On the Laws of Electromagnetic Induction

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Abstract

The theory of electro-magnetic induction is formulated by expressing FARADAY and-AMPÈRE-MAXWELL laws in terms of *pair* and *impair* differential forms to be integrated respectively over *inner* and *outer oriented* surfaces with boundaries. It is shown that frame-invariance of basic electromagnetic fields and invariance of LIE derivatives with respect to relative motions, imply frame-invariance of the induction laws. From the formulation in the space-time manifold it is deduced that frame-invariance of FARADAY and AMPÈRE two forms is equivalent to frame-invariance of all spatial electromagnetic forms, under any transformation (GALILEI or LORENTZ). A noteworthy outcome of the theory is that the *so called* LORENTZ *force term* on a charged particle is not a law of electromagnetic induction. As a consequence most applications of electromagnetism, such as homopolar induction, HALL effect and railgun functioning, are to be suitably reinterpreted.

Key words: Electromagnetism, Ampère law, Faraday law, Lorentz force, Homopolar induction, Hall effect, Railgun.

1. Introduction

A geometric approach to the laws of electromagnetism reveals the need for considering, in integral formulations, arbitrarily moving material circuits, so that every-day engineering applications can be investigated by the theory and well-posedness and frame-invariance properties can be correctly deduced. This revisitation shows that the laws of electromagnetic induction, when correctly formulated, are in fact GALILEI invariant, if the basic fields are assumed to be such. In the light of the proposed formulation, it is further shown that the *so called* LORENTZ *force* acting a charged particle is rather an expression of the electric field evaluated, according to FARADAY law of induction, by an observer which tests a body in translational motion across a region of spatially-invariant magnetic *vortex* (an alternative name for the *magnetic induction* which underlines that it is a *pair* two-form, or equivalently a *impair* vector field) and of time-invariant magnetic potential one-form.

A critical discussion of previous treatments is performed and some important issues of classical electromagnetism are reconsidered in the new perspective. In particular GALILEI invariance provides a simple direct answer to the troubles concerning the induction effects due to the relative motion of a magnet and a conductor loop, as expressed long ago by Einstein (1905) and still lasting in literature, see e.g. (Griffiths, 1999, p. 477). Moreover the elimination of the not GALILEI invariant LORENTZ *force* restores to classical electrodynamics the scientific flavor of a well-conceived theory, in fulfillment of the auspices expressed by RICHARD PHILLIPS FEYNMAN in remarking the unpleasant situation faced in dealing with the laws of electromagnetic induction (Feynman et al., 1964, II.17-1).

Some basic issues of integration on manifolds and of exterior differential calculus are preliminarily summarized for the reader's convenience. Integration of forms on inner oriented submanifolds and of odd forms on outer oriented submanifolds in an oriented ambient manifold are illustrated in detail as basic tools for the development of the theory.

The connection between the exterior calculus and the more usual vector calculus is recalled and the basics of classical electromagnetism are reformulated according to both formats. This treatment is propaedeutic to the main sections dealing with GALILEI invariance and with the electromagnetics of moving bodies, where the exterior differential calculus format is adopted, being basic for a treatment of induction laws independent of metric properties of the ambient space.

A careful attention to the roles played by inner and outer orientations in the integration over surfaces and along their boundary cycles leads naturally to propose a new terminology. The *electric field* one-form and the *magnetic vortex* two-form are involved in FARADAY law of induction, where an inner orientation of the involved surface and of its boundary circuit is considered. The *electric displacement flux* and *electric current impair* two-form, and the *magnetic winding impair* one-form, are involved in AMPÈRE law of induction, where an outer orientation of the involved surface and of its boundary circuit is adopted. The former choice provides a clear physical interpretation of the *emf* as circulation. The latter provides a better physical description, as a flux rule, of the induction law and of the equivalent condition of charge balance.

The formulation of electromagnetism in classical space-time, an affine four-dimensional manifold, provides an impressively simple expression of balance laws for electric and magnetic charges as closedness conditions of three-forms in space-time. Induction laws are expressed as exactness conditions of the same forms. Charge balance and induction laws are thus simply expressed, in an equivalent format, as integrability conditions for exterior forms and exactness conditions in terms of potential forms, according to POINCARÉ Lemma. The observer-dependent splitting into space and time components of two and three-forms over space-time, which is well-known in literature, is here extended to consider motion along the trajectory. This extension shows that the electric and magnetic spatial fields involved in the theory are spatially frame-invariant if and only if the space-time forms are space-time frame-invariant. According to this general property, the common affirmation, see e.g. (Stratton, 1941, 1.22, p.77), that the electromagnetic fields *are not* invariant according to GALILEI changes of observer but *are* invariant according to LORENTZ changes of observer, looses any significance. Indeed GALILEI invariance will holds according to the classical point of view while LORENTZ invariance will holds in the relativistic context. Which one of these two theoretical assumptions should be deemed as the best suitable to describe physical reality being just a task for experimental verification.

A final discussion points out the merits of the present approach in providing a firm basis to the laws of electromagnetic induction, in the context of classical electrodynamics.

2. Tensors and push-pull operations

At a point $\mathbf{x} \in \mathbb{M}$ of a manifold $\mathbb{M}$ the linear space of 0th order tensors (scalars) is denoted by $\mathrm{FUN}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$, the linear space of tangent vectors (velocity of curves on $\mathbb{M}$) is denoted by $\mathbb{T}_{\mathbf{x}}\mathbb{M}$ and the dual linear space of cotangent vectors (real valued linear maps on the tangent space) by $\mathbb{T}^*_{\mathbf{x}}\mathbb{M}$. By reflexivity, the duality operation is involutive, so that $\mathbb{T}^{**}_{\mathbf{x}}\mathbb{M} = \mathbb{T}_{\mathbf{x}}\mathbb{M}$.

Covariant, *contravariant* and *mixed* second order tensors, henceforth simply called *tensors*, are scalar-valued bilinear maps over the product of two tangent or cotangent spaces. Second order tensors can be equivalently characterized as linear operators between tangent or cotangent spaces, so that suitable compositions are meaningful. We will consider the following tensors and relevant linear tensors spaces

$$
\begin{aligned}
\mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \textsc{Cov}(\mathbb{T}_{\mathbf{x}}\mathbb{M}) &= BL\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,,\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathcal{R}) = BL\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}^*_{\mathbf{x}}\mathbb{M})\,,\\
\mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} \in \textsc{Con}(\mathbb{T}_{\mathbf{x}}\mathbb{M}) &= BL\,(\mathbb{T}^*_{\mathbf{x}}\mathbb{M}\,,\mathbb{T}^*_{\mathbf{x}}\mathbb{M}\,;\mathcal{R}) = BL\,(\mathbb{T}^*_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})\,,\\
\mathbf{s}_{\mathbf{x}}^{\mathrm{Mix}} \in \textsc{Mix}(\mathbb{T}_{\mathbf{x}}\mathbb{M}) &= BL\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,,\mathbb{T}^*_{\mathbf{x}}\mathbb{M}\,;\mathcal{R}) = BL\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})\,.
\end{aligned}
$$

A covariant tensor $\boldsymbol{\gamma}_{\mathbf{x}}^{\mathrm{Cov}} \in \textsc{Cov}_{\mathbf{x}}(\mathbb{TM})$ is *non-degenerate* if

$$\boldsymbol{\gamma}_{\mathbf{x}}^{\mathrm{Cov}}(\mathbf{a}\,,\mathbf{b}) = 0 \quad \forall\, \mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathbb{M} \quad \Longrightarrow \quad \mathbf{a} = \mathbf{o}.$$

Then $\boldsymbol{\gamma}_{\mathbf{x}}^{\mathrm{Cov}} \in BL\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}^*_{\mathbf{x}}\mathbb{M})$ is an isomorphism (linear and invertible) with a contravariant inverse $(\boldsymbol{\gamma}_{\mathbf{x}}^{\mathrm{Cov}})^{-1} \in \textsc{Con}(\mathbb{T}_{\mathbf{x}}\mathbb{M}) = BL\,(\mathbb{T}^*_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})$. These tensors can be composed with covariant and contravariant tensors to transform (alterate) them into mixed tensors

$$(\boldsymbol{\gamma}_{\mathbf{x}}^{\mathrm{Cov}})^{-1} \cdot \mathbf{s}_{\mathbf{x}}^{\mathrm{Cov}} \in \textsc{Mix}(\mathbb{T}_{\mathbf{x}}\mathbb{M})\,, \qquad \mathbf{s}_{\mathbf{x}}^{\mathrm{Con}} \cdot \boldsymbol{\gamma}_{\mathbf{x}}^{\mathrm{Cov}} \in \textsc{Mix}(\mathbb{T}_{\mathbf{x}}\mathbb{M})\,.$$

The generic tensor fiber is denoted by $\textsc{Tens}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$. Linear spaces of symmetric covariant and contravariant tensors at $\mathbf{x} \in \mathbb{M}$ are denoted $\textsc{Sym}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$, $\textsc{Sym}^*(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ and positive definite symmetric covariant tensors by $\textsc{Pos}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$. A metric tensor $\mathbf{g}_{\mathbf{x}} \in \textsc{Pos}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ is the natural candidate to be adopted for alteration of tensors.

The pull-back of a scalar $f_{\boldsymbol{\zeta}(\mathbf{x})} \in \mathcal{R}$ along a map $\boldsymbol{\zeta} \in \mathrm{C}^0(\mathbb{M}\,;\mathbb{N})$ between differentiable manifolds $\mathbb{M}$ and $\mathbb{N}$, is the scalar $(\boldsymbol{\zeta}{\downarrow}f)_{\mathbf{x}} \in \mathcal{R}$ defined by the equality

$$(\boldsymbol{\zeta}{\downarrow}f)_{\mathbf{x}} := f_{\boldsymbol{\zeta}(\mathbf{x})}\,.$$

Given a differentiable curve $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,;\mathbb{M})$, with $\mathbf{x} = \mathbf{c}(0)$, and a differentiable map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, the associated *tangent map* at $\mathbf{x} \in \mathbb{M}$, denoted by $T_{\mathbf{x}}\boldsymbol{\zeta} \in BL\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,;\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})$ is defined by the linear corresponcence

$$\mathbf{v}_{\mathbf{x}} = \partial_{\lambda=0}\,\mathbf{c}(\lambda) \mapsto T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{v}_{\mathbf{x}} = \partial_{\lambda=0}\,(\boldsymbol{\zeta} \circ \mathbf{c})(\lambda)\,.$$

If the map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ is invertible, the co-tangent map

$$T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} := (T_{\mathbf{x}}\boldsymbol{\zeta})^* \in BL\,(\mathbb{T}^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}(\mathbb{M})\,;\mathbb{T}^*_{\mathbf{x}}\mathbb{M})\,,$$

is defined, for every $\mathbf{w}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ and $\mathbf{v}^*_{\boldsymbol{\zeta}(\mathbf{x})} \in \mathbb{T}^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}(\mathbb{M})$, by

$$\langle\, \mathbf{v}^*_{\boldsymbol{\zeta}(\mathbf{x})}, T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{w}_{\mathbf{x}} \,\rangle = \langle\, T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \cdot \mathbf{v}^*_{\boldsymbol{\zeta}(\mathbf{x})}, \mathbf{w}_{\mathbf{x}} \,\rangle\,,$$

and the inverse tangent map is denoted by

$$T_{\boldsymbol{\zeta}(\mathbf{x})}^{-1}\boldsymbol{\zeta} := (T_{\mathbf{x}}\boldsymbol{\zeta})^{-1} \in BL\,(\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta}(\mathbb{M})\,;\mathbb{T}_{\mathbf{x}}\mathbb{M})\,.$$

The push-forward of a tangent vector $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ is defined by the formula

$$(\boldsymbol{\zeta}{\uparrow}\mathbf{v})_{\boldsymbol{\zeta}(\mathbf{x})} := T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N}\,.$$

The pull-back of a cotangent vector $\mathbf{v}^*_{\boldsymbol{\zeta}(\mathbf{x})}$, along an invertible differentiable map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, is the cotangent vector $(\boldsymbol{\zeta}{\downarrow}\mathbf{v}^*)_{\mathbf{x}}$ defined by invariance

$$\langle\,(\boldsymbol{\zeta}{\downarrow}\mathbf{v}^*)_{\mathbf{x}}, \mathbf{v}_{\mathbf{x}}\,\rangle = \langle\,\mathbf{v}^*_{\boldsymbol{\zeta}(\mathbf{x})}, (\boldsymbol{\zeta}{\uparrow}\mathbf{v})_{\boldsymbol{\zeta}(\mathbf{x})}\,\rangle\,,$$

so that

$$(\boldsymbol{\zeta}{\downarrow}\mathbf{v}^*)_{\mathbf{x}} := T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \cdot \mathbf{v}^*_{\boldsymbol{\zeta}(\mathbf{x})}\,.$$

Pull-back and push forward, if both defined, are inverse operations. Push-pull operations for tensors are defined by invariance.

For instance, the pull-back of a twice-covariant tensor $\mathbf{s}_{\boldsymbol{\zeta}(\mathbf{x})} \in \mathrm{Cov}(\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})$ is the a twice-covariant tensor $\boldsymbol{\zeta}{\downarrow}\mathbf{s}_{\boldsymbol{\zeta}(\mathbf{x})} \in \mathrm{Cov}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ explicitly defined, for any pair of tangent vectors $\mathbf{u}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$, by

$$\begin{aligned}
\boldsymbol{\zeta}{\downarrow}\mathbf{s}^{\mathrm{Cov}}_{\boldsymbol{\zeta}(\mathbf{x})}(\mathbf{u}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}}) &:= \mathbf{s}^{\mathrm{Cov}}_{\boldsymbol{\zeta}(\mathbf{x})}(T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{u}_{\mathbf{x}}, T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{w}_{\mathbf{x}}) \\
&= \langle\,\mathbf{s}^{\mathrm{Cov}}_{\boldsymbol{\zeta}(\mathbf{x})} \cdot T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{u}_{\mathbf{x}}, T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{w}_{\mathbf{x}}\,\rangle \\
&= \langle\,T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \cdot \mathbf{s}^{\mathrm{Cov}}_{\boldsymbol{\zeta}(\mathbf{x})} \cdot T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{u}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}}\,\rangle\,.
\end{aligned}$$

Push-pull relations for covariant, contravariant and mixed tensors, along a map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, are then given by

$$\begin{aligned}
\boldsymbol{\zeta}{\downarrow}\mathbf{s}^{\mathrm{Cov}}_{\boldsymbol{\zeta}(\mathbf{x})} &= T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \cdot \mathbf{s}^{\mathrm{Cov}}_{\boldsymbol{\zeta}(\mathbf{x})} \cdot T_{\mathbf{x}}\boldsymbol{\zeta} \in \mathrm{Cov}(\mathbb{T}_{\mathbf{x}}\mathbb{M})\,, \\
\boldsymbol{\zeta}{\uparrow}\mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} &= T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} \cdot T^*_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \in \mathrm{Con}(\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})\,, \\
\boldsymbol{\zeta}{\uparrow}\mathbf{s}^{\mathrm{Mix}}_{\mathbf{x}} &= T_{\mathbf{x}}\boldsymbol{\zeta} \cdot \mathbf{s}^{\mathrm{Mix}}_{\mathbf{x}} \cdot T^{-1}_{\boldsymbol{\zeta}(\mathbf{x})}\boldsymbol{\zeta} \in \mathrm{Mix}(\mathbb{T}_{\boldsymbol{\zeta}(\mathbf{x})}\mathbb{N})\,.
\end{aligned}$$

The linear spaces of covariant and contravariant tensors are in separating duality[1] by the pairing

$$\langle\,\mathbf{s}^{\mathrm{Con}}_{\mathbf{x}}, \mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}}\,\rangle := J^1_{\mathbf{x}}(\mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} \cdot (\mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}})^A)\,,$$

[1] A *separating* duality pairing between linear spaces is a bilinear form such that vanishing for any value of one of its arguments implies vanishing of the other argument.

where $J^1_{\mathbf{x}}$ denotes the linear invariant and the adjoint tensor $(\mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}})^A$ is defined by the identity

$$(\mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}})^A(\mathbf{a}\,,\mathbf{b}) := \mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}}(\mathbf{b}\,,\mathbf{a})\,, \qquad \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}\,.$$

Scalar-valued k-linear, alternating maps on $\mathbb{T}_{\mathbf{x}}\mathbb{M}$ are called *k-covectors* at $\mathbf{x} \in \mathbb{M}$ with linear span $\mathrm{ALT}^k(\mathbb{T}_{\mathbf{x}}\mathbb{M})$, where $k \leq m = \dim \mathbb{M}$. *Maximal-covectors* are m-covectors spanning a one-dimensional linear space denoted by $\mathrm{MXF}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$. Covectors of order greater than m vanish identically. Forms of order k are sections $\boldsymbol{\omega}^k \in \boldsymbol{\Lambda}^k(\mathbb{T}\mathbb{M}\,;\mathcal{R}) := \mathrm{C}^1(\mathbb{M}\,;\mathrm{ALT}^k(\mathbb{T}\mathbb{M}))$.

3. Stokes' formula

The modern way to integral transformations is to consider maximal-forms as geometric objects to be integrated over a (orientable) manifold. For any given manifold with boundary, the notion of exterior differential of a form is conceived to transform the integral of a form over the boundary into an integral over the manifold (G. Romano, 2007). The resulting formula is the generalization of the fundamental formula of integral calculus to manifolds of finite dimension higher than one.

As quoted by de Rham (1955), according to Segre (1951), this general integral transformation was considered by Volterra (1889); Poincaré (1895); Brouwer (1906). It includes as special cases the classical formulae due to Gauss[2], Green[3], Ostrogradski[4] and to Ampère[5], Kelvin[6], Hamel.[7] The formula for surfaces in 3D space was communicated by Kelvin to Stokes[8] and was taught by him at Cambridge. In its modern general formulation Stokes formula could rather be renamed Volterra[9]-Poincaré[10] -Brouwer[11] (Stokes) formula.

[2] Carl Friedrich Gauss (1789-1857) German mathematician.
[3] George Green (1793-1841) British mathematical physicist.
[4] Mikhail Vasilevich Ostrogradski (1801-1862) Russian mathematician.
[5] André-Marie Ampère (1775-1836) French mathematical physicist.
[6] William Thomson, Lord Kelvin (1824-1907) Scottish mathematical physicist.
[7] Georg Karl Wilhelm Hamel (1877-1954) German mathematician.
[8] George Gabriel Stokes (1819-1903) British mathematical physicist.
[9] Vito Volterra (1860-1940) Italian mathematical physicist.
[10] Jules Henri Poincaré (1854-1912) French mathematician and theoretical physicist.
[11] Luitzen Egbertus Jan Brouwer (1881-1966) Dutch mathematician.

Definition 3.1 (Stokes formula for the exterior derivative). *In a m-dimensional manifold $\mathbb{M}$, let $\mathbf{\Omega}$ be any n-dimensional submanifold ($m \geq n$) with $(n-1)$-dimensional boundary manifold $\partial\mathbf{\Omega}$. The exterior derivative $d\boldsymbol{\omega} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{A{\small LT}}^{(n+1)}(\mathbb{TM}))$ of a n-form $\boldsymbol{\omega} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{A{\small LT}}^{n}(\mathbb{TM}))$ is the $(n+1)$-form such that*

$$\int_{\mathbf{\Omega}} d\boldsymbol{\omega} = \int_{\partial\mathbf{\Omega}} \boldsymbol{\omega}\,.$$

To underline duality between the boundary operator and the exterior differentiation, Stokes formula may be rewritten as

$$\langle\, d\boldsymbol{\omega}, \mathbf{\Omega} \,\rangle = \langle\, \boldsymbol{\omega}, \partial\mathbf{\Omega} \,\rangle\,.$$

Being $\partial\partial\mathbf{\Omega} = 0$ for any chain of manifolds $\mathbf{\Omega}$ it follows that also $dd\boldsymbol{\omega} = 0$ for any form $\boldsymbol{\omega}$.

Proposition 3.1 (Geometric homotopy formula). *The boundary chain of the extrusion of a manifold may be evaluated by the following* geometric homotopy formula

$$\partial(J_{\boldsymbol{\zeta}}(\mathbf{\Omega}, \lambda)) = \boldsymbol{\zeta}_\lambda(\mathbf{\Omega}) - \mathbf{\Omega} - J_{\boldsymbol{\zeta}}(\partial\mathbf{\Omega}, \lambda)\,.$$

Proof. As depicted in figg.1,2, the signs in the formula are due to the following choice. The orientation of the $(n+2)$-dimensional flow tube $J_{\boldsymbol{\zeta}}(\mathbf{\Omega}, \lambda)$ induces an orientation on its boundary $\partial(J_{\boldsymbol{\zeta}}(\mathbf{\Omega}, \lambda))$. Assuming on $\boldsymbol{\zeta}_\lambda(\mathbf{\Omega})$ this orientation, it follows that $\boldsymbol{\zeta}_0(\mathbf{\Omega}) = \mathbf{\Omega}$ has the opposite orientation and the same holds for $J_{\boldsymbol{\zeta}}(\partial\mathbf{\Omega}, \lambda)$. ■

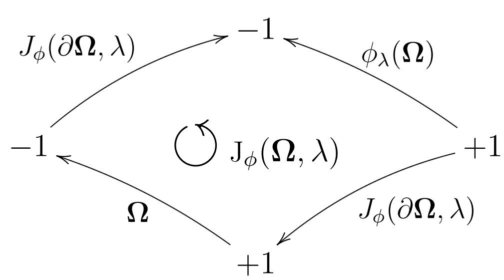

Figure 1: Geometric homotopy formula (n=1)

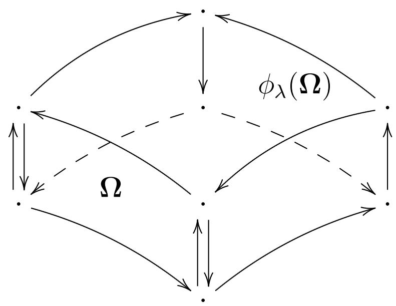

Figure 2: Geometric homotopy formula (n=2)

Proposition 3.2 (Extrusion formula). *Let* $\boldsymbol{\zeta} \in \mathrm{C}^1(\boldsymbol{\Omega} \times \mathcal{R}\,; \mathbb{M} \times \mathcal{R})$ *be an extrusion-map defined by*

$$\begin{tikzcd} \boldsymbol{\Omega} \times \mathcal{R} \arrow[r, "\boldsymbol{\zeta}_\lambda"] \arrow[d, "\boldsymbol{\pi}_{\mathcal{R},\boldsymbol{\Omega}\times\mathcal{R}}"'] & \mathbb{M} \times \mathcal{R} \arrow[d, "\boldsymbol{\pi}_{\mathcal{R},\mathbb{M}\times\mathcal{R}}"] \\ \mathcal{R} \arrow[r, "\mathrm{SH}_\lambda"] & \mathcal{R} \end{tikzcd} \quad \Longleftrightarrow \quad \boldsymbol{\pi}_{\mathcal{R},\mathbb{M}\times\mathcal{R}} \circ \boldsymbol{\zeta}_\lambda = \mathrm{SH}_\lambda \circ \boldsymbol{\pi}_{\mathcal{R},\boldsymbol{\Omega}\times\mathcal{R}}\,,$$

with $\lambda \in \mathcal{R}$ *extrusion-time and* $\mathbf{v}_{\boldsymbol{\zeta}} := \partial_{\lambda=0}\,\boldsymbol{\zeta}_\lambda$ *relevant velocity field. Then the following* extrusion formula *holds*

$$\boxed{\partial_{\lambda=0} \int_{\boldsymbol{\zeta}_\lambda(\boldsymbol{\Omega})} \boldsymbol{\omega} = \int_{\boldsymbol{\Omega}} (d\boldsymbol{\omega}) \cdot \mathbf{v}_{\boldsymbol{\zeta}} + \oint_{\partial\boldsymbol{\Omega}} \boldsymbol{\omega} \cdot \mathbf{v}_{\boldsymbol{\zeta}}\,.}$$

Proof. The first item is the *geometric homotopy formula* depicted in figg.1,2 relating the chain generated by the extrusion of a manifold and its boundary chain

$$\partial(J_{\boldsymbol{\zeta}}(\boldsymbol{\Omega}, \lambda)) = \boldsymbol{\zeta}_\lambda(\boldsymbol{\Omega}) - \boldsymbol{\Omega} - J_{\boldsymbol{\zeta}}(\partial\boldsymbol{\Omega}, \lambda)\,,$$

The signs in the formula are due to the following choice. The orientation of the $(n+2)$-dimensional flow tube $J_{\boldsymbol{\zeta}}(\boldsymbol{\Omega}, \lambda)$ induces an orientation on its boundary $\partial(J_{\boldsymbol{\zeta}}(\boldsymbol{\Omega}, \lambda))$. Assuming on $\boldsymbol{\zeta}_\lambda(\boldsymbol{\Omega})$ this orientation, it follows that $\boldsymbol{\zeta}_0(\boldsymbol{\Omega}) = \boldsymbol{\Omega}$ has the opposite orientation and the same holds for $J_{\boldsymbol{\zeta}}(\partial\boldsymbol{\Omega}, \lambda)$.

Let us consider a $(n-1)$-form $\boldsymbol{\omega}$ defined in the manifold spanned by extrusion of the manifold $\boldsymbol{\Omega}$, so that the geometric homotopy formula gives

$$\int_{\boldsymbol{\zeta}_\lambda(\boldsymbol{\Omega})} \boldsymbol{\omega} = \int_{\partial(J_{\boldsymbol{\zeta}}(\boldsymbol{\Omega},\lambda))} \boldsymbol{\omega} + \int_{J_{\boldsymbol{\zeta}}(\partial\boldsymbol{\Omega},\lambda)} \boldsymbol{\omega} + \int_{\boldsymbol{\Omega}} \boldsymbol{\omega}\,,$$

Differentiation with respect to the extrusion-time yields

$$\partial_{\lambda=0} \int_{\boldsymbol{\zeta}_\lambda(\boldsymbol{\Omega})} \boldsymbol{\omega} = \partial_{\lambda=0} \int_{\partial(J_{\boldsymbol{\zeta}}(\boldsymbol{\Omega},\lambda))} \boldsymbol{\omega} + \partial_{\lambda=0} \int_{J_{\boldsymbol{\zeta}}(\partial\boldsymbol{\Omega},\lambda)} \boldsymbol{\omega} \,.$$

Then, denoting by $\mathbf{v}_{\boldsymbol{\zeta}} := \partial_{\lambda=0}\, \boldsymbol{\zeta}_\lambda$ the velocity field of the extrusion, applying STOKES formula and taking into account that by FUBINI[12] theorem

$$\partial_{\lambda=0} \int_{J_{\boldsymbol{\zeta}}(\boldsymbol{\Omega},\lambda)} d\boldsymbol{\omega} = \int_{\boldsymbol{\Omega}} (d\boldsymbol{\omega}) \cdot \mathbf{v}_{\boldsymbol{\zeta}} \,, \qquad \partial_{\lambda=0} \int_{J_{\boldsymbol{\zeta}}(\partial\boldsymbol{\Omega},\lambda)} \boldsymbol{\omega} = \oint_{\partial\boldsymbol{\Omega}} \boldsymbol{\omega} \cdot \mathbf{v}_{\boldsymbol{\zeta}} \,,$$

we get the result. ■

Proposition 3.3 (Differential homotopy formula). *The differential homotopy formula, also named* H. CARTAN[13] magic formula*, reveals that* LIE[14] *derivative* $\mathcal{L}$ *and exterior derivative* d *are related by*

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega} = d(\boldsymbol{\omega} \cdot \mathbf{v}) + (d\boldsymbol{\omega}) \cdot \mathbf{v} \,.$$

Proof. Applying STOKES formula to the last term in the *extrusion formula* we get

$$\partial_{\lambda=0} \int_{\boldsymbol{\zeta}_\lambda(\boldsymbol{\Omega})} \boldsymbol{\omega} = \int_{\boldsymbol{\Omega}} (d\boldsymbol{\omega}) \cdot \mathbf{v}_{\boldsymbol{\zeta}} + \int_{\boldsymbol{\Omega}} d(\boldsymbol{\omega} \cdot \mathbf{v}_{\boldsymbol{\zeta}}) \,.$$

On the other hand, the time-rate of the integral pull-back transformation leads to REYNOLDS[15] formula

$$\partial_{\lambda=0} \int_{\boldsymbol{\zeta}_\lambda(\boldsymbol{\Omega})} \boldsymbol{\omega} = \int_{\boldsymbol{\Omega}} \partial_{\lambda=0}\, (\boldsymbol{\zeta}_\lambda {\downarrow} \boldsymbol{\omega}) = \int_{\boldsymbol{\Omega}} \mathcal{L}_{\mathbf{v}_{\boldsymbol{\zeta}}}\, \boldsymbol{\omega} \,.$$

Equating r.h.s. of both formulas, setting $\mathbf{v} = \mathbf{v}_{\boldsymbol{\zeta}}$, gives the result. ■

[12] GUIDO FUBINI (1879-1943) Italian mathematician.
[13] HENRI CARTAN (1904-2008) French mathematician.
[14] MARIUS SOPHUS LIE (1842-1899) Norwegian mathematician.
[15] OSBORNE REYNOLDS (1842-1913) English physicist.

This recursive formula for the exterior derivative of a n-form $\boldsymbol{\omega}$ in terms of LIE derivative of forms of decreasing order, associated with the recursive LEIBNIZ formula

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}\cdot\mathbf{w} := \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}\cdot\mathbf{w}) - \boldsymbol{\omega}\cdot\mathcal{L}_{\mathbf{v}}\mathbf{w}\,,$$

yields a recursive formula for the exterior derivative of a n-form $\boldsymbol{\omega}$ in terms of LIE brackets between vector fields (G. Romano, 2007)

$$d\boldsymbol{\omega}\cdot\mathbf{v}\cdot\mathbf{w} = \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}\cdot\mathbf{w}) - d(\boldsymbol{\omega}\cdot\mathbf{v})\cdot\mathbf{w} - \boldsymbol{\omega}\cdot[\mathbf{v}\,,\mathbf{w}]\,.$$

The recursion from the $(n+1)$-form $d\boldsymbol{\omega}\cdot\mathbf{v}\cdot\mathbf{w}_1\ldots\cdot\mathbf{w}_n$ till the 0-form $d(\boldsymbol{\omega}\cdot\mathbf{w}_1\ldots\cdot\mathbf{w}_n)\cdot\mathbf{v} = \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega}\cdot\mathbf{w}_1\ldots\cdot\mathbf{w}_n)$ yields PALAIS[16] formula (Palais, 1954) which for $n=1$ writes

$$\begin{aligned} d\boldsymbol{\omega}^1\cdot\mathbf{v}\cdot\mathbf{w} &= (\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^1)\cdot\mathbf{w} - d(\boldsymbol{\omega}^1\cdot\mathbf{v})\cdot\mathbf{w} \\ &= d_{\mathbf{v}}\,(\boldsymbol{\omega}^1\cdot\mathbf{w}) - \boldsymbol{\omega}^1\cdot[\mathbf{v},\mathbf{w}] - d_{\mathbf{w}}\,(\boldsymbol{\omega}^1\cdot\mathbf{v})\,. \end{aligned}$$

The exterior derivative of a differential 1-form is a two-form which is well-defined by PALAIS formula because the expression at the r.h.s. fulfills the tensoriality criterion. The value of the exterior derivative at a point is independent of the extension of argument vectors to vector fields, extension needed to compute the involved directional and LIE derivatives. Boundary-less surfaces are said to be *closed*, and hence differential n-forms such that $d\boldsymbol{\omega} = 0$ are called *closed* forms.

An m-dimensional manifold $\mathbb{M}$ is a *star-shaped manifold* if there exists a point $\mathbf{x}_0 \in \mathbb{M}$ and a *homotopy* $\mathbf{h}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$, continuous in $\lambda \in [0,1]$, such that $\mathbf{h}_1$ is the identity map, i.e. $\mathbf{h}_1(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{M}$, and $\mathbf{h}_0$ is the constant map $\mathbf{h}_0(\mathbf{x}) = \mathbf{x}_0$ for all $\mathbf{x} \in \mathbb{M}$. This homotopy is called a *contraction* to $\mathbf{x}_0 \in \mathbb{M}$. The proof of the following result may be found in (G. Romano, 2007).

Lemma 3.1 (Poincaré formula). *Let* $\boldsymbol{\omega}^k$ *be a form and* $\mathbf{h}_\lambda \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ *an homotopy on* $\mathbb{M}$ *with velocity* $\mathbf{v}_\mu = \partial_{\mu=\lambda}\,\mathbf{h}_\mu \circ \mathbf{h}_\lambda^{-1} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$. *Then we have the formula*

$$\boldsymbol{\omega}^k = d\boldsymbol{\alpha}^{(k-1)} + \boldsymbol{\beta}^k\,,$$

[16] RICHARD SHELDON PALAIS (1931-) American mathematician.

$$\boldsymbol{\alpha}^{(k-1)} = \int_0^1 \mathbf{h}_\lambda\!\downarrow\!(\boldsymbol{\omega}^k \cdot \mathbf{v})\, d\lambda\,, \qquad \boldsymbol{\beta}^k = \int_0^1 \mathbf{h}_\lambda\!\downarrow\!(d\boldsymbol{\omega}^k \cdot \mathbf{v})\, d\lambda\,.$$

If $d\boldsymbol{\omega}^k = 0$ *the form* $\boldsymbol{\omega}^k$ *is exact being* $\boldsymbol{\omega}^k = d\boldsymbol{\alpha}^{(k-1)}$. *This is known as* Poincaré *Lemma: in a star-shaped manifold any closed form is exact.*

Lemma 3.2 (Commutation of exterior derivatives and pushes). *The pull back of a form by an injective immersion* $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ *and the exterior derivative of differential forms commute*

$$d_{\mathbb{M}} \circ \boldsymbol{\varphi}\!\downarrow = \boldsymbol{\varphi}\!\downarrow \circ\, d_{\mathbb{N}}\,.$$

Proof. For any k-form $\boldsymbol{\omega}^k \in \boldsymbol{\Lambda}^k(\mathbb{N}\,;\mathcal{R})$ we have that $\boldsymbol{\varphi}\!\downarrow\!\boldsymbol{\omega}^k \in \boldsymbol{\Lambda}^k(\mathbb{M}\,;\mathcal{R})$ and the image of any $(k+1)$-dimensional chain $\boldsymbol{\Sigma}^{k+1} \subset \mathbb{M}$ by the injective immersion $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ is still a $(k+1)$-dimensional chain $\boldsymbol{\varphi}(\boldsymbol{\Sigma}^{k+1}) \subset \mathbb{N}$. Then, by Stokes and integral pull-back formulas, the equality

$$\begin{aligned}\int_{\boldsymbol{\Sigma}^{k+1}} d_{\mathbb{M}}(\boldsymbol{\varphi}\!\downarrow\!\boldsymbol{\omega}^k) &= \oint_{\partial\boldsymbol{\Sigma}^{k+1}} \boldsymbol{\varphi}\!\downarrow\!\boldsymbol{\omega}^k = \oint_{\boldsymbol{\varphi}(\partial\boldsymbol{\Sigma}^{k+1})} \boldsymbol{\omega}^k \\ &= \oint_{\partial\boldsymbol{\varphi}(\boldsymbol{\Sigma}^{k+1})} \boldsymbol{\omega}^k = \int_{\boldsymbol{\varphi}(\boldsymbol{\Sigma}^{k+1})} d_{\mathbb{N}}\,\boldsymbol{\omega}^k = \int_{\boldsymbol{\Sigma}^{k+1}} \boldsymbol{\varphi}\!\downarrow\!(d_{\mathbb{N}}\,\boldsymbol{\omega}^k)\,,\end{aligned}$$

following from the property $\boldsymbol{\varphi}(\partial\boldsymbol{\Sigma}^{k+1}) = \partial\boldsymbol{\varphi}(\boldsymbol{\Sigma}^{k+1})$, yields the result. ■

4. Homologies and cohomologies

The exterior differentiation d^n operates on the linear space of forms $\mathrm{C}^1(\mathbb{M}\,;\mathrm{Alt}^n(\mathbb{TM}))$ and the boundary operator $\partial^{(n+1)}$, operates on $(n+1)$-chains. It is then convenient to write Stokes formula as follows:

$$\langle\, \boldsymbol{\Omega}^{n+1}, d^n\boldsymbol{\omega}^n \,\rangle = \langle\, \partial^{n+1}\boldsymbol{\Omega}^{n+1}, \boldsymbol{\omega}^n \,\rangle\,.$$

In general, $\boldsymbol{\Omega}^n$ is a chain and ∂^n is a boundary operator. Hence $\boldsymbol{\omega}^n$ is called a co-chain and d^n is the co-boundary operator. The relevant theory, first outlined by de Rham[17] in his famous 1931 thesis, is exposed in (de Rham, 1931, 1955). The basic results are expressed by the following annihilation

[17] Georges de Rham (1903-1990) Swiss mathematician.

relations which extend to chain and co-chains well-known formulae for dual operators in linear algebra:

$$\begin{cases} \mathrm{Ker}\partial^k = (\mathrm{Im}d^{k-1})^0 \,, \\ \mathrm{Ker}d^k = (\mathrm{Im}\partial^{k+1})^0 \,, \end{cases} \qquad \begin{cases} \mathrm{Im}\partial^{k+1} = (\mathrm{Ker}d^k)^0 \,, \\ \mathrm{Im}d^{k-1} = (\mathrm{Ker}\partial^k)^0 \,, \end{cases}$$

where the annihilators are defined as exemplified by:

$$(\mathrm{Im}\partial^k)^0 := \{ \boldsymbol{\omega}^{k-1} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{A\scriptstyle LT}^{k-1}(\mathbb{TM})) \, : \, \langle \boldsymbol{\omega}^{k-1}, \partial^k \boldsymbol{\Omega}^k \rangle = 0 \quad \forall \, \boldsymbol{\Omega}^k \} \,.$$

Homologies and *cohomologies* of degree k are the quotient spaces:

$$H_k(\mathbb{M}) := \mathrm{Ker}\partial^k / \mathrm{Im}\partial^{k+1} \quad \text{and} \quad H^k(\mathbb{M}) := \mathrm{Ker}d^k / \mathrm{Im}d^{k-1} \,,$$

Duality between *Homologies* and *cohomologies* is expressed by the *period*, the integral of a cocycle (closed cochain) over a cycle (closed chain). The Stokes formula provides the invariance property:

$$\oint_{\mathbf{c}^k} \boldsymbol{\omega}^k = \oint_{\mathbf{c}^k + \mathbf{l}^k} \boldsymbol{\omega}^k + \boldsymbol{\alpha}^k \,,$$

with $\mathbf{c}^k \in \mathrm{Ker}\partial^k$ and $\boldsymbol{\omega}^k \in \mathrm{Ker}d^k$, for all $\mathbf{l}^k \in \mathrm{Im}\partial^{k+1}$ and $\boldsymbol{\alpha}^k \in \mathrm{Im}d^{k-1}$.

The de Rham annihilations reveal that duality provided by the *period* is separating and this ensures the existence of an isomorphism between the spaces of homologies and cohomologies of degree k. Accordingly these will have the same finite dimension, the k-dimensional Betti[18] number of $\mathbb{M}$.

Currents introduced by de Rham are the k-dimensional extension of *scalar distributions* of Schwartz.[19] Currents are linear functionals on the linear space of smooth exterior forms with compact support on a manifold. These topological notions are gaining a rapidly increasing attention in theoretical and computational aspects of electromagnetics (Bossavit, 1991, 2004, 2005), (Tonti, 1995, 2002), (Gross and Kotiuga, 2004), (Auchmann and Kurz, 2007a), (Auchmann and Kurz, 2007b), (Kurz, Auchmann and Flemisch, 2009).

[18] Enrico Betti (1823-1892) Italian mathematician.
[19] Laurent-Moïse Schwartz (1915-2002) French mathematician.

5. Classical integral transformations

Let $(\mathbb{M}^m, \boldsymbol{\mu}^m)$ be a n-dimensional volume manifold. The *divergence* of a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}; \mathbb{TM})$ is defined as the constant of proportionality between the LIE derivative of the volume form along the flow of the vector field and the volume form itself:

$$\mathcal{L}_{\mathbf{v}}\boldsymbol{\mu} = (\operatorname{div}\mathbf{v})\,\boldsymbol{\mu}\,.$$

The divergence may be equivalently defined in terms of the exterior derivative by the relation

$$d(\boldsymbol{\mu}\mathbf{v}) = (\operatorname{div}\mathbf{v})\,\boldsymbol{\mu}\,.$$

Indeed, $d\boldsymbol{\mu} = 0$ identically as $d\boldsymbol{\mu}$ is an $(m+1)$-form in an n-dimensional manifold, so that by the homotopy formula:

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\mu} = (d\boldsymbol{\mu})\mathbf{v} + d(\boldsymbol{\mu}\mathbf{v}) = d(\boldsymbol{\mu}\mathbf{v})\,.$$

From the STOKES formula, introduced in section 3, we may derive all classical integral transformation formulas, as special cases. Indeed being:

gradient:	$d\,f \;\; = \mathbf{g}\cdot\nabla f\,,$	$\dim\mathbb{M} = m$
curl:	$d(\mathbf{g}\mathbf{v}) = (\operatorname{rot}\mathbf{v})\,\boldsymbol{\mu}\,,$	$\dim\mathbb{M} = 2$
curl:	$d(\mathbf{g}\mathbf{v}) = \boldsymbol{\mu}\cdot(\operatorname{rot}\mathbf{v})\,,$	$\dim\mathbb{M} = 3$
divergence:	$d(\boldsymbol{\mu}\mathbf{v}) = (\operatorname{div}\mathbf{v})\boldsymbol{\mu}\,,$	$\dim\mathbb{M} = m$

we get the following statements:

- $\boldsymbol{\Sigma}^1 \subset \mathbb{M}^m$: the *gradient formula*:

  $$\int_{\boldsymbol{\Sigma}^1} df = \int_{\boldsymbol{\Sigma}^1} \mathbf{g}\nabla f = \int_{\boldsymbol{\Sigma}^1} \mathbf{g}(\nabla f, \mathbf{t})\;(\mathbf{g}\,\mathbf{t}) = \int_{\partial\boldsymbol{\Sigma}^1} f = f(\mathbb{B}) - f(\mathbb{A})\,,$$

  with $\mathbb{A}$, $\mathbb{B}$ end points of the curve $\boldsymbol{\Sigma}^1$ oriented from $\mathbb{A}$ to $\mathbb{B}$ and $\mathbf{g}\,\mathbf{t}$ volume form (the signed-length) induced along the curve $\boldsymbol{\Sigma}$.

- $\boldsymbol{\Sigma}^2 \subset \mathbb{M}^3$: the *curl formula*:

$$\int_{\mathbf{\Sigma}} d(\mathbf{g}\mathbf{v}) = \int_{\mathbf{\Sigma}} \boldsymbol{\mu}(\operatorname{rot}\mathbf{v}) = \int_{\mathbf{\Sigma}} \mathbf{g}(\operatorname{rot}\mathbf{v},\mathbf{n})\,(\boldsymbol{\mu}\mathbf{n}) = \int_{\partial\mathbf{\Sigma}} \mathbf{g}\mathbf{v} = \int_{\partial\mathbf{\Sigma}} \mathbf{g}(\mathbf{v},\mathbf{t})\,(\mathbf{g}\,\mathbf{t})\,,$$

with $\mathbf{n}$ piecewise smooth field of unit normals to the surface $\mathbf{\Sigma}$ and $\mathbf{t}$ unit tangent to the boundary of the surface. For $\dim \mathbb{M} = 3\,,\quad \dim \mathbf{\Sigma} = 2$ the *curl theorem* writes:

$$\int_{\mathbf{\Sigma}} \boldsymbol{\mu}\cdot(\operatorname{rot}\mathbf{v}) = \int_{\mathbf{\Sigma}} d(\mathbf{g}\cdot\mathbf{v}) = \int_{\partial\mathbf{\Sigma}} \mathbf{g}\cdot\mathbf{v}\,.$$

It is evident that the *curl* vector or scalar fields in the formulas above are orientation dependent.

- $\dim \mathbb{M} = m\,, \dim \mathbf{\Sigma} = k \le m$ the *divergence formula*:

$$\int_{\mathbf{\Sigma}} (\operatorname{div}\mathbf{v})\,\boldsymbol{\mu} = \int_{\mathbf{\Sigma}} d(\boldsymbol{\mu}\cdot\mathbf{v}) = \int_{\partial\mathbf{\Sigma}} \boldsymbol{\mu}\cdot\mathbf{v} = \int_{\partial\mathbf{\Sigma}} \mathbf{g}(\mathbf{v},\mathbf{n})\,(\boldsymbol{\mu}\cdot\mathbf{n})\,,$$

  with $\mathbf{n}$ unit normal to the boundary $\partial\mathbf{\Sigma}$.

Remark 5.1. *The definition of gradient, curl and divergence in $\mathcal{R}^3$ given above are based on the following algebraic results (G. Romano, 2007).*

- *To any one-form df on $\mathcal{R}^n$ there correspond a unique vector ∇f in $\mathcal{R}^n$ such that $df = \mathbf{g}\cdot\nabla f$.*
- *To any two-form $\boldsymbol{\omega}^2$ on $\mathcal{R}^3$ there correspond a unique vector $\mathbf{w}$ in $\mathcal{R}^3$ such that $\boldsymbol{\omega}^2 = \boldsymbol{\mu}\cdot\mathbf{w}$, with $\boldsymbol{\mu}$ a given volume form.*
- *All volume forms $\boldsymbol{\mu}$ on $\mathcal{R}^n$ are proportional one another.*

A noteworthy formula, due to Helmholtz,[20] is also a direct consequence of the homotopy formula, see (Deschamps, 1970, 1981). To see this, given a time-dependent tangent vector field $\mathbf{u} \in C^2(\mathbb{M}\times\mathcal{Z}\,;\mathbb{TM})$, we set $\boldsymbol{\omega}^2 = \boldsymbol{\mu}\cdot\mathbf{u}$. To evaluate the flux of the field $\mathbf{u} \in C^2(\mathbb{M}\,;\mathbb{TM})$ through a surface $\mathbf{\Sigma}^2$ drifted by a flow $\boldsymbol{\varphi}_\alpha \in C^2(\mathbb{M}\,;\mathbb{M})$, we set $\dot{\boldsymbol{\omega}}^2 := \partial_{\alpha=0}\,\boldsymbol{\omega}^2_\alpha$, and apply the homotopy formula to get:

$$\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\mathbf{\Sigma}^2)} \boldsymbol{\omega}^2_\alpha = \int_{\mathbf{\Sigma}^2} \dot{\boldsymbol{\omega}}^2 + \mathcal{L}_{\mathbf{v}_\varphi}\,\boldsymbol{\omega}^2 = \int_{\mathbf{\Sigma}^2} \dot{\boldsymbol{\omega}}^2 + d(\boldsymbol{\omega}^2\cdot\mathbf{v}) + (d\boldsymbol{\omega}^2)\cdot\mathbf{v}\,.$$

[20] Hermann von Helmholtz (1821-1894) German physician and physicist.

Translating into the language of vector analysis, recalling that

$$\boldsymbol{\mu} \cdot \mathbf{u} \cdot \mathbf{v} = \mathbf{g} \cdot (\mathbf{u} \times \mathbf{v}) \,,$$
$$d(\mathbf{g} \cdot (\mathbf{u} \times \mathbf{v})) = \boldsymbol{\mu} \cdot (\operatorname{rot} (\mathbf{u} \times \mathbf{v})) \,,$$

we have:

$$d(\boldsymbol{\omega}^2 \cdot \mathbf{v}) = d(\boldsymbol{\mu} \cdot \mathbf{u} \cdot \mathbf{v}) = \boldsymbol{\mu} \cdot (\operatorname{rot} (\mathbf{u} \times \mathbf{v})) \,,$$
$$(d\boldsymbol{\omega}^2) \cdot \mathbf{v} = d(\boldsymbol{\mu} \cdot \mathbf{u}) \cdot \mathbf{v} = (\operatorname{div} \mathbf{u})\boldsymbol{\mu} \cdot \mathbf{v} \,.$$

Substituting into the first expression, we get Helmholtz's formula

$$\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Sigma}^2)} \boldsymbol{\omega}^2_\alpha = \int_{\boldsymbol{\Sigma}^2} \boldsymbol{\mu} \cdot (\dot{\mathbf{u}} + \operatorname{rot} (\mathbf{u} \times \mathbf{v})) + (\operatorname{div} \mathbf{u})\boldsymbol{\mu} \cdot \mathbf{v} \,.$$

6. Inner and outer orientations, odd forms

The reader interested in the issues of orientation of manifolds and integration over compact manifolds, whether orientable or not, is addressed to the mathematical treatment given in (Abraham et al., 2002). A presentation of basic aspects and a discussion with applications to electromagnetism is provided in (Bossavit, 1991, 2004), (Tonti, 1995, 2002) and references therein.

A treatment of *pair* (or *plain*, *even*) and *impair* (or *twisted*, *odd*) forms in oriented affine manifolds, with emphasis on formulation of Clerk-Maxwell[21] equations in the 4D space-time and in Minkowski[22] relativistic space-time, has been provided in (Hehl and Obukhov, 2003) and revisited with a punctual analysis in (Marmo et al., 2005; Marmo and Tulczyjew, 2006).

Due to orientability of space-time, the relevance of odd forms in physics has been questioned in a recent article by da Rocha and Rodrigues (2010), with an ongoing controversy (Itin et al., 2010; da Rocha and Rodrigues, 2010).

In fact, the notion of *even and odd k-covectors* and of *even and odd k-forms*, introduced in (de Rham, 1931, 1955; Schouten, 1951), is required not only to perform integration over non-orientable manifolds, but also to define the flux of a field across a surface or the winding of a field around a cycle, in such a way that the result depends only on the outer orientation of the

[21] James Clerk-Maxwell (1831-1879) Scottish mathematical physicist.
[22] Hermann Minkowski (1864-1909) Russian mathematician.

integration manifold, but neither on the inner orientation of the manifold nor on the orientation of the ambient manifold.

In the context of electromagnetic induction theory, integration over non-orientable manifold is required, for instance, to evaluate the global electric charge on a MÖBIUS[23] strip or on a KLEIN[24] bottle. On the other hand, integration over outer oriented manifold and on its boundary is required to properly formulate the AMPÈRE law of induction, see Section 19.1.

Let us preliminarily provide the definition of immersed manifold.

Definition 6.1 (Immersion). *A smooth map $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,;\mathbb{M}^n)$ is called an immersion, of the k-manifold $\boldsymbol{\Sigma}^k$ into the n-manifold $\mathbb{M}^n$ with $k \le n$, if for any $\mathbf{x} \in \boldsymbol{\Sigma}^k$ the tangent map $T_{\mathbf{x}}u \in \mathrm{C}^1(\mathbb{T}_{\mathbf{x}}\boldsymbol{\Sigma}^k\,;\mathbb{T}_{u(\mathbf{x})}\mathbb{M}^n)$ is injective.*

The range of an injective immersion $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,;\mathbb{M}^n)$, of a compact and connected k-dimensional manifold $\boldsymbol{\Sigma}^k$ with boundary into an n-dimensional manifold $\mathbb{M}^n$ without boundary, is a connected k-dimensional submanifold $u(\boldsymbol{\Sigma}^k)$ of $\mathbb{M}^n$. Denoting by $\{\,\partial_1,\ldots,\partial_k\,\}$ the standard basis of $\mathcal{R}^k$, let us consider a tesselation of $\boldsymbol{\Sigma}^k$ whose simplicial map $s \in \mathrm{C}^1(\mathrm{SIMP}^k\,;\mathbb{M})$ at $\mathbf{x} = s(0^k) \in \boldsymbol{\Sigma}^k$ has domain is the reference simplex

$$\mathrm{SIMP}^k = \{\, x \in \mathcal{R}^n \;:\; x_i \ge 0 \quad \forall\, i\,,\; \sum_{i=1,k} x_i \le 1\,\}\,,$$

and maps the basis of $\mathcal{R}^k$ in the basis $\{\,\mathbf{e}_1,\ldots,\mathbf{e}_k\,\}_{\mathbf{x}}$ of $\mathbb{T}_{\mathbf{x}}\boldsymbol{\Sigma}^k$ with: $\mathbf{e}_i = s(\partial_i)$.

Definition 6.2 (Volumes and point-orientations). *In a n-dimensional manifold $\mathbb{M}^n$, a volume $\boldsymbol{\mu}^n(\mathbf{x}) \in \mathrm{ALT}^n(\mathbb{T}_{\mathbf{x}}\mathbb{M}^n)$ is a non-null n-covector at $\mathbf{x} \in \mathbb{M}^n$. Being the linear space of n-covectors at $\mathbf{x} \in \mathbb{M}^n$ one dimensional, the equivalence relation of positive proportionality defines, at $\mathbf{x} \in \mathbb{M}^n$, two disjoint classes of volumes $\{\,\mathrm{OR}^+_{\mathbf{x}}, \mathrm{OR}^-_{\mathbf{x}}\,\}$, named point-orientations.*

Definition 6.3 (Inner orientation, volume manifolds). *A manifold $\mathbb{M}^n$ endowed with a smooth volume form, viz. with a nowhere vanishing section $\boldsymbol{\mu}^n \in \mathrm{C}^1(\mathbb{M}^n\,;\mathrm{VOL}(\mathbb{TM}^n))$ of the bundle $(\mathrm{VOL}(\mathbb{TM}^n), \boldsymbol{\pi}_{\mathrm{VOL}}, \mathbb{M}^n)$ is said to be inner oriented. The pair $(\mathbb{M}^n\,, \boldsymbol{\mu}^n)$ is called a smooth volume manifold.*

Let us adopt the redundant terminology of *pair* (or *plain*) form, to contrast *impair* (or *twisted*) form.

[23] AUGUST FERDINAND MÖBIUS (1790-1868) German mathematician.
[24] CHRISTIAN FELIX KLEIN (1849-1925) German mathematician.

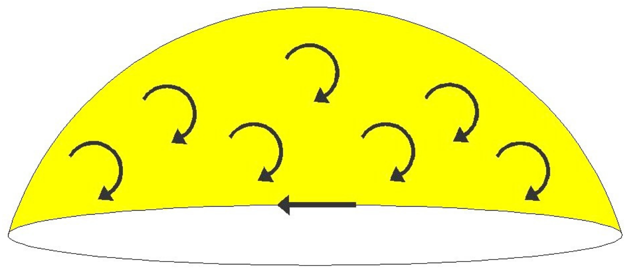

Figure 3: inner-oriented surface and boundary

Definition 6.4 (Pair and impair covectors). *In a n-dimensional manifold $\mathbb{M}^n$, k-covectors $\boldsymbol{\omega}^k(\mathbf{x}) \in \mathrm{ALT}^k(\mathbb{T}_{\mathbf{x}}\mathbb{M}^n)$, with $k \leq n$ are assumed to be function of the orientation of the manifold.* Pair *covectors are invariant with respect to the orientation, while* impair *covectors change sign as the orientation changes.*

Definition 6.5 (Integral over inner oriented submanifolds). *Given a even k-form $\boldsymbol{\omega}^k : \mathbb{M}^n \mapsto \mathrm{ALT}^k(\mathbb{T}\mathbb{M}^n)$ in an n-manifold $\mathbb{M}^n$, the integral, over an inner oriented k-manifold $(\boldsymbol{\Sigma}^k_{\mathrm{IN}}, \boldsymbol{\mu}^k_{\mathrm{IN}})$ with immersion $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k; \mathbb{M}^n)$, of the pull-back k-form $u{\downarrow}\boldsymbol{\omega}^k : \boldsymbol{\Sigma}^k \mapsto \mathrm{ALT}^k(\mathbb{T}\boldsymbol{\Sigma}^k)$, is denoted by:*

$$\int_{\boldsymbol{\Sigma}^k_{\mathrm{IN}}} u{\downarrow}\boldsymbol{\omega}^k$$

and is defined, à la RIEMANN[25] *, as inductive limit, along a family of simplicial tesselations directed by refinement, of finite sums of scalar terms:*

$$\mathrm{SIGN}(\boldsymbol{\mu}^k_{\mathrm{IN}}(\mathbf{e}_1, \dots, \mathbf{e}_k)_{\mathbf{x}})\,(k!)^{-1}\,\boldsymbol{\omega}^k(u{\uparrow}\mathbf{e}_1, \dots, u{\uparrow}\mathbf{e}_k)_{u(\mathbf{x})}\,, \qquad \mathbf{x} \in \boldsymbol{\Sigma}^k\,.$$

The sign of integral, as defined above, is independent of permutations of the basis $\{\,\mathbf{e}_1, \dots, \mathbf{e}_k\,\}_{\mathbf{x}}$ in $\mathbb{T}_{\mathbf{x}}\boldsymbol{\Sigma}^k$, the significant property being the following:

- Changing the inner orientation results in changing the integral of a form into its opposite.

This definition is suitable to compare the value of a global vortex on an inner oriented surface in the EUCLID[26] 3-space, with the corresponding value of the global circulation around its inner oriented boundary circuit, see fig. 3.

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[25] BERNHARD RIEMANN (1826-1866) German mathematician.
[26] EUCLID OF ALEXANDRIA (325-265) BC.

Definition 6.6 (Volume manifolds, induced measures and densities). *A* volume form *in a n-manifold $\mathbb{M}^n$ is a field of volumes $\boldsymbol{\mu}^n \in \boldsymbol{\Lambda}^n(\mathbb{T}\mathbb{M}^n\,;\mathcal{R})$. The pair $(\mathbb{M}^n\,,\boldsymbol{\mu}^n)$ is called a volume manifold. The induced measure is defined by the map:*

$$\mathrm{MEAS}(\boldsymbol{\mu}^n) := \mathrm{SIGN}(\boldsymbol{\mu}^n)\,\boldsymbol{\mu}^n\,.$$

The density associated with a scalar field $\rho : \mathbb{M}^n \mapsto \mathcal{R}$ and a volume form $\boldsymbol{\mu}^n \in \boldsymbol{\Lambda}^n(\mathbb{T}\mathbb{M}^n\,;\mathcal{R})$ is the product $\rho\,\mathrm{MEAS}(\boldsymbol{\mu}^n)$

Definition 6.7 (Integral of a density). *Let us consider in a compact n-manifold $\mathbb{M}^n$ a density $\rho\,\mathrm{MEAS}(\boldsymbol{\mu}^n) \in \boldsymbol{\Lambda}^n(\mathbb{T}\mathbb{M}^n\,;\mathcal{R})$. Then, its integral over a manifold $\boldsymbol{\Sigma}^k$ with immersion $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,;\mathbb{M}^n)$:*

$$\int_{\boldsymbol{\Sigma}^k} u{\downarrow}(\rho\,\mathrm{MEAS}(\boldsymbol{\mu}^k))$$

is defined, à la RIEMANN, *as the inductive limit of finite sums of scalar terms:*

$$(n!)^{-1}\,\rho(u(\mathbf{x}))\,\mathrm{MEAS}(\boldsymbol{\mu}^k)\cdot(u{\uparrow}\mathbf{e}_1,\ldots,u{\uparrow}\mathbf{e}_k)_{u(\mathbf{x})}\,,\quad \mathbf{x}\in\boldsymbol{\Sigma}^k\,,$$

along a family of simplicial tesselations directed by refinement. The integral is then independent of permutations of the basis vectors $u{\uparrow}\mathbf{e}_i, i = 1,\ldots,k$.

Densities can be integrated over non-orientable manifolds, since arbitrary changes of point-orientations do not affect the integral. The next notion provides a generalization of densities to exterior forms of lower order.

Definition 6.8 (Impair forms). *In a volume manifold $(\mathbb{M}^n\,,\boldsymbol{\mu}^n)$, a map assigning, to a point $\mathbf{x}\in\mathbb{M}$ and to a point-orientation $\mathrm{Or}_{\mathbf{x}} \in \{\,\mathrm{Or}^+_{\mathbf{x}},\mathrm{Or}^-_{\mathbf{x}}\,\}$ of the tangent space $\mathbb{T}_{\mathbf{x}}\mathbb{M}$, a k-covector $\boldsymbol{\omega}^k_{\mathbf{x}} \in \mathrm{Alt}^k(\mathbb{T}_{\mathbf{x}}\mathbb{M}^n)$, $k \le n$, or its opposite depending on whether $\mathrm{Or}_{\mathbf{x}} = \mathrm{Or}^+_{\mathbf{x}}$ or $\mathrm{Or} = \mathrm{Or}^-_{\mathbf{x}}$, is called an* impair *$k$-form, and is written as:*

$$\boldsymbol{\omega}^k_{\mathrm{ODD}} := \mathrm{SIGN}(\boldsymbol{\mu}^n)\,\boldsymbol{\omega}^k\,.$$

Accordingly, densities are *impair* volume forms. In an analogous way, the notion of *impair* vector fields may be introduced as follows.

Definition 6.9 (Impair vector fields). *An* impair *vector field on a n-manifold $\mathbb{M}^n$ is a map which assigns to a point $\mathbf{x}\in\mathbb{M}$ and to a point-orientation $\mathrm{Or}_{\mathbf{x}} \in \{\,\mathrm{Or}^+_{\mathbf{x}},\mathrm{Or}^-_{\mathbf{x}}\,\}$ of the tangent space $\mathbb{T}_{\mathbf{x}}\mathbb{M}$ a vector $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}^n$ or its opposite depending on whether $\mathrm{Or}_{\mathbf{x}} = \mathrm{Or}^+_{\mathbf{x}}$ or $\mathrm{Or} = \mathrm{Or}^-_{\mathbf{x}}$, and may then be written as: $\mathrm{SIGN}(\boldsymbol{\mu}^n)\,\mathbf{v}$.*

Definition 6.10 (Outer orientability). *In a volume manifold* $(\mathbb{M}^n \,, \boldsymbol{\mu}^n)$ *, a* k*-manifold* $\boldsymbol{\Sigma}^k$ *with immersion* $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathbb{M}^n)$ *is outer orientable if there exists a* $(n-k)$*-tuple of linearly independent smooth vector fields* $\mathbf{n}_i \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathbb{T}_{u(\boldsymbol{\Sigma}^k)}\mathbb{M}^n)$ *along* $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathbb{M}^n)$ *, that is vector fields fulfilling the commutative diagram:*

$$\begin{tikzcd} & \mathbb{TM} \arrow[d, "\boldsymbol{\pi}_{\mathrm{TAN}}"] \\ \boldsymbol{\Sigma}^k \arrow[ur, "\mathbf{n}_i"] \arrow[r, "u"] & \mathbb{M}^n \end{tikzcd} \qquad \Longleftrightarrow \qquad \boldsymbol{\pi}_{\mathrm{TAN}} \circ \mathbf{n}_i = u \,,$$

whose values $\mathbf{n}_i(\mathbf{x}) \in \mathbb{T}_{(u(\mathbf{x}))}\mathbb{M}^n$ *, at each point* $\mathbf{x} \in \boldsymbol{\Sigma}^k$ *, are transversal to* $\mathbb{T}_{u(\mathbf{x})}u(\boldsymbol{\Sigma}^k) = u\!\uparrow\!(\mathbb{T}_{\mathbf{x}}\boldsymbol{\Sigma}^k)$ *, i.e. are such that* $\mathbb{T}_{(u(\mathbf{x}))}\mathbb{M}^n = \mathbb{T}_{u(\mathbf{x})}u(\boldsymbol{\Sigma}^k) \oplus \mathbf{N}_{\mathbf{x}}$ *, where* $\mathbf{N}_{\mathbf{x}}$ *is the linear span of the vectors* $\mathbf{n}_i(\mathbf{x})$ *.*

Definition 6.11 (Global, inner and outer volume forms). *Let* $(\mathbb{M}^n \,, \boldsymbol{\mu}^n)$ *be a smooth volume manifold and* $\boldsymbol{\Sigma}^k$ *,* $k \le n$ *, an outer orientable immersed manifold with immersion* $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathbb{M}^n)$ *. A smooth pair of related outer volume form:*

$$\boldsymbol{\mu}^{n-k}_{\mathrm{OUT}} \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathrm{ALT}^{n-k}(\mathbb{T}_{u(\boldsymbol{\Sigma}^k)}\mathbb{M}^n)) \,,$$

and inner volume form $\boldsymbol{\mu}^k_{\mathrm{IN}} \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathrm{VOL}(\mathbb{T}\boldsymbol{\Sigma}^k))$ *may be defined by setting, at each* $\mathbf{x} \in \boldsymbol{\Sigma}^k$ *:*

$$\begin{aligned} &\boldsymbol{\mu}^k_{\mathrm{IN}}\,(\mathbf{e}_1, \dots, \mathbf{e}_k)_{\mathbf{x}}\ \boldsymbol{\mu}^{n-k}_{\mathrm{OUT}}\,(\mathbf{n}_1, \dots, \mathbf{n}_{(n-k)})_{u(\mathbf{x})} \\ &\qquad\qquad = \boldsymbol{\mu}^n\,(\mathbf{n}_1 \dots, \mathbf{n}_{(n-k)}, u\!\uparrow\!\mathbf{e}_1, \dots, u\!\uparrow\!\mathbf{e}_k)_{u(\mathbf{x})} \,, \end{aligned}$$

for any basis $\{\,\mathbf{e}_1, \dots, \mathbf{e}_k\,\}_{\mathbf{x}}$ *of* $\mathbb{T}_{\mathbf{x}}\boldsymbol{\Sigma}^k$ *and for any list* $\{\,\mathbf{n}_{k+1} \dots, \mathbf{n}_n\,\}_{u(\mathbf{x})}$ *of* $n-k$ *vector fields fulfilling the requirement of Definition 6.10.*

Definition 6.12 (Integral over outer oriented submanifolds). *Let a volume* n*-manifold* $(\mathbb{M}^n \,, \boldsymbol{\mu}^n)$ *and an* impair k*-form* $\boldsymbol{\omega}^k_{\mathrm{ODD}} : \mathbb{M}^n \mapsto \mathrm{ALT}^k(\mathbb{M}^n)$ *be given. The integral, over a connected outer oriented* k*-manifold* $(\boldsymbol{\Sigma}^k_{\mathrm{OUT}} \,, \boldsymbol{\mu}^k_{\mathrm{OUT}})$ *with immersion* $u \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathbb{M}^n)$ *, of the* k*-form* $u\!\downarrow\!\boldsymbol{\omega}^k : \boldsymbol{\Sigma}^k \mapsto \mathrm{ALT}^k(\mathbb{T}\boldsymbol{\Sigma}^k)$ *is denoted by:*

$$\int_{\boldsymbol{\Sigma}^k_{\mathrm{OUT}}} u\!\downarrow\!\boldsymbol{\omega}^k_{\mathrm{ODD}} \,,$$

and is defined, à la Riemann*, as the inductive limit, along a family of simplicial tesselations directed by refinement, of finite sums of scalar terms:*

$$\mathrm{SIGN}(\boldsymbol{\mu}^k_{\mathrm{IN}}(\mathbf{e}_1, \dots, \mathbf{e}_k)_{\mathbf{x}})\,(k!)^{-1}\,\boldsymbol{\omega}^k_{\mathrm{ODD}}(u\!\uparrow\!\mathbf{e}_1, \dots, u\!\uparrow\!\mathbf{e}_k)_{u(\mathbf{x})} \,, \qquad \mathbf{x} \in \boldsymbol{\Sigma}^k_{\mathrm{OUT}} \,,$$

the volume form $\boldsymbol{\mu}^k_{\text{IN}} \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathrm{Vol}(\mathbb{T}\boldsymbol{\Sigma}^k))$ *being the one induced by the forms:*

$$\boldsymbol{\mu}^{n-k}_{\text{OUT}} \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathrm{Alt}^{n-k}(\mathbb{T}_{u(\boldsymbol{\Sigma}^k)}\mathbb{M}^n))\,,$$

$$\boldsymbol{\mu}^{n} \in \mathrm{C}^1(\boldsymbol{\Sigma}^k\,; \mathrm{Vol}^{n}(\mathbb{T}_{u(\boldsymbol{\Sigma}^k)}\mathbb{M}^n))\,,$$

according to Definition 6.10.

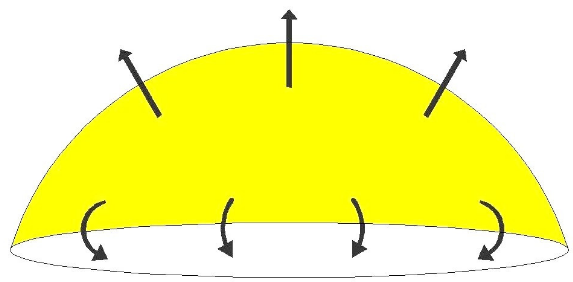


Figure 4: outer-oriented surface and boundary

- Changing the orientation of the ambient manifold results in changing the induced inner orientation of the integration manifold but not the value of the integral, because the integrand is an *impair* form which also changes sign.

- Changing the outer orientation of the integration manifold results in changing the inner orientation induced by the ambient orientation. The integral is then changed into its opposite.

This definition is suitable to define the global flux across an outer oriented surface in the Euclid 3-space, and, likewise, to define the global winding around its outer oriented boundary circuit, see fig. 4. In inner oriented volume manifolds, the integral of an *impair* form over an outer oriented hypersurface as the physical meaning of a flux across the hypersurface, because its sign depends only upon the surface outer orientation, and the integral over the outer oriented boundary cycle has the meaning of winding around the circuit, see fig. 4.

Let us now consider an orientable compact and connected k-manifold $\boldsymbol{\Sigma}^k$ and the canonical immersion $\partial u \in \mathrm{C}^1(\partial\boldsymbol{\Sigma}^k\,; \boldsymbol{\Sigma}^k)$ of its $(k-1)$-dimensional boundary manifold $\partial\boldsymbol{\Sigma}^k$ into the k-manifold $\boldsymbol{\Sigma}^k$.

For an inner oriented surface $\mathbf{\Sigma}^k_{\text{IN}}$, see fig. 3, the STOKES formula of Sect.3 for an *pair* k-form $\boldsymbol{\omega}^k$ writes:

$$\int_{\mathbf{\Sigma}^k_{\text{IN}}} d(u{\downarrow}\boldsymbol{\omega}^k) = \oint_{\partial u(\partial \mathbf{\Sigma}^k_{\text{IN}})} u{\downarrow}\boldsymbol{\omega}^k = \oint_{\partial \mathbf{\Sigma}^k_{\text{IN}}} \partial u{\downarrow}(u{\downarrow}\boldsymbol{\omega}^k)\,.$$

- Changing the inner orientation of the surface $\mathbf{\Sigma}_{\text{IN}}$, all integrals in the equality will change sign, so that the equality is still valid.

By definition 6.12, the STOKES formula holds also for the integrals of *impair* forms over outer oriented manifolds. In fact, let an outer orientation across a k-manifold $\mathbf{\Sigma}^k_{\text{OUT}}$, see fig. 4, (a crossing direction for the flux) and the induced outer orientation around the boundary circuit (a turning sense for the winding) be given. The STOKES formula writes:

$$\int_{\mathbf{\Sigma}^k_{\text{OUT}}} d(u{\downarrow}\boldsymbol{\omega}^k_{\text{ODD}}) = \oint_{\partial u(\partial \mathbf{\Sigma}^k_{\text{OUT}})} u{\downarrow}\boldsymbol{\omega}^k_{\text{ODD}} = \oint_{\partial \mathbf{\Sigma}^k_{\text{OUT}}} \partial u{\downarrow}(u{\downarrow}\boldsymbol{\omega}^k_{\text{ODD}})\,.$$

- Changing the orientation of the ambient manifold results in changing the induced inner orientations of the integration manifolds but not the value of the integrals due to the sign change of the *impair* integrand form.

- Changing the outer orientation of the surface, and the associated outer orientation on the boundary circuit, all integrals will change sign and the equality still holds.

7. Calculus on manifolds

An *immersion* (*submersion*) is a map with injective (surjective) associated tangent map.

A *fibration* of a manifold $\mathbb{M}$ is a *projection* (surjective submersion) $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{B})$ on a base manifold $\mathbb{B}$.

A *fiber* $\mathbb{M}(\mathbf{x})$ is the inverse image of a point $\mathbf{x} \in \mathbb{B}$ by the projection.

A *section* of a fibration $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{B})$ is a map $\mathbf{s} \in \mathrm{C}^1(\mathbb{B}\,;\mathbb{M})$ that is a right inverse of the projection, i.e. such that $\boldsymbol{\pi} \circ \mathbf{s} = \mathbf{id}_{\mathbb{B}}$.

A *fiber-bundle* is a fibration with diffeomorphic fibers.

A *vector-bundle* has linear fibers.

A *morphism* is a fiber preserving map between fiber-bundles.

An *endomorphism* is a morphism between a fiber-bundle and itself.

A *homomorphism* is a fiberwise linear morphism between vector-bundles and an *isomorphism* is an invertible homomorphism.

A *pull-back bundle* $\boldsymbol{\pi}_{\mathbb{N},\mathbb{M}} = \mathbf{f}_{\mathbb{B},\mathbb{N}}{\downarrow}\boldsymbol{\pi}_{\mathbb{B},\mathbb{M}}$ of a fiber-bundle $\boldsymbol{\pi}_{\mathbb{B},\mathbb{M}}$ by an invertible map $\mathbf{f}_{\mathbb{B},\mathbb{N}} : \mathbb{N} \mapsto \mathbb{B}$ is defined by the commutative diagram

$$\begin{tikzcd} & \mathbb{M} \arrow[dl, "\boldsymbol{\pi}_{\mathbb{N},\mathbb{M}}"'] \arrow[d, "\boldsymbol{\pi}_{\mathbb{B},\mathbb{M}}"] \\ \mathbb{N} \arrow[r, "\mathbf{f}_{\mathbb{B},\mathbb{N}}"] & \mathbb{B} \end{tikzcd} \quad \Longleftrightarrow \quad \mathbf{f}_{\mathbb{B},\mathbb{N}} \circ \boldsymbol{\pi}_{\mathbb{N},\mathbb{M}} = \boldsymbol{\pi}_{\mathbb{B},\mathbb{M}} \,.$$

The tangent bundle $\mathbb{TM}$ is the disjoint union of tangent spaces (linear fibers) $\mathbb{T}_{\mathbf{x}}\mathbb{M}$. Vector *fields* are sections of a tangent bundle.

A tensor bundle $\text{Tens}(\mathbb{TM})$ is the disjoint union of tensor spaces (linear fibers) $\text{Tens}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$. Tensor *fields* are sections of a tensor bundle.

Fields of k-covectors are called differential k-forms, or simply *k-forms*.

Push-forward of contravariant tensor fields is well defined for any differentiable morphism $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$. Pull-back of covariant tensor fields requires that the morphism is injective. Push (or pull) transformations of other fields require that this map is a diffeomorphism.

From chain rule of calculus it follows that push-pull operations by diffeomorphisms enjoy the following commutativity property with composition

$$\boldsymbol{\zeta}{\uparrow}(\mathbf{s}_{\text{Con}} \circ \mathbf{s}_{\text{Cov}}) = (\boldsymbol{\zeta}{\uparrow}\mathbf{s}_{\text{Con}}) \circ (\boldsymbol{\zeta}{\uparrow}\mathbf{s}_{\text{Cov}}) \,.$$

The tangent functor T, when applied to a differentiable map $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$, defines a *homomorphism* between the relevant tangent bundles, according to the commutative diagram

$$\begin{tikzcd} \mathbb{TM} \arrow[r, "T\boldsymbol{\zeta}"] \arrow[d, "\boldsymbol{\pi}_{\mathbb{M},\mathbb{TM}}"'] & \mathbb{TN} \arrow[d, "\boldsymbol{\pi}_{\mathbb{N},\mathbb{TN}}"] \\ \mathbb{M} \arrow[r, "\boldsymbol{\zeta}"] & \mathbb{N} \end{tikzcd} \quad \Longleftrightarrow \quad \boldsymbol{\pi}_{\mathbb{N},\mathbb{TN}} \circ T\boldsymbol{\zeta} = \boldsymbol{\zeta} \circ \boldsymbol{\pi}_{\mathbb{M},\mathbb{TM}} \,.$$

7.1. Lie derivatives

The Lie[27] derivative of a vector field $\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ according to a vector field $\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ is defined by considering the flow $\mathbf{Fl}^{\mathbf{u}}_{\lambda}$ generated

[27] Marius Sophus Lie (1842-1899) Norwegian mathematician **?**.

by solutions of the differential equation $\mathbf{u} = \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{u}}_{\lambda}$ and by differentiating the pull-back along the flow

$$\mathcal{L}_{\mathbf{u}}\mathbf{w} := \partial_{\lambda=0}\, (\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\!\mathbf{w}) = \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\!(\mathbf{w} \circ \mathbf{Fl}^{\mathbf{u}}_{\lambda})\,.$$

Let us recall that push forward along the flow $\mathbf{Fl}^{\mathbf{u}}_{\lambda}$ is defined in terms of the tangent functor T as

$$(\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\uparrow\!\mathbf{w}) \circ \mathbf{Fl}^{\mathbf{u}}_{\lambda} := T\mathbf{Fl}^{\mathbf{u}}_{\lambda} \cdot \mathbf{w}\,,$$

and that the pull back is defined by $\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\, := \mathbf{Fl}^{\mathbf{u}}_{-\lambda}\!\uparrow$. Push-pull of scalar fields are just change of base points and hence the Lie derivative of scalar fields coincides with the directional derivative.

The commutator of tangent vector fields $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ is the skew-symmetric tangent-vector valued operator defined by

$$[\mathbf{u}\,, \mathbf{w}]f := (\mathcal{L}_{\mathbf{u}}\mathcal{L}_{\mathbf{w}} - \mathcal{L}_{\mathbf{w}}\mathcal{L}_{\mathbf{u}})f\,,$$

with $f \in \mathrm{C}^1(\mathbb{M}\,;\mathcal{R})$ a scalar field. A basic theorem concerning Lie derivatives states that $\mathcal{L}_{\mathbf{u}}\mathbf{w} = [\mathbf{u}\,, \mathbf{w}]$ and hence the commutator of tangent vector fields is called the Lie bracket. For any injective morphism $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{N})$ the Lie bracket enjoys the following push-naturality property (G. Romano, 2007)

$$\boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{u}}\,\mathbf{w}) = \boldsymbol{\zeta}\!\uparrow\![\mathbf{u}\,, \mathbf{w}] = [\boldsymbol{\zeta}\!\uparrow\!\mathbf{u}\,, \boldsymbol{\zeta}\!\uparrow\!\mathbf{w}] = \mathcal{L}_{\boldsymbol{\zeta}\uparrow\mathbf{u}}\,\boldsymbol{\zeta}\!\uparrow\!\mathbf{w}\,.$$

For a tensor field $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{Tens}(\mathbb{TM}))$ with Lie derivative

$$\mathcal{L}_{\mathbf{u}}\,\mathbf{s} := \partial_{\lambda=0}\, (\mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\!\mathbf{s}) = \partial_{\lambda=0}\, \mathbf{Fl}^{\mathbf{u}}_{\lambda}\!\downarrow\!(\mathbf{s} \circ \mathbf{Fl}^{\mathbf{u}}_{\lambda})\,,$$

the push-naturality property extends to

$$\boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{u}}\,\mathbf{s}) = \mathcal{L}_{\boldsymbol{\zeta}\uparrow\mathbf{u}}\,\boldsymbol{\zeta}\!\uparrow\!\mathbf{s}\,.$$

By the commutativity property between push and composition, Leibniz[28] rule for the $\partial_{\lambda=0}$ derivative yields the analogous Leibniz rule for Lie derivatives of tensor fields

$$\mathcal{L}_{\mathbf{u}}\, (\mathbf{s}_{\mathrm{Con}} \circ \mathbf{s}_{\mathrm{Cov}}) = (\mathcal{L}_{\mathbf{u}}\, \mathbf{s}_{\mathrm{Con}}) \circ \mathbf{s}_{\mathrm{Cov}} + \mathbf{s}_{\mathrm{Con}} \circ (\mathcal{L}_{\mathbf{u}}\, \mathbf{s}_{\mathrm{Cov}})\,.$$

[28] Gottfried Wilhelm von Leibniz (1646-1716) German scientist.

7.2. Connection and parallel derivatives

A linear connection ∇ in a manifold $\mathbb{M}$ fulfills the characteristic properties of a point derivation (Dieudonne, 1969, vol. III, XVII-18)

$$\begin{aligned}\nabla_{\mathbf{w}}(\mathbf{u}_1 + \mathbf{u}_2) &= \nabla_{\mathbf{w}}\mathbf{u}_1 + \nabla_{\mathbf{w}}\mathbf{u}_2\,,\\ \nabla_{(\mathbf{w}_1+\mathbf{w}_2)}\mathbf{u} &= \nabla_{\mathbf{w}_1}\mathbf{u} + \nabla_{\mathbf{w}_2}\mathbf{u}\,,\\ \nabla_{\mathbf{w}}(f\mathbf{u}) &= f\,\nabla_{\mathbf{w}}\mathbf{u} + (\nabla_{\mathbf{w}}f)\mathbf{u}\,,\\ \nabla_{(f\,\mathbf{w})}\mathbf{u} &= f\,\nabla_{\mathbf{w}}\mathbf{u}\,,\end{aligned}$$

where $f \in \mathrm{C}^1(\mathbb{M}\,;\mathcal{R})$ and $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2 \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ and $\nabla_{\mathbf{w}}f$ is the standard derivative of scalar fields. In terms of parallel transport along a curve $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,;\mathbb{M})$, with $\mathbf{u} = \partial_{\lambda=0}\,\mathbf{c}(\lambda)$, the derivative according to a connection is defined by

$$\nabla_{\mathbf{u}}\mathbf{w} := \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow(\mathbf{w}\circ\mathbf{c})(\lambda)\,.$$

Parallel transported vector fields $(\mathbf{w}\circ\mathbf{c})(\lambda) = \mathbf{c}(\lambda)\Uparrow\mathbf{w}_0$ have a null parallel derivative, because

$$\nabla_{\mathbf{u}}\mathbf{w} := \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow(\mathbf{w}\circ\mathbf{c})(\lambda) = \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow\mathbf{c}(\lambda)\Uparrow\mathbf{w}_0 = \partial_{\lambda=0}\,\mathbf{w}_0 = 0\,.$$

The curvature of the connection is the *tensorial*[29] map $\textsc{curv}$, which acting on a vector field $\mathbf{s} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ gives a tangent-vector valued two-form $\textsc{curv}(\mathbf{s})$ defined by[30]

$$\textsc{curv}(\mathbf{s})(\mathbf{u}\,,\mathbf{w}) := ([\nabla_{\mathbf{u}}\,,\nabla_{\mathbf{w}}] - \nabla_{[\mathbf{u}\,,\mathbf{w}]})(\mathbf{s})\,,$$

and the torsion $\textsc{tors}$ is the tangent-vector valued two-form defined by

$$\textsc{tors}(\mathbf{u}\,,\mathbf{w}) := \nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} - [\mathbf{u}\,,\mathbf{w}]\,.$$

Mixed tensor fields $\textsc{Tors}(\mathbf{u})$ and $\textsc{Curv}(\mathbf{s}\,,\mathbf{u})$ are defined by the identities

$$\begin{aligned}\textsc{Tors}(\mathbf{u})\cdot\mathbf{w} &:= \textsc{tors}(\mathbf{u}\,,\mathbf{w}) = -\textsc{tors}(\mathbf{w}\,,\mathbf{u})\,,\\ \textsc{Curv}(\mathbf{s}\,,\mathbf{u})\cdot\mathbf{w} &:= \textsc{curv}(\mathbf{s})(\mathbf{u}\,,\mathbf{w}) = -\textsc{curv}(\mathbf{s})(\mathbf{w}\,,\mathbf{u})\,.\end{aligned}$$

[29] Tensoriality of a multilinear map, acting on vector fields and generating a vector field, means that point values of the image field depends only on the values of the source fields at the same point. A form is then a vector-valued, tensorial, alternating multilinear map.

[30] The curvature form for connection on a fiber bundle and the relevant expression in terms of parallel derivatives are treated in G. Romano (2007).

A connection with vanishing torsion is named *torsion-free* or *symmetric*, and a connection with vanishing curvature is said to be *curvature-free* or *flat*.

A RIEMANN manifold $(\mathbb{M}\,, \mathbf{g})$ is endowed with a field of metric tensors $\mathbf{g} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{POS}(\mathbb{TM}))$. The associated LEVI-CIVITA[31] connection is uniquely defined by the properties $\nabla\mathbf{g} = 0$ and $\mathrm{TORS} = 0$.

A linear connection in a RIEMANN manifold induces a linear connection in each submanifold by means of pre-composition by the immersion map and post-composition by the orthogonal projection map.

7.3. Lie derivatives in terms of parallel derivatives

Noteworthy formulae provide the LIE derivatives of tensor fields in terms of parallel derivatives. For convenience we set $\mathbf{Y}(\mathbf{v}) := \nabla(\mathbf{v}) + \mathrm{TORS}(\mathbf{v})$.

Proposition 7.1. *Let* $\mathbb{M}$ *be a manifold and* ∇ *a linear connection in* $\mathbb{M}$. *The* LIE *derivative of a covariant tensor field* $\mathbf{s}_{\mathrm{Cov}} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{COV}(\mathbb{TM}))$ *along the flow* $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{M})$ *of a tangent vector field* $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM})$ *is given by*

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} = \nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} + \mathbf{s}_{\mathrm{Cov}} \cdot \mathbf{Y}(\mathbf{v}) + \mathbf{Y}(\mathbf{v})^* \cdot \mathbf{s}_{\mathrm{Cov}}\,.$$

If $\mathbf{s}_{\mathrm{Cov}} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{SYM}(\mathbb{TM}))$, *the formula specializes into*

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} = \nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} + 2\,\mathrm{sym}\,(\mathbf{s}_{\mathrm{Cov}} \cdot \mathbf{Y}(\mathbf{v}))\,.$$

Proof. Applying LEIBNIZ rule to the LIE derivative and to the parallel derivative, we have that, for any $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM})$

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) &= \mathcal{L}_{\mathbf{v}}\,(\mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \mathbf{w})) - \mathbf{s}_{\mathrm{Cov}}\,(\mathcal{L}_{\mathbf{v}}\mathbf{u}, \mathbf{w}) - \mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \mathcal{L}_{\mathbf{v}}\mathbf{w})\,,\\ (\nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) &= \nabla_{\mathbf{v}}\,(\mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \mathbf{w})) - \mathbf{s}_{\mathrm{Cov}}\,(\nabla_{\mathbf{v}}\mathbf{u}, \mathbf{w}) - \mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \nabla_{\mathbf{v}}\mathbf{w})\,.\end{aligned}$$

LIE derivative and parallel derivative of scalar fields coincide, so that

$$\mathcal{L}_{\mathbf{v}}\,(\mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \mathbf{w})) = \nabla_{\mathbf{v}}\,(\mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \mathbf{w}))\,.$$

Hence

$$\begin{aligned}(\mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) = (\nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}})(\mathbf{u}, \mathbf{w}) &+ \mathbf{s}_{\mathrm{Cov}}\,(\nabla_{\mathbf{v}}\mathbf{u}, \mathbf{w}) + \mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \nabla_{\mathbf{v}}\mathbf{w})\\ &- \mathbf{s}_{\mathrm{Cov}}\,(\mathcal{L}_{\mathbf{v}}\mathbf{u}, \mathbf{w}) - \mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u}, \mathcal{L}_{\mathbf{v}}\mathbf{w})\,.\end{aligned}$$

[31] TULLIO LEVI-CIVITA (1873-1941) Italian mathematician.

Moreover, since $\textsc{tors}(\mathbf{v},\mathbf{u}) := \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v}\,,\mathbf{u}]$ we may write

$$
\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\,\mathbf{s}_{\mathrm{Cov}})(\mathbf{u},\mathbf{w}) = (\nabla_{\mathbf{v}}\,\mathbf{s}_{\mathrm{Cov}})(\mathbf{u},\mathbf{w}) + \mathbf{s}_{\mathrm{Cov}}\,(\textsc{tors}(\mathbf{v},\mathbf{u}),\mathbf{w}) + \mathbf{s}_{\mathrm{Cov}}\,(\nabla_{\mathbf{u}}\mathbf{v},\mathbf{w}) \\
+ \mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u},\textsc{tors}(\mathbf{v},\mathbf{w})) + \mathbf{s}_{\mathrm{Cov}}\,(\mathbf{u},\nabla_{\mathbf{w}}\mathbf{v})\,,
\end{aligned}
$$

which, by definition of the torsion field $\mathrm{T\textsc{ors}}(\mathbf{v}) \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{M\textsc{ix}}(\mathbb{TM}))$, gives the result. ■

Analogous proofs, explicitly reported in (G. Romano, 2007), lead to the expressions for $\mathbf{u}^* \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}^*\mathbb{M})$, $\mathbf{s}_{\mathrm{Con}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{C\textsc{on}}(\mathbb{TM}))$ and $\mathbf{s}_{\mathrm{Mix}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{M\textsc{ix}}(\mathbb{TM}))$ listed hereafter

$$
\begin{aligned}
\mathcal{L}_{\mathbf{v}}\,\mathbf{u}^* &= \nabla_{\mathbf{v}}\,\mathbf{u}^* + \mathbf{u}^* \cdot \mathbf{Y}(\mathbf{v})\,, \\
\mathcal{L}_{\mathbf{v}}\,\mathbf{s}_{\mathrm{Con}} &= \nabla_{\mathbf{v}}\,\mathbf{s}_{\mathrm{Con}} - \mathbf{Y}(\mathbf{v}) \cdot \mathbf{s}_{\mathrm{Con}} - \mathbf{s}_{\mathrm{Con}} \cdot \mathbf{Y}(\mathbf{v})^*\,, \\
\mathcal{L}_{\mathbf{v}}\,\mathbf{s}_{\mathrm{Mix}} &= \nabla_{\mathbf{v}}\,\mathbf{s}_{\mathrm{Mix}} + \mathbf{s}_{\mathrm{Mix}} \cdot \mathbf{Y}(\mathbf{v}) - \mathbf{Y}(\mathbf{v}) \cdot \mathbf{s}_{\mathrm{Mix}}\,.
\end{aligned}
$$

The next result provides the Lie derivative of a form in terms of parallel derivatives.

Proposition 7.2. *Let $\mathbb{M}$ be a manifold and ∇ a linear connection in $\mathbb{M}$. The Lie derivative of a k-form $\boldsymbol{\omega} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{A\textsc{lt}}^k(\mathbb{TM}))$ along the flow $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ of a tangent vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ is given by*

$$
\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega} = \nabla_{\mathbf{v}}\,\boldsymbol{\omega} + \mathrm{C\textsc{ycle}}\,(\boldsymbol{\omega} \circ \mathbf{Y}(\mathbf{v}))\,,
$$

where the operator $\mathrm{C\textsc{ycle}}$ evaluates the sum of the values of a k-form over cyclic permutations of the argument vectors and the form $\boldsymbol{\omega} \circ \mathbf{Y}(\mathbf{v}) \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{A\textsc{lt}}^k(\mathbb{TM}))$ is defined by

$$
(\boldsymbol{\omega} \circ \mathbf{Y}(\mathbf{v}))(\mathbf{a},\mathbf{b},\mathbf{c}) = \boldsymbol{\omega}(\mathbf{Y}(\mathbf{v}) \cdot \mathbf{a},\mathbf{b},\mathbf{c})
$$

Proof. Making explicit reference to a 3-form and to a triplet of vector fields $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$, a proof analogous to the one in Prop.7.1 gives

$$
\begin{aligned}
(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega})(\mathbf{a},\mathbf{b},\mathbf{c}) = (\nabla_{\mathbf{v}}\,\boldsymbol{\omega})(\mathbf{a},\mathbf{b},\mathbf{c}) + \boldsymbol{\omega}(\textsc{tors}(\mathbf{v},\mathbf{a}),\mathbf{b},\mathbf{c}) + \boldsymbol{\omega}(\nabla_{\mathbf{a}}\mathbf{v},\mathbf{b},\mathbf{c}) \\
+ \boldsymbol{\omega}(\mathbf{a},\textsc{tors}(\mathbf{v},\mathbf{b}),\mathbf{c}) + \boldsymbol{\omega}(\nabla_{\mathbf{b}}\mathbf{v},\mathbf{c},\mathbf{a}) \\
+ \boldsymbol{\omega}(\textsc{tors}(\mathbf{v},\mathbf{c}),\mathbf{a},\mathbf{b}) + \boldsymbol{\omega}(\nabla_{\mathbf{c}}\mathbf{v},\mathbf{a},\mathbf{b})\,,
\end{aligned}
$$

and the result follows. ■

In terms of the maximal-form $\boldsymbol{\mu}_{\mathbf{g}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{M\scriptsize{XF}}(\mathbb{TM}))$ induced by the metric tensor field $\mathbf{g} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{P\scriptsize{OS}}(\mathbb{TM}))$, the linear invariant J_1 of a mixed tensor $\mathbf{s}_{\mathrm{MIX}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{M\scriptsize{IX}}(\mathbb{TM}))$ is defined by

$$J_1(\mathbf{s}_{\mathrm{MIX}})\,\boldsymbol{\mu}_{\mathbf{g}} := \mathrm{C\scriptsize{YCLE}}\,(\boldsymbol{\mu}_{\mathbf{g}} \circ \mathbf{s}_{\mathrm{MIX}})\,.$$

Defining the 0-th order invariant $J_o(\mathbf{v})$ as the proportionality factor in the equality $\nabla_{\mathbf{v}}\,\boldsymbol{\mu}_{\mathbf{g}} = J_o(\mathbf{v})\,\boldsymbol{\mu}_{\mathbf{g}}$, the formula of Prop.7.2 may be rewritten as

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\mu}_{\mathbf{g}} = \big(J_o(\mathbf{v}) + J_1(\mathbf{Y}(\mathbf{v}))\big)\,\boldsymbol{\mu}_{\mathbf{g}}\,.$$

By definition of divergence $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\mu}_{\mathbf{g}} = (\operatorname{div}\mathbf{v})\,\boldsymbol{\mu}_{\mathbf{g}}$ we get

$$\operatorname{div}\mathbf{v} = J_o(\mathbf{v}) + J_1(\mathbf{Y}(\mathbf{v}))\,.$$

Adopting Levi-Civita connection, the implication $\nabla\mathbf{g} = 0 \Longrightarrow \nabla\boldsymbol{\mu}_{\mathbf{g}} = 0$, leads to the standard formula

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\mu}_{\mathbf{g}} = J_1(\nabla\mathbf{v})\,\boldsymbol{\mu}_{\mathbf{g}}\,,$$

which, by definition of divergence, is equivalent to $\operatorname{div}\mathbf{v} = J_1(\nabla\mathbf{v})$.

8. Continuum Kinematics

Continuum Kinematics (CK) investigates about geometric properties of motions. The natural theoretical framework is provided by a four-dimensional affine *events manifold* $\mathbb{M}$ and by its representations according to observers. In this way, the theory is best developed on the basis of informations available from physical experience and formalized with the mathematical tools offered by differential geometry.

8.1. Events manifolds and observers

The affine structure of the *events manifold* $\mathbb{M}$ permits to choose in the associated linear space of translations V a constant vector field of *time arrows* constructed by translation of a non-null vector of V.

The *events manifold* $\mathbb{M}$ is then fibrated into a family of disjoint *time-lines* generated by the field of *time arrows*. A *time-lines* is an oriented one-dimensional affine manifold isomorphic to a model one-dimensional scalar time-line $\mathcal{Z}$.

To the *time-fibration* there corresponds a complementary *space-fibration* into a family of *spatial-slices* which are disjoint hyperplanes transversal to the *time-arrows* and parallel one another. Transversality means that, at any event, the *spatial-slice* does not contain the *time-arrow*. The *spatial-slices* are three-dimensional affine submanifolds isomorphic to a model three-dimensional affine manifold, the ambient space $\mathcal{S}$.

Two EUCLID observers measure time along two proportional fields of *time arrows* and their *spatial-slices* are parallel one another, so that they give the same judgement of *simultaneity*.

Two EUCLID observers are said to be *synchronized* if they adopt the same field of *time-arrows* and if events in the same *spatial-slice* are assigned the same time-coordinate by the two observers.

The point of view of an observer amounts in the choice of the field of *time-arrows* and of the *spatial-slices*. Each observer $\boldsymbol{\gamma} \in \mathrm{C}^1(\mathbb{M}, \mathcal{S} \times \mathcal{Z})$ detects a diffeomorphism between the events manifold and a *space-time* manifold, the product between the affine space manifold $\mathcal{S}$ and the affine *time-line* $\mathcal{Z}$.

The events manifold is thus fibrated by complementary projections $\boldsymbol{\pi}_{\mathcal{S},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathcal{S})$ and $\boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathcal{Z})$, the latter being independent of the observer, since in classical mechanics *time is absolute*, i.e. the field of *time-arrows* is the same for all synchronized EUCLID observers.

The assumption of an absolute time and the adoption of a positive definite metric tensor are main distinctive issues between classical and relativistic mechanics.

The fibers $\mathbb{M}(t) \equiv (\mathcal{S}\,, t)$ of *simultaneous* events, with $t \in \mathcal{Z}$, are all isomorphic to the ambient space $\mathcal{S}$ and the fibers $\mathbb{M}(\mathbf{x}) \equiv (\mathbf{x}\,, \mathcal{Z})$ of *isotopic* events, with $\mathbf{x} \in \mathcal{S}$, are isomorphic to the *time line* $\mathcal{Z}$.

Both $\mathcal{S}$ and $\mathcal{Z}$ are RIEMANN manifolds endowed with metric fields $\mathbf{g}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{S}\,;\mathrm{POS}(\mathbb{T}\mathcal{S}))$ and $\mathbf{g}_{\mathcal{Z}} \in \mathrm{C}^1(\mathcal{Z}\,;\mathrm{POS}(\mathbb{T}\mathcal{Z}))$, where $\mathrm{POS}(\bullet)$ is the bundle of twice-covariant, symmetric and positive definite tensors. Space and time manifolds are assumed to be endowed with linear connections $\nabla^{\mathcal{S}}$ and ∇^{I}. The associated parallel transports are denoted by $\Uparrow^{\mathcal{S}}$ and $\Uparrow^{I}$.

The ambient space $\mathcal{S}$ considered in CK is the 3D affine EUCLID[32] space endowed with a metric field invariant under translation. The metric field in the affine time-line $\mathcal{Z}$ is also invariant under translation.

[32] EUCLID OF ALEXANDRIA (325-265) BC.

8.2. Trajectory and motion

The trajectory $\mathcal{T}$ is a non-linear manifold characterized by an injective immersion $\mathbf{i}_{\mathbb{M},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathbb{M})$ which is such that the immersed trajectory $\mathcal{T}^{\mathbb{M}} := \mathbf{i}_{\mathbb{M},\mathcal{T}}(\mathcal{T}) \subset \mathbb{M}$ is a submanifold of the events manifold.[33]

Each observer $\boldsymbol{\gamma} \in \mathrm{C}^1(\mathbb{M}, \mathcal{S} \times \mathcal{Z})$ induces space and time projections on the trajectory manifold, defined by

$$\boldsymbol{\pi}_{\mathcal{S},\mathcal{T}} := \boldsymbol{\pi}_{\mathcal{S},\mathbb{M}} \circ \mathbf{i}_{\mathbb{M},\mathcal{T}}\,, \quad \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} := \boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} \circ \mathbf{i}_{\mathbb{M},\mathcal{T}}\,.$$

The time projection $\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{Z})$ generates a fibration of the trajectory manifold $\mathcal{T}$ over the base $\mathcal{Z}$ and the *placement* at time $t \in \mathcal{Z}$ is the fiber of simultaneous events in the trajectory

$$\mathcal{T}(t) = \{\, \mathbf{e} \in \mathcal{T} \mid \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}(\mathbf{e}) = t \,\}\,.$$

The *spatial placement* is the projection $\boldsymbol{\Omega} = \boldsymbol{\pi}_{\mathcal{S},\mathcal{T}}(\mathcal{T}(t)) \subset \mathcal{S}$, assumed to be a compact connected submanifold. The time fibration makes the trajectory manifold $\mathcal{T}$ a fiber-bundle.

The space projection $\boldsymbol{\pi}_{\mathcal{S},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{S})$ generates a fibration of the trajectory manifold whose base manifold is the (non-disjoint) union of the spatial placements $\cup_{t\in\mathcal{Z}}\boldsymbol{\Omega} \subset \mathcal{S}$, the *wake* of the trajectory. The fiber based at $\mathbf{x} \in \mathcal{S}$ is a set of *isotopic* events in the trajectory

$$\mathcal{T}(\mathbf{x}) = \{\, \mathbf{e} \in \mathcal{T} \mid \boldsymbol{\pi}_{\mathcal{S},\mathcal{T}}(\mathbf{e}) = \mathbf{x} \,\}\,,$$

which in general is not a manifold.

It follows that the space fibration fails in general to make the trajectory manifold $\mathcal{T}$ a fiber-bundle.[34] The time-projection $\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}(\mathcal{T}(\mathbf{x})) \subset \mathcal{Z}$ is the set of time instants at which there is a body particle crossing the spatial point $\mathbf{x} \in \mathcal{S}$, possibly a singleton.

Fibre regularity in space fibration is assured if the trajectory manifold has the same dimensionality of the space-time manifold, and we will call this the *Maximal Dimensionality Property* (MDP).

[33] Events in the immersed trajectory are represented by coordinates in the events manifold, whose dimensionality may be higher than the one of the trajectory manifold.
[34] Fibre regularity in spatial fibration of the trajectory does not hold in general, for lower dimensional trajectories, see Rem.15.1.

The motion is a one-parameter family of automorphisms[35] $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ of the trajectory time-bundle over the time shift $\mathrm{SH}_\alpha \in \mathrm{C}^1(\mathcal{Z}\,;\mathcal{Z})$, defined by $\mathrm{SH}_\alpha(t) := t + \alpha$ with $t \in \mathcal{Z}$ *time-instant* and $\alpha \in \mathcal{R}$ *time-lapse*, as described by the commutative diagrams

$$
\begin{tikzcd}
\mathcal{T}^{\mathbb{M}} \arrow[r, "\boldsymbol{\varphi}_\alpha^{\mathbb{M}}"] \arrow[dd, bend right=60, "\boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}}"'] & \mathcal{T}^{\mathbb{M}} \\
\mathcal{T} \arrow[u, "\mathbf{i}_{\mathbb{M},\mathcal{T}}"] \arrow[r, "\boldsymbol{\varphi}_\alpha"] \arrow[d, "\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}"'] & \mathcal{T} \arrow[u, "\mathbf{i}_{\mathbb{M},\mathcal{T}}"'] \arrow[d, "\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}"] \\
\mathcal{Z} \arrow[r, "\mathrm{SH}_\alpha"] & \mathcal{Z}
\end{tikzcd}
\qquad \Longleftrightarrow \qquad
\begin{aligned}
\boldsymbol{\varphi}_\alpha^{\mathbb{M}} \circ \mathbf{i}_{\mathbb{M},\mathcal{T}} &= \mathbf{i}_{\mathbb{M},\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha \,, \\
\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha &= \mathrm{SH}_\alpha \circ \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} \,.
\end{aligned}
$$

which provides a formal expression of the property of conservation of simultaneity, that is: the motion sends simultaneous events into time-shifted simultaneous events. The map $\boldsymbol{\varphi}_\alpha^{\mathbb{M}} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}}\,;\mathcal{T}^{\mathbb{M}})$ is called the *immersed motion*.

The *trajectory velocity* $\mathbf{v}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathbb{T}\mathcal{T})$ and its immersion in the events manifold, the *space-time velocity* $\mathbf{v}_{\mathcal{T}}^{\mathbb{M}} = \mathbf{i}_{\mathbb{M},\mathcal{T}}\!\uparrow\!\mathbf{v}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}}\,;\mathbb{T}\mathcal{T}^{\mathbb{M}})$ are defined by

$$
\mathbf{v}_{\mathcal{T}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \,, \qquad \mathbf{v}_{\mathcal{T}}^{\mathbb{M}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha^{\mathbb{M}} \,.
$$

Since the motion is parametrized by time, velocities have a unit time-projection

$$
\begin{aligned}
T\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} \cdot \mathbf{v}_{\mathcal{T}} &= \partial_{\alpha=0}\, \mathrm{SH}_\alpha \circ \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} = 1 \,, \\
T\boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} \cdot \mathbf{v}_{\mathcal{T}}^{\mathbb{M}} &= \partial_{\alpha=0}\, \mathrm{SH}_\alpha \circ \boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} = 1 \,.
\end{aligned}
$$

All informations concerning the trajectory velocity are conveyed by its spatial-projection. However, in performing time-differentiations of tensor fields along the motion, the whole trajectory velocity must be considered. Indeed, the evaluation of time-derivatives along temporal and spatial directions could in general bring the base point outside the trajectory, at events where the relevant field is undefined.

8.3. Body and particles

According to our point of view, the standard notions of *body* and *material particles* are not primary, as in most standard presentations of the matter,

[35] An *automorphism* is an invertible morphism from a fiber-bundle onto itself.

but are rather deduced as secondary to that of *trajectory* and *motion*. To this end, we observe that events related by the space-time motion along the trajectory, i.e.

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{M} \mid \exists\, \alpha \in \mathcal{R} \,:\, \mathbf{e}_2 = \boldsymbol{\varphi}_\alpha(\mathbf{e}_1)\,,$$

form a class of equivalence and that the equivalence relation foliates the trajectory manifold. We may then give the following definitions.

A material particle is a line (a one-dimensional manifold) whose elements are motion-related events in the trajectory.

The body is the disjoint union of the trajectory material particles, a *quotient manifold* induced by the foliation of the trajectory manifold.

Remarkably, the geometric approach to CK does not require, the introduction of a fixed reference configuration, an observer dependent notion. All kinematic operations are in fact defined on the trajectory time-bundle immersed in space-time. This is a distinctive feature of the new approach which, as we will see, entails significant conceptual and methodological consequences. The issue of referential formulations will be introduced in proper geometric terms in Sect.8.5, as a mathematical trick useful to perform linear operations on a straightened out trajectory in which the motion is a simple time-translation.

8.4. Conservation law

The trajectory time-bundle is characterized by a *conservation law* concerning a maximal form defined on the time-vertical tangent bundle $\mathbb{V}(\mathcal{T}) \subset \mathbb{T}(\mathcal{T})$, that is the subbundle of vectors tangent to placements. The conservation law states that the integral of the maximal form over any placement is left invariant by the motion.

The conservation law of interest in electromagnetics is concerns the electric charge form $\boldsymbol{\rho}^3 \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{M}\text{XF}(\mathbb{T}\mathcal{T}))$, whose time-invariance along the motion is expressed by the condition

$$\int_{\boldsymbol{\Omega}_{t_1}} \boldsymbol{\rho}^3 = \int_{\boldsymbol{\Omega}_{t_2}} \boldsymbol{\rho}^3\,,$$

which, by the integral transformation formula

$$\int_{\boldsymbol{\Omega}_{t_2}} \boldsymbol{\rho}^3 = \int_{\boldsymbol{\Omega}_{t_1}} \boldsymbol{\varphi}_{(t_2-t_1)}{\downarrow}\boldsymbol{\rho}^3\,,$$

is equivalent to pull-back condition $\boldsymbol{\rho}^3 = \boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\rho}^3$ and to LIE differential condition $\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\boldsymbol{\rho}^3 := \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\rho}^3 = 0$ (G. Romano, 2007).

8.5. *Trajectory straightening*

According to the geometric viewpoint, the standard *referential formulation* is translated in the following *straightening out* statement concerning the trajectory. This is an observer dependent notion.

A *straight* trajectory is a product manifold $\boldsymbol{\Omega} \times \mathcal{Z}$, with $\boldsymbol{\Omega} \subset \mathcal{S}$. It is a time-bundle under the cartesian projection $\boldsymbol{\pi}_{\mathcal{Z},\boldsymbol{\Omega}\times\mathcal{Z}} \in \mathrm{C}^1(\boldsymbol{\Omega}\times\mathcal{Z}\,;\mathcal{Z})$. In a *straight* trajectory we may introduce a special motion, the *time-translation* $\mathrm{SHIFT}_\alpha \in \mathrm{C}^1(\boldsymbol{\Omega}\times\mathcal{Z}\,;\boldsymbol{\Omega}\times\mathcal{Z})$, defined by $\mathrm{SHIFT}_\alpha(\mathbf{x}\,,t) := (\mathbf{x}\,, t+\alpha)$.

Lemma 8.1 (Trajectory straightening). *Any nonlinear trajectory may be straightened by means of a time-bundle isomorphism* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,;\boldsymbol{\Omega}\times\mathcal{Z})$, *according to the commutative diagram*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\boldsymbol{\zeta}"] \arrow[d, "\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}"'] & \boldsymbol{\Omega}\times\mathcal{Z} \arrow[d, "\boldsymbol{\pi}_{\mathcal{Z},\boldsymbol{\Omega}\times\mathcal{Z}}"] \\ \mathcal{Z} \arrow[r, "\boldsymbol{id}_{\mathcal{Z}}"] & \mathcal{Z} \end{tikzcd} \quad \Longleftrightarrow \quad \boldsymbol{\pi}_{\mathcal{Z},\boldsymbol{\Omega}\times\mathcal{Z}} \circ \boldsymbol{\zeta} = \boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}\,.$$

in such a way that the motion is transformed into the time-translation $\mathrm{SHIFT}_\alpha \in \mathrm{C}^1(\boldsymbol{\Omega}\times\mathcal{Z}\,;\boldsymbol{\Omega}\times\mathcal{Z})$, *as depicted in the commutative diagram*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\boldsymbol{\varphi}_\alpha"] \arrow[d, "\boldsymbol{\zeta}"'] \arrow[dd, bend right=60, "\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}}"'] & \mathcal{T} \arrow[d, "\boldsymbol{\zeta}"'] \\ \boldsymbol{\Omega}\times\mathcal{Z} \arrow[r, "\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha"] \arrow[d, "\boldsymbol{\pi}_{\mathcal{Z},\boldsymbol{\Omega}\times\mathcal{Z}}"'] & \boldsymbol{\Omega}\times\mathcal{Z} \arrow[d, "\boldsymbol{\pi}_{\mathcal{Z},\boldsymbol{\Omega}\times\mathcal{Z}}"] \\ \mathcal{Z} \arrow[r, "\mathrm{SH}_\alpha"] & \mathcal{Z} \end{tikzcd} \quad \Longleftrightarrow \quad \begin{aligned} & \boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha \circ \boldsymbol{\zeta} = \boldsymbol{\zeta} \circ \boldsymbol{\varphi}_\alpha\,, \\ & \boldsymbol{\pi}_{\mathcal{Z},\boldsymbol{\Omega}\times\mathcal{Z}} \circ \boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha = \mathrm{SH}_\alpha \circ \boldsymbol{\pi}_{\mathcal{Z},\boldsymbol{\Omega}\times\mathcal{Z}}\,, \\ & \textit{with} \quad (\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha) = \mathrm{SHIFT}_\alpha\,. \end{aligned}$$

Proof. A placement $\boldsymbol{\Omega} \in \mathcal{T}(t)$ is mapped isomorphically by the motion $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ to another placement $(\boldsymbol{\Omega}_{(t+\alpha)}\,, t+\alpha) = \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}\,,t)$ in the trajectory. Choosing a map $\boldsymbol{\zeta} \in \mathrm{C}^1((\boldsymbol{\Omega}\,,t)\,;\mathbb{M}(t))$ with image $\boldsymbol{\zeta}(\boldsymbol{\Omega}\,,t) = (\boldsymbol{\Omega}\,,t)$, the isomorphism is defined on each time-fiber by

$$\boldsymbol{\zeta} := \mathrm{SHIFT}_\alpha \circ \boldsymbol{\zeta} \circ \boldsymbol{\varphi}_{-\alpha}\,,$$

which sends $(\boldsymbol{\Omega}_{(t+\alpha)}\,, t+\alpha)$ onto $(\boldsymbol{\Omega}\,, t+\alpha)$. ∎

In the standard terminology the *time-translation* over a *straight trajectory* is referred to as a *fixed reference placement*.

LIE time-derivatives along the motion are transformed by push according to the straightening isomorphism into parallel time-derivative according to the time-translation transport, i.e. in classical partial time derivatives, taken at fixed spatial point.

Indeed, for any $\mathbf{u} \in \mathrm{C}^1(\mathcal{T}\,; \mathbb{V}\mathcal{T})$ and $\mathbf{s} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{T}\mathrm{ENS}(\mathbb{V}\mathcal{T}))$, with push forward to the straightened trajectory $\mathbf{u}_{\mathrm{REF}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{u} \in \mathrm{C}^1(\boldsymbol{\Omega} \times \mathcal{Z}\,; \mathbb{T}\boldsymbol{\Omega})$ and $\mathbf{s}_{\mathrm{REF}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{s} \in \mathrm{C}^1(\boldsymbol{\Omega} \times \mathcal{Z}\,; \mathrm{T}\mathrm{ENS}(\mathbb{T}\boldsymbol{\Omega}))$, we have

$$(\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\varphi}_\alpha)\!\downarrow\!(\mathbf{s}_{\mathrm{REF}} \circ (\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\varphi}_\alpha)) = (\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\varphi}_\alpha)\!\Downarrow(\mathbf{s}_{\mathrm{REF}} \circ (\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\varphi}_\alpha)) = \mathbf{s}_{\mathrm{REF}} \circ \mathrm{SHIFT}_\alpha \,,$$

and hence, by the push-naturality property of LIE derivatives, it follows that

$$\boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{u}}\,\mathbf{s}) = \mathcal{L}_{\mathbf{u}_{\mathrm{REF}}}\,\mathbf{s}_{\mathrm{REF}} = \partial_{\alpha=0}\,(\mathbf{s}_{\mathrm{REF}} \circ \mathrm{SHIFT}_\alpha) \,.$$

Referential formulations provide the key tool to perform linear operations in a nonlinear geometric context. The time-rate of a material tensor field along the motion may be conveniently evaluated by pushing the tensor field to a straight trajectory by means of a straightening out isomorphism, by performing on the pushed tensor field the partial time-derivative and eventually pulling the result back to the actual trajectory.

9. Spatial and material fields

Let us now introduce a nomenclature motivated by the importance of making a carefully distinction between different kinds of tensor fields in CK which have peculiar physical meanings and are to be treated by different geometric tools. All the fields of interest in CK are based on the trajectory, with the only exception of the space-time metric field, see Sect.10.

Space-time bundle $\mathbb{T}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}$ is the tangent bundle $\mathbb{T}\mathbb{M}$ to the space-time manifold, with the fibration induced by the time-projection $\boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,; \mathcal{Z})$, and the base restricted to the immersed trajectory $\mathcal{T}^{\mathbb{M}}$.

Spatial bundle $\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}} \subset \mathbb{T}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}$ is the sub-bundle of *time-vertical* space-time vectors $\mathbf{u}^{\mathbb{M}}_{\mathcal{T}} \in \mathbb{V}_{\mathbf{e}}\mathbb{M} \subset \mathbb{T}_{\mathbf{e}}\mathbb{M}$, having a null time-projection, i.e. $T_{\mathbf{e}}\boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} \cdot \mathbf{u}^{\mathbb{M}}_{\mathcal{T}} = 0$.

Trajectory bundle $\mathbb{T}\mathcal{T}$ is the tangent bundle $\mathbb{T}\mathcal{T}$ to the trajectory, with the fibration induced by the time-projection $\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{Z})$.

Material bundle $\mathbb{V}\mathcal{T} \subset \mathbb{T}\mathcal{T}$ is the sub-bundle of vectors $\mathbf{u}_{\mathcal{T}} \in \mathbb{T}_{\mathbf{e}}\mathcal{T}(t)$ that are time-vertical $T_{\mathbf{e}}\boldsymbol{\pi}_{\mathcal{Z},\mathcal{T}} \circ \mathbf{u}_{\mathcal{T}} = 0$.

In the sequel the physical terminology *field* $\mathbf{s} \in \mathrm{C}^1(\mathbb{B}\,;\mathbb{M})$ will be adopted, corresponding to the geometric terminology *section of the fiber-bundle* $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{B})$, see Sect.7. This means that $\mathbf{s}(\mathbf{x}) \in \mathbb{M}(\mathbf{x})$.

Space-time tensor fields $\mathbf{s}_{\mathbb{M}} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}}\,;\mathrm{T{\scriptsize ENS}}(\mathbb{T}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}))$ are sections of the space-time tensor bundle $\mathrm{T{\scriptsize ENS}}(\mathbb{T}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}})$.

Spatial tensor fields $\mathbf{s}^{\mathbb{M}}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}}\,;\mathrm{T{\scriptsize ENS}}(\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}))$ are sections of the spatial tensor bundle $\mathrm{T{\scriptsize ENS}}(\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}})$.

Spatial vector fields can be considered as time-dependent tangent fields to the ambient space manifold defined on the current placement $\mathbf{u}^{\mathcal{S}}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}\,;\mathbb{T}\mathcal{S})$. Sections of pull-back bundles $\mathbb{T}\mathbb{M}_{\mathcal{T}} = \mathbf{i}_{\mathbb{M},\mathcal{T}}{\downarrow}\mathbb{T}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}$ and $\mathbb{V}\mathbb{M}_{\mathcal{T}} = \mathbf{i}_{\mathbb{M},\mathcal{T}}{\downarrow}\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}$ will still be called *space-time* and *spatial* tensor fields.

Trajectory tensor fields $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{T{\scriptsize ENS}}(\mathbb{T}\mathcal{T}))$ are sections of the trajectory tensor bundle $\mathrm{T{\scriptsize ENS}}(\mathbb{T}\mathcal{T})$.

Material tensor fields $\mathbf{u}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{T{\scriptsize ENS}}(\mathbb{V}\mathcal{T}))$ are sections of the material tensor bundle $\mathrm{T{\scriptsize ENS}}(\mathbb{V}\mathcal{T})$.

Material vector fields can be considered as time-dependent tangent fields to the current placement manifold $\mathbf{u}_{\boldsymbol{\varphi},t} \in \mathrm{C}^1(\boldsymbol{\Omega}\,;\mathbb{T}\boldsymbol{\Omega})$. The immersion preserves simultaneity, so $\mathbb{V}\mathcal{T}^{\mathbb{M}} = \mathbb{V}(\mathbf{i}_{\mathbb{M},\mathcal{T}}{\uparrow}\mathcal{T}) = \mathbf{i}_{\mathbb{M},\mathcal{T}}{\uparrow}(\mathbb{V}\mathcal{T})$. In the sequel we will drop the subscript $()_{\mathcal{T}}$ whenever confusion may not occur.

10. Space-time, spatial and material metrics

The *space-time metric* $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{P{\scriptsize OS}}(\mathbb{T}\mathbb{M}))$ is defined by the pull-back

$$\mathbf{g}_{\mathrm{E}} := \boldsymbol{\pi}_{\mathcal{S},\mathbb{M}}{\downarrow}\mathbf{g}_{\mathcal{S}} + \boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}}{\downarrow}\mathbf{g}_{\mathcal{Z}}\,,$$

explicitly $\mathbf{g}_{\mathbb{M}}(\mathbf{u}\,,\mathbf{w}) := \mathbf{g}_{\mathcal{S}}(T\boldsymbol{\pi}_{\mathcal{S},\mathbb{M}} \cdot \mathbf{u}\,, T\boldsymbol{\pi}_{\mathcal{S},\mathbb{M}} \cdot \mathbf{w}) + \mathbf{g}_{\mathcal{Z}}(T\boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} \cdot \mathbf{u}\,, T\boldsymbol{\pi}_{\mathcal{Z},\mathbb{M}} \cdot \mathbf{w})$ for all vector fields $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M})$ tangent to the events manifold.

The *spatial metric* field $\mathbf{g} \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{Pos}(\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}))$ is the restriction of the space-time metric to spatial vector fields. The time-line metric plays then no role, so that

$$\mathbf{g} := \boldsymbol{\pi}_{\mathcal{S},\mathbb{M}}\!\downarrow\!\mathbf{g}_{\mathcal{S}}\,.$$

In turn, the *material metric* $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{Pos}(\mathbb{V}\mathcal{T}))$ is the pull-back of the spatial metric to the material bundle

$$\mathbf{g}_{\mathcal{T}} := \mathbf{i}_{\mathbb{M},\mathcal{T}}\!\downarrow\!\mathbf{g}\,.$$

The *material metric* $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{Pos}(\mathbb{V}\mathcal{T}))$ is a non-degenerate covariant tensor field that provides the standard tool for alteration of material tensors.

11. Spatial and material projections

The *spatial projection* $\mathbf{P}_{\mathbb{V}\mathbb{M}} \in \mathrm{C}^1(\mathbb{T}\mathbb{M}\,; \mathbb{V}\mathbb{M})$ is a *homomorphism* from the tangent space-time bundle $\mathbb{T}\mathbb{M}$ onto the time-vertical bundle $\mathbb{V}\mathbb{M}$, characterized by the fiberwise $\mathbf{g}_{\mathrm{E}}$-orthogonality property

$$\mathbf{g}_{\mathrm{E}}(\mathbf{a}_{\mathbb{M}} - \mathbf{P}_{\mathbb{V}\mathbb{M}} \cdot \mathbf{a}_{\mathbb{M}}\,, \mathbf{P}_{\mathbb{V}\mathbb{M}} \cdot \mathbf{b}_{\mathbb{M}}) = 0\,, \quad \mathbf{a}_{\mathbb{M}}, \mathbf{b}_{\mathbb{M}} \in \mathbb{T}\mathbb{M}\,,$$

and hence is $\mathbf{g}_{\mathrm{E}}$-symmetric and idempotent $\mathbf{P}_{\mathbb{V}\mathbb{M}} \circ \mathbf{P}_{\mathbb{V}\mathbb{M}} = \mathbf{P}_{\mathbb{V}\mathbb{M}}$, being, $\forall\, \mathbf{a}_{\mathbb{M}}, \mathbf{b}_{\mathbb{M}} \in \mathbb{T}\mathbb{M}$

$$\mathbf{g}_{\mathrm{E}}(\mathbf{P}_{\mathbb{V}\mathbb{M}} \cdot \mathbf{a}_{\mathbb{M}}\,, \mathbf{b}_{\mathbb{M}}) = \mathbf{g}_{\mathrm{E}}(\mathbf{P}_{\mathbb{V}\mathbb{M}} \cdot \mathbf{b}_{\mathbb{M}}\,, \mathbf{a}_{\mathbb{M}}) = \mathbf{g}_{\mathrm{E}}(\mathbf{P}_{\mathbb{V}\mathbb{M}} \cdot \mathbf{a}_{\mathbb{M}}\,, \mathbf{P}_{\mathbb{V}\mathbb{M}} \cdot \mathbf{b}_{\mathbb{M}})\,.$$

The complementary *temporal projection* $\mathbf{P}_{\mathbb{Z}\mathbb{M}} := \mathbf{I} - \mathbf{P}_{\mathbb{V}\mathbb{M}}$ is likewise fiberwise $\mathbf{g}_{\mathrm{E}}$-orthogonal.

The projection $\boldsymbol{\Pi}^{\mathbb{M}} \in \mathrm{C}^1(\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}\,; \mathbb{V}\mathcal{T}^{\mathbb{M}})$, from the spatial bundle onto the immersed material sub-bundle $\mathbb{V}\mathcal{T}^{\mathbb{M}} = \mathbb{V}\mathbb{M} \cap \mathbb{T}\mathcal{T}^{\mathbb{M}}$, is characterized by the $\mathbf{g}$-orthogonality property

$$\mathbf{g}(\mathbf{a}^{\mathbb{M}}_{\mathcal{T}} - \boldsymbol{\Pi}^{\mathbb{M}} \cdot \mathbf{a}^{\mathbb{M}}_{\mathcal{T}}\,, \mathbf{i}_{\mathbb{M},\mathcal{T}}\!\uparrow\!\mathbf{b}_{\mathcal{T}}) = 0\,, \quad \mathbf{a}^{\mathbb{M}}_{\mathcal{T}} \in \mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}\,, \quad \forall\, \mathbf{b}_{\mathcal{T}} \in \mathbb{V}\mathcal{T}\,.$$

The *material projector* $\boldsymbol{\Pi} \in \mathrm{C}^1(\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}\,; \mathbb{V}\mathcal{T})$, from the spatial bundle onto the material bundle, is then defined by the identity

$$\mathbf{g}_{\mathcal{T}}(\boldsymbol{\Pi} \cdot \mathbf{a}^{\mathbb{M}}_{\mathcal{T}}\,, \mathbf{b}_{\mathcal{T}}) = \mathbf{g}(\mathbf{a}^{\mathbb{M}}_{\mathcal{T}}\,, \mathbf{i}_{\mathbb{M},\mathcal{T}}\!\uparrow\!\mathbf{b}_{\mathcal{T}})\,, \quad \forall\, \mathbf{a}^{\mathbb{M}}_{\mathcal{T}} \in \mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}\,, \quad \forall\, \mathbf{b}_{\mathcal{T}} \in \mathbb{V}\mathcal{T}\,,$$

which implies that $\mathbf{\Pi}^A = \mathbf{i}_{\mathbb{M},\mathcal{T}}\!\uparrow \in \mathrm{C}^1(\mathbb{V}\mathcal{T}\,;\mathbb{V}\mathcal{T}^{\mathbb{M}})$ is the $(\mathbf{g}_{\mathcal{T}}, \mathbf{g})$-adjoint transformation fulfilling the commutative diagram

$$\begin{array}{ccc} \mathbb{V}\mathcal{T} & \xrightarrow{\mathbf{\Pi}^A} & \mathbb{V}\mathcal{T}^{\mathbb{M}} \\ {\scriptstyle \boldsymbol{\pi}_{\mathcal{T},\mathbb{T}\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\pi}_{\mathbb{M},\mathbb{T}\mathbb{M}}} \\ \mathcal{T} & \xrightarrow{\mathbf{i}_{\mathbb{M},\mathcal{T}}} & \mathcal{T}^{\mathbb{M}} \end{array} \quad \Longleftrightarrow \quad \mathbf{i}_{\mathbb{M},\mathcal{T}} \circ \boldsymbol{\pi}_{\mathcal{T},\mathbb{T}\mathcal{T}} = \boldsymbol{\pi}_{\mathbb{M},\mathbb{T}\mathbb{M}} \circ \mathbf{\Pi}^A \,.$$

Then $\mathbf{\Pi} := \mathbf{i}_{\mathbb{M},\mathcal{T}}\!\downarrow \circ\, \mathbf{\Pi}^{\mathbb{M}}$ that is $\mathbf{\Pi}^{\mathbb{M}} = \mathbf{\Pi}^A \circ \mathbf{\Pi}$. For any mixed tensor field $\mathbf{L} \in \mathrm{C}^1(\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}\,;\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}})$ and any pair of material vectors $\mathbf{a}_{\mathcal{T}}, \mathbf{b}_{\mathcal{T}} \in \mathbb{V}\mathcal{T}$, we have that

$$\mathbf{g}((\mathbf{L} \circ \mathbf{\Pi}^A)\mathbf{a}_{\mathcal{T}}\,, \mathbf{\Pi}^A \mathbf{b}_{\mathcal{T}}) = \mathbf{g}_{\mathcal{T}}((\mathbf{\Pi} \circ \mathbf{L} \circ \mathbf{\Pi}^A) \cdot \mathbf{a}_{\mathcal{T}}\,, \mathbf{b}_{\mathcal{T}})\,,$$

with $\mathbf{\Pi} \circ \mathbf{L} \circ \mathbf{\Pi}^A \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{Mix}(\mathbb{V}\mathcal{T}))$ mixed material tensor field.

12. Time–rates and time-invariance

The evaluation of time-rates of material and spatial tensors requires the comparison of values of these fields at different times along the motion. This operation is not trivial because the fields to be compared do not belong to the same linear space.

It is therefore compelling to transform the field values at a variable time into fields values pertaining to the evaluation time. This transformation may be carried out separately for material and spatial fields, as dictated by the *geometric paradigm*. We may then give the following definitions.

Definition 12.1 (Time rate and invariance of material fields). *The time rate of a material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,;\mathrm{Tens}(\mathbb{V}\mathcal{T}))$ is given by the* Lie *derivative along the motion*

$$\dot{\mathbf{s}}_{\mathcal{T}} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \mathbf{s}_{\mathcal{T}} = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha\!\downarrow(\mathbf{s}_{\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha)\,.$$

Accordingly, time invariance of a material tensor field along the motion, is expressed by the differential condition $\dot{\mathbf{s}}_{\mathcal{T}} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \mathbf{s}_{\mathcal{T}} = 0$, equivalent to the pull-back property

$$\mathbf{s}_{\mathcal{T}} = \boldsymbol{\varphi}_\alpha\!\downarrow \mathbf{s}_{\mathcal{T}} := \boldsymbol{\varphi}_\alpha\!\downarrow(\mathbf{s}_{\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha)\,.$$

Definition 12.2 (Time rate and invariance of spatial fields). *The time rate of a spatial tensor field* $\mathbf{s}^{\mathbb{M}}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}} ; \mathrm{T\small{ENS}}(\mathbb{V}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}))$ *along the motion is given by the parallel derivative*

$$\dot{\mathbf{s}}^{\mathbb{M}}_{\mathcal{T}} := \nabla_{\mathbf{v}^{\mathbb{M}}_{\mathcal{T}}} \mathbf{s}^{\mathbb{M}}_{\mathcal{T}} = \partial_{\alpha=0}\, \boldsymbol{\varphi}^{\mathbb{M}}_{\alpha}\!\Downarrow (\mathbf{s}^{\mathbb{M}}_{\mathcal{T}} \circ \boldsymbol{\varphi}^{\mathbb{M}}_{\alpha}) \,.$$

Accordingly, invariance of a spatial tensor field along the motion, is expressed by the differential condition $\dot{\mathbf{s}}^{\mathbb{M}}_{\mathcal{T}} := \nabla_{\mathbf{v}^{\mathbb{M}}_{\mathcal{T}}} \mathbf{s}^{\mathbb{M}}_{\mathcal{T}} = 0$, *equivalent to the parallel transport property*

$$\mathbf{s}^{\mathbb{M}}_{\mathcal{T}} = \boldsymbol{\varphi}^{\mathbb{M}}_{\alpha}\!\Downarrow \mathbf{s}^{\mathbb{M}}_{\mathcal{T}} = \boldsymbol{\varphi}^{\mathbb{M}}_{\alpha}\!\Downarrow (\mathbf{s}^{\mathbb{M}}_{\mathcal{T}} \circ \boldsymbol{\varphi}^{\mathbb{M}}_{\alpha}) \,.$$

We underline that, as resulting from the definitions above, invariance of a material field along the motion is a natural notion, i.e. fulfill the principle of Geometric Naturality, being determined only by the motion itself. On the contrary, the notion of invariance of a spatial field along the motion depends on the choice of a linear connection in the space-time manifold.

Contrary to the familiar meaning, material tensors evaluated at different instants of time along the trajectory, might be said to be *the same* if they are related by the push-pull transformation defined by the displacement map.

This geometric notion clarifies and corrects statements of formulations where the expression *the same* is assumed to mean that involved tensors have a *null difference*. Indeed the *difference* of tensors not belonging to *the same* linear tensor fiber is an undefined operation.

13. Frame-invariance

A *change of frame* is an automorphism $\boldsymbol{\zeta}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{M} ; \mathbb{M})$ of the events manifold. A *relative motion* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T} ; \mathcal{T}_{\boldsymbol{\zeta}})$ is a diffeomorphism between trajectory time-bundles, induced by a change of frame according to the commutative diagram

$$\begin{tikzcd} \mathbb{M} \arrow[rrr, bend left, "\boldsymbol{\zeta}_{\mathbb{M}}"] & \mathcal{T} \arrow[l, "\mathbf{i}_{\mathbb{M},\mathcal{T}}"'] \arrow[r, shift left, "\boldsymbol{\zeta}"] & \mathcal{T}_{\boldsymbol{\zeta}} \arrow[l, shift left, "\boldsymbol{\zeta}^{-1}"] \arrow[r, "\mathbf{i}_{\mathbb{M},\mathcal{T}_{\boldsymbol{\zeta}}}"] & \mathbb{M} \end{tikzcd} \quad \Longleftrightarrow \quad \mathbf{i}_{\mathbb{M},\mathcal{T}_{\boldsymbol{\zeta}}} \circ \boldsymbol{\zeta} = \boldsymbol{\zeta}_{\mathbb{M}} \circ \mathbf{i}_{\mathbb{M},\mathcal{T}} \,.$$

Trajectories and body motions $\boldsymbol{\varphi}_{\alpha} \in \mathrm{C}^1(\mathcal{T} ; \mathcal{T})$ and $\boldsymbol{\zeta}{\uparrow}\boldsymbol{\varphi}_{\alpha} \in \mathrm{C}^1(\mathcal{T}_{\boldsymbol{\zeta}} ; \mathcal{T}_{\boldsymbol{\zeta}})$, as seen by observers in relative motion $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T} ; \mathcal{T}_{\boldsymbol{\zeta}})$, are related by the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\boldsymbol{\zeta}} \arrow[r, "\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha"] & \mathcal{T}_{\boldsymbol{\zeta}} \\ \mathcal{T} \arrow[u, "\boldsymbol{\zeta}\uparrow"] \arrow[r, "\boldsymbol{\varphi}_\alpha"] & \mathcal{T} \arrow[u, "\boldsymbol{\zeta}\uparrow"'] \end{tikzcd} \qquad \Longleftrightarrow \qquad (\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha) \circ \boldsymbol{\zeta} = \boldsymbol{\zeta} \circ \boldsymbol{\varphi}_\alpha \,.$$

Definition 13.1 (Frame-invariance of material fields). *A material tensor field* $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{T}\small{\mathrm{ENS}}(\mathbb{V}\mathcal{T}))$ *is frame-invariant if under the action of a relative motion* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}})$ *it varies according to push*

$$\mathbf{s}_{\mathcal{T}_{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}\uparrow\mathbf{s}_{\mathcal{T}} \,.$$

Lemma 13.1 (Frame-invariance of trajectory speed). *The trajectory speed is frame-invariant*

$$\mathbf{v}_{\mathcal{T}_{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}\uparrow\mathbf{v}_{\mathcal{T}} \,.$$

Proof. By definition $\mathbf{v}_{\mathcal{T}} := \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha$ so that $\boldsymbol{\varphi}_\alpha = \mathbf{Fl}_\alpha^{\mathbf{v}_{\mathcal{T}}}$. A direct computation then yields

$$\mathbf{v}_{\mathcal{T}_{\boldsymbol{\zeta}}} := \partial_{\alpha=0}\,\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi}_\alpha = \partial_{\alpha=0}\,(\boldsymbol{\zeta} \circ \boldsymbol{\varphi}_\alpha \circ \boldsymbol{\zeta}^{-1}) = (T\boldsymbol{\zeta} \cdot \mathbf{v}_{\mathcal{T}}) \circ \boldsymbol{\zeta}^{-1} = \boldsymbol{\zeta}\uparrow\mathbf{v}_{\mathcal{T}} \,,$$

gives the formula. ∎

A Euclid *change of observer* means that $\boldsymbol{\zeta}_{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}_{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}})$ is isometric, that is

$$\mathbf{g}_{\mathcal{T}} = \boldsymbol{\zeta}_{\mathrm{ISO}}\downarrow\mathbf{g}_{\mathcal{T}_{\boldsymbol{\zeta}_{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}}} = (T\boldsymbol{\zeta}_{\mathrm{ISO}})^* \cdot \mathbf{g}_{\mathcal{T}_{\boldsymbol{\zeta}_{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}}} \cdot T\boldsymbol{\zeta}_{\mathrm{ISO}}$$

or explicitly

$$\mathbf{g}_{\mathcal{T}}(\mathbf{u}_{\mathcal{T}}\,, \mathbf{w}_{\mathcal{T}}) = \mathbf{g}_{\mathcal{T}_{\boldsymbol{\zeta}_{\mathrm{ISO}}\uparrow\boldsymbol{\varphi}}}(T\boldsymbol{\zeta}_{\mathrm{ISO}} \cdot \mathbf{u}_{\mathcal{T}}\,, T\boldsymbol{\zeta}_{\mathrm{ISO}} \cdot \mathbf{w}_{\mathcal{T}}) \circ \boldsymbol{\zeta}_{\mathrm{ISO}} \,.$$

Invariance under Euclid change of observer is called Euclid *Frame Invariance* (EFI). The material metric tensor is Euclid frame-invariant by definition. That all other material tensors are Euclid frame-invariant is the statement of a constitutive axiom. The following property follows from push-naturality of Lie derivatives and Lemma 13.1.

Lemma 13.2 (Push naturality of Lie time-derivatives). *The* LIE *time-derivative of a material tensor field* $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{TENS}(\mathbb{V}\mathcal{T}))$ *and the* LIE *time-derivative of its push by a relative motion* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}})$ *, are related by push*

$$\mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\zeta}}}}(\boldsymbol{\zeta}\!\uparrow\!\mathbf{s}_{\mathcal{T}}) = \boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\mathbf{s}_{\mathcal{T}})\,.$$

The related result concerning invariance is provided below.

Proposition 13.1 (Invariance of convective time-derivatives). *The invariance of a material tensor field* $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathrm{TENS}(\mathbb{V}\mathcal{T}))$ *with respect to a relative motion* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}})$ *, implies invariance of the convective time-derivative, viz. of its* LIE *derivative along the motion*

$$\mathbf{s}_{\mathcal{T}_{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{s}_{\mathcal{T}} \quad \Longrightarrow \quad \mathcal{L}_{\mathbf{v}_{\mathcal{T}_{\boldsymbol{\zeta}}}}\mathbf{s}_{\mathcal{T}_{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}})\,.$$

Proof. The result is a direct consequence of Lemma 13.2. ■

Material tensors evaluated, at a given time instant along the trajectory, by different observers in relative isometric motion, might be said to appear *the same* when they are related by push-pull transformation according to the relative motion. In literature, *the same* is sometimes assumed more or less explicitly to mean that involved tensors, as seen by different observers, have a *null difference*. But *difference* of tensors, based on trajectories detected by different observers, is an undefined operation.

14. Galilei invariance

Definition 14.1 (Translational relative motion). *A relative motion* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}})$ *is translational according to a spatial connection, if the relevant spatial velocity field* $\mathbf{v}_{\boldsymbol{\zeta},\mathcal{S}} := \mathbf{P}_{\mathbb{VM}} \cdot \mathbf{v}_{\boldsymbol{\zeta}}$ *has a vanishing parallel derivative:*

$$\nabla \mathbf{v}_{\boldsymbol{\zeta},\mathcal{S}} = 0\,.$$

Definition 14.2 (Stationary relative motion). *A relative motion* $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}})$ *is stationary if the partial time-derivative of the relevant spatial velocity field vanishes, viz.:*

$$\nabla_{\mathbf{v}_{\boldsymbol{\zeta},\mathcal{Z}}}\,\mathbf{v}_{\boldsymbol{\zeta},\mathcal{S}} = 0\,.$$

The acceleration field of the relative motion is given by

$$\nabla_{\mathbf{v}_\zeta} \mathbf{v}_\zeta := \partial_{\alpha=0} \, \boldsymbol{\zeta}_\alpha \Downarrow (\mathbf{v}_\zeta \circ \boldsymbol{\zeta}_\alpha) = \nabla_{\mathbf{v}_{\zeta,\mathcal{Z}}} \mathbf{v}_{\zeta,\mathcal{S}} + \nabla_{\mathbf{v}_{\zeta,\mathcal{S}}} \mathbf{v}_{\zeta,\mathcal{S}} \, ,$$

If the relative motion is stationary and translational the acceleration vanishes. By definition, the relative motion $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T} \, ; \mathcal{T}_\zeta)$ between two GALILEI[36] observers is stationary and translational, and parallel transport and push along relative motions are coincident.

A GALILEI transformation is metric-preserving and hence also volume-preserving so that $\boldsymbol{\zeta}{\downarrow}\boldsymbol{\mu} = \boldsymbol{\mu}$. Setting $\boldsymbol{\mu} \cdot \mathbf{F} = \boldsymbol{\omega}^2_{\mathbf{F}}$ we then have:

$$\boldsymbol{\zeta}{\downarrow}(\boldsymbol{\mu} \cdot \mathbf{F}) = (\boldsymbol{\zeta}{\downarrow}\boldsymbol{\mu}) \cdot (\boldsymbol{\zeta}{\downarrow}\mathbf{F}) = \boldsymbol{\mu} \cdot (\boldsymbol{\zeta}{\downarrow}\mathbf{F}) \, .$$

so that the two-form $\boldsymbol{\omega}^2_{\mathbf{F}}$ is GALILEI invariant iff the vector field $\mathbf{F}$ is such. Moreover, taking the time derivative and applying the LEIBNIZ rule:

$$\mathcal{L}_\zeta \, (\boldsymbol{\mu} \cdot \mathbf{F}) = (\mathcal{L}_\zeta \, \boldsymbol{\mu}) \cdot \mathbf{F} + \boldsymbol{\mu} \cdot (\mathcal{L}_\zeta \, \mathbf{F}) \, .$$

Being $\mathcal{L}_\zeta \, \boldsymbol{\mu} = 0$. It follows that:

$$\mathcal{L}_\zeta \, (\boldsymbol{\mu} \cdot \mathbf{F}) = \boldsymbol{\mu} \cdot (\mathcal{L}_\zeta \, \mathbf{F}) \, ,$$

and we may conclude that the LIE derivative of the two-form field $\boldsymbol{\omega}^2_{\mathbf{F}}$ is GALILEI invariant iff the LIE derivative of the vector field $\mathbf{F}$ is such.

15. Space-time and trajectory connections

Connections $\nabla^{\mathcal{S}}$ in the space manifold and $\nabla^{\mathcal{Z}}$ in the time line, naturally induce a connection ∇ in the space-time manifold $\mathbb{M}$ by defining the transports of spatial and temporal projections of a space-time vector.

15.1. Space-time connection

Definition 15.1 (Space-time connection). *The parallel transport of a tangent vector* $\mathbf{u}_{\mathbb{M}}(\mathbf{e}) \in \mathbb{T}_{\mathbf{e}}\mathbb{M}$ *, along a space-time curve* $\mathbf{c} \in \mathrm{C}^1(\mathcal{R} \, ; \mathbb{M})$ *through* $\mathbf{e} = \mathbf{c}(0)$ *, is performed by considering the spatial (time) projection, in transporting the projected vector according to the induced spatial (time) parallel*

[36] GALILEO GALILEI (1564-1642) Italian scientist.

transport along the space-time curve, denoted by $\Uparrow^{\mathbb{M},\mathcal{S}}$ $(\Uparrow^{\mathbb{M},I}$ *) and in defining*

$$\boldsymbol{\pi}\uparrow(\mathbf{c}(\lambda)\Uparrow \mathbf{u}_{\mathbb{M}}(\mathbf{e})) := \mathbf{c}(\lambda)\Uparrow^{\mathbb{M},\mathcal{S}} \mathbf{u}_{\mathbb{M}}(\mathbf{e})\,,$$

$$\boldsymbol{\pi}_{\mathbb{M}_{\mathcal{Z}},\mathbb{M}}\uparrow(\mathbf{c}(\lambda)\Uparrow \mathbf{u}_{\mathbb{M}}(\mathbf{e})) := \mathbf{c}(\lambda)\Uparrow^{\mathbb{M},I} \mathbf{u}_{\mathbb{M}}(\mathbf{e})\,.$$

The parallel derivative, along the tangent vector $\mathbf{v}_{\mathbb{M}}(\mathbf{e}) = \partial_{\lambda=0}\,\mathbf{c}(\lambda) \in \mathbb{T}_{\mathbf{e}}\mathbb{M}$ *based at* $\mathbf{e} = \mathbf{c}(0)$ *, of a tangent vector field* $\mathbf{u}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}\mathbb{M})$ *, defined on a segment of the space-time curve* $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,;\mathbb{M})$ *around* $\mathbf{e} \in \mathbb{M}$ *, is accordingly given by*

$$\nabla_{\mathbf{v}_{\mathbb{M}}(\mathbf{e})}\mathbf{u}_{\mathbb{M}} := \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow(\mathbf{u}_{\mathbb{M}}\circ\mathbf{c})(\lambda)\,.$$

In NLCEM all fields of interest for parallel differentiation, other than the metric tensor field, are spatial fields whose domain is the space-time immersion $\mathcal{T}^{\mathbb{M}} \subset \mathbb{M}$ of the trajectory. The space-time curve $\mathbf{c} \in \mathrm{C}^1(\mathcal{R}\,;\mathcal{T}^{\mathbb{M}})$ must then lie in the immersed trajectory with velocity $\mathbf{v}_{\mathbb{M}}(\mathbf{e}) \in \mathbb{T}_{\mathbf{e}}\mathcal{T}^{\mathbb{M}}$.

Remark 15.1 (Material time-derivatives). *The material time-derivative*

$$\nabla_{\mathbf{v}}\mathbf{s} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}}\,;\mathrm{T\small{ENS}}(\mathbb{T}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}))$$

of a space-time tensor field $\mathbf{s} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}}\,;\mathrm{T\small{ENS}}(\mathbb{T}\mathbb{M}_{\mathcal{T}^{\mathbb{M}}}))$ *is the parallel derivative in direction of the space-time trajectory velocity* $\mathbf{v} \in \mathrm{C}^1(\mathcal{T}^{\mathbb{M}}\,;\mathbb{T}\mathcal{T}^{\mathbb{M}})$ *defined by*

$$\nabla_{\mathbf{v}}\mathbf{s} := \partial_{\alpha=0}\,\boldsymbol{\varphi}_{\alpha}\Downarrow(\mathbf{s}\circ\boldsymbol{\varphi}_{\alpha})\,.$$

Let us consider the orthogonal decomposition

$$\mathbf{v} = \mathbf{v}_{\mathcal{Z}} + \mathbf{v}_{\mathcal{S}}\,,\quad \mathbf{v}_{\mathcal{Z}} := \mathbf{P}_{\mathbb{Z}\mathbb{M}}(\mathbf{v})\,,\quad \mathbf{v}_{\mathcal{S}} := \mathbf{P}_{\mathbb{V}\mathbb{M}}(\mathbf{v}_{\mathbb{M}})\,.$$

The celebrated Euler[37] *formula, for the* material (or substantial) time-derivative *of a space-time field along the motion, is in fact the statement of additivity of the space-time connection* ∇

$$\nabla_{\mathbf{v}}\mathbf{s} = \nabla_{\mathbf{v}_{\mathcal{Z}}+\mathbf{v}_{\mathcal{S}}}\mathbf{s} = \nabla_{\mathbf{v}_{\mathcal{Z}}}\mathbf{s} + \nabla_{\mathbf{v}_{\mathcal{S}}}\mathbf{s}\,.$$

Taking into account that $\mathbf{v}_{\mathcal{Z}} = \mathbf{P}_{\mathbb{Z}\mathbb{M}}(\mathbf{v}) = 1$ *, so that* $\nabla_{\mathbf{v}}\mathbf{v}_{\mathcal{Z}} = 0$ *, the formula for the acceleration writes*

$$\nabla_{\mathbf{v}}\mathbf{v} = \nabla_{(\mathbf{v}_{\mathcal{Z}}+\mathbf{v}_{\mathcal{S}})}\mathbf{v}_{\mathcal{S}} = \nabla_{\mathbf{v}_{\mathcal{Z}}}\mathbf{v}_{\mathcal{S}} + \nabla_{\mathbf{v}_{\mathcal{S}}}\mathbf{v}_{\mathcal{S}}\,.$$

[37] Leonhard Euler (1707-1783) Swiss scientist.

These formulae are however applicable only when the trajectory manifold is, at least locally, a product space-time manifold, because then neither $\mathbf{v}_{\mathcal{Z}}$ *nor* $\mathbf{v}_{\mathcal{S}}$ *can point out of the trajectory and hence temporal and spatial partial derivatives are feasible. In the kinematics of lower dimensional continua these split formulae are therefore not applicable.*

Definition 15.2 (Galilei time-invariance). *Time-invariance of a spatial tensor field* $\boldsymbol{\alpha}$ *according to a* Galilei *observer means that*

$$\nabla_{\mathbf{v}_{\mathcal{Z}}} \boldsymbol{\alpha} = 0 \,.$$

where $\mathbf{v}_{\mathcal{Z}}$ *is the time-component of the measured space-time velocity.*

16. Electromagnetic induction: standard treatment

A noteworthy physical application of the theory of differential forms and integration on manifolds is to the laws of Electromagnetism, see (É. Cartan, 1924), (Deschamps, 1970). In fact this is the natural context for the mathematical modeling of these physical phenomena, more than the usual vector calculus which leads unavoidably to a confused coincident representations of geometrically distinct objects.

Let $\{\mathcal{S}\,,\mathbf{g}\}$ be the Euclid ambient 3-D affine manifold without boundary, endowed with the metric tensor field $\mathbf{g}$ and the induced the volume 3-form $\boldsymbol{\mu}$.

The geometric objects involved in electrodynamics are *impair* and *pair* exterior forms related to vector and scalar fields by linear isomorphisms generated by the metric tensor (for one-forms) and by the volume form (for two-forms and three-forms), as explicitly illustrated in the following lists.

- Faraday law:

$\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{g} \cdot \mathbf{E}$ electric circulation (one-form , vector field) ,

$\boldsymbol{\omega}^2_{\mathbf{B}} = \boldsymbol{\mu} \cdot \mathbf{B}$ magnetic vortex (two-form , *impair* vector field) ,

- Ampère law:

$\boldsymbol{\omega}^1_{\mathbf{H}} = \mathbf{g} \cdot \mathbf{H}$ magnetic winding (*impair* one-form , *impair* vector field) ,

$\boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\mu} \cdot \mathbf{D}$ electric flux (*impair* two-form , vector field) ,

$\boldsymbol{\omega}^2_{\mathbf{J}} = \boldsymbol{\mu} \cdot \mathbf{J}$ current flux (*impair* two-form , vector field) ,

$\boldsymbol{\rho}^3 = \rho\, \boldsymbol{\mu}$ electric charge (*impair* three-form , scalar field) .

In engineering and physics literature, it is customary to express the laws of electromagnetic induction in terms of the spatial vector fields $\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{J}, \mathbf{D} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{V}\mathbb{M}_{\mathcal{T}\mathbb{M}})$ and of the scalar field $\rho \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{FUN}(\mathbb{V}\mathbb{M}_{\mathcal{T}\mathbb{M}}))$, electric charge density per unit volume, by the integral relations

$$\oint_{\partial\boldsymbol{\Sigma}^{\mathrm{IN}}} \mathbf{g} \cdot \mathbf{E} = - \int_{\boldsymbol{\Sigma}^{\mathrm{IN}}} \boldsymbol{\mu} \cdot (\nabla_{\mathbf{v}_{\mathcal{Z}}} \mathbf{B}) \qquad \text{HENRY-FARADAY}(1831)$$

$$\oint_{\partial\boldsymbol{\Omega}^{\mathrm{OUT}}} \boldsymbol{\mu} \cdot \mathbf{B} = 0 \qquad \text{GAUSS}(1831)$$

$$\oint_{\partial\boldsymbol{\Sigma}^{\mathrm{OUT}}} \mathbf{g} \cdot \mathbf{H} = \int_{\boldsymbol{\Sigma}^{\mathrm{OUT}}} \boldsymbol{\mu} \cdot (\nabla_{\mathbf{v}_{\mathcal{Z}}} \mathbf{D} + \mathbf{J}) \qquad \text{MAXWELL}(1861)\text{-AMPÈRE}(1826)$$

$$\oint_{\partial\boldsymbol{\Omega}^{\mathrm{OUT}}} \boldsymbol{\mu} \cdot \mathbf{D} = \int_{\boldsymbol{\Omega}^{\mathrm{OUT}}} \rho\, \boldsymbol{\mu} \qquad \text{GAUSS}(1835)$$

with $\nabla_{\mathbf{v}_{\mathcal{Z}}} \mathbf{D}$ and $\nabla_{\mathbf{v}_{\mathcal{Z}}} \mathbf{B}$ partial time-derivatives, as seen by an observer, $\boldsymbol{\Sigma}$ a bounded connected surface and $\boldsymbol{\Omega}$ bounded connected domain in $\mathcal{S}$. Applying AMPÈRE law to closed surfaces $\boldsymbol{\Sigma} = \partial\boldsymbol{\Omega}$, we infer that

$$\nabla_{\mathbf{v}_{\mathcal{Z}}} \rho + \operatorname{div} \mathbf{J} = 0$$

which expresses the so called *equation of continuity*.

In all equations above, the time-change of the surface $\boldsymbol{\Sigma}$ and of the domain $\boldsymbol{\Omega}$, as seen by a GALILEI observer, are not considered, neither the effect of a change of observer is taken into acount. A critically discussion about these equations, which are customary in literature, will be performed in the sequel, with the equation of continuity discussed in Remark 19.1.

17. Electromagnetic induction in continuous bodies

The standard formulation of the laws of electromagnetic induction is introduced hereafter, with innovative features: GALILEI invariant induction laws are formulated and their well-posedness is discussed, leading to correct expressions of electric and magnetic charge conservation in terms of LIE derivatives. The laws are first introduced as integral laws, over arbitrarily drawn two-dimensional submanifolds, and then translated into equivalent differential expressions. While integral formulas provides a direct tool for the evaluation of electromotive or magnetomotive forces along circuits, the differential formulas opens the way for introduction and evaluation of potential fields, respectively one-forms and zero-forms. Metric independent formulations of electromagnetic induction were introduced by Murnaghan (1921); Kottler (1922); É. Cartan (1924); van Dantzig (1934).

18. Electromotive induction by magnetic vortex rate

18.1. Integral Faraday law

The magnetic vortex $\boldsymbol{\omega}^2_{\mathbf{B}}$ is a GALILEI invariant, material *pair* two-form. In (Tonti, 1995, p. 284) it is said: *Therefore, the magnetic flux is associated with a surface element and inner orientation, i.e. with a prescribed direction along its boundary.* The name *flux* is however not appropriate for an extensive quantity dependent on inner orientation, no crossing direction across the surface being specified.

For this reason, we prefer to adopt the name *magnetic vortex*, suggested by the sketch in fig.3. The name *magnetic flux* is rather apt to describe extensive quantities related to outer oriented surface, as depicted in fig.4, which will be considered with reference to electrical induction, in Section 19.

To introduced the law of induction formulated by FARADAY[38] in 1821, let us consider an inner oriented spatial surface $\boldsymbol{\Sigma}^{\text{IN}}$, with the induced inner orientation on its boundary $\partial\boldsymbol{\Sigma}^{\text{IN}}$ (see fig. 3). FARADAY law of magnetic induction is expressed by

$$\boxed{-\oint_{\partial\boldsymbol{\Sigma}^{\text{IN}}} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Sigma}^{\text{IN}})} \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\boldsymbol{\Sigma}^{\text{IN}}} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}}\,.}$$

Here $\boldsymbol{\omega}^1_{\mathbf{E}}$ is the *pair* spatial electric one-form and $\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}}$ is the LIE derivative of the *pair* spatial magnetic vortex two-form along the motion. By STOKES formula and localization, denoting by $d_{\mathcal{S}}$ the exterior derivative of spatial forms FARADAY law may be expressed by the differential condition

$$\boxed{-d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}}\,.}$$

The electric field $\mathbf{E}$, defined by $\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{g} \cdot \mathbf{E}$, is an *pair* vector field, while the magnetic vector field $\mathbf{B}$, defined by $\boldsymbol{\omega}^2_{\mathbf{B}} = \boldsymbol{\mu} \cdot \mathbf{B}$, is an *impair* vector field.

[38] MICHAEL FARADAY (1791-1867) British physicist.

The LIE derivative of the magnetic vortex two-form, according to Prop.7.1, is expressed in terms of parallel derivatives by

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = \nabla_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + 2\,\mathrm{anti}((\nabla\mathbf{v}_{\mathcal{S}})^* \cdot \boldsymbol{\omega}^2_{\mathbf{B}})\,.$$

Remark 18.1. *In (Schwinger et al., 1998, p.9) and in (Thidé, 2010, p.12-14) the parallel derivative $\nabla_{\mathbf{v}_{\mathcal{S}}}\,\boldsymbol{\omega}^2_{\mathbf{B}}$ (according to translation in EUCLID space) is considered in place of the LIE derivative $\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\,\boldsymbol{\omega}^2_{\mathbf{B}}$. This mistake amounts in assuming that $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = \nabla_{\mathbf{v}_{\mathcal{S}}}\,\boldsymbol{\omega}^2_{\mathbf{B}}$ viz. that $\mathrm{emi}\,(\boldsymbol{\omega}^2_{\mathbf{B}} \circ \nabla\mathbf{v}_{\mathcal{S}}) = 0$. So the treatment is correct for purely translational motions in which $\nabla\mathbf{v}_{\mathcal{S}} = 0$ but will lead to completely wrong conclusions in general. An especially important consequence is the false claim about a proof of the LORENTZ force term.*

18.2. Well-posedness of Faraday law

In order that the integral FARADAY formula be meaningful, its r.h.s. should be proven to be independent of the choice of the surface $\boldsymbol{\Sigma}^{\mathrm{IN}}$, for a given boundary $\partial\boldsymbol{\Sigma}^{\mathrm{IN}}$, and independent of the motion of the surface $\boldsymbol{\Sigma}^{\mathrm{IN}}$ for a given motion of the boundary $\partial\boldsymbol{\Sigma}^{\mathrm{IN}}$. This condition may be formalized by requiring that the time-derivatives of the integrals

$$\int_{\boldsymbol{\varphi}^1_\alpha(\boldsymbol{\Sigma}^{\mathrm{IN}}_1)} \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}}\,, \qquad \int_{\boldsymbol{\varphi}^2_\alpha(\boldsymbol{\Sigma}^{\mathrm{IN}}_2)} \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}}\,,$$

be the same for any motions such that

$$\partial(\boldsymbol{\varphi}^1_\alpha(\boldsymbol{\Sigma}^{\mathrm{IN}}_1)) = \partial(\boldsymbol{\varphi}^2_\alpha(\boldsymbol{\Sigma}^{\mathrm{IN}}_2))\,,$$

which is equivalent to require that, for any spatial control-window $\mathbf{C} \subset \mathcal{S}$

$$\partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\partial\mathbf{C})} \boldsymbol{\omega}^2_{\mathbf{B}} = \oint_{\partial\mathbf{C}} \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\mathbf{C}} d_{\mathcal{S}}\,(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}}) = 0\,.$$

Under the assumption that the

By localizing and recalling the commutation property in Lemma 3.2, this is equivalent to

$$d_{\mathcal{S}}\,(\mathcal{L}^{\mathcal{S}}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}}) = d_{\mathcal{S}}\,(\mathcal{L}^{\mathcal{S}}_{\mathbf{v}}\,(\mathbf{i}_{\mathbb{M},\mathbb{M}_{\mathcal{S}}}\!\downarrow\!\boldsymbol{\omega}^2_{\mathbf{B}})) = d_{\mathcal{S}}\,(\mathbf{i}_{\mathbb{M},\mathbb{M}_{\mathcal{S}}}\!\downarrow\!\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}}) = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,(d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}}) = 0\,,$$

a condition assured by GAUSS law[39] $d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = 0$ for the magnetic vortex. By POINCARÉ Lemma, the closedness condition $d_{\mathcal{S}}\,(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}}) = 0$ assures the existence of a one-form $\boldsymbol{\omega}^1_{\mathbf{E}}$ electric field, fulfilling FARADAY differential law

$$-d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}^{\mathcal{S}}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}}\,.$$

18.3. Differential formulation of Faraday law

The theory of electromotive induction is based on the assumption that the trajectory, in which the electric field $\boldsymbol{\omega}^1_{\mathbf{E}}$ and the magnetic vortex $\boldsymbol{\omega}^2_{\mathbf{B}}$ are defined, may spread over the whole ambient space, being either a material body or the *empty space* (or *aether*).

The *aether* is assumed to be homogeneous, isotropic and mass-free, so that no motion of it can be detected. As a consequence the induction law in the *aether* is written in terms of partial time derivatives by any observer since the *aether* appears as motion-free, to any observer.

A careful attention must be devoted to singularities in the time dependence of the fields at a fixed spatial point in the trajectory (i.e. in a spatial-fiber) at those time instants when sudden changes of material properties occur, as tested by an observer.

To recover a standard form of FARADAY law, we consider the magnetic vortex $\boldsymbol{\omega}^2_{\mathbf{B}}$ form at an event on the trajectory in whose neighborhood the vortex has a smooth spatial and temporal dependence. Then the LIE derivative along the space-time velocity $\mathbf{v}$ can be split as sum of LIE derivatives along the spatial and temporal components $\mathbf{v} = \mathbf{v}_{\mathcal{S}} + \mathbf{v}_{\mathcal{Z}}$

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + \mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\,\boldsymbol{\omega}^2_{\mathbf{B}}\,,$$

so that the spatial description of FARADAY differential law writes

$$\boxed{-d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + \mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\,\boldsymbol{\omega}^2_{\mathbf{B}}\,.}$$

Expressing the LIE derivative of the magnetic vortex $\boldsymbol{\omega}^2_{\mathbf{B}}$ along the spatial velocity $\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\,\boldsymbol{\omega}^2_{\mathbf{B}}$ by the homotopy formula, and recalling that $d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = 0$, we get:

$$\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = d_{\mathcal{S}}\,(\boldsymbol{\omega}^2_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}}) + (d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}})\cdot\mathbf{v}_{\mathcal{S}} = d_{\mathcal{S}}\,(\boldsymbol{\omega}^2_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}})\,,$$

[39] The physical interpretation is that magnetic charge density is vanishing.

and the FARADAY law may be rewritten in integral form as

$$\boxed{\begin{aligned} -\oint_{\partial\boldsymbol{\Sigma}^{\text{IN}}} \boldsymbol{\omega}^1_{\mathbf{E}} &= \int_{\boldsymbol{\Sigma}^{\text{IN}}} (\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^2_{\mathbf{B}} + \mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^2_{\mathbf{B}}) \\ &= \int_{\boldsymbol{\Sigma}^{\text{IN}}} \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^2_{\mathbf{B}} + \oint_{\partial\boldsymbol{\Sigma}^{\text{IN}}} \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}\,, \end{aligned}}$$

where $\boldsymbol{\Sigma}^{\text{IN}}$ is a surface in the body placement $\boldsymbol{\Omega}$. The first integral at the r.h.s. (i.e. the surface integral of $\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^2_{\mathbf{B}}$) is wrongly omitted in the formula proved in (Greiner, 1998, p.240). The differential expression is

$$\boxed{\begin{aligned} -d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{E}} &= \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}} \\ &= \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^2_{\mathbf{B}} + \mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^2_{\mathbf{B}} \\ &= \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^2_{\mathbf{B}} + d_{\mathcal{S}}\, (\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}})\,, \end{aligned}}$$

which in vectorial notation becomes

$$-\text{rot}\, \mathbf{E} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \mathbf{B} + \text{rot}\, (\mathbf{B} \times \mathbf{v}_{\mathcal{S}})\,.$$

The convective term $\text{rot}\, (\mathbf{B} \times \mathbf{v}_{\mathcal{S}})$ is omitted in (Weyl, 1922, §20 (8) p.161) entitled *The Electrodynamics of Moving Fields.*

We observe that $\mathbf{B}$ is an *impair* spatial vector field and $\mathbf{v}_{\mathcal{S}}$ is a *pair* spatial vector field. The cross product $\mathbf{B} \times \mathbf{v}_{\mathcal{S}}$, between *impair* and *pair* spatial vector fields, is a *pair* vector field and the rotor $\text{rot}\, (\mathbf{B} \times \mathbf{v}_{\mathcal{S}})$ of a *pair* spatial vector field, is an *impair* vector field. Also $\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \mathbf{B}$ is *impair* and all this agrees with the fact that $\text{rot}\, \mathbf{E}$ is *impair* too.

18.4. Galilei invariance of Faraday law

Let $\boldsymbol{\zeta} \in \text{C}^1(\mathcal{T}\,; \mathcal{T}_{\boldsymbol{\zeta}})$ be the relative motion between two GALILEI observers. GALILEI invariance of $\boldsymbol{\omega}^1_{\mathbf{E}}$ and $\boldsymbol{\omega}^2_{\mathbf{B}}$ is expressed by

$$(\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\omega}^1_{\mathbf{E}}\,, \qquad (\boldsymbol{\omega}^2_{\mathbf{B}})_{\boldsymbol{\zeta}} = \boldsymbol{\zeta}{\uparrow}\boldsymbol{\omega}^2_{\mathbf{B}}\,.$$

By Lemma 13.2 these invariance properties assure GALILEI invariance of FARADAY law

$$\boxed{-\oint_{\partial\boldsymbol{\Sigma}^{\text{IN}}} \boldsymbol{\omega}^1_{\mathbf{E}} = \int_{\boldsymbol{\Sigma}^{\text{IN}}} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}} \quad \Longleftrightarrow \quad -\oint_{\partial\boldsymbol{\zeta}(\boldsymbol{\Sigma}^{\text{IN}})} (\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}} = \int_{\boldsymbol{\zeta}(\boldsymbol{\Sigma}^{\text{IN}})} \mathcal{L}_{(\boldsymbol{\zeta}\uparrow\mathbf{v})}\, (\boldsymbol{\omega}^2_{\mathbf{B}})_{\boldsymbol{\zeta}}\,.}$$

Indeed it is

$$\oint_{\partial \boldsymbol{\Sigma}^{\text{IN}}} \boldsymbol{\omega}^1_{\mathbf{E}} = \oint_{\partial \boldsymbol{\zeta}(\boldsymbol{\Sigma}^{\text{IN}})} \boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\omega}^1_{\mathbf{E}} = \oint_{\partial \boldsymbol{\zeta}(\boldsymbol{\Sigma}^{\text{IN}})} (\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}} \,,$$

and

$$\int_{\boldsymbol{\Sigma}^{\text{IN}}} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\boldsymbol{\zeta}(\boldsymbol{\Sigma}^{\text{IN}})} \boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}}) = \int_{\boldsymbol{\zeta}(\boldsymbol{\Sigma}^{\text{IN}})} \mathcal{L}_{(\boldsymbol{\zeta}\uparrow\mathbf{v})}\, (\boldsymbol{\omega}^2_{\mathbf{B}})_{\boldsymbol{\zeta}} \,.$$

18.5. Faraday potential one-form

FARADAY law of magnetic induction is expressed by

$$\boxed{-\oint_{\partial \boldsymbol{\Sigma}^{\text{IN}}} \boldsymbol{\omega}^1_{\mathbf{E}} = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Sigma}^{\text{IN}})} \boldsymbol{\omega}^2_{\mathbf{B}} = \int_{\boldsymbol{\Sigma}^{\text{IN}}} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}} \,.}$$

An explicit expression for the electric field $\boldsymbol{\omega}^1_{\mathbf{E}}$ can be got by observing that, being $d\boldsymbol{\omega}^2_{\mathbf{B}} = 0$, POINCARÉ lemma ensures that the closed form of magnetic vortex $\boldsymbol{\omega}^2_{\mathbf{B}}$ admits a potential $\boldsymbol{\omega}^1_{\mathbf{B}}$, the *pair* FARADAY one-form, so that we may set

$$\boldsymbol{\omega}^2_{\mathbf{B}} = d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{B}} \quad \Longleftrightarrow \quad \mathbf{B} = \operatorname{rot} \mathbf{A} \,,$$

where $\mathbf{A}$, defined by $\boldsymbol{\omega}^1_{\mathbf{B}} = \mathbf{g} \cdot \mathbf{A}$, is the *pair* magnetic vector potential. By relying on the commutation property in Lemma 3.2, FARADAY differential law may be written as

$$-d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \mathcal{L}_{\mathbf{v}}\, d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = d_{\mathcal{S}}\, \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^1_{\mathbf{B}} \,,$$

and leads to the following formula, in terms of the *pair* electric scalar potential $V_{\mathbf{E}} \in \mathrm{C}^1(\mathcal{S}\,; \mathrm{FUN}(\mathbb{T}\mathcal{S}))$

$$-\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}}\, V_{\mathbf{E}} \,.$$

To get a GALILEI invariant electric field, the FARADAY one-form $\boldsymbol{\omega}^1_{\mathbf{B}}$ and electric zero-form $V_{\mathbf{E}}$ are assumed to be GALILEI-invariant. Splitting the LIE derivative by additivity along time and space components

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + \mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} \,,$$

and resorting to homotopy formula for the space component, we get

$$-\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} + d_{\mathcal{S}}\, (\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}) + d_{\mathcal{S}}\, V_{\mathbf{E}}$$

and in vector notation

$$\boxed{-\mathbf{E} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \mathbf{A} + \mathbf{B} \times \mathbf{v}_{\mathcal{S}} + d_{\mathcal{S}} \left(\mathbf{g}(\mathbf{A}, \mathbf{v}_{\mathcal{S}})\right) + d_{\mathcal{S}} V_{\mathbf{E}} \,.}$$

This expression should be compared with the not GALILEI invariant formula, see e.g. (Sadiku, 2010, eq. 9.45), which may be obtained by dropping the convective derivative

$$-\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}} V_{\mathbf{E}} \quad \iff \quad -\mathbf{E} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \mathbf{A} + \operatorname{grad} V_{\mathbf{E}} \,.$$

It is important to underline that the scalar field $\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}$ is spatially differentiable only under a regularity assumption on the velocity field which is likely to be violated in applications. Instances of lack of regularity are met in the investigation of induction due the motion of a transverse conductive bar sliding on a pair of parallel rails or in the spinning of a disk. In these situations singular terms due to jumps in the velocity field must be properly taken into account, see Sect. 33, 32.2.

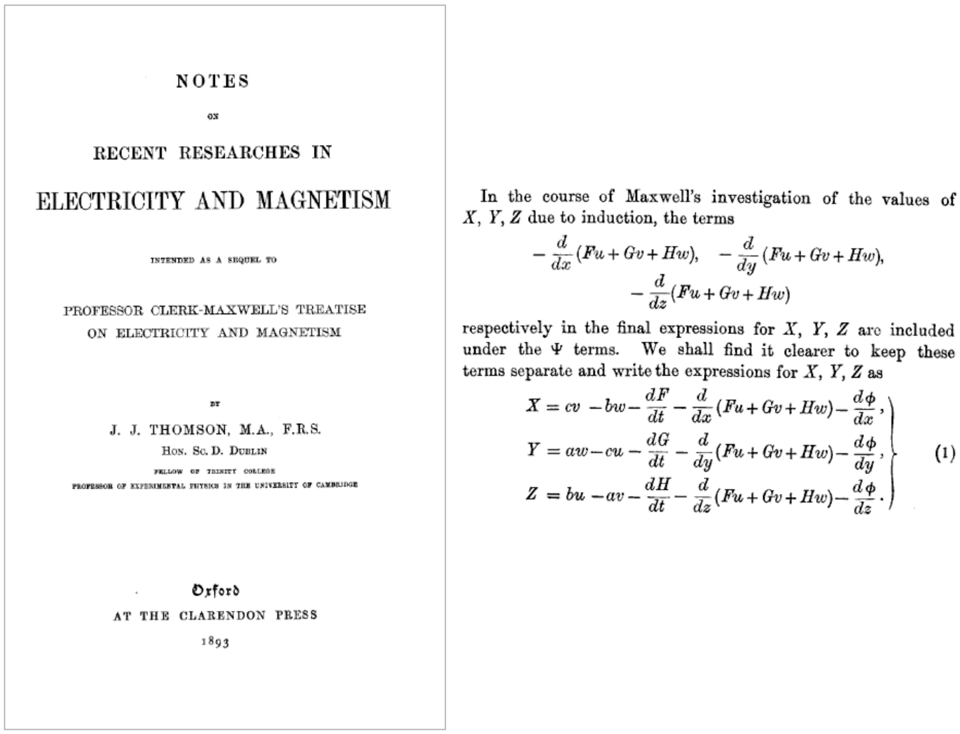
Remark 18.2. *The GALILEI invariant formula for the electric field, in terms of the LIE derivative of the FARADAY one-form $\boldsymbol{\omega}^1_{\mathbf{B}}$ along the motion, should give up with the claim about the fact that two GALILEI observers, one fixed to the magnets and the other drifted by a relative translational motion, should evaluate the electric field induced by a magnetic vortex by resorting to different laws of electrodynamics (see e.g. (Griffiths, 1999, p. 477)), a claim exposed also by ALBERT EINSTEIN at the very beginning of his celebrated paper on electrodynamics of moving bodies (Einstein, 1905).*

Remark 18.3. *After having independently developed the present treatment, in reading the treatise on Electricity and Magnetism by J.J. Thomson (1893), the author became aware of the fact that the same GALILEI invariant formula for the electric field, expressed in cartesian coordinates, was there reported in ch. VII, p. 534, as depicted in fig. 5 below. It easy to check that the formula there exposed, when written in our notations, becomes*

$$\boldsymbol{\omega}^1_{\mathbf{E}} = -\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} - \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \boldsymbol{\omega}^1_{\mathbf{B}} - d_{\mathcal{S}} \left(\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}\right) - d_{\mathcal{S}} V_{\mathbf{E}} \,,$$

or in vector notation: $\mathbf{E} = \mathbf{v}_{\mathcal{S}} \times \mathbf{B} - \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \mathbf{A} - d_{\mathcal{S}} \, \mathbf{g}(\mathbf{A}, \mathbf{v}_{\mathcal{S}}) - d_{\mathcal{S}} V_{\mathbf{E}} \,.$ *As J.J. THOMSON[40] says, he got this expression by a modification of the original*

[40] JOSEPH JOHN THOMSON (1856-1940) British physicist.



NOTES

ON

RECENT RESEARCHES IN

ELECTRICITY AND MAGNETISM

INTENDED AS A SEQUEL TO

PROFESSOR CLERK-MAXWELL'S TREATISE
ON ELECTRICITY AND MAGNETISM

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AT THE CLARENDON PRESS
1893

In the course of Maxwell's investigation of the values of X, Y, Z due to induction, the terms

$$-\frac{d}{dx}(Fu+Gv+Hw), \quad -\frac{d}{dy}(Fu+Gv+Hw), \quad -\frac{d}{dz}(Fu+Gv+Hw)$$

respectively in the final expressions for X, Y, Z are included under the Ψ terms. We shall find it clearer to keep these terms separate and write the expressions for X, Y, Z as

$$\left.\begin{aligned} X &= cv - bw - \frac{dF}{dt} - \frac{d}{dx}(Fu+Gv+Hw) - \frac{d\phi}{dx}, \\ Y &= aw - cu - \frac{dG}{dt} - \frac{d}{dy}(Fu+Gv+Hw) - \frac{d\phi}{dy}, \\ Z &= bu - av - \frac{dH}{dt} - \frac{d}{dz}(Fu+Gv+Hw) - \frac{d\phi}{dz}. \end{aligned}\right\} \quad (1)$$

Figure 5: J.J. Thomson formulation

formula in (Clerk-Maxwell, 1861, (77) p.342) where, setting

$$U_{\mathbf{E}} = V_{\mathbf{E}} + \boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} \,,$$

the electric field was written as

$$\boldsymbol{\omega}^1_{\mathbf{E}} = -\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} - \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} - d_{\mathcal{S}}\, U_{\mathbf{E}} \,,$$

or in vector form

$$\mathbf{E} = -\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \mathbf{A} + \mathbf{v}_{\mathcal{S}} \times \mathbf{B} - \nabla U_{\mathbf{E}} \,,$$

see also (Clerk-Maxwell, 1865, (D) p.485).

It is surprising that engineers and physicists, who had at hand the Galilei *invariant expression of the electric field as formulated by* Clerk-Maxwell *and* J.J. Thomson*, did instead adopt, and still do, a non-invariant expression. The reason may probably be found in that the wave equation in* empty space *is readily obtained from the expression without the convective term. In our opinion, the two seemingly contradictory requirements, i.e.* GALILEI *invariance and recovery of the wave equation in empty space, may be reconciled by observing that the vanishing of the velocity field is a consequence of isotropy and homogeneity of the electromagnetic constitutive properties of the mass-free empty space, a feature that makes any motion of it undetectable.*

Remark 18.4. *In literature, the term* $\mathbf{v}_{\mathcal{S}} \times \mathbf{B}$ *is referred to as the magnetic* LORENTZ[41] *force per unit electric charge on a body in motion (Lorentz, 1899), and most often introduced as a fundamental rule to be assumed in addition to the law of magnetic induction, on the basis of experimental evidence, see e.g. (Barut, 1980, p.88), (Feynman et al., 1964, II.17-2), (Greiner, 1998, p.238) (Jackson, 1999, p.3), (Griffiths, 1999, ch.5.1.2), (Kovetz, 2000, sec.15), (Sadiku, 2010, ch.9.3B) (Lehner, 2010, 6.1.2, p.344). The physical significance of a not* GALILEI*-invariant force is however firmly questionable. As shown above, the term* $\mathbf{v} \times \mathbf{B}$ *was still present in the original treatment by* MAXWELL*, but only as one of two addends, each one not* GALILEI *invariant, the total contribution* $-\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \mathbf{A} + \mathbf{v}_{\mathcal{S}} \times \mathbf{B}$ *being instead* GALILEI *invariant. The issue will be further discussed in Rem. 32.1.*

19. Magnetomotive induction by electric flux rate

19.1. Ampère law

The discovery in 1820 by ØRSTED[42] that a magnetic field was induced by an electric current, was immediately followed by a mathematical formulation of the law of electric induction, due to AMPÈRE (1820), subsequently modified by CLERK-MAXWELL who envisaged the need for the additional term concerning the electric displacement (Clerk-Maxwell, 1861). According to the geometric point of view exposed in this paper, AMPÈRE law is expressed by the integral condition

$$\boxed{\oint_{\partial \boldsymbol{\Sigma}^{\text{OUT}}} \boldsymbol{\omega}^1_{\mathbf{H}} = \partial_{\alpha=0} \int_{\boldsymbol{\varphi}_\alpha(\boldsymbol{\Sigma}^{\text{OUT}})} \boldsymbol{\omega}^2_{\mathbf{D}} + \int_{\boldsymbol{\Sigma}^{\text{OUT}}} \boldsymbol{\omega}^2_{\mathbf{J}} = \int_{\boldsymbol{\Sigma}^{\text{OUT}}} \mathcal{L}_{\mathbf{v}_{\mathcal{S}}} \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}} \,,}$$

[41] HENDRIK ANTOON LORENTZ (1853-1928) Dutch physicist.
[42] HANS CHRISTIAN ØRSTED (1777-1851) Dutch physicist.

for any outer oriented circuit $\partial\mathbf{\Sigma}^{\text{OUT}}$ bounding a correspondingly outer oriented surface, $\mathbf{\Sigma}^{\text{OUT}}$ see fig. 4. The magnetic winding $\boldsymbol{\omega}^1_{\mathbf{H}}$ is an *impair* one-form, the electric displacement flux and the conduction current flux $\boldsymbol{\omega}^2_{\mathbf{D}}, \boldsymbol{\omega}^2_{\mathbf{J}}$ are *impair* two-forms.

The electric displacement $\mathbf{D}$, defined by $\boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\mu}\mathbf{D}$, is a *pair* vector field, and the magnetic field $\mathbf{H}$, defined by $\boldsymbol{\omega}^1_{\mathbf{H}} = \mathbf{gH}$, is an *impair* vector field. The electric current $\mathbf{J}$, defined by $\boldsymbol{\omega}^2_{\mathbf{J}} = \boldsymbol{\mu}\mathbf{J}$, is a *pair* vector field.

19.2. Well-posedness of Ampère law

In order that Ampère law be meaningful, it is to be proven that the r.h.s. is independent of the choice of surface $\mathbf{\Sigma}$, for a given circuit $\partial\mathbf{\Sigma}$, and independent of the motion of surface $\mathbf{\Sigma}$ for a given motion of circuit $\partial\mathbf{\Sigma}$. This condition may be formalized by requiring that the time derivatives of the integrals

$$\int_{\boldsymbol{\varphi}^1_\alpha(\mathbf{\Sigma}^{\text{OUT}}_1)} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}}\,, \qquad \int_{\boldsymbol{\varphi}^2_\alpha(\mathbf{\Sigma}^{\text{OUT}}_2)} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}}\,,$$

be the same for motions such that

$$\partial(\boldsymbol{\varphi}^1_\alpha(\mathbf{\Sigma}^{\text{OUT}}_1)) = \partial(\boldsymbol{\varphi}^2_\alpha(\mathbf{\Sigma}^{\text{OUT}}_2))\,.$$

The chain $\boldsymbol{\varphi}^1_\alpha(\mathbf{\Sigma}^{\text{OUT}}_1) - \boldsymbol{\varphi}^2_\alpha(\mathbf{\Sigma}^{\text{OUT}}_2)$ is then a closed surface at any time. It follows that, for any outer-oriented control-window $\mathbf{C}^{\text{OUT}}$, the flux across its boundary surface $\partial\mathbf{C}^{\text{OUT}}$ should vanish

$$\oint_{\partial\mathbf{C}^{\text{OUT}}} (\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}}) = \int_{\mathbf{C}^{\text{OUT}}} d_{\mathcal{S}}\, (\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}}) = 0\,.$$

Localizing, we get the equivalent closedness condition

$$d_{\mathcal{S}}\, (\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}}) = 0\,.$$

The commutative property in Lemma 3.2 implies that

$$d_{\mathcal{S}}\, (\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{D}}) = \mathcal{L}_{\mathbf{v}}\, (d_{\mathcal{S}}\, \boldsymbol{\omega}^2_{\mathbf{D}})\,.$$

The geometric formulation of Gauss law[43] for the electric displacement flux, consists in the assessment of the exactness condition

$$\boxed{\boldsymbol{\rho}^3 = d_{\mathcal{S}}\, \boldsymbol{\omega}^2_{\mathbf{D}}\,.}$$

[43] This law is the physical interpretation of a mathematical property ensuing from the fact that the Euclid ambient space $\mathcal{S}$ is star shaped. Then Poincaré lemma 3.1 holds true and the law follows from the vanishing of the 4-form $d_{\mathcal{S}}\, \boldsymbol{\rho}^3$ in the 3-space $\mathcal{S}$.

Hence the closedness condition above translates into *electric charge balance law*

$$\boxed{\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} = 0}$$

and, in integral form

$$\begin{aligned}\int_{\mathbf{C}^{\mathrm{OUT}}} \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} &= \partial_{\alpha=0}\int_{\boldsymbol{\varphi}_\alpha(\mathbf{C}^{\mathrm{OUT}})} \boldsymbol{\rho}^3 + \int_{\mathbf{C}^{\mathrm{OUT}}} d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} \\ = \oint_{\partial\mathbf{C}^{\mathrm{OUT}}} \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}} &= \partial_{\alpha=0}\oint_{\boldsymbol{\varphi}_\alpha(\partial\mathbf{C}^{\mathrm{OUT}})} \boldsymbol{\omega}^2_{\mathbf{D}} + \oint_{\partial\mathbf{C}^{\mathrm{OUT}}} \boldsymbol{\omega}^2_{\mathbf{J}} = 0\,.\end{aligned}$$

The electric charge $\boldsymbol{\rho}^3$ is an *impair* three-form which may be integrated over even non-orientable manifolds to evaluate the total charge.

In empty space, i.e. at events not laying in a charged material particle, Gauss law reduces to the closedness condition $d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{D}} = 0$.

Observing that the outer orientations of open 3D manifolds in Euclid space, *spring* and *sink*, respectively correspond to outer orientations *outward* and *inward* for its boundary 2D manifold, the electric charge balance law has to be read as

- The time-rate of increase of the total electric charge, in a traveling control-window, is equal to the inward flux of electric conduction current through the window boundary.

We emphasize that the assumption of absence of bulk sources of electric charge plays a basic role in ensuring well-posedness of Ampère law. The electric charge balance law $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} = 0$ is Galilei invariant since such are by assumption $\boldsymbol{\rho}^3$ and $\boldsymbol{\omega}^2_{\mathbf{J}}$ ad hence the terms $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^3$ and $d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}}$ according to Lemma 13.2 and 3.2.

Remark 19.1. *In literature, the electric charge balance law is usually written as* $\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} = 0$ *or in vectorial notation*

$$\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\rho + \operatorname{div}\mathbf{J} = 0\,,$$

being $\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\rho = \nabla_{\mathbf{v}_{\mathcal{Z}}}\,\rho$. *This is called the equation of continuity (Weyl, 1922, p.161), (Feynman et al., 1964, II.18-1), (Barut, 1980, p.90), (Purcell, 1985, p.127), (Schwinger et al., 1998, p.9), (Greiner, 1998, p.251), (Jackson, 1999, p.238), (Griffiths, 1999, p.345), (Wegner, 2003, p.50), (Thidé, 2010, p.10), (Sadiku, 2010, p.385). The correct expression* $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} = 0$ *introduced* *above, reduces to the incomplete one by assuming a translating body and a* GALILEI *observer sitting on it, so that* $\mathbf{v}_{\mathcal{S}} = 0$ *and hence* $\mathbf{v} = \mathbf{v}_{\mathcal{Z}}$ *. If the formulation of the equation of continuity in terms of partial time derivative of the electric charge is assumed to be (as usually made in literature) a general physical law, this would lead to the completely unsatisfactory conclusion that* AMPÈRE *law of induction is well-posed only for* GALILEI *observers testing time-invariant material circuits.*

When the material tensor $\boldsymbol{\rho}^3$ has a regular spatial and time-dependence, we may write

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\rho}^3 = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\rho}^3 + \mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\rho}^3 \,.$$

Then, by the homotopy formula, being $d_{\mathcal{S}}\, \boldsymbol{\rho}^3 = 0$, we infer that

$$\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\rho}^3 = d_{\mathcal{S}}\, (\boldsymbol{\rho}^3 \cdot \mathbf{v}_{\mathcal{S}}) + (d_{\mathcal{S}}\, \boldsymbol{\rho}^3) \cdot \mathbf{v}_{\mathcal{S}} = d_{\mathcal{S}}\, (\boldsymbol{\rho}^3 \cdot \mathbf{v}_{\mathcal{S}}) \,,$$

and the spatial description of electric charge balance law may be written, in terms of exterior derivatives, as

$$\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\rho}^3 + d_{\mathcal{S}}\, (\boldsymbol{\rho}^3 \cdot \mathbf{v}_{\mathcal{S}}) + d_{\mathcal{S}}\, \boldsymbol{\omega}^2_{\mathbf{J}} = 0 \,.$$

Expressing our formula in vector notations, HELMHOLTZ differential condition expressing *electric charge balance law* is recovered

$$\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \rho + \operatorname{div}(\rho\, \mathbf{v}_{\mathcal{S}}) + \operatorname{div} \mathbf{J} = 0 \,,$$

as quoted in (Darrigol, 2000) who refers to (Helmholtz, 1870). It seems that the correct analysis perfomed in (Clerk-Maxwell, 1861) and in (Helmholtz, 1870) have been neglected in the course of the nineteenth century, probably also due to the influence exerted by the incorrect formulation of classical electromagnetics referred to in (Einstein, 1905).

19.3. Differential formulation of Ampère law

Upon localization, AMPÈRE's law may be formulated in differential terms according to the equivalent notations

$$\boxed{d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{H}} = \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}} \,.}$$

$$d_{\mathcal{S}}\, (\mathbf{g}\, \mathbf{H}) = \mathcal{L}_{\mathbf{v}}\, (\boldsymbol{\mu}\, \mathbf{D}) + \boldsymbol{\mu}\, \mathbf{J} \,,$$

$$\boldsymbol{\mu} \cdot (\operatorname{rot} \mathbf{H}) = \mathcal{L}_{\mathbf{v}}\, (\boldsymbol{\mu}\, \mathbf{D}) + \boldsymbol{\mu}\, \mathbf{J} \,,$$

$$\operatorname{rot}\mathbf{H} = \mathcal{L}_{\mathbf{v}}\,\mathbf{D} + (\operatorname{div}\mathbf{v})\,\mathbf{D} + \mathbf{J}\,.$$

Setting $\boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\rho}^2 + \boldsymbol{\omega}^2_{\mathbf{Z}}$ with $d_{\mathcal{S}}\,\boldsymbol{\rho}^2 = \boldsymbol{\rho}^3$ and $d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{Z}} = 0$, we may introduce the Ampère electric potential one-form $\boldsymbol{\omega}^1_{\mathbf{Z}}$ such that

$$\boldsymbol{\omega}^2_{\mathbf{Z}} = d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{Z}}\,,$$

and the differential form of Ampère law may be written as

$$d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{H}} = \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^2 + d_{\mathcal{S}}\,\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^1_{\mathbf{Z}} + \boldsymbol{\omega}^2_{\mathbf{J}}\,.$$

Being $d_{\mathcal{S}}\,(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^2 + \boldsymbol{\omega}^2_{\mathbf{J}}) = 0$ we may set $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\rho}^2 + \boldsymbol{\omega}^2_{\mathbf{J}} = d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{N}}$ and write

$$\boldsymbol{\omega}^1_{\mathbf{H}} = \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^1_{\mathbf{Z}} + \boldsymbol{\omega}^1_{\mathbf{N}} + d_{\mathcal{S}}\,V_{\mathbf{H}}$$

where all terms at the r.h.s. are Galilei invariant.

If the additive decomposition of Lie derivative is feasible, we get

$$\boldsymbol{\omega}^1_{\mathbf{H}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\boldsymbol{\omega}^1_{\mathbf{Z}} + d_{\mathcal{S}}\,(\boldsymbol{\omega}^1_{\mathbf{Z}}\cdot\mathbf{v}_{\mathcal{S}}) + \boldsymbol{\omega}^2_{\mathbf{Z}}\cdot\mathbf{v}_{\mathcal{S}} + \boldsymbol{\omega}^1_{\mathbf{N}} + d_{\mathcal{S}}\,V_{\mathbf{H}}$$

Ampère law is accordingly written then as

$$\oint_{\partial\boldsymbol{\Sigma}^{\text{OUT}}} \boldsymbol{\omega}^1_{\mathbf{H}} = \int_{\boldsymbol{\Sigma}^{\text{OUT}}} (\boldsymbol{\omega}^2_{\mathbf{J}} + \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\boldsymbol{\omega}^2_{\mathbf{D}}) + \oint_{\partial\boldsymbol{\Sigma}^{\text{OUT}}} \boldsymbol{\omega}^2_{\mathbf{D}}\cdot\mathbf{v}_{\mathcal{S}} + \int_{\boldsymbol{\Sigma}^{\text{OUT}}} \boldsymbol{\rho}^3\cdot\mathbf{v}_{\mathcal{S}}\,,$$

and in differential form

$$d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{H}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\boldsymbol{\omega}^2_{\mathbf{D}} + \boldsymbol{\omega}^2_{\mathbf{J}} + d_{\mathcal{S}}\,(\boldsymbol{\omega}^2_{\mathbf{Z}}\cdot\mathbf{v}_{\mathcal{S}}) + \boldsymbol{\rho}^3\cdot\mathbf{v}_{\mathcal{S}}\,,$$

or in vector analysis notation, setting $\boldsymbol{\omega}^2_{\mathbf{Z}} = \boldsymbol{\mu}\cdot\mathbf{Z}$

$$\operatorname{rot}\mathbf{H} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\mathbf{D} + \mathbf{J} + \operatorname{rot}(\mathbf{Z}\times\mathbf{v}_{\mathcal{S}}) + \rho\,\mathbf{v}_{\mathcal{S}}\,.$$

The customary one, e.g. (Sadiku, 2010, eq. 9.23)

$$\operatorname{rot}\mathbf{H} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\mathbf{D} + \mathbf{J}\,,$$

in which the velocity is assumed to vanish, has not general validity.

19.4. Galilei invariance of Ampère law

Galilei invariance of Ampère law follows from the Galilei invariance of the involved fields $\boldsymbol{\omega}^1_{\mathbf{H}}$, $\boldsymbol{\omega}^2_{\mathbf{D}}$, $\boldsymbol{\omega}^2_{\mathbf{J}}$ and from Lemma 13.2 and 3.2 ensuring invariance under relative motions of the Lie derivative and of the exterior derivative of invariant tensors.

20. Electromagnetic constitutive relations

When expressed in terms of differential forms, the laws of electromagnetic induction do not involve neither the chosen orientation nor the metric properties of the physical space. The constitutive laws, expressing the *electric permittivity* and the *magnetic permeability* of a medium in terms of vector fields, depend on the metric properties of the space.

Indeed, in the standard EUCLID space $(\mathcal{S}\,,\mathbf{g})$ the non-singular metric tensor leads to the one-to-one correspondences

$$\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{g}\cdot\mathbf{E}\,,\quad \boldsymbol{\omega}^2_{\mathbf{B}} = \boldsymbol{\mu}\cdot\mathbf{B}\,,\quad \boldsymbol{\omega}^1_{\mathbf{H}} = \mathbf{g}\cdot\mathbf{H}\,,\quad \boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\mu}\cdot\mathbf{D}\,,$$

The *impair* electric flux two-form $\boldsymbol{\omega}^2_{\mathbf{D}}$ is in one-to-one linear correspondence with the *pair* electric displacement vector field $\mathbf{D}$ according to the relation $\boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\mu}\cdot\mathbf{D}$.

The *impair* magnetic winding one-form $\boldsymbol{\omega}^1_{\mathbf{H}}$ and the magnetic *impair* vector field $\mathbf{B}$ which is in one-to-one linear correspondence with the *pair* magnetic vortex two-form $\boldsymbol{\omega}^2_{\mathbf{B}}$ according to the relation $\boldsymbol{\omega}^2_{\mathbf{B}} = \boldsymbol{\mu}\cdot\mathbf{B}$.

The *empty space* is assumed to be massless and to have have linear, uniform and isotropic electromagnetic constitutive properties.

The *electric permittivity* ε_o is a pointwise relation between the electric field one-form $\boldsymbol{\omega}^1_{\mathbf{E}}$ and the electric flux two-form $\boldsymbol{\omega}^2_{\mathbf{D}}$.

The separating duality induced by the pairing $\langle\boldsymbol{\omega}^1_{\mathbf{E}},\mathbf{D}\rangle$, between dual *pair* geometrical fields, leads to the following electric constitutive equation

$$\varepsilon_o\,\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{D}\,.$$

Analogously, the *magnetic permeability* μ_o is a pointwise relation between

The duality pairing $\langle\boldsymbol{\omega}^1_{\mathbf{H}},\mathbf{B}\rangle$, between dual *impair* geometrical fields, leads to the following magnetic constitutive equation

$$\mu_o\,\boldsymbol{\omega}^1_{\mathbf{H}} = \mathbf{B}\,.$$

The *electric permittivity* and the *magnetic permeability* are then fields of linear maps between dual spaces, which can be represented by scalar fields.

In *empty space* we may set

$$\mathbf{D} = \varepsilon_o\,\mathbf{E}\,,\quad \mathbf{B} = \mu_o\,\mathbf{H}\,,$$

with $\varepsilon_o,\mu_o : \mathcal{S}\times\mathcal{Z}\mapsto\mathcal{R}$ constant scalar fields. such that $c^{-2} = \varepsilon_o,\mu_o$.

Assuming as basic fields $\mathbf{E}$ and $\mathbf{M} = c\,\mathbf{B}$, the law of electromagnetic induction may be written in the symmetric form

$$\begin{cases} \operatorname{rot}\mathbf{E} = c^{-1}\,\mathcal{L}_{\mathbf{v}}\,\mathbf{M} \\ \operatorname{rot}\mathbf{M} = c^{-1}\,\mathcal{L}_{\mathbf{v}}\,\mathbf{E}\,, \end{cases}$$

Due to uniformity and isotropy of its electromagnetic constitutive properties, no spatial motion of the massless *empty space* can be detected and the laws of induction in *empty space* reduce to the standard ones referred to in literature as MAXWELL-HERTZ equations (see e.g. (Einstein, 1905)), where partial time-derivatives are put in place of LIE derivatives along the space-time motion.

20.1. Poynting vector

The total electric and magnetic power expended, per unit volume in a control window in *empty space*, is the *pair* scalar field given by the formula:

$$\langle \boldsymbol{\omega}^1_{\mathbf{E}}, \mathbf{J} + \dot{\mathbf{D}} \rangle + \langle \boldsymbol{\omega}^1_{\mathbf{H}}, \dot{\mathbf{B}} \rangle\,,$$

where $\dot{\mathbf{D}} := \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\mathbf{D}$ and $\dot{\mathbf{B}} := \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\mathbf{B}$.

On the other hand, we have the identity

$$\langle \boldsymbol{\omega}^1_{\mathbf{E}}, \operatorname{rot}\mathbf{H} \rangle - \langle \boldsymbol{\omega}^1_{\mathbf{H}}, \operatorname{rot}\mathbf{E} \rangle = -\operatorname{div}(\mathbf{E}\times\mathbf{H})\,.$$

FARADAY and AMPÈRE laws of induction

$$\operatorname{rot}\mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}\,,$$

$$-\operatorname{rot}\mathbf{E} = \dot{\mathbf{B}}\,,$$

substituted in the identity above, yield POYNTING relation

$$\langle \boldsymbol{\omega}^1_{\mathbf{E}}, \mathbf{J} + \dot{\mathbf{D}} \rangle + \langle \boldsymbol{\omega}^1_{\mathbf{H}}, \dot{\mathbf{B}} \rangle = -\operatorname{div}(\mathbf{E}\times\mathbf{H})\,,$$

whose integral version pertaining to a 3D control window $\mathbf{C}$ writes

$$\int_{\mathbf{C}^{\text{OUT}}} \langle \boldsymbol{\omega}^1_{\mathbf{E}}, \mathbf{J} + \dot{\mathbf{D}} \rangle + \langle \boldsymbol{\omega}^1_{\mathbf{H}}, \dot{\mathbf{B}} \rangle \boldsymbol{\mu} + \oint_{\partial\mathbf{C}^{\text{OUT}}} \boldsymbol{\mu}\cdot(\mathbf{E}\times\mathbf{H}) = 0\,.$$

The introduction of the *pair* vector field $\mathbf{E}\times\mathbf{H}$ is due to POYNTING[44] in (Poynting, 1884) and to HEAVISIDE[45] in the same year, see (Stratton, 1941,

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[44] JOHN HENRY POYNTING (1852-1914) British physicist.

[45] OLIVER HEAVISIDE (1850-1925) British physicist.

ch.II, p.132). The relation may be so read: The total electric and magnetic power expended, per unit volume of a control window in *empty space*, is equal to the incoming flux of POYNTING vector field through its boundary.

21. Events manifold and framings

The events manifold $\mathbb{M}$ is a 4-dimensional star shaped orientable manifold without boundary.

Definition 21.1 (Framing). *A* framing *is a section*

$$(\mathbf{u}\,,\boldsymbol{\alpha}) \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM} \times_{\mathbb{M}} \mathbb{T}^*\mathbb{M})$$

of the WHITNEY *product between tangent and cotangent bundles, such that*

$$\langle \boldsymbol{\alpha}, \mathbf{u} \rangle \neq 0\,,$$

$$d\boldsymbol{\alpha} = 0\,.$$

The values of the tangent vector field $\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ are called *time-arrows* and the closed one-form $\boldsymbol{\alpha} \in \Lambda^1(\mathbb{TM})$ is called the *slicing*.

Lemma 21.1 (Space-time split). *Tangent vectors* $\mathbf{w} \in \mathbb{TM}$ *are uniquely split into a spatial and a temporal component such that*

$$\begin{cases} \mathbf{w} = \mathbf{w}_{\mathcal{S}} + \mathbf{w}_{\mathcal{Z}}\,, \\ \mathbf{w}_{\mathcal{Z}} = k\,\mathbf{u}\,, \\ \langle \boldsymbol{\alpha}, \mathbf{w}_{\mathcal{S}} \rangle = 0\,. \end{cases}$$

Proof. The evaluation

$$\langle \boldsymbol{\alpha}, \mathbf{w} \rangle = \langle \boldsymbol{\alpha}, \mathbf{w}_{\mathcal{S}} \rangle + \langle \boldsymbol{\alpha}, \mathbf{w}_{\mathcal{Z}} \rangle = k\,\langle \boldsymbol{\alpha}, \mathbf{u} \rangle\,,$$

gives the result. ■

Lemma 21.2 (Spatial foliation). *Tangent vectors in the kernel of* $\boldsymbol{\alpha}$ *draw an integrable distribution which foliates the events manifold into* 3*-dimensional leaves called* spatial slices*.*

Proof. By star-shapedness and POINCARÉ Lemma, closure is equivalent to exactness so that we may set $\boldsymbol{\alpha} = dt$ with $t \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{FUN}(\mathbb{TM}))$. Integrability of the kernel-distribution is assured by FROBENIUS condition

$$\begin{cases} \langle\, dt, \mathbf{v}_1 \,\rangle = 0 \\ \langle\, dt, \mathbf{v}_2 \,\rangle = 0 \end{cases} \implies \quad \langle\, dt, \mathcal{L}_{\mathbf{v}_1} \mathbf{v}_2 \,\rangle = 0\,,$$

whose fulfillment follows from the equality $\mathcal{L}_{\mathbf{v}_1} \mathbf{v}_2 = -\mathcal{L}_{\mathbf{v}_2} \mathbf{v}_1 = [\mathbf{v}_1\,, \mathbf{v}_2]$ and the expression of LIE bracket

$$\langle\, dt, [\mathbf{v}_1\,, \mathbf{v}_2] \,\rangle = [\mathbf{v}_1\,, \mathbf{v}_2]\, t = (\mathbf{v}_1 \mathbf{v}_2 - \mathbf{v}_2 \mathbf{v}_1)\, t\,,$$

since by assumption $\mathbf{v}_1\, t = \langle\, dt, \mathbf{v}_1 \,\rangle = 0\,,\ \mathbf{v}_2\, t = \langle\, dt, \mathbf{v}_2 \,\rangle = 0\,.$ ■

Under the action of a framing $(\mathbf{u}\,, \boldsymbol{\alpha}) \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM} \times_{\mathbb{M}} \mathbb{T}^*\mathbb{M})$ the tangent manifold $\mathbb{TM}$ is split into a WHITNEY bundle $\mathbb{HM} \times_{\mathbb{M}} \mathbb{ZM}$ of horizontal and vertical vectors. The 3-D fibers of $\mathbb{HM}$ are in the kernel of $\boldsymbol{\alpha}$ while the 1-D fibers of $\mathbb{ZM}$ are lines generated by $\mathbf{u}$.

Both subbundles of $\mathbb{TM}$ are integrable. The 3-D leaves of the horizontal foliation are level set for the time 0-form $t \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{FUN}(\mathbb{TM}))$. Each 1-D leaf of the vertical foliation defines a spatial point.

The oriented lines of events envelops of the time-arrows, are called *time-lines* and define equivalence classes of *isotopic* events. The corresponding quotient 3D manifold $\mathbb{M}_{\mathcal{S}}$ is isomorpic to a typical fiber $\mathcal{S}$ called the *ambient manifold*. The projector on the spatial manifold $\mathbb{M}_{\mathcal{S}}$ will be denoted by $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{M}_{\mathcal{S}})$.

The complementary time-fibration defines equivalence classes of *simultaneous* events, also called *spatial slices* and the relevant quotient manifold, denoted by $\mathbb{M}_{\mathcal{Z}}$ is isomorphic to the time-line $\mathcal{Z}$ (motivated by the German word *zeit* for *time*).

Definition 21.2 (Tuned framing). *In a framing*

$$(\mathbf{u}\,, \boldsymbol{\alpha}) \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM} \times_{\mathbb{M}} \mathbb{T}^*\mathbb{M})\,,$$

the time-arrows *vector field* $\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM})$ *and the* time 0-*form (scalar field)* $t \in \mathrm{C}^1(\mathbb{M}\,; \mathrm{FUN}(\mathbb{TM}))$ *are said to be* tuned *if* $\langle\, dt, \mathbf{u} \,\rangle = 1$.

Lemma 21.3 (Tunability). *Any framing* $(\mathbf{u}\,, \boldsymbol{\alpha}) \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM} \times_{\mathbb{M}} \mathbb{T}^*\mathbb{M})$ *is tunable.*

Proof. FROBENIUS integrability of the kernel-distribution of $\boldsymbol{\alpha}$ may be equivalently expressed by the condition

$$\boldsymbol{\alpha} \wedge d\boldsymbol{\alpha} = 0\,.$$

For any scalar field $f \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{FUN}(\mathbb{TM}))$ we have that

$$d(f\boldsymbol{\alpha}) = f\,d\boldsymbol{\alpha} + df \wedge \boldsymbol{\alpha}\,.$$

This relation ensures integrability of the kernel-distribution of $f\boldsymbol{\alpha}$ since

$$d(f\boldsymbol{\alpha}) \wedge (f\boldsymbol{\alpha}) = f\,d(f\boldsymbol{\alpha}) \wedge \boldsymbol{\alpha} = f(f d\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} + df \wedge \boldsymbol{\alpha} \wedge \boldsymbol{\alpha}) = 0\,.$$

Tuning is realized by setting $f = \langle \boldsymbol{\alpha}, \mathbf{u} \rangle^{-1}$. ■

22. Special framings

The events manifold $\mathbb{M}$ is a m-dimensional pseudo-RIEMANN manifold with pseudo-metric tensor field $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathbb{M}\,;\mathrm{SYM}(\mathbb{TM}))$ having the MINKOWSKI signature $(+++-)$. An observer is defined by a pair $(\mathbf{u}\,, dt)$ made of a field $\mathbf{u} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ of tangent vectors called *time-arrows* and of an exact *time* one-form $dt \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{T}^*\mathbb{M})$ such that $\langle dt, \mathbf{u} \rangle = 1$.

The subbundle of the tangent bundle $\mathbb{TM}$ whose fibers are the null spaces of the *time* one-form is integrable, which means that there exists a submanifold (the integral manifold) whose tangent manifold gives the subbundle.

In classical physics the events manifold $\mathbb{M}$ is a four-dimensional affine manifold with model linear space V, see e.g. (É. Cartan, 1924) and an observer is defined by a constant vector field of *time-arrows* generated by translations of a vector $\mathbf{u} \in V$.

Two observers are synchronized if the corresponding classes of simultaneous events are parallel each other and the time origin event $\mathbf{e}_0$ of each of them belongs to the same class.

Each observer generates an isomorphism $\boldsymbol{\gamma} \in \mathrm{C}^1(\mathbb{M}\,;\mathcal{S} \times \mathcal{Z})$ defined by

$$\boldsymbol{\gamma}(\mathbf{e}) = (\mathbf{x}\,, t) \quad \Longleftrightarrow \quad \mathbf{x} \equiv t\mathbf{u} + \mathbf{e}_0 \in \mathbb{M}\,, \qquad \mathbf{x} \in \mathcal{S}\,, t \in \mathcal{Z}\,.$$

which assigns, to any event in $\mathbb{M}$, the corresponding location and time-instant as detected by the observer. An observes splits then the tangent bundle $\mathbb{TM}$ into complementary subbundles, the spatial bundle $\mathbb{VM}$ and a temporal bundle $\mathbb{ZM}$. The projectors on spatial slices and on time-lines are denoted by $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M}_{\mathcal{S}})$ and $\boldsymbol{\pi}_{\mathbb{M}_{\mathcal{Z}},\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M}_{\mathcal{Z}})$ so that $T\boldsymbol{\pi} \in \mathrm{C}^0(\mathbb{TM}\,;\mathbb{VM})$ and $T\boldsymbol{\pi}_{\mathbb{M}_{\mathcal{Z}},\mathbb{M}} \in \mathrm{C}^0(\mathbb{TM}\,;\mathbb{ZM})$ are the corresponding tangent.

23. Space-Time and Material-Time splits

The next Lemma shows that a k-form on the events m-manifold $\mathbb{M}$ is seen by an observer $(\mathbf{u}\,, dt)$ as equivalent to a pair of forms, respectively of degree k and $k-1$, in spatial subbundle $\mathbb{VM}$. The result will enable one to compare formulations of electrodynamics in the four-dimensional space-time with the standard one in three-dimensional space.

Let us consider the quotient manifold $\mathbb{M}_{\mathcal{S}}$ and its injective immersion $\mathbf{i} \in \mathrm{C}^1(\mathbb{M}_{\mathcal{S}}\,; \mathbb{M})$ in the space-time manifold such that $\boldsymbol{\pi} \circ \mathbf{i} = \mathbf{id}_{\mathbb{M}_{\mathcal{S}}}$.

Lemma 23.1 (Space-time split). *A framing*

$$(\mathbf{u}\,, dt) \in \mathrm{C}^1(\mathbb{M}\,; \mathbb{TM} \times_{\mathbb{M}} \mathbb{T}^*\mathbb{M})$$

induces a one-to-one bilinear correspondence between forms $\boldsymbol{\omega}^k_{\mathbb{TM}} \in \boldsymbol{\Lambda}^k(\mathbb{TM}\,; \mathcal{R})$ *in the events manifold and pairs of spatial forms, according to the relations*

$$\begin{aligned} \boldsymbol{\omega}^k_{\mathbb{VM}} &:= \mathbf{i}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}} \in \boldsymbol{\Lambda}^k(\mathbb{VM}\,; \mathcal{R})\,, \\ \boldsymbol{\omega}^{k-1}_{\mathbb{VM}} &:= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^k_{\mathbb{TM}} \cdot \mathbf{u}) \in \boldsymbol{\Lambda}^{k-1}(\mathbb{VM}\,; \mathcal{R})\,, \end{aligned}$$

with the inverse split formula

$$\boldsymbol{\omega}^k_{\mathbb{TM}} = \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{VM}} + \tfrac{1}{\langle dt, \mathbf{u}\rangle} dt \wedge (\boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^{k-1}_{\mathbb{VM}})\,.$$

In a tuned framing the formula becomes

$$\boldsymbol{\omega}^k_{\mathbb{TM}} = \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{VM}} + dt \wedge (\boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^{k-1}_{\mathbb{VM}})\,.$$

Proof. Setting $k = 2$ for simplicity and $\boldsymbol{\pi}{\uparrow}\delta\mathbf{e}_i = \delta\mathbf{x}_i$, $\langle dt, \delta\mathbf{e}_i\rangle = \delta t_i$, for $i = 1, 2$, we have that

$$\delta\mathbf{e}_i = \frac{\delta t_i}{\langle dt, \mathbf{u}\rangle}\,\mathbf{u} + \mathbf{i}{\uparrow}\delta\mathbf{x}_i \in \mathbb{TM}\,,$$

being $\boldsymbol{\pi}{\uparrow}\mathbf{u} = 0$. Then

$$\begin{aligned} (\boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{VM}}) \cdot (\delta\mathbf{e}_1\,, \delta\mathbf{e}_2) &= \boldsymbol{\omega}^2_{\mathbb{VM}} \cdot (\delta\mathbf{x}_1\,, \delta\mathbf{x}_2) \\ &= (\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{TM}}) \cdot (\delta\mathbf{x}_1\,, \delta\mathbf{x}_2) \\ &= \boldsymbol{\omega}^2_{\mathbb{TM}} \cdot (\mathbf{i}{\uparrow}\delta\mathbf{x}_1\,, \mathbf{i}{\uparrow}\delta\mathbf{x}_2)\,. \end{aligned}$$

On the other hand the definition of exterior product gives

$$
\begin{aligned}
& dt \wedge (\boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^1_{\mathbb{VM}}) \cdot (\delta\mathbf{e}_1 \,, \delta\mathbf{e}_2) \\
& = (\boldsymbol{\omega}^2_{\mathbb{TM}} \cdot \mathbf{u} \cdot (\mathbf{i}{\uparrow}\delta\mathbf{x}_2))\, \delta t_1 - (\boldsymbol{\omega}^2_{\mathbb{TM}} \cdot \mathbf{u} \cdot (\mathbf{i}{\uparrow}\delta\mathbf{x}_1))\, \delta t_2 \,.
\end{aligned}
$$

Then the evaluation

$$
\begin{aligned}
\boldsymbol{\omega}^2_{\mathbb{TM}} \cdot (\delta\mathbf{e}_1 \,, \delta\mathbf{e}_2) &= \boldsymbol{\omega}^2_{\mathbb{TM}} \cdot \left(\tfrac{\delta t_1}{\langle dt, \mathbf{u}\rangle}\, \mathbf{u} + \mathbf{i}{\uparrow}\delta\mathbf{x}_1 \,, \tfrac{\delta t_2}{\langle dt, \mathbf{u}\rangle}\, \mathbf{u} + \mathbf{i}{\uparrow}\delta\mathbf{x}_2\right) \\
&= \tfrac{\delta t_1\, \delta t_2}{\langle dt, \mathbf{u}\rangle^2}\, (\boldsymbol{\omega}^2_{\mathbb{TM}} \cdot \mathbf{u} \cdot \mathbf{u}) + \boldsymbol{\omega}^2_{\mathbb{TM}} \cdot (\mathbf{i}{\uparrow}\delta\mathbf{x}_1) \cdot (\mathbf{i}{\uparrow}\delta\mathbf{x}_2) \\
&+ \tfrac{1}{\langle dt, \mathbf{u}\rangle} \Big[(\boldsymbol{\omega}^2_{\mathbb{TM}} \cdot \mathbf{u} \cdot (\mathbf{i}{\uparrow}\delta\mathbf{x}_2))\, \delta t_1 - (\boldsymbol{\omega}^2_{\mathbb{TM}} \cdot \mathbf{u} \cdot (\mathbf{i}{\uparrow}\delta\mathbf{x}_1))\, \delta t_2\Big] \,,
\end{aligned}
$$

taking into account that $\boldsymbol{\omega}^2_{\mathbb{TM}} \cdot \mathbf{u} \cdot \mathbf{u} = 0$, yields the result. ■

Let us now assume the trajectory velocity $\mathbf{v}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}\,; \mathbb{T}\mathcal{T})$ as time-arrows field, so that

$$
\langle dt, \mathbf{v}_{\mathcal{T}} \rangle \neq 0 \,, \qquad \boldsymbol{\pi}{\uparrow}\mathbf{v}_{\mathcal{T}} = 0 \,, \qquad \delta\mathbf{e}_i = \frac{\delta t_i}{\langle dt, \mathbf{v}_{\mathcal{T}} \rangle}\, \mathbf{v}_{\mathcal{T}} + \mathbf{i}{\uparrow}\delta\mathbf{x}_i \in \mathbb{TM} \,.
$$

Lemma 23.2 (Material-Time split). *A tuned trajectory framing*

$$
(\mathbf{v}_{\mathcal{T}} \,, dt) \in \mathrm{C}^1(\mathcal{T}\,; \mathbb{T}\mathcal{T} \times_{\mathbb{M}} \mathbb{T}^*\mathcal{T})
$$

induces a one-to-one bilinear correspondence between forms $\boldsymbol{\omega}^k_{\mathbb{TM}} \in \boldsymbol{\Lambda}^k(\mathbb{T}\mathcal{T}\,; \mathcal{R})$ *in the trajectory manifold and a pair of material forms, according to the relations*

$$
\boxed{
\begin{aligned}
\boldsymbol{\omega}^k_{\mathbb{V}\mathcal{T}} &:= \mathbf{i}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{T}\mathcal{T}} \in \boldsymbol{\Lambda}^k(\mathbb{VM}\,; \mathcal{R}) \,, \\
\boldsymbol{\omega}^{k-1}_{\mathbb{V}\mathcal{T}} &:= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^k_{\mathbb{T}\mathcal{T}} \cdot \mathbf{v}_{\mathcal{T}}) \in \boldsymbol{\Lambda}^{k-1}(\mathbb{VM}\,; \mathcal{R}) \,,
\end{aligned}
}
$$

with the inverse split formula

$$
\boxed{\boldsymbol{\omega}^k_{\mathbb{TM}} = \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{VM}} + dt \wedge (\boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^{k-1}_{\mathbb{VM}}) \,.}
$$

The next result follows from Lemma 3.2 with the proviso that the chains involved in the proof must lie in a spatial slice.

Lemma 23.3 (Spatialization of exterior derivative). *The pull-back according to the canonical immersion* $\mathbf{i} \in \mathrm{C}^1(\mathbb{M}_{\mathcal{S}}\,;\mathbb{M})$ *and the exterior derivatives,* d *in the space-time manifolds* $\mathbb{M}$ *and* $d_{\mathcal{S}}$ *in the spatial fibers of* $\mathbb{M}_{\mathcal{S}}$ *fulfill the commutative diagram*

$$\begin{tikzcd}
\boldsymbol{\Lambda}^{k+1}(\mathbb{TM}\,;\mathcal{R}) \arrow[r, "\mathbf{i}\downarrow"] & \boldsymbol{\Lambda}^{k+1}(\mathbb{VM}\,;\mathcal{R}) \\
\boldsymbol{\Lambda}^{k}(\mathbb{TM}\,;\mathcal{R}) \arrow[u, "d"] \arrow[r, "\mathbf{i}\downarrow"] & \boldsymbol{\Lambda}^{k}(\mathbb{VM}\,;\mathcal{R}) \arrow[u, "d_{\mathcal{S}}"']
\end{tikzcd} \quad \iff \quad \boxed{\mathbf{i}\downarrow \circ\, d = d_{\mathcal{S}} \circ \mathbf{i}\downarrow .}$$

Lemma 23.4 (Spatialization of Lie derivative). *The pull-back according to the canonical immersion* $\mathbf{i} \in \mathrm{C}^1(\mathbb{M}_{\mathcal{S}}\,;\mathbb{M})$ *of the* LIE *derivative* $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k_{\mathbb{TM}}$ *of a form* $\boldsymbol{\omega}^k_{\mathbb{TM}} \in \boldsymbol{\Lambda}^k(\mathbb{TM}\,;\mathcal{R})$ *in the space-time manifold is equal to the* LIE *derivative* $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k_{\mathbb{VM}}$ *of its spatialization* $\boldsymbol{\omega}^k_{\mathbb{VM}} = \mathbf{i}\downarrow\boldsymbol{\omega}^k_{\mathbb{TM}} \in \boldsymbol{\Lambda}^k(\mathbb{VM}\,;\mathcal{R})$

$$\mathbf{i}\downarrow(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k_{\mathbb{TM}}) = \mathcal{L}_{\mathbf{v}}\,(\mathbf{i}\downarrow\boldsymbol{\omega}^k_{\mathbb{TM}})\,.$$

The property can be also enunciated by assessing that pull-back according to the immersion $\mathbf{i} \in \mathrm{C}^1(\mathbb{M}_{\mathcal{S}}\,;\mathbb{M})$ *and* LIE *derivative* $\mathcal{L}_{\mathbf{v}}$ *along the motion, fulfill the commutative diagram*

$$\begin{tikzcd}
\boldsymbol{\Lambda}^{k}(\mathbb{TM}\,;\mathcal{R}) \arrow[r, "\mathbf{i}\downarrow"] & \boldsymbol{\Lambda}^{k}(\mathbb{VM}\,;\mathcal{R}) \\
\boldsymbol{\Lambda}^{k}(\mathbb{TM}\,;\mathcal{R}) \arrow[u, "\mathcal{L}_{\mathbf{v}}"] \arrow[r, "\mathbf{i}\downarrow"] & \boldsymbol{\Lambda}^{k}(\mathbb{VM}\,;\mathcal{R}) \arrow[u, "\mathcal{L}_{\mathbf{v}}"']
\end{tikzcd} \quad \iff \quad \boxed{\mathbf{i}\downarrow \circ\, \mathcal{L}_{\mathbf{v}} = \mathcal{L}_{\mathbf{v}} \circ \mathbf{i}\downarrow .}$$

Proof. The motion preserves simultaneity and hence the following commutation property holds

$$\begin{tikzcd}
\mathbb{M} & \mathbb{M}_{\mathcal{S}} \arrow[l, "\mathbf{i}"'] \\
\mathbb{M} \arrow[u, "\boldsymbol{\varphi}_\alpha"] & \mathbb{M}_{\mathcal{S}} \arrow[l, "\mathbf{i}"'] \arrow[u, "\boldsymbol{\varphi}_\alpha"']
\end{tikzcd} \quad
\begin{tikzcd}
\mathbb{TM} & \mathbb{VM} \arrow[l, "\mathbf{i}\uparrow"'] \\
\mathbb{TM} \arrow[u, "\boldsymbol{\varphi}_\alpha\uparrow"] & \mathbb{VM} \arrow[l, "\mathbf{i}\uparrow"'] \arrow[u, "\boldsymbol{\varphi}_\alpha\uparrow"']
\end{tikzcd} \quad \iff \quad \boxed{\begin{cases} \mathbf{i} \circ \boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_\alpha \circ \mathbf{i}\,, \\ \mathbf{i}\uparrow \circ\, \boldsymbol{\varphi}_\alpha\uparrow = \boldsymbol{\varphi}_\alpha\uparrow \circ \mathbf{i}\uparrow . \end{cases}}$$

Hence a direct computation

$$
\begin{aligned}
(\mathbf{i}{\downarrow}(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k_{\mathbb{TM}}))(\mathbf{a}) &= (\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^k_{\mathbb{TM}})(\mathbf{i}{\uparrow}\mathbf{a}) \\
&= \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}})(\mathbf{i}{\uparrow}\mathbf{a}) \\
&= \partial_{\alpha=0}\,\boldsymbol{\omega}^k_{\mathbb{TM}}(\boldsymbol{\varphi}_\alpha{\uparrow}\mathbf{i}{\uparrow}\mathbf{a}) \\
&= \partial_{\alpha=0}\,\boldsymbol{\omega}^k_{\mathbb{TM}}(\mathbf{i}{\uparrow}\boldsymbol{\varphi}_\alpha{\uparrow}\mathbf{a}) \\
&= \partial_{\alpha=0}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}})(\boldsymbol{\varphi}_\alpha{\uparrow}\mathbf{a}) \\
&= \mathcal{L}_{\mathbf{v}}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}})(\mathbf{a})\,,
\end{aligned}
$$

yields the result. ■

Lemma 23.5 (Spatialization of a contracted form). *Let us consider a form* $\boldsymbol{\omega}^k_{\mathbb{TM}} \in \boldsymbol{\Lambda}^k(\mathbb{TM}\,;\mathcal{R})$ *on the space-time manifold and a time-vertical vector field* $\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ *so that* $\mathbf{w} = \mathbf{i}{\uparrow}\mathbf{w}_{\mathcal{S}}$ *with* $\mathbf{w}_{\mathcal{S}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{VM})$ *. Then*

$$
\mathbf{i}{\downarrow}(\boldsymbol{\omega}^k_{\mathbb{TM}} \cdot \mathbf{w}) = (\mathbf{i}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}}) \cdot \mathbf{w}_{\mathcal{S}}\,.
$$

Proof. For any $\mathbf{a}_{\mathcal{S}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{VM})$

$$
\begin{aligned}
\mathbf{i}{\downarrow}(\boldsymbol{\omega}^k_{\mathbb{TM}} \cdot \mathbf{w})(\mathbf{a}_{\mathcal{S}}) = (\boldsymbol{\omega}^k_{\mathbb{TM}} \cdot \mathbf{w})(\mathbf{i}{\uparrow}\mathbf{a}_{\mathcal{S}}) &= \boldsymbol{\omega}^k_{\mathbb{TM}}(\mathbf{i}{\uparrow}\mathbf{w}_{\mathcal{S}}\,,\mathbf{i}{\uparrow}\mathbf{a}_{\mathcal{S}}) \\
&= (\mathbf{i}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}})(\mathbf{w}_{\mathcal{S}}\,,\mathbf{a}_{\mathcal{S}})\,.
\end{aligned}
$$

which gives the result. ■

24. Space-time formulations of electromagnetics

The expressions of electric and magnetic induction rules, according to Faraday and Ampère laws, take their most concise and elegant form when expressed, in the four-dimensional space-time manifold $\mathbb{M}$, in terms of Faraday and Ampère electromagnetic two-forms $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}, \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \in \boldsymbol{\Lambda}^2(\mathbb{TM}\,;\mathcal{R})$ and of the 4-current three-form $\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \in \boldsymbol{\Lambda}^3(\mathbb{TM}\,;\mathcal{R})$.

The treatment developed below extends classical results, where body motion is not taken into account (É. Cartan, 1924, p. 17-19).

FARADAY and AMPÈRE electromagnetic two-forms and of the current three-form in the space-time manifold are expressed in terms of corresponding spatial fields by

$$\begin{aligned}
\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} &= \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^2_{\mathbf{B}} - dt \wedge \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^1_{\mathbf{E}}\,,\\
\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} &= \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^2_{\mathbf{D}} + dt \wedge \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^1_{\mathbf{H}}\,,\\
\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} &= \boldsymbol{\pi}{\downarrow}\,\boldsymbol{\rho}^3 \;\, - dt \wedge \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^2_{\mathbf{J}}\,.
\end{aligned}$$

The former pair of forms is usually referred to as electromagnetic *field strength* and electromagnetic *excitation* (Hehl and Obukhov, 2003) or electromagnetic *field* and electromagnetic *induction* (Marmo et al., 2005).

The formulation of FARADAY induction law is expressed by the closedness of FARADAY *pair* two-form $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$, equivalent to vanishing of its integral on the boundary of any three-dimensional submanifold $\boldsymbol{\Sigma}^3_{\mathbb{M}} \subset \mathbb{M}$

$$\oint_{\partial\boldsymbol{\Sigma}^3_{\mathbb{M}}} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = \int_{\boldsymbol{\Sigma}^3_{\mathbb{M}}} d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \quad \Longleftrightarrow \quad d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{F}}\,.$$

In the same way, AMPÈRE induction law is expressed, in terms of the *impair* two-form $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}$ by the condition

$$\oint_{\partial\boldsymbol{\Sigma}^3_{\mathbb{M}}} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} = \int_{\boldsymbol{\Sigma}^3_{\mathbb{M}}} \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \quad \Longleftrightarrow \quad d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} = \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}\,.$$

The manifold $\mathbb{M}$ being star-shaped, according to POINCARÉ Lemma these conditions are equivalent to the closedness properties

$$d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = 0\,, \quad d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{F}} = 0\,,$$

which are expressions of the conservation of electric and magnetic charges, respectively. To esplicate the relation between these conditions and the standard ones in the three-dimensional EUCLID space, we resort to the split induced by an EUCLID observer.

24.1. Faraday law in space-time manifold

Let $\mathbf{v}_{\mathcal{T}} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,; \mathbb{T}\mathcal{T})$ be the trajectory velocity.

Definition 24.1 (Electric field and magnetic vortex). *The electric field and the magnetic vortex in the body in motion may be set equal to the following pull-backs, to the time-vertical subbundle* $\mathbb{VM}$*, of the electromagnetic two-form* $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$ *in the space-time bundle* $\mathbb{TM}$ *(*spatializations*)*

$$\begin{aligned} \boldsymbol{\omega}^2_{\mathbf{B}} &= \mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \in \boldsymbol{\Lambda}^2(\mathbb{VM}\,;\mathcal{R})\,, \\ -\boldsymbol{\omega}^1_{\mathbf{E}} &= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\cdot\mathbf{v}_{\mathcal{T}}) \in \boldsymbol{\Lambda}^1(\mathbb{VM}\,;\mathcal{R})\,. \end{aligned}$$

Proposition 24.1 (Faraday law). *Closedness of* Faraday *two-form in space-time manifold is equivalent to the spatial* Gauss *law for the magnetic vortex and to the spatial* Faraday *induction law, i.e.*

$$d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = 0 \quad \iff \quad \begin{cases} d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}} = 0\,, \\ \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}} = 0\,. \end{cases}$$

Proof. Recalling the commutativity properties stated in Lemmata 23.3,23.4 and the homotopy formula of Sect. 3

$$(d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})\cdot\mathbf{v}_{\mathcal{T}} = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} - d(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\cdot\mathbf{v}_{\mathcal{T}})\,,$$

from Lemma 23.1 we infer that

$$\begin{cases} \mathbf{i}{\downarrow}(d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{B}}\,, \\ \mathbf{i}{\downarrow}((d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})\cdot\mathbf{v}_{\mathcal{T}}) = \mathbf{i}{\downarrow}(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} - d(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\cdot\mathbf{v}_{\mathcal{T}})) \\ \qquad = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) - d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\cdot\mathbf{v}_{\mathcal{T}})) \\ \qquad = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^2_{\mathbf{B}} + d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{E}}\,, \end{cases}$$

and the result follows. ■

In defining the electric field and the magnetic vortex the following gauges should be taken into account

$$\begin{aligned} -\boldsymbol{\omega}^1_{\mathbf{E}} &:= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\cdot\mathbf{v}_{\mathcal{T}}) + d_{\mathcal{S}}\,\boldsymbol{\omega}^0_{\mathbf{E}}\,, \\ \boldsymbol{\omega}^2_{\mathbf{B}} &:= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) + d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{B}}\,. \end{aligned}$$

Then no definite value may be assigned to electric field and to magnetic vortex on the basis of Def. 24.1.

24.2. Ampere law in space-time manifold

Let us now turn to the AMPÈRE induction law.

Definition 24.2 (Electric flux, magnetic winding, charge, current). *Magnetic winding* $\boldsymbol{\omega}^1_{\mathbf{H}}$ *, electric flux* $\boldsymbol{\omega}^2_{\mathbf{D}}$ *, electric current flux* $\boldsymbol{\omega}^2_{\mathbf{J}}$ *and electric charge* $\boldsymbol{\rho}^3$ *may be may be set equal to the pull-backs (*spatializations*)*

$$\begin{aligned} \boldsymbol{\omega}^2_{\mathbf{D}} &= \mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \in \boldsymbol{\Lambda}^2(\mathbb{VM}\,;\mathcal{R})\,, \\ \boldsymbol{\omega}^1_{\mathbf{H}} &= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}}) \in \boldsymbol{\Lambda}^1(\mathbb{VM}\,;\mathcal{R})\,, \\ \boldsymbol{\rho}^3 &= \mathbf{i}{\downarrow}\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \in \boldsymbol{\Lambda}^3(\mathbb{VM}\,;\mathcal{R})\,, \\ -\boldsymbol{\omega}^2_{\mathbf{J}} &= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}}) \in \boldsymbol{\Lambda}^2(\mathbb{VM}\,;\mathcal{R})\,. \end{aligned}$$

Proposition 24.2 (Charge conservation law). *Closedness of the* AMPÈRE *three-form in space-time manifold is equivalent to spatial conservation law for the electric charge, i.e.*

$$d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = 0 \quad \Longleftrightarrow \quad \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} = 0\,.$$

Proof. By homotopy formula we have that

$$(d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}) \cdot \mathbf{v}_{\mathcal{T}} = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} - d(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}})\,.$$

Recalling the commutation property $d_{\mathcal{S}} \circ \mathbf{i}{\downarrow} = \mathbf{i}{\downarrow} \circ d$ stated in Lemma 23.3 , the pull-back of the LIE derivative and of the exterior derivative at the r.h.s. may be written as

$$\begin{cases} \mathbf{i}{\downarrow}(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}) = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}) = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\rho}^3\,, \\ \mathbf{i}{\downarrow}d(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}}) = d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}})) = -d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}}\,. \end{cases}$$

According to Lemma 23.1, the condition $d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = 0$ is equivalent to the conditions

$$\begin{cases} \mathbf{i}{\downarrow}(d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}) = 0\,, \\ \mathbf{i}{\downarrow}(d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}}) = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}) - d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}})) \\ \qquad\qquad = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} = 0\,. \end{cases}$$

The former holds by trivial vanishing of the 4-form $\mathbf{i}{\downarrow}(d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}})$ in the 3D spatial slice, while the latter is the charge spatial conservation law. ■

We underline that partial time derivatives, such as the one appearing in literature in the so called *equation of continuity* $\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\,\boldsymbol{\rho}^3 + d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{J}} = 0$ for electric charges, may be not defined due to abrupt changes, with respect to time, of the electric charge at a spatial point crossed by an electrically charged body.

Proposition 24.3 (Ampère law). AMPÈRE *law in space-time manifold is equivalent to the spatial* GAUSS *law for the electric displacement flux and to the spatial* AMPÈRE *induction law, i.e.*

$$d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} = \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \quad \iff \quad \begin{cases} d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{D}} = \boldsymbol{\rho}^3\,, \\ \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^2_{\mathbf{D}} - d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{H}} = -\boldsymbol{\omega}^2_{\mathbf{J}}\,. \end{cases}$$

Proof. By Lemmata 23.1, 23.3,23.4 and the homotopy formula, we get the equalities

$$\begin{cases} \mathbf{i}{\downarrow}\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = \boldsymbol{\rho}^3\,, \\ \mathbf{i}{\downarrow}(d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}) = d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}) = d_{\mathcal{S}}\,\boldsymbol{\omega}^2_{\mathbf{D}}\,, \\ \mathbf{i}{\downarrow}(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}}) = -\boldsymbol{\omega}^2_{\mathbf{J}}\,, \\ \mathbf{i}{\downarrow}(d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}}) = \mathbf{i}{\downarrow}(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}) - \mathbf{i}{\downarrow}(d(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}})) \\ \qquad = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,(\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}) - d_{\mathcal{S}}\,(\mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_{\mathcal{T}})) \\ \qquad = \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\,\boldsymbol{\omega}^2_{\mathbf{D}} - d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{H}}\,, \end{cases}$$

and hence the result. ■

25. Electromagnetic potentials in space-time manifold

In conclusion, we see that the laws of electrodynamic induction are written and discussed in the simplest way, from the geometric point of view, when formulated in a 4-dimensional space-time manifold $\mathbb{M}$. The physical interpretation is however more cryptic than in the standard 3-dimensional spatial treatment, since the familiar picture, provided by the everyday space-time splitting, is lost.

The mathematical expressions of magnetic and electric charge balance laws in the space-time manifold are respectively given by

$$\begin{cases} d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{F}} = 0 & \iff \quad \displaystyle\oint_{\partial\boldsymbol{\Omega}^4_{\mathbb{M}}} \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{F}} = 0\,, \\ d\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = 0 & \iff \quad \displaystyle\oint_{\partial\boldsymbol{\Omega}^4_{\mathbb{M}}} \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = 0\,, \end{cases}$$

to hold for all 4-dimensional submanifold $\boldsymbol{\Omega}^4_{\mathbb{M}} \subset \mathbb{M}$.

These closedness properties are respectively equivalent to assume that absence of bulk sources of magnetic or electric charges is found by any observer testing the charge balance laws.

By POINCARÉ Lemma, the closedness conditions above are equivalent to the potentiality requirements

$$\begin{cases} \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{F}} = d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\,, \\ \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}\,, \end{cases}$$

which in turn have been previously shown to be equivalent to the differential FARADAY and AMPÈRE induction laws in space-time. The integral expression are given by

$$\begin{cases} \displaystyle\int_{\boldsymbol{\Omega}^3_{\mathbb{M}}} \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{F}} = \oint_{\partial\boldsymbol{\Omega}^3_{\mathbb{M}}} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\,, \\ \displaystyle\int_{\boldsymbol{\Omega}^3_{\mathbb{M}}} \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} = \oint_{\partial\boldsymbol{\Omega}^3_{\mathbb{M}}} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}\,, \end{cases}$$

to hold for all 3-dimensional submanifold $\boldsymbol{\Omega}^3_{\mathbb{M}} \subset \mathbb{M}$.

It is usually assumed that $\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{F}} = 0$, a condition inferred from the experimental fact that magnetic monopoles and magnetic currents are still undiscovered. FARADAY law of electromagnetic induction may accordingly be expressed as

$$\oint_{\partial\boldsymbol{\Omega}^3_{\mathbb{M}}} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = 0 \quad \iff \quad 0 = d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \quad \iff \quad \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = d\boldsymbol{\omega}^1_{\mathbb{M},\mathbf{F}}\,,$$

or as an *action principle* for the space-time FARADAY two form

$$\partial_{\alpha=0} \int_{\delta\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}^2_{\mathbb{M}})} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = \oint_{\partial\boldsymbol{\Omega}^2_{\mathbb{M}}} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \delta\mathbf{v}\,,$$

where $\delta\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\boldsymbol{\Omega}^2_{\mathbb{M}}\,;\mathbb{M})$ is a virtual motion in the events manifold with velocity $\delta\mathbf{v} = \partial_{\alpha=0}\,\delta\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\boldsymbol{\Omega}^2_{\mathbb{M}}\,;\mathbb{TM})$.

The space-time potential one-form $\boldsymbol{\omega}^1_{\mathbb{M},\mathbf{F}} \in \boldsymbol{\Lambda}^1(\mathbb{TM}\,;\mathcal{R})$, called *electromagnetic potential*, is related to the spatial magnetic potential one-form $\boldsymbol{\omega}^1_{\mathbf{B}} \in \boldsymbol{\Lambda}^1(\mathbb{VM}\,;\mathcal{R})$ and to the scalar potential $V_{\mathbf{E}}$ by the pull-backs

$$\begin{aligned} \boldsymbol{\omega}^1_{\mathbf{B}} &= \mathbf{i}{\downarrow}\boldsymbol{\omega}^1_{\mathbb{M},\mathbf{F}}\,, \\ V_{\mathbf{E}} &= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^1_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}_{\mathcal{T}})\,, \end{aligned}$$

so that

$$\boldsymbol{\omega}^1_{\mathbb{M},\mathbf{F}} = \boldsymbol{\pi}{\downarrow}\boldsymbol{\omega}^1_{\mathbf{B}} - dt \wedge \boldsymbol{\pi}{\downarrow}V_{\mathbf{E}}\,.$$

26. Frame invariance

A space-time change of framing $\boldsymbol{\zeta}_{\mathbb{M}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{M})$ is *compatible* with a spatial change of framing $\boldsymbol{\zeta}_{\mathcal{S}} \in \mathrm{C}^1(\mathbb{M}_{\mathcal{S}}\,;\mathbb{M}_{\boldsymbol{\zeta},\mathcal{S}})$ if the following commutative diagram holds

$$\begin{tikzcd} \mathbb{M} & \mathbb{M}_{\boldsymbol{\zeta},\mathcal{S}} \arrow[l, "\mathbf{i}_{\boldsymbol{\zeta}}"'] \\ \mathbb{M} \arrow[u, "\boldsymbol{\zeta}_{\mathbb{M}}"] & \mathbb{M}_{\mathcal{S}} \arrow[l, "\mathbf{i}"'] \arrow[u, "\boldsymbol{\zeta}_{\mathcal{S}}"'] \end{tikzcd} \iff \mathbf{i}_{\boldsymbol{\zeta}} \circ \boldsymbol{\zeta}_{\mathcal{S}} = \boldsymbol{\zeta}_{\mathbb{M}} \circ \mathbf{i}\,.$$

Acting on with the tangent functor, we get the commutative diagram

$$\begin{tikzcd} \mathbb{TM} & \mathbb{TM}_{\boldsymbol{\zeta},\mathcal{S}} \arrow[l, "T\mathbf{i}_{\boldsymbol{\zeta}}"'] \\ \mathbb{TM} \arrow[u, "T\boldsymbol{\zeta}_{\mathbb{M}}"] & \mathbb{TM}_{\mathcal{S}} \arrow[l, "T\mathbf{i}"'] \arrow[u, "T\boldsymbol{\zeta}_{\mathcal{S}}"'] \end{tikzcd} \iff T\mathbf{i}_{\boldsymbol{\zeta}} \circ T\boldsymbol{\zeta}_{\mathcal{S}} = T\boldsymbol{\zeta}_{\mathbb{M}} \circ T\mathbf{i}\,.$$

In the sequel all space-time change of framing will be assumed to be compatible with a spatial change of framing.

Lemma 26.1 (Commutation between spatialization and push). *Let us consider a form $\boldsymbol{\omega}^k_{\mathbb{TM}} \in \boldsymbol{\Lambda}^k(\mathbb{TM}\,;\mathcal{R})$ in the space-time manifold and a time-vertical tangent vector field $\mathbf{w} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{TM})$ so that $\mathbf{w} = \mathbf{i}{\uparrow}\mathbf{w}_{\mathcal{S}}$ with $\mathbf{w}_{\mathcal{S}} \in \mathrm{C}^1(\mathbb{M}\,;\mathbb{VM})$. Then*

$$\boldsymbol{\zeta}_{\mathcal{S}}{\downarrow}\mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}} = \mathbf{i}{\downarrow}\boldsymbol{\zeta}_{\mathbb{M}}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}}\,.$$

Proof. Assuming $k = 2$ and $\mathbf{a}, \mathbf{b} \in \mathbb{VM}$, we get

$$\begin{aligned}(\boldsymbol{\zeta}_{\mathcal{S}}{\downarrow}\mathbf{i}_{\zeta}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}})(\mathbf{a}\,,\mathbf{b}) &= \boldsymbol{\omega}^k_{\mathbb{TM}}(\mathbf{i}_{\zeta}{\uparrow}\boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\mathbf{a}\,,\mathbf{i}_{\zeta}{\uparrow}\boldsymbol{\zeta}_{\mathcal{S}}{\uparrow}\mathbf{b})\\ &= \boldsymbol{\omega}^k_{\mathbb{TM}}(\boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\mathbf{i}{\uparrow}\mathbf{a}\,,\boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\mathbf{i}{\uparrow}\mathbf{b})\\ &= (\mathbf{i}{\downarrow}\boldsymbol{\zeta}_{\mathbb{M}}{\downarrow}\boldsymbol{\omega}^k_{\mathbb{TM}})(\mathbf{a}\,,\mathbf{b})\,,\end{aligned}$$

which gives the result. ■

Proposition 26.1 (Frame invariance of Faraday and Ampère laws). *The space-time frame invariance of* Faraday *and* Ampère *electromagnetic two-forms and of the current three-form*

$$\begin{cases}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\zeta_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\,,\\ (\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}})_{\zeta_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}\,,\\ (\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}})_{\zeta_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}\,,\end{cases}$$

imply the space-time frame invariance of Faraday *and* Ampère *laws of induction*

$$\begin{aligned} d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = 0 \quad &\Longleftrightarrow \quad d(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\zeta_{\mathbb{M}}} = 0\,,\\ d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} = \boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \quad &\Longleftrightarrow \quad d(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}})_{\zeta_{\mathbb{M}}} = (\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}})_{\zeta_{\mathbb{M}}}\,,\end{aligned}$$

Proof. The result is a direct consequence of the commutativity between exterior derivative and push by a diffeomorphism, see Lemma 3.2. Indeed

$$d(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\zeta_{\mathbb{M}}} = d(\boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}(d\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})\,,$$

and similarly for the second equivalence. ■

The next result proves the equivalence between frame-invariance of events four-forms and spatial frame-invariance of their spatial components, under any change of frame. We underline that a frame is just a chart for the events manifold which assigns Gauss coordinates to each event in it (Einstein, 1916, Part II: The General Theory of Relativity - 25. Gaussian Co-ordinates, p.75). Consequently, a change of frame is just an automorphism of the events manifold, as enunciated in the definition given in Sect.13, in accord with the spirit of general relativity. Nevertheless neither *relativity theory*, special or general, nor Minkowski pseudo-metric, play any role in the treatment of frame invariance.

Proposition 26.2 (Space-time and spatial frame-invariance). *Space time frame invariance of* FARADAY *and* AMPÈRE *electromagnetic two-forms* $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}, \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \in \boldsymbol{\Lambda}^2(\mathbb{TM}\,;\mathcal{R})$ *and of the current three-form* $\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \in \boldsymbol{\Lambda}^3(\mathbb{TM}\,;\mathcal{R})$ *is equivalent to spatial frame invariance of all spatial electromagnetic forms*

$$\begin{cases} (\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\boldsymbol{\zeta}_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \\ (\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}})_{\boldsymbol{\zeta}_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \\ (\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}})_{\boldsymbol{\zeta}_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \end{cases} \iff \begin{cases} (\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}_{\mathcal{S}}} = \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\omega}^1_{\mathbf{E}} \\ (\boldsymbol{\omega}^2_{\mathbf{B}})_{\boldsymbol{\zeta}_{\mathcal{S}}} = \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbf{B}} \\ (\boldsymbol{\omega}^1_{\mathbf{H}})_{\boldsymbol{\zeta}_{\mathcal{S}}} = \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\omega}^1_{\mathbf{H}} \\ (\boldsymbol{\omega}^2_{\mathbf{D}})_{\boldsymbol{\zeta}_{\mathcal{S}}} = \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbf{D}} \\ (\boldsymbol{\omega}^2_{\mathbf{J}})_{\boldsymbol{\zeta}_{\mathcal{S}}} = \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbf{J}} \\ (\boldsymbol{\rho}^3)_{\boldsymbol{\zeta}_{\mathcal{S}}} = \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\rho}^3 \end{cases}$$

Proof. Let us assume space-time frame invariance $(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\boldsymbol{\zeta}_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$ of FARADAY two-form. Then, by space-time frame invariance of the trajectory speed $\mathbf{v}_{\mathcal{T}_{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{v}_{\mathcal{T}}$, stated in Lemma 13.1, and the commutativity property stated in Lemma 26.1, we infer the material frame invariance of the electric field one-form $\boldsymbol{\omega}^1_{\mathbf{E}}$, since

$$\begin{aligned} (\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}_{\mathcal{S}}} &= \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!((\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\boldsymbol{\zeta}_{\mathbb{M}}} \cdot \mathbf{v}_{\mathcal{T}_{\boldsymbol{\zeta}}}) = \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}) \\ &= \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}_{\mathcal{T}})\,, \\ \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\omega}^1_{\mathbf{E}} &= \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\mathbf{i}\!\downarrow\!(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}_{\mathcal{T}}) = \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}_{\mathcal{T}})\,, \end{aligned}$$

and spatial frame invariance of the magnetic vortex two-form $\boldsymbol{\omega}^2_{\mathbf{B}}$, follows by a similar evaluation

$$\begin{aligned} (\boldsymbol{\omega}^2_{\mathbf{B}})_{\boldsymbol{\zeta}_{\mathcal{S}}} &= \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\boldsymbol{\zeta}_{\mathbb{M}}} = \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\,, \\ \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbf{B}} &= \boldsymbol{\zeta}_{\mathcal{S}}\!\uparrow\!\mathbf{i}\!\downarrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = \mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}\,. \end{aligned}$$

The converse implications follow from Lemma 23.1. The same procedure leads to the conclusion that space-time frame invariance of AMPÈRE two-form and of the 4-current three-form is equivalent to spatial frame invariance of magnetic winding $\boldsymbol{\omega}^1_{\mathbf{H}}$, electric flux $\boldsymbol{\omega}^2_{\mathbf{D}}$, electric current flux $\boldsymbol{\omega}^2_{\mathbf{J}}$, and electric charge $\boldsymbol{\rho}^3$. ∎

27. Frame changes in components

Let us represent the FARADAY two-form $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$ by its GRAM matrix with respect to a space-time framing $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with the time arrow as first vector, i.e. $\mathbf{e}_0 = \mathbf{u}$, and the tangent vector fields $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ got by immersion of an orthonormal frame $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ in the spatial slice. Running indexes are assumed to be $\alpha, \beta = 0, 1, 2, 3$ and $i, j, k = 1, 2, 3$. The dual coframing $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ in the cotangent bundle $\mathbb{T}^*\mathbb{M}$ of the framing $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the tangent bundle $\mathbb{TM}$ is defined by $\langle \mathbf{e}^\alpha, \mathbf{e}_\beta \rangle = \delta^\alpha_{.\beta}$.

Then, according to Def.24.1, being $\mathbf{e}_i = \mathbf{i}{\uparrow}\mathbf{s}_i$, the elements of the GRAM matrix of FARADAY two-form are given by

$$\begin{aligned} \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}(\mathbf{e}_i\,,\mathbf{e}_j) &= \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{s}_i\,,\mathbf{s}_j) = \boldsymbol{\mu}(\mathbf{B}, \mathbf{s}_i, \mathbf{s}_j) = \epsilon_{i,j,k}\, B_k\,, \\ \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}(\mathbf{e}_0\,,\mathbf{e}_i) &= \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}(\mathbf{v}_{\mathcal{T}}\,,\mathbf{e}_i) = -\boldsymbol{\omega}^1_{\mathbf{E}}(\mathbf{s}_i) = -E_i\,. \end{aligned}$$

By assumed orthonormality, $\epsilon_{i,j,k} = \boldsymbol{\mu}(\mathbf{s}_i, \mathbf{s}_j, \mathbf{s}_k)$ is the RICCI[46] permutator giving 0 if two indices are equal and otherwise $+1$ or -1, depending on the parity of the indices permutation.

Denoting the space-time velocity of the test particle by

$$\mathbf{v}_{\mathcal{T}} = \begin{bmatrix} 1 \\ v_{\mathcal{T}} \\ 0 \\ 0 \end{bmatrix},$$

the matrix expression of the FARADAY two form is given by

$$\text{GRAM}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = \begin{bmatrix} 0 & -E_1 & -E_2 - v_{\mathcal{T}} B_3 & -E_3 + v_{\mathcal{T}} B_2 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 + v_{\mathcal{T}} B_3 & -B_3 & 0 & B_1 \\ E_3 - v_{\mathcal{T}} B_2 & B_2 & -B_1 & 0 \end{bmatrix}.$$

Indeed from the relation $-\boldsymbol{\omega}^1_{\mathbf{E}} = \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}_{\mathcal{T}})$, being

$$\begin{bmatrix} 0 & -E_1 & -E_2 - v_{\mathcal{T}} B_3 & -E_3 + v_{\mathcal{T}} B_2 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 + v_{\mathcal{T}} B_3 & -B_3 & 0 & B_1 \\ E_3 - v_{\mathcal{T}} B_2 & B_2 & -B_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ v_{\mathcal{T}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -v_{\mathcal{T}}\, E_1 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}^T$$

[46] GREGORIO RICCI CURBASTRO (1853-1925) Italian mathematician.

we infer that

$$\textsc{Gram}(\boldsymbol{\omega}^1_{\mathbf{E}}) = \begin{bmatrix} -v_{\mathcal{T}}\, E_1 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}^T .$$

28. Classical Electrodynamics

The Galilei transformations for a translational motion with relative spatial velocity $v_{\mathcal{S}}$ in the x direction and the associated Jacobi matrix in the framing $\{\, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \,\}$ are given by

$$\boldsymbol{\zeta}_{\mathbb{M}} : \begin{cases} t \mapsto t \\ x \mapsto x - v_{\mathcal{S}}\, t \\ y \mapsto y \\ z \mapsto z \end{cases} \qquad [T\boldsymbol{\zeta}_{\mathbb{M}}] = \begin{bmatrix} 1 & & & \\ -v_{\mathcal{S}} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

The pushed Faraday two form $\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$ is defined by

$$(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})(\mathbf{a}_{\boldsymbol{\zeta}_{\mathbb{M}}}, \mathbf{b}_{\boldsymbol{\zeta}_{\mathbb{M}}}) = \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}(T\boldsymbol{\zeta}_{\mathbb{M}}^{-1} \cdot \mathbf{a}_{\boldsymbol{\zeta}_{\mathbb{M}}}, T\boldsymbol{\zeta}_{\mathbb{M}}^{-1} \cdot \mathbf{b}_{\boldsymbol{\zeta}_{\mathbb{M}}}) ,$$

or, shortly

$$\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} = (T\boldsymbol{\zeta}_{\mathbb{M}}^{-1})^* \circ \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \circ T\boldsymbol{\zeta}_{\mathbb{M}}^{-1} .$$

In the dual coframing $\{\, \mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3 \,\}$ of the framing $\{\, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \,\}$, defined by $\langle \mathbf{e}^\alpha, \mathbf{e}_\beta \rangle = \delta^{\alpha}_{.\beta}$, the matrix $[(T\boldsymbol{\zeta}_{\mathbb{M}}^{-1})^*]$ of the dual map $(T\boldsymbol{\zeta}_{\mathbb{M}}^{-1})^*$ is the transpose of the matrix of the map $T\boldsymbol{\zeta}_{\mathbb{M}}^{-1}$ i.e. $[(T\boldsymbol{\zeta}_{\mathbb{M}}^{-1})^*] = [(T\boldsymbol{\zeta}_{\mathbb{M}}^{-1})]^T$.

Denoting by $[\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}]$ the matrix of the operator $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}(\mathbf{x}) \in BL\,(\mathbb{T}_{\mathbf{x}}\mathbb{M}\,; \mathbb{T}^*_{\mathbf{x}}\mathbb{M})$ it is easy to check that

$$\textsc{Gram}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = [\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}]^T , \quad \textsc{Gram}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = [\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}]^T .$$

The relation $[\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}] = [T\boldsymbol{\zeta}_{\mathbb{M}}^{-1}]^T \circ [\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}] \circ [T\boldsymbol{\zeta}_{\mathbb{M}}^{-1}]$ may then also be written

$$\textsc{Gram}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = [T\boldsymbol{\zeta}_{\mathbb{M}}^{-1}]^T \circ \textsc{Gram}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \circ [T\boldsymbol{\zeta}_{\mathbb{M}}^{-1}] .$$

Performing the computation and setting $v_{\mathcal{TS}} = v_{\mathcal{T}} - v_{\mathcal{S}}$ we get

$$\textsc{Gram}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = \begin{bmatrix} 0 & -E_1 & -E_2 - v_{\mathcal{TS}}\, B_3 & -E_3 + v_{\mathcal{TS}}\, B_2 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 + v_{\mathcal{TS}}\, B_3 & -B_3 & 0 & B_1 \\ E_3 - v_{\mathcal{TS}}\, B_2 & B_2 & -B_1 & 0 \end{bmatrix} .$$

The particle velocity in the new framing is given by the push $\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}$ and has the matrix expression

$$[T\boldsymbol{\zeta}_{\mathbb{M}}] \cdot [\mathbf{v}_{\mathcal{T}}] = \begin{bmatrix} 1 & & & \\ -v_{\mathcal{S}} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ v_{\mathcal{T}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ v_{\mathcal{T}} - v_{\mathcal{S}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ v_{\mathcal{TS}} \\ 0 \\ 0 \end{bmatrix}.$$

The contraction $\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}$ has the Gram matrix representation

$$\text{Gram}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \cdot [\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}] = \begin{bmatrix} -v_{\mathcal{TS}}\, E_1 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}^T.$$

The components of the spatial electric field $(\boldsymbol{\omega}^1_{\mathbf{E}})_{\zeta_{\mathcal{S}}} = -\mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}})$ in the new framing may be evaluated by applying the Gram matrix above to the pushed basis vectors $\{\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{e}_1, \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{e}_2, \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{e}_3\}$. This is equivalent to compute the matrix expression of $(T\boldsymbol{\zeta}_{\mathbb{M}})^* \cdot (\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}})$, given by

$$[T\boldsymbol{\zeta}_{\mathbb{M}}]^T \cdot \text{Gram}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \cdot [\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}] = \begin{bmatrix} -(v_{\mathcal{TS}} + v_{\mathcal{S}})E_1 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}^T$$

Hence

$$\text{Gram}((\boldsymbol{\omega}^1_{\mathbf{E}})_{\zeta_{\mathcal{S}}}) = \begin{bmatrix} -(v_{\mathcal{TS}} + v_{\mathcal{S}})E_1 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}^T$$

in accord with the invariance property assessed in Prop.26.2.

We underline that the matrix $\text{Gram}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})$ of the Faraday two form, given at the beginning of the section, differs from the one reproduced in most treatments of electromagnetism. According to these treatments, it is assumed *ab initio* that the spatial velocity of the test particle vanishes, that is $\mathbf{v}_{\mathcal{T}} = \mathbf{u}$. As a matter of fact, this needless assumption led to the wrong statement that the value of electric field in the new framing can be recovered from the matrix

$$\mathrm{GRAM}(\boldsymbol{\zeta}_{\mathbb{M}}\uparrow\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = \begin{bmatrix} 0 & -E_1 & -E_2 + v_{\mathcal{S}}\, B_3 & -E_3 - v_{\mathcal{S}}\, B_2 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 - v_{\mathcal{S}}\, B_3 & -B_3 & 0 & B_1 \\ E_3 + v_{\mathcal{S}}\, B_2 & B_2 & -B_1 & 0 \end{bmatrix},$$

by contracting with the vector $\mathbf{u} = [1, 0, 0, 0]$ and then performing the spatialization to get the incorrect result

$$\mathrm{GRAM}((\boldsymbol{\omega}^1_{\mathbf{E}})_{\zeta_{\mathcal{S}}}) = \begin{bmatrix} 0 \\ E_1 \\ E_2 - v_{\mathcal{S}}\, B_3 \\ E_3 + v_{\mathcal{S}}\, B_2 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 - v_{\mathcal{S}}\, B_3 \\ E_3 + v_{\mathcal{S}}\, B_2 \end{bmatrix}^T .$$

The right evaluation requires instead to perform the contraction with the vector $\boldsymbol{\zeta}_{\mathbb{M}}\uparrow\mathbf{u} = [1, -v_{\mathcal{S}}, 0, 0]$ and then performing the spatialization to get the final correct result

$$\mathrm{GRAM}((\boldsymbol{\omega}^1_{\mathbf{E}})_{\zeta_{\mathcal{S}}}) = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}^T .$$

29. Relativistic Electrodynamics

Let us consider a LORENTZ transformation and the associated JACOBI matrix

$$\boldsymbol{\zeta}_{\mathbb{M}} : \begin{cases} c\,t \mapsto \beta(c\,t - v_{\mathcal{S}}\, x) \\ x \mapsto \beta(x - v_{\mathcal{S}}\, c\,t) \\ y \mapsto y \\ z \mapsto z \end{cases} \qquad [T\boldsymbol{\zeta}_{\mathbb{M}}] = \begin{bmatrix} \beta & -\beta\, v_{\mathcal{S}} & & \\ -\beta\, v_{\mathcal{S}} & \beta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Then, assuming $c = 1$ and setting $\frac{1}{\beta^2} = 1 - v^2_{\mathcal{S}}$ and $\psi = 1 - v_{\mathcal{T}} v_{\mathcal{S}}$, the matrix

$$\mathrm{GRAM}(\boldsymbol{\zeta}_{\mathbb{M}}\uparrow\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) = [T\boldsymbol{\zeta}^{-1}_{\mathbb{M}}]^T \circ \mathrm{GRAM}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \circ [T\boldsymbol{\zeta}^{-1}_{\mathbb{M}}]$$

is given by

$$\begin{bmatrix} 0 & -E_1 & -\beta(E_2 + v_{\mathcal{TS}}\ B_3) & -\beta(E_3 - v_{\mathcal{TS}}\ B_2) \\ E_1 & 0 & \beta(\psi\, B_3 - v_{\mathcal{S}}\, B_2) & \beta(-\psi\, B_3 - v_{\mathcal{S}}\, B_3) \\ \beta(E_2 + v_{\mathcal{TS}}\, B_3) & -\beta(\psi\, B_3 - v_{\mathcal{S}}\, B_2) & 0 & B_1 \\ \beta(E_3 - v_{\mathcal{TS}}\, B_2) & -\beta(-\psi\, B_3 - v_{\mathcal{S}}\, B_3) & -B_1 & 0 \end{bmatrix}.$$

The pushed particle velocity is

$$[\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}] = \begin{bmatrix} \beta\,\psi \\ \beta\, v_{\mathcal{TS}} \\ 0 \\ 0 \end{bmatrix}$$

We see that the time component of the velocity is not equal to unity. The ratio between the spatial component in the x-direction and the time component gives the velocity expression

$$\frac{\beta\, v_{\mathcal{TS}}}{\beta\,\psi} = \frac{v_{\mathcal{TS}}}{\psi} = \frac{v_{\mathcal{T}} - v_{\mathcal{S}}}{1 - v_{\mathcal{T}} v_{\mathcal{S}}},$$

which is EINSTEIN's formula for composition of velocities having the same direction and opposite sense.

Being $\beta^2(\psi + v_{\mathcal{S}}\, v_{\mathcal{TS}}) = 1$, the spatial electric field $(\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}_{\mathcal{S}}}$ in the new framing is evaluated by the following contraction

$$\mathrm{GRAM}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \cdot [\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}] = \begin{bmatrix} -\beta\, v_{\mathcal{TS}}\, E_1 \\ \beta\,\psi\, E_1 \\ E_2 \\ E_3 \end{bmatrix}^T.$$

The components of the spatial electric field $(\boldsymbol{\omega}^1_{\mathbf{E}})_{\boldsymbol{\zeta}_{\mathcal{S}}} = -\mathbf{i}_{\boldsymbol{\zeta}}\!\downarrow\!(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}})$ in the new framing may be evaluated by applying the GRAM matrix above to the pushed basis vectors $\{\,\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{e}_1, \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{e}_2, \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{e}_3\,\}$. This is equivalent to compute the matrix expression of $(T\boldsymbol{\zeta}_{\mathbb{M}})^* \cdot (\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}})$, which, recalling that $\beta^2(\psi + v_{\mathcal{S}}\, v_{\mathcal{TS}}) = 1$, is given by

$$[T\boldsymbol{\zeta}_{\mathbb{M}}]^T \cdot \mathrm{GRAM}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \cdot [\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\mathbf{v}_{\mathcal{T}}] = \begin{bmatrix} \beta^2\, v_{\mathcal{T}}(v_{\mathcal{S}} - 1)E_1 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}^T$$

Hence

$$\textsc{Gram}((\boldsymbol{\omega}^1_{\mathbf{E}})_{\zeta_{\mathcal{S}}}) = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}^T$$

in accord with the invariance property assessed in Prop.26.2.

The behavior of electric and magnetic fields under LORENTZ transformations was considered in EINSTEIN's seminal paper[47] on the principle of relativity and his conclusions have propagated in all subsequent literature, see e.g. (Weyl, 1922, p.194), (É. Cartan, 1924, p.17), (Stratton, 1941, p.72,79), (Panofsky and Phillips, 1962, p.330), (Feynman et al., 1964, 26.4, p.840), (Purcell, 1985, p.108,128), (Schwinger et al., 1998, 10.3 p.119), (Greiner, 1998, 22.33 p.465), (Jackson, 1999, 11.10 p.558), (Griffiths, 1999, 12.3 p.531), (Wegner, 2003, p.86), (Vanderlinde, 2004, p.316-317), (Thidé, 2010, p.173), (Lehner, 2010, p.628).

Neither in EINSTEIN's paper, nor in any of the subsequent reproductions of his result, an explicit calculation was however performed. The original statement in (Einstein, 1905) is as follows.
Now the principle of relativity requires that if the Maxwell-Hertz equations for empty space hold good in system K, they also hold good in system k ... Evidently the two systems of equations found for system k must express exactly the same thing, since both systems of equations are equivalent to the Maxwell-Hertz equations for system K. Since, further, the equations of the two systems agree, with the exception of the symbols for the vectors, it follows that the functions occurring in the systems of equations at corresponding places must agree, with the exception of a factor $\psi(v)$, which is common for all functions of the one system of equations, and is independent of ξ, η, ζ and τ but depends upon v. Thus we have the relations

$$\begin{array}{lcllcl} \mathrm{X}' & = & \mathrm{X}, & \mathrm{L}' & = & \mathrm{L}, \\ \mathrm{Y}' & = & \beta\left(\mathrm{Y} - \frac{v}{c}\mathrm{N}\right), & \mathrm{M}' & = & \beta\left(\mathrm{M} + \frac{v}{c}\mathrm{Z}\right), \\ \mathrm{Z}' & = & \beta\left(\mathrm{Z} + \frac{v}{c}\mathrm{M}\right), & \mathrm{N}' & = & \beta\left(\mathrm{N} - \frac{v}{c}\mathrm{Y}\right). \end{array}$$

This same result might be deduced by performing an improper geometric operation of contraction between pushed two-form $\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$ and the unpushed vector field $\mathbf{u}$. Expressed in terms of the matrix $\textsc{Gram}(\boldsymbol{\zeta}_{\mathbb{M}}\!\uparrow\!\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})$ and of

[47] (Einstein, 1905) 6. Transformation of the Maxwell-Hertz Equations for Empty Space. On the Nature of the Electromotive Forces Occurring in a Magnetic Field During Motion.

the vector $[\mathbf{u}] = [1,0,0,0]$, the contraction leads to the incorrect result

$$\text{GRAM}(\boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \cdot [\mathbf{u}] = \begin{bmatrix} 0 \\ E_1 \\ \beta(E_2 + v_{\mathcal{TS}}\, B_3) \\ \beta(E_3 - v_{\mathcal{TS}}\, B_2) \end{bmatrix} .$$

For a vanishing velocity of the test particle $\mathbf{v}_{\mathcal{T}} = \mathbf{u}$ it is $v_{\mathcal{TS}} = v_{\mathcal{S}}$ and the previous wrong formula specializes to coincide with EINSTEIN's result.

In all treatments in literature reference is made to relativistic electrodynamics and to LORENTZ transformations. As seen above, the issue is however of a more general character and consists in detecting the proper geometric transformation of the spatial representative of space-time electromagnetic differential forms, under an automorphic change of framing in the space-time manifold. The issue is in line with the spirit of general relativity in that it is required that the laws of electromagnetic induction should be equally valid in any framing. Equal validity means that the involved fields should transform by push according to diffeomorphic transformations between observers, the *molluscs* of EINSTEIN.[48]

30. When and how velocity comes into play

In line with the requirements of general relativity, we want to state the laws of electromagnetic induction so that they will be equally valid in any frame. This means that they should transform by push according to diffeomorphic transformations between observers, the *molluscs* of (Einstein, 1916, Part II: The General Theory of Relativity - 28. Exact Formulation of the General Principle of Relativity, p.84).

In this respect, the natural choice is the one made in Sect.24, since closedness of FARADAY and AMPÈRE electromagnetic two-forms is an invariant property under diffeomorphic transformations if FARADAY and AMPÈRE two-forms are assumed to be invariant, as stated in Prop.26.1.

To get the spatial form of the laws of electromagnetic induction, the electric field $\boldsymbol{\omega}^1_{\mathbf{E}}$, magnetic winding $\boldsymbol{\omega}^1_{\mathbf{H}}$ and electric current flux $\boldsymbol{\omega}^2_{\mathbf{J}}$ have been defined as spatialization of the contraction of between the space-time forms and the velocity of the particle.

[48] (Einstein, 1916, Part II: The General Theory of Relativity - 28. Exact Formulation of the General Principle of Relativity, p.84).

These definitions of spatial electromagnetic forms are however subject to a degree of indeterminacy and could in general be reformulated by adding the exterior derivative of potential forms. Indeed, the spatial laws of electromagnetic induction

$$\begin{cases} \mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \boldsymbol{\omega}^2_{\mathbf{B}} + d_{\mathcal{S}} \boldsymbol{\omega}^1_{\mathbf{E}} = 0 \,, \\ \mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \boldsymbol{\omega}^2_{\mathbf{D}} - d_{\mathcal{S}} \boldsymbol{\omega}^1_{\mathbf{H}} = -\boldsymbol{\omega}^2_{\mathbf{J}} \,, \\ \mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \boldsymbol{\rho}^3 + d_{\mathcal{S}} \boldsymbol{\omega}^2_{\mathbf{J}} = 0 \,, \end{cases}$$

leave the fields $\boldsymbol{\omega}^1_{\mathbf{E}}, \boldsymbol{\omega}^1_{\mathbf{H}}, \boldsymbol{\omega}^2_{\mathbf{J}}$ determined to within the exterior derivative of zero-forms to $\boldsymbol{\omega}^1_{\mathbf{E}}$ and $\boldsymbol{\omega}^1_{\mathbf{H}}$, and of a one-form to $\boldsymbol{\omega}^2_{\mathbf{J}}$. Any conclusion about the evaluation of these fields must involve the determination of these potential forms. This conclusion is in contrast to the claim of a *motional* term to be added to the *inductive* electric field in moving bodies and of an analogous (unnamed) term to be added to the magnetic winding in moving bodies.

To better enlighten the issue, and see how these additive terms make their appearance into the theory, let us consider two trajectory velocity fields $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{C}^1(\mathcal{T}\,; \mathbb{T}\mathcal{T})$ measured in a frame. Denoting the change of speed by

$$\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 \in \mathrm{C}^1(\mathcal{T}\,; \mathbb{T}\mathcal{T}) \,,$$

observing that $\Delta\mathbf{v} = \mathbf{i}{\uparrow}\Delta\mathbf{v}_{\mathcal{S}}$ with $\Delta\mathbf{v}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{T}\,; \mathbb{V}\mathcal{T})$ and recalling Lemma 23.5, the change $\Delta\boldsymbol{\omega}^1_{\mathbf{E}}$ of spatial electric field $\boldsymbol{\omega}^1_{\mathbf{E}}$ is given by

$$\begin{aligned} -\Delta\boldsymbol{\omega}^1_{\mathbf{E}} &= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}_2) - \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}_1) = \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \Delta\mathbf{v}) \\ &= (\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \cdot \Delta\mathbf{v}_{\mathcal{S}} = \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \Delta\mathbf{v}_{\mathcal{S}} \,, \end{aligned}$$

Analogously, the change $\Delta\boldsymbol{\omega}^1_{\mathbf{H}}$ of magnetic winding $\boldsymbol{\omega}^1_{\mathbf{H}}$ is given by

$$\begin{aligned} \Delta\boldsymbol{\omega}^1_{\mathbf{H}} &= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_2) - \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_1) = \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}} \cdot \Delta\mathbf{v}) \\ &= (\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{A}}) \cdot \Delta\mathbf{v}_{\mathcal{S}} = \boldsymbol{\omega}^2_{\mathbf{D}} \cdot \Delta\mathbf{v}_{\mathcal{S}} \,, \end{aligned}$$

and the change $\Delta\boldsymbol{\omega}^2_{\mathbf{J}}$ of spatial current $\boldsymbol{\omega}^2_{\mathbf{J}}$ is given by

$$\begin{aligned} -\Delta\boldsymbol{\omega}^2_{\mathbf{J}} &= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_2) - \mathbf{i}{\downarrow}(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \mathbf{v}_1) = \mathbf{i}{\downarrow}(\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}} \cdot \Delta\mathbf{v}) \\ &= (\mathbf{i}{\downarrow}\boldsymbol{\omega}^3_{\mathbb{M},\mathbf{A}}) \cdot \Delta\mathbf{v}_{\mathcal{S}} = \boldsymbol{\rho}^3 \cdot \Delta\mathbf{v}_{\mathcal{S}} \,. \end{aligned}$$

One could thus conclude that the expressions of the electric field $\boldsymbol{\omega}^1_{\mathbf{E}}$, magnetic winding $\boldsymbol{\omega}^1_{\mathbf{H}}$ and electric current $\boldsymbol{\omega}^2_{\mathbf{J}}$, as posited in Def. 24.1 and 24.2, are dependent on particle velocity and also give the expression of the induced variations in these electromagnetic fields by a variation of spatial velocity. Such a conclusion has however no physical basis because other equivalent definitions of these fields may be given with suitable gauges. This basic warning does not seem to have been taken into account in literature. There the addition of velocity dependent terms is motivated either by transformations similar to the ones given above or attributed to experimental evidences. In literature on electromagnetics, the additive *motional* term to the *transformer* electric field is commonly called LORENTZ *force* (Lorentz, 1895, 1899) although the term is reported as customary in (Hertz, 1892, XIV-2, p.248) where the equations of electrodynamics of moving bodies developed by Helmholtz (1874) are referred to. Improper attribution to Hertz (1892) of first attempts to extend FARADAY induction law to moving bodies, was made by Weyl (1922, p.191-192) who was evidently unaware of the comprehensive general formulation given by Clerk-Maxwell (1861, (77) p.342) and referred to by J.J. Thomson (1893, (1) p.534).

31. About relativistic Electromagnetics

The consequences of the previous results are of the utmost importance: if either GALILEI or LORENTZ space-time invariance is assumed to be fulfilled by FARADAY and AMPÈRE space-time two-forms, the corresponding spatial invariance will be fulfilled by all spatial electromagnetic forms.

Classical electrodynamics and relativistic electrodynamics differ just by the assumed group of transformations under which FARADAY and AMPÈRE space-time two-forms are considered to be invariant. Relativistic electrodynamics collapses naturally into classical electrodynamics when speeds slower and slower than light speed are considered, because in the limit GALILEI and LORENTZ groups of transformations tend to coincide.

The GALILEI and LORENTZ transformations for a translation with relative velocity $v_{\mathcal{S}}$ in the x direction are respectively expressed by the replacements

$$\begin{cases} t \mapsto t \\ x \mapsto x - v_{\mathcal{S}}\, t \\ y \mapsto y \\ z \mapsto z \end{cases} \qquad\qquad \begin{cases} c\,t \mapsto \beta(c\,t - \gamma\, x) \\ x \mapsto \beta(x - \gamma\, c\, t) \\ y \mapsto y \\ z \mapsto z \end{cases}$$

where $\gamma = v_{\mathcal{S}}/c$ and β given in terms of the scalar speed $v_{\mathcal{S}}$ between observers, by

$$\frac{1}{\beta^2} = 1 - \left(\frac{v_{\mathcal{S}}}{c}\right)^2 = 1 - \gamma^2\,,$$

so that $\beta^2(1-\gamma^2) = 1$.

The corresponding JACOBI matrices are

$$\begin{bmatrix} 1 & & & \\ -v_{\mathcal{S}} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} \beta & -\beta\,\gamma & & \\ -\beta\,\gamma & \beta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

with inverses

$$\begin{bmatrix} 1 & & & \\ v_{\mathcal{S}} & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \begin{bmatrix} \beta & \beta\,\gamma & & \\ \beta\,\gamma & \beta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

When $c \to +\infty$ the transformations are equal in the limit.

Let us represent the FARADAY two form by its GRAM matrix with respect to a basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in space-time with the time arrow as first basis vector, i.e. $\mathbf{e}_0 = \mathbf{u} \in \mathbb{TM}$, and any basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the spatial slice

$$\begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

In this respect we quote the following comments from (Einstein, 1905)

1. If a unit electric point charge is in motion in an electromagnetic field, there acts upon it, in addition to the electric force, an “electromotive force” which, if we neglect the terms multiplied by the second and higher powers of v/c, is equal to the vector-product of the velocity of the charge and the magnetic force, divided by the velocity of light. (Old manner of expression.)

2. If a unit electric point charge is in motion in an electromagnetic field, the force acting upon it is equal to the electric force which is present at the locality of the charge, and which we ascertain by transformation of the field to a system of co-ordinates at rest relatively to the electrical charge. (New manner of expression.)

Furthermore it is clear that the asymmetry mentioned in the introduction as arising when we consider the currents produced by the relative motion of a magnet and a conductor, now disappears. Moreover, questions as to the "seat" of electrodynamic electromotive forces (unipolar machines) now have no point.

The formulae derived in (Einstein, 1905), motivated by LORENTZ transformation of space-time coordinates between observers in relative motion, differ from the former two ones derived in Sect.30 just by a relativistic effect consisting in the multiplicative factor β.

While the formulae in (Einstein, 1905) refer to a change of observer according to a LORENTZ transformation, the issue of Sect.30 refers to a change of electromagnetic fields due to a change of the body velocity, as detected by a given observer, under the action of given associated electromagnetic fields and appear to be new.

The term $\boldsymbol{\omega}_{\mathbf{B}}^2 \cdot \Delta \mathbf{v}_{\mathcal{S}}$ has nothing to do with the seemingly similar LORENTZ force term introduced in literature.

As we have shown, the asymmetry mentioned by EINSTEIN is only due to an improper statement of FARADAY law of induction. The relativity principle cannot be invoked to be capable of symmetrizing FARADAY law of induction.

The important observations stem rather from purely kinematical considerations concerning the finite maximal velocity of light, playing the role of event communication tool, and concerning the independency of light scalar speed in empty space from the velocity of the source.

Precisely they the principle of relativity and on the principle of the constancy of the velocity of light, are there enunciated as follows.

1. *The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.*
2. *Any ray of light moves in the "stationary" system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or*

by a moving body. Hence

$$\text{velocity} = \frac{\text{light path}}{\text{time interval}}$$

where time interval is to be taken in the sense of the definition in § 1.

The conclusions of Prop.26.2 have no effect on most issues investigated in (Einstein, 1905), such as *7. Theory of Doppler's Principle and of Aberration*, *8. Transformation of the Energy of Light Rays. Theory of the Pressure of Radiation Exerted on Perfect Reflectors*, *9. Transformation of the Maxwell-Hertz Equations when Convection-Currents are Taken into Account*, *10. Dynamics of the Slowly Accelerated Electron*, this last with the exception of items 1. and 3. where the electron is considered to be acted upon by magnetic forces which are absent in our theory, the only force acting on the electron being the electric field.

All of Sect. *6. Transformation of the Maxwell-Hertz Equations for Empty Space. On the Nature of the Electromotive Forces Occurring in a Magnetic Field During Motion*, should be obviously corrected.

Let us now examine the following statements in (Einstein, 1905).

Now the principle of relativity requires that if the Maxwell-Hertz equations for empty space hold good in system K, they also hold good in system k

and that

Evidently the two systems of equations found for system k must express exactly the same thing, since both systems of equations are equivalent to the Maxwell-Hertz equations for system K. Since, further, the equations of the two systems agree, with the exception of the symbols for the vectors, it follows that the functions occurring in the systems of equations at corresponding places must agree, with the exception of a factor $\psi(v)$, which is common for all functions of the one system of equations, and is independent of ξ, η, ζ and τ but depends upon v. Thus we have the relations

$$\begin{aligned} \mathrm{X}' &= \mathrm{X}, & \mathrm{L}' &= \mathrm{L}, \\ \mathrm{Y}' &= \beta\left(\mathrm{Y} - \tfrac{v}{c}\mathrm{N}\right), & \mathrm{M}' &= \beta\left(\mathrm{M} + \tfrac{v}{c}\mathrm{Z}\right), \\ \mathrm{Z}' &= \beta\left(\mathrm{Z} + \tfrac{v}{c}\mathrm{M}\right), & \mathrm{N}' &= \beta\left(\mathrm{N} - \tfrac{v}{c}\mathrm{Y}\right). \end{aligned}$$

No explicit algebra for the derivation of these formulae is reported in (Einstein, 1905) but we can argue that the result should follow by an algebra performed according to a reasoning akin to the one exposed in our Sect.30, as explicated below.

$$
\begin{aligned}
-\Delta\boldsymbol{\omega}^1_{\mathbf{E}} &= \mathbf{i}{\downarrow}((\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\boldsymbol{\zeta}_{\mathbb{M}}} \cdot \mathbf{v}_{\boldsymbol{\zeta}_{\mathbb{M}}}) - \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}) \\
&= \mathbf{i}{\downarrow}(\boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\mathbf{v}) - \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}) \\
&= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\mathbf{v}) - \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{v}) \\
&= (\mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}) \cdot \Delta\mathbf{v}_{\mathcal{S}} \\
&= \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \Delta\mathbf{v}_{\mathcal{S}} \,,
\end{aligned}
$$

where $\Delta\mathbf{v}_{\mathcal{S}}$ is the spatial component of the relative velocity between observers $\mathbf{v}_{\boldsymbol{\zeta}_{\mathbb{M}}} - \mathbf{v} = \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\mathbf{v} - \mathbf{v}$. This computation is however based on the wrong condition $(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\boldsymbol{\zeta}_{\mathbb{M}}} = \boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$ which pretends to express equality between forms detected by distinct observers and the wrong evaluation $\mathbf{v}_{\boldsymbol{\zeta}_{\mathbb{M}}} - \mathbf{v}$ of the relative velocity. Frame-invariance is instead expressed by the meaningful equality $(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}})_{\boldsymbol{\zeta}_{\mathbb{M}}} = \boldsymbol{\zeta}_{\mathbb{M}}{\uparrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$ between forms detected by the same observer.

When the spatial velocity vanishes so that $\mathbf{v} = \mathbf{v}_{\mathcal{Z}} = \mathbf{u}$, these relations become

$$
\begin{aligned}
\boldsymbol{\omega}^2_{\mathbf{B}} &:= \mathbf{i}{\downarrow}\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \in \boldsymbol{\Lambda}^2(\mathbb{VM}\,;\mathcal{R}) \,, \\
-\boldsymbol{\omega}^1_{\mathbf{E}} &:= \mathbf{i}{\downarrow}(\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}} \cdot \mathbf{u}) \in \boldsymbol{\Lambda}^1(\mathbb{VM}\,;\mathcal{R}) \,.
\end{aligned}
$$

In components with respect to a synchronized observer $\mathbf{u} = (0,0,0,1)$. Hence an equivalent expression in terms of vectors $\mathbf{E}$ and $\mathbf{B}$ is obtained by the following matrix form of $\boldsymbol{\omega}^2_{\mathbb{M},\mathbf{F}}$, as reported in most classical formulations, see e.g. (Stratton, 1941)

$$
\begin{bmatrix}
0 & B_3 & -B_2 & -E_1 \\
-B_3 & 0 & B_1 & -E_2 \\
B_2 & -B_1 & 0 & -E_3 \\
E_1 & E_2 & E_3 & 0
\end{bmatrix}
$$

If this matrix expression is retained also for a non-vanishing spatial velocity, the following expression is got

$$
\begin{bmatrix}
0 & B_3 & -B_2 & -E_1 \\
-B_3 & 0 & B_1 & -E_2 \\
B_2 & -B_1 & 0 & -E_3 \\
E_1 & E_2 & E_3 & 0
\end{bmatrix}
\cdot
\begin{bmatrix} v_1 \\ v_1 \\ v_1 \\ 1 \end{bmatrix}
=
\begin{bmatrix}
v_1\,B_3 - B_2\,v_1 - E_1 \\
-v_1\,B_3 + v_1\,B_1 - E_2 \\
v_1\,B_2 - v_1\,B_1 - E_3 \\
v_1\,E_1 + v_1\,E_2 + v_1\,E_3
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{v}_{\mathcal{S}} \times \mathbf{B} - \mathbf{E} \\
\\
\mathbf{g}_{\mathbb{S}}(\mathbf{E}, \mathbf{v}_{\mathcal{S}})
\end{bmatrix}
$$

This formulation is responsible for the introduction of the velocity-dependent term in the evaluation of the electric field, see e.g. (Weyl, 1922).

$$\begin{bmatrix} 0 & B_3 & -B_2 & -v_1\,B_3 + B_2\,v_1 - E_1 \\ -B_3 & 0 & B_1 & v_1\,B_3 - v_3\,B_1 - E_2 \\ B_2 & -B_1 & 0 & -v_1\,B_2 + v_2\,B_1 - E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{bmatrix} = \begin{bmatrix} -E_1 \\ -E_2 \\ -E_3 \\ v_1\,E_1 + v_2\,E_2 + v_3\,E_3 \end{bmatrix}$$

$$= \begin{bmatrix} \\ -\mathbf{E} \\ \\ \mathbf{g}_{\mathbb{S}}(\mathbf{E}, \mathbf{v}_{\mathcal{S}}) \end{bmatrix}$$

32. Examples of applications of Faraday law

32.1. Charged body translating in a uniform magnetic vortex

Let a material body in a translational motion $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ with respect to an observer be crossing a region with a spatially constant value of the magnetic vortex, according to the standard Euclid connection, so that:

$$\nabla \boldsymbol{\omega}^2_{\mathbf{B}} = 0\,.$$

Let us first explain the idea in discursive terms. The vector potential $\mathbf{A}$ associated with the odd-vector field $\mathbf{B}$ of magnetic vortices may be assumed to have transversal circular envelope lines around the point of a longitudinal axis with the direction of the magnetic field. Then, at any istant of time, the vector potential intensity is linearly varying along any straight line. Let the body velocity field be orthogonal to the magnetic vortex odd vector field $\mathbf{B}$. Accordingly, the parallel derivative of the vector potential, along the motion velocity, will have the direction of the vector potential and intensity given by the product of half the intensity of the rotor times the intensity of the velocity. Taking into account the usual orientations, and evaluating the parallel derivative of the magnetic vortex potential, the electric field due to magnetic induction is given by one-half the standard expression of the Lorentz force (per unit electric charge):

$$\tfrac{1}{2}\mathbf{v}_{\mathcal{S}} \times \mathbf{B}\,.$$

To see this result expressed in formulae, we rely on the expression of the Lie derivative of a spatial tensor field in terms of parallel derivatives, which for a covariant tensor field writes (G. Romano, 2007):

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\alpha}^{\mathrm{Cov}} = \nabla_{\mathbf{v}}\,\boldsymbol{\alpha}^{\mathrm{Cov}} + \boldsymbol{\alpha}^{\mathrm{Cov}} \cdot \nabla \mathbf{v}_{\mathcal{S}} + (\nabla \mathbf{v}_{\mathcal{S}})^* \cdot \boldsymbol{\alpha}^{\mathrm{Cov}}\,,$$

and on the following results.

Lemma 32.1 (Linear Faraday potential). *A magnetic vortex field which is spatially constant, according to the standard connection of Euclid space, so that* $\nabla \boldsymbol{\omega}^2_{\mathbf{B}} = 0$, *admits a linear Faraday potential one-form* $\boldsymbol{\omega}^1_{\mathbf{B}} \in \boldsymbol{\Lambda}^1(\mathbb{T}\mathcal{S}\,;\mathcal{R})$, *that is* $\boldsymbol{\omega}^2_{\mathbf{B}} = d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{B}}$ *with*

$$\boldsymbol{\omega}^1_{\mathbf{B}} := \tfrac{1}{2}\boldsymbol{\mu} \cdot \mathbf{B} \cdot \mathbf{r} = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{r}\,,$$

to within a Galilei invariant scalar potential. Here $\boldsymbol{\mu}$ *is the standard volume form and* $\mathbf{r}(\mathbf{x}) := \mathbf{x}$.

Proof. Being $\nabla \boldsymbol{\omega}^2_{\mathbf{B}} = 0$, $\nabla \mathbf{r} = \mathbf{I}$, $\nabla^* \mathbf{r} = \mathbf{I}^*$, and recalling that $d_{\mathcal{S}}\, \boldsymbol{\omega}^2_{\mathbf{B}} = 0$, the homotopy formula (see Section 7.3) and the above quoted expression of the LIE derivative in terms of parallel derivative, give

$$d_{\mathcal{S}}\, (\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{r}) = \mathcal{L}_{\mathbf{r}}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \nabla_{\mathbf{r}}\, \boldsymbol{\omega}^2_{\mathbf{B}} + \boldsymbol{\omega}^2_{\mathbf{B}} \circ \nabla \mathbf{r} + \nabla^* \mathbf{r} \circ \boldsymbol{\omega}^2_{\mathbf{B}} = 2\, \boldsymbol{\omega}^2_{\mathbf{B}}\,,$$

which is the formula to be proved. ■

Proposition 32.1 (Electric field in a translating body). *A body with a translational motion, across a region of spatially uniform magnetic vortex, experiences an electric field given by*

$$-\boldsymbol{\omega}^1_{\mathbf{E}} = \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + \tfrac{1}{2} \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} + d_{\mathcal{S}}\, V_{\mathbf{E}}\,.$$

If the electric zero-form $V_{\mathbf{E}}$ has a null gradient, then a GALILEI observer which measures a time-invariant FARADAY one-form $\boldsymbol{\omega}^1_{\mathbf{B}}$ will detect, in the translating body, an electric field which admits a velocity-dependent scalar potential according to the formula

$$\boldsymbol{\omega}^1_{\mathbf{E}} = -\tfrac{1}{2} \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} = d_{\mathcal{S}}\, (\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}) \quad \Longleftrightarrow \quad \mathbf{E} = \tfrac{1}{2}\, (\mathbf{v}_{\mathcal{S}} \times \mathbf{B})\,.$$

Proof. Let us consider a GALILEI observer which detects a translational motion $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}\,;\mathcal{T})$ and measures the space-time velocity $\mathbf{v} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha$, which has a uniform spatial component, so that $\nabla \mathbf{v}_{\mathcal{S}} = 0$. From the formula for the LIE derivative in terms of parallel derivatives, we get

$$\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = \nabla_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + \boldsymbol{\omega}^1_{\mathbf{B}} \cdot \nabla \mathbf{v}_{\mathcal{S}} = \nabla_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}}\,.$$

Being $\nabla \boldsymbol{\omega}^2_{\mathbf{B}} = 0$, Lemma 32.1 gives $\boldsymbol{\omega}^1_{\mathbf{B}} = \tfrac{1}{2} \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{r}$ and

$$\nabla_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = \tfrac{1}{2} \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}$$

hence

$$\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = \nabla_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = \tfrac{1}{2} \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}\,.$$

The electric field is then given by

$$\begin{aligned} -\boldsymbol{\omega}^1_{\mathbf{E}} &= \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}}\, V_{\mathbf{E}} \\ &= \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + \mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + d_{\mathcal{S}}\, V_{\mathbf{E}} \\ &= \mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} + \tfrac{1}{2} \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} + d_{\mathcal{S}}\, V_{\mathbf{E}}\,. \end{aligned}$$

The term $-\frac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}} = -\frac{1}{2}\boldsymbol{\mu}\cdot\mathbf{B}\cdot\mathbf{v}_{\mathcal{S}} = \frac{1}{2}\mathbf{g}\cdot(\mathbf{v}_{\mathcal{S}}\times\mathbf{B})$ is the one-form providing the velocity-dependent part of the electric field (that is, *force* per unit electric charge) as detected by an observer which measures a time-invariant magnetic potential $\boldsymbol{\omega}^1_{\mathbf{B}}$ and a spatially constant scalar potential $V_{\mathbf{E}}$. To see that a Galilei observer, which measures a time-invariant Faraday one-form $\boldsymbol{\omega}^1_{\mathbf{B}}$, detects an electric field admitting a velocity-dependent potential, we observe that the homotopy formula

$$\begin{aligned}\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\boldsymbol{\omega}^1_{\mathbf{B}} &= (d_{\mathcal{S}}\,\boldsymbol{\omega}^1_{\mathbf{B}})\cdot\mathbf{v}_{\mathcal{S}} + d_{\mathcal{S}}\,(\boldsymbol{\omega}^1_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}})\\ &= \boldsymbol{\omega}^2_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}} + d_{\mathcal{S}}\,(\boldsymbol{\omega}^1_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}})\,,\end{aligned}$$

and the previously proved formula $\mathcal{L}_{\mathbf{v}_{\mathcal{S}}}\boldsymbol{\omega}^1_{\mathbf{B}} = \frac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}}$ together yield the potentiality property: $-\frac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}} = d_{\mathcal{S}}\,(\boldsymbol{\omega}^1_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}})$. ■

Remark 32.1. *It is manifest that the so-called* Lorentz *force law* $\boldsymbol{\omega}^2_{\mathbf{B}}\cdot\mathbf{v}_{\mathcal{S}}$ *is contradicted by the previous calculation which instead agrees with the* 1881 *findings by* J.J. Thomson*. His result was subsequently modified by* Heaviside *in* 1885 – 1889 *and by* Lorentz *in* 1892*, who eliminated the factor one-half. These historical notes, taken from (Darrigol, 2000), came to the attention of the author just after the present theory had been independently developed. The same expression was introduced as a well-known formula in Hertz (1892, XVI-2, p.248) with a brief discussion and a warning against its interpretation as an electric force, see fn.52.*

32.2. Faraday's paradox

Faraday disk: the classical device is constructed from a brass or copper disk that can rotate in front of a circular magnet. The induction EM force between the center of the disk and a point on its rim is measured by closing the circuit with the aid of brush contacts.

- 1^{st} experiment: The magnet is held to prevent it from rotating, while the disc is spun on its axis. The result is that the galvanometer registers a direct current.

- 2^{nd} experiment: The disc is held stationary while the magnet is spun on its axis. The result is that the galvanometer registers no current.

- 3^{rd} experiment: The disc and magnet are spun together. The galvanometer registers a current, as it did in step 1.

These experiments are commonly referred to as a *paradox* as it violates the standard spatial version of FARADAY's law of electromagnetic induction.

In fact, according to (Feynman et al., 1964, II.17.2): *as the disc rotates, the "circuit", in the sense of the place in space where the currents are, is always the same. But the part of the "circuit" in the disc is in material which is moving. Although the flux through the "circuit" is constant, there is still an EMF, as can be observed by the deflection of the galvanometer. Clearly, here is a case where the* $\mathbf{v} \times \mathbf{B}$ *force in the moving disc gives rise to an EMF which cannot be equated to a change of flux.*

The conviction that there are evidences of failure of FARADAY's flux rule has been taken for granted in literature as witnessed by the recent comments in (Lehner, 2010, 6.1.4. p.349).

A perfectly similar situation is provided by the experiment of the *homopolar generator* where a cylindrical magnet itself is spinning around its axis and two brush contacts, at the axel and on the rim, are placed to close the conducting circuit. These and others, real or thought, experiments have repeatedly been proposed in literature to confirm the possible failure of FARADAY's flux rule. What really emerges from these examples is the inadequacy of the standard formulation of the induction law in which the motion of the material circuit is not properly taken into account.

HERING's experiment, discussed in (Lehner, 2010, 6.1.4. p.349), can be interpreted according to FARADAY's flux rule by observing that there is a circuit including the galvanometer through which the magnetic flux is vanishing at all times during the opening phase of the experiment. An *emf* is induced between the sliding contacts but this may well give rise to eddy currents in the magnet and not in the controlled circuit. All these experiments are thus adducing no evidences against FARADAY's flux rule but rather they are warnings against incorrect interpretations of it.

Let us discuss the *paradox* by applying the formula for the spatial description of the induced electric field, illustrated in Section 18.5:

$$\boldsymbol{\omega}^1_{\mathbf{E}} = -\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} - d(\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}) - \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} + dV_{\mathbf{E}}\,.$$

In FARADAY experiments the spatial description of the magnetic vortex is time-independent, when measured by the GALILEI observer sitting on the support of the disk axis. The same observer will measure also a time-invariant FARADAY potential, so that: $\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^1_{\mathbf{B}} = 0$ and a velocity field of the spinning disk which, in terms of the angular velocity antisymmetric tensor $\mathbf{W} = \omega\, \mathbf{R}$, is given by:

$$\mathbf{v}_{\mathcal{S}}(\mathbf{x}) = \mathbf{W} \cdot \mathbf{r}(\mathbf{x}) = \omega \, \mathbf{R} \cdot \mathbf{r}(\mathbf{x})$$

with $\mathbf{R}$ rotation of $\pi/2$ in the disk plane, $\mathbf{x}$ a radius vector with origin at the disk axis and $\mathbf{r}(\mathbf{x}) := \mathbf{x}$. Then $\nabla \mathbf{v}_{\mathcal{S}} = \mathbf{W}$. Assuming that the magnetic flux $\boldsymbol{\omega}^2_{\mathbf{B}}$ is spatially constant in the disk, i.e. $\nabla \, \boldsymbol{\omega}^2_{\mathbf{B}} = 0$, from Lemma 32.1 we know that the Faraday potential is given by: $\boldsymbol{\omega}^1_{\mathbf{B}} = \frac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{r}$, so that:

$$\begin{aligned} \mathcal{L}_{\mathbf{v}_{\mathcal{S}}} \, \boldsymbol{\omega}^1_{\mathbf{B}} &= \nabla_{\mathbf{v}_{\mathcal{S}}} \boldsymbol{\omega}^1_{\mathbf{B}} + \boldsymbol{\omega}^1_{\mathbf{B}} \cdot \nabla \mathbf{v}_{\mathcal{S}} \\ &= \nabla_{\mathbf{v}_{\mathcal{S}}} \, \boldsymbol{\omega}^1_{\mathbf{B}} + \tfrac{1}{2}(\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{r}) \cdot \mathbf{W} \, . \end{aligned}$$

The parallel derivative of the magnetic potential, being $\nabla \, \boldsymbol{\omega}^2_{\mathbf{B}} = 0$ by assumption, evaluates to:

$$2 \, \nabla_{\mathbf{v}_{\mathcal{S}}} \, \boldsymbol{\omega}^1_{\mathbf{B}} = \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} + (\nabla_{\mathbf{v}_{\mathcal{S}}} \, \boldsymbol{\omega}^2_{\mathbf{B}}) \cdot \mathbf{r} = \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}} \, .$$

Being $\mathbf{R}^T = \mathbf{R}^{-1} = -\mathbf{R}$, for an arbitrary spatial vector field $\mathbf{h}$ in the disk plane, we get

$$\boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{W} \cdot \mathbf{r}, \mathbf{h}) = \omega \, \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{R} \cdot (\mathbf{R} \cdot \mathbf{r}), \mathbf{R} \cdot \mathbf{h}) = -\boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{r}, \mathbf{W} \cdot \mathbf{h}) \, ,$$

and hence

$$\begin{aligned} 2 \, \langle \mathcal{L}_{\mathbf{v}_{\mathcal{S}}} \, \boldsymbol{\omega}^1_{\mathbf{B}}, \mathbf{h} \rangle &= 2 \, \langle \nabla_{\mathbf{v}_{\mathcal{S}}} \, \boldsymbol{\omega}^1_{\mathbf{B}}, \mathbf{h} \rangle + \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{r}, \mathbf{W} \cdot \mathbf{h}) \\ &= \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{W} \cdot \mathbf{r}, \mathbf{h}) + \boldsymbol{\omega}^2_{\mathbf{B}}(\mathbf{r}, \mathbf{W} \cdot \mathbf{h}) = 0 \, . \end{aligned}$$

The analysis reveals that the magnetically induced electric vector field in the disk vanishes identically, when the magnetic vortex in the disk is spatially uniform. However, to compute the electromotive force in the circuit we must take into account the jump discontinuity of the velocity at the axis and at the rib brush contacts. These provide concentrated contributions to the *emf* whose sum is equal to:

$$\begin{aligned} & -\boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_1) \cdot (\mathbf{W} \cdot \mathbf{x}_1) + \boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_2) \cdot (\mathbf{W} \cdot \mathbf{x}_2) \\ &= -\tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{x}_1 \cdot (\mathbf{W} \cdot \mathbf{x}_1) + \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{x}_2 \cdot (\mathbf{W} \cdot \mathbf{x}_2) \, . \end{aligned}$$

The global *emf* so evaluated is coincident with the one provided by the integral formula of Faraday for moving bodies, see Section 18.1, when the spinning velocity of the material disk radius closing the circuit is taken into account. Indeed the expression above is exactly equal to the magnetic induction times the rate at which the spatially fixed radius is spanning the disk area with an angular velocity opposite to that of the wheel.

The discussion of the FARADAY disk performed in (Lehner, 2010, 6.1.4. p.350) provides an evidence of the inadequacy of the flux rule to discuss problems of magnetic induction involving a discontinuous velocity field.

33. Sliding bar on rails under a uniform magnetic vortex

Let us consider the problem concerning the electromotive force *(emf)* generated in a conductive bar sliding on two fixed parallel rails under the action of a magnetic vortex which is spatially uniform, time-independent and complanar. An observer sitting on the rails measures a time independent FARADAY potential field and may thus evaluate the *emf* due to the electric field distributed along the bar is found by integration along the line from $\mathbf{x}_1$ to $\mathbf{x}_2$:

$$-\boldsymbol{\omega}^1_{\mathbf{E}} \cdot \mathbf{l} = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} \cdot \mathbf{l}\,.$$

On the other hand, by the integral formula of FARADAY, the total *emf* in a circuit, obtained by closing the loop by another transversal bar fixed to the rails, is evaluated to be:

$$\oint \boldsymbol{\omega}^1_{\mathbf{E}} = -\oint \boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} = -\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} \cdot \mathbf{l}\,.$$

So one-half of the total *emf* is lost as a result of our previous evaluation of the contribution provided by the electric field distributed along the bar. To resolve this puzzling result we have to consider that, in this thought experiment, the velocity field is no more uniform in space. Moreover, being uniform in the bar and vanishing in the rails, it undergoes two points of jump discontinuities at the sliding contacts. Then, the observer sitting on the rails measures the distributed electric field in the bar, as evaluated before, plus two impulses of *emf* concentrated at the sliding contacts, whose sum is opposite to the sum of the jumps given by:

$$(\boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_1) - \boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_2)) \cdot \mathbf{v} = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} \cdot \mathbf{l}$$

where $\mathbf{x}_1, \mathbf{x}_2$ are the positions of the sliding contacts and $\mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1$. Indeed the velocity jumps, in going from 1 to 2 , are $\mathbf{v}$ and $-\mathbf{v}$, respectively, and

FARADAY potentials at 1 and 2 are given by

$$\boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_1) = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{r}(\mathbf{x}_1) = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{x}_1\,,$$
$$\boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_2) = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{r}(\mathbf{x}_2) = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{x}_2\,,$$

so that

$$\boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_1) - \boldsymbol{\omega}^1_{\mathbf{B}}(\mathbf{x}_2) = \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{x}_1 - \tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{x}_2 == -\tfrac{1}{2}\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{l}\,.$$

Thus, the two impulses of *emf* concentrated at the sliding contacts provide just the lost one-half of the total *emf* in the circuit, which therefore amounts to $-\boldsymbol{\omega}^2_{\mathbf{B}} \cdot \mathbf{v} \cdot \mathbf{l}$ and is equal to the one previously computed in one stroke by the integral rule of FARADAY. The instructive problem illustrated above is discussed in (Sadiku, 2010, C. Moving Loop in Time-Varying Field, Example 9.1, p. 375), by tacitly assuming a GALILEI observer sitting on the rails and adopting the LORENTZ force expression. The same problem with one bar fixed and the other one translating is discussed in (Feynman et al., 1964, II.17.1, fig.17.1) both in terms of the flux rule and in terms of the LORENTZ force (also with a tacit choice of the suitable GALILEI observer). Both analyses, and similar ones in literature, make no distinction between distributed and concentrated contributions to the *emf* and are based on the non-invariant LORENTZ force expression. The right value of the total *emf* in the circuit is however found, because the doubled value of the distributed electric field is equivalent to the addition of the impulses of *emf* at the sliding contacts.

33.1. The railgun: a weapon application

Let two parallel conductive rails and a sliding or rolling conductive projectile be subject to a high intensity electric current. The magnetic vortex field $\boldsymbol{\omega}^2_{\mathbf{B}}$ generated by the electric current act on the current itself by trying pushing away one from the other the two rails, which should then be properly fixed to remain in place, and pushes forward the sliding projectile, see fig.6.

33.2. The Hall effect

The HALL[49] effect consists in detecting a potential difference (Hall voltage) on opposite sides of a thin sheet of conducting or semiconducting material through which an electric current is flowing in presence of a coplanar

[49] EDWIN HERBERT HALL (1855-1938) American physicist.

Figure 6: US Navy Railgun 2008

magnetic vortex, fig.7. The experiments were first carried out on a thin gold sheet mounted on a glass plate at Johns Hopkins University under the guidance of Rowland[50] (Hall, 1879). The motivation for the experiment adduced in Hall's paper is a reasoning on a statement in (Clerk-Maxwell, 1873, vol.II p.144). The effect is commonly explained in terms of Lorentz force, but should be properly interpreted on the basis of the formula exposed here in Prop.32.1 which differs by a factor one-half. Since the quantized Hall effect is currently adopted as the standard for the definition of the electrical resistance, a corresponding revision should be made.

34. Discussion

According to the new treatment developed before, in both induction laws, the motion of material particles could be measured by any Galilei observer, without changing the evaluation of the electric field and of the magnetic winding. In this respect, confusions are still made in the recent literature, when dealing with the general laws of electromagnetic induction, as can be verified by inspecting several exposition of the fundamentals of electromagnetism.

The treatment of *Galileian Electromagnetism* by Le Bellac, Leblond (1972) considers two nonrelativistic limits (electric and magnetic) with arguments based on a non covariant formulation of the laws of electromagnetism.

In the introduction and survey of (Jackson, 1999, p.3) it is said:

[50] Henry Augustus Rowland (1848-1901) American physicist.

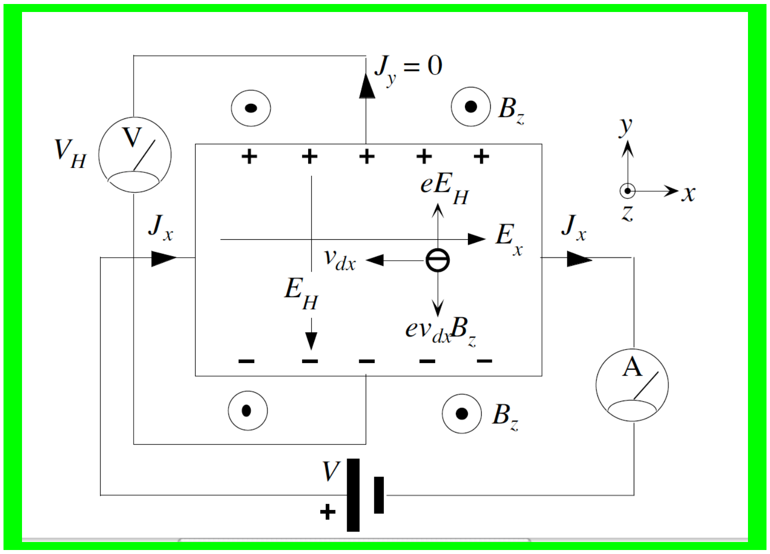

Figure 7: Hall effect - standard interpretation

Also essential for consideration of charged particle motion is the Lorentz force equation, $\mathbf{F} = q(\mathbf{E} + \mathbf{v}_{\mathcal{S}} \times \mathbf{B})$ *, which gives the force acting on a point charge q in the presence of electromagnetic fields.*

In FARADAY's law of induction (Jackson, 1999, p.209) the electric field is denoted by $\mathbf{E}'$ which is so described, ibid. p.210:

It is important to note, however, that the electric field $\mathbf{E}'$ *is the electric field at* $d\boldsymbol{l}$ (an infinitesimal piece of circuit) *in the coordinate system or medium in which* $d\boldsymbol{l}$ *is at rest, since is that field that causes current to flow if a circuit is actually present.*

Then, ibid. p.21,1 in writing: $\mathbf{E}' = \mathbf{E} + \mathbf{v}_{\mathcal{S}} \times \mathbf{B}$ it is said that

$\mathbf{E}$ is *the electric field in the laboratory* and $\mathbf{E}'$ is *the electric field at* $d\boldsymbol{l}$ *in its rest frame of coordinates.*

So an infinite number of observers would be needed to measure $\mathbf{E}'$ in a material circuit in arbitrary motion. Moreover, how to define univocally the *rest* frame of reference for an infinitesimal piece of of circuit? The same formula is reported in (Post, 1962, p.71-72), (Misner, Thorne, Wheeler, 1973, p.73), (Barut, 1980, p.88) and (Wegner, 2003, p.43). In all these treatments, no convincing strategy is exposed to choose the observer measuring the velocity which appears in the expression of the *so called* LORENTZ force.

The formula providing the spatial description of FARADAY law for mobile circuits is reported, without motivations, in (Sadiku, 2010, eq. 9.16) but a similar extension to mobile circuits is not considered for AMPÈRE law. Moreover, ibid. ch. 9.5, devoted to MAXWELL equations, it is literally said:

it is worthwhile to mention other equations that go hand in hand with Maxwell's equations. The LORENTZ *force equation* $\mathbf{F} = q(\mathbf{E} + \mathbf{v}_{\mathcal{S}} \times \mathbf{B})$ *is associated with Maxwell's equations. Also the equation of continuity is implicit in Maxwell's equations.*

In (Griffiths, 1999, p.475), introductory remark to *Electrodynamics and Relativity*, it is affirmed that:

Does it (GALILEI principle of relativity) *also apply to the laws of electrodynamics? At first glance the answer would seem to be no.*

A discussion, on the effect of relative motion between a conducting loop moving with a train and a magnet fixed on the rails, follows, but the whole analysis contains unmotivated affirmations.

In (Phipps, 1993), and in (Schwinger et al., 1998, p.9) the electromotive force induced by a magnetic field on a moving (translating) circuit, is evaluated by means of the material time-derivative, (expressed as sum of partial time-derivative plus parallel derivative at frozen time), according to the formula (in our notations):

$$-d_{\mathcal{S}}\, \boldsymbol{\omega}^1_{\mathbf{E}} = \nabla_{\mathbf{v}}\, \boldsymbol{\omega}^2_{\mathbf{B}} = \nabla_{\mathbf{v}_{\mathcal{Z}}}\, \boldsymbol{\omega}^2_{\mathbf{B}} + \nabla_{\mathbf{v}_{\mathcal{S}}}\, \boldsymbol{\omega}^2_{\mathbf{B}}\,,$$

in which parallel derivatives are improperly considered instead of LIE derivatives. The same treatment is reproduced in (Thidé, 2010, p.12-14).

HERTZ[51] is wrongly credited by Darrigol (2000) to have first proposed, in the monograph on *Electric waves*, an *ad hoc* modification of electromagnetic induction laws to recover GALILEI invariance. However, as quoted by HERTZ himself in that monograph (Hertz, 1892, XIV, fn.2 p.247), the correct GALILEI invariant statement of the law in terms of convective derivatives was already contributed in (Clerk-Maxwell, 1861, (77) p.342). A general formulation of the equations of electrodynamics of moving bodies was developed by Helmholtz (1874) and quoted in the treatment by Hertz (1892, XVI-2, p.248) where a statement concerning what is now called the LORENTZ *force* was made in the following terms:[52]

[51] HEINRICH RUDOLF HERTZ (1857-1894) German physicist.

[52] The triplet (X_1, X_2, X_3) are the components of the vector $\rho\,(\mathbf{v}_{\mathcal{S}} \times \mathbf{B})$.

Now the resultant of (X_1, X_2, X_3) *is an electric force which arises as soon as a body moves in the magnetic field. It is that force which in a narrower sense we are accustomed to denote as the electromotive force induced through the motion. But it should be observed that, according to our views, the separation of this from the total force can have no physical meaning.*

This warning about the interpretation of the velocity dependent term as an electric force was however ignored in later treatments and is presently completely neglected.

Later on, in his treatise on Space-Time-Matter (*Raum-Zeit-Materie*), Weyl (1922, p.191-192)[53] still attributed to HERTZ the first attempts to extend FARADAY induction law to moving bodies, seemingly unaware of the fact that a treatment in terms of scalar and vector potentials was first contributed in (Clerk-Maxwell, 1861, (77) p.342) as quoted in the book (J.J. Thomson, 1893, (1) p.534) who says: *In the course of Maxwell's investigation of the values of* X, Y, Z *due to induction, the terms...respectively in the final expressions for* X, Y, Z *are included under the* Ψ *terms. We shall find it clearer to keep these terms separate and write the expressions for* X, Y, Z *as....* This last reference provides the first explicit formulation of the differential laws of induction in which the velocity dependent scalar potential appears as separated from the GALILEI invariant electrostatic potential.

Consideration of LORENTZ force term seems to emerge in a natural way in the treatment by Hertz (1892, p.248), later reproduced by (Weyl, 1922, p.191-192), being deduced from the induction law written as $\operatorname{rot}\mathbf{E} = \operatorname{rot}(\mathbf{v}_{\mathcal{S}} \times \mathbf{B})$ under the assumption that $\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \mathbf{B} = 0$. There the motion is correctly taken into account by the convective term. The fault was to assume that the undetermined scalar potential was the electrostatic potential. The analysis developed in the present paper have instead shown that a further velocity dependent scalar potential $\boldsymbol{\omega}^1_{\mathbf{B}} \cdot \mathbf{v}_{\mathcal{S}}$ must be taken into account. This leads naturally to get a GALILEI invariant theory with results in perfect accord to the formulation by (J.J. Thomson, 1893, (1) p.534).

FEYNMAN[54] in *The Feynman Lectures on Physics* (Feynman et al., 1964, II.17-1), while illustrating FARADAY law of induction, says:

[53] HERMANN KLAUS HUGO WEYL (1885-1955) German mathematician.

[54] RICHARD PHILLIPS FEYNMAN (1918-1988) American physicist.

We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the rule as the combined effect of two quite separate phenomena. Moreover, ibid. ch. II.17-2, as a comment to the paradoxes of Faraday disk and of the circuit with rocking contacts, envisaged for discussing the applicability of Faraday law of magnetic induction (referred to as the *flux rule*), it is said that: *The "flux rule" does not work in this case. It must be applied to circuits in which the material of the circuit remains the same. When the material of the circuit is changing, we must return to the basic laws. The correct physics is always given by the two basic laws* $\mathbf{F} = q(\mathbf{E} + \mathbf{v}_{\mathcal{S}} \times \mathbf{B})$ *and* $\operatorname{rot} \mathbf{E} = -\mathcal{L}_{\mathbf{v}_{\mathcal{Z}}} \mathbf{B}$.

According to the point of view exposed in the present paper, the opposite conclusion, that neither one of the previous laws can be considered as a basic law of magnetic induction, must be drawn. The expression of the Lorentz *force* law (with a correction factor one-half) and the expression of the induction law in terms of partial time-derivative are simply additive terms entering into the evaluation of the electric field according to Faraday law. These terms do not enjoy Galilei invariance and hence cannot be given the physical interpretation of forces but might be at most defined as pseudo-forces, detected by special observers in special circumstances. The basic position in the theory must be reserved to Faraday law and to the consequent expression of the electric field in terms of the magnetic potential.

When dealing with *the relativity of magnetic and electric fields* in (Feynman et al., 1964, II.13-6) it is written:

When we said that the magnetic force on a charge was proportional to its velocity, you may have wondered: "What velocity? With respect to which reference frame?" It is, in fact, clear from the definition of $\mathbf{B}$ *given at the beginning of this chapter that what this vector is will depend on what we choose as a reference frame for our specification of the velocity of charges. But we have said nothing about which is the proper frame for specifying the magnetic field.*

Feynman's answer to the question is based on a relativity argument, A relativity argument is also resorted to in the treatment developed in (Purcell, 1985, ch.5). The same approach is taken in a recent book by Crowell (2010).

Anyway, it is hardly acceptable that experiments in classical electrodynamics should require relativistic arguments for their interpretation. Our treatment shows that the Galilei invariant formulation, the one naturally set up in the present paper, does the job, without any recourse to special relativity. Feynman definition of $\mathbf{B}$ is based on the Lorentz *force* law exerted on an electrically charged body in motion, a magnetic force which, as he says, has a *strange directional character* (Feynman et al., 1964, II.13-1). The same approach is taken in (Purcell, 1985, ch.6). In this respect the treatments, of moving conductors or dielectrics in magnetic fields, performed in (Landau and Lifshits, 1984) should also be consulted. These views concerning the Lorentz *force* law seem again to originate from the treatment given by Hermann Weyl in his treatise on *Raum-Zeit-Materie* (Weyl, 1922, p.191-192).

The recent treatments of classical electrodynamics in (Hehl and Obukhov, 2003; Lindell, 2004) is performed in terms of differential forms and adopts the elegant and synthetic geometric approach in the 4-dimensional space-time manifold. However body motions are still ignored and, in the expression of the induction laws, partial time derivatives at fixed points in the Euclid space are considered instead of Lie time-derivatives along the body motion, with the consequence that the laws of induction are not covariant and hence Galilei invariance does not follows.

In (Kovetz, 2000, sec. 8), when illustrating Faraday law, the magnetic induction flux is considered through a *fixed*, *open* surface. An *open* surface probably there stands for a surface with boundary, but the meaning of *fixed* is not (and could hardly be) clarified. In (Sadiku, 2010, ch. 8.2) the force acting on an electrically charged particle is said to be the sum of two terms. The former is the electric field and the second is the Lorentz force due to the magnetic induction and to the charged body velocity. But the electric field is just defined as the field providing the force acting on the unit point charge, so that a contradiction is apparent. The only way of picking the electric field out of the total force would indeed be to consider a fixed charged body, but again fixed with respect to what Galilei observer? A critical discussion on Lorentz force is reported by Smid (2010), although in somewhat naïve terms. The intrinsic strangeness of Lorentz law and the unanswered question about what Galilei observer is measuring the body velocity, both quoted by Feynman, may be overcome, as illustrated in this paper, by considering the correct form of the magnetic induction law for moving material circuits. The analysis, performed in Sections 33 and 32.2, of two well-known examples of a magnetically induced *emf* puts moreover into evidence that due attention to jump discontinuities of the velocity field must be paid, to evaluate concentrated impulses of the induction *emf* there located.

The space-time 4D formulation of the electromagnetic induction laws, in terms of conservation laws of two basic tensor fields, was proposed by Bateman (1910)[55] on the basis of earlier work by Hargreaves (1908). The theory, which was formulated in terms of differential forms by ÉLIE CARTAN[56] in (É. Cartan, 1924, p. 17-19) and is also reported in (Truesdell and Toupin, 1960, Ch. F). A formulation in terms of differerential forms in space-time is revisited in the context of relativity theory in (Misner, Thorne, Wheeler, 1973) and has recently been revisited in the textbooks (Hehl and Obukhov, 2003; Lindell, 2004) on the foundations of classical electrodynamics. In line with traditional treatment in literature, a generalized LORENTZ *force* relation is however there introduced as a further assumption, see the second of the six axioms in (Hehl and Obukhov, 2003, B2 p.121) and the treatment in (Lindell, 2004, ch. 5.4, p.151). A geometric treatment of electromagnetism in space-time, with a careful distinction between even and odd forms, has been contributed in (Marmo et al., 2005; Marmo and Tulczyjew, 2006).

New features of the approach developed in the four dimensional treatment introduced in Sect. **??** of the present paper, are

1. no additional law is assumed, the induction laws and the constitutive relations being the only ones considered as basic,
2. body motion is taken into account in induction laws.

Constitutive relations in the four-dimensional formalism are treated in (Marmo et al., 2005) and in (Lindell, 2004, 2006).

35. Conclusions

The fundamentals of electromagnetism have been revisited by a proper formulation of the electromagnetic induction laws for material bodies in motion. We emphasize that considering the motion of a body is an unavoidable task since the absence of motion would imply a restriction to consideration of a translating body as seen by an observer sitting on it. Then, bodies in relative translational motion and, more in general, deforming bodies, which are

[55] HARRY BATEMAN (1882-1946) British mathematician.
[56] ÉLIE CARTAN (1869-1951) French mathematician.

dealt with in everyday engineering applications of electromagnetism, would be ruled out. The differential geometric approach, performed in terms of integrals of exterior forms, leads to a formulation involving the Lie derivative, along the space-time motion, of the magnetic vortex (Faraday) and of the electric displacement flux (Ampère). The well-posedness conditions (independence of the considered surfaces and of their spatial motion) have been investigated and explicated in terms of balance laws. Galilei invariance of the new form of the induction laws is discussed and assessed by push naturality of Lie derivatives. The Lorentz force law, concerning the non-Galilei invariant *force* acted by a magnetic vortex upon a moving electrically charged particle, usually introduced as an independent axiom motivated by experience, has been critically addressed. The Galilei invariant formula, which differs by a one-half multiplicative factor in the velocity dependent term and by an additional term expressing the time-rate of the magnetic vortex potential, has been deduced as a direct consequence of Faraday law, when applied to detection of the electric field induced in a body translating in a region of spatially uniform magnetic vortex and of time-invariant Faraday magnetic potential. Constitutive relations have been briefly discussed in the Euclid framework. The formulation of electromagnetics in the four-dimensional space-time affine manifold has been extended to moving bodies, thus providing a clear picture of the following fundamental result. The balance laws for the electric and the magnetic charges, expressed by the closedness conditions on two electromagnetic 3-forms, are equivalent to the laws of electromagnetic induction which state the existence of corresponding potential 2-forms. Motions of involved bodies are taken into account by considering the description provided by an observer. Galilei invariance of the basic fields laws is a natural consequence of the observer-independent space-time formulation.

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