, \mathbf{b}_{\mathcal{T}_\varphi}](\mathbf{e}).$$

For example's sake, we consider a two-dimensional body in motion in a plane, a polar system of coordinates and the two induced reference systems, the *natural* reference system, in which the basis vectors are the velocities of the coordinate lines, and the *engineering mobile* reference system, with unitary radial and circumferential basis vectors at each point. We denote by (ρ, θ) the pair of radial and circumferential coordinates. In the following sections, the induced natural and normalized connections introduced in Sect. 6.3 will be adopted to evaluate the GRAM matrix associated with the covariant stretching tensor, viz. the matrix whose (i, j) entry is the evaluation of the tensor on the corresponding pair of base vectors.

8.1 Natural connection

Let us now consider, in polar coordinates, a natural reference system with radial and circumferential basis vectors $\mathbf{a}_i$, $i = 1, 2$ with $\mathbf{a}_1 = \mathbf{a}_\rho$ and $\mathbf{a}_2 = \mathbf{a}_\theta$ at each point, see Fig. 1. Adopting the natural connection, say ∇, by the vanishing of parallel derivatives and Lie brackets of basis vector fields,

$$\nabla\mathbf{a}_\rho = 0, \quad \nabla\mathbf{a}_\theta = 0, \quad [\mathbf{a}_\rho, \mathbf{a}_\theta] = 0,$$

we have:

$$\mathrm{TORS}(\mathbf{a}_\rho, \mathbf{a}_\theta) = \nabla_{\mathbf{a}_\rho}\mathbf{a}_\theta - \nabla_{\mathbf{a}_\theta}\mathbf{a}_\rho - [\mathbf{a}_\rho, \mathbf{a}_\theta] = 0,$$

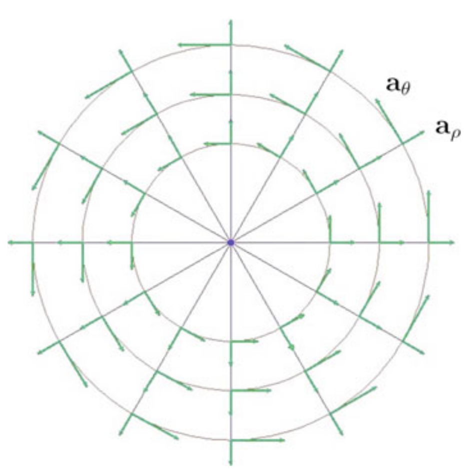

Fig. 1 Natural reference system

and Euler's formula writes:

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi} = \tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi} + \mathrm{SYM}\,(\mathbf{g}_{\mathcal{T}_\varphi} \circ \nabla\mathbf{v}_{\mathcal{T}_\varphi}).$$

To evaluate this expression, we set $\mathbf{v}_{\mathcal{T}_\varphi} = v_\rho\,\mathbf{a}_\rho + v_\theta\,\mathbf{a}_\theta$ and observe that:

$$\begin{aligned}
&\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\rho) = 1, && \mathbf{g}_{\mathcal{T}_\varphi}(\nabla_{\mathbf{a}_\rho}\mathbf{v}_{\mathcal{T}_\varphi}, \mathbf{a}_\rho) = \nabla_{\mathbf{a}_\rho} v_\rho,\\
&\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\theta) = 0, && \mathbf{g}_{\mathcal{T}_\varphi}(\nabla_{\mathbf{a}_\rho}\mathbf{v}_{\mathcal{T}_\varphi}, \mathbf{a}_\theta) = \rho^2\,\nabla_{\mathbf{a}_\rho} v_\theta,\\
&\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\rho) = 0, && \mathbf{g}_{\mathcal{T}_\varphi}(\nabla_{\mathbf{a}_\theta}\mathbf{v}_{\mathcal{T}_\varphi}, \mathbf{a}_\rho) = \nabla_{\mathbf{a}_\theta} v_\rho,\\
&\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\theta) = \rho^2, && \mathbf{g}_{\mathcal{T}_\varphi}(\nabla_{\mathbf{a}_\theta}\mathbf{v}_{\mathcal{T}_\varphi}, \mathbf{a}_\theta) = \rho^2\,\nabla_{\mathbf{a}_\theta} v_\theta.
\end{aligned}$$

Setting $\mathbf{a}_1 = \mathbf{a}_\rho$ and $\mathbf{a}_2 = \mathbf{a}_\theta$, we have, for $i, j = 1, 2$:

$$\begin{aligned}
\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_i, \mathbf{a}_j) &= \nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_i, \mathbf{a}_j)) - \mathbf{g}_{\mathcal{T}_\varphi}(\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{a}_i, \mathbf{a}_j) - \mathbf{g}_{\mathcal{T}_\varphi}(\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{a}_j, \mathbf{a}_i)\\
&= \nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_i, \mathbf{a}_j)) = \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_i, \mathbf{a}_j)).
\end{aligned}$$

Then,

$$\begin{aligned}
\tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\rho) &= \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\rho)) = 0,\\
\tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\theta) &= \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\theta)) = 0,\\
\tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\rho) &= \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\rho)) = 0,\\
\tfrac{1}{2}\nabla_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\theta) &= \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,(\mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\theta)) = \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\rho^2 = \rho\,v_\rho.
\end{aligned}$$

The GRAM matrix of the stretching is thus given by the following:

$$\mathrm{GRAM}(\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi}) = \begin{bmatrix} \nabla_{\mathbf{a}_\rho} v_\rho & \tfrac{1}{2}(\rho^2\,\nabla_{\mathbf{a}_\rho} v_\theta + \nabla_{\mathbf{a}_\theta} v_\rho) \\ \tfrac{1}{2}(\rho^2\,\nabla_{\mathbf{a}_\rho} v_\theta + \nabla_{\mathbf{a}_\theta} v_\rho) & \rho^2\,\nabla_{\mathbf{a}_\theta} v_\theta + \rho\,v_\rho \end{bmatrix}.$$

8.1.1 Examples

A constant radial velocity field $\mathbf{v}_{\mathcal{T}_\varphi} := (v_\rho\,\mathbf{a}_\rho, 1)$, corresponding to $v_\rho = \alpha$ and $v_\theta = 0$, generates the stretching:

$$\mathrm{GRAM}(\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\,\mathbf{g}_{\mathcal{T}_\varphi}) = \begin{bmatrix} 0 & 0 \\ 0 & \rho\,\alpha \end{bmatrix},$$

which corresponds to a purely circumferential rate of elongation equal to α/ρ, vanishing at infinity and going to infinity at the origin. A linear circumferential velocity field $\mathbf{v}^{\mathcal{S}}_{\boldsymbol{\varphi}} = \alpha\,\mathbf{a}_\theta$, corresponding to $v_\rho = 0$ and $v_\theta = \alpha$, generates a vanishing stretching and in fact represents an infinitesimal act of rotation about the origin.

8.2 Normalized connection

Let us now consider, in the polar system of coordinates, a mobile reference system with unitary radial and circumferential basis vectors $\overline{\mathbf{a}}_i$, $i = 1, 2$ with $\overline{\mathbf{a}}_1 = \overline{\mathbf{a}}_\rho = \mathbf{a}_\rho$ and $\overline{\mathbf{a}}_2 = \overline{\mathbf{a}}_\theta = \mathbf{a}_\theta/\rho$ at each point, see Fig. 2.

Preliminarily, we provide the evaluation of the Lie brackets of the basis vector fields.

Lemma 7 (Lie brackets) *The Lie brackets of the basis vector fields of the mobile reference system are given by the following:*

$$[\overline{\mathbf{a}}_\theta, \overline{\mathbf{a}}_\rho](\rho, \theta) = (1/\rho)\,\overline{\mathbf{a}}_\theta(\rho, \theta).$$

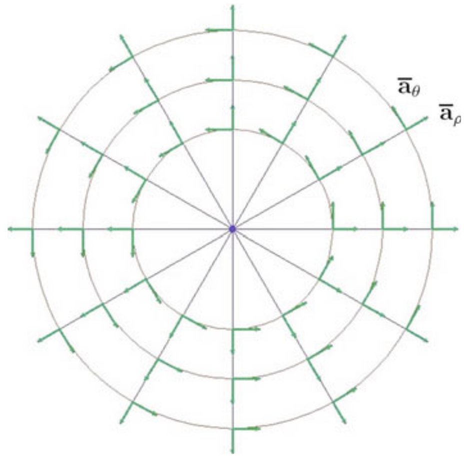

Fig. 2 Engineering reference system

Proof. The integral flows generated by unitary vector fields tangent to the polar coordinate lines are:

$$\begin{aligned}\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\rho}(\rho,\theta) &= (\rho+\lambda,\theta),\\ \mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\theta}(\rho,\theta) &= (\rho,\theta+\lambda/\rho),\end{aligned}$$

so that, in the natural polar reference system:

$$\begin{aligned}\overline{\mathbf{a}}_\rho(\rho,\theta) &= \partial_{\lambda=0}\,\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\rho}(\rho,\theta) = (1,0),\\ \overline{\mathbf{a}}_\theta(\rho,\theta) &= \partial_{\lambda=0}\,\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\theta}(\rho,\theta) = (0,1/\rho).\end{aligned}$$

The latter vector is unitary since it has component $1/\rho$ on the circumferential tangent vector at (ρ,θ) whose length is ρ. Let us now evaluate the Lie derivative:

$$\mathcal{L}_{\overline{\mathbf{a}}_\theta}\overline{\mathbf{a}}_\rho = [\overline{\mathbf{a}}_\theta,\overline{\mathbf{a}}_\rho] = \partial_{\lambda=0}\,T(\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\theta})^{-1}\cdot(\overline{\mathbf{a}}_\rho\circ\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\theta}).$$

Being $(\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\theta})^{-1}(\rho,\theta) = (\rho,\theta-\lambda/\rho)$, we get:

$$\begin{aligned}T(\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\theta})^{-1}\cdot(1,0) &= \partial_{\mu=0}\,(\mathbf{Fl}_\lambda^{\overline{\mathbf{a}}_\theta})^{-1}(\rho+\mu,\theta)\\ &= \partial_{\mu=0}\,(\rho+\mu,\theta-\lambda/(\rho+\mu)) = (1,\theta+\lambda/\rho^2),\end{aligned}$$

so that: $[\overline{\mathbf{a}}_\theta,\overline{\mathbf{a}}_\rho](\rho,\theta) = \partial_{\lambda=0}\,(1,\theta+\lambda/\rho^2) = (0,1/\rho^2) = (1/\rho)\,\overline{\mathbf{a}}_\theta(\rho,\theta)$ and the result is proven. □

Adopting the normalized reference system and the associated connection, say $\overline{\nabla}$, the parallel derivatives of the normalized basis vector fields vanish:

$$\overline{\nabla}\overline{\mathbf{a}}_\rho = 0, \qquad \overline{\nabla}\overline{\mathbf{a}}_\theta = 0.$$

Setting $\mathbf{v}_{\mathcal{T}_\varphi} = \overline{v}_k\,\overline{\mathbf{a}}_k$, we have $\overline{v}_\rho = v_\rho$ and $\overline{v}_\theta = \rho\,v_\theta$.
Then,

$$\overline{\text{TORS}}(\overline{\mathbf{a}}_\rho,\overline{\mathbf{a}}_\theta) = \overline{\nabla}_{\overline{\mathbf{a}}_\rho}\overline{\mathbf{a}}_\theta - \overline{\nabla}_{\overline{\mathbf{a}}_\theta}\overline{\mathbf{a}}_\rho - [\overline{\mathbf{a}}_\rho,\overline{\mathbf{a}}_\theta] = -[\overline{\mathbf{a}}_\rho,\overline{\mathbf{a}}_\theta].$$

With $\mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_i,\overline{\mathbf{a}}_j) = \delta_{ij}$ for $i,j = 1,2$,

$$\overline{\nabla}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_i,\overline{\mathbf{a}}_j) = \overline{\nabla}_{\mathbf{v}_{\mathcal{T}_\varphi}}(\mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_i,\overline{\mathbf{a}}_j)) = \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}(\mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_i,\overline{\mathbf{a}}_j)) = 0,$$

and Euler's formula writes:

$$\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\mathbf{g}_{\mathcal{T}_\varphi} = \text{SYM}\,(\mathbf{g}_{\mathcal{T}_\varphi}\circ(\overline{\text{TORS}}+\overline{\nabla})\mathbf{v}_{\mathcal{T}_\varphi}),$$

that is:

$$\begin{aligned}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_j) &= \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathrm{TORS}}(\mathbf{v}_{\mathcal{T}_\varphi}, \overline{\mathbf{a}}_i), \overline{\mathbf{a}}_j) + \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathrm{TORS}}(\mathbf{v}_{\mathcal{T}_\varphi}, \overline{\mathbf{a}}_j), \overline{\mathbf{a}}_i)\\ &+\mathbf{g}_{\mathcal{T}_\varphi}(\overline{\nabla}\mathbf{v}_{\mathcal{T}_\varphi} \cdot \overline{\mathbf{a}}_i, \overline{\mathbf{a}}_j) + \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\nabla}\mathbf{v}_{\mathcal{T}_\varphi} \cdot \overline{\mathbf{a}}_j, \overline{\mathbf{a}}_i).\end{aligned}$$

Taking into account the vanishing of parallel derivatives of basis vector fields and the formula for the torsion, we have:

$$\begin{aligned}\overline{\mathrm{TORS}}(\mathbf{v}_{\mathcal{T}_\varphi}, \overline{\mathbf{a}}_i) = \overline{v}_k\, \overline{\mathrm{TORS}}(\overline{\mathbf{a}}_k, \overline{\mathbf{a}}_i) &= \overline{v}_k\, (\overline{\nabla}_{\overline{\mathbf{a}}_k}\overline{\mathbf{a}}_i - \overline{\nabla}_{\overline{\mathbf{a}}_i}\overline{\mathbf{a}}_k - [\overline{\mathbf{a}}_k, \overline{\mathbf{a}}_i])\\ &= \overline{v}_k\, [\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_k],\\ \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathrm{TORS}}(\mathbf{v}_{\mathcal{T}_\varphi}, \overline{\mathbf{a}}_i), \overline{\mathbf{a}}_j) &= \overline{v}_k\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathrm{TORS}}(\overline{\mathbf{a}}_k, \overline{\mathbf{a}}_i), \overline{\mathbf{a}}_j)\\ &= \overline{v}_k\, \mathbf{g}_{\mathcal{T}_\varphi}([\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_k], \overline{\mathbf{a}}_j),\\ \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\nabla}\mathbf{v}_{\mathcal{T}_\varphi} \cdot \overline{\mathbf{a}}_i, \overline{\mathbf{a}}_j) &= \overline{\nabla}_{\overline{\mathbf{a}}_i}\overline{v}_k\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_k, \overline{\mathbf{a}}_j) = \overline{\nabla}_{\overline{\mathbf{a}}_i}\overline{v}_j.\end{aligned}$$

Alternatively, the expression for the stretching may be obtained by the following direct computation:

$$\begin{aligned}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_j) &= \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, (\mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_j)) - \mathbf{g}_{\mathcal{T}_\varphi}(\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_j) - \mathbf{g}_{\mathcal{T}_\varphi}(\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\overline{\mathbf{a}}_j, \overline{\mathbf{a}}_i)\\ &= +(\mathcal{L}_{\overline{\mathbf{a}}_j}\overline{v}_k)\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_k, \overline{\mathbf{a}}_i) + (\mathcal{L}_{\overline{\mathbf{a}}_i}\overline{v}_k)\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_k, \overline{\mathbf{a}}_j)\\ &\quad - \overline{v}_k\, \mathbf{g}_{\mathcal{T}_\varphi}(\mathcal{L}_{\overline{\mathbf{a}}_k}\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_j) - \overline{v}_k\, \mathbf{g}_{\mathcal{T}_\varphi}(\mathcal{L}_{\overline{\mathbf{a}}_k}\overline{\mathbf{a}}_j, \overline{\mathbf{a}}_i)\\ &= (\mathcal{L}_{\overline{\mathbf{a}}_j}\overline{v}_i) + (\mathcal{L}_{\overline{\mathbf{a}}_i}\overline{v}_j)\\ &\quad + \overline{v}_k\, \mathbf{g}_{\mathcal{T}_\varphi}([\overline{\mathbf{a}}_i, \overline{\mathbf{a}}_k], \overline{\mathbf{a}}_j) + \overline{v}_k\, \mathbf{g}_{\mathcal{T}_\varphi}([\overline{\mathbf{a}}_j, \overline{\mathbf{a}}_k], \overline{\mathbf{a}}_i).\end{aligned}$$

The GRAM matrix of the stretching is then given by the following:

$$\begin{aligned}\overline{\mathrm{GRAM}}(\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}) &= \begin{bmatrix} \overline{\nabla}_{\overline{\mathbf{a}}_\rho}\overline{v}_\rho & \frac{1}{2}(\overline{\nabla}_{\overline{\mathbf{a}}_\rho}\overline{v}_\theta + \overline{\nabla}_{\overline{\mathbf{a}}_\theta}\overline{v}_\rho)\\ \frac{1}{2}(\overline{\nabla}_{\overline{\mathbf{a}}_\rho}\overline{v}_\theta + \overline{\nabla}_{\overline{\mathbf{a}}_\theta}\overline{v}_\rho) & \overline{\nabla}_{\overline{\mathbf{a}}_\theta}\overline{v}_\theta \end{bmatrix}\\ &+ \begin{bmatrix} 0 & -\frac{1}{2}(\overline{v}_\theta/\rho)\\ -\frac{1}{2}(\overline{v}_\theta/\rho) & \overline{v}_\rho/\rho \end{bmatrix}.\end{aligned}$$

This formula shows that, in this orthonormal reference system, the GRAM matrix of the stretching is the sum of a familiar term, which is the symmetric part of the Jacobi matrix of the components of the velocity field and of a corrective term which takes into account the curvilinear nature of the coordinates.

The equivalence of the expressions of the stretching, in terms of the connections induced by the natural and by the normalized reference system, may be deduced by taking account of the relations:

$$\begin{aligned}\overline{v}_\rho &= v_\rho, \qquad \mathbf{a}_\rho = \overline{\mathbf{a}}_\rho\\ \overline{v}_\theta &= \rho\, v_\theta, \quad \mathbf{a}_\theta = \rho\, \overline{\mathbf{a}}_\theta\\ \overline{\nabla}_{\overline{\mathbf{a}}_\rho}\overline{v}_\theta &= \overline{\nabla}_{\mathbf{a}_\rho}(\rho\, v_\theta) = v_\theta + \rho\, \overline{\nabla}_{\mathbf{a}_\rho}v_\theta,\\ \overline{\nabla}_{\overline{\mathbf{a}}_\theta}\overline{v}_\rho &= (1/\rho)\, \overline{\nabla}_{\mathbf{a}_\theta}v_\rho,\end{aligned}$$

and of the consequent equalities:

$$\begin{aligned}\tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\rho) &= \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_\rho, \overline{\mathbf{a}}_\rho),\\ \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\rho, \mathbf{a}_\theta) &= \rho\, \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_\rho, \overline{\mathbf{a}}_\theta),\\ \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\rho) &= \rho\, \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_\theta, \overline{\mathbf{a}}_\rho),\\ \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\mathbf{a}_\theta, \mathbf{a}_\theta) &= \rho^2\, \tfrac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}_\varphi}}\, \mathbf{g}_{\mathcal{T}_\varphi}(\overline{\mathbf{a}}_\theta, \overline{\mathbf{a}}_\theta).\end{aligned}$$

9 Conclusions

Continuum mechanics is primarily concerned with time changes of metric properties due to an evolution process along a trajectory. This is the very realm of Kinematics, and investigating on the convective time-rate of change of metric properties is fundamental for the statement of basic principles of CM. So, according to Johann Bernoulli (letter on 26 February 1715 to Pierre Varignon), equilibrium has to be tested by ascertaining the vanishing of the virtual power performed by the force system, acting on the body at a given time, against any virtual velocity field which is an infinitesimal isometry, characterized by the vanishing of the associate stretching. The notion of stretching is of central importance in CM and such is then Euler's formula for its computation as the symmetric part of the parallel derivative of the velocity field, since it provides the implicit representation of the subspace of infinitesimal isometries (rigidity virtual velocities). The Lagrange multipliers corresponding to the constraint of rigidity are stress fields in the body [7]. Euler's original statement of the formula was performed according to Euclid connection, corresponding to the path-independent parallel transport by translation. The formula for the stretching has here been revisited in the context of Riemann manifolds endowed with an arbitrary linear connection. This treatment extends also the one by Killing who considered the metric and torsion-free Levi-Civita connection. In the extended formula, the torsion of the connection and the parallel derivative of the metric field are thus involved. An analogous extended formula has been provided for the volumetric stretching. The results have been applied to the evaluation of the GRAM matrix of the stretching, either in a natural system of polar coordinates or in the corresponding engineering normalized mobile reference system in which the induced connection is not torsion-free. The geometric approach adopted in this paper is in line with a geometrization program of CM carried out in [28,29,25,26,30,31] based on mathematical tools collected in [7].

Acknowledgments The support of consortium ReLUIS—Research line 2.2.2—Triennium 2010–2013—is gratefully acknowledged.

References

1. Euler, L.: Principia motus fluidorum. Novi Comm. Acad. Sci. Petrop. **6**, 271–311 (1761)
2. Euler, L.: Institutionum calculi integralis volumen tertium. Opera Omnia, Series 1, Vol. **13** (1770)
3. Euler, L.: Sectio secunda de principiis motus fluidorum. Novi Comm. Acad. Sci. Petrop. **14**, 270–386 (1770)
4. Truesdell, C., Toupin, R.A.: The classical field theories. In: Flügge, S. (ed.) Encyclopedia of Physics, vol. III/1, pp. 226–793. (1960)
5. Killing, W.: Über die Grundlagen der Geometrie. J. Reine Angew. Math. **109**, 121–186 (1892)
6. Abraham, R., Marsden, J. E., Ratiu, T.: Manifolds, Tensor Analysis, and Applications, 2nd ed. (3rd ed. 2002). Springer, New York (1988)
7. Romano, G.: Continuum Mechanics on Manifolds. Lecture Notes, University of Naples Federico II, Naples, pp. 1–695 (2007). http://wpage.unina.it/romano
8. Romano, G., Barretta, R.: Covariant hypo-elasticity. Eur. J. Mech. A Solids **30**, 1012–1023 (2011)
9. Truesdell, C.: A first Course in Rational Continuum Mechanics, vol. I, 1st ed. (2nd ed. 1991). Academic Press, New York (1977)
10. Romano, G., Diaco, M.: A Functional Framework for Applied Continuum Mechanics, In: New Trends in Mathematical Physics. World Scientific, Singapore 193-204 (2004). Posted in http://wpage.unina.it/romano
11. Romano, G., Diaco, M., Barretta, R.: Variational Formulation of the First Principle of Continuum Thermodynamics. Continuum Mech. Thermodyn. **22**(3), 177–187 (2010)
12. Christov, C.I.: On frame indifferent formulation of the Maxwell-Cattaneo model of finite speed heat conduction. Mech. Res. Commun. **36**, 481–486 (2009)
13. Christov, C.I.: Frame indifferent formulation of Maxwell's elastic-fluid model and the rational continuum mechanics of the electromagnetic field. Mech. Res. Commun. **38**, 334–339 (2011)
14. Morro, A.: Evolution equations and thermodynamic restrictions for dissipative solids. Math. Comput. Modell. **52**, 1869–1876 (2010)
15. Morro, A.: Evolution equations for non-simple viscoelastic solids. J. Elast. **105**, 93–105 (2011)
16. Straughan, B.: Thermal convection with the Cattaneo-Christov model. Int. J. Heat Mass Transf. **53**, 95–98 (2010)
17. Straughan, B.: Porous convection with Cattaneo heat flux. Int. J. Heat Mass Transf. **53**, 2808–2812 (2010)
18. Straughan, B.: Acoustic waves in a Cattaneo-Christov gas. Phys. Lett. A **374**, 2667–2669 (2010)
19. Straughan, B.: Tipping points in Cattaneo-Christov thermohaline convection. Proc. R. Soc. A **467**, 7–18 (2011)
20. Ciarletta, M., Straughan, B., Tibullo, V.: Christov-Morro theory for non-isothermal diffusion. Nonlinear Anal. Real World Appl. **13**, 1224–1228 (2012)
21. Saunders, D.J.: The Geometry of Jet Bundles. London Mathematical Society Lecture Note Series, 142. Cambridge University Press (1989)
22. Marsden, J.E., Hughes, T.J.R.: Mathematical Foundations of Elasticity. Prentice-Hall, Redwood City (1983)

23. Lie, M.S., Engel, F.: Theorie der Transformationsgruppen, Vol. 3. B.G. Teubner, Leipzig (1888-1890-1893). Chelsea, New York (1970)
24. Petersen, P.: Riemannian Geometry. Springer, New York (1998)
25. Romano, G., Barretta, R., Diaco, M.: On the general form of the law of dynamics. Int. J. Non-Linear Mech. **44**(6), 689–695 (2009)
26. Romano, G., Barretta, R., Diaco, M.: On continuum dynamics. J. Math. Phys. **50**, 102903-1-26 (2009)
27. Truesdell, C., Noll, W.: The Non-Linear Field Theories of Mechanics. Handbuch der Physik, Springer, Berlin, 1–591 (1965)
28. Romano, G., Diaco, M., Barretta, R.: On the theory of material inhomogeneities. Mech. Res. Commun. **33**(6), 758–763 (2006)
29. Romano, G., Sellitto, C., Barretta, R.: Nonlinear shell theory: a duality approach. J. Mech. Mater. Struct. **2**(7), 1207–1230 (2007)
30. Romano, G., Barretta, R., Barretta, A.: On Maupertuis principle in dynamics. Rep. Math. Phys. **63**(3), 331–346 (2009)
31. Romano, G., Barretta, R., Diaco, M.: Algorithmic tangent stiffness in elastoplasticity and elastoviscoplasticity: a geometric insight. Mech. Res. Commun. **37**(3), 289–292 (2010)