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Covariant hypo-elasticity

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ABSTRACT

The theory of constitutive behavior, with explicit reference to hypo-elastic materials, is addressed with a geometric approach which, following physical arguments, leads to a covariant formulation. The essential role played by a careful distinction between spatial vectors, material-based spatial vectors and material vectors is emphasized. Definite answers to debated issues, such as the proper definition of stress rate, the formulation of integrability conditions, the fulfilment of material frame-indifference, and the task of evaluating the stress state evolution, are given. Simple shear and extension of a specimen of a hypo-elastic material are investigated as applications of the theory. Improper statements and unsound physical responses of hypo-elastic materials are overcome by the covariant theory, thus restoring the proper role to this constitutive model, for both theoretical and computational purposes.

1. Introduction

The notion of hypo-elastic material was introduced by Truesdell (1955) and subsequently thoroughly investigated by himself and other renowned researchers, as quoted in Truesdell and Noll (1965). The definition of stress rate originally proposed by Truesdell (1955) takes into account the rate of volume variation by a modification of the convective rate introduced earlier by Oldroyd (1950a,b) and investigated, in connection with plasticity, by A.E. Green (1956). Notably, Prager (1951), Noll (1955) and Thomas (1955) proposed instead for the stress tensor the adoption of the co-rotational rate (Zaremba, 1903; Jaumann, 1911). Several stress rates were investigated by Sedov (1960) and a further objective rate was proposed a little later by Green and Naghdi (1965). The question concerning the choice of a suitable rate for the stress tensor was in fact the overwhelming feature of subsequent contributions to hypo-elasticity. In the presentation given by Truesdell and Noll (1965), co-rotational and convective rates were considered as suitable and equally scored candidates. As a consequence the constitutive law resulted to be determinate only to within an additive isotropic bilinear term in $\mathbf{T}$ (Cauchy mixed stress) and $\mathbf{D}$ (mixed stretching). This indeterminacy was imputed to be responsible of entailing implications in the analytical formulation of the hypo-elastic constitutive law.

The issue was further investigated by Casey and Naghdi (1988) in the context of finite rigid plasticity. Many other proposal of stress rates have been set forth in subsequent years and in recent papers by Bruhns et al. (1999) and Liu (2004) on finite strain elastic–plastic models, as many as ten proposals of objective stress rates have been listed.

The problematics concerning the formulation of a hypo-elastic material behavior are addressed here in a general framework designed to provide a rational formulation of constitutive relations in continuum mechanics.

The turning point is that the question about the choice of an objective stress rate is replaced by the more basic one concerning the criterion to be adopted in comparing tangent vector fields at two configurations of the body in space. A physical–geometrical reasoning, based on a careful distinction between spatial vectors, material-based spatial vectors and material vectors, leads in Section 2 to the statement of the *covariance paradigm* according to which material tangent vectors are transformed and compared by push along the material displacement. The statement of the *covariance paradigm* extends naturally to material tensors at displaced configurations, the push of tensors being defined by invariance as detailed in Section 3. It follows that the unique, natural definition of material tensor rate is the *convective time-derivative* (or Lie time-derivative (Lie and Engel, 1888)) along the motion. As a consequence, any indeterminacy about the stress-rate formulation is overcome. The convective time-derivative of the stress is here called the *stressing*.

An issue deserving a careful treatment concerns the *spatial description* of a material tensor field, which is a material-valued tensor field defined on the body's trajectory in space. When a consistent geometric framework is set up, it comes out that, as

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stated by the *covariance paradigm*, the spatial description cannot be acted upon by a covariant time-differentiation along the trajectory. Moreover, even if this inherent difficulty is ignored, covariant time-derivatives cannot be expressed as sum of partial time and spatial derivatives because the regularity required by partial time-differentiation, at a fixed point in space, is in general lacking. The needed regularity properties are conceivable in special problems of hydrodynamics, to which Euler's original treatment was devoted, but are usually not fulfilled in solid mechanics. The issue will be discussed in Section 6.1 with reference to hypo-elasticity where it plays a decisive role. Most improper statements in the formulation of constitutive relations have been induced from treating the spatial description of a material tensor field as it were a time-dependent spatial tensor field. This confusion is made possible by the shadowing coincidence of dimensionality of material and spatial tangent spaces for three-dimensional bodies.

The covariance paradigm provides a general framework for the discussion of *time-independence*, *invariance* and *frame-indifference* of material tensors, as illustrated in Section 5. Time-independence is defined as *variance by push under material displacements*, invariance is defined as *variance by push under relative motions*, and *frame-indifference* is invariance under *isometric* relative motions. The covariance paradigm, and the physical assumption that the stress is frame-indifferent, assure that hypo-elastic constitutive relations fulfill *material frame indifference* (M.F.I.), if the constitutive operators, as seen by two Euclid observers, are related by push along the isometric relative motion, a result assessed in Section 6. This is a methodological change with respect to the statement of *form-invariance* found in literature, according to which equality between constitutive operators, with different domains and codomains, should be involved.

The integrability conditions to be imposed on a hypo-elastic constitutive operator to ensure the existence of strain-valued or scalar-valued stress potentials, are investigated in Section 7. The issue was first addressed by Bernstein (1960) and quoted by Truesdell and Noll (1965), with an analysis which has been a guideline for subsequent investigations. The formulation of the hypo-elastic law there adopted did not take however properly into account the geometry of the problem, as discussed in Section 6.1. The topic is here investigated *ex novo* by relying on standard results of potential theory in linear spaces. A two-steps procedure is envisaged to show that two symmetry conditions, on the derivative of the hypo-elastic operator and on the operator itself, ensure respectively Cauchy and Green integrability of the hypo-elastic constitutive operator. Explicit integral formulas for the relevant potentials are provided. The integrability conditions and the proper definition of *time-independence* of the constitutive operator, allow to recover an elastic or hyper-elastic behavior in a fixed natural reference configuration. The integrability property is shown to be independent of the choice of a reference configuration and of alterations of the involved dual tensors, thus leading to a consistent theory of material behavior. The simplest hypo-elastic law, which has been widely adopted for computational purposes, is analyzed in Section 8 and found to be integrable to a hyper-elastic law, contrary to opposite conclusions drawn in literature on the basis of incorrect integrability conditions. The theory is illustrated by benchmark examples of a hypo-elastic specimen subject to simple shear and extension.

2. Physical-geometrical premise

The physical reasoning, which leads to the formulation of a geometrically covariant theory of material behavior in continuum mechanics, is carried out by a careful investigation of the implications consequent to the basic distinction between material particles and spatial points, as detailed hereafter.

The *ambient space* of classical continuum mechanics is the Euclid 3-D space $\mathcal{S}$ in which motions of bodies take place. The theory developed in this paper is however applicable without changes to bodies whose placements are manifolds immersed in a Riemann ambient space, endowed with the Levi-Civita connection. Illustrations of the relevant geometrical issues may be found in Abraham et al. (1988), Romano (2007). In the sequel, when needed, maps with co-domain in the ambient space and spatial-valued tensor fields will be denoted by the apex $^{\mathrm{SP}}$, whose dropping specifies that the spatial-valued map is co-restricted to its image.

The *material body* $\mathcal{B}$ is a set of labels, the *particles* $\mathrm{p} \in \mathcal{B}$, which become available to physical experience in their *motion* $\boldsymbol{\varphi}^{\mathrm{SP}} : \mathcal{B} \times I \mapsto \mathcal{S}$ through the ambient space during an open observation time interval I.

1. *Spatial vectors* are tangent to parametrized lines drawn in the ambient space. The comparison between spatial vectors based at different points along a curve is made by a *parallel transport* along that curve. The resulting differentiation leads to the notion of *parallel* (or *covariant*) parametric derivative along a curve, which depends linearly on the parametrization velocity at the relevant point, as described in Section 3. In the Euclid space, the translation defines a parallel transport which depends only upon the start and end points and not on the particular curve joining them. The corresponding derivative is the usual directional derivative.
2. *Material-based spatial vectors* are tangent to time-parametrized curves tracked by material particles in motion in the ambient space. The comparison between material-based spatial vectors at different times is made by *parallel transport* along the motion. The resulting time-differentiation leads to the notion of *parallel* (or *covariant*) time derivative along the motion, which depends linearly on the motion velocity at the relevant time, as described in Section 3. The velocity of a particle in motion in the ambient space provides a paradigmatic example of a material-based spatial vector and its parallel time-derivative is the acceleration.
3. *Material vectors* are tangent to lines drawn in the placement of a body in a given *material configuration* $\boldsymbol{\varphi}_t : \mathcal{B} \mapsto \boldsymbol{\Omega}_t$, with $\boldsymbol{\Omega}_t := \boldsymbol{\varphi}_t(\mathcal{B})$. A material line in a source placement $\boldsymbol{\Omega}_t$ is transformed to a material line in the target placement $\boldsymbol{\Omega}_t$ by the relative *material displacement* $\boldsymbol{\varphi}_{\tau,t} := \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \in \mathrm{C}^1(\boldsymbol{\Omega}_t; \boldsymbol{\Omega}_\tau)$, which is a *diffeomorphism*, i.e. an invertible map continuously differentiable with the inverse. Accordingly, material tangent vectors at the positions of the same material particle in the two material configurations are linearly related by means of the *tangent displacement*, called *transplacement gradient* in Truesdell (1991). In differential geometry this is named a transformation by *push* and extends to any material tensor, as illustrated in Section 3. Accordingly, the time-rate of variation of a material vector field is described by the time-derivative of its pull-back, along the motion, to the actual configuration. This is indeed the definition of *convective time-derivative*.

Material tangent vectors can be merged into the ambient space, but still cannot be acted upon by parallel transport along the motion. This observation assumes a clear evidence if reference is made to lower dimensional continua, such as wires or membranes moving in the three-dimensional Euclid space, as exemplified in Figs. 1 and 2 where the red arrows are material tangent vectors while the black ones denote parallel transported vectors that may not be tangent to the body's placement. Indeed parallel transported

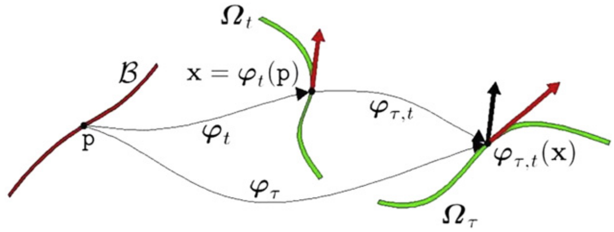

Fig. 1. Push of a 1D material vector and parallel transport of its spatial immersion.

vectors will not retain in general their original nature of *spatial immersions of material vectors*.

Once notions and reasonings developed above have been firmly established, the criterion for the comparison between material tangent vectors at displaced configurations of a body, can be stated as follows.

Proposition 2.1 (Covariance paradigm). Material tangent vectors, based at positions of the same material particle in displaced configurations, are compared by push along the material displacement.

Basic geometric objects in continuum mechanics are second-order *spatial tensors* and *material tensors* which are bilinear real-valued maps, the former on pairs of spatial tangent vectors, the latter on pairs of material tangent vectors or covectors.

According to the covariance paradigm, Proposition 2.1, material tensors at different locations of the same particle can be compared only after they have been brought to have the same domain. This is accomplished by pushing the material vectors, tangent at the location of one of them, to the location of the other, along the displacement. The resulting transformation is the definition of pull-back of a material tensor, as formalized in Section 3 and its time derivative is the *convective time-derivative* of the material tensor.

A *tensor field* assigns to each point in its domain a tensor based at that point.

The basic *spatial tensor field* in continuum mechanics is the *metric tensor field* $\mathbf{g}$ which endows the spatial points $\mathbf{x}^{\mathrm{SP}} \in \mathcal{S}$ with symmetric positive definite tensors $\mathbf{g}_{\mathbf{x}^{\mathrm{SP}}} \in \mathrm{C}^1(\mathbb{T}_{\mathbf{x}^{\mathrm{SP}}}\mathcal{S}, \mathbb{T}_{\mathbf{x}^{\mathrm{SP}}}\mathcal{S}; \mathcal{R}) = \mathrm{C}^1(\mathbb{T}_{\mathbf{x}^{\mathrm{SP}}}\mathcal{S}, \mathbb{T}^*_{\mathbf{x}^{\mathrm{SP}}}\mathcal{S})$, where $\mathbf{T}_{\mathbf{x}^{\mathrm{SP}}}\mathcal{S}$ is the tangent space at $\mathbf{x}^{\mathrm{SP}} \in \mathcal{S}$ and $\mathbb{T}^*_{\mathbf{x}^{\mathrm{SP}}}\mathcal{S}$ the dual cotangent space. The definition of a metric tensor field makes the ambient space a Riemann manifold.

A *material tensor field* assigns to the position of each particle, at an instant along the motion, a material tensor based at that position.

The *material metric tensor* is defined at each configuration of a body by applying the spatial metric tensor to the spatial immersion of material tangent vectors, as detailed in Section 3. Material metric tensor fields at a two time instants along the motion are compared by pull-back, according to the material displacement map. Their difference (in fact one half of) provides the most natural definition of the relevant *stretch* or Cauchy-Green strain tensor field. This is a material field which measures the variation in the metric properties of the body when displaced along the motion.

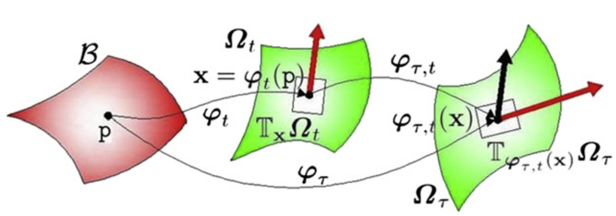

Fig. 2. Push of a 2D material vector and parallel transport of its spatial immersion.

A referential configuration endowed with the pull-back of the material metric tensor at a given time, provides a paradigmatic example of Riemann manifold in continuum mechanics.

Material tensors are the ones entering in constitutive relations describing the material behavior, such being stretch, stress, temperature, internal energy and so on. The convective time-derivatives of the material metric tensor and of the material stress tensor are respectively called *stretching* and *stressing*, as defined in Section 4.

According to the *covariance paradigm*, the comparison, between constitutive laws of a material body at any pair of time instants along a motion, or between the constitutive laws of a body, as detected by two observers in relative motion, must be performed by push, respectively according to the material displacement map or according to the relative motion between the observers.

A pushed constitutive law is defined by requiring that material fields, fulfilling the constitutive relation in a source configuration, must be still related, by the pushed law, when they are transformed by push to the target configuration, as described in Section 6 with reference to hypo-elasticity. Theoretical notions and properties, required to develop a treatment of the hypo-elastic model in the physical–geometrical framework illustrated above, are provided in the next sections.

3. Push and time-derivatives

At a point $\mathbf{x} \in \Omega_t$, the linear space of 0th order material tensors (scalars) is denoted by $\mathrm{FUN}_{\mathbf{x}}(\Omega_t)$, the dual spaces of tangent and cotangent material vectors by $\mathbb{T}_{\mathbf{x}}\Omega_t$ and $\mathbb{T}^*_{\mathbf{x}}\Omega_t$.

Covariant, contravariant and mixed second-order material tensors belong to linear spaces of scalar-valued bilinear maps (or linear operators):

$$\mathrm{COV}_{\mathbf{x}}(\Omega_t) = L(\mathbb{T}_{\mathbf{x}}\Omega_t, \mathbb{T}_{\mathbf{x}}\Omega_t; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\Omega_t; \mathbb{T}^*_{\mathbf{x}}\Omega_t),$$
$$\mathrm{CON}_{\mathbf{x}}(\Omega_t) = L(\mathbb{T}^*_{\mathbf{x}}\Omega_t, \mathbb{T}^*_{\mathbf{x}}\Omega_t; \mathcal{R}) = L(\mathbb{T}^*_{\mathbf{x}}\Omega_t; \mathbb{T}_{\mathbf{x}}\Omega_t),$$
$$\mathrm{MIX}_{\mathbf{x}}(\Omega_t) = L(\mathbb{T}_{\mathbf{x}}\Omega_t, \mathbb{T}^*_{\mathbf{x}}\Omega_t; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\Omega_t; \mathbb{T}_{\mathbf{x}}\Omega_t).$$

A generic material tensor space is denoted by $\mathrm{TENS}_{\mathbf{x}}(\Omega_t)$.

At a given fixed time $t \in I$, a map $\zeta^{\mathrm{SP}}_t \in \mathrm{C}^1(\Omega_t; \mathcal{S})$, with the co-restriction $\zeta_t \in \mathrm{C}^1(\Omega_t; \zeta^{\mathrm{SP}}_t(\Omega_t))$ a diffeomorphism, will be called a *geometric displacement* to contrast its physical interpretation, of displacement at fixed time, in comparison with the one of a *material displacement* $\varphi_{\tau,t} := \varphi_\tau \circ \varphi_t^{-1} \in \mathrm{C}^1(\Omega_t; \Omega_\tau)$ along the motion. This distinction will become significant in the discussion about time-independence and invariance in Section 5.

The push of a material scalar $f_{\varphi,t}(\mathbf{x}) \in \mathrm{FUN}_{\mathbf{x}}(\Omega_t)$, along a *geometric displacement* $\zeta^{\mathrm{SP}}_t \in \mathrm{C}^1(\Omega_t; \mathcal{S})$, is a change of its base point:

$$(\zeta_t \uparrow f_{\varphi,t})(\zeta_t(\mathbf{x})) = f_{\varphi,t}(\mathbf{x}).$$

The push of a tangent material vector $\mathbf{v}_{\varphi,t}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\Omega_t$ is the evaluation of the *tangent geometric displacement* $T_{\mathbf{x}}\zeta_t \in L(\mathbb{T}_{\mathbf{x}}\Omega_t; \mathbb{T}_{\zeta_t(\mathbf{x})}\zeta_t(\Omega_t))$, by the formula:

$$\zeta_t \uparrow (\mathbf{v}_{\varphi,t}(\mathbf{x})) := T_{\mathbf{x}}\zeta_t \cdot \mathbf{v}_{\varphi,t}(\mathbf{x}),$$

and the push of a material cotangent vector $\mathbf{v}^*_{\varphi,t}(\mathbf{x}) \in \mathbb{T}^*_{\mathbf{x}}\Omega_t$ is defined by invariance:

$$\langle \zeta_t \uparrow \mathbf{v}^*_{\varphi,t}, \zeta_t \uparrow \mathbf{v}_{\varphi,t} \rangle = \zeta_t \uparrow \langle \mathbf{v}^*_{\varphi,t}, \mathbf{v}_{\varphi,t} \rangle.$$

The push of a tensor is also defined by invariance. For a twice-covariant material tensor field $\mathbf{s}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{COV}(\Omega_t))$, the push is explicitly defined, for any pair of material tangent vector fields $\mathbf{u}_{\varphi,t}, \mathbf{w}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathbb{T}\Omega_t)$, by:

$$(\zeta_t \uparrow \mathbf{s}_{\varphi,t})(\zeta_t \uparrow \mathbf{u}_{\varphi,t}, \zeta_t \uparrow \mathbf{w}_{\varphi,t}) := \zeta_t \uparrow (\mathbf{s}_{\varphi,t}(\mathbf{u}_{\varphi,t}, \mathbf{w}_{\varphi,t})).$$

Introducing the cotangent map $T^*_{\zeta_t(\mathbf{x})}\zeta_t \in L(\mathbb{T}^*_{\zeta_t(\mathbf{x})}\zeta_t(\Omega_t); \mathbb{T}^*_{\mathbf{x}}\Omega_t)$ such that:

$$\langle \mathbf{u}_{\zeta_t(\mathbf{x})}, T_{\mathbf{x}}\zeta_t \cdot \mathbf{w}_{\mathbf{x}} \rangle = \langle T^*_{\zeta_t(\mathbf{x})}\zeta_t \cdot \mathbf{u}_{\zeta_t(\mathbf{x})}, \mathbf{w}_{\mathbf{x}} \rangle,$$

for every $\mathbf{w}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\Omega_t$ and $\mathbf{u}_{\zeta_t(\mathbf{x})} \in \mathbb{T}^*_{\zeta_t(\mathbf{x})}\zeta_t(\Omega_t)$. With the abridged notation $\mathbf{s}_{\mathbf{x}} := \mathbf{s}_{\varphi,t}(\mathbf{x})$, the pushes of covariant, contravariant and mixed material tensors are given by:

$$\zeta_t \uparrow \mathbf{s}^{\mathrm{COV}}_{\mathbf{x}} = T^*_{\mathbf{x}}\zeta_t^{-1} \circ \mathbf{s}^{\mathrm{COV}}_{\mathbf{x}} \circ T_{\zeta_t(\mathbf{x})}\zeta_t^{-1},$$
$$\zeta_t \uparrow \mathbf{s}^{\mathrm{CON}}_{\mathbf{x}} = T_{\mathbf{x}}\zeta_t \circ \mathbf{s}^{\mathrm{CON}}_{\mathbf{x}} \circ T^*_{\zeta_t(\mathbf{x})}\zeta_t,$$
$$\zeta_t \uparrow \mathbf{s}^{\mathrm{MIX}}_{\mathbf{x}} = T_{\mathbf{x}}\zeta_t \circ \mathbf{s}^{\mathrm{MIX}}_{\mathbf{x}} \circ T_{\zeta_t(\mathbf{x})}\zeta_t^{-1}.$$

The *material metric field* $\mathbf{g}_{\varphi,t} \in C^1(\Omega_t; \mathrm{COV}(\Omega_t))$ at time $t \in I$ is induced in the configuration $\varphi_t : \mathcal{B} \mapsto \Omega_t$ by the spatial metric field $\mathbf{g} \in C^1(\mathcal{S}; \mathrm{COV}(\mathcal{S}))$ according to the definition:

$$\mathbf{g}_{\varphi,t}(\mathbf{u}_{\varphi,t}, \mathbf{w}_{\varphi,t}) := \mathbf{i}_{\varphi,t} \downarrow (\mathbf{g}(\mathbf{i}_{\varphi,t} \uparrow \mathbf{u}_{\varphi,t}, \mathbf{i}_{\varphi,t} \uparrow \mathbf{w}_{\varphi,t})),$$

where $\mathbf{i}_{\varphi,t} \in C^1(\Omega_t; \mathcal{S})$, is the *inclusion map* which places the image of the body onto the submanifold $\Omega^{\mathrm{SP}}_t = \mathbf{i}_{\varphi,t}(\Omega_t) \subset \mathcal{S}$ of the ambient space. The spatial tangent vector $\mathbf{i}_{\varphi,t} \uparrow (\mathbf{v}_{\varphi,t}(\mathbf{x})) = T_{\mathbf{x}}\mathbf{i}_{\varphi,t} \cdot \mathbf{v}_{\varphi,t}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}^{\mathrm{SP}}}\mathcal{S}$ is the spatial *immersion* of the material tangent vector $\mathbf{v}_{\varphi,t}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\Omega_t$. Alteration of tensors is defined by the relations:

$$\mathbf{s}^{\mathrm{MIX}}_{\mathbf{x}} = \mathbf{g}^{-1}_{\mathbf{x}} \circ \mathbf{s}^{\mathrm{COV}}_{\mathbf{x}} = \mathbf{s}^{\mathrm{CON}}_{\mathbf{x}} \circ \mathbf{g}_{\mathbf{x}},$$

which, in components form, correspond to lowering and rising of indexes.

The adjoint $T^A_{\zeta_t(\mathbf{x})}\zeta_t \in L(\mathbb{T}_{\zeta_t(\mathbf{x})}\zeta_t(\Omega_t); \mathbb{T}_{\mathbf{x}}\Omega_t)$ of the tangent map is defined by $T^A_{\zeta_t(\mathbf{x})}\zeta_t := \mathbf{g}^{-1}_{\mathbf{x}} \circ T^*_{\zeta_t(\mathbf{x})}\zeta_t \circ \mathbf{g}_{\zeta_t(\mathbf{x})}$. Then:

$$(\zeta_t \downarrow \mathbf{g}_{\zeta_t(\mathbf{x})})^{\mathrm{MIX}}_{\mathbf{x}} = T^A_{\zeta_t(\mathbf{x})}\zeta_t \circ T_{\mathbf{x}}\zeta_t,$$

a formula that will be referred to in evaluating the mixed form of the stretch. The pull is the push along the inverse diffeomorphism: $\zeta_t \downarrow = \zeta_t^{-1} \uparrow$. All these definitions extend directly to the push along a material displacement.

The *convective time-derivative* at time $t \in I$ of a *material tensor field* $\mathbf{s}_{\varphi,t} \in C^1(\Omega_t; \mathrm{TENS}(\Omega_t))$, along $\varphi^{\mathrm{SP}} : \mathcal{B} \times I \mapsto \mathcal{S}$, is defined by:

$$\mathcal{L}_{\varphi,t}\mathbf{s}_{\varphi} := \partial_{\tau=t}\, \varphi_{\tau,t} \downarrow \mathbf{s}_{\varphi,\tau}.$$

The pulled-back tensors $(\varphi_{\tau,t} \downarrow \mathbf{s}_{\varphi,\tau})(\mathbf{x})$ belong, for all $\tau \in I$, to the same linear tensor space $\mathrm{TENS}_{\mathbf{x}}(\Omega_t)$ so that the derivative $\partial_{\tau=t}$ makes sense. A simple but quite important property is that the pull-back of a convective time-derivative is equal to the time-derivative of the pull-back:

$$\varphi_{t,s} \downarrow (\mathcal{L}_{\varphi,t}\ \mathbf{s}_{\varphi}) = \partial_{\tau=t}\, \varphi_{\tau,s} \downarrow \mathbf{s}_{\varphi,\tau}.$$

The parallel transport of a spatial tensor along a curve $\mathbf{c}^{\mathrm{SP}} \in C^1(\mathcal{R}; \mathcal{S})$ in space is deduced, from the definition of parallel transport of a tangent spatial vector, by invariance, for a twice-covariant tensor $\mathbf{s}^{\mathrm{SP}} \in C^1(\mathcal{S}; \mathrm{COV}(\mathcal{S}))$, according to the formula:

$$(\mathbf{c}^{\mathrm{SP}}_{\lambda} \Uparrow \mathbf{s}^{\mathrm{SP}})(\mathbf{c}^{\mathrm{SP}}_{\lambda} \Uparrow \mathbf{u}, \mathbf{c}^{\mathrm{SP}}_{\lambda} \Uparrow \mathbf{w}) := \mathbf{s}^{\mathrm{SP}}(\mathbf{u}, \mathbf{w}) \circ \mathbf{c}^{\mathrm{SP}}_{\lambda}, \quad \lambda \in \mathcal{R}, \quad \mathbf{u}, \mathbf{w} \in \mathbb{T}\mathcal{S},$$

and similarly for other spatial tensors.

The *parallel* (or *covariant*) derivative along a curve $\mathbf{c}^{\mathrm{SP}} \in C^1(\mathcal{R}; \mathcal{S})$ of a *spatial tensor field* $\mathbf{s}^{\mathrm{SP}} \in C^1(\mathcal{S}; \mathrm{TENS}(\mathcal{S}))$, is accordingly defined by:

$$\nabla_{\dot{\mathbf{c}}^{\mathrm{SP}}_{\lambda}} \mathbf{s}^{\mathrm{SP}} := \partial_{\lambda=0}\, \mathbf{c}^{\mathrm{SP}}_{\lambda} \Downarrow (\mathbf{s}^{\mathrm{SP}} \circ \mathbf{c}^{\mathrm{SP}}_{\lambda}),$$

where $\dot{\mathbf{c}}^{\mathrm{SP}}_{\lambda} := \partial_{\lambda=0}\mathbf{c}^{\mathrm{SP}}_{\lambda}$ is the parametrization velocity of the curve at $\lambda = 0$ and $\mathbf{c}_{\lambda}^{\mathrm{SP}} \Downarrow$ denotes the parallel transport from $\mathbf{c}^{\mathrm{SP}}(\lambda)$ to $\mathbf{c}^{\mathrm{SP}}(0)$. If the curve is time-parametrized the definition of *parallel time-derivative* is got.

The *parallel time-derivative* along the motion

$$\nabla_{\varphi,t}\mathbf{s}^{\mathrm{SP}}_{\varphi} := \partial_{\tau=t}\, \varphi^{\mathrm{SP}}_{\tau,t} \Downarrow \mathbf{s}^{\mathrm{SP}}_{\varphi,\tau},$$

of a material-based spatial tensor field $\mathbf{s}^{\mathrm{SP}}_{\varphi,t} \in C^1(\Omega_t; \mathrm{TENS}(\mathcal{S}))$ is called the *material time-derivative*. The paradigmatic example is the *acceleration* of the motion: $\mathbf{a}^{\mathrm{SP}}_{\varphi,t} := \partial_{\tau=t}\, \varphi^{\mathrm{SP}}_{\tau,t} \Downarrow \mathbf{v}^{\mathrm{SP}}_{\varphi,\tau} \in C^1(\Omega_t; \mathbb{T}\mathcal{S})$ which is the material time-derivative of the motion velocity field $\mathbf{v}^{\mathrm{SP}}_{\varphi,t} := \partial_{\tau=t}\, \varphi^{\mathrm{SP}}_{\tau,t} \in C^1(\Omega_t; \mathbb{T}\mathcal{S})$.

Remark 3.1. The spatial description of a material-based spatial tensor field is defined on the body's trajectory:

$$\mathcal{T}(\mathcal{B}, \varphi) := \{(\mathbf{x}^{\mathrm{SP}}, t) | \mathbf{x}^{\mathrm{SP}} = \varphi^{\mathrm{SP}}(p, t), p \in \mathcal{B}, t \in I\},$$

by:

$$\mathbf{s}^{\mathrm{SP}}_{\mathcal{T}(\mathcal{B},\varphi)}(\mathbf{x}^{\mathrm{SP}}, t) := \mathbf{s}^{\mathrm{SP}}_{\varphi}(p, t), \quad \mathbf{x}^{\mathrm{SP}} = \varphi(p, t).$$

As a rule, neither the convective time-derivative nor the parallel time-derivative along the motion may be evaluated by resorting to Leibniz rule to get a split into the sum of partial time and space derivatives. This is due to the possible highly irregular time dependence of a material tensor field, at a fixed point of space, as a function of time. In this respect, we quote that, although often misguidedly considered of general validity and also taken as the definition of acceleration (see (Marsden and Hughes, 1983), Prop. 1.10, p. 32 and (Truesdell, 1991), II. 6.4, p. 104), the celebrated d'Alembert-Euler split formula for the acceleration:

$$\mathbf{a}^{\mathrm{SP}}_{\mathcal{T}(\mathcal{B},\varphi),t} = \partial_{\tau=t}\, \mathbf{v}^{\mathrm{SP}}_{\mathcal{T}(\mathcal{B},\varphi),\tau} + \nabla_{\mathbf{v}^{\mathrm{SP}}_{\mathcal{T}(\mathcal{B},\varphi),t}} \mathbf{v}^{\mathrm{SP}}_{\mathcal{T}(\mathcal{B},\varphi),t},$$

is in fact applicable only in investigations about continuous flows of a fluid in a region of space, as in problems of hydrodynamics, where it was originarily conceived. With reference to hypo-elasticity the issue will be discussed in detail in Section 6.1.

4. Stretch, stretching, stress and stressing

To a pair of material configurations $\varphi_t : \mathcal{B} \mapsto \Omega_t$ and $\varphi_\tau : \mathcal{B} \mapsto \Omega_\tau$ there corresponds a Cauchy-Green strain (*stretch*) material tensor field:

$$\epsilon_{\varphi_{\tau,t}} := \frac{1}{2}(\varphi_{\tau,t} \downarrow \mathbf{g}_{\varphi,\tau} - \mathbf{g}_{\varphi,t}) \in C^1(\Omega_t; \mathrm{COV}(\Omega_t)),$$

which is a symmetric covariant tensor field. The mixed form is denoted by $\mathbf{E}_{\varphi_{\tau,t}} := \mathbf{g}^{-1}_{\varphi,t} \circ \epsilon_{\varphi_{\tau,t}} = 1/2(\mathbf{C}_{\varphi_{\tau,t}} - \mathbf{I}_{\varphi,t}) \in C^1(\Omega_t; \mathrm{MIX}(\Omega_t))$ where $\mathbf{I}_{\varphi,t}$ is the identity tensor field and:

$$\mathbf{C}_{\varphi_{\tau,t}} := T^A\varphi_{\tau,t} \circ T\varphi_{\tau,t},$$

is the *right* Cauchy-Green tensor field, extending to lower dimensional bodies the one defined in Truesdell and Noll (1965).

The time-derivative $\partial_{\tau=t}\, \varphi_{\tau,t} \downarrow \mathbf{g}_{\varphi,\tau}$ is a tensor field too because, for any $\tau \in I$, the tensors $(\varphi_{\tau,t} \downarrow \mathbf{g}_{\varphi,\tau})(\mathbf{x})$ are all based at the same point $\mathbf{x} \in \Omega_t$. We may then give the following definition.

Definition 4.1 (Stretching field). *The material stretching at time $t \in I$ is the covariant symmetric tensor field $\dot{\epsilon}_{\varphi,t} \in C^1(\Omega_t; \mathrm{COV}(\Omega_t))$ defined as one-half the convective time-derivative of the material metric tensor field:*

$$\dot{\epsilon}_{\varphi,t} := \partial_{\tau=t}\, \epsilon_{\varphi_{\tau,t}} = \partial_{\tau=t} \frac{1}{2}(\varphi_{\tau,t} \downarrow \mathbf{g}_{\varphi_\tau} - \mathbf{g}_{\varphi,t}) = \frac{1}{2} \mathcal{L}_{\varphi,t}\, \mathbf{g}_{\varphi}.$$

The expression of the *stretching* in terms of the velocity field of the body in motion, is provided by a celebrated formula due to Euler for 3D bodies. Our treatment extends the result to lower dimensional bodies.

Preliminarily we observe that the parallel derivative of the material-based spatial velocity field $\mathbf{v}^{\mathrm{SP}}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathbb{T}\mathcal{S})$, along a material tangent vector field $\mathbf{u}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathbb{T}\Omega_t)$, is a material-based spatial vector field $\nabla \mathbf{v}^{\mathrm{SP}}_{\varphi,t} \cdot \mathbf{u}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathbb{T}\mathcal{S})$. To get a material vector field, the spatial vectors are projected over the material tangent spaces. The result yields the *material parallel derivative* $\nabla^{\mathrm{MAT}} \mathbf{v}^{\mathrm{SP}}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{MIX}(\Omega_t))$ defined at $\mathbf{x} \in \Omega_t$ by:

$$\mathbf{g}_{\varphi,t}(\nabla^{\mathrm{MAT}}_{\mathbf{x}} \mathbf{v}^{\mathrm{SP}}_{\varphi,t} \cdot \mathbf{u}_{\varphi,t},\ \mathbf{w}_{\varphi,t}) := \mathbf{g}\big(\nabla_{\mathbf{x}} \mathbf{v}^{\mathrm{SP}}_{\varphi,t} \cdot \mathbf{u}_{\varphi,t},\ T_{\mathbf{x}} \mathbf{i}_{\varphi,t} \cdot \mathbf{w}_{\varphi,t}\big),$$

where $\mathbf{u}_{\varphi,t}, \mathbf{w}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathbb{T}\Omega_t)$ are material tangent vector fields. The generalized expression of Euler's formula then reads (Romano, 2007):

$$\frac{1}{2} \mathcal{L}_{\varphi,t}\, \mathbf{g}_{\varphi} = \mathbf{g}_{\varphi,t} \circ \mathrm{sym}\Big(\nabla^{\mathrm{MAT}} \mathbf{v}^{\mathrm{SP}}_{\varphi,t}\Big).$$

The mixed form of the stretching tensor is accordingly provided by the symmetric part of the *material parallel derivative* of the velocity field:

$$\mathbf{D}_{\varphi,t} := \partial_{\tau=t}\, \mathbf{E}_{\varphi_{\tau,t}} = \mathbf{g}^{-1}_{\varphi,t} \circ \frac{1}{2} \mathcal{L}_{\varphi,t}\, \mathbf{g}_{\varphi} = \mathrm{sym}\Big(\nabla^{\mathrm{MAT}} \mathbf{v}^{\mathrm{SP}}_{\varphi,t}\Big).$$

The symbol $\mathbf{D}$ was adopted by Truesdell and Noll (1965) to denote the symmetric part of the velocity gradient for a 3D body. In our treatment the result and the notation are extended to lower dimensional bodies. The stress field is introduced by duality:

Definition 4.2 (Stress field). *The stress field at time $t \in I$ is a contravariant symmetric material tensor field $\boldsymbol{\sigma}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{CON}(\Omega_t))$.*

Definition 4.3 (Duality pairing). *The duality pairing between the stretching $\dot{\boldsymbol{\epsilon}}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{COV}(\Omega_t))$ and the stress $\boldsymbol{\sigma}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{CON}(\Omega_t))$ fields is the scalar material field $\langle \boldsymbol{\sigma}_{\varphi,t}, \dot{\boldsymbol{\epsilon}}_{\varphi,t} \rangle := J_1(\boldsymbol{\sigma}_{\varphi,t} \circ \dot{\boldsymbol{\epsilon}}_{\varphi,t}) \in \mathrm{C}^1(\Omega_t; \mathrm{FUN}(\Omega_t))$ of linear invariants of the mixed tensor field $\boldsymbol{\sigma}_{\varphi,t} \circ \dot{\boldsymbol{\epsilon}}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{MIX}(\Omega_t))$.*

In continuum mechanics, contravariant Cauchy stress fields are material fields whose duality pairing with the covariant stretching provides the stress power per unit volume in the actual configuration. The mixed tensor field, called the Cauchy *true* stress because its boundary flux provides the boundary tractions field, is expressed by:

$$\mathbf{T}_{\varphi,t} = \boldsymbol{\sigma}_{\varphi,t} \circ \mathbf{g}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{MIX}(\Omega_t)).$$

The previous formula may be written in components as $\mathbf{T}^{i}_{.j} = \boldsymbol{\sigma}^{ik} \mathbf{g}_{kj}$ with $i, j, k = 1, \ldots, n$ for a body of dimensionality $n = 1, 2, 3$.

Definition 4.4 (Stressing field). *The stressing field is the convective time-derivative, along the motion $\varphi^{\mathrm{SP}} : \mathcal{B} \times I \mapsto \mathcal{S}$, of the material stress field, i.e.:*

$$\mathcal{L}_{\varphi,t}\, \boldsymbol{\sigma}_{\varphi} := \partial_{\tau=t}\, \varphi_{\tau,t} \downarrow \boldsymbol{\sigma}_{\varphi,\tau} \in \mathrm{C}^1(\Omega_t; \mathrm{CON}(\Omega_t)).$$

5. Time-independence, invariance and frame-indifference

As a consequence of the *covariance paradigm*, Proposition 2.1, we give the following definitions of *time-independence* and *invariance*.

Definition 5.1 (Time-independence of material tensor fields). *Time-independence of a material tensor field $\mathbf{s}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{TENS}\ (\Omega_t))$ under the action of a material displacement $\varphi_{\tau,t} \in \mathrm{C}^1(\Omega_t; \Omega_\tau)$ means that the material tensor field transforms by push along the motion:*

$$\mathbf{s}_{\varphi,\tau} = \varphi_{\tau,t} \uparrow \mathbf{s}_{\varphi,t}, \quad \forall\, \tau, t \in I.$$

A relative motion, with respect to the motion $\varphi^{\mathrm{SP}} : \mathcal{B} \times I \mapsto \mathcal{S}$, is a family of geometric displacements $\zeta^{\mathrm{SP}}_t \in \mathrm{C}^1(\Omega_t; \mathcal{S})$ with a smooth dependence on time $t \in I$.

Definition 5.2 (Invariance of material tensor fields). *Invariance of a material tensor field $\mathbf{s}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{TENS}(\Omega_t))$ under the action of a geometric displacement $\zeta^{\mathrm{SP}}_t \in \mathrm{C}^1(\Omega_t; \mathcal{S})$ means that the material tensor field transforms by push along the relative motion, at fixed time:*

$$\mathbf{s}_{\zeta\uparrow\varphi,t} = \zeta_t \uparrow \mathbf{s}_{\varphi,t}, \quad \forall\, t \in I.$$

The material displacement in the pushed motion $\zeta \uparrow \varphi$ is defined by the commutative diagram:

$$\begin{tikzcd} \zeta_t(\Omega_t) \arrow[r, "(\zeta\uparrow\varphi)_{\tau,t}"] & \zeta_\tau(\Omega_\tau) \\ \Omega_t \arrow[u, "\zeta_t"] \arrow[r, "\varphi_{\tau,t}"] & \Omega_\tau \arrow[u, "\zeta_\tau"'] \end{tikzcd} \iff \begin{aligned} (\zeta\uparrow\varphi)_{\tau,t} &:= (\zeta_\tau \circ \varphi_\tau) \circ (\zeta_t \circ \varphi_t)^{-1} \\ &= \zeta_\tau \circ \varphi_{\tau,t} \circ \zeta_t^{-1}. \end{aligned}$$

Accordingly, the push along the material displacement in the pushed motion is expressible by the chain composition:

$$(\zeta \uparrow \varphi)_{\tau,t} \uparrow = \zeta_\tau \uparrow \circ\, \varphi_{\tau,t} \uparrow \circ\, \zeta_t^{-1} \uparrow.$$

The next Lemma yields a basic result for the assessment of invariance of convective time-derivatives.

Lemma 5.1 (Covariance of convective time-derivatives). *The convective time-derivative of a material tensor field $\mathbf{s}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{TENS}(\Omega_t))$ and the convective time-derivative of its push by a relative motion $\zeta^{\mathrm{SP}}_t \in \mathrm{C}^1(\Omega_t; \mathcal{S})$ are related by the covariance rule:*

$$\mathcal{L}_{(\zeta\uparrow\varphi),t}(\zeta \uparrow \mathbf{s}_{\varphi}) = \zeta_t \uparrow \mathcal{L}_{\varphi,t}\, \mathbf{s}_{\varphi}.$$

Proof. By definition of material displacement from time $t \in I$ to time $\tau \in I$ along the pushed motion, we have that:

$$\begin{aligned} (\zeta \uparrow \varphi)_{\tau,t} \downarrow (\zeta_\tau \uparrow \mathbf{s}_{\varphi,\tau}) &= (\zeta_\tau \circ \varphi_{\tau,t} \circ \zeta_t^{-1}) \downarrow (\zeta_\tau \uparrow \mathbf{s}_{\varphi,\tau}) \\ &= \zeta_t \uparrow \big((\varphi_{\tau,t} \downarrow \circ\, \zeta_\tau \downarrow)(\zeta_\tau \uparrow \mathbf{s}_{\varphi,\tau})\big) \\ &= \zeta_t \uparrow (\varphi_{\tau,t} \downarrow \mathbf{s}_{\varphi,\tau}). \end{aligned}$$

The result follows by definition of convective time-derivative:

$$\begin{aligned} &\mathcal{L}_{\varphi,t}\, \mathbf{s}_{\varphi} := \partial_{\tau=t}\, \varphi_{\tau,t} \downarrow \mathbf{s}_{\varphi,\tau}, \\ &\mathcal{L}_{(\zeta\uparrow\varphi),t}\, (\zeta \uparrow \mathbf{s}_{\varphi}) := \partial_{\tau=t}\, (\zeta \uparrow \varphi)_{\tau,t} \downarrow (\zeta \uparrow \mathbf{s}_{\varphi})_\tau, \end{aligned}$$

and by the linearity of the map $\zeta_t \uparrow \in \mathrm{C}^0(\mathbb{T}\Omega_t; \mathbb{T}\mathcal{S})$ at each $\mathbf{x} \in \Omega_t$, which provides the commutativity property $\partial_{\tau=t} \zeta_t \uparrow = \zeta_t \uparrow \partial_{\tau=t}$.

In Marsden and Hughes (1983, Th. 6.19, p. 101) a statement resembling the one in Lemma 1 is formulated and proved with reference to time-dependent spatial tensor fields. Both the statement and the proof are however different and that result, pertaining to spatial fields, cannot be applied to material tensor fields involved in constitutive relations.

A result concerning invariance, which will be resorted to in the analysis of hypo-elastic materials, is explicitly stated hereafter.

Proposition 5.1 (Convective time-derivatives of invariant tensors). *Invariance of a material tensor field $\mathbf{s}_{\varphi,t} \in \mathrm{C}^1(\Omega_t; \mathrm{TENS}(\Omega_t))$ with respect to a relative motion $\zeta^{\mathrm{SP}}_t \in \mathrm{C}^1(\Omega_t; \mathcal{S})$ implies invariance of its convective time-derivative:*

$$\mathbf{s}_{\zeta\uparrow\varphi,t} = \zeta_t \uparrow \mathbf{s}_{\varphi,t} \quad \Rightarrow \quad \mathcal{L}_{(\zeta\uparrow\varphi),t}\, \mathbf{s}_{\zeta\uparrow\varphi,t} = \zeta_t \uparrow \mathcal{L}_{\varphi,t}\, \mathbf{s}_{\varphi}.$$

Proof. The result follows directly from Lemma 1 and the Definition 6 of invariance.

A *change of observer* is a time-dependent family of diffeomorphic maps $\zeta_t^{\mathrm{OBS}} \in \mathrm{C}^1(\mathcal{S};\mathcal{S})$ of the ambient space onto itself. It induces a relative motion from any given motion.

A *change of* Euclid *observer* requires that the change of observer $\zeta_t^{\mathrm{OBS}} \in \mathrm{C}^1(\mathcal{S};\mathcal{S})$ be an isometry: $\mathbf{g}_{\zeta_t^{\mathrm{ISO}}\uparrow\varphi,t} = \zeta_t^{\mathrm{ISO}}\uparrow\mathbf{g}_{\varphi,t}$. Invariance under change of Euclid observer is called *frame-indifference*. The material metric tensor is frame-indifferent by definition.

Remark 5.1. *A basic physical assumption is that the stress tensor is frame-indifferent, i.e. invariant under a change of Euclid observer. Of course, in general, the stress tensor will not be time-independent, that is invariant under a material displacement of the body along the motion, even if the material displacement $\boldsymbol{\varphi}_{\tau,t} := \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \in \mathrm{C}^1(\boldsymbol{\Omega}_t;\boldsymbol{\Omega}_\tau)$ is isometric, i.e. even if the material metric tensor is time-independent: $\mathbf{g}_{\varphi,\tau} = \boldsymbol{\varphi}_{\tau,t}\uparrow\mathbf{g}_{\varphi,t}$. Time-independence of the stress tensor under isometric material displacements holds however for elastic materials, as will be discussed in* Remark 4.

6. Hypo-elasticity

A hypo-elastic response of a body $\mathcal{B}$ in motion $\boldsymbol{\varphi}^{\mathrm{SP}}: \mathcal{B}\times I \mapsto \mathcal{S}$ is expressed, at each time $t \in I$, by assuming that the stretching at the configuration $\boldsymbol{\varphi}_t : \mathcal{B} \mapsto \boldsymbol{\Omega}_t$ is a function of stress and stressing.

Definition 6.1 (Hypo-elastic law). *The hypo-elastic response at time $t \in I$ is governed by a stress-dependent constitutive linear operator $\mathbf{H}_{\varphi,t}$ which provides the stretching $1/2\ \mathcal{L}_{\varphi,t}\,\mathbf{g}_\varphi$ corresponding to the stressing $\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_\varphi$:*

$$\dot{\boldsymbol{\epsilon}}_{\varphi,t} := \frac{1}{2}\mathcal{L}_{\varphi,t}\,\mathbf{g}_\varphi = \mathbf{H}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t})\cdot\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_\varphi.$$

The operator $\mathbf{H}_{\varphi,t}$ is defined on the linear space $\mathrm{CON}(\boldsymbol{\Omega}_t)$ and takes values in the linear space $L\,(\mathrm{CON}(\boldsymbol{\Omega}_t);\ \mathrm{COV}(\boldsymbol{\Omega}_t))$ whose elements are linear operators between the domain space $\mathrm{CON}(\boldsymbol{\Omega}_t)$ and its dual $\mathrm{COV}(\boldsymbol{\Omega}_t)$. At each $\boldsymbol{\sigma}_{\varphi,t} \in \mathrm{CON}(\boldsymbol{\Omega}_t)$ the tangent compliance $\mathbf{H}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t})$ is assumed to be an invertible linear operator.

In Truesdell and Noll (1965) the symbol $\overset{\Delta}{\mathrm{T}}$ has been proposed for the mixed form of the *stressing*, there called *convected stress rate*. In their notation, the hypo-elastic law is written in components as:

$$\mathrm{D}^{i\cdot}_{\cdot j} = \mathrm{H}^{i\cdot\cdot s}_{\cdot jr\cdot}\left(\mathrm{T}^{p\cdot}_{\cdot q}\right)\overset{\Delta}{\mathrm{T}}{}^{r\cdot}_{\cdot s}$$

Definition 6.2 (Time-independent hypo-elasticity). *A hypo-elastic constitutive operator is time-independent in a time-interval I if the instantaneous operators at any pair of time instants $\tau, t \in I$ are related by push along the material displacement:*

$$\mathbf{H}_{\varphi,\tau} = \boldsymbol{\varphi}_{\tau,t}\uparrow\mathbf{H}_{\varphi,t},$$

the pushed operator, being defined by:

$$\begin{aligned}&\boldsymbol{\varphi}_{\tau,t}\uparrow\left(\mathbf{H}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t})\cdot\mathcal{L}_{\varphi,t}\boldsymbol{\sigma}_\varphi\right)\\ &\quad = (\boldsymbol{\varphi}_{\tau,t}\uparrow\mathbf{H}_{\varphi,t})(\boldsymbol{\varphi}_{\tau,t}\uparrow\boldsymbol{\sigma}_{\varphi,t})\cdot\boldsymbol{\varphi}_{\tau,t}\uparrow(\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_\varphi).\end{aligned}$$

This means that time-invariant material tensor fields, fulfilling the constitutive relation at time $t \in I$, are still related by the law at time $\tau \in I$.

To endow the mathematical definition of hypo-elastic law with a physical meaning apt to describe a material behavior, it is compelling to show independence of the change of Euclid observer of the motion, which is the meaning of *material frame-indifference* (M.F.I.).

The physical assumption, quoted in Remark 5.1, is that the stress is frame-indifferent, i.e.: $\boldsymbol{\sigma}_{\zeta_t^{\mathrm{ISO}}\uparrow\varphi,t} = \zeta_t^{\mathrm{ISO}}\uparrow\boldsymbol{\sigma}_{\varphi,t}$ for any change of Euclid observer. Frame-indifference of the material metric tensor and Preposition 5.1 assure that stressing and stretching are frame-indifferent too, i.e.

$$\mathcal{L}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t}\,\boldsymbol{\sigma}_{\zeta^{\mathrm{ISO}}\uparrow\varphi} = \zeta_t^{\mathrm{ISO}}\uparrow\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_\varphi,$$

$$\mathcal{L}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t}\,\mathbf{g}_{\zeta^{\mathrm{ISO}}\uparrow\varphi} = \zeta_t^{\mathrm{ISO}}\uparrow\mathcal{L}_{\varphi,t}\,\mathbf{g}_\varphi,$$

so that:

$$\begin{aligned}\zeta_t^{\mathrm{ISO}}\uparrow\frac{1}{2}\mathcal{L}_{\varphi,t}\,\mathbf{g}_\varphi &= \frac{1}{2}\mathcal{L}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t}\,\mathbf{g}_{\zeta^{\mathrm{ISO}}\uparrow\varphi}\\ &= \mathbf{H}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t}\left(\boldsymbol{\sigma}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t}\right)\cdot\mathcal{L}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t}\,\boldsymbol{\sigma}_{\zeta^{\mathrm{ISO}}\uparrow\varphi}\\ &= \mathbf{H}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t}(\zeta_t^{\mathrm{ISO}}\uparrow\boldsymbol{\sigma}_{\varphi,t})\cdot\zeta_t^{\mathrm{ISO}}\uparrow\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_\varphi.\end{aligned}$$

Hence M.F.I. is expressed by the following condition of invariance under change of Euclid observer on the hypo-elastic constitutive operator:

$$\mathbf{H}_{\zeta^{\mathrm{ISO}}\uparrow\varphi,t} = \zeta_t^{\mathrm{ISO}}\uparrow\mathbf{H}_{\varphi,t}.$$

6.1. Non-covariant stress rates

In literature the following expression is exposed for the *convected stress rate* (Truesdell and Noll, 1965, formula 36.20, p. 97):

$$\overset{\Delta}{\mathrm{T}} = \dot{\mathbf{T}} + \mathbf{L}^T\mathbf{T} + \mathbf{TL}.$$

According to our notation, when written for the contravariant stress field $\boldsymbol{\sigma}_{\varphi,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t;\mathrm{CON}(\boldsymbol{\Omega}_t))$, this definition reads:

$$\overset{\Delta}{\boldsymbol{\sigma}}_{\varphi,t} = \dot{\boldsymbol{\sigma}}_{\varphi,t} - 2\,\mathrm{sym}\left(\nabla\mathbf{v}^{\mathrm{SP}}_{\varphi,t}\circ\boldsymbol{\sigma}_{\varphi,t}\right),$$

where $\nabla\mathbf{v}^{\mathrm{SP}}_{\varphi,t}$ is the covariant derivative of the velocity field, for instance the one induced by a co-ordinate system, and $\dot{\boldsymbol{\sigma}}_{\varphi,t}$ should be the material time-derivative, defined by:

$$\dot{\boldsymbol{\sigma}}_{\varphi,t} := \partial_{\tau=t}\ \boldsymbol{\varphi}^{\mathrm{SP}}_{\tau,t}\Downarrow\left(\boldsymbol{\sigma}_{\varphi,\tau}\circ\boldsymbol{\varphi}^{\mathrm{SP}}_{\tau,t}\right) \in \mathrm{C}^1(\boldsymbol{\Omega}_t;\mathrm{CON}(\boldsymbol{\Omega}_t)).$$

However, in evaluating this formula, the particle is held fixed and the comparison of material stress tensors at two instants along the motion is performed by parallel transport, an operation which is in contrast with the prescription of the covariance paradigm, Proposition 2.1. Most often, by considering the *spatial description* of the stress field, defined on the body's trajectory $\mathcal{T}(\mathcal{B},\boldsymbol{\varphi})$ by: $\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\varphi)}(\mathbf{x}^{\mathrm{SP}},t) := \boldsymbol{\sigma}_\varphi(\mathrm{p},t)$, $\mathbf{x}^{\mathrm{SP}} = \boldsymbol{\varphi}(\mathrm{p},t)$, the time-derivative along the motion is split into the sum of partial time and spatial derivatives:

$$\dot{\boldsymbol{\sigma}}_{\mathcal{T}(\mathcal{B},\varphi),t} = \partial_{\tau=t}\,\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\varphi),\tau} + \nabla_{\mathbf{v}_{\mathcal{T}(\mathcal{B},\varphi),t}}\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\varphi),t}.$$

Expressions like these ones have been adopted, in components, in Oldroyd (1950a,b), Truesdell (1955), Bernstein (1960), Sedov (1960), Truesdell and Noll (1965) and in subsequent treatments, reproductions or variants, e.g. (Dienes, 1979; Gurtin, 1981; Marsden and Hughes, 1983; Pinsky et al., 1983; Atluri, 1984; Simó and Pister, 1984; Johnson and Bammann, 1984; Moss, 1984; Sowerby and Chu, 1984; Lo, 1988; Sansour and Bednarczyk, 1993; Xiao et al., 1997; Bruhns et al., 2006; Yavari et al., 2006) and references therein. Essential difficulties are involved in the evaluation of the formulas above. In fact, the covariant differentiation $\nabla_{\mathbf{v}_{\mathcal{T}(\mathcal{B},\varphi),t}}$ is an operation not allowed on the material-valued tensor field $\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\varphi),t}$. Moreover, the partial time-derivative $\partial_{\tau=t}\,\boldsymbol{\sigma}_{\mathcal{T}(\mathcal{B},\varphi),\mathcal{T}}$, evaluated at a fixed point in space, is not feasible in general, because the domain of the spatial description of a material field is the trajectory, so that time-dependence, at a fixed point in space, will be quite irregular, in general. This is witnessed by the example of a spinning cogwheel in which material particles will cross the peripheral spatial points at isolate instants of time. The same difficulty holds for material-based spatial tensor fields.

The source of confusion is that material-based spatial tensor fields, whose spatial description is defined on the trajectory, are treated as they were time-dependent spatial tensor fields, defined over the whole ambient space.

Indeed, a formula, similar to the incorrect one for the convective stress-rate reported above, holds true for the Lie time-derivative, along the motion, of a time-dependent spatial tensor field $\mathbf{s}_t^{\mathrm{SP}} \in \mathrm{C}^1(\boldsymbol{\mathcal{S}};\mathrm{TENS}(\boldsymbol{\mathcal{S}}))$ with $t \in I$. For a twice contravariant spatial tensor field, in our intrinsic notation, it writes (Marsden and Hughes, 1983; Romano, 2007):

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\varphi},t}\mathbf{s}^{\mathrm{SP}} := \partial_{\tau=t}\, \boldsymbol{\varphi}_{\tau,t}^{\mathrm{SP}} \downarrow (\mathbf{s}_\tau^{\mathrm{SP}} \circ \boldsymbol{\varphi}_{\tau,t}^{\mathrm{SP}}) = \dot{\mathbf{s}}_t^{\mathrm{SP}} - 2\,\mathrm{sym}(\nabla \mathbf{v}_{\boldsymbol{\varphi},t}^{\mathrm{SP}} \circ \mathbf{s}_t^{\mathrm{SP}}),$$

with the *parallel time-derivative* along the motion, of the time-dependent spatial tensor field given by:

$$\dot{\mathbf{s}}_t^{\mathrm{SP}} := \partial_{\tau=t}\, \boldsymbol{\varphi}_{\tau,t}^{\mathrm{SP}} \Downarrow (\mathbf{s}_\tau^{\mathrm{SP}} \circ \boldsymbol{\varphi}_{\tau,t}^{\mathrm{SP}}) = \partial_{\tau=t}\, \mathbf{s}_\tau^{\mathrm{SP}} + \nabla_{\mathbf{v}_{\boldsymbol{\varphi},t}}\, \mathbf{s}_t^{\mathrm{SP}}.$$

6.2. *Expression in terms of mixed tensors*

As stated in Marsden and Hughes (1983), alterations performed with the spatial metric tensor field and push according to a displacement map do not commute. This lack of commutativity is at the root of the list of expressions, of convective time-derivatives evaluated in terms various alterations of the stress tensor field, exposed there on page 100, box 6.1. As a consequence, the adoption of different representations by altered tensors, would lead to different constitutive laws. This unpleasant feature may be overcome if a fully covariant approach is taken, by requiring that alterations of the material tensors, pushed from a natural configuration, should be carried out according to the pushed material metric tensor. Then push and alteration will commute, as illustrated, with reference to contravariant and mixed stress tensors, in the next statement.

Proposition 6.1 (Push and alteration). *Push of a mixed stress tensor is equal to the mixed stress tensor stemming from the alteration of the pushed contravariant stress according to the pushed material metric:*

$$\boldsymbol{\varphi}_{\tau,t} \uparrow (\boldsymbol{\sigma}_{\boldsymbol{\varphi},t} \circ \mathbf{g}_{\boldsymbol{\varphi},t}) = (\boldsymbol{\varphi}_{\tau,t} \uparrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \circ (\boldsymbol{\varphi}_{\tau,t} \uparrow \mathbf{g}_{\boldsymbol{\varphi},t}).$$

Proof. The result, which is a special expression of the naturality of push transformations with respect to tensor composition, follows from the formulas:

$$\boldsymbol{\varphi}_{\tau,t} \uparrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},t} = T\boldsymbol{\varphi}_{\tau,t} \circ \boldsymbol{\sigma}_{\boldsymbol{\varphi},t} \circ T^*\boldsymbol{\varphi}_{\tau,t},$$

$$\boldsymbol{\varphi}_{\tau,t} \uparrow \mathbf{g}_{\boldsymbol{\varphi},t} = T^*\boldsymbol{\varphi}_{\tau,t}^{-1} \circ \mathbf{g}_{\boldsymbol{\varphi},t} \circ T\boldsymbol{\varphi}_{\tau,t}^{-1},$$

$$\boldsymbol{\varphi}_{\tau,t} \uparrow (\boldsymbol{\sigma}_{\boldsymbol{\varphi},t} \circ \mathbf{g}_{\boldsymbol{\varphi},t}) = T\boldsymbol{\varphi}_{\tau,t} \circ \boldsymbol{\sigma}_{\boldsymbol{\varphi},t} \circ \mathbf{g}_{\boldsymbol{\varphi},t} \circ T\boldsymbol{\varphi}_{\tau,t}^{-1},$$

provided in Section 3.

Once the hypo-elastic law has been formulated in terms of dual covariant metric and contravariant stress tensors in a natural reference configuration, its expression in terms of mixed tensors in the actual configuration may then be got in two equivalent ways: either by altering with the referential material metric tensor and then pushing forward the result, or by pushing forward, the stress tensor and the referential material metric tensor, and then performing the alteration in the actual configuration with this pushed material metric tensor.

6.3. *Evaluation of the stress field*

In most computational algorithms, a basic issue concerns the evaluation of the stress field along the motion in terms of the hypo-elastic tangent stiffness at a time $t \in I$:

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\varphi},t}\boldsymbol{\sigma}_{\boldsymbol{\varphi}} = \left(\mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t})\right)^{-1} \cdot \frac{1}{2}\, \boldsymbol{\mathcal{L}}_{\boldsymbol{\varphi},t}\mathbf{g}_{\boldsymbol{\varphi}}.$$

The computation is conveniently carried out in terms of a reference configuration $\boldsymbol{\chi}_o : \mathcal{B} \mapsto \boldsymbol{\Omega}_o$. A schematic view of the relations between the body, a reference configuration and the material configurations along the motion, is provided by the diagram below:

$$\begin{tikzcd}
\boldsymbol{\Omega}_o \arrow[dr, squiggly, "\boldsymbol{\varphi}_t^{\mathrm{REF}}"'] \arrow[drr, squiggly, bend left, "\boldsymbol{\varphi}_\tau^{\mathrm{REF}}"] & & \\
& \boldsymbol{\Omega}_t \arrow[r, "\boldsymbol{\varphi}_{\tau,t}"] & \boldsymbol{\Omega}_\tau \\
\mathcal{B} \arrow[uu, "\boldsymbol{\chi}_o"] \arrow[ur, "\boldsymbol{\varphi}_t"] \arrow[urr, bend right, "\boldsymbol{\varphi}_\tau"'] & &
\end{tikzcd}$$

$$\iff \begin{cases} \boldsymbol{\varphi}_t = \boldsymbol{\varphi}_t^{\mathrm{REF}} \circ \boldsymbol{\chi}_o \\ \boldsymbol{\varphi}_{\tau,t} = \boldsymbol{\varphi}_\tau \circ \boldsymbol{\varphi}_t^{-1} \\ \boldsymbol{\varphi}_{\tau,t} = \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \circ (\boldsymbol{\varphi}_t^{\mathrm{REF}})^{-1} = \boldsymbol{\varphi}_{\tau,t}^{\mathrm{REF}} \end{cases}$$

By pulling back to a reference configuration $\chi_o : \mathcal{B} \mapsto \boldsymbol{\Omega}_o$ the hypo-elastic law writes:

$$\partial_{\tau=t}\, \frac{1}{2}\, \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \downarrow \frac{1}{2}\, \mathbf{g}_{\boldsymbol{\varphi},\tau} = (\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \mathbf{H}_{\boldsymbol{\varphi},t})(\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},t}) \cdot \partial_{\tau=t}\, \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}.$$

Its inverse is given by:

$$\partial_{\tau=t}\, \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau} = \left((\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \mathbf{H}_{\boldsymbol{\varphi},t})(\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},t})\right)^{-1} \cdot \partial_{\tau=t}\, \frac{1}{2}\, \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \downarrow \frac{1}{2}\, \mathbf{g}_{\boldsymbol{\varphi},\tau},$$

so that:

$$\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \left(\mathbf{H}_{\boldsymbol{\varphi},t}(\boldsymbol{\sigma}_{\boldsymbol{\varphi},t})\right)^{-1} = \left((\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \mathbf{H}_{\boldsymbol{\varphi},t})(\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},t})\right)^{-1}.$$

To evaluate the referential stress increment in a time interval $[s,t]$, the following integral equation should be solved:

$$\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},t} - \boldsymbol{\varphi}_s^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},s} = \int_s^t \left(\left(\boldsymbol{\varphi}_\theta^{\mathrm{REF}} \downarrow \mathbf{H}_{\boldsymbol{\varphi},\theta}\right)\left(\boldsymbol{\varphi}_\theta^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},\theta}\right)\right)^{-1} \cdot \left(\frac{1}{2}\partial_{\mathcal{T}=\theta}\, \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \downarrow \mathbf{g}_{\boldsymbol{\varphi},\tau}\right) \mathrm{d}\theta.$$

The strategy adopted by Pinsky et al. (1983), for the numerical integration of the rate constitutive equation, should be mentioned as an iterative algorithm for the solution of the discretized integral equation.

The converse problem of evaluating the strain field along the motion is readily solved by integrating the direct hypo-elastic law pulled back to a reference configuration:

$$\frac{1}{2}\boldsymbol{\varphi}_t^{\mathrm{REF}} \downarrow \mathbf{g}_{\boldsymbol{\varphi},t} - \frac{1}{2}\boldsymbol{\varphi}_s^{\mathrm{REF}} \downarrow \mathbf{g}_{\boldsymbol{\varphi},s} = \int_s^t (\boldsymbol{\varphi}_\theta^{\mathrm{REF}} \downarrow \mathbf{H}_{\boldsymbol{\varphi},\theta})\left(\boldsymbol{\varphi}_\theta^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},\theta}\right) \cdot \partial_{\tau=\theta}\, \boldsymbol{\varphi}_\tau^{\mathrm{REF}} \downarrow \boldsymbol{\sigma}_{\boldsymbol{\varphi},\tau}\, \mathrm{d}\theta.$$

7. Integration of the hypo-elastic law

Let us now deal with the question about the integrability of a time-independent hypo-elastic constitutive operator, so that: $\mathbf{H}_{\boldsymbol{\varphi},\tau} = \boldsymbol{\varphi}_{\tau,t} \uparrow \mathbf{H}_{\boldsymbol{\varphi},t}$ for any $\tau, t \in I$. Reference will be made to the tangent linear space $\mathbf{T}_{\mathbf{x}}\boldsymbol{\Omega}$ to a generic configuration $\boldsymbol{\varphi} : \mathcal{B} \mapsto \boldsymbol{\mathcal{S}}$ with $\boldsymbol{\varphi}(\mathcal{B}) = \boldsymbol{\Omega}$. In the sequel the subscript in d_F and C_F refers to derivatives taken by holding the base point $\mathbf{x} \in \boldsymbol{\Omega}$, of the involved tensor fields, fixed.

The standard potential theory deals with operators defined in a normed linear space and taking values into the dual linear space. The mathematical tool to be invoked is the symmetry lemma concerning the differential of the operator (Volterra, 1913; Vainberg, 1964). It is a special case of Poincar lemma about exactness of closed forms on differentiable manifolds (Abraham et al., 1988; Marsden and Hughes, 1983; Romano, 2007).

Integrability in hypo-elasticity is a more involved issue since the constitutive operator $\mathbf{H}$ is defined on the linear space $\mathrm{CON}_{\mathbf{x}}(\Omega)$ and takes values in the linear space $L\,(\mathrm{CON}_{\mathbf{x}}(\Omega);\,\mathrm{COV}_{\mathbf{x}}(\Omega))$.

Definition 7.1 (Integrability). *The constitutive operator* $\mathbf{H}$ *of a hypo-elastic material is Cauchy-integrable if there exists a strain-valued stress-potential* $\boldsymbol{\Phi}$ *such that* $\mathbf{H} = d_F\boldsymbol{\Phi}$ *and Green-integrable if there exists a scalar-valued stress-potential* E^* *such that* $\boldsymbol{\Phi} = d_F E^*$ *and hence* $\mathbf{H} = d_F^2 E^*$.

Let us illustrate in detail the two-steps procedure to be followed in applying the standard symmetry lemma to the assessment of the integrability conditions on the hypo-elastic constitutive operator.

Proposition 7.1 (Integrability conditions). *The constitutive operator* $\mathbf{H}$ *of a hypo-elastic material is Cauchy-integrable if and only if the following symmetry condition holds:*

$$\langle d_F\mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}}\cdot\delta_1\boldsymbol{\sigma}_{\mathbf{x}},\delta_2\boldsymbol{\sigma}_{\mathbf{x}}\rangle = \langle d_F\mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}}\cdot\delta_2\boldsymbol{\sigma}_{\mathbf{x}},\delta_1\boldsymbol{\sigma}_{\mathbf{x}}\rangle,$$

for all $\delta\boldsymbol{\sigma}_{\mathbf{x}},\delta_1\boldsymbol{\sigma}_{\mathbf{x}},\delta_2\boldsymbol{\sigma}_{\mathbf{x}}\in\mathrm{CON}_{\mathbf{x}}(\Omega)$. *The further symmetry condition:*

$$\langle \mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta_1\boldsymbol{\sigma}_{\mathbf{x}},\delta_2\boldsymbol{\sigma}_{\mathbf{x}}\rangle = \langle \mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta_2\boldsymbol{\sigma}_{\mathbf{x}},\delta_1\boldsymbol{\sigma}_{\mathbf{x}}\rangle,$$

ensures Green-integrability

Proof. In the linear tensor space $\mathrm{CON}_{\mathbf{x}}(\Omega)$ the symmetry condition for the derivative:

$$\langle d_F\mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}}\cdot\delta_1\boldsymbol{\sigma}_{\mathbf{x}},\delta_2\boldsymbol{\sigma}_{\mathbf{x}}\rangle = \langle d_F\mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}}\cdot\delta_2\boldsymbol{\sigma}_{\mathbf{x}},\delta_1\boldsymbol{\sigma}_{\mathbf{x}}\rangle,$$

for any fixed $\delta\boldsymbol{\sigma}_{\mathbf{x}}\in\mathrm{CON}_{\mathbf{x}}(\Omega)$ and for all $\delta_1\boldsymbol{\sigma}_{\mathbf{x}},\delta_2\boldsymbol{\sigma}_{\mathbf{x}}\in\mathrm{CON}_{\mathbf{x}}(\Omega)$, is equivalent to the vanishing, along any loop in the space $\mathrm{CON}_{\mathbf{x}}(\Omega)$, of the integral (Vainberg, 1964):

$$\oint \mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}} = 0.$$

In coordinates, setting $(\mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}})_{ij} = H_{ijkl}\delta\sigma^{kl}$, the symmetry condition writes: $d_{pq}H_{ijkl}\delta\sigma^{kl}\bullet\delta_1\sigma^{pq}\bullet\delta_2\sigma^{ij} = d_{ij}H_{pqkl}\delta\sigma^{kl}\bullet\delta_2\sigma^{ij}\bullet\delta_1\sigma^{pq}$, that is:

$$d_{pq}H_{ijkl} = d_{ij}H_{pqkl}$$

The vanishing of loop integrals implies that, for any fixed $\delta\boldsymbol{\sigma}_{\mathbf{x}}\in\mathrm{CON}_{\mathbf{x}}(\Omega)$, there exists a scalar potential defined, up to an additive constant, by the integral:

$$\Phi_{\delta\sigma_{\mathbf{x}}}(\boldsymbol{\sigma}_{\mathbf{x}}) = \int_0^1 \left\langle (\mathbf{H}\circ\mathbf{c}^*_{\mathbf{x}})(\lambda)\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}},\dot{\mathbf{c}}^*_{\mathbf{x}}(\lambda)\right\rangle \mathrm{d}\lambda,$$

where $\mathbf{c}_{\mathbf{x}}{}^* \in C^1(I;\,\mathrm{CON}_{\mathbf{x}}(\boldsymbol{\Omega}))$ with $I = [0,1]$ is any path between the points $\mathbf{c}_{\mathbf{x}}{}^*(0) = \mathbf{o}_{\mathbf{x}}$ and $\mathbf{c}_{\mathbf{x}}{}^*(1) = \sigma_{\mathbf{x}}$. Hence:

$$d_F\Phi_{\delta\sigma_{\mathbf{x}}}(\boldsymbol{\sigma}_{\mathbf{x}}) = \mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}}.$$

The dual operators $\mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\in L\,(\mathrm{CON}_{\mathbf{x}}(\Omega);\mathrm{COV}_{\mathbf{x}}(\Omega))$ and $\mathbf{H}^*(\boldsymbol{\sigma}_{\mathbf{x}}) \in L\,(\mathrm{CON}_{\mathbf{x}}(\Omega);\mathrm{COV}_{\mathbf{x}}(\Omega))$ are related by the identity:

$$\left\langle \mathbf{H}^*(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta_2\boldsymbol{\sigma}_{\mathbf{x}},\delta_1\boldsymbol{\sigma}_{\mathbf{x}}\right\rangle = \left\langle \mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta_1\boldsymbol{\sigma}_{\mathbf{x}},\delta_2\boldsymbol{\sigma}_{\mathbf{x}}\right\rangle,$$

and hence we have that:

$$\Phi_{\delta\sigma_{\mathbf{x}}}(\boldsymbol{\sigma}_{\mathbf{x}}) = \int_0^1 \left\langle (\mathbf{H}^*\circ\mathbf{c}^*_{\mathbf{x}})(\lambda)\cdot\dot{\mathbf{c}}^*_{\mathbf{x}}(\lambda),\delta\boldsymbol{\sigma}_{\mathbf{x}}\right\rangle \mathrm{d}\lambda = \boldsymbol{\Phi}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}},$$

with the strain-valued stress-potential $\boldsymbol{\Phi}$, defined on each tensor space $\mathrm{CON}_{\mathbf{x}}(\Omega)$ by:

$$\boldsymbol{\Phi}(\boldsymbol{\sigma}_{\mathbf{x}}) := \int_0^1 (\mathbf{H}^*\circ\mathbf{c}^*_{\mathbf{x}})(\lambda)\cdot\dot{\mathbf{c}}^*_{\mathbf{x}}(\lambda)\ \mathrm{d}\lambda.$$

Being $d_F\boldsymbol{\Phi}_{\delta\sigma_{\mathbf{x}}}(\boldsymbol{\sigma}_{\mathbf{x}}) = \mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}})\cdot\delta\boldsymbol{\sigma}_{\mathbf{x}}$, for any fixed $\delta\boldsymbol{\sigma}_{\mathbf{x}}\in\mathrm{CON}_{\mathbf{x}}(\Omega)$, we infer that:

$$d_F\boldsymbol{\Phi} = \mathbf{H}.$$

The fulfillment of the symmetry property $\mathbf{H}^*(\boldsymbol{\sigma}_{\mathbf{x}}) = \mathbf{H}(\boldsymbol{\sigma}_{\mathbf{x}}) = d_F\boldsymbol{\Phi}(\boldsymbol{\sigma}_{\mathbf{x}})$, implies the vanishing, in the tensor space $\mathrm{CON}_{\mathbf{x}}(\Omega)$, of any loop integral:

$$\oint \boldsymbol{\Phi}(\boldsymbol{\sigma}_{\mathbf{x}}) = 0.$$

This ensures existence of a scalar-valued stress-potential E^* whose values on the tensor space $\mathrm{CON}_{\mathbf{x}}(\Omega)$ are given by:

$$E^*(\boldsymbol{\sigma}_{\mathbf{x}}) = \int_0^1 (\boldsymbol{\Phi}\circ\mathbf{c}^*_{\mathbf{x}})(\lambda)\cdot\dot{\mathbf{c}}^*_{\mathbf{x}}(\lambda)\,\mathrm{d}\lambda = \int_0^1 \mathrm{d}\lambda\int_0^\lambda (\mathbf{H}\circ\mathbf{c}^*_{\mathbf{x}})(\xi)\cdot\dot{\mathbf{c}}^*_{\mathbf{x}}(\xi)\,\mathrm{d}\xi,$$

so that $\boldsymbol{\Phi} = d_F E^*$ and $\mathbf{H} = d_F\boldsymbol{\Phi} = d_F{}^2E^*$.

Remark 7.1. *The symmetry condition* $d_{pq}H_{ijkl} = d_{ij}H_{pqkl}$ *differs from the one reported, with reference to finite elasticity, in Simó and Pister (1984, formula 3.4) which, in our notations, would read* $d_{pq}H_{klij} = d_{ij}H_{klpq}$.

Proposition 7.2 (Integrability and time-independence). *Integrability of a time-independent hypo-elastic constitutive operator, at a given time, implies integrability at every time. By pushing the potentials at a fixed reference configuration, time-independent potentials are got and potentials at displaced reference configurations are related by push.*

Proof. Let $\varphi_{\tau,t}\in C^1(\Omega_t;\Omega_\tau)$ be a material displacement. Time-independence and the integral transformation formula imply that, at $\mathbf{x}\in\Omega_t$, for any given $\delta\boldsymbol{\sigma}_{\varphi,t}\in\mathrm{CON}_{\mathbf{x}}(\Omega_t)$ and for any cycle in $\mathbf{c}^*\subset\mathrm{CON}_{\mathbf{x}}(\Omega_t)$:

$$\begin{aligned}\oint_{\varphi_{\tau,t}(\mathbf{c}^*)} \mathbf{H}_{\varphi,\tau}(\varphi_{\tau,t}\uparrow\boldsymbol{\sigma}_{\varphi,t})\cdot\varphi_{\tau,t}\uparrow\delta\boldsymbol{\sigma}_{\varphi,t} &= \oint_{\varphi_{\tau,t}(\mathbf{c}^*)} \varphi_{\tau,t}\uparrow(\mathbf{H}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t})\cdot\delta\boldsymbol{\sigma}_{\varphi,t})\\ &= \oint_{\mathbf{c}^*} \mathbf{H}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t})\cdot\delta\boldsymbol{\sigma}_{\varphi,t},\end{aligned}$$

$$\begin{aligned}\oint_{\varphi_{\tau,t}(\mathbf{c}^*)} \boldsymbol{\Phi}_{\varphi,\tau}(\varphi_{\tau,t}\uparrow\boldsymbol{\sigma}_{\varphi,t}) &= \oint_{\varphi_{\tau,t}(\mathbf{c}^*)} \varphi_{\tau,t}\uparrow(\boldsymbol{\Phi}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t}))\\ &= \oint_{\mathbf{c}^*} \boldsymbol{\Phi}_{\varphi,t}(\boldsymbol{\sigma}_{\varphi,t}).\end{aligned}$$

The first part of the statement follows. The last part is a simple consequence of the chain composition of pushes according to composite maps.

Taking into account that the pull-back of a convective time-derivative is equal to the time-derivative of the pull-back, the hypo-elastic law of a Green-integrable hypo-elastic material in a reference configuration $\boldsymbol{\chi}_o:\mathcal{B}\mapsto\boldsymbol{\Omega}_o$, setting $\varphi_t^{\mathrm{REF}} = \varphi_t\circ\boldsymbol{\chi}_o^{-1}$, is given by:

$$\begin{aligned}\partial_{\tau=t}\,\frac{1}{2}\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{\tau}\downarrow\mathbf{g}_{\varphi,\tau}\right) &= d_F^2E^*\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{t}\downarrow\boldsymbol{\sigma}_{\varphi,t}\right)\cdot\partial_{\tau=t}\,\boldsymbol{\varphi}^{\mathrm{REF}}_{\tau}\downarrow\boldsymbol{\sigma}_{\varphi,\tau}\\ &= \partial_{\tau=t}\,d_FE^*\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{\tau}\downarrow\boldsymbol{\sigma}_{\varphi,\tau}\right),\end{aligned}$$

and may be readily integrated along the motion, in the interval $[s,t]$, to yield the referential hyper-elastic law:

$$\frac{1}{2}\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{t}\downarrow\mathbf{g}_{\varphi,t}-\boldsymbol{\varphi}^{\mathrm{REF}}_{s}\downarrow\mathbf{g}_{\varphi,s}\right)=d_FE^*\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{t}\downarrow\boldsymbol{\sigma}_{\varphi,t}\right)-d_FE^*\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{s}\downarrow\boldsymbol{\sigma}_{\varphi,s}\right).$$

If $\boldsymbol{\varphi}^{\mathrm{REF}}_{s}:\mathcal{B}\mapsto\boldsymbol{\Omega}_s$ is a natural configuration, where the stress is assumed to vanish, the referential hyper-elastic law, being $\boldsymbol{\varphi}^{\mathrm{REF}}_{t}=\boldsymbol{\varphi}_{t,s}\circ\boldsymbol{\varphi}^{\mathrm{REF}}_{s}$, takes the form:

$$\begin{aligned}\frac{1}{2}\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{t}\downarrow\mathbf{g}_{\varphi,t}-\boldsymbol{\varphi}^{\mathrm{REF}}_{s}\downarrow\mathbf{g}_{\varphi,s}\right) &= \boldsymbol{\varphi}^{\mathrm{REF}}_{s}\downarrow\frac{1}{2}\left(\boldsymbol{\varphi}_{t,s}\downarrow\mathbf{g}_{\varphi,t}-\mathbf{g}_{\varphi,s}\right)\\ &= d_FE^*\left(\boldsymbol{\varphi}^{\mathrm{REF}}_{t}\downarrow\boldsymbol{\sigma}_{\varphi,t}\right).\end{aligned}$$

The integrability condition in Proposition 7.1, concerning the symmetry of the derivative of the hypo-elastic operator, is trivially verified if the operator is independent of the stress state. Moreover, time-independence of the hypo-elastic constitutive operator assures that independence of the stress state, once verified at a given time, holds at any other one. The following chain of inclusions holds true:

time-independent hypo-elasticity

$\supset$ elasticity

$\supset$ hyper-elasticity.

It can be shown that integrability is also independent of the alteration of the tensors chosen to formalize the hypo-elastic response and that integrability to a hyper-elastic law is equivalent to conservation of mechanical energy. In this respect we quote the interesting discussion, in a thermodynamical context, by Casey (2005).

Remark 7.2. *A Cauchy-elastic constitutive model is well defined as a time-independent, integrable hypo-elastic model whose strain-valued stress potential is strictly monotone. An isometric material displacement of an elastic body along the motion, leaves the stress tensor unchanged in time, that is changed by push. To see this, we observe that an isometric material displacement of a Cauchy-elastic body along the motion, does not change the Cauchy-Green stretch in a reference configuration where the elastic law is expressed by an invertible smooth relation between pulled-back stress and metric material tensors. The stress tensors at the isometrically displaced configurations along the motion, are obtained by pushing forward the same corresponding referential elastic stress and will therefore be related by push along the isometric material displacement. In accord with the covariance paradigm and Definition 5.1, the conclusion is that the elastic stress is time-independent.*

8. The simplest hypo-elastic model

The simplest hypo-elastic model, corresponding to the rate form of the standard linear isotropic elasticity model adopted in the small displacements range, has been most widely adopted in computational mechanics, (see e.g. Key and Krieg, 1982). The model was investigated in Simó and Pister (1984), Sansour and Bednarczyk (1993) who, by adopting the incorrect integrability conditions provided in Bernstein (1960), found that this hypo-elastic material is not hyper-elastic. On the contrary, on the basis of the covariant theory and of the correct integrability conditions provided above, it will be shown that the simplest hypo-elastic model is indeed hyper-elastic. Denoting the mixed forms of stretching and stressing at time $t\in I$ by:

$$\mathbf{D}_{\varphi,t}:=\mathbf{g}^{-1}_{\varphi,t}\circ\frac{1}{2}\left(\mathcal{L}_{\varphi,t}\,\mathbf{g}_{\varphi}\right),\quad \overset{\Delta}{\mathbf{T}}_{\varphi,t}:=\frac{1}{2}\left(\mathcal{L}_{\varphi,t}\,\boldsymbol{\sigma}_{\varphi}\right)\circ\mathbf{g}_{\varphi,t},$$

the simplest hypo-elastic model is described by the linear, isotropic rate law:

$$\mathbf{D}_{\varphi,t}=\frac{1}{2\mu}\overset{\Delta}{\mathbf{T}}_{\varphi,t}-\frac{\nu}{E}\,J_1\left(\overset{\Delta}{\mathbf{T}}_{\varphi,t}\right)\mathbf{I}_{\varphi,t}=\mathbf{H}^{\mathrm{MIX}}_{\varphi,t}(\mathbf{T}_{\varphi,t})\cdot\overset{\Delta}{\mathbf{T}}_{\varphi,t},$$

so that the hypo-elastic constitutive operator is given by:

$$\mathbf{H}^{\mathrm{MIX}}_{\varphi,t}(\mathbf{T}_{\varphi,t}):=\frac{1}{2\,\mu}\mathbb{I}_{\varphi,t}-\frac{\nu}{E}\,\mathbf{I}_{\varphi,t}\otimes\mathbf{I}_{\varphi,t},$$

with E Euler (or Young) modulus, ν Poisson ratio and $\mu=E/(2(1+\nu))$ Lamé shear modulus. Here $\mathbf{I}_{\varphi,t}(\mathbf{x})\in\mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t)$ is the identity tensor, $\otimes$ is the tensor product in the inner product tensor space $\mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t)$ and $\mathbb{I}_{\varphi,t}(\mathbf{x})\in L(\mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t);\mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t))$ is the identity operator.

The simplest hypo-elastic constitutive operator is evidently stress independent and symmetric. The integrability conditions of Proposition 7.1 are then fulfilled and we may conclude that the simplest model is Green-hypo-elastic. Integrating along a ray from the origin of the tensor space $\mathrm{MIX}_{\mathbf{x}}(\boldsymbol{\Omega}_t)$ to a stress value $\mathbf{T}_{\varphi,t}$, the strain-valued stress-potential $\boldsymbol{\Phi}^{\mathrm{MIX}}_{\varphi,t}=d_FE^{*\mathrm{MIX}}_{\varphi,t}$ at the material configuration $\boldsymbol{\varphi}_t:\mathcal{B}\mapsto\boldsymbol{\Omega}_t$ is readily computed:

$$\boldsymbol{\Phi}^{\mathrm{MIX}}_{\varphi,t}(\mathbf{T}_{\varphi,t})=\int_0^1\mathbf{H}^{\mathrm{MIX}}_{\varphi,t}\left(\lambda\,\mathbf{T}_{\varphi,t}\right)\cdot\mathbf{T}_{\varphi,t}\,d\lambda=\frac{1}{2\mu}\mathbf{T}_{\varphi,t}-\frac{\nu}{E}J_1(\mathbf{T}_{\varphi,t})\mathbf{I}_{\varphi,t}.$$

Time-independency of the simplest hypo-elastic constitutive operator, is expressed, according to Definition 6.2, by the equality:

$$\frac{1}{2\mu}\,\mathbb{I}_{\varphi,\tau}-\frac{\nu}{E}\,\mathbf{I}_{\varphi,\tau}\otimes\mathbf{I}_{\varphi,\tau}=\boldsymbol{\varphi}_{\tau,t}\uparrow\left(\frac{1}{2\mu}\mathbb{I}_{\varphi,t}-\frac{\nu}{E}\,\mathbf{I}_{\varphi,t}\otimes\mathbf{I}_{\varphi,t}\right),$$

which is inferred from the formulas: $\boldsymbol{\varphi}_{\tau,t}\uparrow\mathbb{I}_{\varphi,t}=\mathbb{I}_{\varphi,\tau}$, $\boldsymbol{\varphi}_{\tau,t}\uparrow\mathbf{I}_{\varphi,t}=\mathbf{I}_{\varphi,\tau}$ and:

$$\boldsymbol{\varphi}_{\tau,t}\uparrow\left(\mathbf{I}_{\varphi,t}\otimes\mathbf{I}_{\varphi,t}\right)=\left(\boldsymbol{\varphi}_{\tau,t}\uparrow\mathbf{I}_{\varphi,t}\right)\otimes\left(\boldsymbol{\varphi}_{\tau,t}\uparrow\mathbf{I}_{\varphi,t}\right).$$

Let us now assume that $\boldsymbol{\varphi}_t:\mathcal{B}\mapsto\boldsymbol{\Omega}_t$ is a natural, stress-free reference configuration. The hyper-elastic law may then be written in terms of the mixed Green's strain tensor, as:

$$\mathbf{E}_{\varphi_{\tau,t}}=\boldsymbol{\Phi}^{\mathrm{MIX}}_{\varphi,t}(\boldsymbol{\varphi}_{\tau,t}\downarrow\mathbf{T}_{\varphi,\tau})=\frac{1}{2\mu}\,\boldsymbol{\varphi}_{\tau,t}\downarrow\mathbf{T}_{\varphi,\tau}-\frac{\nu}{E}J_1(\boldsymbol{\varphi}_{\tau,t}\downarrow\mathbf{T}_{\varphi,\tau})\mathbf{I}_{\varphi,t},$$

or, in inverse form:

$$\frac{1}{2\mu}\,\boldsymbol{\varphi}_{\tau,t}\downarrow\mathbf{T}_{\varphi,\tau}=\mathbf{E}_{\varphi_{\tau,t}}+\frac{\nu}{1-2\nu}\,J_1\left(\mathbf{E}_{\varphi_{\tau,t}}\right)\mathbf{I}_{\varphi,t}.$$

The Cauchy true stress $\mathbf{T}_{\varphi,\tau}\in\mathrm{C}^1(\boldsymbol{\Omega}_\tau;\mathrm{MIX}(\boldsymbol{\Omega}_\tau))$ is recovered from the reference one $\boldsymbol{\varphi}_{\tau,t}\downarrow\mathbf{T}_{\varphi,\tau}\in\mathrm{C}^1(\boldsymbol{\Omega}_t;\mathrm{MIX}\{\boldsymbol{\Omega}_t\})$ by push forward:

$$\mathbf{T}_{\varphi,\tau}=T\boldsymbol{\varphi}_{\tau,t}\circ\left(\boldsymbol{\varphi}_{\tau,t}\downarrow\mathbf{T}_{\varphi,\tau}\right)\circ T\boldsymbol{\varphi}^{-1}_{\tau,t}.$$

8.1. Simple shear

Let us consider a unit cube as a natural stress-free configuration of a body and a Cartesian reference system. A simple shear, see Fig. 3, is described by a material displacement whose expression in the reference system, setting $\gamma_{\tau,t}:=\gamma\,(\tau-t)$, is given by:

$$\boldsymbol{\varphi}_{\tau,t}(x,y,z) = (x + \gamma_{\tau,t}\, y)\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3.$$

The matrices of the relevant tangent map and of its inverse are given by:

$$[T\boldsymbol{\varphi}_{\tau,t}] = \begin{bmatrix} 1 & \gamma_{\tau,t} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [T\boldsymbol{\varphi}^{-1}_{\tau,t}] = \begin{bmatrix} 1 & -\gamma_{\tau,t} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

The matrix of the mixed Green's strain $\mathbf{E}_{\varphi_{\tau,t}}$ writes:

$$\left[\mathbf{E}_{\varphi_{\tau,t}}\right] = \frac{1}{2}\begin{bmatrix} 0 & \gamma_{\tau,t} & 0 \\ \gamma_{\tau,t} & \gamma^2_{\tau,t} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently, the matrix of the referential Cauchy true stress will be:

$$\frac{1}{\mu}[\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\varphi,\tau}] = \begin{bmatrix} 0 & \gamma_{\tau,t} & 0 \\ \gamma_{\tau,t} & \gamma^2_{\tau,t} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\nu}{1-2\nu}\,\gamma^2_{\tau,t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The initial, linearized law is expressed by the usual relation between shearing stressing and stretching, i.e.

$$\overset{\Delta}{\mathbf{T}}_{\varphi,t} = \partial_{\tau=t}\left[\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\varphi,\tau}\right] = \mu\gamma\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2\mu\,\partial_{\tau=t}\,\mathbf{E}_{\varphi_{\tau,t}}$$

$$= 2\mu\,\mathbf{D}_{\varphi,t}.$$

Being:

$$\begin{bmatrix} 1 & \gamma_{\tau,t} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & \gamma_{\tau,t} & 0 \\ \gamma_{\tau,t} & \gamma^2_{\tau,t} & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & -\gamma_{\tau,t} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \gamma^2_{\tau,t} & \gamma_{\tau,t} & 0 \\ \gamma_{\tau,t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the matrix of the Cauchy true stress is given by

$$\frac{1}{\mu}[\mathbf{T}_{\varphi,\tau}] = \begin{bmatrix} \gamma^2_{\tau,t} & \gamma_{\tau,t} & 0 \\ \gamma_{\tau,t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\nu}{1-2\nu}\,\gamma^2_{\tau,t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

8.2. Homogeneous extension

A homogeneous extension, see Fig. 4, is described by a material displacement, of a unitary cube given, in a cartesian reference system, setting $\alpha_{\tau,t} := \alpha\,(\tau - t)$ and $\beta_{\tau,t} := \beta\,(\tau - t)$, by:

$$\boldsymbol{\varphi}_{\tau,t}(x,y,z) = \alpha_{\tau,t}\,x\,\mathbf{e_x} + \beta_{\tau,t}\,y\,\mathbf{e_y} + z\,\mathbf{e_z}.$$

The matrix of the tangent map $T\boldsymbol{\varphi}_{\tau,t}$ and the matrix of the mixed Green strain are:

$$[T\boldsymbol{\varphi}_{\tau,t}] = \begin{bmatrix} \alpha_{\tau,t} & 0 & 0 \\ 0 & \beta_{\tau,t} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[\mathbf{E}_{\varphi_{\tau,t}}] = \frac{1}{2}\begin{bmatrix} \alpha^2_{\tau,t} - 1 & 0 & 0 \\ 0 & \beta^2_{\tau,t} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

According to the simplest rate law, the matrix of the referential Cauchy true stress, setting $k_{\tau,t} = \nu\,((\alpha^2_{\tau,t} + \beta^2_{\tau,t}) - 2)/(1-2\nu)$, is given by

$$\frac{1}{\mu}[\boldsymbol{\varphi}_{\tau,t}\!\downarrow\!\mathbf{T}_{\varphi,\tau}] = \begin{bmatrix} \alpha^2_{\tau,t} - 1 & 0 & 0 \\ 0 & \beta^2_{\tau,t} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - k_{\tau,t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that for the Cauchy true stress we get:

$$\frac{1}{\mu}[\mathbf{T}_{\varphi,\tau}] = \begin{bmatrix} \alpha^2_{\tau,t} - 1 & 0 & 0 \\ 0 & \beta^2_{\tau,t} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - k_{\tau,t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Assuming $\nu = 0$ and $\beta_{\tau,t} = (\alpha_{\tau,t})^{-1}$, which corresponds to a vanishing Poisson effect and to an isochoric displacement, the normal stress $\mathbf{T}_{11}(\tau)$ and the resultant axial force $N(\tau) = A(\tau)\,\mathbf{T}_{11}(\tau) = \mu\,(\alpha_{\tau,t} - 1/(\alpha_{\tau,t}))$, where $A(\tau) = 1/(\alpha_{\tau,t})$ is the transversal area, are plotted in Fig. 4.

8.3. Comparison with other treatments

The two examples presented in Sections 8.1 and 8.2 were treated, with a non-covariant definition of the simplest hypo-elastic law, in Pinsky et al. (1983, examples 6.1 and 6.2) where spatial expressions for the stress rate, according to Oldroyd, Truesdell and Jaumann proposals, have been considered. The former example (simple shear) was also discussed in Sansour and Bednarczyk (1993). Explicit calculations of the same non-covariant examples were exposed in Lo (1988, Appendix) by performing a time integration of the spatial form of the constitutive law, at a fixed point of the ambient space. The analysis for the simple shear led to a Cauchy true stress depending only of Lam shear modulus. For the axial elongation the true stress was evaluated to have an exponential expression. Spurious oscillating responses under monotone shearing were detected in Dienes (1979) by adopting the Jaumann stress rate. The attainment of a limit value of the shear stress (the hypo-elastic yield of Truesdell (1955), see also (Truesdell and Noll, 1965, p. 420) was displayed also by the logarithmic model proposed in Xiao et al. (1997). A maximum load effect was reported in Pinsky et al. (1983, example 6.2) for homogeneous monotone extension of a square block modeled in terms of Jaumann stress rate. All these physically

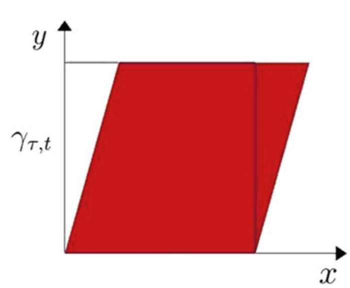

Fig. 3. Simple shear.

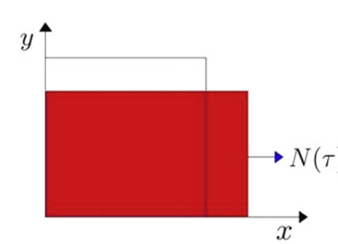

Fig. 4. Homogeneous extension.

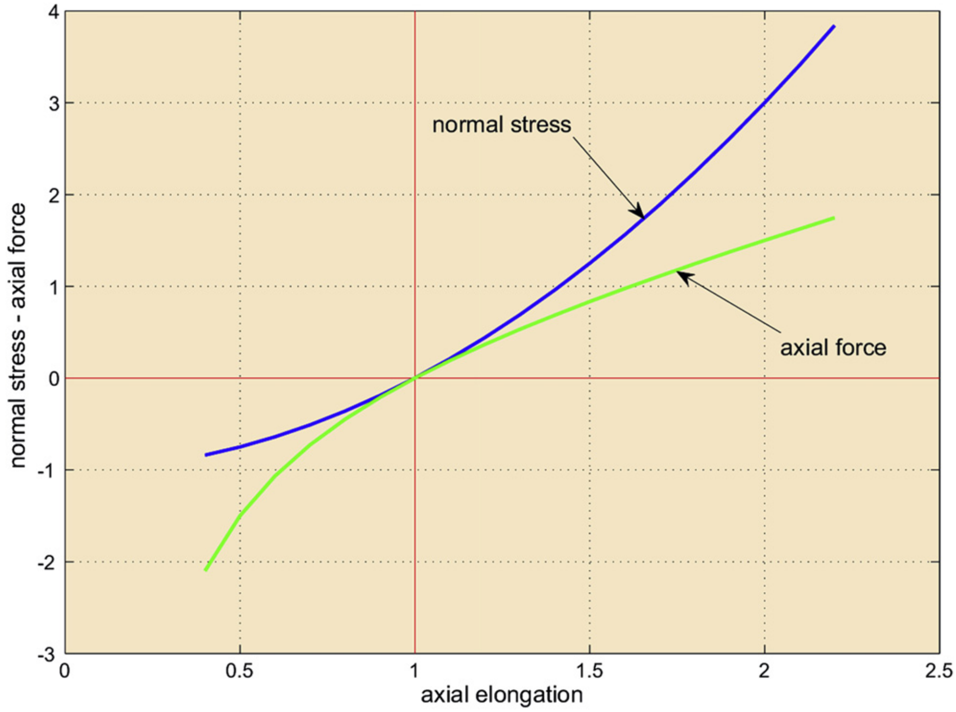

Fig. 5. Normal stress and axial force vs elongation.

unsound effects are absent in the constitutive model formulated according to the covariance paradigm (Fig. 5).

9. Conclusions

Although many partial contributions may be found here and there in literature, an explicit conversion to a fully covariant theory of material behavior was never accomplished. A clear hint towards a covariant theory, with a declared computational bias, was provided by the treatment of finite strain visco-elasto-plasticity developed in Pinsky et al. (1983), Simó and Ortiz (1985), in the wake of the treatment of elasticity theory exposed in Marsden and Hughes (1983).

The *covariance paradigm*, introduced in the present paper on the basis of a punctual distinction between spatial vectors, material-based spatial vectors and material vectors, is an innovative contribution which provides the comparison criterion between material tensor fields in displaced configurations of a body. The involved geometric tool is the push along the relevant displacement. This is in contrast with the requirement of *form-invariance* in which an incorrect equality between constitutive maps, with different domains, is involved, see e.g. (Svendsen and Bertram, 1999). The basic role of the covariance paradigm, in developing the theory of constitutive relations in the geometrically nonlinear range, is especially manifest when applied to continua of lower dimensionality, such as wires and membranes. In the covariant theory, elastic materials are characterized as integrable time-independent hypo-elastic materials. The usual definition is recovered, by pull-back to a reference configuration and integration in time, which leads to a time-independent elastic relation between the pull-back of the stress tensor and the pull-back of the material metric tensor. The result is independent of the choice of a reference configuration. The geometric approach to nonlinear continuum mechanics developed in this paper, with explicit application to the theory of hypo-elastic constitutive models, is a major step in a geometrization program started with the formulation of material inhomogeneities, shell models, and continuum dynamics in Romano et al. (2006, 2007, 2009a,b,c). Background mathematical tools are collected in Romano (2007).

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