

Electrodynamics without Lorentz force

Giovanni Romano

University of Naples Federico II
Naples, Italy

Abstract

This communication is devoted to a brief historical framework and to a comprehensive critical discussion concerning foundational issues of Electrodynamics. Attention is especially focused on the events which, about the end of XIX century, led to the notion of **LORENTZ** force, still today ubiquitous in literature on Electrodynamics. Is this a noteworthy instance of a rule which, generated by an improper simplification of **MAXWELL-J.J. THOMSON** formulation, is in fact physically untenable but, this notwithstanding, highly successful. Modelling of electromagnetic fields and fluxes in spacetime respectively as *even* and *odd* spatial differential forms and the formulation of induction laws by means of exterior and **LIE** derivatives, make their covariance manifest under any smooth spacetime transformations, contrary to the usual affirmation in literature which confines this property to relativistic frame-changes. A remarkable consequence is that there is no entanglement between electric and magnetic fields and fluxes under special relativity transformations. In particular, relativistic support to **LORENTZ** force rule is thus deactivated. For translational motions of charged bodies immersed in a uniform and constant magnetic field, the induced electric field in such a frame, is equal to one half the **LORENTZ** force term. The qualitative successful application of the **LORENTZ** force rule to experimental evidence of special observers is therefore explained.

Keywords: Electromagnetic induction; Lineal rule; Vortex rule; Frame covariance; Lorentz force; Relativistic entanglement.

1 Introduction

The story I am going to tell you, provides a sound confirmation of a sentence by Samuel Langhorne **CLEMENS** (1835–1910) best known by pen name, **MARK TWAIN**:

When even the brightest mind in our world has been trained up from childhood in a superstition of any kind, it will never be possible for that mind, in its maturity, to examine sincerely, dispassionately, and conscientiously any evidence or any circumstance which shall seem to cast a doubt upon the validity of that superstition.

Autobiography (1959).

The difficulty evidenced in the sentence is even greater for those who might have actively contributed in disseminating that superstition.¹

Physically biased readers would deem the present treatment to be rather mathematical than physical in style. In this respect the famous words by **Galilei** (1623) about the essential role of Mathematics in modelling physical phenomena, might however be remembered.²

¹ The author himself graduated in electronic engineering in 1965, an epoch where a training in differential geometry was not included in educational plans, even at postgraduate level.

² **GALILEO GALILEI**, *ad litteram*, in Italian: “La filosofia naturale è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi, io dico l’universo, ma non si può intendere se prima non s’impara a intender la lingua e conoscer i caratteri nei quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto.”

In this respect, we will show that it is precisely the improper mathematical treatment of basic laws of electrodynamics which is responsible for long standing issues and vain debates concerning tentative and flawed interpretations of improper formulations.

For instance, a common misdeed is the confusion between convective and parallel derivatives, with consequent geometric misstatements about frame covariance of electromagnetic induction laws.

A related source of improper statements is the usual representation of fields and fluxes in terms of vector fields and partial derivatives rather than in terms of differential forms and exterior derivatives. These differential geometric notions are in fact the ones directly stemming from integral formulations of physical laws and the ones naturally susceptible of a clean geometric treatment.

An early temptative title for this paper was conceived as *The Tragicomical History of Lorentz Force*, boldly borrowed from *The Tragicomical History of Thermodynamics* (Truesdell, 1980), authored by Clifford Ambrose TRUESDELL III, a master of style and a passionate scholar and historian of Science, credited with an exceptional range of knowledge, as witnessed by treatises (Truesdell and Toupin, 1960; Truesdell and Noll, 1965) and by widely ranging interests and publications.

This title was eventually dismissed since somebody was deceptively brought to think that the paper was an historical essay rather than an original contribution to foundational aspects of Electrodynamics.

The LORENTZ force was so named after the influential dutch physicist Hendrik Antoon LORENTZ (1853–1928), who displayed it in (Lorentz, 1892, 1899, 1904).

The relevant mathematical expression was however formulated much before by James Clerk-Maxwell (1855) in his wonderful completion of the pioneering formulations of electromagnetism by André-Marie Ampère (1826), Michael Faraday (1838) and Franz Ernst Neumann (1846).³

³ The celebrated Scottish scientist James CLERK-MAXWELL (1831–1879), commonly abridged to MAXWELL in literature since the end of XIXth century, is too often improperly cited as C. MAXWELL, or MAXWELL, J.C., even in historical essays (Katz, 1979; Darrigol, 2000; Bucci, 2014).

Still nowadays the role of LORENTZ force is universally considered to be so basic in Electrodynamics that anyone casting a doubt upon the validity of this notion may seriously run the risk of being deemed a heretic.

In this respect we must keep in mind an often verifiable aspect of human belief effectively commented by Bertrand RUSSEL by the crude words (Russell, 1929):

The fact that an opinion has been widely held is no evidence whatever that it is not utterly absurd; indeed in view of the silliness of the majority of mankind, a widespread belief is more likely to be foolish than sensible.

Occasional criticisms in the past years had to face harshly against a widely spread resort to the notion of LORENTZ force, spanning from research papers and treatises on Electrodynamics, to high-school and university textbooks and presently repeated also in a multitude of web sites (Munley, 2004).

Yet difficulties concerning the notion of LORENTZ force have been early raised in literature, by authoritative scholars, see fn.10.

Vain relativistic arguments have also been brought in support of LORENTZ force (Feynman et al., 1964, II.13-6), (Purcell, 1965, Ch.5).

These relativistic interpretations are in manifest contrast with the evidence of tests concerning the action of magnetic fields upon a beam of electric point charges (the cathodic tube rays of Joseph John THOMSON).⁴

There in fact speeds far below the limit value pertaining to light *in vacuo* are involved.

Accordingly, it is quite reasonable to sustain that relativistic arguments should not play any role in this matter.

This is the conclusion of the present analysis, as we shall eventually see.

To bring a contribution aimed at clarifying the issue, we will deal with a spacetime formulation of Electrodynamics and with the relativistic phenomenon of electromagnetic entanglement taken as well-founded in literature.

⁴ Successor of MAXWELL as Cavendish Professor of Physics at Cambridge. His name is usually abridged to J.J. THOMSON to avoid confusion with William THOMSON (Lord KELVIN).

In special relativity, the transformation of electromagnetic fields under **LORENTZ** frame-changes (Lorentz, 1904; Poincaré, 1905; Einstein, 1905a) were conceived in the wake of early treatments by Oliver Heaviside (1885, 1892) and Heinrich Hertz (1892).⁵

These pioneering analyses were later reproduced in literature without significant modifications (Sommerfeld, 1952; Panofski and Phillips, 1962; Feynman et al., 1964; Purcell, 1965; Misner, Thorne & Wheeler, 1973; Parrott, 1987; Landau and Lifšits, 1987; Jackson, 1999).

The involved entanglements of electric and magnetic fields were accordingly assumed to persist in the classical limit, that is for a vanishing ratio between boost speed and speed of light *in vacuo*.

Inspection of early treatments (Lorentz, 1892, 1904; Poincaré, 1905; Einstein, 1905a,b) reveals however that entanglements of electric and magnetic fields were stipulated under the assumptions of an alleged *form-invariance* of electromagnetic induction laws and of conservation of electric charge under **LORENTZ** transformation rule, both contrasted by geometric evidence, as will be here shown in §18.

In particular, transformations of **MAXWELL-HERTZ** equations for empty space due to action of a **LORENTZ** frame-change, were stipulated without explicit proof in (Einstein, 1905a, Part II, §4) as outcome of a simple substitution, by appealing to *form-invariance*.⁶ With any evidence the procedure there sketched was a partial rephrasing of seminal treatments pioneered at the same time by Lorentz (1904) and Poincaré (1905). The differential geometric analysis developed in §18 of the present contribution, leads to the outstanding conclusion that the laws of electromagnetic induction, when properly formulated in terms of integrals of differential forms, are fully covariant under any frame-change and that no electromagnetic entanglement does occur.

⁵ According to (Minkowski, 1908) the transformations proposed by Lorentz (1904) and reformulated as a mathematical group by Henri Poincaré (1905), were first conceived by Wolde-mar Voigt (1887a,b).

⁶ EINSTEIN statement was literally: "If we apply to these (**MAXWELL-HERTZ**) equations the (**LORENTZ**) transformation developed in §3, by referring the electromagnetic processes to the system of co-ordinates there introduced, moving with the velocity v , we obtain the equations....".

More precisely, the resulting transformation rule may be enunciated in the following neat simple terms.

The components of the transformed electromagnetic fields, represented by time-vertical exterior forms, when evaluated in the original framing, obey the following rule.

Either they undergo an amplification by the scalar relativistic factor or otherwise remain invariant.

The alternative depending on whether, in evaluating the component under investigation, the direction of **LORENTZ** boost is included in the list of argument vectors or does not.

The relativistic factor goes to infinity when the boost speed tends to the limit value of light speed *in vacuo*, and to unity in the classical limit, for a boost speed smaller and smaller than the light speed *in vacuo*.

Consequently, in the classical limit the transformed electromagnetic fields tend rapidly to coincide with those evaluated under the action of the standard **GALLEI** transformation group. This is exactly what was to be expected on a physico-mathematical ground, due to the continuous dependence of **LORENTZ** transformations upon the relativistic factor. The contributed amendment to the alleged transformation rules of special relativity deactivates any relevance of relativistic effects in support of **LORENTZ** force.

Basic features of electrodynamics up to the sixties are comprehensively illustrated in the monumental treatise (Truesdell and Toupin, 1960, Ch. F). Formulations in terms of differential forms were carried out in (Misner, Thorne & Wheeler, 1973) and in (Parrott, 1987).

In the recent book (Hehl and Obukhov, 2003) on foundations of classical electrodynamics, the **LORENTZ** force law is introduced as an additional axiom. Accurate overviews on formulations of Electrodynamics and historical spotlights have also been contributed in (Hehl and Obukhov, 2000; Hehl, 2010).

The present treatment starts by observing that the law of electrical induction, usually stated by means of the well-known *flux-rule*, is rather formulated in the most general and clearest way in terms of path integrals of electric and magnetic one-forms along ar-

bitrary piecewise smooth paths.

The *flux-rule*, restated here as *vorticity rule*, being expressed in terms of *inner* oriented surfaces and *even* exterior forms, is applicable only to circuits which are boundaries of surfaces undergoing regular motions.

Consequently the *vorticity rule* can be adopted only in simple model cases of scant applicative interest, or at best as a convenient approximation in describing the functioning of technical devices such as *solenoids*.

The approach in terms of magnetic vector potential was the one originally undertaken by [Ampère \(1826\)](#), [Clerk-Maxwell \(1855, 1861\)](#) and later adopted also by Hermann von [Helmholtz \(1870, 1873, 1874\)](#) and by [J.J. Thomson \(1881, 1893\)](#).

Simplifying modifications, introduced soon later by [Heaviside \(1885, 1892\)](#), [Hertz \(1892\)](#) and [Lorentz \(1892\)](#), were highly successful in the engineering community due to the computationally convenient substitution of the magnetic vector potential field (named *magnetic momentum* by [MAXWELL](#)) with its curl, the magnetic induction vector field.⁷

This convenience may be naïvely illustrated by considering the analogy with the kinematics of an act of rigid rotation, characterised by a non-uniform velocity field with a uniform field of curls (the skew symmetric part of the derivative).

In introducing the simplification, a velocity dependent scalar potential in the expression of the electric vector field was consequently and correctly ignored as not influential for the computation of the curl.

The resulting induction law was however improperly still applied to evaluate the electric field itself and not just its curl, so that the whole affair went unexpectedly along a wrong way.

As a consequence the expression of the electric vector field became velocity dependent, even under the action of [GALILEI](#) frame transformations.

[GALILEI](#) relativity principle of Classical Dynamics, stating covariance of the law of motion when passing from one inertial frame to another still inertial, was thus violated.

⁷ [HEAVISIDE](#) deemed potentials to be treacherous and useless, due to intrinsic lack of uniqueness ([Deschamps, 1981](#)).

To find a way out of this embarrassing situation, the answer sometimes given to naturally arising questions about who is measuring the velocity of a charged particle moving in the magnetic vortex field, is that “measurements ought to be made in the laboratory frame”! But which lab?

When the full spacetime expression of the electric vector field induced by a magnetic momentum is adopted, the resulting scenario becomes by far different and full covariance of electric induction laws, under any change of spacetime frame, is attained.

A differential geometric treatment reveals in fact that the expression of the electric field induced by a magnetic field is given by the negative⁸ of the [LIE](#) derivative of magnetic momentum along the spacetime motion detected in the observer frame.

This convective derivative behaves in a natural way, being covariant under any spacetime frame-change transformation. When the involved vector fields and the motion are transformed by covariance according to frame-change, the spacetime velocity and the convective derivative along it also transform by covariance so that the law of electric induction is still fulfilled ([G. Romano, 2013](#)).

The general formula for the electric field, in terms of [LIE](#) derivative along the spacetime motion, may be split into three additive terms.

The first term, given by the partial time-derivative of the magnetic momentum vortex, is covariant under any frame-change. The remaining two terms are both dependent on the spatial velocity which is not covariant in the general group of diffeomorphic transformations.

None of these two terms is separately covariant under any frame-change but their sum is such.

One of them is what in literature has been abusively labeled “[LORENTZ force](#)”.⁹

The other term is the differential of a functional given by the inner product between magnetic momentum and spatial velocity.

⁸ This is Emil [Lenz \(1834\)](#) law.

⁹ Attribution to [LORENTZ](#) is historically unfounded. The term $\mathbf{v}_\phi \times \mathbf{B}$ was in fact introduced, without giving it the meaning of force, by James [CLERK-MAXWELL](#) in ([Clerk-Maxwell, 1855](#)) when he was twenty-four and Hendrik [LORENTZ](#) was only two years old.

In [MAXWELL](#)'s original treatment ([Clerk-Maxwell, 1865](#), p.485) this last term was merged with the functional expressing the electrostatic potential, to simplify the resulting formula ([Bucci, 2014](#), Eq. D. table 5.2). The undesired serious collateral effect was however that to most scholars this velocity dependent term remained hidden behind a symbolic curtain.

Here is located the very beginning of our story.

However, when the attention turned to evaluation of the curl of the electric field, the last term disappeared being the curl of a gradient (or, more in general, because the exterior derivative is nilpotent, which means that iterated exterior derivatives do vanish).

It is to be ascribed to merits of [J.J. Thomson \(1893\)](#), the discoverer of the electron, to point out the importance of bringing this velocity dependent functional back to full visibility.

The independent analysis performed by the author in ([G. Romano, 2012](#)), in the context of a differential geometric formulation in spacetime, and reproduced for convenience below, fully confirmed the expression contributed in ([J.J. Thomson, 1893](#), Ch VII, Eq(1) p.538).

This early findings were still unknown to me at the time when the related theoretical developments were independently carried out by me relying on the tools of differential geometry.

A practical advantage of the simple [LORENTZ](#) force formula is that it yields the electric field acting on a moving charge just as cross product of magnetic momentum and velocity at the point of evaluation.

The further neglected term requires in addition to compute the spatial differential of the inner product between magnetic momentum and spatial velocity.

This differential does not vanish even in case of a uniform magnetic vortex since the magnetic momentum, being its potential form, is not spatially uniform.

Therefore knowledge of the involved fields in a neighbourhood of that point is needed.

Electrical engineers certainly will not be glad of having lost the possibility of adopting the convenient and simple, but untenable formula, provided by the [LORENTZ](#) force rule, for their computations.

An effective, even if partial, remedy to their disappointment, and a motivation of the many qualitative successful applications of [LORENTZ](#) force rule to experimental evidence will be given in [§9.1](#).

Indeed the explicit calculation of the force acting on a beam of charged particles, in motion with uniform velocity under the action of a time-constant and uniform magnetic momentum, will be carried out therein.

When the correct local expression of the induction law is applied by an observer who evaluates a time-constant and uniform magnetic momentum acting on a beam of charged particles in motion with uniform velocity, a simple result is shown to hold true.

In fact in this special case the partial time-derivative of magnetic momentum vanishes and the sum of the remaining two velocity dependent terms give a result equal to one-half the expression of [LORENTZ](#) force.

This correction factor of one-half is in accord with findings in ([J.J. Thomson, 1881](#)), by a different procedure, as discussed in ([Darrigol, 2000](#), p.430).

The limited validity of halved [LORENTZ](#) force rule, which is confined just to special observers, deprives the term of the physical meaning of force, a caveat about [LORENTZ](#) force rule already clearly expressed in ([Hertz, 1892](#), XVI-2, p.248), but completely ignored ever since.¹⁰ The relevance of the factor one-half is evident since the unit of measure for the magnetic vortex is still presently fixed on the basis of the [LORENTZ](#) force rule.

2 Exterior derivatives

A multilinear function, mapping a list of vector fields on a manifold to a target linear space, is said to be *tensorial* if the value at a point depends only on the values of the argument vector fields at that point, viz. the map “lives at points” ([Spivak, 1970](#)).

¹⁰ In ([Hertz, 1892](#), XVI-2, p.248) the statement concerning the [LORENTZ](#) force was: “Now the resultant of (X_1, X_2, X_3) is an electric force which arises as soon as a body moves in the magnetic field. It is that force which in a narrower sense we are accustomed to denote as the electromotive force induced through the motion. But it should be observed that, according to our views, the separation of this from the total force can have no physical meaning.”

Differential forms (or simply *forms*) are fields of piecewise smooth alternating k -tensors on a manifold \mathcal{M} with finite geometric dimension $\dim(\mathcal{M}) = m$.

The linear space of k -forms on \mathcal{M} will be denoted by $\Lambda^k(T\mathcal{M})$, with T tangent functor.

All k -forms ω^k on \mathcal{M} , due to the alternating property (their value change sign when to arguments are swapped) vanish if the argument list vectors are linear dependent.

Therefore forms with $k > m$ vanish identically.

Forms of maximal degree ω^n on a manifold \mathcal{N} ($\dim(\mathcal{N}) = n$) (volume forms) are proportional one another and are the geometric objects that can be integrated on a n D compact manifold \mathcal{N} .

The exterior derivative of a $(n-1)$ -form ω^{n-1} on \mathcal{N} is the n -form $d\omega^{n-1}$ on \mathcal{N} fulfilling **KELVIN-STOKES-VOLTERRA** integral formula:¹¹

$$\int_{\mathcal{N}} d\omega^{n-1} = \oint_{\partial\mathcal{N}} \omega^{n-1}. \quad (1)$$

for any compact n -submanifold $\mathcal{N} \subset \mathcal{M}$ with

$$\dim(\mathcal{N}) = n \leq m, \quad (2)$$

with boundary $\partial\mathcal{N}$ ($\dim(\partial\mathcal{N}) = n-1$), and any differentiable form ω^{n-1} on \mathcal{N} .

Iterated boundary operator of any manifold generates a null manifold, and the iterated exterior derivative of any differential form generates a zero form:

$$\partial\partial = \mathbf{0} \iff dd = \mathbf{0}. \quad (3)$$

This equivalence may be deduced by rewriting Eq.(1) as a duality relation between exterior derivative d and boundary operator ∂ :

$$\langle d\omega^{n-1}, \mathcal{N} \rangle = \langle \omega^{n-1}, \partial\mathcal{N} \rangle. \quad (4)$$

Then:

$$\begin{aligned} \langle \omega^{n-2}, \partial\partial\mathcal{N} \rangle &= \langle d\omega^{n-2}, \partial\mathcal{N} \rangle \\ &= \langle dd\omega^{n-2}, \mathcal{N} \rangle = 0. \end{aligned} \quad (5)$$

¹¹ The general theorem is due to Vito **Volterra** (1889a,b) with subsequent reformulations by Henri **Poincaré** and Élie **Cartan** (**Poincaré**, 1887; **Cartan**, 1899), see Victor Joseph **Katz** (1979) and Hans **Samelson** (2001). By extending a 2D formula due to André-Marie **AMPÈRE**, William **THOMSON** (lord **KELVIN**) communicated the 3D result in a letter on July 1850 to Gabriel **STOKES** who lectured on it in Cambridge. It is commonly referred to as **STOKES**' formula, even sometimes with awful typo, as **STOKE**'s formula.

3 Lie and covariant derivatives

The flow:

$$\mathbf{FI}_\lambda^\mathbf{v} : \mathcal{M} \mapsto \mathcal{M}, \quad (6)$$

of a tangent vector field $\mathbf{v} : \mathcal{M} \mapsto T\mathcal{M}$ is composed of integral envelopes parametrised so that:

$$\mathbf{v} = \partial_{\lambda=0} \mathbf{FI}_\lambda^\mathbf{v}. \quad (7)$$

The **LIE** or *convective* derivative¹² of a vector field $\mathbf{u} : \mathcal{M} \mapsto T\mathcal{M}$ along a vector field $\mathbf{v} : \mathcal{M} \mapsto T\mathcal{M}$ is the λ -derivative of the pull-back along the flow $\mathbf{FI}_\lambda^\mathbf{v}$:

$$\begin{aligned} \mathcal{L}_\mathbf{v}(\mathbf{u}) &:= \partial_{\lambda=0} (\mathbf{FI}_\lambda^\mathbf{v} \downarrow \mathbf{u}) \\ &= \partial_{\lambda=0} \left(T\mathbf{FI}_\lambda^\mathbf{v} \cdot (\mathbf{u} \circ \mathbf{FI}_\lambda^\mathbf{v}) \right). \end{aligned} \quad (8)$$

The letter T denotes the tangent functor which to a smooth map between two manifolds associates the corresponding differential map relating the relevant tangent bundles, in a fiberwise linear manner.

The symbols \uparrow, \downarrow are push, pull operations on tensors induced by tangent maps.

The **LIE** derivative $\mathcal{L}_\mathbf{v}$ and the *parallel* (also named *covariant*) derivative $\nabla_\mathbf{v}$ along a vector field $\mathbf{v} : \mathcal{M} \mapsto T\mathcal{M}$ differs in the way backward evaluation is performed.

In **LIE** derivatives the evaluation tool is a pull-back \downarrow along the flow $\mathbf{FI}_\lambda^\mathbf{v} : \mathcal{M} \mapsto \mathcal{M}$ so that the values of the field $\mathbf{v} : \mathcal{M} \mapsto T\mathcal{M}$ in a neighbourhood of the evaluation point are involved.

Dependence on $\mathbf{v} : \mathcal{M} \mapsto T\mathcal{M}$ is therefore not tensorial. In *parallel* derivatives the evaluation tool is a backward parallel transport \downarrow along the curve $\mathbf{FI}_\lambda^\mathbf{v} : \mathcal{M} \mapsto \mathcal{M}$ and the only restriction to that curve of the field to be differentiated is significant.

Moreover the result at a point $\mathbf{x} \in \mathcal{M}$ depends linearly on the sole vector $\mathbf{v}_\mathbf{x} \in T_\mathbf{x}\mathcal{M}$, so that the parallel derivative $\nabla_\mathbf{v}$ is tensorial in \mathbf{v} .

¹² *Convective* derivatives were first considered by Clerk-Maxwell (1855) and Helmholtz (1858). The **LIE** derivatives of general tensor fields were introduced by Ślebodziński (1931). The naming after **LIE** is due to David van Dantzig (1932).

LIE derivative \mathcal{L} , covariant derivative ∇ and exterior derivative d are coincident for scalar fields.

The LIE derivative and the covariant derivative of tensor fields are defined by a formal application of LEIBNIZ rule, taking into account invariance of scalar fields under pull-back by a flow and under backward parallel transport along a curve (Spivak, 1970).

4 Spacetime framings

The ambient of a proper electromagnetic analysis is the 4D spacetime manifold \mathcal{E} without boundary and its tangent bundle $T\mathcal{E}$.

Each observer endows the tangent bundle $T\mathcal{E}$ with two geometric fields:

1. A nowhere vanishing field of tangent time-arrows $\mathbf{Z} : \mathcal{E} \mapsto T\mathcal{E}$, pointing towards the future and named *rigging* (Friedman, 1983) or *observer field* (Fecko, 1997) according to the suggestive language of physicists.
2. A *clock* one-form¹³ $\theta : \mathcal{E} \mapsto (T\mathcal{E})^*$ which is closed, i.e. such that:

$$d\theta = 0. \quad (9)$$

It is convenient to stipulate, between the *clock* and the future pointing *observer field*, fulfilment of the *tuning* relation:

$$\langle \theta, \mathbf{Z} \rangle = 1. \quad (10)$$

VOLTERRA's theorem (POINCARÉ Lemma) states that in star shaped manifolds closed forms are exact.¹⁴ The potential $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ is defined to within an additive constant by the requirement:

$$\theta = dt_{\mathcal{E}}. \quad (11)$$

The map $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ defines time-projection (surjective submersion) onto an oriented 1D *time-axis* \mathcal{Z} .¹⁵

¹³ As customary, a superscript $*$ denotes duality.

¹⁴ This theorem, first proved by Vito Volterra (1889a,b), and subsequently quoted by Henri Poincaré (1899), is known in literature as POINCARÉ Lemma (Samelson, 2001).

¹⁵ A submersion is a map with a surjective differential at any point. \mathcal{Z} is initial of the German word *Zeit* meaning *time*.

DEAHNA-FROBENIUS theorem,¹⁶ provides the condition for integrability of the time-vertical tangent distribution, composed of tangent vector fields $\mathbf{V} : \mathcal{E} \mapsto T\mathcal{E}$ fulfilling the PFAFF condition $\langle \theta, \mathbf{V} \rangle = 0$, in the form of vanishing of the exterior derivative in Eq.(9). For a proof see (Kolar et al., 1993; Marsden et al., 2003).

The spacetime manifold \mathcal{E} is doubly foliated into:

1. Leaves of *isochronous* events (3D *spatial slices*), integral manifolds of the kernel distribution of $dt_{\mathcal{E}}$.
2. Lines of *isotopic* events (1D *spatial positions*).

They are mutually transversal due to *tuning* Eq.(10).

By item 1 it is licit to consider the *time-vertical* subbundle $V\mathcal{E}$ of the tangent bundle $T\mathcal{E}$ whose fibers are slices of *isochronous* events (*spatial slices*).

Spacetime tensor fields of degree greater than zero are *time-vertical* if they vanish when any of their arguments is time-horizontal, i.e. tangent to a time-line, and are *time-horizontal* if they vanish when any of their arguments is time-vertical, i.e. tangent to a spatial slice.

Definition 4.1 (Framing). *An observer is described in geometrical terms by a field of rank-one linear projectors on the time-rigging $\mathbf{Z} : \mathcal{E} \mapsto T\mathcal{E}$, named a framing:*

$$\mathbf{R} := \theta \otimes \mathbf{Z}. \quad (12)$$

Then, for all $\mathbf{X} \in T\mathcal{E}$:

$$\mathbf{R}\mathbf{X} = (\theta \otimes \mathbf{Z})\mathbf{X} = \langle dt_{\mathcal{E}}, \mathbf{X} \rangle \mathbf{Z}. \quad (13)$$

The idempotency property, characteristic of linear projectors, is equivalent to tuning:

$$\mathbf{R}\mathbf{R} = \mathbf{R} \iff \langle \theta, \mathbf{Z} \rangle = 1_{\mathcal{Z}} \circ t_{\mathcal{E}}. \quad (14)$$

The time-vertical complementary projector defined by $\mathbf{P} = \mathbf{I} - \mathbf{R}$, is then idempotent too:

$$\begin{aligned} \mathbf{P}\mathbf{P} &= (\mathbf{I} - \mathbf{R})(\mathbf{I} - \mathbf{R}) = \mathbf{I} - \mathbf{R} - \mathbf{R} + \mathbf{R}\mathbf{R} \\ &= \mathbf{I} - \mathbf{R} = \mathbf{P}. \end{aligned} \quad (15)$$

¹⁶ First proved by Feodor Deahna (1840), and later investigated upon and simplified by Ferdinand Georg Frobenius (1875), the result is commonly known as FROBENIUS Theorem (Samelson, 2001).

Then also:

$$\begin{cases} \mathbf{PR} = \mathbf{RP} = \mathbf{0}, \\ \mathbf{RZ} = \mathbf{Z}, \\ \mathbf{PZ} = \mathbf{0}, \\ \mathbf{Ker}(\theta) = \mathbf{Im}(\mathbf{P}). \end{cases} \quad (16)$$

5 Trajectory and motion

A *motion* along a trajectory submanifold $\mathcal{T}_{\mathcal{E}} \subset \mathcal{E}$ is a one-parameter ($\alpha = \text{time-lapse}$) commutative group of automorphic *movements* $\phi_{\alpha} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$, with ϕ_0 the identity and the group composition rule:

$$\phi_{\alpha} \circ \phi_{\beta} = \phi_{\beta} \circ \phi_{\alpha} = \phi_{(\alpha+\beta)}. \quad (17)$$

A *movement* ϕ_{α} is a trajectory automorphism covering the time translation $\theta_{\alpha}(t) := t + \alpha$ so that:

$$t_{\mathcal{E}} \circ \phi_{\alpha} = \theta_{\alpha} \circ t_{\mathcal{E}}. \quad (18)$$

This means isochronous events at time t are mapped into isochronous events at time $t + \alpha$.

Taking the derivative $\partial_{\alpha=0}$ of Eq.(18), the spacetime velocity:

$$\mathbf{V}_{\phi} = \partial_{\alpha=0} \phi_{\alpha} : \mathcal{E} \mapsto T\mathcal{E}, \quad (19)$$

fulfils the property:

$$\begin{aligned} \langle dt_{\mathcal{E}}, \mathbf{V}_{\phi} \rangle &= \partial_{\alpha=0} (t_{\mathcal{E}} \circ \phi_{\alpha}) \\ &= \partial_{\alpha=0} (\theta_{\alpha} \circ t_{\mathcal{E}}) = 1_{\mathbf{Z}} \circ t_{\mathcal{E}}. \end{aligned} \quad (20)$$

From Eq.(12) and Eq.(20) we get:

$$\mathbf{R}\mathbf{V}_{\phi} = \langle dt_{\mathcal{E}}, \mathbf{V}_{\phi} \rangle \cdot \mathbf{Z} = \mathbf{Z}. \quad (21)$$

The spacetime velocity then splits into time-vertical $\mathbf{v}_{\phi} = \mathbf{P}\mathbf{V}_{\phi}$ and time-horizontal \mathbf{Z} components, according to the formula:

$$\mathbf{V}_{\phi} = \mathbf{P}\mathbf{V}_{\phi} + \mathbf{Z} = \mathbf{v}_{\phi} + \mathbf{Z}. \quad (22)$$

The tangent bundle $T\mathcal{E}$ is accordingly split as direct sum of a time-vertical bundle $V\mathcal{E}$ and a time-horizontal bundle $H\mathcal{E}$, with $V\mathcal{E} = \mathbf{Im}(\mathbf{P})$ and $H\mathcal{E} = \mathbf{Im}(\mathbf{R})$.

6 Splitting the motion

The spacetime motion can be split into commutative chain compositions of time-vertical and time-horizontal flows:

$$\phi_{\alpha} = \phi_{\alpha}^{V\mathcal{E}} \circ \phi_{\alpha}^{H\mathcal{E}} = \phi_{\alpha}^{H\mathcal{E}} \circ \phi_{\alpha}^{V\mathcal{E}}. \quad (23)$$

Taking the derivative $\partial_{\alpha=0}$ we infer:

$$\mathbf{V}_{\phi} = \mathbf{V}_{\phi}^{V\mathcal{E}} + \mathbf{V}_{\phi}^{H\mathcal{E}}, \quad (24)$$

where

$$\begin{cases} \mathbf{V}_{\phi}^{V\mathcal{E}} = \partial_{\alpha=0} \phi_{\alpha}^{V\mathcal{E}}, \\ \mathbf{V}_{\phi}^{H\mathcal{E}} = \partial_{\alpha=0} \phi_{\alpha}^{H\mathcal{E}}. \end{cases} \quad (25)$$

Being $\mathbf{V}_{\phi}^{H\mathcal{E}} = \mathbf{Z}$, by uniqueness of the additive decomposition Eq.(22) we infer:

$$\mathbf{V}_{\phi}^{V\mathcal{E}} = \mathbf{P}\mathbf{V}_{\phi} = \mathbf{v}_{\phi}. \quad (26)$$

7 Spacetime and spatial homotopy formulae

To a spacetime 1-form $\Omega^1 \in \Lambda^1(T\mathcal{E})$ there corresponds a spatial 1-form $\omega^1 = \mathbf{P}\downarrow\Omega^1 \in \Lambda^1(V\mathcal{E})$ defined by:

$$\begin{aligned} \omega^1(\mathbf{V}) &= (\mathbf{P}\downarrow\Omega^1)(\mathbf{V}) \\ &:= \Omega^1(\mathbf{P}\mathbf{V}), \quad \forall \mathbf{V} \in T\mathcal{E}, \end{aligned} \quad (27)$$

which is time-vertical since:

$$\omega^1(\mathbf{Z}) = 0. \quad (28)$$

Similarly for any k-form.

We will need some basic results concerning convective and exterior derivatives in spacetime, posted below in Eq.(29), Eq.(30), Eq.(31) and Eq.(32).¹⁷

¹⁷ For one-forms the homotopy formula is due to James Clerk-Maxwell (1861, 1873) and for two-forms and three-forms to Hermann von Helmholtz (1874). According to Andrzej Trautman (2008), the proof for differential forms of any degree is due to Élie Cartan (1922). The statement in terms of spatial forms was introduced in (G. Romano, 2017).

Lemma 7.1 (Extrusion in spacetime). *Let us consider a spacetime motion $\phi_\alpha : \mathcal{E} \mapsto \mathcal{E}$ with velocity $\mathbf{V}_\phi := \partial_{\alpha=0} \phi_\alpha : \mathcal{E} \mapsto T\mathcal{E}$. The time-rate of variation of the integral of a spacetime form Ω^k with $k \leq \dim(\mathcal{E})$, over the moving image of a kD compact submanifold Σ , is expressed in terms of LIE-derivative by the transport formula:*

$$\begin{aligned} \partial_{\alpha=0} \int_{\phi_\alpha(\Sigma)} \Omega^k &= \partial_{\alpha=0} \int_{\Sigma} \phi_{\alpha\downarrow} \Omega^k \\ &= \int_{\Sigma} \mathcal{L}_{\mathbf{V}_\phi}(\Omega^k). \end{aligned} \quad (29)$$

In terms of spacetime exterior derivatives we get the extrusion formula:

$$\partial_{\alpha=0} \int_{\phi_\alpha(\Sigma)} \Omega^k = \int_{\Sigma} (d\Omega^k) \cdot \mathbf{V}_\phi + \int_{\Sigma} d(\Omega^k \cdot \mathbf{V}_\phi). \quad (30)$$

Lemma 7.2 (Spacetime homotopy formula). *Substituting the transport formula Eq.(29) into the extrusion formula Eq.(30) and localising we get:*

$$\mathcal{L}_{\mathbf{V}_\phi}(\Omega^k) = (d\Omega^k) \cdot \mathbf{V}_\phi + d(\Omega^k \cdot \mathbf{V}_\phi). \quad (31)$$

This gives the expression of the LIE derivative of a spacetime form Ω^k with $k \leq n+1 = \dim(T_e\mathcal{E})$ for all $e \in \mathcal{E}$, along the motion velocity field $\mathbf{V}_\phi : \mathcal{E} \mapsto T\mathcal{E}$, in terms of spacetime exterior derivatives.

Proposition 7.1 (Spatial homotopy formula). *The LIE derivative, along the spatial motion velocity field $\mathbf{v}_\phi : \mathcal{E} \mapsto V\mathcal{E}$, of a spatial form ω^k with $k \leq n = \dim(V_e\mathcal{E})$, for all $e \in \mathcal{E}$, is expressed in terms of exterior derivatives by:*

$$\mathcal{L}_{\mathbf{v}_\phi}(\omega^k) = (d\omega^k) \cdot \mathbf{v}_\phi + d(\omega^k \cdot \mathbf{v}_\phi). \quad (32)$$

Proof. Setting $\mathbf{V}_\phi = \mathbf{Z}$ and $\Omega^k = \omega^k$ in Eq.(31), by time-verticality of ω^k Eq.(28) we get:

$$\mathcal{L}_{\mathbf{Z}}(\omega^k) = (d\omega^k) \cdot \mathbf{Z} + d(\omega^k \cdot \mathbf{Z}). \quad (33)$$

Splitting $\mathbf{V}_\phi = \mathbf{P}\mathbf{V}_\phi + \mathbf{Z}$ as in Eq.(22) and setting $\Omega^k = \omega^k$ in Eq.(31), by linearity of exterior and LIE derivatives, recalling Eq.(33) we get:

$$\mathcal{L}_{(\mathbf{P}\mathbf{V}_\phi)}(\omega^k) = (d\omega^k) \cdot \mathbf{P}\mathbf{V}_\phi + d(\omega^k \cdot \mathbf{P}\mathbf{V}_\phi). \quad (34)$$

Consequently Eq.(32) follows from Eq.(34) by setting $\mathbf{v}_\phi := \mathbf{P}\mathbf{V}_\phi$. \square

Denoting by \lrcorner the contraction operator, Eq.(32) can be written as a relation between graded derivatives $\mathcal{L}, d, \lrcorner$ respectively of degree 0, 1, -1, (Kolar et al., 1993):

$$\mathcal{L} = \lrcorner \circ d + d \circ \lrcorner. \quad (35)$$

8 Differential forms versus vector fields

Denoting by $\mathbf{g} : V\mathcal{E} \mapsto (V\mathcal{E})^*$ the spatial metric tensor field, the electric field one-form $\omega_{\mathbf{E}}^1$ and the magnetic momentum one-form $\omega_{\mathbf{M}}^1$ may be expressed by:¹⁸

$$\begin{cases} \omega_{\mathbf{E}}^1 = \mathbf{g} \cdot \mathbf{E}, \\ \omega_{\mathbf{M}}^1 = \mathbf{g} \cdot \mathbf{A}. \end{cases} \quad (36)$$

with \mathbf{E} electric vector field and \mathbf{A} magnetic vector potential.

Denoting by $\mu_{\mathbf{g}}$ the volume-form compatible with the metric,¹⁹ the magnetic vortex vector field \mathbf{B} is related to the magnetic vortex two-form $\omega_{\mathbf{M}}^2$ by:

$$\omega_{\mathbf{M}}^2 = \mu_{\mathbf{g}} \cdot \mathbf{B}. \quad (37)$$

Then

$$\omega_{\mathbf{M}}^2 \cdot \mathbf{v}_\phi = \mu_{\mathbf{g}} \cdot \mathbf{B} \cdot \mathbf{v}_\phi = \mathbf{g} \cdot (\mathbf{B} \times \mathbf{v}_\phi). \quad (38)$$

Exterior derivatives of forms and differential operators on vector fields are related by:

$$\begin{cases} d\omega_{\mathbf{E}}^1 = d(\mathbf{g} \cdot \mathbf{E}) = \mu_{\mathbf{g}} \cdot (\text{rot}(\mathbf{E})), \\ d\omega_{\mathbf{M}}^1 = d(\mathbf{g} \cdot \mathbf{A}) = \mu_{\mathbf{g}} \cdot (\text{rot}(\mathbf{A})), \\ d\omega_{\mathbf{M}}^2 = d(\mu_{\mathbf{g}} \cdot \mathbf{B}) = (\text{div}(\mathbf{B})) \cdot \mu_{\mathbf{g}}, \end{cases} \quad (39)$$

The magnetic vortex 2-form $\omega_{\mathbf{M}}^2$ and the magnetic momentum 1-form $\omega_{\mathbf{M}}^1$ are spatial fields related by

$$\omega_{\mathbf{M}}^2 = d\omega_{\mathbf{M}}^1. \quad (40)$$

¹⁸ Uppercase letters here adopted are standard for vector fields in Electrodynamics.

¹⁹ This means the unit cube has unitary volume.

In terms of vector fields:

$$\mathbf{B} = \text{rot}(\mathbf{A}). \quad (41)$$

The Eq.(40)-(41) are expressions of **GAUSS** principle stating nonexistence of magnetic charges:

$$d\omega_{\mathbf{M}}^2 = \mathbf{0} \iff \text{div}(\mathbf{B}) = \mathbf{0}. \quad (42)$$

In **EUCLID** spacetime, push along the flow generated by the time-arrows field and parallel transport along time-lines are coincident so that time-independence of the time-vertical metric tensor field is expressed by

$$\mathcal{L}_{\mathbf{Z}}(\mathbf{g}) = \nabla_{\mathbf{Z}}(\mathbf{g}) = \mathbf{0}. \quad (43)$$

9 Electric induction

Definition 9.1. *The overall electromotive force $\text{EMF}(\Gamma_{\text{INN}})$ along an inner oriented spatial path Γ_{INN} is the integral:*

$$\text{EMF}(\Gamma_{\text{INN}}) := \int_{\Gamma_{\text{INN}}} \omega_{\mathbf{E}}^1. \quad (44)$$

The electric field $\omega_{\mathbf{E}}^1$ is an even one-form.²⁰

Proposition 9.1 (Lineal electric induction). *Along any spatial inner oriented path Γ_{INN} dragged by a piecewise regular spacetime motion $\phi_{\alpha} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$, the induced electromotive force $\text{EMF}(\Gamma_{\text{INN}})$ is given by the negative time-rate of the magnetic momentum along the motion:*

$$\int_{\Gamma_{\text{INN}}} \omega_{\mathbf{E}}^1 = -\partial_{\alpha=0} \int_{\phi_{\alpha}(\Gamma_{\text{INN}})} \omega_{\mathbf{M}}^1. \quad (45)$$

²⁰ Inner and outer oriented manifolds and even and odd (or twisted) forms are treated in (Schouten, 1951; de Rham, 1955; Tonti, 1995; Marmo et al., 2005). Even forms are simply exterior forms to be integrated on inner oriented manifolds. Odd forms are to be integrated on outer oriented manifolds. Their sign changes by changing orientation of ambient manifold. Odd forms are best described by sets made of two opposite pairs, each one made of an exterior form and of an orientation. Even forms represent circulations and vortices, odd forms have the meaning of sources, winding around and flux through. A thorough discussion is offered in (Bossavit, 1998).

Applying **LIE-REYNOLDS** transport formula:²¹

$$\partial_{\alpha=0} \int_{\phi_{\alpha}(\Gamma_{\text{INN}})} \omega_{\mathbf{M}}^1 = \int_{\Gamma_{\text{INN}}} \mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{M}}^1), \quad (46)$$

and localising the integral Eq.(45), we get the rule:

$$\boxed{-\omega_{\mathbf{E}}^1 = \mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{M}}^1)}. \quad (47)$$

The **LIE** derivative of the spatial metric tensor field \mathbf{g} along spacetime motion, taking into account the time-independence in Eq.(43), is given by:

$$\mathcal{L}_{\mathbf{V}_{\phi}}(\mathbf{g}) = \mathcal{L}_{\mathbf{V}_{\phi}}(\mathbf{g}) = \mathbf{g} \cdot 2 \text{EUL}(\mathbf{v}_{\phi}), \quad (48)$$

The **EULER** stretching tensor is given by:

$$\text{EUL}(\mathbf{v}_{\phi}) := \mathbf{g}^{-1} \cdot \frac{1}{2} \mathcal{L}_{\mathbf{V}_{\phi}}(\mathbf{g}) = \text{sym} \nabla(\mathbf{v}_{\phi}), \quad (49)$$

with ∇ the connection in **EUCLID** spacetime.

Proposition 9.2 (Spacetime split). *Splitting the spacetime velocity into the sum of space and time components the law of electric induction Eq.(47) becomes:*

$$\boxed{-\omega_{\mathbf{E}}^1 = \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^1) + \omega_{\mathbf{M}}^2 \cdot \mathbf{v}_{\phi} + d(\omega_{\mathbf{M}}^1 \cdot \mathbf{v}_{\phi})}, \quad (50)$$

and in vectorial terms, by time-independence Eq.(43):

$$\boxed{-\mathbf{E} = \mathcal{L}_{\mathbf{Z}}(\mathbf{A}) + \mathbf{B} \times \mathbf{v}_{\phi} + \nabla(\mathbf{g}(\mathbf{A}, \mathbf{v}_{\phi}))}. \quad (51)$$

The convective derivative of the field \mathbf{A} at r.h.s. is due to **Helmholtz (1892)** and Eq.(51) is the electric induction law exposed in (**J.J. Thomson, 1893**, Eq(1) p.534).

Proof. By additivity of **LIE** derivative, splitting the spacetime velocity into time and spatial components gives:

$$\mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{M}}^1) = \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^1) + \mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{M}}^1). \quad (52)$$

²¹ In most treatments of electromagnetics a parallel derivative along the motion appears in place of the **LIE** derivative. An instance is provided by the treatment in (**Thide, 2012**, §1.3.4, p.12–14) which is consequently erroneous. The formulation of **FARADAY**'s law of induction in (**Panofski and Phillips, 1962**, §9-3, p.160) is emblematic of the physicists' way of deriving the homotopy formula for the **LIE** derivative.

Setting $\omega^k = \omega_M^1$, the spatial homotopy formula Eq.(32) yields Eq.(50).

By Eq.(36)₂ $\omega_M^1 = \mathbf{g} \cdot \mathbf{A}$ and by time-independence Eq.(43) we may rewrite in vectorial terms:

$$\begin{cases} \mathcal{L}_Z(\mathbf{g} \cdot \mathbf{A}) = \mathbf{g} \cdot \mathcal{L}_Z(\mathbf{A}), \\ \mathcal{L}_{\mathbf{v}_\phi}(\mathbf{g} \cdot \mathbf{A}) = \mathbf{d}(\mathbf{g} \cdot \mathbf{A}) \cdot \mathbf{v}_\phi + \mathbf{d}(\mathbf{g} \cdot \mathbf{A} \cdot \mathbf{v}_\phi). \end{cases} \quad (53)$$

Observing that:

$$\begin{cases} \mathbf{d}(\mathbf{g} \cdot \mathbf{A}) \cdot \mathbf{v}_\phi = \boldsymbol{\mu}_g \cdot \mathbf{B} \cdot \mathbf{v}_\phi = \mathbf{g} \cdot (\mathbf{B} \times \mathbf{v}_\phi), \\ \mathbf{d}(\mathbf{g} \cdot \mathbf{A} \cdot \mathbf{v}_\phi) = \mathbf{g} \cdot \nabla(\mathbf{g}(\mathbf{A} \cdot \mathbf{v}_\phi)), \end{cases} \quad (54)$$

and that by Eq.(36)₁ $\omega_E^1 = \mathbf{g} \cdot \mathbf{E}$, we get Eq.(51). \square

Proposition 9.3 (Vorticity rule). *In the special case when the path Γ_{INN} is the boundary of an inner oriented spatial surface Σ_{INN} undergoing a regular motion, we have:*

$$\begin{cases} \Gamma_{\text{INN}} = \partial\Sigma_{\text{INN}}, \\ \partial\Gamma_{\text{INN}} = \partial\partial\Sigma_{\text{INN}} = \mathbf{0}. \end{cases} \quad (55)$$

Applying **STOKES-VOLTERRA** formula to the integral at r.h.s. of Eq.(45) we get the **LENZ-FARADAY** rule:²²

$$\boxed{-\oint_{\Gamma_{\text{INN}}} \omega_E^1 = \partial_{\alpha=0} \int_{\phi_\alpha(\Sigma_{\text{INN}})} \omega_M^2.} \quad (56)$$

Proof. Being $\partial\Gamma_{\text{INN}} = \partial\partial\Sigma_{\text{INN}} = \mathbf{0}$, we get

$$\begin{aligned} \oint_{\phi_\alpha(\partial\Sigma_{\text{INN}})} \omega_M^1 &= \oint_{\partial(\phi_\alpha(\Sigma_{\text{INN}}))} \omega_M^1 \\ &= \int_{\phi_\alpha(\Sigma_{\text{INN}})} \mathbf{d}\omega_M^1. \end{aligned} \quad (57)$$

Then Eq.(40) and the E.I.L. Eq.(45) give Eq.(56). From **GAUSS** law $\mathbf{d}\omega_M^2 = \mathbf{0}$, stating absence of magnetic monopoles, we have for any inner oriented 3D domain \mathcal{V}_{INN} :

$$\oint_{\partial\mathcal{V}_{\text{INN}}} \omega_M^2 = \int_{\mathcal{V}_{\text{INN}}} \mathbf{d}\omega_M^2 = 0. \quad (58)$$

²² The denomination *flux rule* was changed to *vorticity rule* to conform with the assumption of an inner oriented surface and boundary path, with a clearer physical meaning and in accord with **MAXWELL** point of view.

so that independence of the choice of an inner oriented surface Σ_{INN} such that $\Gamma_{\text{INN}} = \partial\Sigma_{\text{INN}}$ is inferred. \square

Localisation of the integrals in Eq.(56) and recalling Eq.(37), we get the differential law:

$$-\mathbf{d}\omega_E^1 = \mathcal{L}_{\mathbf{v}_\phi}(\omega_M^2) = \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_g \cdot \mathbf{B}). \quad (59)$$

From Eq.(39) applying **LEIBNIZ** rule, Eq.(59) may be expressed, in terms of the magnetic induction vector field \mathbf{B} , as:

$$-\boldsymbol{\mu}_g \cdot \text{rot}(\mathbf{E}) = \boldsymbol{\mu}_g \cdot \mathcal{L}_{\mathbf{v}_\phi}(\mathbf{B}) + \left(\mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_g)\right) \cdot \mathbf{B} \quad (60)$$

Denoting by J_1 the linear invariant, and setting $\mathbf{v} = \mathbf{P}\mathbf{V}$ for any spacetime tangent vector field $\mathbf{V} : \mathcal{E} \mapsto T\mathcal{E}$, we have (**G. Romano, 2017**):

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}(\boldsymbol{\mu}_g) &= J_1\left(\mathbf{g}^{-1} \cdot \frac{1}{2} \mathcal{L}_{\mathbf{v}}(\mathbf{g})\right) \cdot \boldsymbol{\mu}_g \\ &= J_1(\text{EUL}(\mathbf{v})) \cdot \boldsymbol{\mu}_g. \end{aligned} \quad (61)$$

Then by definition of divergence:

$$\mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_g) = \text{div}(\mathbf{v}_\phi) \cdot \boldsymbol{\mu}_g, \quad (62)$$

from Eq.(43) we infer

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_g) &= \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_g) + \cancel{\mathcal{L}_Z(\boldsymbol{\mu}_g)} \\ &= \text{div}(\mathbf{v}_\phi) \cdot \boldsymbol{\mu}_g, \end{aligned} \quad (63)$$

with $\text{div}(\mathbf{v}_\phi) = J_1(\text{EUL}(\mathbf{v}_\phi))$ volumetric stretching. The vorticity rule Eq.(60) then writes:

$$\boxed{-\text{rot}(\mathbf{E}) = \mathcal{L}_{\mathbf{v}_\phi}(\mathbf{B}) + \text{div}(\mathbf{v}_\phi) \cdot \mathbf{B}.} \quad (64)$$

The vorticity rule Eq.(56) is independent of the choice of a surface Σ_{INN} such that $\Gamma_{\text{INN}} = \partial\Sigma_{\text{INN}}$.

The proof is readily got by appealing to Eq.(40) and to **STOKES** formula:

$$\int_{\phi_\alpha(\Sigma_{\text{INN}})} \mathbf{d}\omega_M^1 = \oint_{\phi_\alpha(\partial\Sigma_{\text{INN}})} \omega_M^1 = \oint_{\phi_\alpha(\Gamma_{\text{INN}})} \omega_M^1. \quad (65)$$

Remark 9.1. *Performing the splitting:*

$$\mathcal{L}_{\mathbf{v}_\phi}(\omega_{\mathbf{M}}^2) = \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^2) + \mathcal{L}_{\mathbf{v}_\phi}(\omega_{\mathbf{M}}^2), \quad (66)$$

and applying Eq.(37) and Eq.(32) an alternative expression for Eq.(60) is got.

Indeed, from

$$-\boldsymbol{\mu}_{\mathbf{g}} \cdot \text{rot}(\mathbf{E}) = \mathcal{L}_{\mathbf{Z}}(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B}) + \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B}), \quad (67)$$

recalling that $\mathcal{L}_{\mathbf{Z}}(\boldsymbol{\mu}_{\mathbf{g}}) = \mathbf{0}$, we get

$$-\boldsymbol{\mu}_{\mathbf{g}} \cdot \text{rot}(\mathbf{E}) = \boldsymbol{\mu}_{\mathbf{g}} \cdot \mathcal{L}_{\mathbf{Z}}(\mathbf{B}) + \mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B}), \quad (68)$$

and from the homotopy formula Eq.(32):

$$\mathcal{L}_{\mathbf{v}_\phi}(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B}) = d(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B}) \cdot \mathbf{v}_\phi + d(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B} \cdot \mathbf{v}_\phi). \quad (69)$$

Being

$$d(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B}) = \text{div}(\mathbf{B}) \cdot \boldsymbol{\mu}_{\mathbf{g}}, \quad (70)$$

and

$$\begin{cases} d(\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B} \cdot \mathbf{v}_\phi) = d(\mathbf{g} \cdot (\mathbf{B} \times \mathbf{v}_\phi)) \\ \qquad \qquad \qquad = \boldsymbol{\mu}_{\mathbf{g}} \cdot (\text{rot}(\mathbf{B} \times \mathbf{v}_\phi)), \end{cases} \quad (71)$$

we eventually get:

$$-\text{rot}(\mathbf{E}) = \mathcal{L}_{\mathbf{Z}}(\mathbf{B}) + \text{rot}(\mathbf{B} \times \mathbf{v}_\phi) + \underbrace{\text{div}(\mathbf{B})}_{=0} \cdot \mathbf{v}_\phi. \quad (72)$$

The divergence in the last term of Eq.(72) vanishes due to GAUSS principle expressed by Eqs.(40), (41). The r.h.s of Eq.(72) is the expression of the convective derivative due to Hermann Helmholtz (1858) and Kazimierz Żórawski (1900).

9.1 Motion in a uniform magnetic field

The split lineal electric induction law Eq.(51) reveals that, if the magnetic vortex \mathbf{B} is time-independent ($\mathcal{L}_{\mathbf{Z}}(\mathbf{B}) = \mathbf{0}$) and spatially uniform ($\nabla(\mathbf{B}) = \mathbf{0}$) then the induced electric field is expressed by:

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{v}_\phi \times \mathbf{B})}. \quad (73)$$

The proof of this result is made of two steps aimed at establishing the evaluation:

$$d(\omega_{\mathbf{M}}^1 \cdot \mathbf{v}_\phi) = -\frac{1}{2}(\omega_{\mathbf{M}}^2 \cdot \mathbf{v}_\phi). \quad (74)$$

To this end, we observe that, by an application of LEIBNIZ rule, the LIE derivative of a spatial one-form ω^1 or two-form ω^2 , along the flow of a time-vertical vector field $\mathbf{v} \in V\mathcal{E}$, is expressed, in terms of parallel derivatives ∇ according to a connection with vanishing torsion, as:

$$(\mathcal{L}_{\mathbf{v}} - \nabla_{\mathbf{v}})(\omega^1) = \nabla(\mathbf{v})^* \cdot \omega^1, \quad (75a)$$

$$(\mathcal{L}_{\mathbf{v}} - \nabla_{\mathbf{v}})(\omega^2) = \omega^2 \cdot \nabla(\mathbf{v}) + \nabla(\mathbf{v})^* \cdot \omega^2, \quad (75b)$$

where the star $*$ denotes duality.

Theorem 9.1 (Linear Faraday potential). *A spatially uniform magnetic vortex two-form $\nabla(\omega_{\mathbf{M}}^2) = \mathbf{0}$, admits a magnetic momentum potential one-form $\omega_{\mathbf{M}}^1$, that is $\omega_{\mathbf{M}}^2 = d\omega_{\mathbf{M}}^1$, having the linear distribution:*

$$\boxed{\omega_{\mathbf{M}}^1 := \frac{1}{2}d\omega_{\mathbf{M}}^1 \cdot \mathbf{r} = \frac{1}{2}\omega_{\mathbf{M}}^2 \cdot \mathbf{r} = \frac{1}{2}\boldsymbol{\mu}_{\mathbf{g}} \cdot \mathbf{B} \cdot \mathbf{r}}. \quad (76)$$

to within the differential of a scalar potential. The position vector field \mathbf{r} is defined by

$$\mathbf{r}(\mathbf{p}) := \mathbf{x}, \quad (77)$$

for all $\mathbf{x} = \mathbf{p} - \mathbf{o}$.

Proof. For any increment of position \mathbf{h} we have

$$\begin{aligned} \nabla_{\mathbf{h}}(\mathbf{r}) &= \lim_{\epsilon \rightarrow 0} \epsilon^{-1}(\mathbf{r}(\mathbf{p} + \epsilon\mathbf{h}) - \mathbf{r}(\mathbf{p})) \\ &= \lim_{\epsilon \rightarrow 0} \epsilon^{-1}(\mathbf{x} + \epsilon\mathbf{h} - \mathbf{x}) = \mathbf{h}, \end{aligned} \quad (78)$$

so that, denoting by \mathbf{I} identity map and by \mathbf{I}^* the dual identity map:

$$\begin{cases} \nabla(\mathbf{r}) = \mathbf{I}, \\ (\nabla\mathbf{r})^* = \mathbf{I}^*. \end{cases} \quad (79)$$

Under the assumption $\nabla(\omega_{\mathbf{M}}^2) = \mathbf{0}$, the homotopy formula and the expression in Eq.(75b) of LIE derivative in terms of parallel derivative, recalling GAUSS law Eq.(40) and Eq.(79), give:

$$\begin{aligned} d(\omega_{\mathbf{M}}^2 \cdot \mathbf{r}) &= \mathcal{L}_{\mathbf{r}}(\omega_{\mathbf{M}}^2) - \underbrace{(d\omega_{\mathbf{M}}^2) \cdot \mathbf{r}}_{=0} \\ &= \underbrace{\nabla_{\mathbf{r}}(\omega_{\mathbf{M}}^2)}_{=0} + \omega_{\mathbf{M}}^2 \cdot \nabla\mathbf{r} + (\nabla\mathbf{r})^* \cdot \omega_{\mathbf{M}}^2 = 2\omega_{\mathbf{M}}^2, \end{aligned} \quad (80)$$

which was to be proved. \square

Theorem 9.2 (Electric field on translating charges). *A charged body in translational motion, across a region of spatially uniform magnetic vortex $\nabla\omega_{\mathbf{M}}^2 = 0$, experiences an electric field given by*

$$\begin{aligned} -\omega_{\mathbf{E}}^1 &= \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^1) + \frac{1}{2}\omega_{\mathbf{M}}^2 \cdot \mathbf{v}_\phi, \\ &= \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^1) - \mathbf{d}(\omega_{\mathbf{M}}^1 \cdot \mathbf{v}_\phi). \end{aligned} \quad (81)$$

In vector terms

$$\begin{aligned} -\mathbf{E} &= \mathcal{L}_{\mathbf{Z}}(\mathbf{A}) - \frac{1}{2}(\mathbf{v}_\phi \times \mathbf{B}), \\ &= \mathcal{L}_{\mathbf{Z}}(\mathbf{A}) - \mathbf{d}\mathbf{g}(\mathbf{A}, \mathbf{v}_\phi). \end{aligned} \quad (82)$$

Proof. Let an observer be detecting a translational motion $\phi_\alpha \in C^1(\mathcal{T}_\mathcal{E}; \mathcal{T}_\mathcal{E})$ and measuring the space-time velocity $\mathbf{V}_\phi := \partial_{\alpha=0} \phi_\alpha = \mathbf{v}_\phi + \mathbf{Z}$, whose spatial component is uniform, i.e. $\nabla\mathbf{v}_\phi = \mathbf{0}$.

From the expression of [LIE](#) derivative of a one-form in terms of parallel derivatives [Eq.\(75a\)](#), we get

$$\mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1) = \nabla_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1) + \cancel{(\nabla\mathbf{V}_\phi)^*}(\omega_{\mathbf{M}}^1). \quad (83)$$

Being $\nabla(\omega_{\mathbf{M}}^2) = \mathbf{0}$, [Lemma 9.1](#) gives

$$\omega_{\mathbf{M}}^1 = \frac{1}{2}\omega_{\mathbf{M}}^2 \cdot \mathbf{r}. \quad (84)$$

Then

$$\mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1) = \nabla_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1) = \frac{1}{2}(\omega_{\mathbf{M}}^2) \cdot \mathbf{v}_\phi. \quad (85)$$

By assumption $\mathbf{d}\omega_{\mathbf{E}}^0 = \mathbf{0}$ so that from [\(47\)](#):

$$\begin{aligned} -(\omega_{\mathbf{E}}^1) &= \mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1) \\ &= \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^1)\mathcal{L}_{\mathbf{v}_\phi}(\omega_{\mathbf{M}}^1), \\ &= \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^1) + \frac{1}{2}\omega_{\mathbf{M}}^2 \cdot \mathbf{v}_\phi. \end{aligned} \quad (86)$$

Finally the computation

$$\begin{aligned} \mathbf{d}(\omega_{\mathbf{M}}^1 \cdot \mathbf{v}_\phi) &= \frac{1}{2}\mathbf{d}(\mathbf{d}\omega_{\mathbf{M}}^1 \cdot \mathbf{r} \cdot \mathbf{v}_\phi) \\ &= -\frac{1}{2}\mathbf{d}((\mathbf{d}\omega_{\mathbf{M}}^1) \cdot \mathbf{v}_\phi \cdot \mathbf{r}) \\ &= -\frac{1}{2}(\mathbf{d}\omega_{\mathbf{M}}^1) \cdot \mathbf{v}_\phi, \end{aligned} \quad (87)$$

shows electric field has a velocity linearly dependent potential. \square

10 Magnetic induction

Let us consider an outer oriented surface Σ with boundary $\partial\Sigma$ and define the magnetomotive force $\text{MMF}(\partial\Sigma)$ along an outer oriented spatial path Γ_{OUT} by:

$$\text{MMF}(\Gamma_{\text{OUT}}) := \int_{\Gamma_{\text{OUT}}} \omega_{\mathbf{H}}^1, \quad (88)$$

and adopt the expressions:

$$\begin{cases} \omega_{\mathbf{D}}^2 = \mu_{\mathbf{g}} \cdot \mathbf{D}, \\ \omega_{\mathbf{H}}^1 = \mathbf{g} \cdot \mathbf{H}, \\ \omega_{\mathbf{J}}^2 = \mu_{\mathbf{g}} \cdot \mathbf{J}, \\ \omega_{\rho} = \rho \cdot \mu_{\mathbf{g}}. \end{cases} \quad (89)$$

The spatial vector fields are: \mathbf{D} electric displacement, \mathbf{H} magnetic winding, \mathbf{J} electric current. The scalar field ρ is the electric charge density per unit volume.

Proposition 10.1 (Magnetic induction law). *When the spatial outer oriented circuit Γ_{OUT} is dragged by a piecewise regular spacetime motion $\phi_\alpha : \mathcal{T}_\mathcal{E} \mapsto \mathcal{T}_\mathcal{E}$, a magnetomotive force $\text{MMF}(\Gamma_{\text{OUT}})$ is induced.*

In regions where there are no electric charges and no sources of electric currents, so that:

$$\begin{cases} \mathbf{d}\omega_{\mathbf{D}}^2 = \mathbf{0} \iff \omega_{\mathbf{D}}^2 = \mathbf{d}\omega_{\mathbf{D}}^1, \\ \mathbf{d}\omega_{\mathbf{J}}^2 = \mathbf{0} \iff \omega_{\mathbf{J}}^2 = \mathbf{d}\omega_{\mathbf{J}}^1, \end{cases} \quad (90)$$

the magnetomotive force is given by the time-rate of the global electric flux potential plus the global electric current potential, along the path Γ_{OUT} :

$$\boxed{\int_{\Gamma_{\text{OUT}}} \omega_{\mathbf{H}}^1 = \partial_{\alpha=0} \int_{\phi_\alpha(\Gamma_{\text{OUT}})} \omega_{\mathbf{D}}^1 + \int_{\Gamma_{\text{OUT}}} \omega_{\mathbf{J}}^1.} \quad (91)$$

Out of these regions, the magnetic force induced along the boundary $\Gamma_{\text{OUT}} = \partial\Sigma_{\text{OUT}}$ of a spatial surface Σ_{OUT} can be expressed as time-rate of the global electric displacement flux plus the global electric current flux, by

the formula:

$$\oint_{\partial\Sigma_{\text{OUT}}} \omega_{\mathbf{H}}^1 = \partial_{\alpha=0} \int_{\phi_{\alpha}(\Sigma_{\text{OUT}})} \omega_{\mathbf{D}}^2 + \int_{\Sigma_{\text{OUT}}} \omega_{\mathbf{J}}^2. \quad (92)$$

Independence of the choice of a spatial surface Σ_{OUT} such that $\partial\Sigma_{\text{OUT}} = \mathbf{\Gamma}_{\text{OUT}}$, is inferred from balance of electric charge along the motion:

Proposition 10.2 (Balance of charges and currents). *Through the boundary $\partial\mathcal{V}_{\text{OUT}}$ of an outer oriented spatial domain \mathcal{V}_{OUT} the total outward flux of currents (due to electric displacement and free charges) is vanishing:*

$$\partial_{\alpha=0} \oint_{\phi_{\alpha}(\partial\mathcal{V}_{\text{OUT}})} \omega_{\mathbf{D}}^2 + \oint_{\partial\mathcal{V}_{\text{OUT}}} \omega_{\mathbf{J}}^2 = 0. \quad (93)$$

Proof. Being $\phi_{\alpha}(\partial\mathcal{V}_{\text{OUT}}) = \partial(\phi_{\alpha}(\mathcal{V}_{\text{OUT}}))$, and setting:

$$d\omega_{\mathbf{D}}^2 = \omega_{\rho} = \rho \cdot \boldsymbol{\mu}_{\mathbf{g}}, \quad (94)$$

equivalence between Eq.(93) and COULOMB's balance of electric charge along the motion:

$$\partial_{\alpha=0} \int_{\phi_{\alpha}(\mathcal{V}_{\text{OUT}})} \omega_{\rho} + \oint_{\partial\mathcal{V}_{\text{OUT}}} \omega_{\mathbf{J}}^2 = 0, \quad (95)$$

follows from STOKES formula. \square

Localising Eq.(91) and Eq.(92) we get:

$$\begin{cases} \omega_{\mathbf{H}}^1 = \mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{D}}^1) + \omega_{\mathbf{J}}^1, \\ d\omega_{\mathbf{H}}^1 = \mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{D}}^2) + \omega_{\mathbf{J}}^2. \end{cases} \quad (96)$$

Eq.(96)₁ may be rewritten in terms of the splitting:

$$\mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{D}}^1) = \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{D}}^1) + \omega_{\mathbf{D}}^2 \cdot \mathbf{v}_{\phi} + d(\omega_{\mathbf{D}}^1 \cdot \mathbf{v}_{\phi}). \quad (97)$$

Taking the exterior derivative of Eq.(96)₂ we get

$$d\omega_{\mathbf{J}}^2 = \mathcal{L}_{\mathbf{V}_{\phi}}(d\omega_{\mathbf{D}}^2), \quad (98)$$

which by Eq.(89)₃ and Eq.(94) leads to the vectorial expression:

$$\text{div}(\mathbf{J}) + \mathcal{L}_{\mathbf{V}_{\phi}}(\rho) + \rho \cdot \text{div}(\mathbf{v}_{\phi}) = 0. \quad (99)$$

11 Splitting spacetime forms

Spacetime forms, introduced by Hermann Minkowski (1907), and further discussed by Harry Bateman (1910) elaborating on ideas by Hargreaves (1908), were later investigated by Élie Cartan (1924). More recent treatments and applications to Electrodynamics were contributed in (Thorne & Macdonald, 1982; Fecko, 1997).

Lemma 11.1 (Splitting). *A framing $\mathbf{R} := \boldsymbol{\theta} \otimes \mathbf{Z}$ induces a representation for spacetime k -forms Ω^k in terms of the time-vertical projector $\mathbf{P} = \mathbf{I} - \mathbf{R}$, the time-arrows \mathbf{Z} and the time differential $\boldsymbol{\theta}$:*

$$\Omega^k = \mathbf{P}\downarrow\Omega^k + \boldsymbol{\theta} \wedge (\Omega^k \cdot \mathbf{Z}). \quad (100)$$

Proof. The spatial restriction $\mathbf{P}\downarrow\Omega^k$ of a spacetime k -form Ω^k is defined, for $\mathbf{a}_1, \dots, \mathbf{a}_k \in V\mathcal{E}$ by:

$$(\mathbf{P}\downarrow\Omega^k)(\mathbf{a}_1, \dots, \mathbf{a}_k) = \Omega^k(\mathbf{P}\mathbf{a}_1, \dots, \mathbf{P}\mathbf{a}_k). \quad (101)$$

Let us denote spacetime tangent vectors by

$$\mathbf{Y}_i : \mathcal{E} \mapsto T\mathcal{E}, \quad i = 1, \dots, k-1. \quad (102)$$

Adopting the notations:

$$\begin{aligned} \mathbf{Y} &= \{\mathbf{Y}_1, \dots, \mathbf{Y}_{k-1}\}, \\ \mathbf{P}\mathbf{Y} &= \{\mathbf{P}\mathbf{Y}_1, \dots, \mathbf{P}\mathbf{Y}_{k-1}\}, \\ (\mathbf{P})_i\mathbf{Y} &= \{\mathbf{P}\mathbf{Y}_1, \dots, \mathbf{P}\mathbf{Y}_{i-1}, \mathbf{P}\mathbf{Y}_{i+1}, \dots, \mathbf{P}\mathbf{Y}_k\}, \\ (\mathbf{R})_i\mathbf{Y} &= \{\mathbf{P}\mathbf{Y}_1, \dots, \mathbf{R}\mathbf{Y}_i, \dots, \mathbf{P}\mathbf{Y}_{k-1}\}. \end{aligned} \quad (103)$$

and taking into account the skew character of forms, the result is given by the computation:

$$\begin{aligned} \Omega^k(\mathbf{X}, \mathbf{Y}) &= \Omega^k(\mathbf{P}\mathbf{X} + \mathbf{R}\mathbf{X}, \mathbf{Y}) \\ &= \Omega^k(\mathbf{P}\mathbf{X}, \mathbf{P}\mathbf{Y}) + \Omega^k(\mathbf{R}\mathbf{X}, \mathbf{P}\mathbf{Y}) \\ &\quad + \sum_{i=1, k-1} \Omega^k(\mathbf{P}\mathbf{X}, (\mathbf{R})_i\mathbf{Y}) \\ &= (\mathbf{P}\downarrow\Omega^k)(\mathbf{X}, \mathbf{Y}) + \langle \boldsymbol{\theta}, \mathbf{X} \rangle \Omega^k(\mathbf{Z}, \mathbf{P}\mathbf{Y}) \\ &\quad + \sum_{i=1, k-1} (-1)^i \langle \boldsymbol{\theta}, \mathbf{Y}_i \rangle \Omega^k(\mathbf{Z}, \mathbf{P}\mathbf{X}, (\mathbf{P})_i\mathbf{Y}) \\ &= \left(\mathbf{P}\downarrow\Omega^k + \boldsymbol{\theta} \wedge (\Omega^k \cdot \mathbf{Z}) \right) (\mathbf{X}, \mathbf{Y}). \end{aligned} \quad (104)$$

This concludes the proof. \square

An alternative proof of Lemma 11.1 may be given in terms of components by evaluating the spacetime form on a basis made of exterior products of differential of adapted coordinates on $T\mathcal{E}$.

Grouping the ones that do include the differential $dt_{\mathcal{E}}$ and those that do not, gives the result (Bateman, 1910; Parrott, 1987; Benn and Tucker, 1987; Fecko, 2014).

This 3 + 1 decomposition of the graded algebra $\Lambda(T\mathcal{E})$ of spacetime differential forms, was elegantly revisited in (Fecko, 1997, 2014) by introducing the linear operators:

$$\begin{cases} \mathbf{i}_{\mathbf{Z}} : \Lambda^k(T\mathcal{E}) \mapsto \Lambda^{k-1}(T\mathcal{E}) & \text{(contraction),} \\ \mathbf{j}_{\boldsymbol{\theta}} : \Lambda^k(T\mathcal{E}) \mapsto \Lambda^{k+1}(T\mathcal{E}) & \text{(extension),} \end{cases} \quad (105)$$

defined by

$$\begin{cases} \mathbf{i}_{\mathbf{Z}} \Omega^k = \Omega^k \cdot \mathbf{Z}, \\ \mathbf{j}_{\boldsymbol{\theta}} \Omega^k = \boldsymbol{\theta} \wedge \Omega^k. \end{cases} \quad (106)$$

It is easily verified that the swapped compositions:

$$\begin{cases} \mathbf{i}_{\mathbf{Z}} \mathbf{j}_{\boldsymbol{\theta}} : \Lambda^k(T\mathcal{E}) \mapsto \Lambda^k(T\mathcal{E}), \\ \mathbf{j}_{\boldsymbol{\theta}} \mathbf{i}_{\mathbf{Z}} : \Lambda^k(T\mathcal{E}) \mapsto \Lambda^k(T\mathcal{E}), \end{cases} \quad (107)$$

are complementary projectors in $\Lambda(T\mathcal{E})$. In fact:

$$\mathbf{j}_{\boldsymbol{\theta}} \mathbf{i}_{\mathbf{Z}} \Omega^k = \boldsymbol{\theta} \wedge (\Omega^k \cdot \mathbf{Z}), \quad (108)$$

so that, by graded LEIBNIZ rule (contraction is a derivation of degree -1), we have:

$$\begin{aligned} \mathbf{i}_{\mathbf{Z}} \mathbf{j}_{\boldsymbol{\theta}} \Omega^k &= (\boldsymbol{\theta} \wedge \Omega^k) \cdot \mathbf{Z} \\ &= \langle \boldsymbol{\theta}, \mathbf{Z} \rangle \cdot \Omega^k - \boldsymbol{\theta} \wedge (\Omega^k \cdot \mathbf{Z}). \end{aligned} \quad (109)$$

Inserting the tuning property Eq.(10) in Eq.(109) gives the result. A comparison with Eq.(100) leads to the conclusion that the spatial restriction is given by $\mathbf{P}\downarrow = \mathbf{i}_{\mathbf{Z}} \mathbf{j}_{\boldsymbol{\theta}}$.

In view of the applications of spacetime splitting to the theory of electromagnetic induction, it is convenient to rewrite the result in Eq.(100) in terms of the spacetime motion velocity field $\mathbf{V}_{\phi} : \mathcal{E} \mapsto T\mathcal{E}$:

$$\begin{aligned} \Omega^k &= \mathbf{P}\downarrow \Omega^k \\ &+ \boldsymbol{\theta} \wedge \left(\mathbf{P}\downarrow (\Omega^k \cdot \mathbf{V}_{\phi}) - (\mathbf{P}\downarrow \Omega^k) \cdot \mathbf{V}_{\phi} \right). \end{aligned}$$

(110)

To prove the formula in Eq.(110), the time-arrow is expressed as time-horizontal component of the spacetime velocity field $\mathbf{V}_{\phi} : \mathcal{E} \mapsto T\mathcal{E}$ by setting

$$\mathbf{Z} = \mathbf{R} \cdot \mathbf{V}_{\phi} = \mathbf{V}_{\phi} - \mathbf{P} \cdot \mathbf{V}_{\phi}. \quad (111)$$

The split formula in Eq.(100) may then be rewritten as:

$$\begin{aligned} \Omega^k &= \mathbf{P}\downarrow \Omega^k + \boldsymbol{\theta} \wedge \mathbf{P}\downarrow (\Omega^k \cdot \mathbf{Z}) \\ &= \mathbf{P}\downarrow \Omega^k + \boldsymbol{\theta} \wedge \mathbf{P}\downarrow \left(\Omega^k \cdot (\mathbf{V}_{\phi} - \mathbf{P}\mathbf{V}_{\phi}) \right), \end{aligned} \quad (112)$$

and Eq.(110) follows from the trivial equality

$$\mathbf{P}\downarrow \left(\Omega^k \cdot (\mathbf{P}\mathbf{V}_{\phi}) \right) = (\mathbf{P}\downarrow \Omega^k) \cdot \mathbf{V}_{\phi}. \quad (113)$$

The proof exposed in Lemma 11.1 and the expression Eq.(110) of the split formula in terms of the spacetime velocity field were first proposed in (G. Romano, 2013).

12 Spacetime Electromagnetics

The above introduced decomposition may be applied to spacetime electromagnetics forms.

Let us now consider the FARADAY spacetime two-form $\Omega_{\mathbf{F}}^2$ expressing the *electromagnetic induction field* and subsequently in a similar way the AMPÈRE spacetime two-form $\Omega_{\mathbf{A}}^2$ expressing the *electromagnetic induction flux*, respectively *even* and *odd* forms, so named in (Misner, Thorne & Wheeler, 1973).²³

²³ In (Sommerfeld, 1952, p.212) the author exclaimed: "I wish to create the impression in my readers that the true mathematical structure of these entities will appear only now, as in a mountain landscape when the fog lifts."

12.1 Faraday spacetime two-form

The electric induction phenomena are governed by the closed spacetime induction **FARADAY** two-form $\Omega_{\mathbf{F}}^2$ (the *electromagnetic field*) and by its *potential* one-form $\Omega_{\mathbf{F}}^1$ such that

$$\Omega_{\mathbf{F}}^2 = d\Omega_{\mathbf{F}}^1. \quad (114)$$

From Lemma 11.1, setting $\theta = dt_{\mathcal{E}}$, we infer the following statement.

Proposition 12.1 (Electric induction). *Time-vertical restrictions of spacetime **FARADAY** two-form $\Omega_{\mathbf{F}}^2$ and of its potential one-form $\Omega_{\mathbf{F}}^1$ fulfilling $\Omega_{\mathbf{F}}^2 = d\Omega_{\mathbf{F}}^1$, are the even forms:*²⁴

$$\left\{ \begin{array}{ll} \omega_{\mathbf{M}}^1 = \mathbf{P}\downarrow\Omega_{\mathbf{F}}^1, & \text{magnetic momentum field} \\ \omega_{\mathbf{E}}^0 = \mathbf{P}\downarrow(\Omega_{\mathbf{F}}^1 \cdot \mathbf{V}_{\phi}), & \text{electric potential field} \\ \omega_{\mathbf{M}}^2 = \mathbf{P}\downarrow\Omega_{\mathbf{F}}^2, & \text{magnetic vortex field} \\ \omega_{\mathbf{E}}^1 = -\mathbf{P}\downarrow(\Omega_{\mathbf{F}}^2 \cdot \mathbf{V}_{\phi}), & \text{electric field} \end{array} \right. \quad (115)$$

with the representation formulae:

$$\left\{ \begin{array}{l} \Omega_{\mathbf{F}}^1 = \omega_{\mathbf{M}}^1 + dt_{\mathcal{E}} \wedge (\omega_{\mathbf{E}}^0 - \omega_{\mathbf{M}}^1 \cdot \mathbf{v}_{\phi}), \\ \Omega_{\mathbf{F}}^2 = \omega_{\mathbf{M}}^2 - dt_{\mathcal{E}} \wedge (\omega_{\mathbf{E}}^1 + \omega_{\mathbf{M}}^2 \cdot \mathbf{v}_{\phi}). \end{array} \right. \quad (116)$$

where the scalar $P_{\mathbf{E}}$ is the electric potential field.

According to next proposition, the spacetime rule of electric induction, expressed in terms of **FARADAY** spacetime two-form $\Omega_{\mathbf{F}}^2$ amounts to the closedness property $d\Omega_{\mathbf{F}}^2 = \mathbf{0}$ which by **VOLTERRA** theorem is equivalent to existence of a potential 1-form $\Omega_{\mathbf{F}}^1$.

²⁴ The definition of $\omega_{\mathbf{E}}^1$ in terms of \mathbf{V}_{ϕ} instead of \mathbf{Z} is innovative and decisive to recover the spatial rule in Prop.12.2. The minus sign in the expression of $\omega_{\mathbf{E}}^1$ is motivated by **LENZ** rule, see Eq.(120).

Proposition 12.2 (Gauss-Lenz-Henry-Faraday). *Closedness of **FARADAY** spacetime two-form $\Omega_{\mathbf{F}}^2$ is equivalent to the spatial **GAUSS** law for the magnetic vortex and to **LENZ-HENRY-FARADAY** spatial induction law:*

$$\begin{aligned} d\Omega_{\mathbf{F}}^2 = \mathbf{0} &\iff \begin{cases} d\omega_{\mathbf{M}}^2 = \mathbf{0}, \\ d\omega_{\mathbf{E}}^1 + \mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{M}}^2) = \mathbf{0}. \end{cases} \\ &\iff \begin{cases} \operatorname{div}(\mathbf{B}) = 0, \\ \operatorname{rot}(\mathbf{E}) + \mathcal{L}_{\mathbf{V}_{\phi}}(\mathbf{B}) + \operatorname{div}(\mathbf{v}_{\phi}) \cdot \mathbf{B} = \mathbf{0}. \end{cases} \end{aligned} \quad (117)$$

Proof. The spacetime extrusion formula Eq.(30) with $\Omega^k = \Omega_{\mathbf{F}}^2$ gives:

$$\mathcal{L}_{\mathbf{V}_{\phi}}(\Omega_{\mathbf{F}}^2) = (d\Omega_{\mathbf{F}}^2) \cdot \mathbf{V}_{\phi} + d(\Omega_{\mathbf{F}}^2 \cdot \mathbf{V}_{\phi}). \quad (118)$$

By applying **STOKES'** formula Eq.(1) to the boundary of a spatial compact manifold, the following commutativity between exterior derivative and spatial projection may be inferred:

$$d \circ \mathbf{P}\downarrow = \mathbf{P}\downarrow \circ d. \quad (119)$$

Here and in the sequel, to simplify the notation, the exterior derivative acting on a spacetime form and the one acting on a spatial form will both be denoted by the same symbol d . From definitions Eq.(115) we get:

$$\left\{ \begin{array}{l} \mathbf{P}\downarrow(d\Omega_{\mathbf{F}}^2) = d(\mathbf{P}\downarrow\Omega_{\mathbf{F}}^2) = d\omega_{\mathbf{M}}^2, \\ \mathbf{P}\downarrow(d\Omega_{\mathbf{F}}^2 \cdot \mathbf{V}_{\phi}) = \mathbf{P}\downarrow(\mathcal{L}_{\mathbf{V}_{\phi}}(\Omega_{\mathbf{F}}^2) - d(\Omega_{\mathbf{F}}^2 \cdot \mathbf{V}_{\phi})) \\ \quad = \mathcal{L}_{\mathbf{V}_{\phi}}(\mathbf{P}\downarrow\Omega_{\mathbf{F}}^2) - d(\mathbf{P}\downarrow(\Omega_{\mathbf{F}}^2 \cdot \mathbf{V}_{\phi})) \\ \quad = \mathcal{L}_{\mathbf{V}_{\phi}}(\omega_{\mathbf{M}}^2) + d\omega_{\mathbf{E}}^1. \end{array} \right. \quad (120)$$

The implication \implies in Eq.(117) follows.

The converse implication \impliedby is inferred by applying the representation formula Eq.(110) to the three-form $d\Omega_{\mathbf{F}}^2$:

$$\begin{aligned} d\Omega_{\mathbf{F}}^2 &= \mathbf{P}\downarrow(d\Omega_{\mathbf{F}}^2) \\ &+ dt_{\mathcal{E}} \wedge \left(\mathbf{P}\downarrow(d\Omega_{\mathbf{F}}^2 \cdot \mathbf{V}_{\phi}) - (\mathbf{P}\downarrow d\Omega_{\mathbf{F}}^2) \cdot \mathbf{V}_{\phi} \right). \end{aligned} \quad (121)$$

The r.h.s. of Eq.(117) and Eq.(120) yield:

$$\begin{cases} \mathbf{P}\downarrow d\Omega_{\mathbf{F}}^2 = \mathbf{0}, \\ \mathbf{P}\downarrow(d\Omega_{\mathbf{F}}^2 \cdot \mathbf{V}_\phi) = \mathbf{0}, \end{cases} \quad (122)$$

which by Eq.(121) imply $d\Omega_{\mathbf{F}}^2 = \mathbf{0}$. \square

Similarly, from the extrusion formula Eq.(31), setting $\Omega^k = \Omega_{\mathbf{F}}^k$ we get:

$$\mathcal{L}_{\mathbf{V}_\phi}(\Omega_{\mathbf{F}}^1) = (d\Omega_{\mathbf{F}}^1) \cdot \mathbf{V}_\phi + d(\Omega_{\mathbf{F}}^1 \cdot \mathbf{V}_\phi), \quad (123)$$

which by the commutativity property Eq.(119) and definitions Eq.(115) gives:

$$\mathbf{P}\downarrow(d\Omega_{\mathbf{F}}^1) = d(\mathbf{P}\downarrow\Omega_{\mathbf{F}}^1) = d\omega_{\mathbf{M}}^1, \quad (124)$$

and

$$\begin{cases} \mathbf{P}\downarrow(d\Omega_{\mathbf{F}}^1 \cdot \mathbf{V}_\phi) \\ = \mathbf{P}\downarrow(\mathcal{L}_{\mathbf{V}_\phi}(\Omega_{\mathbf{F}}^1) - d(\Omega_{\mathbf{F}}^1 \cdot \mathbf{V}_\phi)) \\ = \mathcal{L}_{\mathbf{V}_\phi}(\mathbf{P}\downarrow\Omega_{\mathbf{F}}^1) - d(\mathbf{P}\downarrow(\Omega_{\mathbf{F}}^1 \cdot \mathbf{V}_\phi)) \\ = \mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1) + d\omega_{\mathbf{E}}^0. \end{cases} \quad (125)$$

Hence:

$$\Omega_{\mathbf{F}}^2 = d\Omega_{\mathbf{F}}^1 \iff \begin{cases} \omega_{\mathbf{M}}^2 = d\omega_{\mathbf{M}}^1, \\ \omega_{\mathbf{E}}^1 + \mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1) = \mathbf{0}. \end{cases} \quad (126)$$

In vector terms the expression becomes:

$$\begin{cases} \mathbf{B} = \text{rot}(\mathbf{A}), \\ \mathbf{E} + \mathcal{L}_{\mathbf{V}_\phi}(\mathbf{A}) + 2 \text{EUL}(\mathbf{v}_\phi) \cdot \mathbf{A} = \mathbf{0}, \end{cases} \quad (127)$$

with the **EULER** stretching tensor given by Eq.(49):

$$\text{EUL}(\mathbf{v}_\phi) := \mathbf{g}^{-1} \cdot \frac{1}{2} \mathcal{L}_{\mathbf{V}_\phi}(\mathbf{g}). \quad (128)$$

12.2 Ampère spacetime two-form

Let us now turn to **AMPÈRE-MAXWELL** induction law. Relevant spacetime differential form is **AMPÈRE** two-form $\Omega_{\mathbf{A}}^2$ (*electromagnetic induction flux*) which is potential for the *current* three-form $\Omega_{\mathbf{A}}^3$.

From Lemma 11.1 we infer the next statement.

Proposition 12.3 (Magnetic induction). *The time vertical restrictions of the spacetime **AMPÈRE** two-form $\Omega_{\mathbf{A}}^2$ and of the four-current three-form:*

$$\Omega_{\mathbf{A}}^3 = d\Omega_{\mathbf{A}}^2 \quad (129)$$

are given by:²⁵

$$\begin{cases} \omega_{\mathbf{D}}^2 = \mathbf{P}\downarrow\Omega_{\mathbf{A}}^2, & \text{electric displacement flux} \\ \omega_{\mathbf{H}}^1 = \mathbf{P}\downarrow(\Omega_{\mathbf{A}}^2 \cdot \mathbf{V}_\phi), & \text{magnetic induction flux} \\ \omega_{\rho} = \mathbf{P}\downarrow\Omega_{\mathbf{A}}^3, & \text{electric charge} \\ \omega_{\mathbf{J}}^2 = \mathbf{P}\downarrow(\Omega_{\mathbf{A}}^3 \cdot \mathbf{V}_\phi), & \text{electric current flux} \end{cases} \quad (130)$$

with the representation formulae:

$$\begin{cases} \Omega_{\mathbf{A}}^2 = \omega_{\mathbf{D}}^2 + dt\varepsilon \wedge (\omega_{\mathbf{H}}^1 - \omega_{\mathbf{D}}^2 \cdot \mathbf{v}_\phi), \\ \Omega_{\mathbf{A}}^3 = \omega_{\rho} + dt\varepsilon \wedge (\omega_{\mathbf{J}}^2 - \omega_{\rho} \cdot \mathbf{v}_\phi). \end{cases} \quad (131)$$

An evaluation analogous to the one in Prop.12.2 yields the next result.

Proposition 12.4 (Coulomb, Ampère, Maxwell). *Equality between the current three-form $\Omega_{\mathbf{A}}^3$ and the exterior derivative of **AMPÈRE-MAXWELL** two-form $\Omega_{\mathbf{A}}^2$ is equivalent to **COULOMB**'s balance law for electric charge and to the magnetic induction law:*

$$d\Omega_{\mathbf{A}}^2 = \Omega_{\mathbf{A}}^3 \iff \begin{cases} d\omega_{\mathbf{D}}^2 = \omega_{\rho}, \\ d\omega_{\mathbf{H}}^1 = \mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{D}}^2) + \omega_{\mathbf{J}}^2, \end{cases} \quad (132)$$

$$\iff \begin{cases} \text{div}(\mathbf{D}) = \rho, \\ \text{rot}(\mathbf{H}) = \mathcal{L}_{\mathbf{V}_\phi}(\mathbf{D}) + \text{div}(\mathbf{v}_\phi) \cdot \mathbf{D} + \mathbf{J}. \end{cases}$$

The following property states a really awesome equivalence.

²⁵ The definition of $\omega_{\mathbf{H}}^1$ and $\omega_{\mathbf{J}}^2$ in terms of \mathbf{V}_ϕ instead of \mathbf{Z} is decisive to recover the spatial rule in Prop.12.4.

Proposition 12.5 (Equivalence between spacetime and spatial formulations). *The pair of closedness properties:*

$$\begin{cases} d\Omega_{\mathbf{F}}^2 = 0, \\ d\Omega_{\mathbf{A}}^3 = 0, \end{cases} \quad (133)$$

of the spacetime forms $\Omega_{\mathbf{F}}^2$ and $\Omega_{\mathbf{A}}^3$, are equivalent to the spatial electromagnetic rules respectively named after **GAUSS-LENZ-HENRY-FARADAY** for electric induction and after **COULOMB-ØRSTED-AMPÈRE-MAXWELL** for magnetic induction.

Proof. The equivalence follows directly from the computations in Prop.12.2 and Prop.12.4. This equivalence holds true in the general case of deforming continuous bodies. \square

13 Electromagnetic power

The power locally expended by the electromagnetic fields is the sum of electric and magnetic powers:

$$\begin{aligned} \omega_{\text{POWER}}^3 &:= \omega_{\mathbf{E}}^1 \wedge \left(\mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{D}}^2) + \omega_{\mathbf{J}}^2 \right) \\ &+ \omega_{\mathbf{H}}^1 \wedge \mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^2). \end{aligned} \quad (134)$$

Lemma 13.1 (Umov electromagnetic power). *In any outer oriented, bounded, compact and connected spatial domain C_{OUT} the global electromagnetic power depends only on the boundary values of electric and magnetic fields through the incoming flux:*²⁶

$$\int_{C_{\text{OUT}}} \omega_{\text{POWER}}^3 = - \int_{\partial C_{\text{OUT}}} \omega_{\text{UMOV}}^2, \quad (135)$$

of Nikolay **UMOV** odd two-form:

$$\omega_{\text{UMOV}}^2 := \omega_{\mathbf{E}}^1 \wedge \omega_{\mathbf{H}}^1 \in \Lambda^2(V\mathcal{E}). \quad (136)$$

²⁶ The vector field $\mathbf{E} \times \mathbf{H} : \mathcal{E} \mapsto V\mathcal{E}$ was first introduced in (Umov, 1874) and later reproduced in (Poynting, 1884) and (Heaviside, 1885).

Proof. The induction laws Eq.(96)₂ and Eq.(59) give to the local electromagnetic power the expression:

$$\begin{aligned} \omega_{\text{POWER}}^3 &:= \omega_{\mathbf{E}}^1 \wedge d\omega_{\mathbf{H}}^1 - \omega_{\mathbf{H}}^1 \wedge d\omega_{\mathbf{E}}^1 \\ &= -d(\omega_{\mathbf{E}}^1 \wedge \omega_{\mathbf{H}}^1), \end{aligned} \quad (137)$$

so that **STOKES** formula yields the result. \square

The vector formalism usually adopted in literature can be recovered by observing that:

$$\omega_{\mathbf{E}}^1 \wedge \omega_{\mathbf{H}}^1 = \boldsymbol{\mu}_{\mathbf{g}} \cdot (\mathbf{E} \times \mathbf{H}), \quad (138)$$

so that:

$$d(\omega_{\mathbf{E}}^1 \wedge \omega_{\mathbf{H}}^1) = \text{div}(\mathbf{E} \times \mathbf{H}) \cdot \boldsymbol{\mu}_{\mathbf{g}}. \quad (139)$$

14 Changes of Frame

A natural axiomatic statement is that tensor fields on the spacetime manifold \mathcal{E} transform by push under the action of a spacetime automorphism $\zeta : \mathcal{E} \mapsto \mathcal{E}$ describing a smooth frame-change.

The **FARADAY** and **AMPÈRE-MAXWELL** two-forms $\Omega_{\mathbf{F}}^2$ and $\Omega_{\mathbf{A}}^2$ will accordingly transform as:

$$\begin{cases} (\Omega_{\mathbf{F}}^2)_\zeta = \zeta \uparrow \Omega_{\mathbf{F}}^2, \\ (\Omega_{\mathbf{A}}^2)_\zeta = \zeta \uparrow \Omega_{\mathbf{A}}^2. \end{cases} \quad (140)$$

All other electromagnetic fields also transform according to the natural rule, by invariance of their scalar value under push of the involved arguments.

This conclusion can be deduced observing that:

$$\begin{cases} (t_{\mathcal{E}})_\zeta = \zeta \uparrow t_{\mathcal{E}} = t_{\mathcal{E}} \circ \zeta^{-1}, \\ d(t_{\mathcal{E}})_\zeta = d(\zeta \uparrow t_{\mathcal{E}}) = \zeta \uparrow dt_{\mathcal{E}}, \\ \mathbf{Z}_\zeta = \zeta \uparrow \mathbf{Z}, \\ \mathbf{I}_\zeta = \zeta \uparrow \mathbf{I}, \\ \mathbf{R}_\zeta = d(t_{\mathcal{E}})_\zeta \otimes \mathbf{Z}_\zeta = \zeta \uparrow (dt_{\mathcal{E}} \otimes \mathbf{Z}) = \zeta \uparrow \mathbf{R}, \\ \mathbf{P}_\zeta = \mathbf{I}_\zeta - \mathbf{R}_\zeta = \zeta \uparrow \mathbf{I} - \zeta \uparrow \mathbf{R} = \zeta \uparrow \mathbf{P}. \end{cases} \quad (141)$$

The second and the next to last rules follow from commutativity of exterior derivative and push by the diffeomorphic frame-change map.

15 Frame covariance

In literature it is usually affirmed that **MAXWELL** equations are not form-invariant under **EUCLID** frame changes, but are such under **LORENTZ** transformations.

In our view, the mathematically unspecified notion of form-invariance must be replaced by the natural requirement of covariance under a change of frame.

Definition 15.1 (Covariance of a rule). *The transformed fields, got by pushing the involved tensor fields along the frame-change mapping, are required to fulfil the transformed rule when the original tensor fields obey the original rule.*

Covariance of the electromagnetic induction rules is based on the following preliminary result.

Lemma 15.1 (Covariance of spacetime velocity). *The spacetime velocity is covariant under any transformation ζ in the group of automorphisms in \mathcal{E} .*

Proof. The expression of the pushed spacetime motion:

$$(\zeta\uparrow\phi)_\alpha = \zeta \circ \phi_\alpha \circ \zeta^{-1}, \quad (142)$$

taking the derivative $\partial_{\alpha=0}$ yields:

$$\mathbf{V}_{(\zeta\uparrow\phi)} = T\zeta \cdot \mathbf{V}_\phi \circ \zeta^{-1} = \zeta\uparrow\mathbf{V}_\phi. \quad (143)$$

which is the pertinent transformation rule. \square

When a full spacetime formulation is adopted, it is readily verified that the following fundamental result holds.

Proposition 15.1 (Covariance of induction laws). *The integral formulation Eq.(45), or the equivalent full differential expression Eq.(47), and similarly Eq.(92) or Eq.(96), are covariant under any spacetime change of frame.*

Proof. The validity of the result relies on basic properties of **LIE** and exterior derivatives under the action of automorphic spacetime frame change. Indeed, spacetime tensor fields $\sigma : \mathcal{E} \mapsto \text{TENS}(T\mathcal{E})$ and spacetime exterior forms $\omega : \mathcal{E} \mapsto \Lambda(T\mathcal{E})$, fulfil the natural transformation and commutativity property:

$$\begin{cases} \zeta\uparrow(\mathcal{L}_{\mathbf{V}_\phi}(\sigma)) = \mathcal{L}_{(\zeta\uparrow\mathbf{V}_\phi)}(\zeta\uparrow\sigma), \\ \mathbf{d} \circ (\zeta\uparrow\omega) = \zeta\uparrow \circ \mathbf{d}\omega. \end{cases} \quad (144)$$

Accordingly, under a frame-change $\zeta : \mathcal{E} \mapsto \mathcal{E}$, Eq.(47) does transform into:

$$-\zeta\uparrow\omega_{\mathbf{E}}^1 = \zeta\uparrow(\mathcal{L}_{\mathbf{V}_\phi}(\omega_{\mathbf{M}}^1)) = \mathcal{L}_{(\zeta\uparrow\mathbf{V}_\phi)}(\zeta\uparrow\omega_{\mathbf{M}}^1). \quad (145)$$

The covariance property of the spacetime velocity expressed by Eq.(143) gives:

$$\mathcal{L}_{(\zeta\uparrow\mathbf{V}_\phi)}(\zeta\uparrow\omega_{\mathbf{M}}^1) = \mathcal{L}_{(\mathbf{V}_{(\zeta\uparrow\phi)})}(\zeta\uparrow\omega_{\mathbf{M}}^1). \quad (146)$$

Covariance of the electric or magnetic induction rules thus follows. \square

Let us note that:

$$\begin{cases} \mathbf{V}_{(\zeta\uparrow\phi)} = \mathbf{P}\mathbf{V}_{(\zeta\uparrow\phi)} + \mathbf{Z}, \\ \zeta\uparrow\mathbf{V}_\phi = \zeta\uparrow(\mathbf{P}\mathbf{V}_\phi + \mathbf{Z}) = \zeta\uparrow(\mathbf{P}\mathbf{V}_\phi) + \zeta\uparrow\mathbf{Z}. \end{cases} \quad (147)$$

Accordingly, in the framing $\mathbf{R} := dt_{\mathcal{E}} \otimes \mathbf{Z}$ the transformation rule of the time-vertical component of the spacetime velocity is given by:

$$\mathbf{P}\mathbf{V}_{(\zeta\uparrow\phi)} = \zeta\uparrow(\mathbf{P}\mathbf{V}_\phi) + \mathbf{V}_{\text{REL}}, \quad (148)$$

with the relative spacetime velocity between framings defined by:

$$\mathbf{V}_{\text{REL}} := \zeta\uparrow\mathbf{Z} - \mathbf{Z}. \quad (149)$$

Under **NEWTON** frame changes clock rates are preserved, i.e. $\zeta\uparrow dt_{\mathcal{E}} = dt_{\mathcal{E}}$ so that the relative spacetime velocity \mathbf{V}_{REL} between framings is time-vertical.

From Eq.(148) and Eq.(149) we may infer that covariance of the spatial component of spacetime velocity holds only with respect to the subgroup of frame transformations inducing no relative velocity (a trivial case).

Here lies the mathematical reason why covariance is lost, even under **GALILEI** changes of frame when, in place of adopting the correct expression in Eq.(47), the following improperly incomplete induction rule is adopted, with the addend $\mathbf{d}(\omega_{\mathbf{M}}^1 \cdot \mathbf{v}_\phi)$ dropped off the split expression Eq.(50):

$$-\omega_{\mathbf{E}}^1 = \mathcal{L}_{\mathbf{Z}}(\omega_{\mathbf{M}}^1) + \omega_{\mathbf{M}}^2 \cdot \mathbf{v}_\phi. \quad (150)$$

For a direct comparison with the formulations in literature we observe that, the definition in Eq.(36)₂ and Eq.(37):

$$\begin{cases} \omega_{\mathbf{M}}^2 = \mu_{\mathbf{g}} \cdot \mathbf{B}, \\ \omega_{\mathbf{M}}^1 = \mathbf{g} \cdot \mathbf{A}, \end{cases} \quad (151)$$

the definition of vector product in Eq.(38) and the time-independence of the metric in Eq.(43), lead to a vectorial expression of Eq.(150) given by:²⁷

$$-\mathbf{E} = \mathcal{L}_{\mathbf{Z}}(\mathbf{A}) + \mathbf{B} \times \mathbf{v}_\phi. \quad (152)$$

16 Observers point of view

We may now deduce in a straightforward way the transformation rules for spacetime fields, due to the action of a spacetime frame-change $\zeta : \mathcal{E} \mapsto \mathcal{E}$ as described by a given framing \mathbf{R} , see Eq.(12).

More precisely we shall compute, for each spatial electromagnetic field or flux, the expression of the components of the transformed field or flux in a coordinate system adapted to the original framing.

This definition is adopted in degree that the push by the transformation map be substituted to the improper requirement of *form-invariance*.

The latter notion is in fact not susceptible of a mathematical definition and is therefore misleading, as witnessed by the manifest contradiction between the suggested procedure and the conclusion drawn in (Einstein, 1905a, Part II, §4).

²⁷ This incomplete expression, adopted by Heaviside (1885, 1892), Hertz (1892) and Lorentz (1892) was reproduced in all subsequent treatments in literature, e.g. (Deschamps, 1981, Eq.(85)) and (Landau and Lifshits, 1987, Eq.(17.2)).

17 Special relativity

Let us consider a spacetime frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ adapted to a given framing $\mathbf{R} = dt_{\mathcal{E}} \otimes \mathbf{Z}$, with the first vector given by $\mathbf{X}_0 = \mathbf{Z}/c$, and the remaining time-vertical.

All basis vectors are dimensionless.

To a **LORENTZ** boost $\zeta_{\mathbf{L}} : \mathcal{E} \mapsto \mathcal{E}$ with velocity \mathbf{w} in the \mathbf{X}_1 direction:

$$\mathbf{w} = w \mathbf{X}_1, \quad (153)$$

dropping the invariant basis vectors $\{\mathbf{X}_2, \mathbf{X}_3\}$, there corresponds the tangent transformation:

$$T\zeta_{\mathbf{L}} : T\mathcal{E} \mapsto T\mathcal{E}, \quad (154)$$

given by:²⁸

$$\begin{bmatrix} \zeta_{\mathbf{L}} \uparrow \mathbf{Z} \\ \zeta_{\mathbf{L}} \uparrow \mathbf{X}_1 \end{bmatrix} = \gamma_w \begin{bmatrix} 1 & w \\ w/c^2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{Z} \\ \mathbf{X}_1 \end{bmatrix}. \quad (155)$$

In terms of the adimensional speed $\beta(w) := w/c$ of the boost (ratio between boost speed w and light speed *in vacuo* c), the relativistic factor has the expression:

$$\gamma_w := (1 - w^2/c^2)^{-1/2} = 1/\sqrt{1 - \beta(w)^2}. \quad (156)$$

Transformations $\zeta_{\mathbf{L}} : \mathcal{E} \mapsto \mathcal{E}$ of the group defined by Eq.(155)-(156) are designed to get invariance of the nonsingular spacetime metric tensor $\mathbf{g}_{\mathbf{M}} : T\mathcal{E} \mapsto (T\mathcal{E})^*$ (Minkowski, 1908; Weyl, 1922):

$$\mathbf{g}_{\mathbf{M}} = \mathbf{P} \downarrow \mathbf{g} - c^2 (dt_{\mathcal{E}} \otimes dt_{\mathcal{E}}). \quad (157)$$

Here $\mathbf{g} : V\mathcal{E} \mapsto (V\mathcal{E})^*$ is the positive definite *spatial metric*. Invariance under the boost $\zeta_{\mathbf{L}} : \mathcal{E} \mapsto \mathcal{E}$ means:

$$\mathbf{g}_{\mathbf{M}} = \zeta_{\mathbf{L}} \downarrow \mathbf{g}_{\mathbf{M}}. \quad (158)$$

The inverse boost is got by replacing w with $-w$, so that the relativistic factor γ_w is unchanged.

²⁸ These transformations were introduced and named after **LORENTZ**, by Poincaré (1905), who provided a partial amendment of the ones proposed by Voigt (1887a,b) and by Lorentz (1904).

In addition to Eq.(155) we have for $\alpha = 2, 3$:

$$\zeta_L \uparrow \mathbf{X}_\alpha = \mathbf{X}_\alpha. \quad (159)$$

Explicitly we write:

$$\begin{cases} \zeta_L \uparrow \mathbf{Z} = \gamma_w (\mathbf{Z} + w \mathbf{X}_1), \\ \zeta_L \uparrow \mathbf{X}_1 = \gamma_w \left((w/c^2) \mathbf{Z} + \mathbf{X}_1 \right), \end{cases} \quad (160)$$

with inverse ($-w$ in place of w) given by:

$$\begin{cases} \zeta_L \downarrow \mathbf{Z} = \gamma_w (\mathbf{Z} - w \mathbf{X}_1), \\ \zeta_L \downarrow \mathbf{X}_1 = \gamma_w \left(- (w/c^2) \mathbf{Z} + \mathbf{X}_1 \right). \end{cases} \quad (161)$$

A vector $\mathbf{V} \in T\mathcal{E}$ has components transformed by the matrix inverse-transpose of the one in Eq.(155):

$$\begin{bmatrix} V_{\zeta_L \uparrow \mathbf{Z}} \\ V_{\zeta_L \uparrow \mathbf{X}_1} \end{bmatrix} = \gamma_w \begin{bmatrix} 1 & -w/c^2 \\ -w & 1 \end{bmatrix} \cdot \begin{bmatrix} V_{\mathbf{Z}} \\ V_{\mathbf{X}_1} \end{bmatrix}. \quad (162)$$

In the limit $w/c \rightarrow 0$ we get:

$$\begin{cases} \gamma_w \rightarrow 1, \\ w/c^2 \rightarrow 0. \end{cases} \quad (163)$$

The **LORENTZ** tangent map Eq.(155) reduces then to the **GALILEI** transformation rule, for a relative translational speed w in direction of \mathbf{X}_1 :

$$\begin{bmatrix} \zeta_G \uparrow \mathbf{Z} \\ \zeta_G \uparrow \mathbf{X}_1 \end{bmatrix} = \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{Z} \\ \mathbf{X}_1 \end{bmatrix}. \quad (164)$$

18 Frame changes in special relativity

A spacetime tensor field σ is transformed by the action of a **LORENTZ** automorphism $\zeta_L : \mathcal{E} \mapsto \mathcal{E}$ into the pushed field $\zeta_L \uparrow \sigma$. A framing \mathbf{R} is likewise sent into the pushed framing $\zeta_L \uparrow \mathbf{R}$.

A spacetime frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ adapted to \mathbf{R} is pushed to a spacetime frame adapted to the pushed framing.

A comparison of a tensor field with its pushed counterpart will be made in the sequel by evaluating the longitudinal and transversal components

of both fields in the *adapted* spacetime frame $\{\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$.

These evaluations are carried out to contrast method and conclusions exposed in literature.

To be honest, there is no real need of this stuff because in electromagnetism all involved fields are fully covariant under any spacetime frame-change and all rules of electromagnetic induction are also fully covariant, when properly expressed in geometric terms, thanks to the commutativity property between push transformation and exterior derivative and of naturality of **LIE** derivatives with respect to push, as expressed in Eq.(144).

These properties are direct consequences of basic mathematical notions concerning the relation in Eq.(1) between integrals over compact manifolds and over their boundaries, and the transformation of integrals under the action of diffeomorphic maps, the fields acted upon by the integrals being exterior forms of maximal degree on the relevant domains.

The transformation of electromagnetic fields and fluxes under spacetime changes of frame has a central role in many treatments and is therefore certainly worth to be explicitly investigated, a task performed in the next subsections.

18.1 Electric induction

18.1.1 Electric field

The electric spacetime time-vertical one-form $\omega_{\mathbf{E}}^1$ is transformed by the **LORENTZ** change of observer into the one-form:

$$\zeta_L \uparrow \omega_{\mathbf{E}}^1, \quad (165)$$

Its longitudinal component, along the direction \mathbf{X}_1 of the boost, is given by:

$$\begin{aligned} (\zeta_L \uparrow \omega_{\mathbf{E}}^1) \cdot \mathbf{X}_1 &= \zeta_L \uparrow \left(\omega_{\mathbf{E}}^1 \cdot (\zeta_L \downarrow \mathbf{X}_1) \right) \\ &= \gamma_w \cdot \zeta_L \uparrow \left(\omega_{\mathbf{E}}^1 \cdot \left(\mathbf{X}_1 - \frac{w}{c^2} \mathbf{Z} \right) \right) \\ &= (\gamma_w \cdot \omega_{\mathbf{E}}^1 \cdot \mathbf{X}_1) \circ \zeta_L^{-1}. \end{aligned} \quad (166)$$

Cancellation is due to time-verticality of $\omega_{\mathbf{E}}^1$.

The frame-transformation for the spacetime transversal component along directions \mathbf{X}_α , with $\alpha = 2, 3$, are:

$$\begin{aligned} (\zeta_L \uparrow \omega_{\mathbf{E}}^1)(\mathbf{X}_\alpha) &= \zeta_L \uparrow \left(\omega_{\mathbf{E}}^1 \cdot (\zeta_L \downarrow \mathbf{X}_\alpha) \right) \\ &= \zeta_L \uparrow \left(\omega_{\mathbf{E}}^1 \cdot \mathbf{X}_\alpha \right) \\ &= (\omega_{\mathbf{E}}^1 \cdot \mathbf{X}_1) \circ \zeta_L^{-1}. \end{aligned} \quad (167)$$

The transversal components of the electric field $\omega_{\mathbf{E}}^1$ along \mathbf{X}_α , with $\alpha = 2, 3$, are then invariant:

$$\omega_{\mathbf{E}}^1 \cdot \mathbf{X}_\alpha \rightarrow (\omega_{\mathbf{E}}^1 \cdot \mathbf{X}_\alpha) \circ \zeta_L^{-1}. \quad (168)$$

18.1.2 Magnetic vortex

The frame-transformation formula for the components of the magnetic vortex $\omega_{\mathbf{M}}^2$ in the longitudinal planes $\{\mathbf{x}_1, \mathbf{x}_\alpha\}$, with $\alpha = 2, 3$, writes:

$$\begin{aligned} (\zeta_L \uparrow \omega_{\mathbf{M}}^2)(\mathbf{X}_1, \mathbf{X}_\alpha) &= \zeta_L \uparrow \left(\omega_{\mathbf{M}}^2 \cdot (\zeta_L \downarrow \mathbf{X}_1) \cdot \mathbf{X}_\alpha \right) \\ &= \zeta_L \uparrow \left(\gamma_w \Omega_{\mathbf{M}}^2 \left(\mathbf{X}_1 - \cancel{(w/c^2) \mathbf{Z}}, \mathbf{X}_\alpha \right) \right). \end{aligned} \quad (169)$$

Cancellation is due to time-verticality of $\omega_{\mathbf{M}}^2$. The components of magnetic vortex in longitudinal planes are amplified by the relativistic factor:

$$\omega_{\mathbf{M}}^2(\mathbf{X}_1, \mathbf{X}_\alpha) \rightarrow \gamma_w \cdot \zeta_L \uparrow \left(\omega_{\mathbf{M}}^2(\mathbf{X}_1, \mathbf{X}_\alpha) \right). \quad (170)$$

On the other hand, the component of $\omega_{\mathbf{M}}^2$ in the transversal plane $\{\mathbf{X}_2, \mathbf{X}_3\}$ is invariant:

$$(\zeta_L \uparrow \omega_{\mathbf{M}}^2)(\mathbf{X}_2, \mathbf{X}_3) = \zeta_L \uparrow \left(\omega_{\mathbf{M}}^2(\mathbf{X}_2, \mathbf{X}_3) \right). \quad (171)$$

18.1.3 Vectorial notation

In terms of spatial vector fields, we have, for $\alpha = 2, 3$:

$$\begin{aligned} \omega_{\mathbf{E}}^1(\mathbf{X}_1) &= \mathbf{g}(\mathbf{E}^\parallel, \mathbf{X}_1), \\ \omega_{\mathbf{E}}^1(\mathbf{X}_\alpha) &= \mathbf{g}(\mathbf{E}^\perp, \mathbf{X}_\alpha), \\ \omega_{\mathbf{M}}^2(\mathbf{X}_2, \mathbf{X}_3) &= \boldsymbol{\mu}(\mathbf{B}, \mathbf{X}_2, \mathbf{X}_3) = \mathbf{g}(\mathbf{B}^\parallel, \mathbf{X}_1), \\ \omega_{\mathbf{M}}^2(\mathbf{X}_1, \mathbf{X}_\alpha) &= \boldsymbol{\mu}(\mathbf{B}, \mathbf{X}_1, \mathbf{X}_\alpha) = \mathbf{g}(\mathbf{B}^\perp, \mathbf{X}_\alpha). \end{aligned} \quad (172)$$

Here above \parallel and \perp denote the components parallel and orthogonal to the boost direction \mathbf{X}_1 .

Eq.(18.42) and (18.43) in (Panofski and Phillips, 1962, p.330) and Table (26.3) in (Feynman et al., 1964, 26.3) contain the currently adopted transformation rules for electric and magnetic spatial vector fields under a **LORENTZ** boost.

In §19 a Synoptic Table offers a comparison of these rules, labeled as *old*, versus the ones contributed here, labeled as *new*.

Agreement holds only for the parallel magnetic induction vector field \mathbf{B}^\parallel (orthogonal to the transversal plane).

On the contrary, all other *old* transformation rules exposed in literature, pertaining to transversal magnetic induction vector field \mathbf{B}^\perp (parallel to the transversal plane) and to electric vector field \mathbf{E} , are not in agreement with the *new* ones. No entanglements are found as outcome of the new analysis.

18.2 Magnetic induction

The electric displacement two-form $\omega_{\mathbf{D}}^2$, magnetic winding one-form $\omega_{\mathbf{H}}^1$, electric charge three-form ω_ρ , and current two-form $\omega_{\mathbf{J}}^2$, all time-vertical and *odd*.

The transformation rules of their components are interpreted in the original framing as:

$$\left\{ \begin{aligned} (\zeta_L \uparrow \omega_{\mathbf{D}}^2) \cdot \mathbf{X}_1 \cdot \mathbf{X}_\alpha &= \zeta_L \uparrow \left(\omega_{\mathbf{D}}^2 \cdot (\zeta_L \downarrow \mathbf{X}_1) \cdot \mathbf{X}_\alpha \right) \\ &= \zeta_L \uparrow \left(\gamma_w \omega_{\mathbf{D}}^2 \cdot \mathbf{X}_1 \cdot \mathbf{X}_\alpha \right) \\ &\quad - \gamma_w (w/c^2) \cancel{\omega_{\mathbf{D}}^2 \cdot \mathbf{Z} \cdot \mathbf{X}_\alpha}, \end{aligned} \right. \quad (173)$$

$$\left\{ \begin{aligned} (\zeta_L \uparrow \omega_{\mathbf{H}}^1) \cdot \mathbf{X}_1 &= \zeta_L \uparrow \left(\omega_{\mathbf{H}}^1 \cdot (\zeta_L \downarrow \mathbf{X}_1) \right) \\ &= \zeta_L \uparrow \left(\gamma_w \omega_{\mathbf{H}}^1 \cdot \left(\mathbf{X}_1 - \cancel{(w/c^2) \mathbf{Z}} \right) \right), \end{aligned} \right. \quad (174)$$

$$\left\{ \begin{aligned} (\zeta_L \uparrow \omega_{\mathbf{J}}^2) \cdot \mathbf{X}_1 \cdot \mathbf{X}_\alpha &= \zeta_L \uparrow \left(\omega_{\mathbf{J}}^2 \cdot (\zeta_L \downarrow \mathbf{X}_1) \cdot \mathbf{X}_\alpha \right) \\ &= \zeta_L \uparrow \left(\gamma_w \omega_{\mathbf{J}}^2 \cdot \mathbf{X}_1 \cdot \mathbf{X}_\alpha \right) \\ &\quad - \gamma_w (w/c^2) \cancel{\omega_{\mathbf{J}}^2 \cdot \mathbf{Z} \cdot \mathbf{X}_\alpha}, \end{aligned} \right. \quad (175)$$

$$\left\{ \begin{aligned} (\zeta_L \uparrow \omega_\rho) \cdot \mathbf{X}_1 \cdot \mathbf{X}_{23} &= \zeta_L \uparrow (\omega_\rho \cdot (\zeta_L \downarrow \mathbf{X}_1) \cdot \mathbf{X}_{23}) \\ &= \zeta_L \uparrow (\gamma_w \omega_\rho \cdot \mathbf{X}_1 \cdot \mathbf{X}_{23}) \\ &\quad - \zeta_L \uparrow (\gamma_w (w/c^2) \omega_\rho \cdot \mathbf{Z} \cdot \mathbf{X}_{23}), \end{aligned} \right. \quad (176)$$

with the shorthand $\mathbf{X}_{23} = \mathbf{X}_2 \cdot \mathbf{X}_3$.

The *new* transformation rules exposed below in the Synoptic Table provide also an *errata corrige* to the rule in (G. Romano, 2013), where the electromagnetic entanglement, although vanishing in the classical limit, was still present due to a trivial lack of cancellation by spatiality.

We may conclude that, between electrodynamical fields, transformed by the action of the **LORENTZ** group and interpreted in the original frame, relativistic entanglements do not occur.

18.3 General transformation rule

A direct inspection of the proofs in §18.1 and §18.2 reveals that resulting transformation rule for space-time (electromagnetic) differential forms under the action of a **LORENTZ** frame-change, depends only on the list of basis vectors relevant to the involved components and not on the spacetime differential forms themselves.

Precisely, the components transformation rules depend on whether the list of basis vector arguments does include the boost direction or does not.

- In the first case the transformation is an amplification by the relativistic factor.
- In the second case the transformation is just by invariance.

Any way, no entanglement does occur.

19 Comparisons

The Synoptic Table below provides a comparison between the *new* transformation rules for electric and magnetic spatial vector fields and the *old* rules.

The manifest outcome is that entanglements involved in the *old* rules do not occur in the *new* ones.

Synoptic Table	
<i>new</i>	<i>old</i>
$(\mathbf{E}^\parallel, \mathbf{E}^\perp) \rightarrow (\gamma_w \mathbf{E}^\parallel, \mathbf{E}^\perp)$	$(\mathbf{E}^\parallel, \gamma_w (\mathbf{E}^\perp + \mathbf{w} \times \mathbf{B}))$
$(\mathbf{B}^\parallel, \mathbf{B}^\perp) \rightarrow (\mathbf{B}^\parallel, \gamma_w \mathbf{B}^\perp)$	$(\mathbf{B}^\parallel, \gamma_w (\mathbf{B}^\perp - \mathbf{w} \times \mathbf{E}))$
$(\mathbf{H}^\parallel, \mathbf{H}^\perp) \rightarrow (\gamma_w \mathbf{H}^\parallel, \mathbf{H}^\perp)$	$(\mathbf{H}^\parallel, \gamma_w (\mathbf{H}^\perp + \mathbf{w} \times \mathbf{D}))$
$(\mathbf{D}^\parallel, \mathbf{D}^\perp) \rightarrow (\mathbf{D}^\parallel, \gamma_w \mathbf{D}^\perp)$	$(\mathbf{D}^\parallel, \gamma_w (\mathbf{D}^\perp + \mathbf{w} \times \mathbf{H}))$
$(\mathbf{J}^\parallel, \mathbf{J}^\perp) \rightarrow (\mathbf{J}^\parallel, \gamma_w \mathbf{J}^\perp)$	$(\mathbf{J}^\parallel, \gamma_w (\mathbf{J}^\perp + \mathbf{w} \times \mathbf{B}))$

The term $\mathbf{w} \times \mathbf{B}$, in the *old* expression for the transversal component of electric field, is responsible for the invalid relativistic support of **LORENTZ** force.

The geometric analysis provides similarly the transformation rules for all other time-vertical forms, as interpreted in the original framing.

Under a **LORENTZ** automorphism, the integral, over a body configuration \mathcal{P} , transform in such a way that:

$$\left\{ \begin{aligned} \int_{\mathcal{P}} \zeta_L \uparrow \omega_\rho &= \gamma_w \cdot \int_{\mathcal{P}} \omega_\rho, & \text{electric charge} \\ \int_{\mathcal{P}} \zeta_L \uparrow \mu_{\mathbf{g}} &= \gamma_w \cdot \int_{\mathcal{P}} \mu_{\mathbf{g}}, & \text{metric volume} \\ \int_{\mathcal{P}} \zeta_L \uparrow \mathbf{m} &= \gamma_w \cdot \int_{\mathcal{P}} \mathbf{m}. & \text{material mass} \end{aligned} \right. \quad (177)$$

As a consequence:²⁹

$$\left\{ \begin{aligned} \rho &\rightarrow \rho, & \text{electric charge per unit volume} \\ \rho_m &\rightarrow \rho_m. & \text{mass per unit volume} \end{aligned} \right. \quad (178)$$

The charge density ρ per unit volume, is an *even* scalar field on the trajectory manifold $\mathcal{T}_{\mathcal{E}} \subset \mathcal{E}$.

Invariance follows from the rules Eq.(177)_{1,2} and from the definition Eq.(89)₄:

$$\omega_\rho = \rho \cdot \mu_{\mathbf{g}}. \quad (179)$$

²⁹ Contrary to our evaluation, the transformation $\rho \rightarrow \gamma_w \rho$ is attributed to the charge density ρ per unit volume, in standard treatments (Weyl, 1922; Jefimenko, 1999).

An analogous reasoning shows invariance of the mass per unit volume, an *even* scalar field ρ_m on the trajectory manifold $\mathcal{T}_{\mathcal{E}} \subset \mathcal{E}$, defined by:

$$\mathbf{m} = \rho_m \cdot \boldsymbol{\mu}_{\mathbf{g}}. \quad (180)$$

A perfect analogy holds between the transformation rules in Eq.(177)_{1,2,3} for electric charge, metric volume and mass, since they are all integrals of spatial forms of maximal degree and the rule in §18.3 does apply.

The formula in Eq.(177)₃ describes in a simple direct way the estimate of the transformed *mass* as detected by an observer in terms of the *mass-form* transformed according to a **LORENTZ** frame-change.

Let us prove explicitly the formula in Eq.(177)₂.

To this end, for sake of simplicity, the time-vertical tangent vector fields $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ are taken mutually \mathbf{g} -orthogonal. The metric on the spatial bundle $\mathbf{g} : V\mathcal{E} \mapsto (V\mathcal{E})^*$ can be extended by means of the projection pull-back to a spacetime singular metric:

$$\mathbf{P}\downarrow\mathbf{g} : T\mathcal{E} \mapsto (T\mathcal{E})^*. \quad (181)$$

The singular metric so got is time-vertical, i.e. vanishing when the time-arrow \mathbf{Z} is an argument.

Recalling Eq.(161), the longitudinal component is transformed as:

$$\begin{aligned} (\zeta_L \uparrow \mathbf{P}\downarrow\mathbf{g})(\mathbf{X}_1, \mathbf{X}_1) &= \zeta_L \uparrow \left((\mathbf{P}\downarrow\mathbf{g})(\zeta_L \downarrow \mathbf{X}_1, \zeta_L \downarrow \mathbf{X}_1) \right) \\ &= \zeta_L \uparrow \left(\mathbf{g}(\gamma_w \cdot \mathbf{X}_1, \gamma_w \cdot \mathbf{X}_1) \right) \\ &= \gamma_w^2 \cdot \zeta_L \uparrow \left(\mathbf{g}(\mathbf{X}_1, \mathbf{X}_1) \right). \end{aligned} \quad (182)$$

The remaining components on the diagonal are left invariant. The metric volume form is given by:

$$\left(\boldsymbol{\mu}_{\mathbf{g}}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \right)^2 = \det \left(\mathbf{G}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \right). \quad (183)$$

With $\mathbf{G}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ **GRAM** matrix of the metric \mathbf{g} . Substituting Eq.(182) into Eq.(183) we get:

$$\boldsymbol{\mu}_{\mathbf{g}}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \rightarrow \gamma_w \cdot \boldsymbol{\mu}_{\mathbf{g}}(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3). \quad (184)$$

According to Eq.(182), the transformed spatial length in the longitudinal direction, evaluated by the original observer, is amplified by the relativistic factor, as first deduced in (G. Romano, 2014c):

$$\mathbf{g}(\mathbf{X}_1, \mathbf{X}_1)^{\frac{1}{2}} \rightarrow \gamma_w \cdot \mathbf{g}(\mathbf{X}_1, \mathbf{X}_1)^{\frac{1}{2}}. \quad (185)$$

Also, according to Eq.(177)₁ the transformed electric charge, evaluated by the original observer, is amplified by the relativistic factor.

The same result applies to the metric volume provided by Eq.(184) and reported in Eq.(177)₂. An analogous proof applies also to the material mass, as reported in Eq.(177)₃ leading to the following consideration.

The **MACLAURIN** series formula for the relativistic factor in terms of the adimensional speed $\beta(w) := w/c$, truncated at the third term:

$$\gamma_w = (1 - w^2/c^2)^{-1/2} \simeq 1 + 0 + \frac{1}{2} w^2/c^2, \quad (186)$$

substituted in Eq.(177)₃ multiplied by c^2 yields:

$$\gamma_w \cdot \int_{\mathcal{P}} \mathbf{m} \cdot c^2 \simeq \int_{\mathcal{P}} \mathbf{m} \cdot c^2 + \int_{\mathcal{P}} \frac{1}{2} \cdot \mathbf{m} \cdot w^2. \quad (187)$$

The expression in Eq.(187) is named *total energy*. The former of the terms:

$$\int_{\mathcal{P}} \mathbf{m} \cdot c^2, \quad \int_{\mathcal{P}} \frac{1}{2} \cdot \mathbf{m} \cdot w^2. \quad (188)$$

is the *rest energy* of the body \mathcal{P} , while the latter is similar to the expression for the *kinetic energy*, but with boost speed w in place of the velocity field of the body.³⁰

An analogous transformation law applies to *kinetic momentum*. The continuum definition is provided by the variational expression:

$$\int_{\mathcal{P}} \delta\omega^1 \wedge (\mathbf{m} \cdot \mathbf{v}_{\phi}) = \int_{\mathcal{P}} \langle \delta\omega^1, \mathbf{v}_{\phi} \rangle \cdot \mathbf{m}, \quad (189)$$

and, according to Eq.(177)₃ is transformed into:

$$\gamma_w \cdot \int_{\mathcal{P}} \langle \delta\omega^1, \mathbf{v}_{\phi} \rangle \cdot \mathbf{m}. \quad (190)$$

³⁰ Confusion between the two is surprisingly spread all over standard physics literature on relativity dealing with particle kinematics.

For a continuous body, the mass is a maximal material form to be integrated on the current configuration \mathcal{P} . The previous analysis reveals that in this context the transformation rules dictated by special relativity are deduced in a simplest way by means of Eq.(190).

These evaluations should be compared with the involved argument exposed in physically biased tracts on special relativity where the transformed quantities appear as tentative definitions to be confirmed by thought collision experiment (Rindler, 1989, Ch.V), (Forshaw and Smith, 2009, §(7.1.1)), (Fernflores, 2019).

For what concerns the clock $dt_{\mathcal{E}} \in (T\mathcal{E})^*$, we observe that the rate of the pushed clock, when evaluated in the original framing, is *faster* by the relativistic factor:

$$\begin{aligned} \langle \zeta_L \uparrow dt_{\mathcal{E}}, \mathbf{Z} \rangle &= \zeta_L \uparrow \langle dt_{\mathcal{E}}, \zeta_L \downarrow \mathbf{Z} \rangle \\ &= \gamma_w \cdot \zeta_L \uparrow \langle dt_{\mathcal{E}}, \mathbf{Z} - \cancel{w \mathbf{X}_1} \rangle \quad (191) \\ &= \gamma_w \geq 1, \end{aligned}$$

since $\langle dt_{\mathcal{E}}, \mathbf{Z} \rangle = 1$. Cancellation is due to time-verticality of \mathbf{X}_1 .

In the classical limit $w/c \rightarrow 0$, from Eq.(156) we get $\gamma_w = 1$ and the result reduces to the one concerning NEWTON frame-changes

$$\zeta_N : \mathcal{E} \mapsto \mathcal{E}, \quad (192)$$

characterised by invariance of the clock rate:

$$\zeta_N \uparrow dt_{\mathcal{E}} = dt_{\mathcal{E}}. \quad (193)$$

By virtue of this property, spatial vector fields are transformed by NEWTON frame-changes into vector fields that are still spatial in the same framing.

20 Conclusions

Three main contributions have been brought to theoretical Electrodynamics and Relativity.

The first contribution concerns the development of a spacetime formulation of electric induction law in terms of electric field and magnetic potential, (*à la*

MAXWELL and J.J. THOMSON) in place of the seemingly convenient but eventually misleading reduced formulation (*à la* HEAVISIDE-HERTZ-LORENTZ). in terms of exterior derivatives.

In the reduced formulations a velocity dependent exact differential term is zeroed by the action of taking the exterior derivative, but this fact destroys frame-covariance of the induction laws.

The integral formulation ought to be made in terms of one-forms integrated along 1D paths in spacetime.

It has been shown (G. Romano, 2014) that this new formulation solves all troubles concerning the *flux-rule* exposed in (Feynman et al., 1964, II.17-2).

Indeed the *flux-rule* (or better the *vorticity-rule*) was there applied out of its range of validity which is limited to boundary paths undergoing regular motions, as assessed here in Prop.9.3.

Moreover, the formulation in terms of differential forms puts into evidence a new term, depending on the stretching of deformable bodies, completely overlooked by standard vectorial formulations.

The second contribution provides a special expression referring to the case of translational motions in a field of constant magnetic momentum and uniform magnetic vorticity.

It is thus possible to apply a simple formula for the electric field in terms of the magnetic vortex, which is exactly one-half of what is usually labeled as LORENTZ force rule.

The third contribution is concerned with the spacetime formulation of Electromagnetism in terms of the electromagnetic FARADAY and AMPÈRE two-forms and the detection of the transformation rules for electric and magnetic fields and fluxes under change of frame and in particular under LORENTZ transformations of special relativity.

The electromagnetic induction laws, expressed in terms of LIE and exterior derivatives, are covariant under any change of frame.

This means they simply transform by push according to the diffeomorphic transformation map, since both these derivatives and all involved geometrical entities transform by push, in a natural way.

This statement amends the claim that MAXWELL laws are not invariant under GALILEI group of

frame transformations (Choquet-Bruhat, 2009, p.21, Eq.(2.6)).

In this respect I observe that invariance is not the proper qualification to be asked for, since tensor fields of degree greater than zero (i.e. other than scalar fields) are involved in the laws of electromagnetic induction and therefore covariance, that is variance by push, should rather be invoked.

The transformation rule by covariance, when applied to LORENTZ frame changes, and interpreted in the original frame, reveals that relativistic transformations previously considered in literature ought to be thoroughly modified.

Indeed, in amendment of standard statements, the conclusion of the new investigation is that electromagnetic entanglements between electric and magnetic spatial fields are completely absent, as clearly depicted in the Synoptic Table of §19.

In the classical limit $w/c \rightarrow 0$ and the relativistic factor tends to unity $\gamma_w \rightarrow 1$.

Moreover, $w/c^2 \rightarrow 0$ in Eq.(155), so that invariance of electric and magnetic forms under spatial GALILEI transformation is recovered in the limit, as expected on physical and mathematical ground, due to continuous dependence of the transformation on the light speed.

In particular, the amended relativistic transformation rule for the electric field deprives the LORENTZ force rule of any relativistic support.

Last but not least, according to the present treatment, all spatial differential forms of maximal degree, such as metric volume form, mass form and charge form, are transformed in the same way, by amplification according to the relativistic factor, as given by Eq.(177).

Length in direction of boost and clock rates is likewise modified by the same amplification. These results are of special relevance and suggest a revision of physical interpretations in special relativity.

Since early contributions (Poincaré, 1905) electric charge conservation under LORENTZ transformations was assumed in deducing transformation rules of electromagnetic fields.

Quite the other way, in relativistic dynamics mass is not assumed to be conserved but rather to be amplified according to the relativistic factor.

According to the geometrical analysis exposed in the present contribution, mass, electric charge and metric volume, which all are spatial differential forms of the maximal degree, have all an identical behaviour under relativistic frame transformations.

The spacetime treatment, performed in terms of differential forms and LIE derivatives along the motion, brings to conceptual and methodological decisive improvements over the still presently ubiquitously adopted standard vectorial expressions.

Theoretical discussions are moreover significantly simplified and clarified by the adopted geometric framework.

This is especially evident in discussing questions about frame covariance of induction laws and in evaluating the transformations induced by frame changes when interpreted in the original framing.

As a matter of fact, all modern treatments of electrodynamics still include inappropriate entanglements borrowed from the analysis in (Lorentz, 1904; Poincaré, 1905; Einstein, 1905a) which were based on the incomplete interpretation of the original formulation exposed in (Clerk-Maxwell, 1861, 1865) and on the consequent misleading simplification brought by Heaviside (1885, 1892); Hertz (1892); Lorentz (1892).

Unfortunately, the clarification contributed by J.J. Thomson (1893) about the original formulation of electromagnetic induction laws by his master James CLERK-MAXWELL was completely overlooked in the pertinent literature of the XX century.

The geometric analysis first carried out in (G. Romano, 2012), and revised and further developed in the present paper, provides an independent confirmation of these clarifications bringing them again under the spotlight after more than a century of oblivion.

The mathematical theory of differential forms and integration on manifolds, largely due to Élie Cartan (1899, 1923, 1924, 1945), Georges de Rham (1955) and Hassler Whitney (1957), as well illustrated in (Marsden et al., 2003; Hitchin, 2003; Fecko, 2006), is presently self-proposing as the suitable tool for spacetime formulation of electromagnetic induction laws.

This comment refers especially to treatments involving deformable bodies in motion, and for the description of spacetime transformations of electromag-

netic fields in special relativity.

A sound evidence of merits of clarity and conciseness of the exterior differential machinery with respect to the standard vectorial one, emerges by comparing the sharp and general reasoning in Prop.15.1 to the involved treatment in (Jefimenko, 1999) relying on questionable electromagnetic entanglements and on form invariance.

Thanks to this powerful theory, a direct recourse to the relevant notions and properties permits to get rid of the alleged assumption of *form-invariance* of electromagnetic induction laws and of *conservation* of electric charge under LORENTZ frame-changes, and to state natural and consistent rules of transformation for physical fields represented by differential forms in spacetime.

21 Some hints for collateral reading

I would draw attention of readers interested in historical and attributional issues in differential geometry, to two nice brief papers that could easily escape to a first search.

One is by Samelson (2001) where evidence about the birth of exterior derivatives and of their powerful properties first investigated by Volterra (1889a,b) are given.

The other one is by Trautman (2008) about the *naissance* of LIE-derivatives theory.

Historical notes on the development of the laws of electromagnetic induction were recently contributed by Ovidio Mario Bucci (2014).

For mathematical biased scholars we also suggest the treatments in (Marmo et al., 2005; Marmo and Tulczyjew, 2006; De Nicola and Tulczyjew, 2009) concerned in particular with the notion of orientations in spacetime.

A comprehensive exposition of differential forms, integration on manifolds and orientation, is offered in (Fecko, 1997, 2006) (Marsden et al., 2003) and (Hitchin, 2003).

The role of differential forms in Electrodynamics was well outlined in (Deschamps, 1970, 1981) and is effectively described in (Fecko, 2014).

Discrete topological formulations of Electromagnetics and relevant computational aspects are discussed in (Tonti, 1995, 2002), (Bossavit, 1991, 1998, 2004, 2005), (Gross and Kotiuga, 2004; Kurz and Auchmann, 2012), (Stern et al., 2015) and references therein.

References

- Ampère A.A., 1826. Mémoire sur la théorie des phénomènes électro-dynamiques, uniquement déduite de l'expérience. Méquignon-Marvis, Paris.
- Bateman H., 1910. The transformation of the electro-dynamical equations, Proceedings London Mathematical Society 8(2) 223–264.
- Benn I.M. and Tucker R.W., 1987. Introduction to Spinors and Geometry with Applications in Physics. Adam Hilger, Bristol, England.
- Bucci O.M., 2014. From Electromagnetism to the Electromagnetic Field: The Genesis of Maxwell's Equations. IEEE Antennas and Propagation Magazine, 56(6) 183–207.
- Bossavit, A., 1991. Differential Geometry for the student of numerical methods in Electromagnetism. Électricité de France, Études et Recherches.
- Bossavit, A., 1998. On the geometry of electromagnetism, Lecture 1–10. J. Japan Soc. Appl. Electromagn. & Mech.
- Bossavit, A., 2004. Computational electromagnetism and Whitney forms. Minneapolis, 11–15 May 2004.
- Bossavit, A., 2005. Applied Differential Geometry (A compendium). Électricité de France, Études et Recherches.
- Bossavit, A., 2008. A uniform rationale for Whitney forms on various supporting shapes. Mathematics and Computers in Simulation 80, 1567–1577.
- Cartan É., 1899. Sur certaines expressions différentielle et sur le problème de Pfaff. Annales École Normale Supérieure, (1) 239–332; Oeuvres Part II, Vol.I, pp.303–397.
- Cartan É., 1922. Leçons sur les invariants intégraux. Hermann, Paris.
- Cartan É., 1923. Sur les variétés à connexion affine et la théorie de la relativité généralisé (première partie). Annales Scientifiques de l'École Normale Supérieure 40(3) 325–412.
- Cartan É., 1924. Sur les variétés à connexion affine et la théorie de la relativité généralisé (première par-

- tie) (suite). *Annales Scientifiques de l'École Normale Supérieure* 41(3) 1–25.
- Cartan, É., 1945. *Les systèmes différentiels extérieurs et leur applications géométriques*. Hermann, Paris.
- Choquet-Bruhat Y., 2009. *General Relativity and the Einstein Equations*. Oxford University Press.
- Clerk-Maxwell J., 1855. On Faraday's Lines of Force, *Transactions of the Cambridge Philosophical*, Vol. X, Part I, Dec. 10, 1855, and Feb. 11, 1856. in: *The scientific papers of James Clerk-Maxwell*, W.D. Niven ed., Dover New-York (1890) 155–229.
- Clerk-Maxwell J., 1861. On Physical Lines of Force, *The London, Edinburgh, Dublin Phil. Mag. and Journal of Science*, Fourth series, Part I, II, III, IV.
- Clerk-Maxwell J., 1865. A Dynamical Theory of the Electromagnetic Field. *Philosophical Transactions of the Royal Society of London* 155, 459–512.
- Clerk-Maxwell J., 1873. *A Treatise on Electricity and Magnetism*, Vol. I, II. 2nd ed. 1881. Oxford at Clarendon Press.
- Dantzig D. van, 1932. Zur allgemeinen projektiven Differentialgeometrie I, II. *Proc. Kon. Akad. Amsterdam* 35, 524–534, 535–542
- Dantzig D. van, 1934. The fundamental equations of electromagnetism, independent of metrical geometry. *Mathematical Proceedings of the Cambridge Philosophical Society* 30(4) 1934, 421–427. Published online by Cambridge University Press: 24 October 2008.
- Darrigol O., 2000. *Electrodynamics from Ampère to Einstein*. Oxford University Press.
- Deahna F., 1840. Ueber die Bedingungen der Integrabilität linearer Differentialgleichungen erster Ordnung zwischen einer beliebigen Anzahl veränderlicher Größen. *Journal für die reine und angewandte Mathematik* (20) 340–350.
- De Nicola A., Tulczyjew W.M., 2009. A variational formulation of electrodynamics with external sources. *International Journal of Geometric Methods in Modern Physics*, (6)-1, 173–200.
- Deschamps G.A., 1970. Exterior differential forms. In: *Mathematics Applied to Physics*, Roubine É. (Ed.), Springer-Verlag, Berlin, 111–161.
- Deschamps G.A., 1981. Electromagnetics and differential form, *Proceedings of the IEEE* 69(6) 676–696.
- de Rham G., 1955. *Variétés différentiables*, Hermann, Paris. English translation: *Differentiable Manifolds, Forms, Currents, Harmonic Forms*. Springer-Verlag, Berlin, 1984.
- Einstein A., 1905. Zur Elektrodynamik bewegter Körper. *Annalen der Physik* (322) 10, 891–921. English: *On the Electrodynamics of Moving bodies*.
- Einstein A., 1905. Ist die Trägheit eines Körpers von seinem Energiegehalt Abhängig? *Annalen der Physik* (323) 13, 639–641.
- Einstein A., 1920. *Relativity: The Special and General Theory*, Methuen & Co. London. Transl. from the 1916 deutsche edition: *Über die spezielle und die allgemeine Relativitätstheorie*.
- Einstein A., 1923. *The Meaning of Relativity: Four Lectures delivered at Princeton University on May 1921*. Princeton University Press, Princeton (1923), 5th edn. (1951) 6th edn. (1956).
- Faraday M., 1838. *Experimental Reseaches in Electricity*. Philosophical Transactions, London.
- Fecko M., 1997. On 3+1 decompositions with respect to an observer field via differential forms *J. Math. Phys.* 38 (1997) 4542–4560.
- Fecko M., 2006. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, Cambridge, UK.
- Fecko M., 2014. Modern geometry in not-so-high echelons of physics: Case studies. *Acta Physica Slovaca* 63(5).
- Fernflores F., 2019. *The Equivalence of Mass and Energy*. Stanford Encyclopedia of Philosophy
- Feynman R.P., Leighton R.B., Sands M.L., 1964. *The Feynman Lectures on Physics*. San Francisco, Pearson/Addison-Wesley.
- Forshaw J.R. and Smith A.G., 2009. *Dynamics and Relativity*. The Manchester Physics Series, John Wiley & Sons.
- Friedman, M., 1983. *Foundations of Space-Time Theories*. Princeton University Press.
- Frobenius G., 1875. Über das Pfaffsche Problem. *J. Reine Angew. Math.* 82 230–315 = *Gesammelte Abhandlungen*, Springer-Verlag Berlin, 1966.
- Galilei G., 1623. *Il Saggiatore* (in Italian) Rome, 1623; *The Assayer*, English trans. Stillman Drake and C.D. O'Malley, in *The Controversy on the Comets of 1618* (University of Pennsylvania Press, 1960).
- Gross P.W., Kotiuga P.R., 2004. *Electromagnetic Theory and Computation: a Topological Approach*. Cambridge University Press, New York.
- Hargreaves R., 1908. Integral forms and their connection with physical equations. *Cambr. Phil. Trans.* 21, 107.
- Havas P., 1964. Four-dimensional formulations of Newtonian mechanics and their relation to the special and the general theory of relativity. *Rev. Mod. Phys.* 36 938.

- Heaviside O., 1885. Electromagnetic induction and its propagation *The Electrician*, January 10, 178-180, February 21, 306-307.
- Heaviside O., 1892. On the Forces, Stresses, and Fluxes of Energy in the Electromagnetic Field. *Philosophical Transaction of the Royal Society A* 183, 423-480.
- Hehl F.W., Obukhov Yu.N., 2000. A gentle introduction to the foundations of classical electrodynamics: The meaning of the excitations (D,H) and the field strengths (E,B). arXiv, Physics Class 2000.
- Hehl F.W., Obukhov Yu.N., 2003. *Foundations of Classical Electrodynamics, Charge, Flux and Metric*. Birkhäuser, Boston.
- Hehl F.W., Obukhov Yu.N., 2005. Dimensions and units in electrodynamics. *Gen Relativ Gravit* **37**, 733-749 (2005).
- Hehl F.W., 2010. On the changing form of Maxwell's equations during the last 150 years - spotlights on the history of classical electrodynamics. 22 February 2010, UCL Department of Mathematics, Applied Mathematics Seminar.
- Helmholtz H. von, 1858. Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *Journal für die reine und angewandte Mathematik* 55, 25-55.
- Helmholtz H. von, 1870. Über die Theorie der Elektrodynamik. Erste Abhandlung: Über die Bewegungsgleichungen der Elektrizität für ruhende Körper. *Annalen der Physik und Chemie*, Leipzig.
- Helmholtz H. von, 1873. Über die Theorie der Elektrodynamik. Zweite Abhandlung: Kritische, *Journal für die reine und angewandte Mathematik*.
- Helmholtz H. von, 1874. Über die Theorie der Elektrodynamik. Dritte Abhandlung: Die Elektrodynamischen Kräfte in bewegten Leitern. *Journal für die reine und angewandte Mathematik*.
- Helmholtz H. von, 1892. Das Prinzip der kleinsten Wirkung in der Elektrodynamik. *Annalen der Physik und der Chemie*.
- Hertz H.R., 1892. *Untersuchungen über die Ausbreitung der Elektrischen Kraft*. Teubner, Leipzig. English translation 1893: *Electric waves*.
- Hitchin N., 2003. *Differentiable manifolds*.
- Jackson J.D., 1999. *Classical Electrodynamics*. 3rd edition, Wiley, New York.
- Jefimenko O.D., 1999. On the Relativistic Invariance of Maxwell's Equations. *Z. Naturforsch.* 54a, 637-644.
- Katz V.J., 1979. The History of Stokes' Theorem. *Mathematics Magazine* (Mathematical Association of America) (52) 146-156.
- Kolar I., Michor P.W., Slovák J., 1993. *Natural operations in differential geometry*. Springer-Verlag, Berlin.
- Kotiuga, P.R., 1984. *Hodge Decompositions and Computational Electromagnetics*. PhD Thesis, Dept. of Electrical Engineering, McGill University, Montréal.
- Kurz S., Auchmann B., 2012. *Differential Forms and Boundary Integral Equations for Maxwell-Type Problems*. In: Langer U., Schanz M., Steinbach O., Wendland W. (eds) *Fast Boundary Element Methods in Engineering and Industrial Applications*. Lecture Notes in Applied and Computational Mechanics, vol 63. Springer, Berlin, Heidelberg.
- Landau L.D., Lifšits E.M., 1987. *The Classical Theory Of Fields*. Course of Theoretical Physics, vol. 2 (Fourth revised English ed.) Butterworth-Heinemann, Amsterdam.
- Lenz E., 1834. Über die Bestimmung der Richtung durch elektodynamische Vertheilung erregten galvanischen Ströme. *Annalen der Physik und Chemie* **31** 483.
- Lorentz H.A., 1892. La théorie électromagnétique de Maxwell et son application aux corps mouvants. *Archives néerlandaises des Sciences exactes*, 25: 363-552.
- Lorentz H.A., 1898. Weiterbildung der Maxwellschen Theorie. *Elektronentheorie*. In: *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, 145-280. Leipzig; 1898
- Lorentz H.A., 1899. *Simplified Theory of Electrical and Optical Phenomena in Moving Systems*. Proceedings of the Royal Netherlands Academy (1) 427-442.
- Lorentz H.A., 1904. Electromagnetic phenomena in a system moving with any velocity smaller than that of light. *Proc. Royal Neth. Acad.*, 6: 809-831.
- Marmo G., Parasecoli E., Tulczyjew W.M., 2005. Space-time orientations and Maxwell's equations. *Reports of Mathematical Physics*, 56: 209-248.
- Marmo G., Tulczyjew W.M., 2006. Time reflection and the dynamics of particles and antiparticles. *Reports of Mathematical Physics*, 58: 147-164.
- Marsden J.E., Ratiu T., Abraham R., 2003. *Manifolds, Tensor Analysis, and Applications*. Third edition Springer, NY.
- Minkowski H., 1907. Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern. Vorgelegt in der Sitzung vom 21 Dezember 1907. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1908, 53-111.
- Minkowski H., 1908. *Space and Time*. Lecture given at the 80th Meeting of Natural Scientists in Cologne on

- September 21 1908. German Original: Raum und Zeit (1909), Jahresberichte der Deutschen Mathematiker-Vereinigung, 1-14, B.G. Teubner
- Misner C.W., Thorne K.S., Wheeler J.A., 1973. Gravitation. W.H. Freeman, San Francisco.
- Munley F., 2004. Challenges to Faraday's flux rule. Am. J. Phys. 72(12) December 2004
- Neumann F.E., 1846. Allgemeine Gesetze Der Induzierten Elektrischen Ströme. Annalen der Physik, 143(1):31-44.
- Panofsky W.K.H., Phillips M., 1955-1962. Classical Electricity and Magnetism, second edition. Addison-Wesley, Reading, MA.
- Parrott S., 1987. Relativistic Electrodynamics and Differential Geometry. Springer-Verlag New York.
- Poincaré H., 1887. Sur les residus des integrales doubles, Acta Math. (9):321-380.
- Poincaré H., 1899. Les méthodes nouvelles de la mécanique céleste, vol 1-3. Gauthiers-Villars, Paris (1892-1893-1899)
- Poincaré H., 1905. Sur la dynamique de l'électron. Rendiconti del Circolo matematico di Palermo 21: 129-176. Received July 23, 1905; Printed December 14-16, 1905; Published January 1906.
- Poynting J.H., 1884. On the Transfer of Energy in the Electromagnetic Field. Philosophical Transactions 175, 277. <http://dx.doi.org/10.1098/rstl.1884.0016>
- Purcell E.M., 1965. Berkeley Physics Course, Vol. 2. McGraw-Hill, NY.
- Rindler W., 1989. Introduction to Special Relativity. Oxford University Press.
- Romano, G., 2012. On the Laws of Electromagnetic Induction, *arXiv: 1105.3340. On-line by IOP*, 1-67.
- Romano G., 2013. On Electromagnetic Entanglements under Changes of Frame. Rend. Acc. Naz. Sc. Let. Arti, in Napoli, February.
- Romano G., 2014. Beyond Feynman's troubles in Electromagnetics. Rend. Acc. Naz. Sc. Let. Arti, in Napoli, April 2014. <http://wpage.unina.it/romano/>
- Romano G., 2014. On Time and Length in Special Relativity. Rend. Acc. Naz. Sc. Let. Arti, in Napoli, May 2014. <http://wpage.unina.it/romano/>
- Russell B., 1929. Marriage and Morals. George Allen & Unwin, London.
- Romano G., 2017. Continuum Mechanics and Electrodynamics. <http://wpage.unina.it/romano/>
- Samelson H., 2001. Differential Forms, the Early Days; or the Stories of Deahna's Theorem and of Volterra's Theorem. The American Mathematical Monthly. The Mathematical Association of America, 108(6) 522-530. <http://www.jstor.org/stable/2695706>
- Schouten, J.A., 1951. *Tensor Analysis for Physicists*. Oxford University Press, London.
- Ślebodziński W., 1931. Sur les équations de Hamilton. Bull. Acad. Roy. de Belg. 17 (5) 864-870.
- Sommerfeld, A., 1952. Electrodynamics - Lectures on Theoretical Physics vol. III. Academic Press - New York.
- Spivak M.D., 1970. A comprehensive Introduction to Differential Geometry. Vol.I-V, 3rd edn. rev.. Publish or Perish, Inc., Houston (1979,1999)
- Stern A., Tong Y., Desbrun M., Marsden J.E., 2015. Geometric Computational Electrodynamics with Variational Integrators and Discrete Differential Forms. In: Chang D., Holm D., Patrick G., Ratiu T. (eds) Geometry, Mechanics, and Dynamics. Fields Institute Communications, vol 73. Springer, New York, NY.
- Thidé, B., 2012. Electromagnetic Field Theory (Second ed.). <http://www.plasma.uu.se/CED/Book>
- Thomson J.J., 1881. On the electric and magnetic effects produced by the motion of electrified bodies. The London Philosophical Magazine and Journal of Science 5 11 (68) 229-249.
- Thomson J.J., 1893. Notes on recent researches in electricity and magnetism, intended as a sequel to Professor Clerk-Maxwell's 'Treatise on Electricity and Magnetism', Macmillan, London. <https://archive.org/details/notesonrecentre01thomgoog>
- Thorne K.S., Macdonald D., 1982. Electrodynamics in curved spacetime: 3 + 1 formulation. Monthly Notices of the Royal Astronomical Society, 198(2): 339-343 and Microfiche MN 198/1 (1982).
- Tonti, E., 1995. On the Geometrical Structure of the Electromagnetism. In Gravitation, Electromagnetism and Geometrical Structures, on 80th birthday of Licherowicz, Ed. Ferrarese, Pitagora Bologna, 281-308.
- Tonti, E., 2002. Finite Formulation of Electromagnetic Field, IEEE Transaction on Magnetics (38)2: 333-336.
- Trautman A., 1966. Comparison of Newtonian and Relativistic Theories of Space-Time. In: Hoffmann, B. (ed.) Perspectives in geometry and relativity. Indiana University Press, Bloomington.
- Trautman A., 2008. Remarks on the history of the notion of Lie differentiation. In Variations, Geometry and Physics. O. Krupková and D.J. Saunders (Editors) Nova Science Publishers.
- Truesdell C.A., Toupin R., 1960. The Classical Field Theories, Handbuck der Physik, Ed. by Siegfried Flügge, band III/1, Springer-Verlag, Berlin, 226-793.

- Truesdell C.A., Noll W., 1965. The Non-Linear Field Theories of Mechanics. Handbuch der Physik, Ed. by Siegfried Flügge, Second Ed. 1992, Third Ed. by S. Antman, 2004, Springer, New York.
- Truesdell C.A., 1980. The Tragicomical History of Thermodynamics 1822–1854. Springer, New York.
- Umov N.A., 1874. Ein Theorem über die Wechselwirkungen in Endlichen Entfernungen. Zeitschrift für Mathematik und Physik XIX, 97.
- Voigt W., 1887. Theoretische Studien über die Elastizitätsverhältnisse der Krystalle. Abh. Ges. Wiss. Göttingen, 34.
- Voigt W., 1887. Theorie des Lichts für bewegte Medien. Göttinger Nachrichten (8): 177–238.
- Volterra V., 1889. Delle variabili complesse negli iperspazii, Rend. Accad. dei Lincei, ser. IV, vol. V, Nota I, 158–165, Nota II, 291–299 = Opere Matematiche, Accad. Nazionale dei Lincei, Roma (1954), 1, 403–410, 411–419.
- Volterra V., 1889. Sulle funzioni conjugate, Rend. Accad. dei Lincei, ser. IV, vol. V, 599–611 = Opere Matematiche, Accad. Nazionale dei Lincei, Roma (1954), 1, 420–432.
- Wegner F., 2003. Classical Electrodynamics. Lecture Notes, Institut für Theoretische Physik, Ruprecht-Karls-Universität Heidelberg.
- Weyl H., 1922. Raum-Zeit-Materie. Springer, Berlin, 1922. Temps-Espace-Matière. Blanchard, Paris, 1922. Space-Time-Matter. Dover, New York, 1950.
- Whitney H., 1957. Geometric Integration Theory. Princeton Univ. Press.
- Żórawski K., 1900. Über die Erhaltung der Wirbelbewegung. C. R. Acad. Sci. Cracovie, 335–341.