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On torsion and shear of Saint-Venant beams

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ABSTRACT

Torsion and shear stress fields of a Saint-Venant beam and the relative location of shear and twist centres are investigated for sections of any degree of connectedness. The sliding-torsional compliance tensor of a Timoshenko beam is evaluated by an energy equivalence with Saint-Venant theory. Accordingly, the mutual sliding-torsional term is shown to depend linearly on the relative position of shear and twist centres and the standard definition of shear centre in a Timoshenko beam is found to be coincident with Saint-Venant twist centre. Coincidence of shear and twist centres is assessed for sections with vanishing Poisson ratio and for open, closed and multi-cell thin-walled cross sections. The eigenvalues of the shear factors tensor and the torsion factor are shown to be greater than unity, with the principal directions of shearing and bending compliances non necessarily coincident for non-symmetric cross sections. Numerical examples are developed to provide evidences of the location of the centres and of the principal shearing directions, for non-symmetric L-shaped cross sections with various thickness ratios.

1. Introduction

The linearly elastic isotropic beam investigated by Saint-Venant (1856a, 1856b) is still the reference model for engineers to evaluate strain and stress fields. An extensive list of contributions have been given by researchers on this basic topic of elasticity theory, to deepen both theoretical and applicative issues, among which classical treatments as (Timoshenko, 1940; Southwell, 1941; Timoshenko and Goodier, 1951; Sokolnikoff, 1956; Novozhilov, 1961; Solomon, 1968).

Nevertheless, some results concerning the model could be usefully reconsidered and discussed, still today. This paper deals with the notions of shear centre, twist centre, torsion and shear compliances, as introduced in literature (Timoshenko, 1921, 1922; Weber, 1926; Schwalbe, 1935; Trefftz, 1935; Cicala, 1935; Osgood, 1943; Goodier, 1944; Weinstein, 1947; Timoshenko and Goodier, 1951; Sokolnikoff, 1956) and investigated in (Reissner and Tsai, 1972; Reissner, 1979; Stephen and Maltbaek, 1979; Muller, 1982; Romano et al., 1992; Stronge and Zhang, 1993; Schramm et al., 1994; Gruttmann et al., 1999; Sapountzakis and Mokos, 2005; Lacarbonara and Paolone, 2007). Saint-Venant beam theory is briefly introduced with an intrinsic formulation, in the wake of the treatment provided in (Romano, 2002b), for uniform flexure and torsion, and in (Barretta and Barretta, 2010) for shear and torsion. The intrinsic formulation leads to an elegant presentation with the significant advantage that invariant results and tensorial characters are detected in a natural way. The plan of the paper is the following.

In Section 2.2 we recall basic results of Saint-Venant theory of linearly elastic isotropic beam subject to extension, flexure, shear and torsion. The expression of the axial and transversal components of the displacement field is the starting point to deduce the formulae for normal and tangential stress fields acting on the cross sections. The theory is characterized by the assumption concerning the vanishing of body force field and of surface loading on the lateral mantle of the beam and the vanishing of normal interactions between longitudinal fibres. Equilibrium requirements imply that the resultant axial and shear forces and the resultant torque about the axis, evaluated on the tractions acting on a cross section of the beam, are bound to be constant along the axis. Accordingly, the resultant bending moment is described by an affine law. The field of elongations of the beam longitudinal fibres is an affine function of the position vector and is thus uniquely characterized by a gradient vector and by the elongation of the centroidal fibre. The normal stress is proportional to the elongation of the longitudinal fibre through the Euler modulus E (often called Young modulus and denoted also by Y).

The evaluation of the tangential stress field requires the determination of the following items: the gradient of longitudinal extensions and of its derivative along the beam axis, called the *shearing*; the *twisting* curvature; the *shear warping* vector field and the *twist warping* scalar field. Accordingly, the shear stress field is the split into a shear tangential stress field and a twist tangential stress field, see Section 2.3. The resultant of the former is equal to the shear force while the latter has a null resultant. The definition of the shear tangential stress field leads naturally to the notion of

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shear centre as the unique point in the cross section about which the twisting moment of any shear tangential stress field vanishes. The first conception of a shear centre is attributed by Timoshenko (1953) to Robert Maillart (1921–22), who envisaged it to explain the results of experimental tests on beams with C-shaped sections. A documentation on Maillart, and on his outstanding contributions to the design of reinforced concrete structures, in provided in (Billington, 1998). The shear centre, defined with a coordinate-based formalism in (Timoshenko, 1940, Goodier, 1944; Timoshenko and Goodier, 1951; Sokolnikoff, 1956; Novozhilov, 1961; Solomon, 1968), is introduced with an intrinsic formulation in Prop. 2. Following the original conception by Trefftz (1935) and the treatment by Capurso (1971), the twist centre is introduced in Prop. 4, to define torsion-orthogonal tangential stress fields as the ones which are uncoupled in elastic energy from twist tangential stress fields. A general formula is provided in Section 3 for the relative position of shear and twist centres and it is shown that shear and twist centres coincide for arbitrary cross sections with vanishing Poisson ratio and for all thin-walled sections, whether open or closed and even multi-cell with variable thickness. The evaluation of the shear deformability of Timoshenko beams with thin-walled cross sections was carried out in (Romano et al., 1992). Numerical computations of shear deformability for beams with arbitrary cross sections were performed in (Schramm et al., 1994) where early contributions by Cowper (1966) and Mason and Hermann (1968) were referenced. Initially curved and twisted composite beams have been investigated in (Borri and Merlini, 1986; Borri et al., 1992). A Timoshenko-like model has been recently developed in the same context in (Yu et al., 2002; Hodges, 2006) on the basis of the variational asymptotic method (VAM) proposed in (Berdichevsky, 1976). The energy approach is here adopted in Section 4 to formulate the expression of the symmetric, positive definite sliding-torsional compliance operator. It is shown that the mutual sliding-torsional compliance depends linearly on the relative position of Saint-Venant shear and twist centres and that Timoshenko shear centre, evaluated by Saint-Venant beam theory, is coincident with Saint-Venant twist centre. By a direct application of the Cauchy-Schwarz inequality, it is proved that the principal shear factors are strictly greater than unity and that the torsional factor is not less than unity, being unitary for circular cross sections, whether compact or tubular. The numerical computations, reported in Section 5, for asymmetric L-shaped cross sections, performed according to Saint-Venant theory, confirm that the distance between shear and twist centres is still quite small, for a wide range of thickness ratios and that principal directions of the bending stiffness may be quite distinct from principal directions of the shear factors tensor, in general. The analytical determinations, carried out in Section 6, with reference to open circular thin-walled cross sections, reveal that small corrections due to the curvature of the middle-line may be neglected, so that coincidence of the shear and twist centres may be assessed in this case. Extension of the treatment to non-isotropic and non-homogeneous elastic beams will be developed in forthcoming contributions.

2. Saint-Venant beam theory

2.1. Notations

Let Ω be the cross section of a Saint-Venant linearly elastic isotropic beam (Saint-Venant, 1856a, 1856b), $\partial\Omega$ the boundary with outward normal versor field $\mathbf{n}$, V the two-dimensional linear space of translations in the plane of the cross section, V^* its dual, $\mathbf{k}$ the unit vector along the z axis of the beam and $\mathbf{R}\in C^1(V;V)$ the isometric linear operator which rotates by $\pi/2$ counterclockwise the vector fields in V, so that $\mathbf{R}^T = \mathbf{R}^{-1} = -\mathbf{R}$ and $\mathbf{RR} = -\mathbf{I}$. The following geometric moments of the cross section, with respect to a pole $\mathbf{O}$, are of interest in the theory:

- the area (zeroth geometric moment):
$$A := \int_\Omega dA,$$
- the static moment (first geometric moment):
$$\mathbf{F_O} := \int_\Omega \mathbf{p}\, dA,$$
- the inertia moment (second geometric moment):
$$\mathbf{J_O} := \int_\Omega \mathbf{p}\otimes_{\mathbf{g}}\mathbf{p}\, dA$$

where $\mathbf{p}$ is the position vector of points in Ω with respect to $\mathbf{O}$.

Denoting $\mathbf{g}\in SYM(V) = L(V;V^*)$ the metric tensor in V, the tensor product $\mathbf{u}\otimes_{\mathbf{g}}\mathbf{v}$ of the vectors $\mathbf{u},\mathbf{v}\in V$ is defined by the identity $(\mathbf{u}\otimes_{\mathbf{g}}\mathbf{v})\cdot\mathbf{w} := \mathbf{g}(\mathbf{v},\mathbf{w})\mathbf{u}$ for any vector $\mathbf{w}\in V$. Here and throughout in paper, the dot $\cdot$ denotes linear dependence on the subsequent argument. The *centroid* $\mathbf{G}$ of the section, is the mean point defined by $\mathbf{p_G}{:=}\mathbf{F_O}/A$. In the sequel: $N(z) = \int_\Omega \sigma(\mathbf{r},z)dA$ is the resultant *normal force*, viz. the resultant of the axial component of tractions on the cross section, $\mathbf{S}(z) = \int_\Omega \boldsymbol{\tau}(\mathbf{r},z)dA$ is the resultant *shear force*, viz. the resultant of the in-plane component of the tractions on the cross section, $\mathbf{M_G}(z) = \int_\Omega \sigma(\mathbf{r},z)\mathbf{r}dA$ is the *bending moment*, defined as the $\mathbf{R}$-rotated axial vector of the resultant bending moment (the in-plane component of the resultant moment) of tractions on the cross section, with respect to the *centroid* $\mathbf{G}$ of the section, $\mathcal{M}_\mathbf{G}(z) = \int_\Omega \mathbf{g}(\boldsymbol{\tau}(\mathbf{r},z),\mathbf{Rr})dA$ is the scalar resultant *twisting moment* (the axial component of the resultant moment) of tractions on the cross section, with respect to the *centroid* $\mathbf{G}$ of the section. An apex $()'$ denotes the derivative with respect to z, and $\mathbf{r}$ is the position vector of a point of the cross section with respect to the *centroid* $\mathbf{G}$.

2.2. Displacement and stress fields

Let $\varepsilon_\mathbf{G}$ be the *axial elongation*. The *bending curvature* $\mathbf{d}(z) = \mathbf{d}(0) + \mathbf{d}'z$ is the gradient with respect to $\mathbf{r}$ of the field of the extensions of longitudinal fibres of the beam; the *shearing* $\mathbf{d}'$ is the constant derivative along z of the affine vector function $\mathbf{d}(z)$; the *twist* α is a scalar parameter which will be shown to be the average of the local-twist field over the section; $\varphi(\mathbf{d}',\alpha)\in C^2(\Omega;\mathcal{R})$ is a *warping field*, which is bilinear in their arguments $\mathbf{d}'$ and α; μ and ν are the Lamé shear modulus and the Poisson ratio, respectively; $E = 2\mu(1+\nu)$ is the Euler modulus. The displacement field in the beam subject to extension, bending, shear and torsion, is conveniently split into tangential and normal components to the cross section and expressed as function of the position vector with respect to the centroid. The former is a vector field $\mathbf{u}(\varepsilon_\mathbf{G},\mathbf{d}(0),\mathbf{d}',\alpha,z)\in C^1(\Omega;V)$ in the cross section and the latter is a scalar field $w(\varepsilon_\mathbf{G},\mathbf{d}(0),\mathbf{d}',\alpha,z)\in C^1(\Omega;\mathcal{R})$. The expression of the tangential and normal displacement fields are conveniently written in intrinsic form as (Barretta and Barretta, 2010):

$$\begin{aligned}
\mathbf{u}(\varepsilon_\mathbf{G},\mathbf{d}(0),\mathbf{d}',\alpha,\mathbf{r},z) &= \frac{\nu}{2}\left(\mathbf{Rr}\otimes_\mathbf{g}\mathbf{Rr}-\mathbf{r}\otimes_\mathbf{g}\mathbf{r}\right)\mathbf{d}(z)\\
&\quad -\nu\varepsilon_\mathbf{G}\mathbf{r}-\frac{z^2}{2}\mathbf{d}(0)-\frac{z^3}{6}\mathbf{d}'+\alpha z\mathbf{Rr},\\
w(\varepsilon_\mathbf{G},\mathbf{d}(0),\mathbf{d}',\alpha,\mathbf{r},z) &= \varphi(\mathbf{d}',\alpha,\mathbf{r})+(\mathbf{g}(\mathbf{d}(0),\mathbf{r})+\varepsilon_\mathbf{G})\,z+\mathbf{g}(\mathbf{d}',\mathbf{r})\frac{z^2}{2}.
\end{aligned}$$

Remark 2.1. *In Cartesian components* $\mathbf{r}\equiv\{x,y\}$ *and* $\mathbf{Rr}\equiv\{-y,x\}$ *so that:*

$$\mathbf{r}\otimes_{\mathbf{g}}\mathbf{r}\equiv\begin{vmatrix} x^2 & xy \\ xy & y^2 \end{vmatrix}, \quad \mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr}\equiv\begin{vmatrix} y^2 & -xy \\ -xy & x^2 \end{vmatrix}.$$

The previous intrinsic expression of the tangential component of the displacement field, when expressed in components, is easily checked to be equal to the one in (Sokolnikoff, 1956) where $\mathbf{d}' \equiv \{K_x, K_y\}$, $\varepsilon_{\mathbf{G}}=0$ *and* $\mathbf{d}(0)=0$. *On the other hand, the warping field* $\varphi(\mathbf{d}',\alpha)\in C^2(\Omega;\mathcal{R})$ *appearing in the axial component has a different definition, due to the choice of a different particular integral of the compatibility equation. This different choice is suggested by the application of Cesàro formula (Romano, 2002a, b) for the integration of the strain field and is motivated by the difficulty in translating the standard coordinate-form into an intrinsic expression.*

To compute the strain field as symmetric part of the displacement gradient, according to Euler formula, we first provide the explicit expression of the derivative of the displacement field. To this end, we recall the formulae:

$$\nabla\big[(\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr})\mathbf{d}(z)\big] = \mathbf{g}(\mathbf{Rr},\ \mathbf{d}(z))\mathbf{R} + (\mathbf{Rr})\otimes_{\mathbf{g}}\big(\mathbf{R}^T\mathbf{d}(z)\big),$$

$$\nabla\big[(\mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{d}(z)\big] = \mathbf{g}(\mathbf{r},\mathbf{d}(z))\mathbf{I} + \mathbf{r}\otimes_{\mathbf{g}}\mathbf{d}(z),$$

where $\mathbf{I}\in L(V;V)$ is the identity, $\nabla = \nabla_{\mathbf{r}}$ is the derivative in the cross section, $\nabla_z = ()'$ is the derivative along the beam axis and the transposition $()^T$ is taken with respect to the metric tensor $\mathbf{g}$.

Let us denote by $\mathbf{gv}\in V^*$ the covector associated with the vector $\mathbf{v}\in V$, so that $\mathbf{g}(\mathbf{v},\mathbf{h}) = (\mathbf{gv})\cdot\mathbf{h}, \quad \forall\mathbf{h}\in V$. The derivative of the displacement is then described by the four-blocks operator defined by the identity:

$$\begin{vmatrix} \nabla_{\mathbf{r}}\mathbf{u} & \nabla_z\mathbf{u} \\ \mathbf{g}\nabla_{\mathbf{r}}w & \nabla_z w \end{vmatrix}\begin{vmatrix} \mathbf{h}_1 \\ \alpha_2 \end{vmatrix}\cdot\begin{vmatrix} \mathbf{gh}_2 \\ \alpha_2 \end{vmatrix} = \mathbf{g}(\nabla_{\mathbf{r}}\mathbf{u}\cdot\mathbf{h}_1,\mathbf{h}_2) + \mathbf{g}(\nabla_{\mathbf{r}}w\cdot\mathbf{h}_1,\mathbf{h}_2) + \mathbf{g}(\nabla_z\mathbf{u}\cdot\alpha_1,\mathbf{h}_2) + \mathbf{g}(\nabla_z w\cdot\alpha_1,\alpha_2),$$

for all $\mathbf{h}_1,\mathbf{h}_2\in V$, $\alpha_1,\alpha_2\in\mathcal{R}$, with the expressions of the component blocks given by:

$$\begin{aligned}
\nabla_{\mathbf{r}}\mathbf{u} &= \frac{\nu}{2}\Big(\mathbf{g}(\mathbf{Rr},\mathbf{d}(z))\mathbf{R} + (\mathbf{Rr})\otimes_{\mathbf{g}}\Big(\mathbf{R}^T\mathbf{d}(z)\Big) \\
&\quad -\mathbf{g}(\mathbf{r},\mathbf{d}(z))\mathbf{I} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{d}(z)\Big) - \nu\varepsilon_{\mathbf{G}}\mathbf{I} + \alpha z\mathbf{R}, \\
\nabla_z\mathbf{u} &= \frac{\nu}{2}(\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\ \mathbf{d}' + \alpha\mathbf{Rr} - z\mathbf{d}(0) - \frac{z^2}{2}\mathbf{d}', \\
\nabla_{\mathbf{r}}w &= \nabla\varphi(\mathbf{d}',\alpha,\mathbf{r})\ + z\mathbf{d}(0) + \frac{z^2}{2}\mathbf{d}', \\
\nabla_z w &= \mathbf{g}(\mathbf{r},\mathbf{d}(z)) + \varepsilon_{\mathbf{G}}.
\end{aligned}$$

Taking the symmetric part of the displacement derivative, the axial elongation and the tangential strain fields are given by:

$$\begin{cases} \varepsilon(\varepsilon_{\mathbf{G}},\mathbf{d}(z),\mathbf{r}) = \mathbf{g}(\mathbf{r},\mathbf{d}(z)) + \varepsilon_{\mathbf{G}}, \\ \gamma(\mathbf{d}',\alpha,\mathbf{r}) = \dfrac{\nu}{2}(\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{d}' + \alpha\mathbf{Rr} + \nabla\varphi(\mathbf{d}',\alpha,\mathbf{r}). \end{cases}$$

Moreover, for any versor $\mathbf{h}\in V$ $(\mathbf{g}(\mathbf{h},\mathbf{h}) = 1)$:

$$\begin{aligned}
\mathbf{g}(\mathrm{sym}(\nabla_{\mathbf{r}}\mathbf{u})\cdot\mathbf{h},\mathbf{h}) &= \frac{\nu}{2}(\mathbf{g}(\mathbf{Rr},\mathbf{d}(z))\mathbf{g}(\mathbf{Rh},\mathbf{h}) - \mathbf{g}(\mathbf{d}(z),\mathbf{Rh})\mathbf{g}(\mathbf{r},\mathbf{Rh}) \\
&\quad -\mathbf{g}(\mathbf{r},\mathbf{d}(z)) - \mathbf{g}(\mathbf{d}(z),\mathbf{h})\mathbf{g}(\mathbf{r},\mathbf{h}) - 2\varepsilon_{\mathbf{G}}) \\
&= -\nu(\mathbf{g}(\mathbf{r},\mathbf{d}(z)) + \varepsilon_{\mathbf{G}}).
\end{aligned}$$

The tangential strain tensor is then circular and the transversal principal strain is due to the Poisson effect. Accordingly, the normal and tangential stress fields on Ω, solution of the elastostatic problem, are provided by the formulae:

$$\begin{cases} \sigma(\varepsilon_{\mathbf{G}},\mathbf{d}(z),\mathbf{r}) = E(\mathbf{g}(\mathbf{d}(z),\mathbf{r}) + \varepsilon_{\mathbf{G}}), \\ \tau(\mathbf{d}',\alpha,\mathbf{r}) = \dfrac{\mu\nu}{2}(\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{d}' + \mu\alpha\mathbf{Rr} + \mu\nabla\varphi(\mathbf{d}',\alpha,\mathbf{r}). \end{cases}$$

The field of local-rotations about the beam axis is given by:

$$\begin{aligned}
\mathbf{g}(\mathrm{skew}(\nabla_{\mathbf{r}}\mathbf{u})\cdot\mathbf{h},\mathbf{Rh}) &= \frac{\nu}{4}(\mathbf{g}(\mathbf{Rr},\mathbf{d}) + \mathbf{g}(\mathbf{d},\mathbf{Rh})\mathbf{g}(\mathbf{r},\mathbf{h}) \\
&\quad -\mathbf{g}(\mathbf{r},\mathbf{Rh})\mathbf{g}(\mathbf{d},\mathbf{h}) + \mathbf{g}(\mathbf{d},\mathbf{h})\mathbf{g}(\mathbf{r},\mathbf{Rh}) \\
&\quad +\mathbf{g}(\mathbf{r},\mathbf{h})\mathbf{g}(\mathbf{d},\mathbf{Rh})) + \alpha z \\
&= -\nu\mathbf{g}(\mathbf{r},\mathbf{Rd}(z)) + \alpha z.
\end{aligned}$$

The local-twist field is the derivative with respect to z:

$$\mathbf{g}(\mathrm{skew}(\nabla_{\mathbf{r}}\mathbf{u})'\cdot\mathbf{h},\mathbf{Rh}) = -\nu\mathbf{g}(\mathbf{r},\mathbf{Rd}')\ + \alpha = \frac{1}{2}(\mathrm{curl}\gamma(\mathbf{d}',\alpha,\mathbf{r}))$$

where the last equality is inferred by direct computation. Since

$$\int_\Omega (-\nu\mathbf{g}(\mathbf{r},\mathbf{Rd}') + \alpha)dA = A\alpha,$$

the *twist* α, which is the local-twist at the centroid ($\mathbf{r} = 0$), is equal to the average of the local-twist field over the cross section (Sokolnikoff, 1956).

The fulfilment of the bulk and boundary differential conditions of equilibrium:

$$\begin{aligned}
&\mathrm{div}\,\tau(\mathbf{d}',\alpha,\mathbf{r}) + \sigma'(\mathbf{d}',\mathbf{r}) = 0 \quad \text{in} \quad \Omega, \\
&\mathbf{g}(\tau(\mathbf{d}',\alpha,\mathbf{r}),\mathbf{n}(\mathbf{r})) = 0 \qquad \text{on} \quad \partial\Omega,
\end{aligned}$$

is equivalent to the requirement that the warping field $\varphi(\mathbf{d}',\alpha)\in C^2(\Omega;\mathscr{R})$ be solution of the Poisson-Neumann problem:

$$\begin{cases} \Delta_2\varphi(\mathbf{d}',\alpha,\mathbf{r}) = -2\mathbf{g}(\mathbf{d}',\mathbf{r}), & \mathbf{r}\in\Omega, \\ d_{\mathbf{n}}\varphi(\mathbf{d}',\alpha,\mathbf{r}) = -\dfrac{\nu}{2}\mathbf{g}((\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{d}',\mathbf{n}(\mathbf{r})) & \\ \qquad -\alpha\mathbf{g}(\mathbf{Rr},\mathbf{n}(\mathbf{r})), & \mathbf{r}\in\partial\Omega. \end{cases}$$

Remark 2.2. *A discussion concerning the existence of the warping field* $\varphi(\mathbf{d}',\alpha)$ *for any cross section* Ω*, whether simply or multiply connected, and for any shear force* $\mathbf{S}$ *and twisting moment* $\mathcal{M}_{\mathbf{G}}$*, and its uniqueness up to a constant, may be found in (Barretta and Barretta, 2010) where the treatment in (Romano, 2002b), based on an elastostatic analogy, is referred to.*

2.3. Shear and twist

The classical Poisson-Neumann problem for the warping field is conveniently split into a shear problem (in which α=0) and a torsion problem (in which $\mathbf{d}' = 0$). Their solution fields will be called the shear warping $\varphi^{SH}(\mathbf{d}')$ and the twist warping $\varphi^{TW}(\alpha)$. The warping field depends linearly on the arguments $\mathbf{d}'$ and α, so that:

$$\varphi(\mathbf{d}',\alpha) = \varphi^{SH}(\mathbf{d}') + \varphi^{TW}(\alpha).$$

The shear warping field $\varphi^{SH}(\mathbf{d}')\in C^2(\Omega;\mathcal{R})$ is the solution of the Poisson-Neumann problem:

$$\begin{cases} \Delta_2\varphi^{SH}(\mathbf{d}',\mathbf{r}) = -2\mathbf{g}(\mathbf{d}',\mathbf{r}), & \mathbf{r}\in\Omega, \\ d_{\mathbf{n}}\varphi^{SH}(\mathbf{d}',\mathbf{r}) = -\dfrac{\nu}{2}\mathbf{g}((\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{d}',\mathbf{n}(\mathbf{r})), & \mathbf{r}\in\partial\Omega, \end{cases}$$

and the twist warping field $\varphi^{TW}(\alpha)\in C^2(\Omega;\mathcal{R})$ is solution of the Laplace-Neumann problem:

$$\begin{cases} \Delta_2\varphi^{TW}(\alpha,\mathbf{r}) = 0, & \mathbf{r}\in\Omega, \\ d_{\mathbf{n}}\varphi^{TW}(\alpha,\mathbf{r}) = -\alpha\mathbf{g}(\mathbf{Rr},\mathbf{n}(\mathbf{r})), & \mathbf{r}\in\partial\Omega. \end{cases}$$

Setting:

$$\varphi^{SH}(\mathbf{d}',\mathbf{r}) = \mathbf{g}\Big(\phi^{SH}(\mathbf{r}),\mathbf{d}'\Big),$$

the shear tangential stress field on Ω is provided by the formula:

$$\tau^{SH}(\mathbf{d}',\mathbf{r}) := \mu\Big(\frac{\nu}{2}\big(\mathbf{Rr}\otimes_{\mathbf{g}}\mathbf{Rr} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r}\big) + \nabla\boldsymbol{\phi}^{SH}(\mathbf{r})^{T}\Big)\cdot\mathbf{d}' = \mathbf{F}^{SH}(\mathbf{r})\cdot\mathbf{d}',$$

which defines the tensor $\mathbf{F}^{SH}(\mathbf{r})$ having the physical dimension of a force. Setting $\varphi^{TW}(\alpha,\mathbf{r}) = \alpha\phi^{TW}(\mathbf{r})$, the twist tangential stress field on Ω is provided by the formula:

$$\tau^{TW}(\alpha,\mathbf{r}) := \mu\alpha\Big(\mathbf{Rr} + \nabla\phi^{TW}(\mathbf{r})\Big) = \alpha\mathbf{f}^{TW}(\mathbf{r}),$$

which defines the vector $\mathbf{f}^{TW}(\mathbf{r})$ having the physical dimension of a force times the inverse of a length, $\phi^{TW}(\mathbf{r})$ is a squared length and hence $\nabla\phi^{TW}(\mathbf{r})$ is a length.

Lemma 2.1. *The resultant shear force* $\mathbf{S}$ *may be evaluated by the formula:*

$$\mathbf{S} := \int_{\Omega} \tau(\mathbf{r})dA = -\int_{\Omega} \mathbf{r}\,\mathrm{div}\,(\tau(\mathbf{r}))dA.$$

Proof. By virtue of the identity: $\mathrm{div}(\mathbf{r}\otimes_{\mathbf{g}}\tau(\mathbf{r})) = \mathbf{r}\,\mathrm{div}\,\tau(\mathbf{r}) + \tau(\mathbf{r})$, the divergence theorem gives:

$$\begin{aligned}\int_{\Omega} \tau(\mathbf{r})dA &= \int_{\Omega} \mathrm{div}\big(\mathbf{r}\otimes_{\mathbf{g}}\tau(\mathbf{r})\big)dA - \int_{\Omega} \mathbf{r}\,\mathrm{div}(\tau(\mathbf{r}))dA \\ &= \oint_{\partial\Omega} \mathbf{g}(\tau(\mathbf{r}),\mathbf{n}(\mathbf{r}))\cdot\mathbf{r}ds - \int_{\Omega} \mathbf{r}\,\mathrm{div}(\tau(\mathbf{r}))dA,\end{aligned}$$

and the first integral vanishes since $\mathbf{g}(\tau(\mathbf{r}),\mathbf{n}(\mathbf{r})) = 0$ on $\partial\Omega$.

Since $\mathrm{div}(\tau^{TW}(\alpha,\mathbf{r}))=0$, from Lemma 2.1 it follows that the resultant of the twist tangential stress field vanishes and that the resultant of the shear tangential stress field is equal to the shear force.

Proposition 2.1. *The shear force is given by:* $\mathbf{S} = E\,\mathbf{J_G}\,\mathbf{d}'$.

Proof. A direct computation gives:

$$\begin{aligned}\mathbf{S} &= \int_{\Omega} \tau^{SH}(\mathbf{d}',\mathbf{r})dA = -\int_{\Omega} \mathbf{r}\,\mathrm{div}\Big(\tau^{SH}(\mathbf{d}',\mathbf{r})\Big)dA \\ &= \int_{\Omega} E\mathbf{r}\otimes_{\mathbf{g}}\mathbf{r}\,dA\cdot\mathbf{d}' = E\,\mathbf{J_G}\cdot\mathbf{d}',\end{aligned}$$

where $\mathbf{J_G} := \int_{\Omega}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{r}\,dA$ is the second geometric moment of Ω with respect to the centroid.

The same conclusion may be arrived at by a direct integration of the expression of the field $\tau^{SH}(\mathbf{d}',\mathbf{r})$ in terms of the warping, but at the cost of a much more involved algebra.

The shear tangential stress field may alternatively be expressed in terms of the shear force as:

$$\tau^{SH}(\mathbf{S},\mathbf{r}) = \mathbf{T}^{SH}(\mathbf{r})\cdot\frac{\mathbf{S}}{A}, \quad \mathbf{r}\in\Omega,$$

with $\mathbf{T}^{SH}(\mathbf{r}) = A\mathbf{F}^{SH}(\mathbf{r})(E\mathbf{J_G})^{-1}$ a non-dimensional tensor whose mean value is equal to the identity:

$$\frac{1}{A}\int_{\Omega} \mathbf{T}^{SH}(\mathbf{r})dA = \mathbf{I}.$$

2.4. Shear centre

The global elastic constitutive relations between the kinematic parameters $\{\varepsilon_{\mathbf{G}}, \mathbf{d}(z)\}$ and their dual static counterparts $\{N, \mathbf{M_G}(z)\}$ are uncoupled, due to the choice of the centroid as origin of positions, and may be evaluated by a direct computation of the normal force and of the bending moment:

$$\begin{cases} N = \displaystyle\int_{\Omega} \sigma(\mathbf{d}(z),\varepsilon_{\mathbf{G}},\mathbf{r})dA = EA\varepsilon_{\mathbf{G}}, \\ \mathbf{M_G}(z) = \displaystyle\int_{\Omega} \sigma(\mathbf{d}(z),\varepsilon_{\mathbf{G}},\mathbf{r})\mathbf{r}dA = E\,\mathbf{J_G}\cdot\mathbf{d}(z). \end{cases}$$

Here EA is the scalar axial stiffness and $E\,\mathbf{J_G}$ is the bending stiffness tensor. Taking the derivative of the latter equation along the beam axis we infer that $\mathbf{M}'_{\mathbf{G}}(z) = \mathbf{S}$. The constitutive relations between the kinematic parameters $\{\mathbf{d}',\alpha\}$ and their dual static counterparts $\{\mathbf{S},\mathcal{M}_{\mathbf{C}^{SH}}\}$ may be put in uncoupled form by introducing the *shear centre* $\mathbf{C}^{SH}$.

Proposition 2.2. (Shear centre). *Any shear tangential stress field has a vanishing twisting moment about a pole of the cross section* Ω*, called the shear centre* $\mathbf{C}^{SH}$ *whose position is given by the formula:*

$$\mathbf{r}_{\mathbf{C}^{SH}} = -\frac{1}{A}\mathbf{R}\int_{\Omega} \mathbf{T}^{SH}(\mathbf{r})^{T}\mathbf{Rr}dA = -\mathbf{R}(E\mathbf{J_G})^{-1}\int_{\Omega} \mathbf{F}^{SH}(\mathbf{r})^{T}\mathbf{Rr}dA.$$

Proof. Being $\tau^{SH}(\mathbf{d}',\mathbf{r}) = \mathbf{T}^{SH}(\mathbf{r})\mathbf{S}/A$, the vanishing of the resultant moment about the shear centre $\mathbf{C}^{SH}$ is expressed by:

$$\begin{aligned}\int_{\Omega} \mathbf{g}\Big(\mathbf{Rr}-\mathbf{Rr}_{\mathbf{C}^{SH}},\tau^{SH}(\mathbf{d}',\mathbf{r})\Big)dA &= \int_{\Omega} \mathbf{g}\Big(\mathbf{Rr},\mathbf{T}^{SH}(\mathbf{r})\frac{\mathbf{S}}{A}\Big)dA \\ &\quad -\mathbf{g}(\mathbf{Rr}_{\mathbf{C}^{SH}},\mathbf{S}) = 0.\end{aligned}$$

The first equality then follows by the arbitrariness of $\mathbf{S}$ and the second from the relation $\mathbf{F}^{SH}(\mathbf{r})\mathbf{d}' = \mathbf{T}^{SH}(\mathbf{r})\frac{\mathbf{S}}{A}$, where $E\mathbf{J_G}\mathbf{d}' = \mathbf{S}$.

In (Sokolnikoff, 1956) the shear centre is named *center of flexure*.

Proposition 2.3. *The twist* α *of the beam is proportional to the twisting moment about shear centre* $\mathbf{C}^{SH}$

$$\mathcal{M}_{\mathbf{C}^{SH}} = \mu\,A^2K\alpha,$$

with the non-dimensional factor K is given by:

$$K := \frac{1}{\mu A^2}\int_{\Omega} \mathbf{g}\Big(\mathbf{Rr},\mathbf{f}^{TW}(\mathbf{r})\Big)dA.$$

Proof. By Prop. 2.2 the twisting moment about $\mathbf{C}^{SH}$ is given by

$$\mathcal{M}_{\mathbf{C}^{SH}} = \int_{\Omega} \mathbf{g}\Big(\mathbf{Rr},\tau^{TW}(\alpha,\mathbf{r})\Big)dA = \left(\int_{\Omega} \mathbf{g}\Big(\mathbf{Rr},\mathbf{f}^{TW}(\mathbf{r})\Big)dA\right)\cdot\alpha = \mu A^2K\alpha,$$

because only twist tangential stress fields are involved.

The twist tangential stress field may be written as:

$$\tau^{TW}(\alpha,\mathbf{r}) = \alpha\mathbf{f}^{TW}(\mathbf{r}) = \frac{\mathcal{M}_{\mathbf{C}^{SH}}}{A}\mathbf{t}^{TW}(\mathbf{r}),$$

where $\mathbf{t}^{TW}(\mathbf{r})$ is the inverse of a length. Then

$$\mathbf{f}^{TW}(\mathbf{r}) = \mu\,AK\mathbf{t}^{TW}(\mathbf{r}),$$

and

$$\frac{1}{A}\int_{\Omega} \mathbf{g}\Big(\mathbf{Rr},\mathbf{t}^{TW}(\mathbf{r})\Big)dA = 1.$$

For a circular cross section, or a circular annulus, it is $\mathbf{f}^{TW}(\mathbf{r}) = \mu\mathbf{Rr}$ so that: $K = J_P/A^2$, where $J_P := \int_{\Omega}\mathbf{g}(\mathbf{r},\mathbf{r})\,dA$ is the polar geometric moment of the cross section about the centroid $\mathbf{G}$.

2.5. Twist centre

To introduce the notion of twist centre, the following expression of mutual elastic energy per unit length is preliminarily provided.

Lemma 2.2. (Shear-twist mutual elastic energy). *A tangential stress field* $\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1)$, *with a null resultant moment about a pole* $\mathbf{C}$, *when interacting with a twist tangential strain field* $\boldsymbol{\gamma}^{\mathrm{TW}}(\alpha_2)$, *performs an internal mutual work per unit length given by*:

$$\begin{aligned} u_{12} &:= \int_{\Omega} \mathbf{g}\Big(\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}), \boldsymbol{\gamma}^{\mathrm{TW}}(\alpha_2, \mathbf{r})\Big) dA \\ &= \alpha_2 \mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}}, \mathbf{S}_1) + \alpha_2 \int_{\Omega} \sigma'(\mathbf{d}'_1, \mathbf{r}) \phi^{\mathrm{TW}}(\mathbf{r})\, dA \\ &= \alpha_2 \mathbf{g}\left(\mathbf{R}\mathbf{r}_{\mathbf{C}} + \mathbf{J}_{\mathbf{G}}^{-1} \int_{\Omega} \phi^{\mathrm{TW}}(\mathbf{r}) \mathbf{r}\, dA\right) \cdot \mathbf{S}_1. \end{aligned}$$

Proof. Recalling the formula: $\boldsymbol{\gamma}^{\mathrm{TW}}(\alpha_2, \mathbf{r}) = \alpha_2(\mathbf{R}\mathbf{r} + \nabla\phi^{\mathrm{TW}}(\mathbf{r}))$, the internal mutual work per unit length is given by:

$$\begin{aligned} u_{12} &= \alpha_2 \int_{\Omega} \mathbf{g}(\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}), \mathbf{R}\mathbf{r}) dA \\ &\quad + \alpha_2 \int_{\Omega} \mathbf{g}\Big(\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}), \nabla\phi^{\mathrm{TW}}(\mathbf{r})\Big) dA. \end{aligned}$$

The evaluation of the twisting moment about the centroid $\mathbf{G}$ gives:

$$\int_{\Omega} \mathbf{g}(\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}), \mathbf{R}\mathbf{r}) dA = \mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}}, \mathbf{S}_1).$$

On the other hand, the differential and boundary equations of equilibrium:

$$\begin{cases} -\mathrm{div}\, \boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}) = \sigma'(\mathbf{d}'_1, \mathbf{r}) = E\mathbf{g}(\mathbf{d}'_1, \mathbf{r}) = \mathbf{g}(\mathbf{S}_1, \mathbf{J}_{\mathbf{G}}^{-1}\mathbf{r}), & \text{on } \Omega, \\ \mathbf{g}(\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}), \mathbf{n}(\mathbf{r})) = 0, & \text{on } \partial\Omega, \end{cases}$$

inserted in Green's formula give:

$$\begin{aligned} \int_{\Omega} g\Big(\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}), \nabla\phi^{\mathrm{TW}}(\mathbf{r})\Big) dA &= -\int_{\Omega} \mathrm{div}\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}) \phi^{\mathrm{TW}}(\mathbf{r})\, dA + \oint_{\partial\Omega} \mathbf{g}(\boldsymbol{\tau}(\mathbf{d}'_1, \alpha_1, \mathbf{r}), \mathbf{n}(\mathbf{r})) \phi^{\mathrm{TW}}(\mathbf{r}) ds = \int_{\Omega} \sigma'(\mathbf{d}'_1, \mathbf{r}) \phi^{\mathrm{TW}}(\mathbf{r})\, dA \\ &= \mathbf{g}\, \mathbf{J}_{\mathbf{G}}^{-1} \left(\int_{\Omega} \phi^{\mathrm{TW}}(\mathbf{r}) \mathbf{r}\, dA \right) \cdot \mathbf{S}_1 \end{aligned}$$

which is the result.

Proposition 2.4. (Twist centre). *The twist centre* $\mathbf{C}^{\mathrm{TW}}$ *is the pole in the plane of the cross section* Ω, *such that a tangential stress field* $\boldsymbol{\tau}(\mathbf{d}', \alpha)$ *with vanishing resultant moment about* $\mathbf{C}^{\mathrm{TW}}$ *performs no mutual work when interacting with any twist tangential strain field* $\boldsymbol{\gamma}^{\mathrm{TW}}(\alpha, \mathbf{r}) = \alpha(\mathbf{R}\mathbf{r} + \nabla\phi^{\mathrm{TW}}(\mathbf{r}))$. *The position of* $\mathbf{C}^{\mathrm{TW}}$ *is given by the formula:*

$$\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} = \mathbf{R}\mathbf{J}_{\mathbf{G}}^{-1} \int_{\Omega} \phi^{\mathrm{TW}}(\mathbf{r}) \mathbf{r}\, dA,$$

where $\phi^{\mathrm{TW}} := \varphi^{\mathrm{TW}}(1) \in C^2(\Omega; \mathcal{R})$.

Proof. The result is a direct consequence of the formula in Lemma 2.2.

This definition of twist centre was introduced in (Trefftz, 1935) where the tangential stress fields $\boldsymbol{\tau}(\mathbf{d}', \alpha)$ with vanishing resultant moment about $\mathbf{C}^{\mathrm{TW}}$ were said to induce bending without torsion. We will call them torsion-orthogonal tangential stress fields, as suggested by the definition in Prop. 4. An analogous treatment in coordinates was developed in (Capurso, 1971). Another definition of torsion-free bending was given in (Goodier, 1944).

An alternative definition of twist centre $\mathbf{C}^{\mathrm{TW}}$ was introduced by Cicala (1935); Trefftz (1935); Weinstein (1947). It is based on the observation that the warping ϕ^{TW} due to twist is unique to within an affine scalar function of the position vector. Following the treatment in (Sokolnikoff, 1956), we assume the expression:

$$\phi^{\mathrm{TW}}(\mathbf{r}) + \mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}}, \mathbf{r}) - c.$$

Requiring that zeroth and first moments of the twist warping are zero:

$$\begin{aligned} &\int_{\Omega} \Big(\phi^{\mathrm{TW}}(\mathbf{r}) - c\Big) dA = 0, \\ &\int_{\Omega} \Big(\phi^{\mathrm{TW}}(\mathbf{r}) + \mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}}, \mathbf{r})\Big) \mathbf{r}\, dA = 0, \end{aligned}$$

the value of c and the position of the twist centre are evaluated.

3. Shear centre vs twist centre

The next result concerns the relative position of twist and shear centres and will be resorted to in Section 4 dealing with the mutual sliding-torsional compliance and the notion of shear centre in Timoshenko beam theory. The non-dimensional torsion factor χ^{TW}, defined by:

$$\chi^{\mathrm{TW}} := \frac{J_P}{A^2} \int_{\Omega} \mathbf{g}\Big(\mathbf{t}^{\mathrm{TW}}(\mathbf{r}), \mathbf{t}^{\mathrm{TW}}(\mathbf{r})\Big) dA,$$

and the noteworthy formula:

$$\frac{\chi^{\mathrm{TW}} A^2}{J_P} K = 1,$$

are referred to in the next proposition.

Proposition 3.1. *The relative position of twist and shear centres,* $\mathbf{C}^{\mathrm{TW}}$ *and* $\mathbf{C}^{\mathrm{SH}}$, *is provided by the formula:*

$$\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}} = K\mathbf{R} \int_{\Omega} \mathbf{T}^{\mathrm{SH}}(\mathbf{r})^T \mathbf{t}^{\mathrm{TW}}(\mathbf{r})\, dA = \frac{1}{\mu E} \mathbf{R}\mathbf{J}_{\mathbf{G}}^{-1} \int_{\Omega} \mathbf{F}^{\mathrm{SH}}(\mathbf{r})^T \mathbf{f}^{\mathrm{TW}}(\mathbf{r}) dA.$$

Proof. By definition of twist centre (see Prop. 2.4) the integral:

$$\int_\Omega \mathbf{g}\Big(\tau(\mathcal{M}_{\mathbf{C}^{\mathrm{SH}}},\mathbf{S},\mathbf{r}),\mathbf{t}^{\mathrm{TW}}(\mathbf{r})\Big)dA,$$

vanishes for any tangential stress field such that:

$$\mathcal{M}_{\mathbf{C}^{\mathrm{SH}}} = \mathbf{g}(\mathbf{R}(\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}}),\mathbf{S}).$$

Setting $\tau(\mathcal{M}_{\mathbf{C}^{\mathrm{SH}}},\mathbf{S},\mathbf{r}) = \mathbf{T}^{\mathrm{SH}}(\mathbf{r})\cdot\dfrac{\mathbf{S}}{A} + \mathbf{t}^{\mathrm{TW}}(\mathbf{r})\dfrac{\mathcal{M}_{\mathbf{C}^{\mathrm{SH}}}}{A}$, a direct computation of the integrand gives, for any $\mathbf{S}$:

$$\begin{aligned}
&\mathbf{g}\big(\tau(\mathcal{M}_{\mathbf{C}^{\mathrm{SH}}},\mathbf{S},\mathbf{r}),\mathbf{t}^{\mathrm{TW}}(\mathbf{r})\big)\\
&= \mathbf{g}\Big(\mathbf{T}^{\mathrm{SH}}(\mathbf{r})\cdot\frac{\mathbf{S}}{A} + \mathbf{t}^{\mathrm{TW}}(\mathbf{r})\mathbf{g}\Big(\mathbf{R}(\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}}),\frac{\mathbf{S}}{A}\Big),\mathbf{t}^{\mathrm{TW}}(\mathbf{r})\Big)\\
&= \mathbf{g}\Big(\frac{\mathbf{S}}{A},\mathbf{T}^{\mathrm{SH}}(\mathbf{r})^T\mathbf{t}^{\mathrm{TW}}(\mathbf{r}) + \mathbf{g}\Big(\mathbf{t}^{\mathrm{TW}}(\mathbf{r}),\mathbf{t}^{\mathrm{TW}}(\mathbf{r})\Big)\mathbf{R}(\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}})\Big).
\end{aligned}$$

The former equality in the statement follows from the noteworthy formula above. The latter equality is got by the formulae $\mathbf{T}^{\mathrm{SH}}(\mathbf{r}) = A\mathbf{F}^{\mathrm{SH}}(\mathbf{r})(E\mathbf{J_G})^{-1}$ and $\mu A K\mathbf{t}^{\mathrm{TW}}(\mathbf{r}) = \mathbf{f}^{\mathrm{TW}}(\mathbf{r})$.

The result of Prop. 3.1 reveals that the shear and twist centres coincide if and only if shear τ^{SH} and twist τ^{TW} tangential stress fields are orthogonal in elastic energy. An explicit expression of the relative position of twist and shear centres in terms of Prandtl stress function $\Psi\in C^1(\Omega;\mathcal{R})$ may be given. Preliminarily we recall that, according to Prandtl (1903) torsion theory, in a multiply connected cross section with n holes, see Fig. 1, being:

$$\begin{cases} \operatorname{div}\tau^{\mathrm{TW}} = \operatorname{curl}\mathbf{R}\tau^{\mathrm{TW}} = 0, & \text{in } \Omega,\\ \mathbf{g}(\tau^{\mathrm{TW}},\mathbf{n}) = \mathbf{g}(\mathbf{R}\tau^{\mathrm{TW}},\boldsymbol{t}) = 0, & \text{on } \partial\Omega, \end{cases}$$

the rotated field of twist tangential stresses admits a potential $\Psi\in C^1(\Omega;\mathcal{R})$:

$$\mathbf{R}\tau^{\mathrm{TW}}(\alpha,\mathbf{r}) = \alpha\mathbf{R}\mathbf{f}^{\mathrm{TW}}(\mathbf{r}) = \alpha\mu\nabla\Psi(\mathbf{r}), \quad \mathbf{r}\in\Omega,$$

the Prandtl stress function, solution of the Poisson-Dirichlet problem:

$$\begin{cases} \Delta_2\Psi(\mathbf{r}) = -2 \quad \text{on} \quad \Omega,\\ \Psi(\mathbf{r}) = 0 \quad \text{on} \quad \partial\Omega_0,\\ \Psi(\mathbf{r}) = k_i \quad \text{on} \quad \partial\Omega_i, \quad k_i\in\mathcal{R}, \quad i = 1,2,\ldots,n, \end{cases}$$

with $\partial\Omega_o$, exterior boundary of the cross section and $\partial\Omega_i$ boundary of the hole $i = 1,\ldots,n$. A linear system of n boundary integrability conditions provides the values of the scalar constants k_i (Sokolnikoff, 1956).

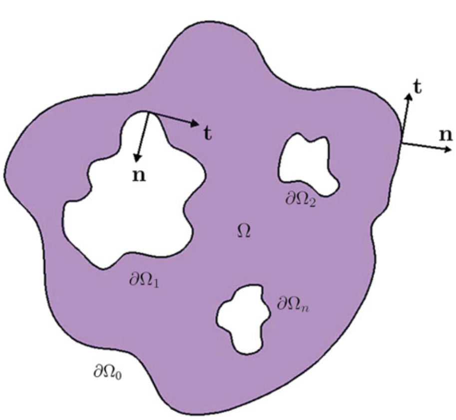

Fig. 1. Multiply connected cross section.

Proposition 3.2. *In terms of the Prandtl stress function $\Psi\in C^1(\Omega;\mathcal{R})$, the relative position of twist and shear centres $\mathbf{C}^{\mathrm{TW}}$ and $\mathbf{C}^{\mathrm{SH}}$ is given by:*

$$\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}} = \frac{\nu}{1+\nu}\mathbf{R}\,\mathbf{J}_{\mathbf{G}}^{-1}\mathbf{R}\left(\int_\Omega \Psi(\mathbf{r})\mathbf{r}dA + \sum_{k=1}^{n} k_i\int_{\Omega_i}\mathbf{r}dA\right).$$

Proof. Recalling the definition:

$$\mathbf{F}^{\mathrm{SH}}(\mathbf{r}) := \mu\Big(\frac{\nu}{2}(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r}) + \nabla\boldsymbol{\phi}^{\mathrm{SH}}(\mathbf{r})^T\Big),$$

the formula in Prop. 3.1 may be rewritten as:

$$\begin{aligned}
\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}} &= \frac{\nu}{4(1+\nu)}\mathbf{R}\mathbf{J}_{\mathbf{G}}^{-1}\mu^{-1}\int_\Omega(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{f}^{\mathrm{TW}}(\mathbf{r})\,dA\\
&\quad + \frac{1}{2(1+\nu)}\mathbf{R}\mathbf{J}_{\mathbf{G}}^{-1}\mu^{-1}\int_\Omega\nabla\boldsymbol{\phi}^{\mathrm{SH}}(\mathbf{r})\mathbf{f}^{\mathrm{TW}}(\mathbf{r})dA.
\end{aligned}$$

The former integral is evaluated as follows. Taking into account the relations:

$$\begin{cases} \mathbf{f}^{\mathrm{TW}}(\mathbf{r}) = -\mu\mathbf{R}\nabla\Psi(\mathbf{r}),\\ \operatorname{div}\big((\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r})\mathbf{R}\big) = 3\mathbf{R}\mathbf{r},\\ \operatorname{div}\big((\mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\big) = -\mathbf{R}\mathbf{r}, \end{cases}$$

by the Leibniz rule we have the formula:

$$\begin{aligned}
(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\nabla\Psi(\mathbf{r}) &= \operatorname{div}\big(\Psi(\mathbf{r})(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\big)\\
&\quad -\Psi(\mathbf{r})\operatorname{div}\big((\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\big)\\
&= \operatorname{div}\big(\Psi(\mathbf{r})(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\big)\\
&\quad - 4\Psi(\mathbf{r})\mathbf{R}\mathbf{r}.
\end{aligned}$$

Recalling the boundary conditions of the Poisson-Dirichlet problem a double application of the divergence theorem gives:

$$\begin{aligned}
&\mu^{-1}\int_\Omega(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{f}^{\mathrm{TW}}(\mathbf{r})dA\\
&= -\int_\Omega(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\nabla\Psi(\mathbf{r})dA\\
&= 4\mathbf{R}\int_\Omega\Psi(\mathbf{r})\mathbf{r}dA - \int_\Omega\operatorname{div}\big(\Psi(\mathbf{r})(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\big)dA\\
&= 4\mathbf{R}\int_\Omega\Psi(\mathbf{r})\mathbf{r}dA - \oint_{\partial\Omega}\Psi(\mathbf{r})(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\mathbf{n}(\mathbf{r})ds\\
&= 4\mathbf{R}\int_\Omega\Psi(\mathbf{r})\mathbf{r}dA - \sum_{k=1}^{n}k_i\oint_{\partial\Omega_i}(\mathbf{R}\mathbf{r}\otimes_{\mathbf{g}}\mathbf{R}\mathbf{r} - \mathbf{r}\otimes_{\mathbf{g}}\mathbf{r})\mathbf{R}\mathbf{n}(\mathbf{r})ds\\
&= 4\left(\mathbf{R}\int_\Omega\Psi(\mathbf{r})\mathbf{r}dA + \sum_{k=1}^{n}k_i\int_{\Omega_i}\mathbf{r}dA\right).
\end{aligned}$$

The latter integral in the expression of $\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}}$ vanishes. Indeed, resorting to the divergence theorem and imposing the differential and boundary conditions:

$$\begin{cases} \operatorname{div}\mathbf{f}^{\mathrm{TW}}(\mathbf{r}) = 0, \quad \mathbf{r}\in\Omega,\\ \mathbf{g}(\mathbf{f}^{\mathrm{TW}}(\mathbf{r}),\mathbf{n}(\mathbf{r})) = 0, \quad \mathbf{r}\in\partial\Omega, \end{cases}$$

Green's formula gives:

$$\int_{\Omega} \nabla \phi^{SH}(\mathbf{r})\mathbf{f}^{TW}(\mathbf{r})dA = -\int_{\Omega} \phi^{SH}(\mathbf{r})\mathrm{div}\,\mathbf{f}^{TW}(\mathbf{r})dA + \oint_{\partial\Omega} \mathbf{g}\left(\mathbf{f}^{TW}(\mathbf{r}),\mathbf{n}(\mathbf{r})\right)\phi^{SH}(\mathbf{r})ds = 0,$$

and the result follows.

From Prop. 3.2 it follows that a vanishing Poisson ratio implies coincidence of shear and twist centres.

Remark 3.1.. *Stephen and Maltbaek (1979) have contributed a formula for the relative position of twist and shear centres in terms of Prandtl stress function, which in coordinate-free form writes*:

$$\mathbf{r}_{\mathbf{C}^{TW}} - \mathbf{r}_{\mathbf{C}^{SH}} = \frac{\nu}{1+\nu}\mathbf{R}\mathbf{J}_{\mathbf{G}}^{-1}\mathbf{R}\int_{\Omega} \Psi(\mathbf{r})\mathbf{r}dA.$$

The result in Prop. 3.2 reveals that in their formula the presence of holes is not taken into account. Its validity is confined to simply connected cross sections and to tubular sections with the hole centered at the centroid of the cross section, since then the additional term vanishes.

Remark 3.2. *The formula in Prop. 3.1 reveals that the property of coincidence of shear and twist centres is equivalent to the vanishing of the mutual elastic energy between twist and shear tangential stress fields. For open thin-walled sections of constant thickness, the vanishing of this mutual elastic energy follows from the fact that the shear tangential strain field is constant through the thickness, while the twist tangential stress field is linear through the thickness with a null mean value (a butterfly-shaped diagram). Contrary to computations and conclusions reported in (Andreaus and Ruta, 1998, Section 6.2), for multi-cellular cross sections the coincidence of the two centres may be inferred by the same argument. To see this, it is expedient to split the flux of twist tangential stresses as the sum of Kirchhoff fluxes in a maximal set of independent loops (Romano, 2002b). The mutual work is then the sum of the mutual works performed, in each loop, by the relevant Kirchhoff flux, which is constant along the loop, times the shear tangential strain, which has a vanishing circulation due to the kinematic compatibility requirement on the shear warping of the section middle line.*

4. Timoshenko beam compliance and shear centre

The elastic compliance of a Timoshenko beam subject to shear and torsion is defined by the symmetric linear operator $\mathbf{D}$ relating the static pair shear force and twisting moment around the shear centre $\{\mathbf{S},\mathcal{M}_{\mathbf{C}^{SH}}\}$ and the dual pair shear sliding, torsional curvature $\{\mathbf{s}^{SH},c^{TW}\}$. The elastic compliance operator may be written as a block array:

$$\begin{vmatrix} \mathbf{s}^{SH} \\ c^{TW} \end{vmatrix} = \begin{vmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{g}\mathbf{D}_{12} & D_{22} \end{vmatrix} \cdot \begin{vmatrix} \mathbf{S} \\ \mathcal{M}_{\mathbf{C}^{SH}} \end{vmatrix},$$

where $\mathbf{g}\mathbf{D}_{12}$ is the covector associated with the vector $\mathbf{D}_{12}$ according the Euclid metric tensor $\mathbf{g}$. In Timoshenko beam model a shear centre $\mathbf{C}^{SH}_{TIMO}$ may be defined as follows.

Definition 4.1. *The Timoshenko shear centre $\mathbf{C}^{SH}_{TIMO}$ is the pole such that the vanishing of the resultant twisting moment about it implies the vanishing of the torsional curvature c^{TW}.*

Proposition 4.1. *The position vector of the Timoshenko shear centre with respect to the Saint-Venant shear centre is given by*

$$\mathbf{r}_{\mathbf{C}^{SH}_{TIMO}} = \mathbf{r}_{\mathbf{C}^{SH}} + (D_{22})^{-1}\mathbf{R}\mathbf{D}_{12}.$$

Proof. The vanishing of the resultant twisting moment about $\mathbf{C}^{SH}_{TIMO}$:

$$\mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}^{SH}} - \mathbf{R}\mathbf{r}_{\mathbf{C}^{SH}_{TIMO}},\mathbf{S}) + \mathcal{M}_{\mathbf{C}^{SH}} = 0,$$

and the condition $c^{TW} = \mathbf{g}(\mathbf{D}_{12},\mathbf{S}) + D_{22}\mathcal{M}_{\mathbf{C}^{SH}} = 0$, yield the expression $\mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}^{SH}} - \mathbf{R}\mathbf{r}_{\mathbf{C}^{SH}_{TIMO}},\mathbf{S}) = (D_{22})^{-1}\mathbf{g}(\mathbf{D}_{12},\mathbf{S})$ which, holding for any $\mathbf{S}$, gives the result.

A coordinate expression of the formula above is given in (Yu et al., 2002).

4.1. Evaluation by Saint-Venant theory

The shear sliding $\mathbf{s}^{SH}$ and the torsional curvature c^{TW} can be evaluated by an energy equivalence relation expressed by the identity:

$$\mathbf{g}\left(\mathbf{S}^*,\mathbf{s}^{SH}\right) + \mathcal{M}^*_{\mathbf{C}^{SH}}c^{TW} := \int_{\Omega} \mathbf{g}\left(\boldsymbol{\tau}\left(\mathbf{S}^*,\mathcal{M}^*_{\mathbf{C}^{SH}},\mathbf{r}\right), \frac{\boldsymbol{\tau}(\mathbf{S},\mathcal{M}_{\mathbf{C}^{SH}},\mathbf{r})}{\mu}\right)dA,$$

where $\mathbf{S}^*,\mathcal{M}^*_{\mathbf{C}^{SH}}$ are arbitrary shear force and twisting moment and the tangential stress fields are solutions of elastostatic problems according to Saint-Venant beam theory, so that:

$$\boldsymbol{\tau}(\mathbf{S},\mathcal{M}_{\mathbf{C}^{SH}},\mathbf{r}) = \mathbf{T}^{SH}(\mathbf{r})\cdot\frac{\mathbf{S}}{A} + \mathbf{t}^{TW}(\mathbf{r})\cdot\frac{\mathcal{M}_{\mathbf{C}^{SH}}}{A}.$$

The shear sliding vector $\mathbf{s}^{SH}$ is non-dimensional and the torsional curvature c^{TW} is the inverse of a length. Introducing the non-dimensional shear factor tensor:

$$\boldsymbol{\chi}^{SH} := \frac{1}{A}\int_{\Omega} \mathbf{T}^{SH}(\mathbf{r})^T\mathbf{T}^{SH}(\mathbf{r})dA,$$

and recalling the formula $\chi^{TW}A^2K = J_P$, the sliding-torsion compliance blocks are expressed by:

$$\begin{cases} \mathbf{D}_{11} = \dfrac{\boldsymbol{\chi}^{SH}}{\mu A}, \\ D_{22} = \dfrac{\chi^{TW}}{\mu J_P} = \dfrac{1}{\mu A^2 K}, \\ \mathbf{D}_{12} = \dfrac{\chi^{TW}}{\mu J_P}\mathbf{R}(\mathbf{r}_{\mathbf{C}^{SH}} - \mathbf{r}_{\mathbf{C}^{TW}}) = D_{22}\mathbf{R}(\mathbf{r}_{\mathbf{C}^{SH}} - \mathbf{r}_{\mathbf{C}^{TW}}). \end{cases}$$

A significant new outcome of this analysis is the following.

Proposition 4.2. *Timoshenko shear centre evaluated by Saint-Venant beam theory is coincident with Saint-Venant twist centre.*

Proof. The expressions of compliance blocks evaluated by Saint-Venant theory give:

$$\mathbf{r}_{\mathbf{C}^{SH}_{TIMO}} = \mathbf{r}_{\mathbf{C}^{SH}} + (D_{22})^{-1}\mathbf{R}\mathbf{D}_{12} = \mathbf{r}_{\mathbf{C}^{SH}} + (\mathbf{r}_{\mathbf{C}^{TW}} - \mathbf{r}_{\mathbf{C}^{SH}}) = \mathbf{r}_{\mathbf{C}^{TW}}.$$

Recalling the formulae $\mathcal{M}_{\mathbf{C}^{SH}} = \mu A^2 K\alpha$ and $\mathbf{S} = E\,\mathbf{J}_{\mathbf{G}}\mathbf{d}'$, the expression of the pair $(\mathbf{s}^{SH},c^{TW})$ (shear sliding, torsional curvature) in terms of the pair $(\mathbf{d}',\alpha)$ (shearing, twist) is given by:

$$\begin{cases} \mathbf{s}^{SH} = \dfrac{\boldsymbol{\chi}^{SH}}{\mu A}E\mathbf{J}_{\mathbf{G}}\cdot\mathbf{d}' + \mathbf{R}(\mathbf{r}_{\mathbf{C}^{SH}} - \mathbf{r}_{\mathbf{C}^{TW}})\cdot\alpha, \\ c^{TW} = \mathbf{g}\mathbf{R}(\mathbf{r}_{\mathbf{C}^{SH}} - \mathbf{r}_{\mathbf{C}^{TW}})\cdot\mathbf{d}' + \alpha. \end{cases}$$

This linear system uncouples and the mutual sliding-torsion compliance $\mathbf{D}_{12}$ vanishes if and only if coincidence of shear and twist centres occurs, a result not explicitly quoted in literature.

Proposition 4.3. (Principal shear factors). *The eigenvalues of the shear factors tensor $\boldsymbol{\chi}^{SH}$ are strictly greater than unity.*

Proof. Let $\mathbf{e}$ be a unit eigenvector of $\boldsymbol{\chi}^{\mathrm{SH}}$. By the mean value formula in Section 2.3 we infer that:

$$\frac{1}{A}\int_{\Omega}\mathbf{T}^{\mathrm{SH}}(\mathbf{r})\,dA = \mathbf{I} \Rightarrow \frac{1}{A}\int_{\Omega}\mathbf{g}\Big(\mathbf{T}^{\mathrm{SH}}(\mathbf{r})\mathbf{e},\mathbf{e}\Big)dA = \mathbf{g}(\mathbf{e},\mathbf{e}) = 1.$$

Then Cauchy-Schwarz inequality:

$$\left(\int_{\Omega}\mathbf{g}\Big(\mathbf{T}^{\mathrm{SH}}(\mathbf{r})\mathbf{e},\mathbf{T}^{\mathrm{SH}}(\mathbf{r})\mathbf{e}\Big)dA\right)\left(\int_{\Omega}\mathbf{g}(\mathbf{e},\mathbf{e})dA\right) \geq \left(\int_{\Omega}\mathbf{g}\Big(\mathbf{T}^{\mathrm{SH}}(\mathbf{r})\mathbf{e},\mathbf{e}\Big)dA\right)^2,$$

yields the result since equality holds if and only if the shear tangential stress field $\mathbf{T}^{\mathrm{SH}}(\mathbf{r})\mathbf{e}$ is proportional to the constant field $\mathbf{e}$.

The proof of Prop. 4.3 is based on the sole mean value property of the operator $\mathbf{T}^{\mathrm{SH}}$. The result concerning the eigenvalues of the shear factors tensor $\boldsymbol{\chi}^{\mathrm{SH}}$ is therefore valid even if approximate shear tangential stress fields are adopted for its evaluation, according to the pure equilibrium theory due to Jourawski (1856); Rankine (1858); Grashof (1878), as considered in (Romano et al., 1992).

Proposition 4.4. (Torsion factor). *The torsion factor χ^{TW} is not less than unity and is equal to unity if and only if the section is a circle or a circular annulus.*

Proof. Observing that $J_P := \int_{\Omega}\mathbf{g}(\mathbf{r},\mathbf{r})dA = \int_{\Omega}\mathbf{g}(\mathbf{Rr},\mathbf{Rr})dA$ and applying the Cauchy-Schwarz inequality, we get:

$$\chi^{\mathrm{TW}} := \frac{1}{A^2}\left(\int_{\Omega}\mathbf{g}(\mathbf{Rr},\mathbf{Rr})dA\right)\left(\int_{\Omega}\mathbf{g}\Big(\mathbf{t}^{\mathrm{TW}}(\mathbf{r}),\mathbf{t}^{\mathrm{TW}}(\mathbf{r})\Big)dA\right) \geq \left(\frac{1}{A}\int_{\Omega}\mathbf{g}\Big(\mathbf{Rr},\mathbf{t}^{\mathrm{TW}}(\mathbf{r})\Big)dA\right)^2 = 1,$$

where the last equality follows from the formula reported before Prop. 2. Equality holds only under proportionality of the fields $\mathbf{t}^{\mathrm{TW}}(\mathbf{r})$ and $\mathbf{Rr}$ and this requires that the cross section boundary is made of a circle or of two concentric circles.

Remark 4.1. *The analytical expression of the shear factors tensor $\boldsymbol{\chi}^{\mathrm{SH}}$ for thin-walled, open and closed cross sections, was given in (Romano et al., 1992). There the Cauchy-Schwarz inequality was applied to prove that the eigenvalues of $\boldsymbol{\chi}^{\mathrm{SH}}$ are strictly greater than unity, a proof reproduced also in (Romano, 2002b). This result has been reformulated in (Favata et al., 2010) with no mention of these previous contributions, notwithstanding a reference to (Romano, 2002b) in a footnote. The treatment in (Favata et al., 2010) is limited to coincidence of the principal axes of $\boldsymbol{\chi}^{\mathrm{SH}}$ and $\mathbf{J}_{\mathbf{G}}$ and adopts a needlessly involved argument to assess the results here proven in Prop. 4.3 and 4.4.*

5. Numerical computations

Let us consider a cross section Ω whose boundary is a closed polyline, with vertices $\mathbf{P}_1,\mathbf{P}_2,\ldots,\mathbf{P}_n,\mathbf{P}_{n+1} = \mathbf{P}_1$ (see Fig. 2). The position vector, with respect to a pole $\mathbf{O}$, of the i-vertex is denoted by $\mathbf{p}_i$. The divergence theorem provides the following formulae to evaluate (Romano, 2002b):

- the area:

$$A := \int_{\Omega} dA = \frac{1}{2}\oint_{\partial\Omega}\mathbf{g}(\mathbf{p},\mathbf{n}(\mathbf{p}))\,ds = \frac{1}{2}\sum_{i=1}^{n}\mathbf{g}(\mathbf{p}_i,\mathbf{Rp}_{i+1}).$$

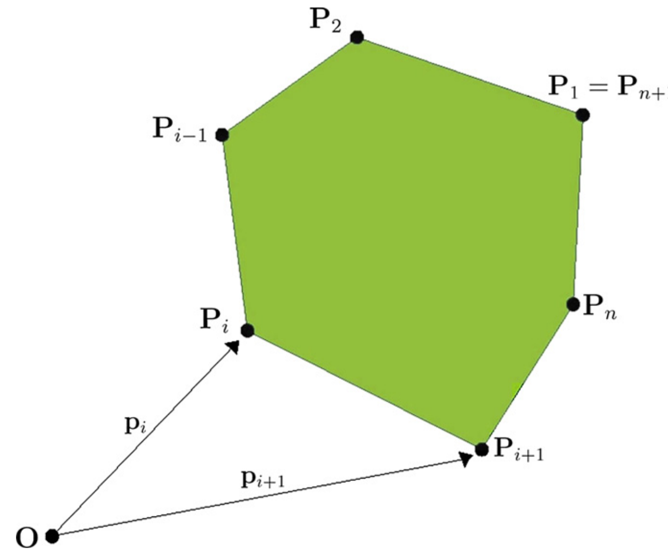

Fig. 2. Polygonal cross section.

- the first geometric moment with respect to the pole $\mathbf{O}$:

$$\mathbf{F_O} := \int_{\Omega}\mathbf{p}dA = \frac{1}{3}\oint_{\partial\Omega}(\mathbf{p}\otimes_{\mathbf{g}}\mathbf{p})\mathbf{n}(\mathbf{p})ds = \frac{1}{6}\sum_{i=1}^{n}\mathbf{g}(\mathbf{p}_i,\mathbf{Rp}_{i+1})(\mathbf{p}_i+\mathbf{p}_{i+1}).$$

- the second geometric moment with respect to the pole $\mathbf{O}$:

$$\mathbf{J_O} := \int_{\Omega}\mathbf{p}\otimes_{\mathbf{g}}\mathbf{p}dA = \frac{1}{4}\oint_{\partial\Omega}\mathbf{g}(\mathbf{p},\mathbf{n}(\mathbf{p}))\mathbf{p}\otimes_{\mathbf{g}}\mathbf{p}ds$$
$$= \frac{1}{12}\sum_{i=1}^{n}\mathbf{g}(\mathbf{p}_i,\mathbf{Rp}_{i+1})\left[\mathbf{p}_i\otimes_{\mathbf{g}}\mathbf{p}_i + \mathrm{sym}(\mathbf{p}_i\otimes_{\mathbf{g}}\mathbf{p}_{i+1}) + \mathbf{p}_{i+1}\otimes_{\mathbf{g}}\mathbf{p}_{i+1}\right].$$

A Cartesian coordinates system $\{x,y\}$ with origin in the centroid $\mathbf{G}$ will be adopted in the sequel.

5.1. L-shaped sections

Let us compute the positions of the shear and twist centres (see Prop. 2.2 and 2.4) and the shear factors tensor of the L-shaped cross section drawn in Fig. 3. Table 1 provides, in components, the outward normal vector $\mathbf{n}$ and the parametric representation of the sides.

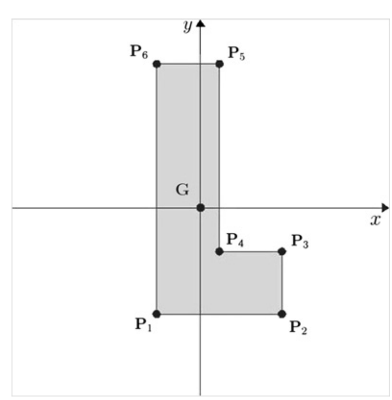

Fig. 3. L-shaped cross section.

Table 1
Geometric data of the boundary of the L-section.

Side	$\lvert\mathbf{n}\rvert$	$\lvert\mathbf{r}\rvert$
$\mathbf{P}_1\mathbf{P}_2$	$\{0, -1\}$	$\{x_{\mathbf{P}_1} \leq x \leq x_{\mathbf{P}_2}, y_{\mathbf{P}_1\mathbf{P}_2}\}$
$\mathbf{P}_2\mathbf{P}_3$	$\{1, 0\}$	$\{x_{\mathbf{P}_2\mathbf{P}_3}, y_{\mathbf{P}_2} \leq y \leq y_{\mathbf{P}_3}\}$
$\mathbf{P}_3\mathbf{P}_4$	$\{0, 1\}$	$\{x_{\mathbf{P}_4} \leq x \leq x_{\mathbf{P}_3}, y_{\mathbf{P}_3\mathbf{P}_4}\}$
$\mathbf{P}_4\mathbf{P}_5$	$\{1, 0\}$	$\{x_{\mathbf{P}_4\mathbf{P}_5}, y_{\mathbf{P}_4} \leq y \leq y_{\mathbf{P}_5}\}$
$\mathbf{P}_5\mathbf{P}_6$	$\{0, 1\}$	$\{x_{\mathbf{P}_6} \leq x \leq x_{\mathbf{P}_5}, y_{\mathbf{P}_5\mathbf{P}_6}\}$
$\mathbf{P}_6\mathbf{P}_1$	$\{-1, 0\}$	$\{x_{\mathbf{P}_6\mathbf{P}_1}, y_{\mathbf{P}_1} \leq y \leq y_{\mathbf{P}_6}\}$

5.1.1. *First computation*

- Lamé shear modulus: μ=1, Poisson ratio: ν=0,3, shearing vector: $|\mathbf{d}'| = \{1, 0\}$.

Matlab helps in solving the Poisson equation: $\Delta_2\varphi^{\mathrm{SH}}(x,y) = -2x$, with the Neumann conditions on the boundary of the L-shaped cross section:

- side $\mathbf{P}_1\mathbf{P}_2 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})(x, y_{\mathbf{P}_1\mathbf{P}_2}) = -y_{\mathbf{P}_1\mathbf{P}_2}\nu x$,
- side $\mathbf{P}_2\mathbf{P}_3 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})\ (x_{\mathbf{P}_2\mathbf{P}_3}, y) = -\dfrac{\nu(y^2 - x^2_{\mathbf{P}_2\mathbf{P}_3})}{2}$,
- side $\mathbf{P}_3\mathbf{P}_4 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})(x, y_{\mathbf{P}_3\mathbf{P}_4}) = y_{\mathbf{P}_3\mathbf{P}_4}\nu x$,
- side $\mathbf{P}_4\mathbf{P}_5 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})\ (x_{\mathbf{P}_4\mathbf{P}_5}, y) = -\dfrac{\nu(y^2 - x^2_{\mathbf{P}_4\mathbf{P}_5})}{2}$,
- side $\mathbf{P}_5\mathbf{P}_6 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})(x, y_{\mathbf{P}_5\mathbf{P}_6}) = y_{\mathbf{P}_5\mathbf{P}_6}\nu x$,
- side $\mathbf{P}_6\mathbf{P}_1 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})\ (x_{\mathbf{P}_6\mathbf{P}_1}, y) = \dfrac{\nu(y^2 - x^2_{\mathbf{P}_6\mathbf{P}_1})}{2}$.

In Fig. 4 the shear tangential stress field:

$$\begin{cases} \tau_x^{\mathrm{SH}}(x,y) = F_{11}^{\mathrm{SH}}(x,y) = \dfrac{\mu\nu}{2}\left(y^2 - x^2\right) + \mu\dfrac{\partial\varphi^{\mathrm{SH}}}{\partial x}(x,y), \\ \tau_y^{\mathrm{SH}}(x,y) = F_{21}^{\mathrm{SH}}(x,y) = -\mu\nu xy + \mu\dfrac{\partial\varphi^{\mathrm{SH}}}{\partial y}(x,y) \end{cases}$$

is plotted. The colour spectrum measures the norm of the elastic shear stress field:

$$\left\|\boldsymbol{\tau}^{\mathrm{SH}}(x,y)\right\| := \left[\left(\tau_x^{\mathrm{SH}}\right)^2(x,y) + \left(\tau_y^{\mathrm{SH}}\right)^2(x,y)\right]^{\frac{1}{2}}.$$

5.1.2. *Second computation*

- Lamé shear modulus: μ=1, Poisson ratio: ν=0,3, shearing vector: $|\mathbf{d}'| = \{0, 1\}$.

Matlab helps in solving the Poisson equation:

$$\Delta_2\varphi^{\mathrm{SH}}(x,y) = -2y,$$

with the Neumann conditions on the boundary of the L-cross section:

- side $\mathbf{P}_1\mathbf{P}_2 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})\ (x, y_{\mathbf{P}_1\mathbf{P}_2}) = \dfrac{\nu(x^2 - y^2_{\mathbf{P}_1\mathbf{P}_2})}{2}$,
- side $\mathbf{P}_2\mathbf{P}_3 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})(x_{\mathbf{P}_2\mathbf{P}_3}, y) = x_{\mathbf{P}_2\mathbf{P}_3}\nu y$,
- side $\mathbf{P}_3\mathbf{P}_4 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})\ (x, y_{\mathbf{P}_3\mathbf{P}_4}) = \dfrac{\nu(y^2_{\mathbf{P}_3\mathbf{P}_4} - x^2)}{2}$,
- side $\mathbf{P}_4\mathbf{P}_5 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})(x_{\mathbf{P}_4\mathbf{P}_5}, y) = x_{\mathbf{P}_4\mathbf{P}_5}\nu y$,
- side $\mathbf{P}_5\mathbf{P}_6 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})\ (x, y_{\mathbf{P}_5\mathbf{P}_6}) = \dfrac{\nu(y^2_{\mathbf{P}_5\mathbf{P}_6} - x^2)}{2}$,
- side $\mathbf{P}_6\mathbf{P}_1 \Rightarrow (d_{\mathbf{n}}\varphi^{\mathrm{SH}})(x_{\mathbf{P}_6\mathbf{P}_1}, y) = -x_{\mathbf{P}_6\mathbf{P}_1}\nu y$.

In Fig. 5 the shear tangential stress field:

$$\begin{cases} \tau_x^{\mathrm{SH}}(x,y) = F_{12}^{\mathrm{SH}}(x,y) = -\mu\nu xy + \mu\dfrac{\partial\varphi^{\mathrm{SH}}}{\partial x}(x,y), \\ \tau_y^{\mathrm{SH}}(x,y) = F_{22}^{\mathrm{SH}}(x,y) = \dfrac{\mu\nu}{2}\left(x^2 - y^2\right) + \mu\dfrac{\partial\varphi^{\mathrm{SH}}}{\partial y}(x,y). \end{cases}$$

is plotted. The colour spectrum measures the norm of the elastic shear tangential stress field:

$$\left\|\boldsymbol{\tau}^{\mathrm{SH}}(x,y)\right\| := \left[\left(\tau_x^{\mathrm{SH}}\right)^2(x,y) + \left(\tau_y^{\mathrm{SH}}\right)^2(x,y)\right]^{\frac{1}{2}}.$$

5.1.3. *Third computation*

- Lamé shear modulus: μ=1, Poisson ratio: ν=0,3, twist: α=1.

Matlab helps in solving the Laplace equation:

$$\Delta_2\varphi^{\mathrm{TW}}(x,y) = 0,$$

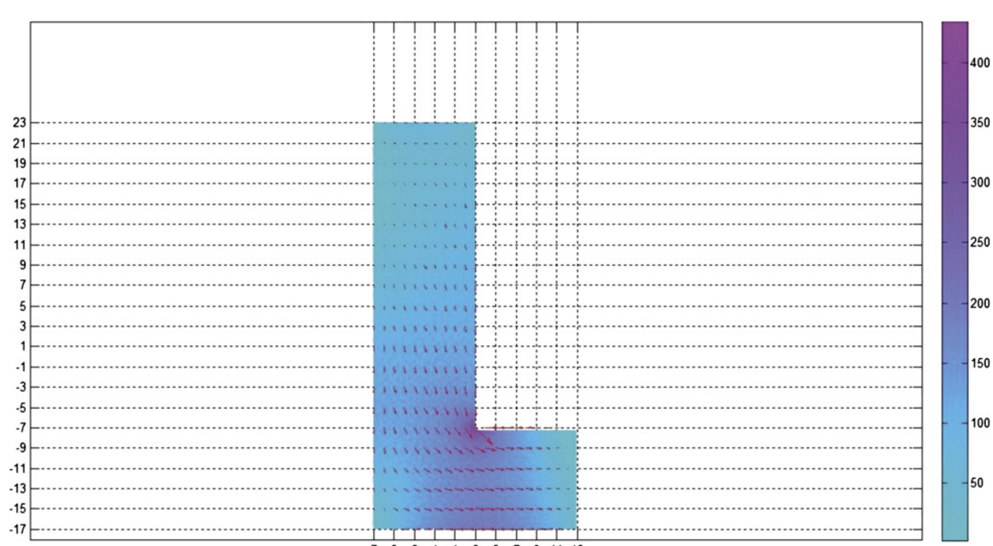

Fig. 4. Shear tangential stress field τ^{SH}; shearing $|\mathbf{d}'| = \{1; 0\}$.

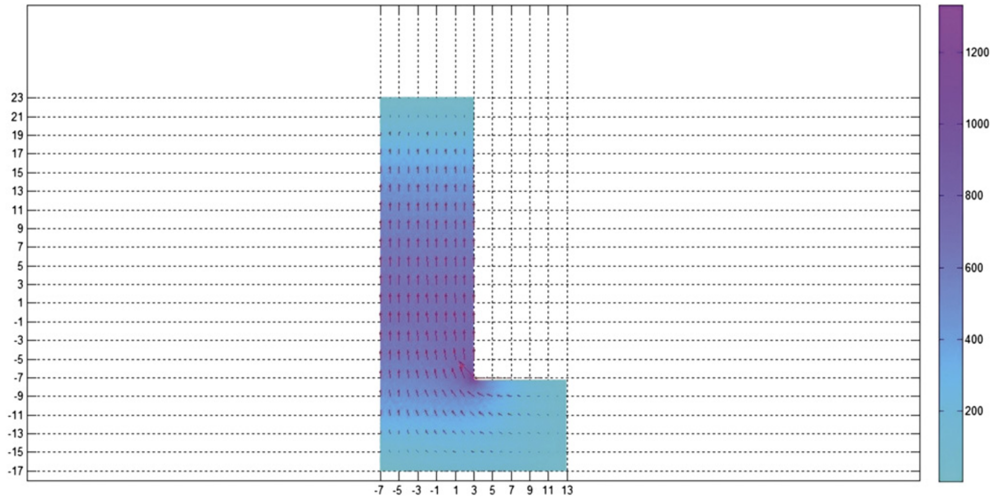

Fig. 5. Shear tangential stress field τ^{SH}; shearing $|\mathbf{d}'| = \{0;1\}$.

with the Neumann conditions on the boundary of the L-cross section:

- side $\mathbf{P}_1\mathbf{P}_2 \Rightarrow (d_\mathbf{n}\varphi^{TW})(x, y_{\mathbf{P}_1\mathbf{P}_2}) = x,$
- side $\mathbf{P}_2\mathbf{P}_3 \Rightarrow (d_\mathbf{n}\varphi^{TW})(x_{\mathbf{P}_2\mathbf{P}_3}, y) = y,$
- side $\mathbf{P}_3\mathbf{P}_4 \Rightarrow (d_\mathbf{n}\varphi^{TW})(x, y_{\mathbf{P}_3\mathbf{P}_4}) = -x,$
- side $\mathbf{P}_4\mathbf{P}_5 \Rightarrow (d_\mathbf{n}\varphi^{TW})(x_{\mathbf{P}_4\mathbf{P}_5}, y) = y,$
- side $\mathbf{P}_5\mathbf{P}_6 \Rightarrow (d_\mathbf{n}\varphi^{TW})(x, y_{\mathbf{P}_5\mathbf{P}_6}) = -x,$
- side $\mathbf{P}_6\mathbf{P}_1 \Rightarrow (d_\mathbf{n}\varphi^{TW})(x_{\mathbf{P}_6\mathbf{P}_1}, y) = -y.$

In Fig. 6 the twist tangential stress field:

$$\begin{cases} \tau_x^{TW}(x,y) = f_x^{TW}(x,y) = \mu\left(-y + \dfrac{\partial\varphi^{TW}}{\partial x}(x,y)\right), \\ \tau_y^{TW}(x,y) = f_y^{TW}(x,y) = \mu\left(x + \dfrac{\partial\varphi^{TW}}{\partial y}(x,y)\right). \end{cases}$$

The colour spectrum measures the norm of the elastic twist tangential stress field:

$$\left\|\boldsymbol{\tau}^{TW}(x,y)\right\| := \left[\left(\tau_x^{TW}\right)^2(x,y) + \left(\tau_y^{TW}\right)^2(x,y)\right]^{\frac{1}{2}}.$$

In Figs. 7–11, the shear centre $\mathbf{C}^{SH}$, the principal directions of the bending stiffness tensor $E\,\mathbf{J_G}$ and of the shear factors tensor χ^{SH}, for different values of the thickness ratio, defined as the ratio between the thickness and the length of the middle-line of the L-shaped section, are drawn. In Table 2, the coordinates of the shear and twist centres of the L-sections, drawn in Figs. 7–11, are reported. In these figures the twist centre $\mathbf{C}^{TW}$ is not drawn since its coordinates are substantially coincident with the ones of the shear centre $\mathbf{C}^{SH}$, in accordance with the analytical evaluation for thin-walled sections, reported in Section 6.

6. Discussion

- The issue of the relative position of twist and shear centres, dealt with here in Section 3, was discussed by Stephen and Maltbaek

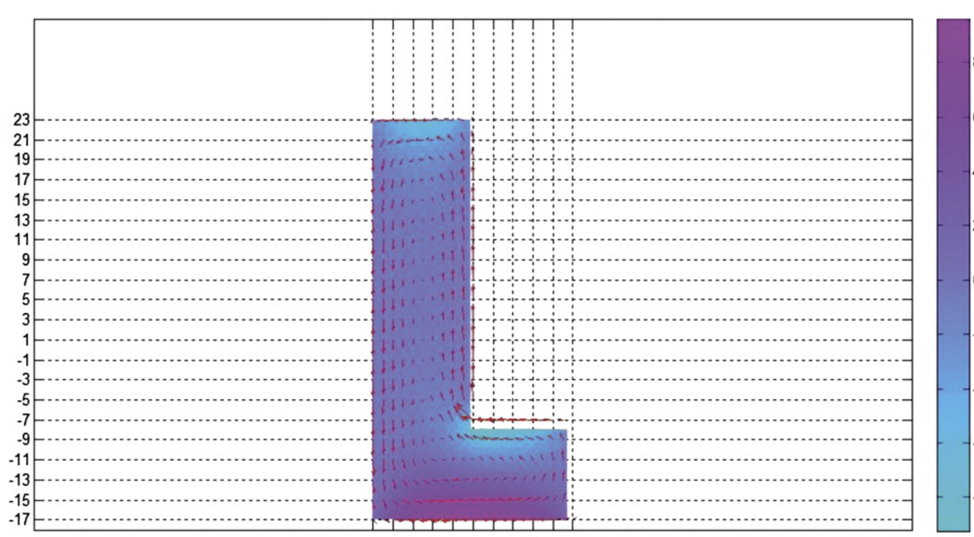

Fig. 6. Torsion tangential stress field τ^{TW}; twist α=1.

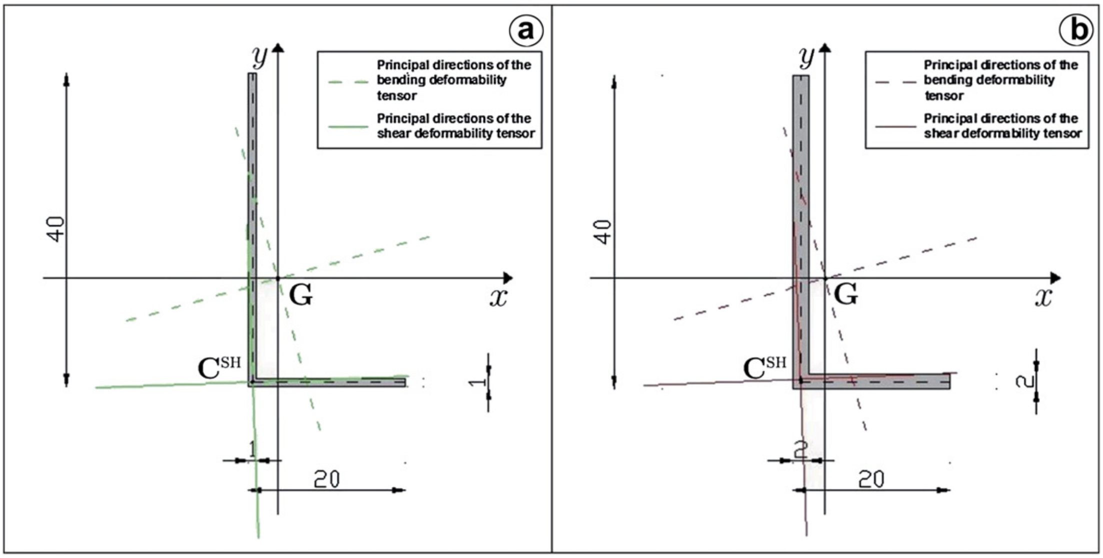

Fig. 7. a: Thickness factor 1/59, b: thickness factor 2/58.

(1979) who, following a methodology introduced by Novozhilov (1961), provided an expression of the relative position of twist and shear centres in terms of Prandtl stress function Ψ. Anyway this methodology has a limited validity when applied to multiply connected cross sections. Indeed, following the steps which led to their formula for the relative position of twist and shear centre, it turns out that an essential role is played by the vanishing of the integrals $\oint_{\partial\Omega}\Psi(\mathbf{r})d_{\mathbf{n}}\psi_1(\mathbf{r})ds$ and $\oint_{\partial\Omega}\Psi(\mathbf{r})\,d_{\mathbf{n}}\psi_2(\mathbf{r})ds$ with ψ_1,ψ_2 harmonic functions conjugate to the harmonic functions ϕ_1, ϕ_2, respectively fulfilling the Neumann boundary conditions $d_{\mathbf{n}}\phi_1(\mathbf{r}) = [(1+\nu)x^2 - \nu y^2]n_x$ and $d_{\mathbf{n}}\phi_2(\mathbf{r}) = [(1+\nu)y^2 - \nu x^2]n_y$ with x,y inertia principal coordinates. The vanishing of these integrals is in fact assured by the constancy of the function Ψ on each connected boundary, since by conjugacy:

$$\oint_{\partial\Omega}\nabla\psi_1(\mathbf{r})\cdot\mathbf{n}ds = \oint_{\partial\Omega}\nabla\phi_1(\mathbf{r})\cdot d_s\mathbf{r}ds = \oint_{\partial\Omega}d_s\phi_1(s)ds = 0.$$

What however remains to be checked is the existence of the conjugate harmonic functions. For tubular sections the existence condition for the conjugate ψ_1 to ϕ_1 writes:

$$\oint_{\partial\Omega_{\text{hole}}} d_{\mathbf{n}}\phi_1(\mathbf{r})ds = \int_{\Omega_{\text{hole}}} d_x\Big[(1+\nu)x^2 - \nu y^2\Big]dA = 2(1+\nu)\int_{\Omega_{\text{hole}}} xdA = 0.$$

Imposing the analogous condition for ψ_2, we see that the centroid of the hole must coincide with the centroid of the section. The non-trivial existence conditions for the conjugate functions was overlooked in (Stephen and Maltbaek, 1979) where an example of application of the formula to closed thin-walled sections is reported.

- To estimate the correction due to the curvature of the middle-line on the evaluation of the distance between shear and twist

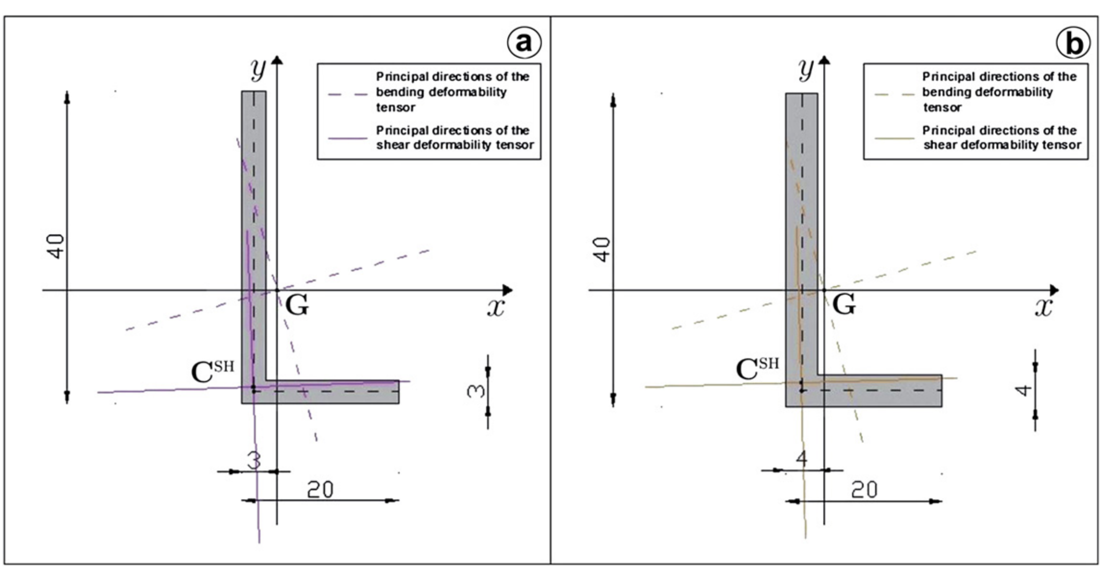

Fig. 8. a: Thickness factor 3/57, b: thickness factor 4/56.

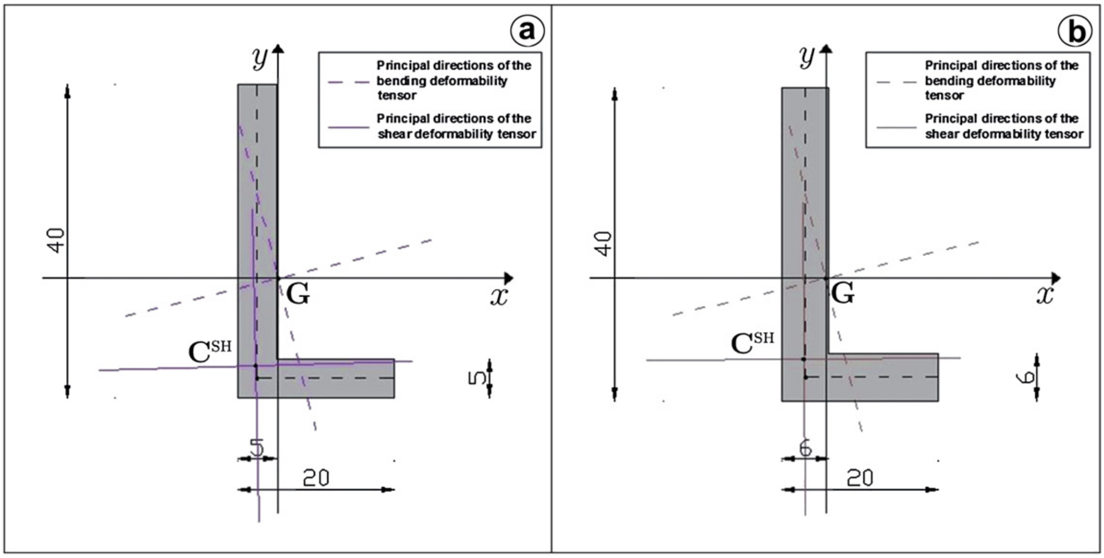

Fig. 9. a: Thickness factor 5/55, b: thickness factor 6/54

centres, let us consider a thin-walled open circular section, as depicted in Fig. 12. By symmetry, the vertical component of the difference $\mathbf{r}_{\mathbf{C}^{TW}} - \mathbf{r}_{\mathbf{C}^{SH}}$ vanishes, while the horizontal component is given by:

$$\mathbf{g}(\mathbf{r}_{\mathbf{C}^{TW}} - \mathbf{r}_{\mathbf{C}^{SH}}, \mathbf{h}) \;=\; \frac{J_P}{\chi^{TW} A^2} \int_{\Omega} \mathbf{g}\Big(\mathbf{t}^{TW}(\mathbf{r}), \mathbf{T}^{SH}(\mathbf{r})\mathbf{R}^T\mathbf{h}\Big)\, dA,$$

being $\mathbf{h}$ the horizontal versor from the centre towards the opening.

Due to the curvature of the middle-line, the length, normalized to unity at the middle-line, assumes the value $1 + n/R$, along each chord at the point of outward positive abscissa n from the middle-line. It may then be considered as sum of a constant unit diagram and of a butterfly-shaped diagram with maximum value $\delta/(2R)$. Along each chord the twist tangential stress $\mathbf{t}^{TW}(\mathbf{r})$ is normal to the chord and has a butterfly-shaped diagram with maximum value $3/\delta$. Then, to evaluate the factor:

$$\frac{J_P}{\chi^{TW} A^2} \;=\; \left(\int_{\Omega} \mathbf{g}\Big(\mathbf{t}^{TW}(\mathbf{r}), \mathbf{t}^{TW}(\mathbf{r})\Big)\, dA \right)^{-1},$$

we observe that the diagram of $\mathbf{g}(\mathbf{t}^{TW}(\mathbf{r}), \mathbf{t}^{TW}(\mathbf{r}))$ on each chord is a parabola, symmetric about the middle-line, vanishing there and with maxima located at the extreme points and equal to $9/\delta^2$. Its area is then equal to $(9/\delta^2)\delta/3 \;=\; 3/\delta$. The length of the middle-line is $\ell \;=\; 2\pi R$ so that the integral at the r.h.s. is given by $6\pi R/\delta$, because, by symmetry, the contribution of the butterfly-shaped length diagram n/R vanishes. On the other hand, the diagram of $\mathbf{T}^{SH}(\mathbf{r})\mathbf{R}^T\mathbf{h}$ along any chord is constant and its integral over the section is equal to the area $2\pi R\delta$ of the cross section. The product of the butterfly-shaped diagram of $\mathbf{t}^{TW}(\mathbf{r})$ with maximum value $3/\delta$ and of the butterfly-shaped length diagram with maximum value $\delta/(2R)$ yields a symmetric parabola whose area is given by $(1/3)3/(2R) \;=\; 1/(2R)$. The integral $\int_{\Omega}\mathbf{g}(\mathbf{t}^{TW}(\mathbf{r}), \mathbf{T}^{SH}(\mathbf{r})\mathbf{R}^T\mathbf{h})dA$ is thus evaluated to be equal to $(2\pi R\delta)/(2R) \;=\; \pi\delta$. At last we get that

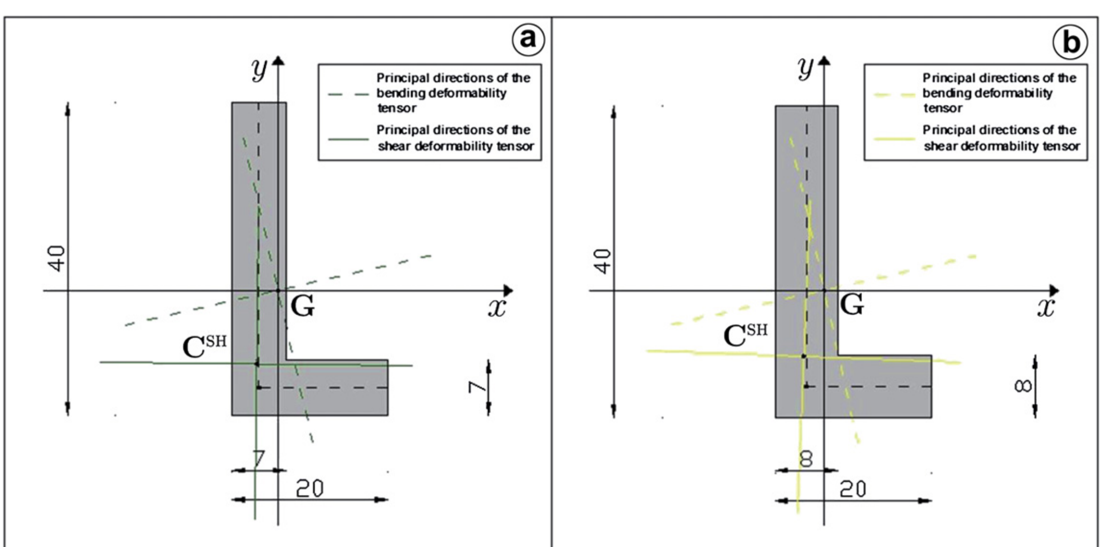

Fig. 10. a: Thickness factor 7/53, b: thickness factor 8/52.

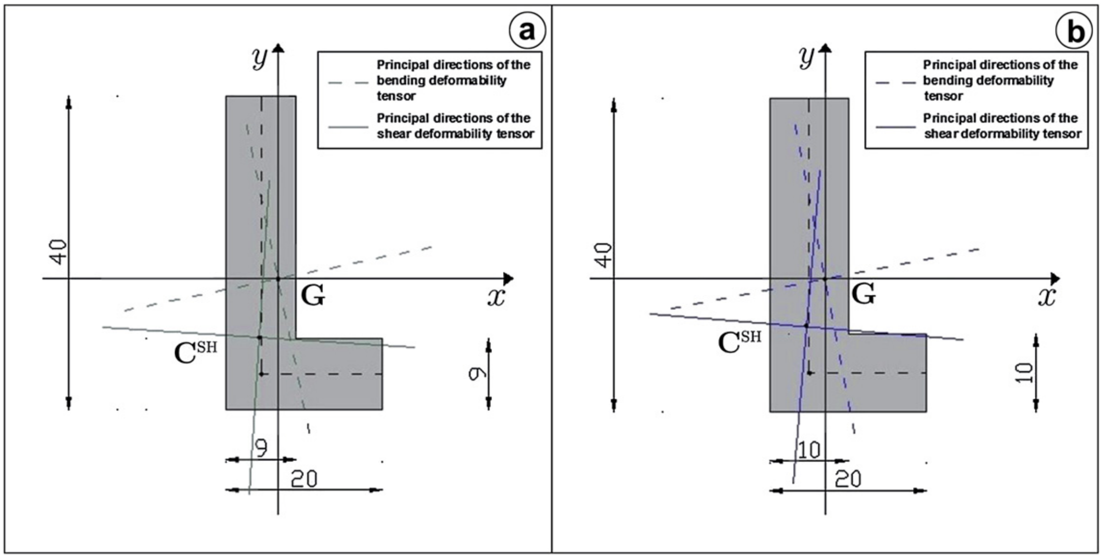

Fig. 11. a: Shape factor 9/51, b: shape factor 10/50.

$\mathbf{g}(\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}} - \mathbf{r}_{\mathbf{C}^{\mathrm{SH}}}, \mathbf{h})$ is equal to $\pi\delta\delta/(6\pi R) = \delta^2/(6R)$. The torsion factor may be easily computed by the knowledge of the values $J_P = 2\pi R^3\delta$ and $A = 2\pi R\delta$ so that $\chi^{\mathrm{TW}} = 1/2\pi(R/\delta)6\pi(R/\delta) = 3(R/\delta)^2$. For typical thickness to radius ratios of about $1/10$, the distance between the two centres is estimated to be as small as $\delta/60$ and the torsion factor is $\chi^{\mathrm{TW}} = 300$. For a circular tube the torsion factor is unitary, $\chi^{\mathrm{TW}} = 1$.

- The mutual work performed in a Saint-Venant beam by a tangential stress field interacting with a torsion elastic strain field, is evaluated by integrating along the beam axis expression of the virtual work per unit length provided in Lemma 2.2. Being: $\sigma'(\mathbf{d}', \mathbf{r})\phi^{\mathrm{TW}}(\mathbf{r}) = (\sigma(\mathbf{d}', \mathbf{r})\phi^{\mathrm{TW}}(\mathbf{r}))'$, the integration yields:

$$U_{12} := \int_0^L u_{12} dz = L\alpha_2 \mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}}, \mathbf{S}_1) + \alpha_2 \left(\int_0^L \sigma(\mathbf{d}'_1, \mathbf{r})\phi^{\mathrm{TW}}(\mathbf{r}) dA \right)' dz$$
$$= L\alpha_2 \mathbf{g}(\mathbf{R}\mathbf{r}_{\mathbf{C}}, \mathbf{S}_1) + \alpha_2 \int_\Omega (\sigma(\mathbf{d}'_1, \mathbf{r}, L) - \sigma(\mathbf{d}'_1, \mathbf{r}, 0))\phi^{\mathrm{TW}}(\mathbf{r}) dA,$$

where $\mathbf{C}$ is a point of the cross section where the tangential stress field $\tau(\mathbf{d}'_1, \alpha_1, \mathbf{r})$ has vanishing twisting moment. Special cases, of the expression for U_{12} have been reported in literature. For instance, formula 5.7 in (Andreaus and Ruta, 1998):

$$U_{12}^{A-R} = W(F, T) = \mathbf{q} \cdot \mathbf{v}^T \Big|_{\mathbf{y}=\mathbf{d}} + l \int_{\mathrm{D}} \sigma w^T,$$

and formula 3.2 in (Ecsedi, 2000):

$$U_{12}^{E} = \theta \left((x_S Q - y_S P)L - \int_A \phi(x, y)\sigma''_z(x, y, 0)\, dA \right),$$

when written in our notations, respectively become:

$$U_{12}^{A-R} = L\alpha_2 \mathbf{g}(\mathbf{S}_1, \mathbf{R}\mathbf{r}_{\mathbf{C}}) + L\alpha_2 \int_\Omega \sigma(\mathbf{d}'_1, \mathbf{r}, L)\phi^{\mathrm{TW}}(\mathbf{r}) dA,$$
$$U_{12}^{E} = L\alpha_2 \mathbf{g}(\mathbf{S}_1, \mathbf{R}\mathbf{r}_{\mathbf{C}}) - \alpha_2 \int_\Omega \sigma(\mathbf{d}'_1, \mathbf{r}, 0)\phi^{\mathrm{TW}}(\mathbf{r}) dA.$$

The former refers to a beam with a vanishing bending moment at $z = 0$, so that $\sigma(\mathbf{d}', \mathbf{r}, 0) = 0$ in Ω, with a misprinted factor L. The latter refers instead to a beam whose bending moment vanishes at $z = L$, so that $\sigma(\mathbf{d}', \mathbf{r}, L) = 0$ in Ω.

- In a recent paper by Dong et al. (2010) it is sustained that principal directions of the shear factors tensor and of the bending stiffness should coincide and that "*when two forces are applied simultaneously to a cross section, it leads to an inconsistency. Only one force should be used at a time, and two sets of calculations are needed to establish the shear correction factors for a non-symmetrical cross-section*". Both claims are in contrast with the theoretical and numerical evidences adduced in (Romano et al., 1992) for thin-walled sections, and in (Schramm et al., 1994) and in the present paper for arbitrary sections.

Table 2
Positions of shear and twist centres vs. thickness ratio in L-section.

Thickness ratio	$\lvert\mathbf{r}_{\mathbf{C}^{\mathrm{SH}}}\rvert$	$\lvert\mathbf{r}_{\mathbf{C}^{\mathrm{TW}}}\rvert$
1/59	{−3.23, −13.13}	{−3.23, −13.19}
2/58	{−3.13, −12.84}	{−3.14, −12.86}
3/57	{−3.05, −12.40}	{−3.05, −12.39}
4/56	{−2.97, −11.82}	{−2.98, −11.81}
5/55	{−2.90, −11.11}	{−2.90, −11.09}
6/54	{−2.82, −10.30}	{−2.82, −10.26}
7/53	{−2.73, −9.40}	{−2.73, −9.35}
8/52	{−2.61, −8.45}	{−2.62, −8.37}
9/51	{−2.46, −7.48}	{−2.47, −7.38}
10/50	{−2.35, −6.07}	{−2.36, −5.94}

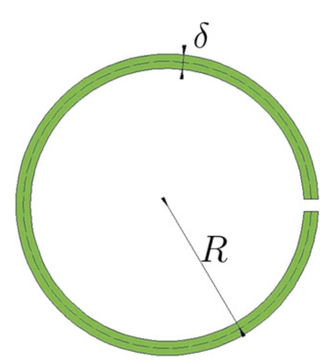

Fig. 12. Thin-walled open circular cross section.

Moreover they are inconsistent with the assumed linearity of the elastostatic problem.

7. Conclusions

The paper is devoted to a presentation of Saint-Venant beam theory under shear and torsion, with special attention to the notions of shear and twist centres and on the evaluation of the sliding-torsional elastic compliance to be adopted in Timoshenko beam theory. Previous contributions in literature are discussed and new evidences and corrections are exposed. The treatment adopts an intrinsic (coordinate-free) formalism which provides the suitable framework for theoretical investigations and for a direct implementation of numerical computation methods. The relative location of shear and twist centres, and the expression in terms of Prandtl stress function, are formulated in full generality, extending the validity of a previous contribution by Stephen and Maltbaek (1979) to multiply connected cross sections. The investigation of the sliding-torsional elastic compliance of Timoshenko beam theory and the numerical computations performed on L-shaped cross sections of various thickness, show that principal directions of shear and bending are in general different. The sliding-torsional coupling term is found to be linearly dependent on the relative position of shear and twist centres and the Timoshenko shear centre is shown to be coincident with the Saint-Venant twist centre, when evaluated by energy equivalence with Saint-Venant beam theory, results not quoted before in literature. The theory exposed in the paper can be extended to include beams whose elastic properties are orthotropic and non-homogeneous fibre by fibre. This extension, which is needed for in the elastic analysis of fibre reinforced beams, will be developed in detail in a forthcoming contribution. The Timoshenko-like modelling of initially curved and twisted composite beams developed in (Yu et al., 2002; Hodges, 2006) and successfully tested in numerical experiments, provides valuable benchmarks.

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