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THE NOTION OF ELASTIC STATE AND APPLICATION TO NONLOCAL MODELS

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Abstract. *A geometric approach to the nonlinear theory of elasticity leads in a natural way to the notion of elastic-state (a covariant tensor) defined as output of the constitutive law in duality with the input natural stress-state (a contravariant tensor). Elastic constitutive behaviours are conveniently described by an incremental formulation. Time-rates of natural stress-state and of elastic-state fields are Lie-derivatives along the motion. According to the rate elastic law, these dual tensor fields are related by a tangent compliance operator fulfilling Green's integrability, expressed by existence of a local stress-potential. Under the assumption of time-invariance the elastic law can be integrated to a relation between natural stress-state and elastic-state fields. Natural stress fields perform, by duality with virtual stretching, a virtual power per unit mass in the continuous body. Existence of a global elastic-state potential energy can be proven by relying upon conservation of mass. Finite elastic strains are just computational tools deprived of physical interpretation and evaluable only in reference configurations where linear operations can be performed. The primary role of natural stress fields as state variables finds a significant application in the context of nonlocal elasticity where the integral convolution law may generate two distinct nonlocal elastic models, a stress-driven and a strain-driven model, by swapping input and output fields. It is shown that only the former leads to well-posed continuous elastic problems, while the opposite holds for the latter model, originally proposed by Eringen and widely but often improperly referred to literature.*

1 INTRODUCTION

Elasticity is a classical subject and certainly by far the most important and well-established in Continuum Mechanics. This is true for the linearised theory in which small displacements are assumed so that all configurations of the body can be assumed to be coincident with a given one. On the contrary, general treatments of elasticity in the nonlinear geometric range appear still not satisfactorily formulated.

We present here evidence of unphysical assumptions and results placed at the basis of presentations of the theory, as it is outlined in standard textbooks and articles.

The outcomes of a recent research on a rate formulation of elasticity theory, bringing advancements and changes in notions and nomenclature, are illustrated.

The plan of presentation may conveniently start with a *résumé* of principal contributions to the notion of nonlinear elasticity, as it was evolving in the course of time.

The whole story may conventionally be considered to have started in 1660 with the celebrated anagram by Robert Hooke CEIIINOSSSTTVV decrypted by Hooke himself in 1678 as

VT TENSIO SIC VIS

meaning *as the extension, so the force*.

On this early ground, the notion of elastic modulus was introduced by Leonhard Euler in 1727 and the notion of strain potential was first conceived by George Green in 1839 [1].

Relevant advancements were brought by Gabriel Lamé [2] and Gustav Kirchhoff [3] in 1952, by Alfred Clebsch in 1863 [4] and by Barré de Saint Venant in 1883 [5].

The theory of large elastic deformations was also in the focus of several important contributions around the middle of the past century.

We limit ourselves to cite here the treatises by Albert Edward Green and Wolfgang Zerna [6] in 1954, the paper by A.E. Green and Paul Mansour Naghdi [7] in 1965, the article in Flügge's Handbuch der Physik by Clifford Truesdell and Walter Noll [8] in 1965 and the book by Truesdell [9] in 1977.

The treatment in [6, 7], is based on the formal assumption of an elastic energy function of the finite strain, point-wise defined by the Lagrange-Green-St.Venant tensor evaluated in terms of the placement mapping of a body from a reference configuration to the actual one.

This finite strain tensor is one-half the difference between the pull-back of the metric tensor by the placement map and the metric tensor itself, both evaluated on a reference configuration.

The treatment in [8] is instead based on the assumption that the Cauchy stress tensor is a function of the deformation gradient, which is the tangent placement map.

Both formulations cannot be considered as completely satisfactory since the statement of an elastic law in terms of a finite geometric strain leaves it open the basic question of how the finite elastic strain is defined.

In fact the notion of a finite elastic strain would require that a transformation from one to another of two distinct configurations of the body be considered, and that the effects of all other sources of deformation be separated by means of some physically acceptable reasoning.

As a matter of fact, anyone engaged in an elastic computation due to large displacements of a body, will readily find himself in big conceptual and experimental troubles, looking for physically meaningful criteria apt to choose a suitable reference configuration as starting point, while envisaging reasonable separation tools to account for non-elastic phenomena.

Moreover, an apparent contrast appears in the elastic law considered in [8] since the static counterpart of the finite elastic strain is a stress tensor acting on the actual configuration, so that there is no need of reference configurations for its definition.

Similarly, the potential elastic energy considered in [6, 7] depends on the geometric strain defined by Lagrange-Green-St.Venant strain tensor.

This tensor measures a geometric strain which is not necessarily purely elastic so that again one is called to face the unfeasible task of conceiving an effective reference configuration and of performing the elimination of all other sources of finite strain.

An original idea by Augustin-Louis Cauchy was revived as early as 1955 by Truesdell who undertook the proposal of a hypo-elastic law, relating the co-rotational (or the convective) stress rate of Cauchy stress field to Euler stretching, by means of a linear stiffness operator nonlinearly dependent on the stress [11].

The analysis by Barry Bernstein in [12], led however to the conclusion that a hypo-elastic law, as formulated in [11], was not integrable to get a corresponding finite strain formulation, a conclusion accepted also in [8].

Hypo-elasticity was therefore considered a non-elastic model and no further rationale was conceived to overcome this obstruction.

Moreover an incorrect application of the principle of material frame indifference led Walter Noll to the false conclusion that a hypo-elastic law is necessarily isotropic [8, (99.5) p.403], notwithstanding adverse opinions expressed by Rodney Hill [13] and William Prager [14].

Consequently, an elastic rate constitutive theory was considered to be unfeasible.

At this point of the history, the research efforts took an adverse direction, when Erastus Henry Lee [10] in 1969, on the basis of the treatment in [8], suggested to perform a multiplicative decomposition of the deformation gradient into a chain of a former plastic local transformation (between tangent spaces) from a reference configuration to an intermediate one, with a subsequent elastic local transformation to the actual configuration.

The unphysical character of an ordered sequence of subsequent non-commutative transformations becomes ever more evident when other straining effects, such as thermal, magnetic or electric, are taken into account.

What is the order of the sequence?

Notwithstanding this manifest mis-formulation, the multiplicative decomposition was accepted ever more by the mechanical community as a remedy necessary to avoid the dramatic effects of a complete *impasse*. At last, it was deemed to be the only remaining way to proceed, even in computational approaches [15].

The consequent damages, adduced to the development of the theory of material behaviour in the nonlinear range, are lasting since more than half a century and are still in action in theoretical and computational mechanics.

And yet, there is a natural route towards rate constitutive relations, which, based on the additive (and commutative) combination of elastic and *anelastic* contributions to the geometric stretching, does not require to give any physical meaning to a class of reference configurations and does not pretend to set up an unnatural ordering in the list of distinct formal descriptions of material *formänderung*.

Putting this route on a firm basis is greatly simplified by recourse to elements of differential geometry, with special concern to definitions and properties of Lie-derivatives, and brings naturally to the fundamental notion of elastic-state, surprisingly absent in standard treatments.

The whole story confirms the essential role of differential geometry in continuum mechanics, since a successful formulation of a rate theory of elasticity could not have been developed without this mathematical basis.

2 RATE ELASTICITY

Had Hooke enunciated his law as

VT VIS, SIC ELASTICA TENSIO

to be read: *as the traction, so the elastic state*, subsequent troubles would have been avoided. A still better statement would have been made by stating, in incremental terms, the anagram AAACCEEIIIILMNOSSSSUTTTTTVV to be decrypted as

VT VIS MUTAT, SIC ELASTICA TENSIO

whose translation reads *as the traction changes, so does the elastic extension*.

A geometric approach to continuum mechanics detects the virtual stretching as Lie-derivative

$$\mathcal{L}_{\delta\mathbf{v}}(\mathbf{g}) := \partial_{\lambda=0}\,\mathbf{Fl}_{\lambda}^{\delta\mathbf{v}} \downarrow \mathbf{g}\,, \tag{1}$$

of the (twice covariant) metric tensor field $\mathbf{g}$ along a virtual velocity field $\delta\mathbf{v}$. Here $\mathbf{Fl}_{\lambda}^{\delta\mathbf{v}}$ is the flow generated by the vector field $\delta\mathbf{v}$ and $\downarrow$ denoted the pull-back operation.

The equilibrium condition is variationally expressed in terms of equality between external virtual power and internal virtual power per unit mass performed at a configuration $\boldsymbol{\Omega}$ by the dual (twice contravariant) stress field $\boldsymbol{\sigma}$, hereafter named the *natural stress*:

$$\langle\, \mathbf{f}, \delta\mathbf{v}\,\rangle = \int_{\boldsymbol{\Omega}} \langle\, \boldsymbol{\sigma}, \mathcal{L}_{\delta\mathbf{v}}(\mathbf{g})\,\rangle \cdot \mathbf{m}\,. \tag{2}$$

The variational condition Eq.(2) is well-known as *principle of virtual powers* and is equivalent to the original equilibrium condition expressed by Johann Bernoulli in 1715 as vanishing of the virtual power of the force system $\mathbf{f}$ for infinitesimal isometries:

$$\langle\, \mathbf{f}, \delta\mathbf{v}\,\rangle = 0\,, \quad \text{for all} \quad \delta\mathbf{v} : \mathcal{L}_{\delta\mathbf{v}}(\mathbf{g}) = \mathbf{0}\,. \tag{3}$$

The implication (2) $\Longrightarrow$ (3) is trivial, while the converse implication (3) $\Longrightarrow$ (2) amounts in considering the natural stress field as Lagrange multiplier of the rigidity constraint.

A rigorous proof of the latter implication can be carried out for 3-D structural models by means of tools of functional analysis (Korn's inequality and Banach's closed range theorem) [16]. For lower dimensional models, the proof is still a challenging open problem.

The Lie-derivative of the metric field $\mathbf{g}$ along the motion is the *stretching*:

$$\dot{\mathbf{g}} := \mathcal{L}_{\mathbf{v}_{\phi}}(\mathbf{g})\,, \tag{4}$$

and the Lie-derivative of the natural stress along the motion is the *stressing*:

$$\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{v}_{\phi}}(\boldsymbol{\sigma})\,, \tag{5}$$

where $\mathbf{v}_{\phi}$ is the velocity of spacetime motion $\boldsymbol{\phi}$.

The introduction of these natural definitions puts an end to a long debate about which stress rate was to be adopted in rate constitutive relations.

Convective and co-rotational derivatives where the first competitors, proposed by Stanisław Zaremba [17] and Gustav Jaumann [18] at the beginning of the past century.

Further investigations were contributed by James Gardner Oldroyd [19], Leonid Ivanovich Sedov [20, 21] and William Prager [14] about half a century later.

All these proposals were formulated in terms of derivatives of the Cauchy stress tensor field according to the standard spacetime connection, with expressions in components.

As observed by Marsden and Hughes [22], all proposed expressions of the stress rate are in fact Lie-derivatives of different alterations of the stress tensor or of tensor products between stress and volume form, along the spacetime motion.

In [22] it was also pointed out that the choice of an alteration of the stress tensor (mixed, covariant or contravariant) is not inessential, since alteration and pull-back do not commute.

A decisive step was recently undertaken in [23] by bringing basic geometric arguments in favour of the introduction of the *stressing* in Eq.(5) as Lie-derivative, along the motion, of the contravariant stress tensor field per unit mass, the *natural stress*.

The *elastic stretching* is similarly assumed to be the Lie-derivative, along the motion, of the elastic-state field:

$$\dot{\mathrm{e}} := \mathcal{L}_{\mathbf{v}_\phi}(\mathrm{e})\,, \tag{6}$$

The rate-elastic law is formulated in terms of elastic stretching and stressing, by means of a linear tangent compliance operator $\mathbf{H}(\boldsymbol{\sigma})$ nonlinearly dependent on the natural stress:

$$\xrightarrow{\dot{\boldsymbol{\sigma}}} \bullet \xrightarrow{\;\mathbf{H}(\boldsymbol{\sigma})\;} \bullet \xrightarrow{\dot{\mathrm{e}}} \qquad \Longleftrightarrow \qquad \dot{\mathrm{e}} = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}\,. \tag{7}$$

The tangent elastic compliance $\mathbf{H}(\boldsymbol{\sigma})$ is assumed to be positive definite

$$\delta\boldsymbol{\sigma} \neq \mathbf{0} \implies \langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}, \delta\boldsymbol{\sigma} \rangle > 0\,, \tag{8}$$

and to fulfil Cauchy's integrability at each point of each configuration:

$$\mathbf{H}(\boldsymbol{\sigma}) = \mathbf{d}_{\mathcal{F}}\boldsymbol{\Psi}(\boldsymbol{\sigma})\,, \tag{9}$$

where the fibre derivative $\mathbf{d}_{\mathcal{F}}$ is taken at a fixed time instant (i.e. in each fibre of the time-bundle constructed by time projection from the spacetime manifold onto the time-line [24]).

The basic property of an elastic model is time-invariance of the compliance operator $\mathbf{H}$ along the motion, expressed by vanishing of its Lie-derivative:

$$\mathcal{L}_{\mathbf{v}_\phi}(\mathbf{H}) = \mathbf{0}\,, \tag{10}$$

which, by commutativity between pull-back and fibre-derivative, is equivalent to vanishing of the Lie-derivative of Cauchy potential

$$\mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\Psi}) = \mathbf{0}\,. \tag{11}$$

Leibniz rule for the Lie-derivative gives:

$$\mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\Psi} \circ \boldsymbol{\sigma}) = \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\Psi}) \circ \boldsymbol{\sigma} + \mathbf{d}_{\mathcal{F}}\boldsymbol{\Psi}(\boldsymbol{\sigma}) \cdot \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\sigma})\,, \tag{12}$$

and hence the rate-elastic law may be written as

$$\mathcal{L}_{\mathbf{v}_\phi}(\mathrm{e}) = \mathbf{d}_{\mathcal{F}}\boldsymbol{\Psi}(\boldsymbol{\sigma}) \cdot \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\sigma}) = \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\Psi} \circ \boldsymbol{\sigma})\,. \tag{13}$$

Integration yields, to within an invariant tensor field, the nonlinear relation mapping the natural stress to the elastic state along the motion:

$$\xrightarrow{\boldsymbol{\sigma}} \bullet \overset{\boldsymbol{\Psi}}{\rightsquigarrow} \bullet \xrightarrow{\mathbf{e}} \quad \Longleftrightarrow \quad \mathbf{e} = \boldsymbol{\Psi}(\boldsymbol{\sigma})\,. \tag{14}$$

The further basic assumption is fulfilment of Green's integrability at each point of each configuration:

$$\boldsymbol{\Psi}(\boldsymbol{\sigma}) = \mathbf{d}_{\mathcal{F}}\Xi(\boldsymbol{\sigma})\,, \tag{15}$$

so that the nonlinear elastic relation may be written as in terms of a stress scalar potential Ξ:

$$\xrightarrow{\boldsymbol{\sigma}} \bullet \overset{\mathbf{d}_{\mathcal{F}}\Xi}{\rightsquigarrow} \bullet \xrightarrow{\mathbf{e}} \quad \Longleftrightarrow \quad \mathbf{e} = \mathbf{d}_{\mathcal{F}}\Xi(\boldsymbol{\sigma})\,. \tag{16}$$

The elastic-state potential Ξ^* is conjugate, according to the Euler-Legendre-Fenchel transform, to the stress potential Ξ:

$$\begin{cases} \Xi(\boldsymbol{\sigma}) + \Xi^*(\mathbf{e}) = \langle \boldsymbol{\sigma}, \mathbf{e} \rangle\,, \\ \mathbf{e} = \mathbf{d}_{\mathcal{F}}\Xi(\boldsymbol{\sigma})\,, \\ \boldsymbol{\sigma} = \mathbf{d}_{\mathcal{F}}\Xi^*(\mathbf{e})\,, \end{cases} \tag{17}$$

with the property that the sum of their Lie-derivatives vanishes:

$$\dot{\Xi}(\boldsymbol{\sigma}) + \dot{\Xi}^*(\mathbf{e}) = 0\,. \tag{18}$$

Conservation of mass and time invariance of the elastic compliance along the motion Eq.(10), denoting by α the time lapse and by $\downarrow$ the pull-back, are expressed by the properties

$$\begin{cases} \mathcal{L}_{\mathbf{v}_\phi}(\mathbf{m}) = \mathbf{0} & \Longleftrightarrow \boldsymbol{\phi}_\alpha \downarrow \mathbf{m} = \mathbf{m}\,, \\ \mathcal{L}_{\mathbf{v}_\phi}(\Xi) = \mathbf{0} & \Longleftrightarrow \boldsymbol{\phi}_\alpha \downarrow \Xi = \Xi\,, \\ \mathcal{L}_{\mathbf{v}_\phi}(\Xi^*) = \mathbf{0} & \Longleftrightarrow \boldsymbol{\phi}_\alpha \downarrow \Xi^* = \Xi^*\,. \end{cases} \tag{19}$$

Since the metric is time independent, the spacetime velocity $\mathbf{v}_\phi$ can be replaced with its spatial component $\mathbf{v}$, so that in a purely elastic process the rate of elastic state is given by

$$\dot{\mathbf{e}} = \mathcal{L}_{\mathbf{v}_\phi}(\mathbf{g}) = \mathcal{L}_{\mathbf{v}}(\mathbf{g})\,, \tag{20}$$

so that, from equilibrium Eq.(2), setting $\delta\mathbf{v} = \mathbf{v}$, we deduce the expression of the elastic mechanical power:

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \mathcal{L}_{\mathbf{v}}(\mathbf{g}) \rangle = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \dot{\mathbf{e}} \rangle\,. \tag{21}$$

Then, integrating over a time interval of duration Δt, the mechanical work performed in an elastic process is equal to the variation of the global elastic-state functional [23]:

$$\begin{aligned} \int_0^{\Delta t} \langle \mathbf{f}, \mathbf{v} \rangle \, d\alpha &= \int_0^{\Delta t} \int_{\boldsymbol{\phi}_\alpha(\boldsymbol{\Omega})} \langle \boldsymbol{\sigma}, \dot{\mathbf{e}} \rangle \cdot \mathbf{m} \, d\alpha \\ &= \int_{\boldsymbol{\phi}_{\Delta t}(\boldsymbol{\Omega})} \Xi^*(\mathbf{e}) \cdot \mathbf{m} - \int_{\boldsymbol{\Omega}} \Xi^*(\mathbf{e}) \cdot \mathbf{m} \\ &= \int_{\boldsymbol{\Omega}} \Big(\Xi^* \circ (\boldsymbol{\phi}_{\Delta t} \downarrow \mathbf{e}) - (\Xi^* \circ \mathbf{e}) \Big) \cdot \mathbf{m} \, . \end{aligned} \tag{22}$$

The last equality entails that variations of the global elastic state potential vanish along any push-closed path of elastic-states in the spacetime trajectory, i.e. paths fulfilling by the condition:

$$\boldsymbol{\phi}_{\Delta t} \downarrow \mathbf{e} = \mathbf{e} \, . \tag{23}$$

By property Eq.(18) push-closed paths of elastic-states are also push-closed paths of stress-states and vice versa. Existence of a global elastic-state potential is the characteristic property of elastic bodies.

3 CONSIDERATIONS

What can indeed be theoretically assessed and experimentally tested with a simplest 1-D model à la Hooke is in fact a linear relation between an increment of traction and the consequent increment of elastic elongation, which, assuming absence of other phenomena, such as variation of temperature, electric and magnetic fields, dislocation movements and phase changes, is equal to the increment of geometric elongation.

On this basis a general theory of elasticity can be built. Extension to 2-D or 3-D models is achieved by replacing the traction-rate with a twice contravariant tensor of stress rate (or stressing) and the elongation-rate with a twice covariant tensor of elastic-state rate (or elastic stretching) [23].

In a purely elastic model the elastic stretching is equal to the geometric stretching which is the rate-of-change of the material metric tensor in the process under consideration.

The mathematical notion apt to the evaluation of these tensor rates is the Lie-derivative (or convected derivative) along a flow.

This basic notion underlies the whole Continuum Mechanics, but, rather disappointingly, is not adequately considered and also not even referred to, in most textbooks.

And yet the notion was introduced by the norwegian geometer Marius Sophus Lie around 1888 and named after him by David van Dantzig in 1932 [25]. Lie derivatives of general tensors were first considered by Władysław Ślebodziński in 1931 [26].

The rate formulation in terms of Lie-derivatives leads to the following properties:

- The elastic law is formulated at each configuration as a linear relation between time-rates along the motion of material state variables, that is variables represented by fields on the current configuration.

- The constitutive elastic operator acts as a tangent compliance with the natural stress-rate as input of the elastic law and the elastic-state rate (a geometrical notion only in pure elasticity) as output, as depicted in diagram (7). Both are Lie-derivatives. The former is the Lie-derivative of the natural stress. The latter is the Lie-derivative of the elastic-state.

- The tangent elastic compliance is time-invariant along the motion and admits a Green's potential.

In a geometric approach to the theory of elasticity the elastic-state of a material is defined as output of a time-invariant, monotone and fibre differentiable compliance operator acting on the stress-state. This operator is the fibre derivative of a time-invariant, convex elastic stress potential Ξ [23]:

$$\mathrm{e} = \boldsymbol{\Psi}(\boldsymbol{\sigma}) = \mathbf{d}_{\mathcal{F}}\Xi(\boldsymbol{\sigma})\,. \tag{24}$$

Indeed, taking the time derivative of Eq.(24) along the motion, by time-invariance Eq.(12) and integrability Eq.(9), we get the rate law in diagram (7):

$$\begin{aligned} \dot{\mathrm{e}} &= \mathbf{d}_{\mathcal{F}}\boldsymbol{\Psi}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \\ &= \mathbf{d}^{\mathbf{2}}_{\mathcal{F}}\Xi(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \\ &= \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}\,. \end{aligned} \tag{25}$$

Finite elastic strains, defined in an arbitrarily fixed reference configuration, play the role of computational tools with no direct physical meaning.

Constitutive parameters of rate elastic models are to be detected by laboratory experiments performed at a testing configuration of the body and thence the rate elastic law is defined by time-invariance along the motion.

The physical requirement of conservativeness is fulfilled, due to time independence and mass conservation.

A variational principle can be formulated for the elastic-state solution of elastodynamics. The new constitutive theory provides a consistent and computationally effective framework for nonlinear elasticity and elasto-visco-plasticity, and a suitable model also for investigating problems concerning elastic membranes, lattice materials and soft matter.

4 NONLOCAL ELASTICITY

Nonlocal models in elasticity provide evidence of agreement between a theoretical analysis based on a physico-geometric rationale, and a non-classical scheme originally conceived to analyse dislocation problems and presently widely adopted for investigations of nano-structures.

A direct application, of the described rate theory of elasticity to nonlocal elastic problems, leads naturally to the formulation of a stress-driven integral elastic model [27, 28].

The well-posed nonlocal elastostatic problem so generated provides a valid alternative to the currently adopted strain-driven model [29, 30], which has been revealed to be ill-posed [31, 32].

Let us describe the theory in the original context of a referential description of elasticity in a body undergoing small displacements.

Let $\mathbb{T}$ be a generalised function or distribution, according to Sobolev-Schwartz [33], [34]. A standard reference is [35].

Denoting by $*$ the linear convolution operator, an abstract nonlocal model is described by a convolution between a source field s and an output field f, as depicted by the diagram

$$\xrightarrow{\;s\;} \bullet \quad \mathbb{T}* \quad \bullet \xrightarrow{\;f\;} \tag{26}$$

Accordingly, the strain-driven and the stress-driven models are respectively represented by:

1. Strain-driven: $s = E \cdot \Delta\mathbf{e}\,, \quad f = \Delta\boldsymbol{\sigma}\ ,$

$$\xrightarrow{E\cdot\Delta\mathbf{e}} \bullet \;\; \mathbb{T}* \;\; \bullet \xrightarrow{\Delta\boldsymbol{\sigma}} \tag{27}$$

2. Stress-driven: $s = C \cdot \Delta\boldsymbol{\sigma}\,, \quad f = \Delta\mathbf{e}\ ,$

$$\xrightarrow{C\cdot\Delta\boldsymbol{\sigma}} \bullet \;\; \mathbb{T}* \;\; \bullet \xrightarrow{\Delta\mathbf{e}} \tag{28}$$

In the diagrams (27) and (28) $\Delta\boldsymbol{\sigma}$ and $\Delta\mathbf{e}$ are increments of natural stress-state and elastic-state fields, and E, $C = E^{-1}$ are the standard, stress independent, local elastic stiffness and compliance operators. We underline that the laws expressed by diagrams (27) and (28) are not one the inverse of the other.

If the distribution is generated by a locally integrable scalar field φ, the convolution can be written as:

$$f = \mathbb{T}_\varphi * s = \int_{\boldsymbol{\Omega}} \varphi(\mathbf{y} - \mathbf{x}) \cdot s(\mathbf{x})\, d\mathbf{y}\,. \tag{29}$$

When $\boldsymbol{\Omega} \subset \Re^n$, the kernel

$$\varphi : \Re^n \mapsto [0, +\infty)\,, \tag{30}$$

is usually assumed to be given by the fundamental solution of a differential problem in $\Re^n$ under homogeneous boundary conditions at infinity [29].

This feature led to the incorrect conclusion that, in formulating a nonlocal elastic law, the integral convolution could be replaced with a differential equation, even when the configuration $\boldsymbol{\Omega}$ is a bounded (connected, compact) domain in $\Re^n$ [36].

This wrong statement is still spread over a multitude of contributions aimed to simulate the elastic behaviour of nano-structures.

It is important to underline that nonlocal constitutive laws are expressed by functional operators generating the output field f from a source field s. These operators are not invertible by simply swapping source and output fields since the latter are more regular than the former.

The increment of elastic-state $\Delta\mathbf{e}$ may well be piecewise square integrable. On the contrary, the stress-state is required to fulfil equilibrium conditions.

It is this requirement that renders the strain-driven model ill-posed, since no stress field generated by the nonlocal elastic law (27) is able to fulfil the equilibrium conditions and consequently the elastostatic problem admits no solution [32], [27], [28].

5 CONCLUSIONS

After a *résumé* of principal contributions to the formulation of nonlinear elasticity and a critical overview of its development in the course of the past century, a recently proposed new theory based on an incremental definition in terms of Lie-derivatives of material fields has been briefly illustrated.

The theory considers the *natural stress* field as fundamental state variable in duality which the geometric stretching and performing virtual power per unit mass.

Substitution of the *natural stress* field in place of the Cauchy stress (performing virtual power per unit volume) and introduction of the notion of *elastic-state* are key-points to get integrability to an elastic energy functional.

By fibre integrability and time invariance of the tangent constitutive operator, integration along the motion transforms the rate constitutive law into a relation providing the elastic-state as output of the nonlinear constitutive operator acting on the stress-state.

In physically significant relations the usual denomination of *elastic strain* (related to a stress-state) should be appropriately replaced with the new notion of *elastic-state*. This implies a drastic conceptual innovation by properly stating duality between *stress-states* and *elastic-states* as the characterising feature of an elastic law.

In the context of a geometrically linearised theory, nonlocal elastic laws, widely applied to investigations about peculiar properties of nano-structures, have been introduced.

In line with the general theory of rate elasticity outlined above, it is observed that, contrary to the original proposal adopted in literature, the functional relations between the fields of stress-states and elastic-states, must be reformulated by considering the field of stress-states as input and the field of elastic-states as output.

This modification assures well posedness of the corresponding elasticity problems.

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