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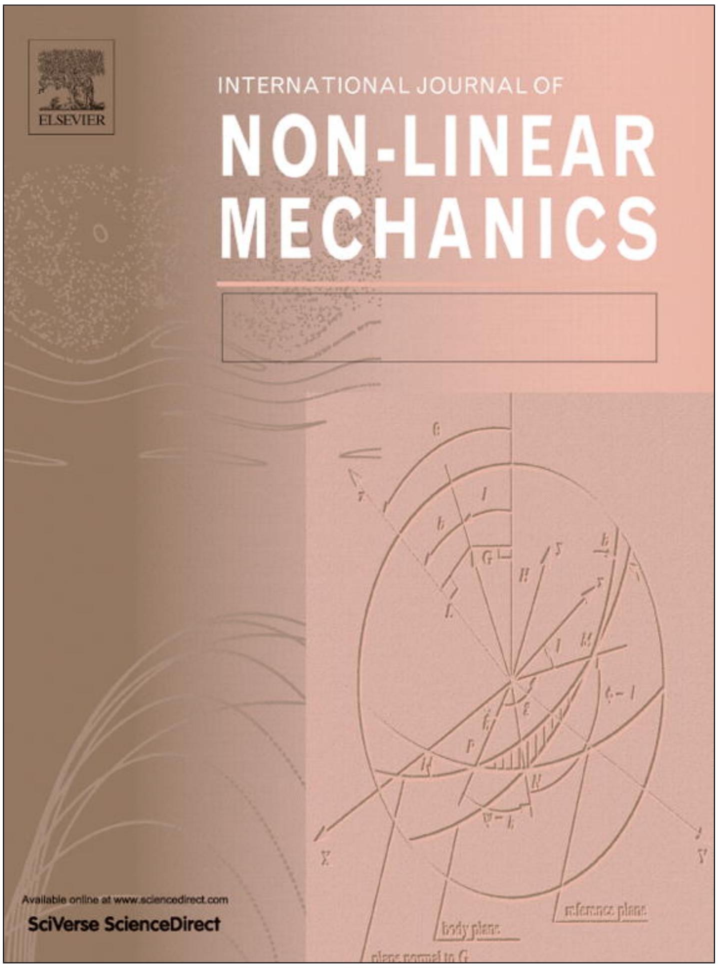

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Geometric constitutive theory and frame invariance

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ABSTRACT

The need for a proper geometric approach to constitutive theory in non-linear continuum mechanics (NLCM) is witnessed by lasting debates about basic questions concerning time-invariance, integrability, conservativeness and frame invariance. Our aim is to bring geometry to play a central role in theoretical and computational issues of NLCM. This demand is imposed by the present state of art, dominated by a mainly algebraic approach which, being a modified heritage of the linearized theory, is inadequate to manage concepts and methods in a non-linear framework. A proper definition of spatial and material fields and the statement of the ensuing covariance paradigm, provide a firm foundation to the theory of constitutive behavior in NLCM. The notion of constitutive frame invariance (CFI) is introduced as geometric correction to the formulation of material frame indifference (MFI). Standard models of constitutive behavior are critically discussed and compared with the ones consistent with the new approach. The outcome is a physically testable theory which eventually results in new effective computation tools for structural engineers.

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1. Introduction

The awkwardness of the notion of material frame-indifference (MFI) as usually stated in the literature, is witnessed by an inspection of past and recent treatments. Contribution [2] by Noll [1, p. 13], entitled *on material frame-indifference* says: *There is a considerable amount of confusion in the literature about the meaning of material frame-indifference, even among otherwise knowledgeable people. This paper is an attempt at clarification.* [1, p. 17 and p. 29], the MFI principle is so stated: *The constitutive laws governing the internal interactions between the parts of the system should not depend on whatever external frame of reference is used to describe them.* The precise meaning of *should not depend on* is however not clarified. A similar statement was made in [2, 3.2.9, p.194] where the axiom of MFI is commented as: *Stated loosely, this axiom means that our constitutive functions are invariant under rotations of the ambient space $\mathcal{S}$ in which our body moves.* Such a *form-invariance* requires the direct comparison of two functions with different domains and co-domains, as discussed below in Section 13. These difficulties were already present in the differential-algebraic treatment developed in *the non-linear field theories of mechanics* (NLFTM) where in Section 14, the MFI principle is expressed by requiring that: *the response of a material is the same for all observers*. The situation has been reproduced in the subsequent literature, until recently, as witnessed by the debates in [3–6]. Critical remarks to the standard reduction procedure were exposed in [7] and discussed in [8]. A further discussion about MFI has been contributed in [9, p. 195] where it is said: *Except perhaps for thermodynamics, few aspects of continuum thermomechanics have received so much attention and been the focus of so much controversy over the years as that of material "objectivity"... In our opinion, one aspect contributing to the subtlety of, and resulting misunderstanding surrounding this issue, is the lack of a clear delineation of the independent concepts involved, as well as a precise mathematical representation for these.* This opinion is agreed upon by the present authors, especially if *independent* is changed into *geometric*.

Indeed, the need for a suitable geometric context to be delineated pertains to the very formulation of constitutive relations in NLCM, not only to the discussion about frame invariance, as detailedly illustrated in the sequel.

This state of affairs renders vain any discussion about special issues, if basic notions are not properly settled. To get rid of these difficulties, a precise statement must be made about what is meant by *material behavior*, about how this concept can be formalized as a mathematical relation between *geometrical objects* involved in its description, and about how *variations* of these objects due to diffeomorphic displacements are to be evaluated.

Then, prior to introducing new definitions and results about constitutive relations, a background presentation of basic geometric issues in NLCM will be developed in the framework of the four-dimensional space–time of events, with an innovative approach. This leads to the formulation of proper notions of *spatial* and

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material tensor fields. These are fields based on the trajectory and acting on tangent vectors with no time-components respectively evaluated according to the time-fibrations of space–time and of trajectory manifolds, as detailed in Section 5. This formulation is propaedeutic to the development of the geometric theory of constitutive behavior in a continuous body.[1] At any event in the trajectory, a constitutive law is in fact described by a relation involving material tensors and their LIE derivatives along the motion. The proper definition of rate of material tensors along the motion and the geometrically consistent notion of frame-invariance are natural consequences of the *covariance paradigm*, according to which constitutive relations at different events on the trajectory are compared by push along the motion. Constitutive relations, as seen by different observers, are also related by push along the relevant space-time frame transformation.

In this framework hypo-elastic, elastic, hyper-elastic and visco-elasto-plastic models are reformulated and new definitions and properties are given. Integrability conditions, conservativeness, time-invariance, frame-invariance are thoroughly discussed. The especially important simplest case of elastic behavior is also analyzed and integrability and conservativeness are assessed. This result modifies negative conclusions stated in the literature [11] on the ground of improper treatments [12] and restores a full physical plausibility to a computationally significant constitutive model [13].

A distinguishing feature of the developed geometric approach is that no observer-dependent reference placement of the body is considered. Only fields pertaining to actual placements of the material body in its dynamical trajectory play a physical role in the theory. Reference placements are in fact relegated into a purely computation realm, deprived of any physical interpretation. Computational issues are briefly outlined in Section 11. The analysis leads to the introduction of the principle of constitutive frame invariance (CFI) as substitute to material frame indifference (MFI) which is geometrically inconsistent. A detailed discussion on this issue and comparisons with the previous formulations are provided in Section 13. The geometric approach adopted in this paper is in line with a geometrization program of NLCM initiated in [14–16] based on mathematical tools collected in [17].

2. Why the geometric formulation is needed

A turning point in the treatment of constitutive models in NLCM was the appearance of the paper on *the non-linear field theories of mechanics* (NLFTM) [18] whose approach has been taken ever more as a standard reference for notions and notations. Mechanical shortcomings are however detectable in that treatment, largely influenced by NOLL's contributions, and in subsequent, even recent, reformulations [19,2,20,1]:

1. The mathematical tools adopted for the theory exposed in [18] are essentially taken from linear algebra and calculus in linear spaces. The suitable context for NLCM is instead differential geometry which provides proper notations, notions and results pertaining to non-linear manifolds. For instance, the improper algebraic approach adopted for the statement of material frame indifference (MFI) is responsible for serious slips and for the confusion between material isotropy and frame-invariance in hypo-elasticity, see [18, (99.5)] and the amendments discussed in Section 13 below.

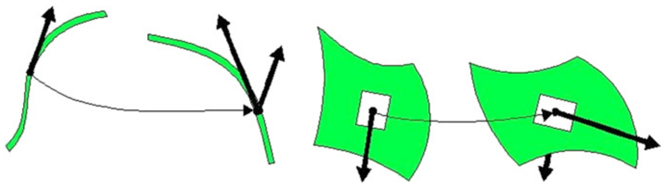

Fig. 1. Lower dimensionality: wire and membrane in motion.

2. Invoking MFI and relying on the algebraic *polar decomposition* a *reduction procedure*, conceived by NOLL [21] and reproduced in [18, Sect. 29], is commonly adopted to replace, in the elastic law, the deformation gradient by the right stretch. This reduction procedure involves an improper equality between constitutive maps which, expressing points of view of observers in relative motion, have domains and co-domains that are based at distinct points in space–time and should therefore be compared by push according to the relative motion. The issue is detailedly discussed in Sections 8 and 13.
3. Constitutive operators are assumed to depend on the *deformation* (or *transplacement*) gradient and possibly on its time rate and to provide the stress or its rate along the motion. The assumption requires the specification of preferred reference placements which cannot have experimental evidence. Moreover time derivatives are taken with respect to the translational connection in the EUCLID ambient space, a procedure which assigns an unmotivated preference to this connection and is definitely not applicable to lower dimensional structural models such as wires or membranes because parallel transport by translation does not preserve tangency to the placements, as sketched in Fig. 1. Even more, the important effects of thermal variations and visco-plastic phenomena may be accounted for only by a suitable (multiplicative) decomposition of the deformation gradient, a decomposition which leads to known conceptual and applicative difficulties [22].
4. The intrinsic formulation discussed in [19,1] considers constitutive relations as definable in an observer-independent way, so that MFI is considered as vacuously satisfied. This trick clashes however against the physical fact that a description of material behavior, by means of constitutive relations involving material tensors, presumes that an observer is making (ideal or factual) experimental tests.[2] Invariance of constitutive relations with respect to a group of transformations from one observer to another must then necessarily be imposed, see Section 9.

Notions such as form-invariance [9] are geometrically untenable since constitutive relations detected by distinct observers are imposed to be equal and not properly compared by push, as prescribed by the *covariance paradigm*. Indifference with respect to superimposed rigid motions is also considered in the literature with improper definitions of relative motions which do not take account of the change of base points, as discussed in Section 13.

The geometric theory restores a general validity to classical models of elasto-visco-plasticity by extending their validity from the simplest linearized framework to the non-linear realm simply by changing partial time-derivatives into LIE derivatives along the motion in the 4D trajectory manifold. Elastic strains belong to the output rather than to the input of the constitutive operator. The input is in fact a set of material tensors with a common base point (*stress* and *stressing*) so that the output will be an instantaneous

[1] The new geometric definitions of *spatial* and *material* tensor fields should not be confused with the homonymous notions introduced in the literature, see e.g. [10] respectively pertaining to fields with domain in a reference and in the current placement.

[2] This criticism is agreed upon by NOLL himself who in [1, p. 20] says: *Also, it seems that the action of the environment on a system cannot be described without using a frame of reference, and hence one must introduce such a frame in the end when dealing with specific problems.*

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elastic stretching. The same holds for *thermal* and *visco-plastic* stretching. The sum of these instantaneous contributions provides the total stretching. The natural requirement of conservativeness of the elastic behavior is shown to be fulfilled by conservation of mass and by Green integrability of the rate elastic law expressed in terms of the Kirchhoff stress tensor, as first assessed by the geometric analysis performed in [13].

The geometric constitutive theory is founded on two essential requirements which, although seemingly trivial, are violated by standard treatment [18,2,20,1]. These requirements may be enunciated as follows:

- *Geometric naturality* (GN) requires that, in introducing basic notions, governing principles and constitutive relations, the *metric* properties and the *motion* along the *trajectory* should be the sole geometric entities of space–time involved in the analysis.
- *Dimensionality independence* (DI) requires that all notions and results of the field theory should be directly applicable to bodies of any dimensionality.

The former requirement excludes formulations in which an unmotivated preference is assigned to a connection in space–time. An instance of its relevance in NLCM is provided by the action principle of continuum dynamics. The variational principle leads to d'Alembert's law of motion in terms of acceleration or to Poincaré law where structure constants appear, depending on whether a standard Euclid connection by translation or the connection induced by a mobile reference frame is chosen [23–25].

With special reference to constitutive theory, GN does not allow for appearance in constitutive relations of the material time-derivative $\dot{\mathbf{F}}$ of the deformation gradient which is evaluated according to the standard connection by translation [13].

With the hope of having given enough support to the geometric point of view in NLCM, we present in the next section a collection of fundamentals from differential geometry which will be referred to in the sequel.

3. Geometric prolegomena

Differential geometry (DG) deals with investigations concerning the notion of differentiable manifold $\mathbb{M}$ which generalizes the idea of regular curve or surface. At a point $\mathbf{x} \in \mathbb{M}$ the linear space of zeroth order tensors (scalars) is denoted by $\mathrm{Fun}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ and the dual spaces of tangent and cotangent vectors by $\mathbb{T}_{\mathbf{x}}\mathbb{M}$ and $\mathbb{T}^*_{\mathbf{x}}\mathbb{M}$ respectively. *Covariant*, *contravariant* and *mixed* second order tensors belong to linear spaces of scalar-valued bilinear maps (or linear operators) as listed hereafter

$$\mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}} \in \mathrm{Cov}(\mathbb{T}_{\mathbf{x}}\mathbb{M}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}, \mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathbb{T}^*_{\mathbf{x}}\mathbb{M}),$$
$$\mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} \in \mathrm{Con}(\mathbb{T}_{\mathbf{x}}\mathbb{M}) = L(\mathbb{T}^*_{\mathbf{x}}\mathbb{M}, \mathbb{T}^*_{\mathbf{x}}\mathbb{M}; \mathcal{R}) = L(\mathbb{T}^*_{\mathbf{x}}\mathbb{M}; \mathbb{T}_{\mathbf{x}}\mathbb{M}),$$
$$\mathbf{s}^{\mathrm{Mix}}_{\mathbf{x}} \in \mathrm{Mix}(\mathbb{T}_{\mathbf{x}}\mathbb{M}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}, \mathbb{T}^*_{\mathbf{x}}\mathbb{M}; \mathcal{R}) = L(\mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathbb{T}_{\mathbf{x}}\mathbb{M}).$$

The linear spaces of covariant and contravariant tensors are in separating duality by the pairing

$$\langle \mathbf{s}^{\mathrm{Con}}_{\mathbf{x}}, \mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}} \rangle := J_1(\mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} \circ (\mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}})^A),$$

where J_1 denotes the linear invariant and the adjoint tensor $(\mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}})^A$ is defined by the identity

$$(\mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}})^A(\mathbf{a},\mathbf{b}) := \mathbf{s}^{\mathrm{Cov}}_{\mathbf{x}}(\mathbf{b},\mathbf{a}), \quad \forall\, \mathbf{a},\mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}.$$

The generic tensor space is denoted by $\mathrm{Tens}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$. Spaces of symmetric covariant and contravariant tensors are denoted by $\mathrm{Sym}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ and $\mathrm{Sym}^*(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ respectively.

- An *immersion* (*submersion*) is a map with injective (surjective) associated tangent map.
- A *fibration* of a manifold $\mathbb{F}$ is a projection (surjective submersion) $\boldsymbol{\pi} \in C^1(\mathbb{F}; \mathbb{M})$ on a base manifold $\mathbb{M}$.
- A *fiber* $\mathbb{F}(\mathbf{x})$ is the inverse image of a point $\mathbf{x} \in \mathbb{M}$ by the projection.
- A *section* of a fibration $\boldsymbol{\pi} \in C^1(\mathbb{F}; \mathbb{M})$ is a map $\mathbf{s} \in C^1(\mathbb{M}; \mathbb{F})$ with the property of being a right inverse of the projection, i.e. $\boldsymbol{\pi} \circ \mathbf{s} = \mathrm{ID}_{\mathbb{M}}$.
- A *fiber-bundle* is a fibration whose fibers are manifolds related by diffeomorphic transformations.
- A *vector-bundle* is a fiber-bundle with linear fibers.
- The *tangent-bundle* $\mathbb{TM}$ to a manifold $\mathbb{M}$ is a vector-bundle, with projection $\boldsymbol{\tau} \in C^1(\mathbb{TM}; \mathbb{M})$, whose fibers are the tangent spaces $\mathbb{T}_{\mathbf{x}}\mathbb{M}$.
- The *tensor-bundle* $\mathrm{Tens}(\mathbb{TM})$ is a bundle whose fibers are tensor spaces.
- A *tensor field* is a section of a tensor-bundle.
- A *bundle morphism* is a fiber preserving map between fiber-bundles. A *diffeomorphism* is a morphism which is invertible with differentiable inverse. An *endomorphism* is a morphism from a fiber-bundle to itself. An *automorphism* is an invertible endomorphism. An *homomorphism* is a morphism between linear bundles which preserves linear operations. An *isomorphism* is an invertible homomorphism.

An intuitive sketch of a fiber bundle is provided by the image of a hairy head, depicted at the l.h.s. of Fig. 2. Each hair is a fiber manifold and the head skin is the base manifold. There are no zones of the skin deprived of hairs and one get a bulb to move infinitesimally in any direction of the head skin by suitably moving among hairs (surjective submersion). The notion of a fiber bundle is quite natural in NLCM whose basic notions are conveniently formulated in terms of tensor bundles over the trajectory manifold. The r.h.s of Fig. 2 depicts a tangent bundle.

Scalar-valued k-linear, alternating maps are called k-covectors. Volumes are non-vanishing k-covectors of maximal order ($k = \dim \mathbb{M}$) and the corresponding linear space is denoted by $\mathrm{Vol}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$.

3.1. Push–pull transformations

The *pull-back* of a scalar $f_{\zeta(\mathbf{x})} \in \mathrm{Fun}(\mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N})$ along a map $\zeta \in C^0(\mathbb{M}; \mathbb{N})$ between differentiable manifolds $\mathbb{M}$ and $\mathbb{N}$, is the scalar $(\zeta{\downarrow}f)_{\mathbf{x}} \in \mathrm{Fun}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ defined by the equality

$$(\zeta{\downarrow}f)_{\mathbf{x}} := f_{\zeta(\mathbf{x})}.$$

Given a differentiable curve $\mathbf{c} \in C^1(\mathcal{R}; \mathbb{M})$, with $\mathbf{x} = \mathbf{c}(0)$, and a differentiable map $\zeta \in C^1(\mathbb{M}; \mathbb{N})$, the associated *tangent map* at $\mathbf{x} \in \mathbb{M}$, denoted by $T_{\mathbf{x}}\zeta \in L(\mathbb{T}_{\mathbf{x}}\mathbb{M}; \mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N})$ is defined by the linear correspondence

$$\mathbf{v}_{\mathbf{x}} = \partial_{\lambda=0}\, \mathbf{c}(\lambda) \mapsto T_{\mathbf{x}}\zeta \cdot \mathbf{v}_{\mathbf{x}} = \partial_{\lambda=0}\, (\zeta \circ \mathbf{c})(\lambda).$$

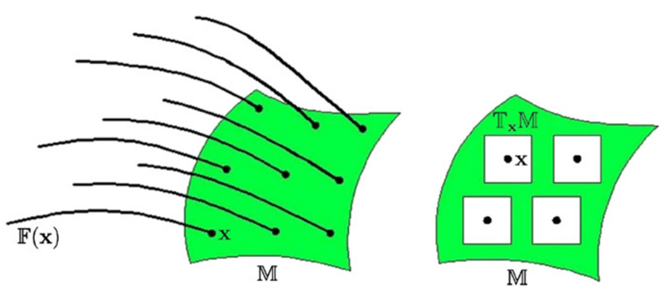

Fig. 2. Fiber bundle and tangent bundle.

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If the map $\zeta \in C^1(\mathbb{M};\mathbb{N})$ is invertible, with inverse map

$T^{-1}_{\zeta(\mathbf{x})}\zeta := (T_{\mathbf{x}}\zeta)^{-1} \in L(\mathbb{T}_{\zeta(\mathbf{x})}\zeta(\mathbb{M});\mathbb{T}_{\mathbf{x}}\mathbb{M})$,

the co-tangent map

$T^{*}_{\zeta(\mathbf{x})}\zeta := (T_{\mathbf{x}}\zeta)^{*} \in L(\mathbb{T}^{*}_{\zeta(\mathbf{x})}\zeta(\mathbb{M});\mathbb{T}^{*}_{\mathbf{x}}\mathbb{M})$

is defined, for every $\mathbf{w}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ and $\mathbf{v}^{*}_{\zeta(\mathbf{x})} \in \mathbb{T}^{*}_{\zeta(\mathbf{x})}\zeta(\mathbb{M})$, by

$\langle \mathbf{v}^{*}_{\zeta(\mathbf{x})}, T_{\mathbf{x}}\zeta \cdot \mathbf{w}_{\mathbf{x}} \rangle = \langle T^{*}_{\zeta(\mathbf{x})}\zeta \cdot \mathbf{v}^{*}_{\zeta(\mathbf{x})}, \mathbf{w}_{\mathbf{x}} \rangle$.

The *push-forward* of a tangent vector $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ is defined by the formula

$(\zeta\uparrow\mathbf{v})_{\zeta(\mathbf{x})} := T_{\mathbf{x}}\zeta \cdot \mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N}$.

The *pull-back* of a cotangent vector $\mathbf{v}^{*}_{\zeta(\mathbf{x})}$, along an invertible differentiable map $\zeta \in C^1(\mathbb{M};\mathbb{N})$, is the cotangent vector $(\zeta\downarrow\mathbf{v}^{*})_{\mathbf{x}}$ defined by invariance

$\langle (\zeta\downarrow\mathbf{v}^{*})_{\mathbf{x}}, \mathbf{v}_{\mathbf{x}} \rangle = \langle \mathbf{v}^{*}_{\zeta(\mathbf{x})}, (\zeta\uparrow\mathbf{v})_{\zeta(\mathbf{x})} \rangle$,

so that

$(\zeta\downarrow\mathbf{v}^{*})_{\mathbf{x}} := T^{*}_{\zeta(\mathbf{x})}\zeta \cdot \mathbf{v}^{*}_{\zeta(\mathbf{x})}$.

Pull-back and push-forward, if both defined, are inverse operations. Push–pull operations for tensors are defined by invariance.

For instance, the pull-back of a twice-covariant tensor $\mathbf{s}_{\zeta(\mathbf{x})} \in \mathrm{Cov}(\mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N})$ is a twice-covariant tensor $\zeta\downarrow\mathbf{s}_{\zeta(\mathbf{x})} \in \mathrm{Cov}(\mathbb{T}_{\mathbf{x}}\mathbb{M})$ explicitly defined, for any pair of tangent vectors $\mathbf{u}_{\mathbf{x}},\mathbf{w}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$, by

$$\begin{aligned}\zeta\downarrow\mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})}(\mathbf{u}_{\mathbf{x}},\mathbf{w}_{\mathbf{x}}) &:= \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})}(T_{\mathbf{x}}\zeta \cdot \mathbf{u}_{\mathbf{x}}, T_{\mathbf{x}}\zeta \cdot \mathbf{w}_{\mathbf{x}}) \\ &= \langle \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \cdot T_{\mathbf{x}}\zeta \cdot \mathbf{u}_{\mathbf{x}}, T_{\mathbf{x}}\zeta \cdot \mathbf{w}_{\mathbf{x}} \rangle \\ &= \langle T^{*}_{\zeta(\mathbf{x})}\zeta \cdot \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \cdot T_{\mathbf{x}}\zeta \cdot \mathbf{u}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}} \rangle.\end{aligned}$$

Push–pull relations for covariant, contravariant and mixed tensors, along a map $\zeta \in C^1(\mathbb{M};\mathbb{N})$, are then given by

$$\begin{aligned}\zeta\downarrow\mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} &= T^{*}_{\zeta(\mathbf{x})}\zeta \circ \mathbf{s}^{\mathrm{Cov}}_{\zeta(\mathbf{x})} \circ T_{\mathbf{x}}\zeta \in \mathrm{Cov}(\mathbb{T}_{\mathbf{x}}\mathbb{M}), \\ \zeta\uparrow\mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} &= T_{\mathbf{x}}\zeta \circ \mathbf{s}^{\mathrm{Con}}_{\mathbf{x}} \circ T^{*}_{\zeta(\mathbf{x})}\zeta \in \mathrm{Con}(\mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N}), \\ \zeta\uparrow\mathbf{s}^{\mathrm{Mix}}_{\mathbf{x}} &= T_{\mathbf{x}}\zeta \circ \mathbf{s}^{\mathrm{Mix}}_{\mathbf{x}} \circ T^{-1}_{\zeta(\mathbf{x})}\zeta \in \mathrm{Mix}(\mathbb{T}_{\zeta(\mathbf{x})}\mathbb{N}).\end{aligned}$$

4. Trajectory manifold

Basic issues in NLCM are most conveniently investigated in the setting of a four dimensional *events manifold* E and of its representation as space–time by an observer. Space–time formulations in NLCM have been considered in the previous treatments (see e.g. [2,9] and references therein). The innovative geometric feature of our presentation is the role played by the trajectory manifold, see Fig. 4.

Extending standard presentations, the treatment allows for considering trajectory manifolds whose dimensionality may be lower than the one of the events manifold. Structural models of wires and membranes can be thus treated in a unitary way together with 3-D bodies.

To this end, it is expedient to introduce the notion of injective immersion $\mathbf{i}_{\mathrm{E},\mathcal{T}}$ of the non-linear trajectory manifold $\mathcal{T}$ into the events manifold E. In classical mechanics, an observer detects the events manifold as a space–time product $\mathcal{S} \times I$, between the affine space manifold $\mathcal{S}$ and the time instants line I. Two fibrations are generated according to the cartesian projections $\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in C^1(\mathrm{E};\mathcal{S})$ and $\boldsymbol{\pi}_{I,\mathrm{E}} \in C^1(\mathrm{E};I)$, the latter being independent of the observer (in classical mechanics *time is absolute*). A *spatial fiber* $\mathrm{E}(t) \equiv (\mathcal{S},t)$ is the set of simultaneous events at a time $t \in I$. The fibrations of the events manifold induce analogous fibrations in the trajectory manifold. The time fibration $\boldsymbol{\pi}_{I,\mathcal{T}} := \boldsymbol{\pi}_{I,\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}}$ associates, with each event in the trajectory, the corresponding absolute time instant. The space fibration $\boldsymbol{\pi}_{\mathcal{S},\mathcal{T}} := \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}}$ associates, with each event in the trajectory, the corresponding spatial location in the space manifold $\mathcal{S}$. The fibers of the trajectory space-fibration are made of localized events and the corresponding time instants will in general belong to a subset of I which fail to be a differentiable manifold. Accordingly the time fibration of the trajectory may not be a fiber bundle.[3]

The *temporal fiber* $\mathcal{T}(t) = \boldsymbol{\Omega}_t \times \{t\}$ in the time fibration $\boldsymbol{\pi}_{I,\mathcal{T}} := \boldsymbol{\pi}_{I,\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}}$ collects simultaneous trajectory-events at time $t \in I$, with $\boldsymbol{\Omega}_t = \boldsymbol{\pi}_{\mathcal{S},\mathcal{T}}(\mathcal{T}(t)) \subset \mathcal{S}$ *spatial placement* of the body assumed to be a compact connected submanifold. Placements are diffeomorphic one-another and time-fibration makes the trajectory manifold $\mathcal{T}$ a fiber-bundle.

5. Spatial and material fields

Time-vertical vectors are tangent to curves of simultaneous events. The time-vertical tangent bundles to the events manifold and to the trajectory manifold are denoted by $\mathbb{V}\mathrm{E}$ and $\mathbb{V}\mathcal{T}$ and are respectively called the *spatial bundle* and the *material bundle*. Indeed, having a null time-component, time-vertical vectors tangent to the events manifold are *spatial tangent vectors* and time-vertical vectors tangent to the trajectory manifold are *material tangent vectors*, see Fig. 3. The physical meaning of the time-verticality condition is that infinitesimal events-variations occur at frozen time, so that the time-vertical tensor spaces are constructed over tangent spaces to the events manifold E and to the trajectory manifold $\mathcal{T}$, at a fixed time. This means that time-vertical tensor spaces are in fact tensor spaces over the ambient space manifold $\mathcal{S}$ and over the spatial placement $\boldsymbol{\Omega}_t$ and the observer can make the identifications

$\mathbb{V}_{(\mathbf{x},t)}(\mathcal{S} \times I) \equiv \mathbb{T}_{\mathbf{x}}\mathcal{S}, \quad \mathbb{V}_{(\mathbf{x},t)}\mathcal{T} \equiv \mathbb{T}_{\mathbf{x}}\boldsymbol{\Omega}_t$.

Three main kinds of tensor fields are involved in NLCM, *spatial fields*, *trajectory based spatial fields* and *material fields*.

Since our primary interest is in the formulation of constitutive laws, we will consider only *spatial fields* and *material fields* because *trajectory based spatial fields* pertain to dynamics (accelerations, kinetic momenta).

Definition 5.1 (*Spatial fields*). A spatial tensor field

$\mathbf{s}_{\mathrm{E}} \in C^1(\mathrm{E};\mathrm{Tens}(\mathbb{V}\mathrm{E}))$

is a section of the spatial tensor bundle $\boldsymbol{\pi}_{\mathrm{E}} \in C^1(\mathrm{Tens}(\mathbb{V}\mathrm{E});\mathrm{E})$ constructed over the time-vertical tangent bundle to the events time-bundle $\boldsymbol{\pi}_{I,\mathrm{E}} \in C^1(\mathrm{E};I)$.

The spatial field of interest in continuum mechanics is the symmetric covariant *spatial metric tensor field* $\mathbf{g}_{\mathrm{E}} \in C^1(\mathrm{E};\mathrm{Sym}(\mathbb{V}\mathrm{E}))$ acting on the spatial bundle. It is related, by definition, to the metric tensor field $\mathbf{g}_{\mathcal{S}} \in C^1(\mathcal{S};\mathrm{Sym}(\mathbb{T}\mathcal{S}))$ acting on the Euclid tangent bundle by

$\mathbf{g}_{\mathrm{E}} = \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}\downarrow\mathbf{g}_{\mathcal{S}} = T^{*}\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \circ \mathbf{g}_{\mathcal{S}} \circ T\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}$

or explicitly

$\mathbf{g}_{\mathrm{E}}(\mathbf{a},\mathbf{b}) := \mathbf{g}_{\mathcal{S}}(T\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \cdot \mathbf{a}_{\mathrm{E}}, T\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \cdot \mathbf{b}_{\mathrm{E}}) \circ \boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}$,

for all spatial (time-vertical) tangent vector fields $\mathbf{a}_{\mathrm{E}},\mathbf{b}_{\mathrm{E}} \in C^1(\mathrm{E};\mathbb{V}\mathrm{E})$.

Positive definiteness of the *spatial metric field* $\mathbf{g}_{\mathrm{E}} \in C^1(\mathrm{E};\mathrm{Sym}(\mathbb{V}\mathrm{E}))$ follows by the same property of the *metric field* $\mathbf{g}_{\mathcal{S}} \in C^1(\mathcal{S};\mathrm{Sym}(\mathbb{T}\mathcal{S}))$ and by injectivity of the tangent maps $T_{\mathbf{e}}\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}} \in L(\mathbb{V}_{\mathbf{e}}\mathrm{E};\mathbb{T}_{\boldsymbol{\pi}_{\mathcal{S},\mathrm{E}}(\mathbf{e})}\mathcal{S})$.

[3] This observation has mechanical implications. For instance, in particle mechanics the formula for acceleration cannot be evaluated as sum of time and spatial derivatives. Indeed partial time-derivatives at a fixed spatial point are then not feasible.

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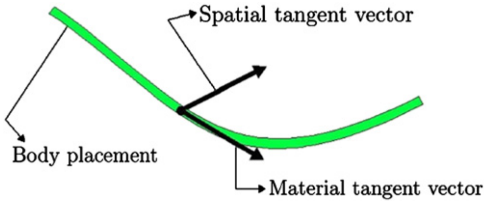

Fig. 3. Spatial and material vectors in a spatial fiber $\mathrm{E}(t)$.

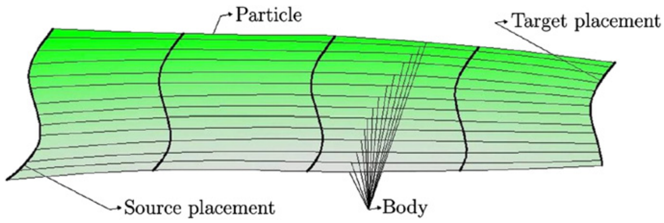

Fig. 4. Trajectory, body and particles in space–time.

Definition 5.2 (*Material fields*). A material tensor field

$$\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{T\small{ENS}}(\mathbb{V}\mathcal{T}))$$

is a section of the material tensor bundle $\boldsymbol{\pi}_{\mathrm{TENS}} \in \mathrm{C}^1(\mathrm{TENS}(\mathbb{V}\mathcal{T}); \mathcal{T})$ constructed over the time-vertical tangent bundle to the trajectory time-bundle $\boldsymbol{\pi}_{I,\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; I)$.

Most fields of primary interest in continuum mechanics are *material fields*. For instance such are *stretch, stretching* (covariant tensors), *stress, stressing* (contavariant tensors), *heat, mass* (volume forms) and *temperature, entropy, thermodynamic potentials* (scalars).

In continuum mechanics an essential role is played by the pull-back of the spatial metric to a *material metric* over the trajectory tangent bundle. The material metric provides the geometric tool to define the stretching entering in constitutive relations.

Definition 5.3 (*Material metric field*). The material metric tensor field $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}, \mathrm{SYM}(\mathbb{V}\mathcal{T}))$ is the pull-back of the spatial metric tensor field $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}; \mathrm{SYM}(\mathbb{V}\mathrm{E}))$ according to the injective immersion $\mathbf{i}_{\mathrm{E},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{E})$

$$\mathbf{g}_{\mathcal{T}} := \mathbf{i}_{\mathrm{E},\mathcal{T}} \downarrow \mathbf{g}_{\mathrm{E}}$$

and is explicitly defined by

$$\mathbf{g}_{\mathcal{T}}(\mathbf{a}_{\mathcal{T}}, \mathbf{b}_{\mathcal{T}}) = \mathbf{g}_{\mathrm{E}}(\mathbf{i}_{\mathrm{E},\mathcal{T}} \uparrow \mathbf{a}_{\mathcal{T}}, \mathbf{i}_{\mathrm{E},\mathcal{T}} \uparrow \mathbf{b}_{\mathcal{T}}),$$

for all $\mathbf{a}_{\mathcal{T}}, \mathbf{b}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathbb{T}\mathcal{T})$.

We will denote by $\boldsymbol{\mu}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{VOL}(\mathbb{V}\mathcal{T}))$ the material volume form associated with the metric field $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{COV}(\mathbb{V}\mathcal{T}))$.

6. Motion

The *motion* is a one-parameter family of orientation preserving automorphisms of the trajectory manifold $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}; \mathcal{T})$ over the time shift $\mathrm{SH}_\alpha \in \mathrm{C}^1(I; I)$ defined by $\mathrm{SH}_\alpha(t) := t + \alpha$ for all $\alpha, t \in I$, as described by the commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\boldsymbol{\varphi}_\alpha} & \mathcal{T} \\ {\scriptstyle \boldsymbol{\pi}_{I,\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\pi}_{I,\mathcal{T}}} \\ I & \xrightarrow{\mathrm{SH}_\alpha} & I \end{array} \quad \Leftrightarrow \quad \boldsymbol{\pi}_{I,\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha = \mathrm{SH}_\alpha \circ \boldsymbol{\pi}_{I,\mathcal{T}}.$$

The *trajectory velocity* is the space–time vector field defined by

$$\mathbf{v}_{\mathcal{T}} := \partial_{\alpha=0} \boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}; \mathbb{T}\mathcal{T}).$$

The LIE derivative of the material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{TENS}(\mathbb{V}\mathcal{T}))$ along to the vector field $\mathbf{v}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathbb{T}\mathcal{T})$ is defined by the time-derivative of the pull-back according to the motion

$$\mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \mathbf{s}_{\mathcal{T}} := \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \downarrow \mathbf{s}_{\mathcal{T}}) = \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \downarrow \circ \mathbf{s}_{\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha) \in \mathrm{C}^1(\mathcal{T}; \mathrm{TENS}(\mathbb{V}\mathcal{T})).$$

Since the motion is parametrized by time, the time-component of the trajectory velocity is equal to the unity. Events related by the motion form a class of equivalence and this equivalence relation foliates the trajectory manifold, as depicted in Fig. 4.

- A *material particle* is a line (a one-dimensional manifold) whose elements are evolution-related trajectory events, see Fig. 4.
- The *body* is the quotient manifold resulting from the induced foliation of the trajectory manifold, see Fig. 4.
- A *body placement* is a fiber of simultaneous trajectory-events. The placement at time $t \in I$ is then $\mathcal{T}(t) = \boldsymbol{\Omega}_t \times \{t\}$.

Remark 6.1. In continuum mechanics it is customary to introduce a manifold $\mathcal{B}$, called the *body*, assumed to be diffeomorphic to all body placements. We will not follow this route whose usefulness is questionable by lack of uniqueness of the choice. The new definition given above, although seemingly abstract, is in fact the direct mathematical transcription of what in the mechanical reality permits to detect the body under investigation. In the same order of ideas, the terms *spatial* and *material* are adopted to denote geometric fields respectively pertaining to the fibers of simultaneous events in space–time and to the fibers of simultaneous events in the trajectory, both testable geometric entities. These new definitions modify the common usage of the word *material* which is attributed to untestable fields defined in an undetectable body manifold.

Definition 6.1 (*Time invariance of material fields*). Time invariance of a material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{TENS}(\mathbb{V}\mathcal{T}))$ along the motion, means variance by push for any time shift $\alpha \in \mathcal{R}$, as described by the commutative diagram

$$\begin{array}{ccc} \mathrm{TENS}(\mathbb{V}\mathcal{T}) & \xleftarrow{\boldsymbol{\varphi}_\alpha \downarrow} & \mathrm{TENS}(\mathbb{V}\mathcal{T}) \\ {\scriptstyle \mathbf{s}_{\mathcal{T}}}\uparrow & & \uparrow{\scriptstyle \mathbf{s}_{\mathcal{T}}} \\ \mathcal{T} & \xrightarrow{\boldsymbol{\varphi}_\alpha} & \mathcal{T} \end{array} \quad \Leftrightarrow \quad \mathbf{s}_{\mathcal{T}} = \boldsymbol{\varphi}_\alpha \downarrow \circ\, \mathbf{s}_{\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha.$$

By definition of pull-back

$$(\boldsymbol{\varphi}_\alpha \downarrow \mathbf{s}_{\mathcal{T}}) := \boldsymbol{\varphi}_\alpha \downarrow \circ \mathbf{s}_{\mathcal{T}} \circ \boldsymbol{\varphi}_\alpha,$$

time invariance may be written as $\mathbf{s}_{\mathcal{T}} = (\boldsymbol{\varphi}_\alpha \downarrow \mathbf{s}_{\mathcal{T}})$ and is equivalent to the vanishing of the LIE derivative $\mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \mathbf{s}_{\mathcal{T}} = 0$.

The trajectory time-bundle of a body is characterized by a conservation law concerning a material volume form defined on the vertical tangent bundle $\mathbb{V}\mathcal{T} \subset \mathbb{T}\mathcal{T}$, that is the subbundle of vectors tangent to placements.

The material mass form $\mathbf{m}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{VOL}(\mathbb{V}\mathcal{T}))$ is proportional to the material volume form $\mathbf{m}_{\mathcal{T}} := \rho_{\mathcal{T}} \boldsymbol{\mu}_{\mathcal{T}}$ by means of the scalar mass density $\rho_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{FUN}(\mathbb{V}\mathcal{T}))$.

Definition 6.2 (*Conservation of mass*). The conservation law states that the integral of the material mass form over any placement is left invariant by the motion, as expressed by the condition

$$\int_{\boldsymbol{\Omega}_{t_1}} (\mathbf{m}_{\mathcal{T}})_{t_1} = \int_{\boldsymbol{\Omega}_{t_2}} (\mathbf{m}_{\mathcal{T}})_{t_2},$$

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which, by the integral transformation formula

$$\int_{\Omega_{t_2}} (\mathbf{m}_{\mathcal{T}})_{t_2} = \int_{\Omega_{t_1}} \boldsymbol{\varphi}_{t_2-t_1} \downarrow (\mathbf{m}_{\mathcal{T}})_{t_1}$$

is equivalent to the pull-back condition $\mathbf{m}_{\mathcal{T}} = \boldsymbol{\varphi}_\alpha \downarrow \mathbf{m}_{\mathcal{T}}$ or to the differential condition $\mathcal{L}_{\mathbf{v}_{\mathcal{T}}} \mathbf{m}_{\mathcal{T}} := \partial_{\alpha=0}(\boldsymbol{\varphi}_\alpha \downarrow \mathbf{m}_{\mathcal{T}}) = 0$ [17].

In mechanics the conservation law assesses time-invariance of the mass form. In electrodynamics the assessment concerns time-invariance of the electric charge form.

7. Trajectory transformations

In NLCM investigations about transformations of the trajectory and of the relevant motion are essential to perform a proper discussion of invariance of constitutive laws under changes of observer and to recover a linear framework suitable for the proof of theoretical results and for numerical computations. It is then convenient to consider the issue from a general point of view and then to specialize the results to the particular context of interest.

Definition 7.1. A trajectory transformation is a diffeomorphism, between a source and a target trajectory time-bundle $\zeta \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_\zeta)$, described by the commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\zeta} & \mathcal{T}_\zeta \\ {\scriptstyle \pi_{I,\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle \pi_{I,\mathcal{T}_\zeta}} \\ I & \xleftrightarrow{\mathrm{ID}_I} & I \end{array} \quad \Leftrightarrow \quad \pi_{I,\mathcal{T}_\zeta} \circ \zeta = \pi_{I,\mathcal{T}}.$$

A correspondence between the motion $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T};\mathcal{T})$ and the pushed motion $(\zeta\uparrow\boldsymbol{\varphi})_\alpha \in \mathrm{C}^1(\mathcal{T}_\zeta;\mathcal{T}_\zeta)$ is induced according to the commutative diagram

$$\begin{array}{ccc} \mathcal{T}_\zeta & \xrightarrow{(\zeta\uparrow\boldsymbol{\varphi})_\alpha} & \mathcal{T}_\zeta \\ {\scriptstyle \zeta}\uparrow & & \uparrow{\scriptstyle \zeta} \\ \mathcal{T} & \xrightarrow{\boldsymbol{\varphi}_\alpha} & \mathcal{T} \end{array} \quad \Leftrightarrow \quad (\zeta\uparrow\boldsymbol{\varphi})_\alpha \circ \zeta = \zeta \circ \boldsymbol{\varphi}_\alpha.$$

Lemma 7.1 (*Trajectory speed of pushed motions*). *Trajectory speeds of pushed motions are related by push*

$$\mathbf{v}_{\mathcal{T}_\zeta} = \zeta\uparrow\mathbf{v}_{\mathcal{T}}.$$

Proof. Being $\mathbf{v}_{\mathcal{T}} := \partial_{\alpha=0}\boldsymbol{\varphi}_\alpha$ and $\mathbf{v}_{\mathcal{T}_\zeta} := \partial_{\alpha=0}(\zeta\uparrow\boldsymbol{\varphi})_\alpha$, the direct computation

$$\mathbf{v}_{\mathcal{T}_\zeta} = \partial_{\alpha=0}(\zeta\circ\boldsymbol{\varphi}_\alpha\circ\zeta^{-1}) = T\zeta\circ\mathbf{v}_{\mathcal{T}}\circ\zeta^{-1} = \zeta\uparrow\mathbf{v}_{\mathcal{T}},$$

gives the result. □

Definition 7.2 (*Invariance of material fields*). A material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{TENS}(\mathbb{V}\mathcal{T}))$ is invariant under the action of a relative motion $\zeta \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_\zeta)$ if it varies according to push

$$\mathbf{s}_{\mathcal{T}_\zeta} = \zeta\uparrow\mathbf{s}_{\mathcal{T}}.$$

The following is a basic property of LIE time-derivatives [26,17].

Lemma 7.2 (*Push of* LIE *derivatives by relative motions*). *The* LIE *derivative of a material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{TENS}(\mathbb{V}\mathcal{T}))$ and the* LIE *time-derivative of its push by a relative motion $\zeta \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_\zeta)$, are related by the push*

$$\zeta\uparrow(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}}) = \mathcal{L}_{\zeta\uparrow\mathbf{v}_{\mathcal{T}}}(\zeta\uparrow\mathbf{s}_{\mathcal{T}}).$$

The related result concerning invariance is provided below.

Proposition 7.1 (*Invariance of* LIE *derivatives*). *Invariance of a material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{TENS}(\mathbb{V}\mathcal{T}))$ with respect to a relative motion $\zeta \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_\zeta)$, implies invariance of the* LIE *derivative along the motion*

$$\mathbf{s}_{\mathcal{T}_\zeta} = \zeta\uparrow\mathbf{s}_{\mathcal{T}} \Rightarrow \mathcal{L}_{\mathbf{v}_{\mathcal{T}_\zeta}}\mathbf{s}_{\mathcal{T}_\zeta} = \zeta\uparrow(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}}).$$

Proof. The result is a direct consequence of Lemmas 7.1 and 7.2. □

As a first application of the previous results we consider a special class of trajectory transformations called *straightening maps* which fulfill the commutative diagram

$$\begin{array}{ccc} \Omega\times I & \xrightarrow{\mathrm{SHIFT}_\alpha} & \Omega\times I \\ {\scriptstyle \zeta}\uparrow & & \uparrow{\scriptstyle \zeta} \\ \mathcal{T}_I & \xrightarrow{\boldsymbol{\varphi}_\alpha} & \mathcal{T}_I \end{array} \quad \Leftrightarrow \quad \mathrm{SHIFT}_\alpha = \zeta\uparrow\boldsymbol{\varphi}_\alpha,$$

with $\mathcal{T}_I$ trajectory segment corresponding to a time interval $I=[t_1,t_2]$, Ω compact manifold and $\mathrm{SHIFT}_\alpha \in \mathrm{C}^1(\Omega\times I;\Omega\times I)$ the *time-translation*

$$\mathrm{SHIFT}_\alpha(\mathbf{x},t) := (\mathbf{x},t+\alpha).$$

The following basic result is a consequence of Lemma 7.2.

Lemma 7.3. *The push of the* LIE *derivative $\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}}$ of a material tensor field by the straightening map $\zeta \in \mathrm{C}^1(\mathcal{T}_I;\Omega\times I)$ is equal to the partial time derivative of the referential tensor field*

$$\zeta\uparrow(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}}) = \partial_{\alpha=0}(\zeta\uparrow\mathbf{s}_{\mathcal{T}})\circ\mathrm{SHIFT}_\alpha.$$

Proof. By fiberwise linearity of the push $\zeta\uparrow \in \mathrm{C}^1(\mathbb{T}\mathcal{T}_I;\mathbb{T}\Omega\times\mathbb{T}I)$ and hence

$$\begin{aligned}\zeta\uparrow(\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{s}_{\mathcal{T}}) &= \zeta\uparrow\partial_{\alpha=0}(\boldsymbol{\varphi}_\alpha\downarrow\mathbf{s}_{\mathcal{T}}) = \partial_{\alpha=0}\,\zeta\uparrow(\boldsymbol{\varphi}_\alpha\downarrow\mathbf{s}_{\mathcal{T}}) \\ &= \partial_{\alpha=0}\,(\zeta\uparrow\boldsymbol{\varphi})_\alpha\downarrow(\zeta\uparrow\mathbf{s}_{\mathcal{T}}) \\ &= \partial_{\alpha=0}\,\mathrm{SHIFT}_\alpha\downarrow\circ(\zeta\uparrow\mathbf{s}_{\mathcal{T}})\circ\mathrm{SHIFT}_\alpha.\end{aligned}$$

According to the usual identification, the push by the time-translation is assumed to be the identity and this gives the result. □

Another basic application of the previous results is to the investigations about invariance under changes of observer. A EUCLID *change of observer* is described by an isometric trajectory transformation $\zeta_{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\zeta_{\mathrm{ISO}}})$, that is such that

$$\mathbf{g}_{\mathcal{T}} = \zeta_{\mathrm{ISO}}\downarrow\mathbf{g}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = (T\zeta_{\mathrm{ISO}})^*\circ\mathbf{g}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}\circ T\zeta_{\mathrm{ISO}}$$

or explicitly $\mathbf{g}_{\mathcal{T}}(\mathbf{u}_{\mathcal{T}},\mathbf{w}_{\mathcal{T}}) = \mathbf{g}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}(T\zeta_{\mathrm{ISO}}\cdot\mathbf{u}_{\mathcal{T}}, T\zeta_{\mathrm{ISO}}\cdot\mathbf{w}_{\mathcal{T}})\circ\zeta_{\mathrm{ISO}}$. EUCLID frame-invariance is the requirement that invariance must hold for all isometric transformations $\zeta_{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\zeta_{\mathrm{ISO}}})$.

8. Constitutive theory

Material tangent vectors at different events of the trajectory, corresponding to the same particle, or at corresponding events at the same time instant of pushed trajectories must be compared by push respectively along the relevant displacement map or along the relative motion. As a consequence, comparison of material tensors must be made after transformation by push to the same tangent space at a trajectory event, as required by the following basic paradigm [13].

Proposition 8.1 (*Covariance paradigm*). *Material tensor fields on a trajectory are compared according to transformation by push along the motion. Material tensor fields pertaining to transformed trajectories are compared according to the relevant push.*

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The covariance paradigm provides the tool to compare material tensor fields corresponding to the same particle at different times along the trajectory or at different placements due to a relative motion. Thus, it is not an invariance requirement but rather reveals the rationale to define invariance of material tensor fields, as introduced in Definition 7.2.

In the *covariance paradigm*, the attribute *covariance* means *variance by convection* (*push*), that is the natural variance induced by the diffeomorphic correspondence between displaced placements of the body. Naturality meaning that no other spurious assumptions are involved.

Standard treatments in the literature are instead based on the adoption of a *parallel transport* (or *connection*) in the ambient manifold as comparison tool, a procedure which is geometrically improper when applied to lower dimensional bodies, like wires or membranes, see Fig. 1, and renders the theory dependent on the arbitrary choice of a connection.

Moreover, according to the *covariance paradigm*, invariance of material tensors is defined as variance by push, so that isometric relative motions do indeed *change* the material metric tensor by push forward, viz. the material metric tensor in the target placement is equal to the push-forward, according to the isometric map, of the material metric tensor in the source placement. *Unchanged*, in the non-linear geometric context, has no definite meaning.

To consider a sufficiently general class of material behavior for engineering applications, we consider a constitutive law as a relation between material tensor fields, governed by a possibly multivalued constitutive operator $\mathbf{H}_{\mathcal{T}}$, according to the following definition.

Definition 8.1 (*Constitutive laws*). A constitutive operator $\mathbf{H}_{\mathcal{T}}$ on the trajectory is a possibly multivalued material tensor bundle morphism whose domain and co-domain are Whitney products[4] of material tensor bundles.

The *bundle morphism* requirement means that material tensors in the domain and co-domain of the constitutive map are evaluated at the same base point in the tangent trajectory bundle, that is, at a common pair (particle position, time instant) in the trajectory.

To simplify the exposition, but without loss of generality, in the forthcoming general discussion, the constitutive operator $\mathbf{H}_{\mathcal{T}}$ will be considered as a tensor bundle morphism between dual material tensor bundles, with the constitutive response denoted by $\boldsymbol{\varepsilon}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}})$.

Definition 8.2 (*Constitutive time invariance (CTI)*). According to the covariance paradigm (Proposition 8.1) the constitutive operator is time invariant if along the motion

$$\mathbf{H}_{\mathcal{T}} = \boldsymbol{\varphi}_{\alpha}\!\uparrow\!\mathbf{H}_{\mathcal{T}},$$

whose definition is given by

$$\mathbf{H}_{\mathcal{T}}(\boldsymbol{\varphi}_{\alpha}\!\uparrow\!\mathbf{s}_{\mathcal{T}}) = (\boldsymbol{\varphi}_{\alpha}\!\uparrow\!\mathbf{H}_{\mathcal{T}})(\boldsymbol{\varphi}_{\alpha}\!\uparrow\!\mathbf{s}_{\mathcal{T}}) \coloneqq \boldsymbol{\varphi}_{\alpha}\!\uparrow\!(\mathbf{H}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}}))$$

and may be expressed by the commutativity property

$$\mathbf{H}_{\mathcal{T}} \circ \boldsymbol{\varphi}_{\alpha}\!\uparrow = \boldsymbol{\varphi}_{\alpha}\!\uparrow \circ \mathbf{H}_{\mathcal{T}}.$$

[4] The Whitney product of two tensor bundles $(\mathbb{N},\boldsymbol{\pi}_{\mathbb{M},\mathbb{N}},\mathbb{M})$ and $(\mathbb{H},\boldsymbol{\pi}_{\mathbb{M},\mathbb{H}},\mathbb{M})$, over the same base manifold $\mathbb{M}$, is the linear bundle defined by [27]

$$\mathbb{N} \times_{\mathbb{M}} \mathbb{H} \coloneqq \{(\mathbf{n},\mathbf{h}) \in \mathbb{N} \times \mathbb{H} \mid \boldsymbol{\pi}_{\mathbb{M},\mathbb{N}}(\mathbf{n}) = \boldsymbol{\pi}_{\mathbb{M},\mathbb{H}}(\mathbf{h})\}.$$

9. Constitutive frame invariance

According to the covariance paradigm (Proposition 8.1) material tensors evaluated at a given event on the trajectory by observers in relative isometric motion appear to be *the same* when they are related by push–pull transformation according to the relative motion. In the literature, *the same* is assumed, more or less explicitly, to mean that involved tensors, as seen by the observers, have a *null difference*. But *difference* of tensors, based on trajectories detected by distinct observers, is an undefined operation.

The material metric tensor is Euclid frame-invariant by definition. For all other material tensors the following physical axiom holds.

Principle 9.1 (*Axiom of frame-invariance (AFI)*). *Material tensors are* Euclid *frame-invariant.*

This axiom leads to the formulation of the new principle of constitutive frame invariance (CFI) as a substitute to the improper formulation of material frame indifference (MFI), to account for the fact that distinct observers will formulate distinct constitutive relations involving distinct material tensors. The formal statement is the following.

Principle 9.2 (*Constitutive frame invariance (CFI)*). *Any constitutive law must conform to the principle of CFI which requires that material fields, fulfilling the law formulated by an observer, when transformed by invariance according to the relative motion, will also fulfill the law formulated by another* Euclid *observer and vice versa. The principle is expressed by the equivalence*

$$\boldsymbol{\varepsilon}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}}) \Leftrightarrow \boldsymbol{\varepsilon}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \mathbf{H}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}(\mathbf{s}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}),$$

for any isometric transformation $\zeta_{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\zeta_{\mathrm{ISO}}})$.

The difference between the new statement of CFI and the material frame indifference (MFI) formulated by Noll may be put into evidence by observing that the MFI consists in the equivalence

$$\boldsymbol{\varepsilon}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}}) \Leftrightarrow \boldsymbol{\varepsilon}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \mathbf{H}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}),$$

where dependence of the constitutive operator on the observer is not taken into account.

The geometric statement of Principle 9.2 provides the precise mathematical expression of the *naïve* physical requirement that material constitutive behavior should be *the same* when evaluated by any Euclid observer.

A first and somewhat vague statement of CFI might be attributed to Stanisław Zaremba who in [28], sustaining invariance under time-dependent rigid transformations, wrote: *C'est là une propriété que doit avoir tout systeme d'équations exprimant les relations qui existent entre les forces intérieures dans une substance et les circonstances de son mouvement.*

In [29] it is moreover written: *L'emploi des variables introduites en hydrodynamique par* Lagrange *permet d'éviter, sans introduire aucune complication dans leséquations, l'usage d'équations incompatibles avec le principe des mouvements relatifs* …. This is probably a first and partial recourse to the idea underlying the *covariance paradigm*, since it suggests the adoption of expressions in convective coordinates to ensure fulfillment of CFI.

This suggestion was resorted to in the proposals later made by Oldroyd [30,31] and by Truesdell [32], in introducing his model of hypo-elasticity, and since then have been reproduced in the relevant literature.

The decisive new contribution provided by the geometric treatment is the adoption of suitable tools to deal with the essentially

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non-linear problems of describing the constitutive response of a material and of detecting the proper rule governing the transformation of the constitutive law formulated by an observer into the one formulated by another observer.

Definition 9.1 (*Pushed constitutive operator*). The pushed constitutive operator $\boldsymbol{\zeta}\!\uparrow\!\mathbf{H}_{\mathcal{T}}$ under a relative motion $\zeta \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_\zeta)$ is defined by the identity

$$(\boldsymbol{\zeta}\!\uparrow\!\mathbf{H}_{\mathcal{T}})(\boldsymbol{\zeta}\!\uparrow\!\mathbf{s}_{\mathcal{T}}) := \boldsymbol{\zeta}\!\uparrow\!(\mathbf{H}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}})),$$

which may be expressed by the equality

$$\boldsymbol{\zeta}\!\uparrow\!\mathbf{H}_{\mathcal{T}} := \boldsymbol{\zeta}\!\uparrow\circ\mathbf{H}_{\mathcal{T}}\circ\boldsymbol{\zeta}\!\downarrow,$$

as depicted by the commutative diagram

$$\begin{tikzcd}
\mathrm{TENS}(\mathbb{V}\mathcal{T}_\zeta) \arrow[r, "\zeta\uparrow\mathbf{H}_{\mathcal{T}}"] \arrow[d, "\zeta\downarrow"'] & \mathrm{TENS}(\mathbb{V}\mathcal{T}_\zeta) \\
\mathrm{TENS}(\mathbb{V}\mathcal{T}) \arrow[r, "\mathbf{H}_{\mathcal{T}}"] & \mathrm{TENS}(\mathbb{V}\mathcal{T}) \arrow[u, "\zeta\uparrow"']
\end{tikzcd}$$

A condition equivalent to fulfillment of the CFI principle is the following.

Proposition 9.1 (*Frame invariance of the constitutive operator*). *A constitutive law conforms to the principle of* CFI *if and only if the constitutive operator is frame invariant, that is*

$$\mathbf{H}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\mathbf{H}_{\mathcal{T}},$$

for any isometric transformation $\boldsymbol{\zeta}_{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\zeta_{\mathrm{ISO}}})$ describing a change of EUCLID *observer. This means that pushed material fields will fulfill the constitutive relation formulated by the pushed observer if and only if the original material fields fulfill the constitutive relation formulated by the original observer.*

Proof. The statement follows, from Definition 7.2 of frame-invariance and Definition 9.1 of pushed operator, by a direct verification of the equivalence with the statement of Principle 9.2. □

10. Hypo-elasticity, elasticity and hyper-elasticity

A paradigmatic and especially important example of constitutive law is provided by the covariant *hypo-elastic* model [13] defined by

$$\mathbf{e}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}})\cdot\dot{\boldsymbol{\sigma}}_{\mathcal{T}},$$

where the *stressing* $\dot{\boldsymbol{\sigma}}_{\mathcal{T}}$ is the LIE (or convective) along the motion, of the material *stress* tensor field $\boldsymbol{\sigma}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{SYM}^*(\mathbb{V}\mathcal{T}))$

$$\dot{\boldsymbol{\sigma}}_{\mathcal{T}} := \mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\boldsymbol{\sigma}_{\mathcal{T}} = \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma}_{\mathcal{T}}) = \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha\!\downarrow\circ\boldsymbol{\sigma}_{\mathcal{T}}\circ\boldsymbol{\varphi}_\alpha)$$

and $\mathbf{e}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{COV}(\mathbb{V}\mathcal{T}))$ is the *elastic stretching*. We explicitly remark that the *elastic stretching* is not the LIE derivative of a material field, unless it is equal to the *total stretching*, as occurs in a purely elastic process where

$$\mathbf{e}_{\mathcal{T}} = \tfrac{1}{2}\,\dot{\mathbf{g}}_{\mathcal{T}} := \tfrac{1}{2}\,\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{g}_{\mathcal{T}} = \tfrac{1}{2}\,\partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{g}_{\mathcal{T}}) = \tfrac{1}{2}\,\partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha\!\downarrow\circ\mathbf{g}_{\mathcal{T}}\circ\boldsymbol{\varphi}_\alpha).$$

A geometric formulation, providing an extension of EULER formula for the stretching to general connections in the ambient space, has recently been contributed in [33]. The WHITNEY product $\mathrm{SYM}^*(\mathbb{V}\mathcal{T})\times_{\mathcal{T}}\mathrm{SYM}^*(\mathbb{V}\mathcal{T})$ is the domain of the constitutive operator $\mathbf{H}_{\mathcal{T}}$ and its co-domain is the tensor bundle $\mathrm{SYM}(\mathbb{V}\mathcal{T})$. The operator $\mathbf{H}_{\mathcal{T}}$ is the hypo-elastic compliance and its value $\mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}})$ is a linear hypo-elastic tangent compliance, assumed to be invertible. The choice originally made in [18], and in most of the subsequent literature, was instead to write the hypo-elastic law in terms of tangent stiffness, according to the non-covariant algebraic formulation there adopted. Our choice is motivated by observing that the hypo-elastic tangent compliance does in fact define the elastic stretching.

The following result proven in [13] provides for the covariant formulation of the hypo-elastic model, the basic integrability conditions. This result differs from the one in [12] which, referring to a geometrically inconsistent hypo-elastic model, has been the source of difficulties in theoretically and computationally oriented treatments of elastoplasticity [11].

The fiber-derivative d_F is the derivative performed in a linear tensor fiber by holding the base point (position and time) fixed in the trajectory.

Lemma 10.1 (*Integrability*). *The conditions of integrability are expressed in terms of the hypo-elastic compliance operator by*

$$\langle d_F\mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}})\cdot\delta\boldsymbol{\sigma}_{\mathcal{T}}\cdot\delta_1\boldsymbol{\sigma}_{\mathcal{T}},\delta_2\boldsymbol{\sigma}\rangle = \langle d_F\mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}})\cdot\delta\boldsymbol{\sigma}_{\mathcal{T}}\cdot\delta_2\boldsymbol{\sigma}_{\mathcal{T}},\delta_1\boldsymbol{\sigma}_{\mathcal{T}}\rangle,$$
$$\langle \mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}})\cdot\delta_1\boldsymbol{\sigma}_{\mathcal{T}},\delta_2\boldsymbol{\sigma}_{\mathcal{T}}\rangle = \langle \mathbf{H}_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}})\cdot\delta_2\boldsymbol{\sigma}_{\mathcal{T}},\delta_1\boldsymbol{\sigma}_{\mathcal{T}}\rangle,$$

for all $\delta\boldsymbol{\sigma}_{\mathcal{T}},\delta_1\boldsymbol{\sigma}_{\mathcal{T}},\delta_2\boldsymbol{\sigma}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T},\mathrm{SYM}^*(\mathbb{V}\mathcal{T}))$. The former condition ensures CAUCHY *integrability, stating the existence of a stretching-valued stress potential $\boldsymbol{\Phi}_{\mathcal{T}} \in \mathrm{C}^1(\mathrm{SYM}^*(\mathbb{V}\mathcal{T});\mathrm{SYM}(\mathbb{V}\mathcal{T}))$ such that $d_F\boldsymbol{\Phi}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}$. Both conditions ensure* GREEN *integrability, stating the existence of a scalar-valued stress potential*

$$E^* \in \mathrm{C}^1(\mathrm{SYM}^*(\mathbb{V}\mathcal{T});\mathrm{FUN}(\mathbb{V}\mathcal{T})),$$

such that $d_F E^* = \boldsymbol{\Phi}_{\mathcal{T}}$ and hence

$$d_F^2 E^* = d_F\boldsymbol{\Phi}_{\mathcal{T}} = \mathbf{H}_{\mathcal{T}}.$$

By virtue of the preceding result the following new definition of elasticity may be given.

Definition 10.1 (*Elasticity*). An elastic (resp. hyper-elastic) constitutive model is a hypo-elastic model characterized by a time-invariant and CAUCHY (resp. GREEN) integrable constitutive operator.

Conservation of the mechanical energy is an essential requirement for hyper-elastic behavior and its mathematical definition provides another significant example of application of the covariance paradigm.

Definition 10.2 (*Conservativeness*). A hypo-elastic constitutive model is conservative if the time-integral of the elastic power along the motion vanishes when the values of the stress field at beginning and at end time-instants are related by push. This means the vanishing of the elastic work, defined as the integral of the specific elastic power over the trajectory segment $\mathcal{T}_I$ corresponding to the time interval $I = [t_1,t_2]$

$$\int_{\mathcal{T}_I} \langle \boldsymbol{\sigma}_{\mathcal{T}},\mathbf{e}_{\mathcal{T}}\rangle\,\mathbf{m}_{\mathcal{T}} = 0,$$

for any closed path $\boldsymbol{\sigma}_{\mathcal{T}}\circ\boldsymbol{\varphi} : I \mapsto \mathrm{C}^1(\mathcal{T};\mathrm{SYM}^*(\mathbb{V}\mathcal{T}))$ of stress fields, i.e. fulfilling the condition

$$\boldsymbol{\sigma}_{\mathcal{T}} = (\boldsymbol{\varphi}_{t_2-t_1}\!\uparrow\!\boldsymbol{\sigma}_{\mathcal{T}}) = \boldsymbol{\varphi}_{t_2-t_1}\!\uparrow\circ\boldsymbol{\sigma}_{\mathcal{T}}\circ\boldsymbol{\varphi}_{t_2-t_1}.$$

Proposition 10.1 (*Conservativeness*). *Conservation of mass and* GREEN *integrability of the hypo-elastic operator, expressed in terms of the* KIRCHHOFF *stress tensor, imply conservation of the elastic work.*

Proof. Conservativeness of the constitutive operator is formulated by performing a push according to a straightening isomorphism $\zeta \in \mathrm{C}^1(\mathcal{T}_I;\boldsymbol{\Omega}_{\mathrm{REF}}\times I)$ in the time interval $I = [t_1,t_2]$. Setting $\boldsymbol{\sigma}_{\mathrm{REF}} := \boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma}_{\mathcal{T}}$, $\mathbf{e}_{\mathrm{REF}} := \boldsymbol{\zeta}\!\uparrow\!\mathbf{e}_{\mathcal{T}}$ and $\mathbf{m}_{\mathrm{REF}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{m}_{\mathcal{T}}$ we have that

$$\begin{aligned}\int_{\mathcal{T}_I} \langle \boldsymbol{\sigma}_{\mathcal{T}},\mathbf{e}_{\mathcal{T}}\rangle\,\mathbf{m}_{\mathcal{T}} &= \int_{\boldsymbol{\Omega}_{\mathrm{REF}}\times I} \langle \boldsymbol{\sigma}_{\mathrm{REF}},\mathbf{e}_{\mathrm{REF}}\rangle\,\mathbf{m}_{\mathrm{REF}} \\ &= \int_{\boldsymbol{\Omega}_{\mathrm{REF}}} \mathbf{m}_{\mathrm{REF}} \int_I \langle \boldsymbol{\sigma}_{\mathrm{REF}},\mathbf{e}_{\mathrm{REF}}\rangle\,dt,\end{aligned}$$

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because conservation of mass, as introduced in Definition 6.2, ensures that $\mathbf{m}_{\text{REF}}$ is time-independent. Conservativeness is then expressed by the condition

$$\int_I \langle \boldsymbol{\sigma}_{\text{REF}}, \mathbf{e}_{\text{REF}} \rangle \, dt = 0.$$

By the assumption of Cauchy integrability, setting $\mathbf{\Phi}_{\text{REF}} = \zeta \uparrow \mathbf{\Phi}_{\mathcal{T}}$ and $\boldsymbol{\sigma}_{\text{REF},\alpha} := \boldsymbol{\sigma}_{\text{REF}} \circ \text{SHIFT}_\alpha$, the referential elastic stretching is defined by

$$\mathbf{e}_{\text{REF}} = \partial_{\alpha=0}\, \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF},\alpha}).$$

The time integral of the referential elastic power may then be written as

$$\begin{aligned}\int_I \langle \boldsymbol{\sigma}_{\text{REF}}, \partial_{\alpha=0}\, \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF},\alpha}) \rangle \, dt \\ = \int_I \partial_{\alpha=0} \langle \boldsymbol{\sigma}_{\text{REF},\alpha}, \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF},\alpha}) \rangle \, dt \\ - \int_I \langle \partial_{\alpha=0}\, \boldsymbol{\sigma}_{\text{REF},\alpha}, \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \rangle \, dt.\end{aligned}$$

Under the assumption of Green integrability, setting $E^*_{\text{REF}} = \zeta \uparrow E^*$ so that $\mathbf{\Phi}_{\text{REF}} = d_F E^*_{\text{REF}}$, we get

$$\langle \partial_{\alpha=0} \boldsymbol{\sigma}_{\text{REF},\alpha}, \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \rangle = \partial_{\alpha=0} E^*_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF},\alpha}).$$

Then, considering the potential E_{REF} defined as Legendre conjugate of the referential stress potentieal E^*_{REF} by the relation

$$E_{\text{REF}}(d_F E^*_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}})) + E^*_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) = \langle \boldsymbol{\sigma}_{\text{REF}}, d_F E^*_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \rangle,$$

the integral takes the expression

$$\int_{\mathbf{\Omega}_{\text{REF}}} \mathbf{m}_{\text{REF}} \int_I \partial_{\alpha=0}\, E_{\text{REF}}(d_F E^*_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF},\alpha})) \, dt,$$

which vanish by the assumed closedness of the stress path. □

Let us now state a computationally important result which restores to the simplest elastic model, long adopted in computational codes, the basic properties of hyper-elasticity and conservativeness. These results resolve the troubles expressed in [11, ex. 5.1], which eventually led to the discard of the geometrically inconsistent model of hypo-elasticity there considered. We denote as usual by $\mu > 0$ the Lamé elastic shear modulus, by $-1 < \nu < 0.5$ the Poisson ratio and by $E > 0$ the Euler elastic modulus. The operators $\mathbb{I}_{\mathcal{T}}$ and $\mathbf{I}_{\mathcal{T}}$ respectively denote the automorphisms of the material bundles $\text{Mix}(\mathbb{V}\mathcal{T})$ and $\mathbb{V}\mathcal{T}$ which are fiberwise identities.

Proposition 10.2 (*Simplest elasticity*). *The simplest elastic operator, expressed in terms of the mixed* Kirchhoff *stress tensor* $\mathbf{K}_{\mathcal{T}} = \boldsymbol{\sigma}_{\mathcal{T}} \circ \mathbf{g}_{\mathcal{T}}$ *by*

$$\begin{aligned}\mathbf{H}^{\text{MIX}}_{\mathcal{T}}(\mathbf{K}_{\mathcal{T}}) &:= \frac{1}{2\mu}\mathbb{I}_{\mathcal{T}} - \frac{\nu}{E}\mathbf{I}_{\mathcal{T}} \otimes \mathbf{I}_{\mathcal{T}} \\ &= \frac{1}{E}\mathbb{I}_{\mathcal{T}} + \frac{\nu}{E}(\mathbb{I}_{\mathcal{T}} - \mathbf{I}_{\mathcal{T}} \otimes \mathbf{I}_{\mathcal{T}}), \quad \left(\frac{1}{2\mu} = \frac{1+\nu}{E}\right)\end{aligned}$$

is Green *integrable and fulfills the principle of* CFI.

Proof. Frame invariance of the operator $\mathbf{H}_{\mathcal{T}}$ follows from the following easily verifiable frame-invariance properties

$$\mathbb{I}_{\mathcal{T}_{\zeta_{\text{ISO}}}} = \zeta_{\text{ISO}} \uparrow \mathbb{I}_{\mathcal{T}}, \quad \mathbf{I}_{\mathcal{T}_{\zeta_{\text{ISO}}}} = \zeta_{\text{ISO}} \uparrow \mathbf{I}_{\mathcal{T}}.$$

By Lemma 10.1, integrability of the simplest hypo-elastic operator is inferred from stress independence and symmetry. □

11. Computation chamber

Numerical computations require to formulate the constitutive problem in a linear context in which all linear operations involved in actual computations are feasible. This task is performed by a *straightening map* which transforms the non-linear trajectory manifold $\mathcal{T}$ into the straight *computation chamber* $\mathbf{\Omega}_{\text{REF}} \times I$ (a trivial time-bundle) and permits to formulate the constitutive relation in terms of partial time-derivatives and to perform time-integration in a suitably chosen time-step. The output of the computation is the referential stress field at the end of the time-step and this field can be pushed back to the trajectory manifold to provide the computation estimate of the actual stress field at the end of the time-step. Intermediate results of the time-integration performed in the *computation chamber* such as the time-integral of the elastic stretching, have no physical meaning and are to be considered as purely computation items. Indeed the push-back to the trajectory manifold is impossible because the information concerning the time-instant is lost as a consequence of the time-integration.

Let us illustrate the procedure in some detail.

In terms of a straightening map $\zeta \in C^1(\mathcal{T}; \mathbf{\Omega}_{\text{REF}} \times I)$ the referential elastic strain field is defined by the integral over a time interval $I = [t_1, t_2]$ of the pull-back of the elastic stretching field

$$\mathbf{e}_{\text{REF},I} := \int_I (\zeta \uparrow \mathbf{e}_{\mathcal{T}})(t) \, dt.$$

Setting

$$\mathbf{\Phi}_{\text{REF}} = \zeta \uparrow \mathbf{\Phi}_{\mathcal{T}}, \quad \boldsymbol{\sigma}_{\text{REF}} = \zeta \uparrow \boldsymbol{\sigma}_{\mathcal{T}},$$
$$\mathbf{e}_{\text{REF}} = \zeta \uparrow \mathbf{e}_{\mathcal{T}}, \quad \mathbf{g}_{\text{REF}} = \zeta \uparrow \mathbf{g}_{\mathcal{T}},$$

the pull-back of the constitutive relation of an elastic material in the reference placement is expressed by

$$\mathbf{e}_{\text{REF}} = d_F \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}) \cdot \partial_{\alpha=0}(\boldsymbol{\sigma}_{\text{REF},\alpha}) = \partial_{\alpha=0} \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF},\alpha}).$$

Integrating in time, we get the expression of the finite increment elastic law

$$\mathbf{e}_{\text{REF},I} = \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}(t_2)) - \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}(t_1)).$$

Under the assumption of invertibility of the stress-potential $\mathbf{\Phi}$, the referential stress at the end of the incremental time-step is given by

$$\boldsymbol{\sigma}_{\text{REF}}(t_2) = \mathbf{\Phi}^{-1}_{\text{REF}}(\mathbf{e}_{\text{REF},I} + \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}(t_1))).$$

The corresponding stress on the trajectory at the end of the time-step is provided by the pull-back $\boldsymbol{\sigma}_{\mathcal{T}}(t_2) = \zeta \downarrow \boldsymbol{\sigma}_{\text{REF}}(t_2)$.

In a purely elastic process $\mathbf{e}_{\mathcal{T}} = \frac{1}{2}\dot{\mathbf{g}}_{\mathcal{T}} = \frac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{g}_{\mathcal{T}}$. Time-integration of $\mathbf{e}_{\text{REF}} = \zeta \uparrow \mathbf{e}$ then gives

$$\mathbf{e}_{\text{REF},I} := \frac{1}{2}\int_I \partial_{\alpha=0}(\mathbf{g}_{\text{REF}} \circ \text{SHIFT}_\alpha) \, dt = \frac{1}{2}(\mathbf{g}_{\text{REF}}(t_2) - \mathbf{g}_{\text{REF}}(t_1)),$$

which is the Green referential strain evaluated in the interval $I = [t_1, t_2]$.

This formula will be resorted to in the next proposition to clarify an issue sometimes vaguely enunciated or also ill-stated in the literature.

The proposition provides a precise statement about the intuitive requirement that the stress response of an elastic material should not change under a rigid body motion, as enunciated by Hooke, Poisson and Cauchy for linearized elasticity, see Section 14.

Proposition 11.1. *If the potential* $\mathbf{\Phi}_{\mathcal{T}} \in C^1(\text{Sym}^*(\mathbb{V}\mathcal{T}); \text{Sym}(\mathbb{V}\mathcal{T}))$ *of an elastic constitutive operator is injective, then any isometric motion leaves the stress tensor field invariant.*

Proof. By definition the Green referential elastic strain

$$\mathbf{e}_{\text{REF}}(t_2, t_1) := \zeta(t_1) \uparrow \tfrac{1}{2}(\boldsymbol{\varphi}_{t_2 - t_1} \downarrow \mathbf{g}_{\mathcal{T}} - \mathbf{g}_{\mathcal{T}})$$

vanishes when the displacement $\boldsymbol{\varphi}_{t_2-t_1}$ is isometric. Then, being $\mathbf{e}_{\text{REF}}(t_2, t_1) = \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}(t_2)) - \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}(t_1))$ we infer that $\mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}(t_2)) = \mathbf{\Phi}_{\text{REF}}(\boldsymbol{\sigma}_{\text{REF}}(t_1))$. Here time-invariance enters in assuring that $\mathbf{\Phi}_{\text{REF}}$ does not depend on time. Injectivity of the

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strain-valued stress-potential $\boldsymbol{\Phi}_{\mathcal{T}} \in \mathrm{C}^1(\mathrm{Con}(\mathbb{V}\mathcal{T});\mathrm{Cov}(\mathbb{V}\mathcal{T}))$ implies that the referential potential $\boldsymbol{\Phi}_{\mathrm{REF}}$ is injective too. Equality $\boldsymbol{\sigma}_{\mathrm{REF}}(t_2) = \boldsymbol{\sigma}_{\mathrm{REF}}(t_1)$ between referential stress tensors at time instants $t_1, t_2 \in I$, follows. Pushing forward by $\zeta(t_2)$ and resorting to the map composition $\zeta(t_2) = \boldsymbol{\varphi}_{t_2 - t_1} \circ \zeta(t_1)$ we get the relation $\boldsymbol{\sigma}_{\mathcal{T}}(t_2) = \boldsymbol{\varphi}_{t_2-t_1} \uparrow \boldsymbol{\sigma}_{\mathcal{T}}(t_1)$ expressing stress time-invariance. □

12. Elasto-visco-plasticity

A more general example of constitutive behavior described by Definition 8.1, and of primary applicative interest in NLCM, is provided by an elasto-visco-plastic material, modeled by the relations

$$\begin{cases} \frac{1}{2}\dot{\mathbf{g}}_{\mathcal{T}} := \frac{1}{2}\mathcal{L}_{\mathbf{v}_{\mathcal{T}}}\mathbf{g}_{\mathcal{T}} = \mathbf{e}_{\mathcal{T}} + \mathbf{p}_{\mathcal{T}}, \\ \mathbf{e}_{\mathcal{T}} = d_F^2 E^*(\boldsymbol{\sigma}_{\mathcal{T}}) \cdot \dot{\boldsymbol{\sigma}}_{\mathcal{T}}, \\ \mathbf{p}_{\mathcal{T}} \in \partial_F \mathcal{F}(\boldsymbol{\sigma}_{\mathcal{T}}), \end{cases}$$

where $\mathcal{F} \subset \mathrm{Fun}(\mathrm{Sym}^*(\mathbb{V}\mathcal{T}))$ is the visco-plastic potential, fiberwise subdifferentiable and convex [17].

The *elastic stretching* $\mathbf{e}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}, \mathrm{Sym}(\mathbb{V}\mathcal{T}))$ and the *visco-plastic stretching* $\mathbf{p}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}, \mathrm{Sym}(\mathbb{V}\mathcal{T}))$ are not convective time derivatives of material fields. Hence, strictly they should not be denoted by a superimposed dot, contrary to the common usage in the literature. Elasto-plasticity is modeled by assuming that the potential is the indicator function of the convex set of admissible stresses $\mathcal{K} \subset \mathrm{Sym}^*(\mathbb{V}\mathcal{T})$, so that

$$\partial_F \mathcal{F}(\boldsymbol{\sigma}_{\mathcal{T}}) = \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathcal{T}}),$$

where $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathcal{T}})$ is the outward normal cone to $\boldsymbol{\sigma}_{\mathcal{T}} \in \mathcal{K}$. The visco-plastic constitutive relation becomes then the plastic flow rule [34]

$$\mathbf{p}_{\mathcal{T}} \in \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathcal{T}}).$$

By a pull-back procedure analogous to the one in Section 11, the elasto-visco-plastic constitutive relations may be formulated in terms of material tensor fields defined in a fixed reference placement. Introducing the referential visco-plastic potential $\mathcal{F}_{\mathrm{REF}} = \zeta \uparrow \mathcal{F}$ and the referential visco-plastic stretching $\mathbf{p}_{\mathrm{REF}} = \zeta \uparrow \mathbf{p}$ and setting $\mathbf{g}_{\mathrm{REF},\alpha} := (\zeta \uparrow \mathbf{g}_{\mathcal{T}}) \circ \mathrm{SHIFT}_\alpha$, we get

$$\begin{cases} \partial_{\alpha=0} \frac{1}{2}\mathbf{g}_{\mathrm{REF},\alpha} = \mathbf{e}_{\mathrm{REF}} + \mathbf{p}_{\mathrm{REF}}, \\ \mathbf{e}_{\mathrm{REF}} = \partial_{\alpha=0}\, d_F E^*_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF},\alpha}), \\ \mathbf{p}_{\mathrm{REF}} \in \partial_F \mathcal{F}_{\mathrm{REF}}(\boldsymbol{\sigma}_{\mathrm{REF}}). \end{cases}$$

In an evolution process, the computation of the referential stress field is conveniently carried out by a discrete time integration scheme and by an iterative algorithm, for the solution, at each time step, of the non-linear discrete constitutive relation, on the basis of trial estimates of the elastic stretching evaluated at a fixed reference placement [22].

The natural physical assumption of frame invariance of material tensors involved in elasto-visco-plastic constitutive relations may be deduced from the assumed frame invariance of stress power and of stress dissipation, as stated in the next proposition.

Proposition 12.1 (*Frame invariance of material tensors*). *Frame invariance of stress power* $\langle \boldsymbol{\sigma}_{\mathcal{T}}, \frac{1}{2}\dot{\mathbf{g}}_{\mathcal{T}} \rangle$ *and of plastic stress dissipation* $\langle \boldsymbol{\sigma}_{\mathcal{T}}, \mathbf{p}_{\mathcal{T}} \rangle$, *for any material stress, total stretching and visco-plastic stretching, implies frame invariance of stress, stressing, elastic and visco-plastic stretching*

$$\boldsymbol{\sigma}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \boldsymbol{\sigma}_{\mathcal{T}}, \quad \dot{\boldsymbol{\sigma}}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \dot{\boldsymbol{\sigma}}_{\mathcal{T}},$$
$$\mathbf{e}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \mathbf{e}_{\mathcal{T}}, \quad \mathbf{p}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \mathbf{p}_{\mathcal{T}}.$$

Proof. Frame invariance of the metric tensor $\mathbf{g}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \mathbf{g}_{\mathcal{T}}$ is the characteristic property of a change of Euclid observer, as defined in Section 7. By Proposition 7.1 frame invariance of the total stretching tensor $\dot{\mathbf{g}}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \dot{\mathbf{g}}_{\mathcal{T}}$ follows. Moreover, for any relative motion $\zeta \in \mathrm{C}^1(\mathcal{T}; \mathcal{T}_\zeta)$, naturality of the push with respect to the duality pairing holds [17]

$$\zeta \uparrow \langle \boldsymbol{\sigma}_{\mathcal{T}}, \dot{\mathbf{g}}_{\mathcal{T}} \rangle = \langle \zeta \uparrow \boldsymbol{\sigma}_{\mathcal{T}}, \zeta \uparrow \dot{\mathbf{g}}_{\mathcal{T}} \rangle.$$

Then frame invariance of the stress power $\zeta_{\mathrm{ISO}} \uparrow \langle \boldsymbol{\sigma}_{\mathcal{T}}, \dot{\mathbf{g}}_{\mathcal{T}} \rangle = \langle \boldsymbol{\sigma}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}, \dot{\mathbf{g}}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} \rangle$ may be expressed as

$$\zeta_{\mathrm{ISO}} \uparrow \langle \boldsymbol{\sigma}_{\mathcal{T}}, \dot{\mathbf{g}}_{\mathcal{T}} \rangle = \langle \zeta_{\mathrm{ISO}} \uparrow \boldsymbol{\sigma}_{\mathcal{T}}, \zeta_{\mathrm{ISO}} \uparrow \dot{\mathbf{g}}_{\mathcal{T}} \rangle = \langle \boldsymbol{\sigma}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}, \zeta_{\mathrm{ISO}} \uparrow \dot{\mathbf{g}}_{\mathcal{T}} \rangle,$$

for all $\dot{\mathbf{g}}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{Cov}(\mathbb{V}\mathcal{T}))$ and this is equivalent to frame invariance of the stress. By a similar argument frame invariance of the stress dissipation leads to the conclusion that the visco-plastic stretching is frame invariant. Frame invariance of elastic stretching is inferred by the additivity rule. □

For integrable hypo-elastic models, frame invariance of the hypo-elastic operator $\mathbf{H}_{\mathcal{T}}$ implies frame invariance of Cauchy-elastic and Green-elastic potentials

$$\boldsymbol{\Phi}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \boldsymbol{\Phi}_{\mathcal{T}}, \quad E^*_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow E^*_{\mathcal{T}}.$$

This property follows at once from the commutativity $\zeta_{\mathrm{ISO}} \uparrow \circ d_F = d_F \circ \zeta_{\mathrm{ISO}} \uparrow$, which holds by fiber-linearity of the push between material tensor bundles.

The proof of the following result is straightforward.

Proposition 12.2 (*CFI in elasto-visco-plasticity*). *The constitutive relation of hypo-elastic visco-plasticity conforms to the principle of* CFI *if the hypo-elastic operator and the visco-plastic constitutive potential are frame invariant*

$$\mathbf{H}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \mathbf{H}_{\mathcal{T}}, \quad \mathcal{F}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \mathcal{F}_{\mathcal{T}}.$$

Instances of fulfillment of the frame invariance conditions in Proposition 12.2 are provided by the simplest hypo-elastic model and by visco-plastic constitutive potentials expressed in terms of invariants of the mixed stress tensor.

13. Comparison with other treatments

According to the analysis performed in Section 3.1, the push of a mixed Kirchhoff tensor field $\mathbf{K}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{Mix}(\mathbb{V}\mathcal{T}))$ involved in Definition 7.2 of frame-invariance, is given, according to the formula in Section 3, by the formula

$$\zeta_{\mathrm{ISO}} \uparrow \mathbf{K}_{\mathcal{T}} = T\zeta_{\mathrm{ISO}} \circ \mathbf{K}_{\mathcal{T}} \circ T\zeta_{\mathrm{ISO}}^{-1}.$$

The treatment performed in [21], [18, 17.3–17.4] and [20, II-14-5] provides a seemingly similar expression written in terms of the linear isometry $\mathbf{Q} \in L(V;V)$ in the linear space V of translations of the Euclid ambient space. Indeed, the formula was there written (in our notations) as

$$\zeta_{\mathrm{ISO}} \uparrow \mathbf{K}_{\mathcal{T}} = \mathbf{Q} \circ \mathbf{K}_{\mathcal{T}} \circ \mathbf{Q}^T.$$

The basic difference is that $\mathbf{Q}, \mathbf{Q}^T \in L(V;V)$ are invertible linear maps while instead the tangent maps

$$T\zeta_{\mathrm{ISO}} \in \mathrm{C}^1(\mathbb{V}\mathcal{T}; \mathbb{V}\mathcal{T}_{\zeta_{\mathrm{ISO}}}) \quad \text{and} \quad T\zeta_{\mathrm{ISO}}^{-1} \in \mathrm{C}^1(\mathbb{V}\mathcal{T}_{\zeta_{\mathrm{ISO}}}; \mathbb{V}\mathcal{T})$$

are non-linear one-to-one correspondences between material bundles which are only fiberwise linear because changes of base points are involved, see Section 3.

The importance of a correct geometric description becomes especially evident if the body is modeled by a lower dimensional manifold, as in the continuum mechanics description of a wire or of a membrane, see Fig. 1.

A purely algebraic treatment, which does not take into account changes of tangent spaces, leads to the notion of *form-invariance* (FI) consisting in the requirement that

$$\mathbf{H}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \mathbf{H}_{\mathcal{T}},$$

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which should be corrected into the equality expressing invariance of the constitutive operator

$$\mathbf{H}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \mathbf{H}_{\mathcal{T}},$$

as considered in Proposition 9.1. In fact FI unwittingly assumes equality between maps having distinct domain and co-domains because

$$\mathbf{H}_{\mathcal{T}} \in \mathrm{C}^1(\mathrm{Tens}(\mathbb{V}\mathcal{T}); \mathrm{Tens}(\mathbb{V}\mathcal{T})),$$

while

$$\mathbf{H}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} \in \mathrm{C}^1(\mathrm{Tens}(\mathbb{V}(\mathcal{T}_{\zeta_{\mathrm{ISO}}})); \mathrm{Tens}(\mathbb{V}(\mathcal{T}_{\zeta_{\mathrm{ISO}}}))).$$

The same criticism applies to the statement of MFI in [21,18,20] which in our notations would require that

$$\mathbf{H}_{\mathcal{T}} = \zeta_{\mathrm{ISO}} \uparrow \mathbf{H}_{\mathcal{T}},$$

instead of the geometrically correct invariance property

$$\mathbf{H}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \zeta_{\mathrm{ISO}} \uparrow \mathbf{H}_{\mathcal{T}},$$

stated in Proposition 9.1. Another consequence of this improper geometric treatment is the confusion between MFI and material isotropy [18, formula 99.5].

An analogous remark applies to the axiom of MFI introduced in [2, 3.2.9, p. 194] by requiring that the elastic strain energy E must fulfill the condition $E = E \circ \mathbf{R}$, for any orthogonal tensor $\mathbf{R}$. A similar condition is reported in [10, 1.16, p. 204]. In both treatments the change of tangent spaces, due to the relative motion, is not taken into account.

A further discussion on the relations between Euclidean frame indifference (EFI), form-invariance (FI) and indifference with respect to superimposed rigid body motions (RMI) has been contributed in [9,35]. Their EFI is indeed similar to our invariance property of Proposition 9.1 but with the push incorrectly substituted by the linear isometry $\mathbf{Q} \in L(V; V)$ which does not account for the change of tangent spaces. The requirement of RMI is instead identical to MFI as introduced by Noll [21] so that the same critical observations do apply.

Despite similarity of nomenclature, the covariance paradigm enunciated in Proposition 8.1 has no relation with the *constitutive covariance*, introduced by Marsden and Hughes [2, chapter 3, p. 199–200], which was intended to be a physical restriction on constitutive relations. In this respect, it should be noted that the *constitutive covariance* requirement is in contrast with the basic rule dictated by the *covariance paradigm* because it refers to an undefined invariance property in constitutive relations and makes no mention of variance by push. This is especially apparent from the following interpretation by Simó [10, p. 203]: *This is an extended (stronger) version of classical material frame indifference. The principle of objectivity only requires proper invariance under rigid-body motions superposed onto the current configuration; that is, invariance under spatial isometries which are characterized by leaving the spatial metric tensor g unchanged. In the covariance requirement, on the other hand, isometries are replaced by arbitrary diffeomorphisms and the metric tensor g no longer remains unchanged but is transformed tensorially (by push-forward)*. Here the spatial metric tensor should be replaced by its *material* pull-back (see Definition 5.3) since the MFI principle is concerned with the relative motion between body trajectories, induced by a change of observer and not with the spatial isometry itself.

14. Some historical notes

Let us make here some considerations on the interesting historical notes provided in [18, 19.A, p. 45]. From there we quote Robert Hooke's proposal in 1678 of a spring scale of forces (*the response of a spring is unaffected by a rigid motion*), the statement by Siméon Denis Poisson in 1829 [36]: *Si l'on fait tourner le corps autour de l'axe des x, et que chacun de ses points décrive un très-petit angle ..., pour un tel déplacement, le corps demeure dans son état naturel, et les pressions intérieures doivent encore être nulles ....* and the acknowledgment of his idea by Augustin Louis Cauchy in 1829 [37]: *le premier état du corps continuera de subsister, si dans le passage du premier état au second, on a déplacé tous les points, en les faisant tourner simultanément autour de l'un des axes coordonnés.* Contrary to the opinion expressed in [18] and reproduced in [9], these are not statements of CFI but rather refer to the constitutive response of an elastic material. Indeed time-invariance of the elastic constitutive operator ensures that the stress field is invariant under isometric motions, as shown in Proposition 11.1, with invariance meaning variance by push, according to the *covariance paradigm* of Proposition 8.1.

15. Conclusions

Constitutive relations are naturally expressed in terms of material tensors and of their time derivatives evaluated at each time instant along the motion, and constitutive frame invariance (CFI) consists in the physical requirement that material behavior is *independent of* the observer. The lack of a geometrically consistent notion of time rates of material tensors and of a statement about what is intended for *independent of* lay at the origin of recurrent difficulties exposed and discussed in the literature.

A full differential geometric approach to NLCM and the ensuing *covariance paradigm* reveal that only material tensor fields and Lie derivatives along the motion are admissible in constitutive relations. It has been further shown that, by virtue of basic properties of Lie derivatives, integrability issues and conservativeness are conveniently discussed by considering an arbitrarily fixed reference placement which plays the role of an observer-dependent *computation chamber*. Numerical evaluations are also best performed stepwise in time by pull-back to a reference placement, fixed at each step, where Lie derivatives along the motion translate into standard linear time derivatives.

In changing Euclid observer, the CFI principle is fulfilled if and only if the constitutive operator is invariant, i.e. transforms by push along the relative motion between observed trajectories.

In contribution [3] of [1] Noll still says: *NLFTM is in many respects obsolete and perhaps should be updated after almost* 40 *years of its original publication. I believe that such an update should be very different from the original.* In following Noll's advise, we have chosen to contribute in the direction of a geometrical approach to NLCM.

The ideas already exposed in [13] and the ones further developed in this paper, lead to a consistent reformulation of NLCM which, relying on basic notions of differential geometry, clarifies debated issues and provides a new effective way of approaching theoretical and computation issues in NLCM.

Only testable fields pertaining to the actual placement of the material body in its dynamical trajectory play a physical role in the theory while arbitrary referential placements are relegated in a purely computation realm which, being observer-dependent, is deprived of physical interpretation.

The analysis leads to a new definition of elastic materials based on a revised notion of hypo-elasticity enunciated in terms of Lie derivatives, allowing for the detection of simple integrability conditions. A new definition of the rate visco-plastic model adopted in engineering applications is also provided and its coupling with the rate formulations of elasticity is illustrated. The notion of constitutive frame invariance (CFI) is a natural

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outcome of the geometric theory which provides the needed correction to the formulation of material frame indifference.

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