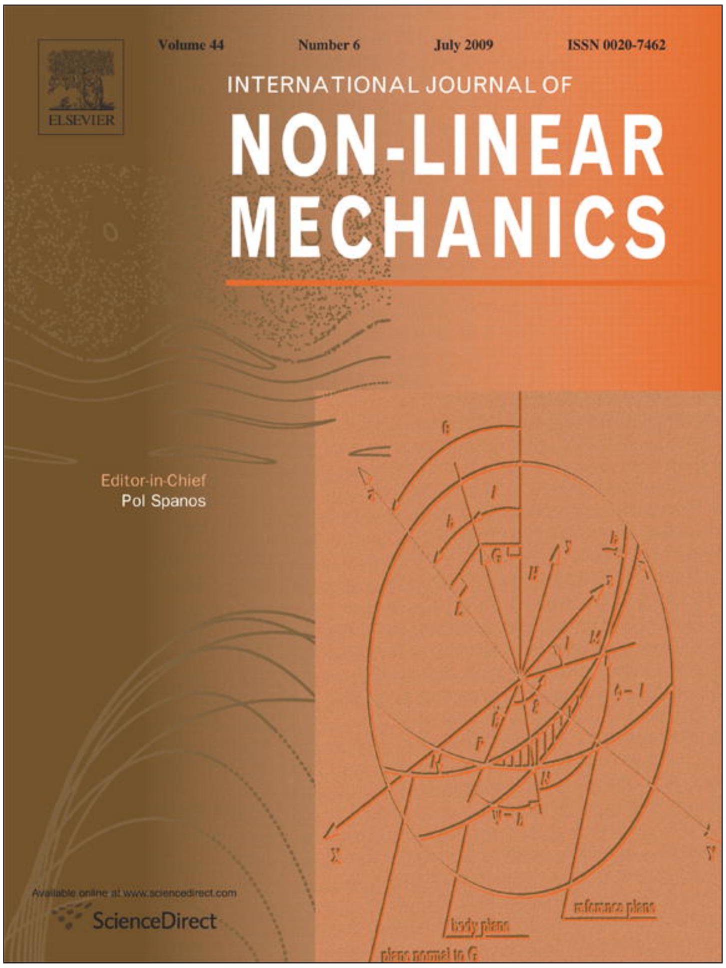
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On the general form of the law of dynamics

Giovanni Romano *, Raffaele Barretta, Marina Diaco

Department of Structural Engineering, University of Naples Federico II, via Claudio 21, 80125 Naples, Italy

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ABSTRACT

The dynamics of Lagrangian systems is formulated with a differential geometric approach and according to a new paradigm of the calculus of variations. Discontinuities in the trajectory, non-potential force systems and linear constraints are taken into account with a coordinate-free treatment. The law of dynamics, characterizing the trajectory in a general non-linear configuration manifold, is expressed in terms of a variational principle and of differential and jump conditions. By endowing the configuration manifold with a connection, the general law is shown to be tensorial in the velocity of virtual flows and to depend on the torsion of the connection. This result provides a general expression of the EULER–LAGRANGE operator. POINCARÉ and LAGRANGE forms of the law are recovered as special cases corresponding, respectively, to the connection induced by natural and mobile reference frames. For free motions, the geodesic property of the trajectory is directly inferred by adopting the LEVI-CIVITA connection induced by the kinetic energy.

1. Introduction

In recent times the interest for geometric formulations of dynamics has considerably grown up in the literature on mathematical and physical aspects of the theory (see, e.g. [1–3]). Anyway most treatments still refer to Newtonian dynamics of a finite system of point-mass particles and are expressed in terms of coordinates, a point of view which prevents to get a clear geometric picture of the theory. We contribute here a treatment of Lagrangian dynamics, in the wake of guidelines and ideas exposed in [4–6], where a new paradigm in variational calculus is illustrated with the purpose to provide a remedy to otherwise unsatisfactory statements of variational principles in dynamics and in optics. The paradigm consists in a new definition of the extremality of the geometric action integral, i.e. the integral of a one-form along a path, according to which it is required that the rate of change of the action integral, when the path is dragged by a virtual flow, must be equal to the boundary integral of the outward flux of the virtual velocity field plus the virtual power performed by the force-forms. This definition, together with a suitably refined definition of virtual velocities and the addition of the terms representing the effect of regular and impulsive forces, covers the case of piecewise regular paths and yields EULER's differential conditions at regular points and the related jump conditions at singular points. The standard format, which substantially reproduces EULER's original treatment [7] by considering the restricted class of variations with fixed end points, is not adequate from the epistemological point of view and does not directly yield the jump conditions at singular points [5,6]. Indeed, extremality of subsequent portions of a path does not imply extremality of their union, a natural requirement to be fulfilled by a well-posed definition. The assumption of fixed end points has naturally suggested to identify the extremality property of the action functional with a stationarity or minimum property [8]. The new definition of extremality for the geometric action integral yields in a natural way the general form of the law of constrained dynamics. By assuming an arbitrary linear connection in the configuration manifold, the extension of the classical EULER's differential condition is directly derived by making recourse to an intrinsic decomposition formula, due to the first author [6]. This coordinate-free formula is valid on a general manifold with a connection and performs the split of the variation of the Lagrangian functional in terms of a fiber-covariant derivative and of a base derivative. It takes the role played by the partial derivative formula in coordinates. No symmetry of the second covariant derivative of scalar functions is assumed and this fact eventually results in the appearance of the torsion of the adopted connection in the expression of the law of dynamics. The usual coordinate form of LAGRANGE law is recovered as a special case by taking the torsion-free connection induced by a coordinate system. ThePOINCARÉ form of the law is got by assuming that the connection is induced by a mobile reference frame. Finally the LEVI-CIVITA connection associated with the metric provided by the kinetic energy is considered. The specialization of the law of dynamics leads to an expression of the law of motion in terms of the WEINGARTEN map of the constraint distribution and, for free motions, in the absence of constraints, yields

* Corresponding author.
E-mail addresses: romano@unina.it (G. Romano), rabarret@unina.it (R. Barretta), diaco@unina.it (M. Diaco).

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the geodesic property of the trajectory. In the conclusions the debated issue of commutativity between the δ and $\dot{()}$ operations is discussed and clarified. This is a further hint that in the framework of geometric formulations of dynamics, principles and variational conditions may be defined and discussed in precise mathematical terms. As a consequence most long debated issues, often affected by ill-defined terms, may be answered.

2. Preliminary notions

A dot $\cdot$ denotes linear dependence on subsequent arguments, a superscript star $*$ denotes the dual quantity and the crochet $\langle , \rangle$ is the duality pairing. For an exposition of calculus on manifolds, we refer, e.g., to [2,6,9]. Let us consider a dynamical system whose configuration manifold $\mathbb{M}$ is modeled on a Banach space and let us denote by $\boldsymbol{\pi}_{\mathbb{M}} \in C^1(\mathbb{TM};\mathbb{M})$ and $\boldsymbol{\pi}^*_{\mathbb{M}} \in C^1(\mathbb{T}^*\mathbb{M};\mathbb{M})$ the dual tangent and cotangent bundles over $\mathbb{M}$. The Whitney sum of two fiber bundles $\mathbf{p}_{\mathbb{E}} \in C^1(\mathbb{E};\mathbb{M})$ and $\mathbf{p}_{\mathbb{F}} \in C^1(\mathbb{F};\mathbb{M})$ over the same base manifold $\mathbb{M}$, denoted by $\mathbb{E}\oplus\mathbb{F}$ (or by $\mathbb{E}\times_{\mathbb{M}}\mathbb{F}$), is the bundle whose fiber over $\mathbf{x} \in \mathbb{M}$ is the Cartesian product $\mathbb{E}_{\mathbf{x}} \times \mathbb{F}_{\mathbf{x}}$. The Pontryagain vector bundle over $\mathbb{M}$ is the Whitney sum of the tangent and the cotangent bundles over $\mathbb{M}$ [10].

The *tangent map* $T\boldsymbol{\varphi} \in C^1(\mathbb{TM};\mathbb{TN})$ to a morphism $\boldsymbol{\varphi} \in C^2(\mathbb{M};\mathbb{N})$ between manifolds is the vector bundle homomorphism (fiber preserving and fiber linear map) defined by the differential

$$(T\boldsymbol{\varphi} \circ \mathbf{v})(\mathbf{x}) = T_{\mathbf{x}}\boldsymbol{\varphi} \cdot \mathbf{v}(\mathbf{x}), \quad \forall \mathbf{v}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\mathbb{M}.$$

Two vector fields $\mathbf{v} \in C^1(\mathbb{M};\mathbb{TM})$ and $\mathbf{u} \in C^1(\mathbb{N};\mathbb{TN})$ are $\boldsymbol{\varphi}$-related if $T\boldsymbol{\varphi} \circ \mathbf{v} = \mathbf{u} \circ \boldsymbol{\varphi}$. Then $\mathbf{u} = \boldsymbol{\varphi} \uparrow \mathbf{v}$ is the *push forward*. Two scalar fields $f \in C^1(\mathbb{M};\Re)$ and $g \in C^1(\mathbb{N};\Re)$ are $\boldsymbol{\varphi}$-related if $f = g \circ \boldsymbol{\varphi}$ and $f = \boldsymbol{\varphi} \downarrow g$ is the *pull back*. Two covector fields $\mathbf{v}^* \in C^1(\mathbb{M};\mathbb{T}^*\mathbb{M})$ and $\mathbf{u}^* \in C^1(\mathbb{N};\mathbb{T}^*\mathbb{N})$ are $\boldsymbol{\varphi}$-related if $\langle \mathbf{v}^*, \mathbf{v}\rangle = \boldsymbol{\varphi} \downarrow \langle \mathbf{u}^*, \boldsymbol{\varphi} \uparrow \mathbf{v}\rangle$ for any $\mathbf{v} \in C^1(\mathbb{M};\mathbb{TM})$. Then the pull back is given by the formula $\boldsymbol{\varphi} \downarrow \mathbf{u}^* = T^*\boldsymbol{\varphi} \circ \mathbf{u}^* \circ \boldsymbol{\varphi}$. Push forward of vectors and pull back of arbitrary tensors by an isomorphism $\boldsymbol{\varphi} \in C^1(\mathbb{M};\mathbb{N})$ may be similarly defined and are denoted by $\boldsymbol{\varphi} \uparrow$ and $\boldsymbol{\varphi} \downarrow$, with $\boldsymbol{\varphi} \downarrow = \boldsymbol{\varphi}^{-1} \uparrow$. The usual notation for push and pull in differential geometry is $\boldsymbol{\varphi}_* = \boldsymbol{\varphi} \uparrow$ and $\boldsymbol{\varphi}^* = \boldsymbol{\varphi} \downarrow$ but then too many stars do appear in the geometrical sky (duality, Hodge star operator). The *natural derivative* $T_{\mathbf{v}}\mathbf{s} \in C^1(\mathbb{M};\mathbb{TE})$ of a section $\mathbf{s} \in C^2(\mathbb{M};\mathbb{E})$ of the fiber bundle $\mathbf{p} \in C^1(\mathbb{E};\mathbb{M})$ along a vector $\mathbf{v} \in \mathbb{TM}$ is defined by $T_{\mathbf{v}}\mathbf{s} = T\mathbf{s} \circ \mathbf{v}$ and meets the property $T\mathbf{p} \circ T_{\mathbf{v}} = \mathbf{v} \circ \mathbf{p}$. The fibers of the *vertical subbundle* $\mathbb{VE}$ of the tangent bundle $\mathbb{TE}$ are the kernels of the tangent fibration map $T\mathbf{p} \in C^1(\mathbb{TE};\mathbb{E})$. Vertical vectors $\mathbf{V} \in \mathbb{VE}$ are then characterized by a null velocity of their base point in $\mathbb{M}$. A *connection* on a fiber bundle is a projector $\mathrm{V} \in C^1(\mathbb{TE};\mathbb{TE})$ on the vertical bundle $\mathbb{VE}$, i.e. a vector bundle homomorphism $\mathbf{P}(\mathbf{e}) \in BL\,(\mathbb{T}_{\mathbf{e}}\mathbb{E};\mathbb{T}_{\mathbf{e}}\mathbb{E})$ such that $\mathbf{P}(\mathbf{e})\circ\mathbf{P}(\mathbf{e})=\mathbf{P}(\mathbf{e})$, $\mathbf{im}(\mathbf{P}(\mathbf{e}))=\mathbb{V}_{\mathbf{e}}\mathbb{E}$. The projector $\mathrm{H}=I-\mathrm{V}$ defines the *horizontal subbundle* $\mathbb{HE} \subset \mathbb{TE}$. The *horizontal lift* $\mathbf{H}_{\mathbf{v}}\mathbf{s} \in C^1(\mathbb{M};\mathbb{HE})$ and the *covariant derivative* $\nabla_{\mathbf{v}}\mathbf{s} \in C^1(\mathbb{M};\mathbb{VE})$ of a section $\mathbf{s} \in C^2(\mathbb{M};\mathbb{E})$ along a vector field $\mathbf{v} \in C^0(\mathbb{M};\mathbb{TM})$ are, respectively, the horizontal and the vertical components of the natural derivative [6,11]:

$$\mathbf{H}_{\mathbf{v}}\mathbf{s} := \mathrm{H} \circ T_{\mathbf{v}} \circ \mathbf{s}, \quad \nabla_{\mathbf{v}}\mathbf{s} := \mathrm{V} \circ T_{\mathbf{v}} \circ \mathbf{s},$$

so that $T_{\mathbf{v}}\mathbf{s} = \mathbf{H}_{\mathbf{v}}\mathbf{s} + \nabla_{\mathbf{v}}\mathbf{s}$ and $T\mathbf{p}(\mathbf{s}) \circ \mathbf{Hs} = \mathbf{id}_{\mathbb{T}_{\mathbf{p}(\mathbf{s})}\mathbb{M}}$, where $\mathbf{Hs} \circ \mathbf{v} = \mathbf{H}_{\mathbf{v}}\mathbf{s}$. The horizontal lift is tensorial in $\mathbf{s}$ and is an isomorphism between the tangent bundle $\mathbb{TM}$ and the horizontal bundle $\mathbb{HE}$. The *push* of a section $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$ along the flows generated by the pair $(\mathbf{v},\mathbf{X})$ with $\mathbf{v} \in C^0(\mathbb{M};\mathbb{TM})$ and $\mathbf{X} \in C^0(\mathbb{E};\mathbb{TE})$, is given by $\mathbf{Fl}^{(\mathbf{v},\mathbf{X})}_{\lambda} \uparrow \mathbf{s} := \mathbf{Fl}^{\mathbf{X}}_{\lambda} \circ \mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_{-\lambda} \in C^1(\mathbb{M};\mathbb{E})$ and the *parallel transport* $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$ of a section $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$ of the fiber bundle $\mathbf{p} \in C^1(\mathbb{E};\mathbb{M})$ along the flow $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in C^1(\mathbb{M};\mathbb{M})$ is defined by [6]

$$\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{s} := \mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda} \circ \mathbf{s} = (\mathbf{Fl}^{(\mathbf{v},\mathbf{H}_{\mathbf{v}})}_{\lambda} \uparrow \mathbf{s}) \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda},$$

so that $\mathbf{p} \circ \mathbf{Fl}^{\mathbf{v}}_{\lambda} \Uparrow \mathbf{s} = \mathbf{p} \circ \mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda} \circ \mathbf{s} = \mathbf{Fl}^{\mathbf{v}}_{\lambda} \circ \mathbf{p} \circ \mathbf{s} = \mathbf{Fl}^{\mathbf{v}}_{\lambda}$. We set $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \Downarrow := \mathbf{Fl}^{\mathbf{v}}_{-\lambda} \Uparrow$. The Legendre transform associated with a Lagrangian $L \in C^1(\mathbb{TM};\Re)$ is the morphism $d_{\mathrm{F}}L \in C^0(\mathbb{TM};\mathbb{T}^*\mathbb{M})$ defined by

$$d_{\mathrm{F}}L(\mathbf{v}) \cdot \mathbf{w} := \partial_{\lambda=0}L(\mathbf{v} + \lambda\mathbf{w}) = TL(\mathbf{v}) \cdot \mathbf{vl}_{\mathbb{TM}}(\mathbf{v}) \cdot \mathbf{w},$$

for all $(\mathbf{v},\mathbf{w}) \in \mathbb{TM}\oplus\mathbb{TM}$. The *vertical lift* at $\mathbf{v} \in \mathbb{TM}$ is the linear map $\mathbf{vl}_{\mathbb{TM}}(\mathbf{v}) \in C^1(\mathbb{T}_{\boldsymbol{\pi}_{\mathbb{M}}(\mathbf{v})}\mathbb{M};\mathbb{T}_{\mathbf{v}}\mathbb{TM})$ defined by $\mathbf{vl}_{\mathbb{TM}}(\mathbf{v}) \cdot \mathbf{w} := \partial_{\lambda=0}(\mathbf{v} + \lambda\mathbf{w})$. It is a fiberwise invertible homomorphism between the bundles $\mathbb{TM}$ and $\mathbb{VTM}$.

The Legendre transform induces a covariant functor *Leg* between the categories of tangent and cotangent bundles over the base manifold $\mathbb{M}$. The Lagrangian functional is regular if its fiber derivative is a diffeomorphism between the bundles $\mathbb{TM}$ and $\mathbb{T}^*\mathbb{M}$. More in general, the Fenchel–Legendre transform relates Hamiltonian $H \in C^0(\mathbb{T}^*\mathbb{M};\Re)$ and Lagrangian $L \in C^0(\mathbb{TM};\Re)$ according to the conjugacy relations [12,13]:

$$H(\mathbf{v}^*) = \sup_{\mathbf{v}\in\mathbb{T}_{\boldsymbol{\pi}^*_{\mathbb{M}}(\mathbf{v}^*)}\mathbb{M}} \{\langle \mathbf{v}^*, \mathbf{v}\rangle - L(\mathbf{v})\},$$
$$L(\mathbf{v}) = \sup_{\mathbf{v}^*\in\mathbb{T}^*_{\boldsymbol{\pi}_{\mathbb{M}}(\mathbf{v})}\mathbb{M}} \{\langle \mathbf{v}^*, \mathbf{v}\rangle - H(\mathbf{v}^*)\}.$$

The Fenchel–Legendre transform holds under the assumption that the functionals are convex and fiber-subdifferentiable. This means that the definition of the fiber derivative must be rewritten as [14]

$$d^+_{\mathrm{F}}L(\mathbf{v}) \cdot \mathbf{w} := \partial_{\lambda=0}L(\mathbf{v} + \lambda\mathbf{w}), \quad \lambda \geqslant 0,$$

and that the unilateral derivative is a sublinear (i.e. positively homogeneous and subadditive) function of the vector $\mathbf{w} \in \mathbb{TM}$. Then conjugacy is equivalent to the subdifferential rules:

$$\mathbf{v}^* \in \partial L(\mathbf{v}), \quad \mathbf{v} \in \partial H(\mathbf{v}^*), \quad (\mathbf{v},\mathbf{v}^*) \in \mathbb{TM} \oplus \mathbb{T}^*\mathbb{M},$$

where the graph of the maps ∂L and ∂H is monotone maximal and conservative [15]. Non-differentiable but fiber-subdifferentiable Lagrangians arise naturally in the analysis of problems of the calculus of variations involving extremality of a length, as in Fermat's principle in optics [5,6]. In continuum mechanics fiber-subdifferentiable Lagrangians must be introduced to simulate anelastic constitutive behaviors of the materials and the most usual kinds of boundary constraints [6].

3. Basic tools of calculus on manifolds

The first tool is the Poincaré–Stokes' formula which states that the integral of a $(k-1)$-form $\boldsymbol{\omega}^{k-1}$ on the boundary chain $\partial\Sigma$ of a kD submanifold Σ of $\mathbb{M}$ is equal to the integral of its exterior derivative $d\boldsymbol{\omega}^{k-1}$, a k-form, on Σ, i.e.

$$\int_{\Sigma} d\boldsymbol{\omega}^{k-1} = \oint_{\partial\Sigma} \boldsymbol{\omega}^{k-1}.$$

This equality can be assumed to be the very definition of the exterior derivative of a k-form. The second tool is Lie's derivative of a vector field $\mathbf{w} \in C^1(\mathbb{M};\mathbb{TM})$ along a flow $\boldsymbol{\varphi}_{\lambda} \in C^1(\mathbb{M};\mathbb{M})$ with velocity $\mathbf{v} = \partial_{\lambda=0}\boldsymbol{\varphi}_{\lambda}$:

$$\mathscr{L}_{\mathbf{v}}\mathbf{w} = \partial_{\lambda=0}(\boldsymbol{\varphi}_{\lambda} \downarrow \mathbf{w}),$$

which is equal to the antisymmetric Lie-bracket: $\mathscr{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v},\mathbf{w}] = -[\mathbf{w},\mathbf{v}]$ defined by $d_{[\mathbf{v},\mathbf{w}]}f = d_{\mathbf{v}}d_{\mathbf{w}}f - d_{\mathbf{w}}d_{\mathbf{v}}f$, for any $f \in C^2(\mathbb{M};\Re)$. The Lie derivative of a differential form $\boldsymbol{\omega}^k \in C^1(\mathbb{M};\Lambda^k(\mathbb{TM}))$ is similarly defined by $\mathscr{L}_{\mathbf{v}}\boldsymbol{\omega}^k = \partial_{\lambda=0}(\boldsymbol{\varphi}_{\lambda} \downarrow \boldsymbol{\omega}^k)$. The third tool is Reynolds' transport formula

$$\int_{\boldsymbol{\varphi}_{\lambda}(\Sigma)} \boldsymbol{\omega}^k = \int_{\Sigma} \boldsymbol{\varphi}_{\lambda} \downarrow \boldsymbol{\omega}^k \Longrightarrow \partial_{\lambda=0} \int_{\boldsymbol{\varphi}_{\lambda}(\Sigma)} \boldsymbol{\omega}^k = \int_{\Sigma} \mathscr{L}_{\mathbf{v}}\boldsymbol{\omega}^k,$$

and the fourth tool is the integral *extrusion formula* [6]

$$\partial_{\lambda=0} \int_{\varphi_\lambda(\Sigma)} \omega^k = \int_\Sigma (d\omega^k) \cdot \mathbf{v} + \int_{\partial\Sigma} \omega^k \cdot \mathbf{v},$$

and the related differential HENRI CARTAN's magic formula [2,3,16] (also called *homotopy formula* [1])

$$\mathscr{L}_{\mathbf{v}} \omega^k = (d\omega^k) \cdot \mathbf{v} + d(\omega^k \cdot \mathbf{v}),$$

where $\omega^k \cdot \mathbf{v}$ denotes the $(k-1)$-form which is the contraction performed by taking $\mathbf{v}$ as the first argument of the form ω^k. The homotopy formula may be readily inverted to get PALAIS formula for the exterior derivative. Indeed, by LEIBNIZ rule for the LIE derivative, we have that, for any two vector fields $\mathbf{v}, \mathbf{w} \in C^1(\mathbb{M}; \mathbb{TM})$

$$\begin{aligned} d\omega^1 \cdot \mathbf{v} \cdot \mathbf{w} &= (\mathscr{L}_{\mathbf{v}} \omega^1) \cdot \mathbf{w} - d(\omega^1 \cdot \mathbf{v}) \cdot \mathbf{w} \\ &= d_{\mathbf{v}}(\omega^1 \cdot \mathbf{w}) - \omega^1 \cdot [\mathbf{v}, \mathbf{w}] - d_{\mathbf{w}}(\omega^1 \cdot \mathbf{v}). \end{aligned}$$

The expression at the r.h.s. of PALAIS formula fulfills the tensoriality criterion, see, e.g. [6,16,17]. The exterior derivative of a differential one-form is thus well defined as a differential two-form, since its value at a point depends only on the values of the argument vector fields at that point. The same algebra may be repeatedly applied to deduce PALAIS formula for the exterior derivative of a k-form [18].

4. Action principle and Euler conditions

Let a status of the system be described by a point of a manifold $\mathbb{M}$, the *state space*. In both theory and applications, there are many instances in which it is compelling to consider fields which are only piecewise regular on $\mathbb{M}$. To this end, we give the following definition.

Definition 1. A *patchwork* $\mathscr{T}(\mathbb{M})$ on $\mathbb{M}$ is a finite family of disjoint open subsets of $\mathbb{M}$ such that the union of their closures is a covering of $\mathbb{M}$. The closure of each subset in the family is called an *element* of the patchwork.

The disjoint union of the boundaries of the elements, deprived of the boundary of $\mathbb{M}$, is the set of *singularity interfaces* $\mathscr{I}(\mathbb{M})$ associated with the patchwork $\mathscr{T}(\mathbb{M})$. A field is said to be *piecewise regular* on $\mathbb{M}$ if it is regular, say C^1, on each element of a patchwork on $\mathbb{M}$ which is called a *regularity patchwork*. In the family of all patchworks on $\mathbb{M}$ we may define a *partial ordering* by saying that a patchwork PAT_1 is *finer than* a patchwork PAT_2 if every element of PAT_1 is included in an element of PAT_2. Given two patchworks it is always possible to find a patchwork *finer than* both by taking as elements the non-empty pairwise intersections of their elements. This property is expressed by saying that the family of all patchworks on $\mathbb{M}$ is an inductive set. Then, let $\mathscr{T}(I)$ be a time-patchwork, i.e. a patchwork of a time interval I. The evolution of the system along a piecewise regular path $\gamma \in C^1(\mathscr{T}(I); \mathbb{M})$ is assumed to be governed by a variational condition on its signed-length, evaluated according to the piecewise regular differential *action one-form* $\omega^1 \in \Lambda^1(\mathscr{T}(\mathbb{M}); \mathbb{T}^*\mathbb{M})$, with $\mathscr{T}(\mathbb{M})$ a regularity patchwork. We assume, without loss in generality, that the trajectory $\gamma \in C^1(\mathscr{T}(I); \mathbb{M})$ is regular in each element of the time-patchwork $\mathscr{T}(I)$. Let us denote by $\boldsymbol{\Gamma} := \gamma(I)$ the geometric trajectory and by $\mathbb{T}_{\boldsymbol{\Gamma}}\mathbb{M}$the vector bundle which is the restriction of the tangent bundle $\mathbb{TM}$ to $\boldsymbol{\Gamma}$.

Definition 2. The *action integral* associated with a geometrical path $\boldsymbol{\Gamma}$ in the state-space $\mathbb{M}$ is the signed-length of the 1D oriented submanifold $\boldsymbol{\Gamma}$, evaluated according to the *action one-form* ω^1 on $\mathbb{M}$: $\int_{\boldsymbol{\Gamma}} \omega^1$.

A general statement of the action principle requires a suitable definition of the *virtual flows* along which the trajectory is assumed to be varied.

Definition 3. The *virtual flows* of $\boldsymbol{\Gamma}$ are flows $\varphi_\lambda \in C^1(\mathbb{M}; \mathbb{M})$ whose velocities $\mathbf{v}_\varphi \in C^1(\boldsymbol{\Gamma}; \mathbb{T}_{\boldsymbol{\Gamma}}\mathbb{M})$ are tangent to interelement boundaries of the regularity patchwork $\mathscr{T}(\mathbb{M})$. Velocities of virtual flows are *virtual velocities*.

In formulating an action principle, the velocities at $\boldsymbol{\Gamma}$ of the test flows are assumed to belong to a vector subbundle $\mathrm{TEST}_{\boldsymbol{\Gamma}}$ of the vector bundle $\mathbb{T}_{\boldsymbol{\Gamma}}\mathbb{M}$. Force systems are represented by a differential two-form α^2 on $\mathbb{T}_{\boldsymbol{\Gamma}}\mathbb{M}$, the *regular force-form*, which provides an abstract description of a possibly non-potential system of forces acting along the trajectory. The force-form α^2 is *potential* if it is defined on a neighbourhood $U(\boldsymbol{\Gamma}) \subset \mathbb{M}$ of the path and there is exact. This amounts toassume that there exists a differential one-form $\beta^1 \in C^1(U(\boldsymbol{\Gamma}); \mathbb{T}^*\mathbb{M})$ such that $\alpha^2 = d\beta^1$, where d is the exterior differentiation. We consider also a differential one-form α^1 on $\mathbb{T}_{\mathscr{I}(\boldsymbol{\Gamma})}\mathbb{M}$, the *impulsive-force-form*, which provides an abstract description of an impulsive system of forces acting at singular points on the trajectory.

Definition 4 (*Geometric action principle*). A trajectory of the system governed by a piecewise regular differential one-form ω^1 on $\mathbb{M}$ is a piecewise regular path $\boldsymbol{\Gamma} \in C^1(\mathscr{T}(I); \mathbb{M})$ such that

$$\partial_{\lambda=0} \int_{\varphi_\lambda \circ \boldsymbol{\Gamma}} \omega^1 = \oint_{\partial\boldsymbol{\Gamma}} \omega^1 \cdot \mathbf{v}_\varphi + \int_{\boldsymbol{\Gamma}} \alpha^2 \cdot \mathbf{v}_\varphi + \int_{\mathscr{I}(\boldsymbol{\Gamma})} \alpha^1 \cdot \mathbf{v}_\varphi,$$

for all virtual flows $\varphi_\lambda \in C^1(\mathbb{M}; \mathbb{M})$ with virtual velocities $\mathbf{v}_\varphi = \partial_{\lambda=0}\varphi_\lambda$ taking values in a test subbundle $\mathrm{TEST}_{\boldsymbol{\Gamma}} \subset \mathbb{T}_{\boldsymbol{\Gamma}}\mathbb{M}$.

This means that the initial rate of increase of the ω^1-length of the trajectory $\boldsymbol{\Gamma}$ along a virtual flow is equal to the outward flux of virtual velocities at end points plus the virtual power performed by the force-forms. Denoting by $\mathbf{x}_1$ and $\mathbf{x}_2$ the initial and final end points of $\boldsymbol{\Gamma}$, it is $\partial\boldsymbol{\Gamma} = \mathbf{x}_2 - \mathbf{x}_1$ (a 0-chain) and the boundary integral may be written as

$$\oint_{\partial\boldsymbol{\Gamma}} \omega^1 \cdot \mathbf{v}_\varphi = (\omega^1 \cdot \mathbf{v}_\varphi)(\mathbf{x}_2) - (\omega^1 \cdot \mathbf{v}_\varphi)(\mathbf{x}_1).$$

The action principle is purely geometrical since it characterizes the trajectory $\boldsymbol{\Gamma}$ to within an arbitrary reparametrization. A necessary and sufficient differential condition for a path to be a trajectory is provided by the next theorem and will be called EULER's condition. The classical result of EULER deals with regular paths and fixed end points and is formulated in coordinates. The new statement introduced in [5,6] deals with the more general case of non-fixed end points and piecewise regular paths, and extremality is expressed in terms of coordinate-free differential and jump conditions.

Theorem 4.1 (*Euler's conditions*). *A path $\boldsymbol{\Gamma} \subset \mathbb{M}$ is a trajectory if and only if the tangent vector field $\mathbf{v}_{\boldsymbol{\Gamma}} \in C^1(\mathscr{T}(\boldsymbol{\Gamma}); \mathbb{T}\boldsymbol{\Gamma})$ meets, in each element of a regularity partition $\mathscr{T}(\boldsymbol{\Gamma})$, the differential condition*

$$(d\omega^1 - \alpha^2) \cdot \mathbf{v}_{\boldsymbol{\Gamma}} \cdot \mathbf{v}_\varphi = 0, \quad \forall \mathbf{v}_\varphi \in \mathrm{TEST}_{\boldsymbol{\Gamma}},$$

and, at the singularity interfaces $\mathscr{I}(\boldsymbol{\Gamma})$, the jump conditions

$$[[\omega^1 \cdot \mathbf{v}_\varphi]] = \alpha^1 \cdot \mathbf{v}_\varphi, \quad \forall \mathbf{v}_\varphi \in \mathrm{TEST}_{\boldsymbol{\Gamma}}.$$

EULER's conditions show that the geometry of the trajectory is uniquely determinate if the exact two-form $d\omega^1$ has a 1D kernel at each point. This is the basic assumption to ensure local existence and uniqueness of the trajectory through a point of the state-space.

5. The law of dynamics

In continuum dynamics, the configurations of the body are depicted as points of a differentiable manifold $\mathbb{C}$ modeled on a

Banach space. The associated tangent bundle is denoted by $\boldsymbol{\pi}_{\mathbb{C}} \in \mathrm{C}^1(\mathbb{TC};\mathbb{C})$. The geometric action principle of dynamics is formulated by considering as state-space the product tangent bundle $\mathbb{TC}\times\mathbb{T}I$ or the product cotangent bundle $\mathbb{T}^*\mathbb{C}\times\mathbb{T}^*I$ to the configuration-time product manifold $\mathbb{C}\times I$. We will denote by $\mathrm{pr}_{\mathbb{TC}}$ and $\mathrm{pr}_{\mathbb{T}I}$ the Cartesian projectors associated with $\mathbb{TC}\times\mathbb{T}I$ and similarly for the product cotangent bundle. The canonical or Liouville one-form on the cotangent bundle over the configuration manifold $\boldsymbol{\theta}\in C^1(\mathbb{T}^*\mathbb{C};\mathbb{T}^*\mathbb{T}^*\mathbb{C})$ is defined by $\langle\boldsymbol{\theta}(\mathbf{v}^*),\mathbf{Y}(\mathbf{v}^*)\rangle=\langle\mathbf{v}^*,T_{\mathbf{v}^*}\boldsymbol{\pi}^*_{\mathbb{C}}\bullet\mathbf{Y}(\mathbf{v}^*)\rangle$, for any $\mathbf{Y}(\mathbf{v}^*)\in\mathbb{T}_{\mathbf{v}^*}\mathbb{T}^*\mathbb{C}$, with the basic property that its exterior derivative is a two-form with a trivial kernel. The counterpart in the tangent bundle is the Poincaré–Cartan one-form $\boldsymbol{\theta}_L := d_{\mathrm{F}}L\downarrow\boldsymbol{\theta}$. In the Hamiltonian description, the action one-form is $\boldsymbol{\omega}^1 := \mathrm{pr}_{\mathbb{T}^*\mathbb{C}}\downarrow\boldsymbol{\theta}-\boldsymbol{\eta}\in\mathbb{T}^*\mathbb{T}^*\mathbb{C}\times\mathbb{T}^*\mathbb{T}^*I$ with $\boldsymbol{\eta}(\mathbf{v}^*,t) := H(\mathbf{v}^*,t)\mathrm{pr}_{\mathbb{T}^*I}\downarrow dt$.The Hamiltonian $H_t\in\mathrm{C}^1(\mathbb{T}^*\mathbb{C};\mathfrak{R})$ is Fenchel–Legendre conjugate to the Lagrangian $L_t\in\mathrm{C}^1(\mathbb{TC};\mathfrak{R})$. In the Lagrangian description, the action one-form is $\boldsymbol{\omega}^1_L := \mathrm{pr}_{\mathbb{TC}}\downarrow\boldsymbol{\theta}_L-\boldsymbol{\eta}_L\in\mathbb{T}^*\mathbb{TC}\times\mathbb{T}^*\mathbb{T}I$, where $\boldsymbol{\eta}_L(\mathbf{v},t) := E(\mathbf{v},t)\mathrm{pr}_{\mathbb{T}I}\downarrow dt$ with $E_t(\mathbf{v})=H_t(d_{\mathrm{F}}L(\mathbf{v}))$ the energy. Let us then consider a compact time interval I, a piecewise regular time-parametrized path $\gamma\in\mathrm{C}^1(I;\mathbb{C})$ in the configuration manifold and its image $\boldsymbol{\Gamma}=\gamma(I)$. The speed along the path is the vector field $\mathbf{v}_\gamma\in\mathrm{C}^1(\boldsymbol{\Gamma};\mathbb{T}\boldsymbol{\Gamma})$ defined by $\mathbf{v}_\gamma(\gamma(t)) := \partial_{\tau=t}\gamma(\tau)$. Conforming virtual speeds of the body are assumed to belong to a vector subbundle $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$ of the tangent bundle $\mathbb{T}_{\boldsymbol{\Gamma}}\mathbb{C}$ to the trajectory. The trajectory in the configuration-time state-space is then given by $(\gamma,\mathbf{id}_I)\in\mathrm{C}^1(I;\mathbb{C}\times I)$, with image $\boldsymbol{\Gamma}_I=(\gamma,\mathbf{id}_I)(I)$, and the lifted trajectory in the velocity–time state-space is$(T\gamma,\mathbf{id}_{\mathbb{T}I})\in\mathrm{C}^1(\mathbb{T}I;\mathbb{TC}\times\mathbb{T}I)$. A virtual flow $\boldsymbol{\varphi}_\lambda\in\mathrm{C}^1(\mathbb{C};\mathbb{C})$ in the configuration manifold induces a synchronous flow $\boldsymbol{\varphi}_\lambda\times\mathbf{id}_I\in\mathrm{C}^1(\mathbb{C}\times I;\mathbb{C}\times I)$ in the configuration-time state-space and a tangent synchronous flow with velocity $(\mathbf{v}_{T\varphi},0)\in\mathbb{TTC}\times\mathbb{TT}I$. In the Hamiltonian formulation, non-potential forces acting on the mechanical system are taken into account by introducing a *force two-form* given by

$$\boldsymbol{\alpha}^2_{\mathbf{f}}(\mathbf{v}^*,t) := -(\mathbf{f}\wedge dt)(\mathbf{v}^*,t),\quad \boldsymbol{\pi}^*_{\mathbb{C}}(\mathbf{v}^*)\in\gamma(t).$$

Given a force one-form $\mathbf{F}_t\in\mathrm{C}^1(\mathbb{C};\mathbb{T}^*\mathbb{C})$ on the configuration manifold, the induced force one form

$$\mathbf{F}_t\in\mathrm{C}^1(\mathbb{T}^*\mathbb{C};\mathbb{T}^*\mathbb{T}^*\mathbb{C})$$

on the cotangent bundle is provided by the formula

$$\mathbf{f}_t(\mathbf{v}^*)\cdot\mathbf{Y}(\mathbf{v}^*) := \langle\mathbf{F}_t(\boldsymbol{\pi}^*_{\mathbb{C}}(\mathbf{v}^*)),T\boldsymbol{\pi}^*_{\mathbb{C}}\cdot\mathbf{Y}(\mathbf{v}^*)\rangle,\quad \forall\mathbf{Y}(\mathbf{v}^*)\in\mathbb{T}_{\mathbf{v}^*}\mathbb{T}^*\mathbb{C}.$$

Our definition differs from the one given in [3,10] where forces fields are considered as fiber preserving maps $\mathbf{F}\in\mathrm{C}^1(\mathbb{TC};\mathbb{T}^*\mathbb{C})$. Indeed, given the configuration manifold $\mathbb{C}$ of a mechanical system, forces are elements of the cotangent manifold $\mathbb{T}^*\mathbb{C}$ and force fields are sections of the cotangent bundle $\boldsymbol{\pi}^*_{\mathbb{C}}\in\mathrm{C}^1(\mathbb{T}^*\mathbb{C};\mathbb{C})$, that is, to any placement $\mathbf{x}\in\mathbb{C}$ they assign a force-covector acting on that placement. The so-called velocity dependent forces acting on a body do in fact depend on relative velocity fields between the body and its surroundings. Dependence of forces on parameters, such as relative velocity fields, friction coefficients, electric charges, and electromagnetic fields, is to be modeled as a constitutive property, for instance a multivalued monotone relation between dual fields of force and velocities, depending on physical parameters. There is no place for velocities in the very definition of force. Impulsive forces at singular points are one-forms $\boldsymbol{\alpha}^1\in\mathbb{T}^*\mathbb{T}^*\mathbb{C}$ defined by

$$\boldsymbol{\alpha}^1\cdot\mathbf{Y}\langle\mathbf{A}_t,T\boldsymbol{\pi}^*_{\mathbb{C}}\cdot\mathbf{Y}\rangle\in\mathrm{C}^1(\mathbb{T}^*\mathbb{C};\mathfrak{R}),$$

where $\mathbf{A}_t(\mathbf{x})\in\mathbb{T}^*_{\mathbf{x}}\mathbb{C}$. Non-potential forces in Lagrangian formulation are similarly introduced or may be deduced with a pull back by the Legendre transform. In the tangent bundle $\mathbb{TC}$, the subbundle of infinitesimal isometries is denoted by rig. These are the tangent vector fields $\mathbf{v}\in\mathrm{C}^1(\mathbb{C};\mathbb{TC})$ fulfilling the condition $\mathscr{L}_{\mathbf{v}}\mathbf{g}=0$, where $\mathbf{g}$ is the euclidean metric. Note that the property of the Lie derivative [2,6]

$$\mathscr{L}_{[\mathbf{u},\mathbf{v}]}\mathbf{g}=[\mathscr{L}_{\mathbf{u}},\mathscr{L}_{\mathbf{v}}]\mathbf{g}$$

ensures that the bundle rig is involutive and hence by Frobenius theorem, integrable, see, e.g. [6,19]. The geometric action principle is stated as follows.

Proposition 5.1 (*Geometric Hamilton's principle*). *The lifted trajectory in the velocity–time state-space, fulfills the action principle:*

$$\partial_{\lambda=0}\int_{(T\boldsymbol{\varphi}_\lambda\times T\mathbf{id}_I)(\boldsymbol{\Gamma}_I)}\boldsymbol{\omega}^1_L=\oint_{\partial\boldsymbol{\Gamma}_I}\boldsymbol{\omega}^1_L\cdot(\mathbf{v}_{T\varphi},0)+\int_{\boldsymbol{\Gamma}_I}\boldsymbol{\alpha}^2\cdot(\mathbf{v}_{T\varphi},0)+\int_{\mathscr{I}(\boldsymbol{\Gamma}_I)}\boldsymbol{\alpha}^1\cdot(\mathbf{v}_{T\varphi},0)$$

for any flow $\boldsymbol{\varphi}_\lambda\in\mathrm{C}^1(\mathbb{C};\mathbb{C})$ whose velocity $\mathbf{v}_{\boldsymbol{\varphi}}=\partial_{\lambda=0}\boldsymbol{\varphi}_\lambda\in\mathrm{C}^1(\boldsymbol{\Gamma};\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}\cap$ rig$)$ is a conforming infinitesimal isometry at $\boldsymbol{\Gamma}$.

In the action principle of Proposition 5.1 the variations of the lifted trajectory in the *velocity–time* state-space are performed by lifted virtual flows which are the differentials of flows in the configuration manifold and no flows along the time axis are considered (synchronous variations). It can be shown that the action principle so formulated is equivalent to the one in which a larger class of flows are considered by allowing fiber-respecting, fiber-linear flows in the *velocity* phase-space and time-flows. A thorough discussion on this topic is performed in [5,6]. On the paths drifted by the flow, the Lagrangian functional is computed by evaluating the velocity of the synchronously varied trajectory which is equal to the push of the velocity of the trajectory. Indeed, by the chain rule we have

$$\partial_{\tau=t}(\boldsymbol{\varphi}_\lambda\circ\gamma)(\tau)=(T\boldsymbol{\varphi}_\lambda\circ\mathbf{v}_\gamma)(\gamma(t))=(\boldsymbol{\varphi}_\lambda\uparrow\mathbf{v}_\gamma\circ\boldsymbol{\varphi}_\lambda)(\gamma(t)).$$

It is convenient to perform the extension of the trajectory speed $\mathbf{v}_\gamma\in\mathrm{C}^1(\boldsymbol{\Gamma};\mathbb{T}\boldsymbol{\Gamma})$ to a vector field

$$\mathscr{F}_{\boldsymbol{\varphi}}(\mathbf{v}_\gamma)\in\mathrm{C}^1(\Sigma_{\boldsymbol{\varphi}}(\boldsymbol{\Gamma});\mathbb{T}_{\Sigma_{\boldsymbol{\varphi}}(\boldsymbol{\Gamma})}\mathbb{C}),$$

where $\Sigma_{\boldsymbol{\varphi}}(\boldsymbol{\Gamma}) := \cup_{|\lambda|\leqslant\varepsilon}\boldsymbol{\varphi}_\lambda(\boldsymbol{\Gamma})$ is the ε-sheet through $\boldsymbol{\Gamma}$ generated by the flow $\boldsymbol{\varphi}_\lambda$ and $\varepsilon>0$. For each λ with $|\lambda|\leqslant\varepsilon$, the extension is defined by the push

$$\mathscr{F}_{\boldsymbol{\varphi}}(\mathbf{v}_\gamma)\circ\boldsymbol{\varphi}_\lambda := \boldsymbol{\varphi}_\lambda\uparrow\mathbf{v}_\gamma\circ\boldsymbol{\varphi}_\lambda=T\boldsymbol{\varphi}_\lambda\circ\mathbf{v}_\gamma.$$

Hence the Lie bracket $[\mathscr{F}_{\boldsymbol{\varphi}}(\mathbf{v}_\gamma),\mathbf{v}_{\boldsymbol{\varphi}}]$ vanishes [6].

The geometric action principle of Proposition 5.1 can be written in a non-geometric form, i.e. in a form depending on the time-parametrization. In the next proposition we show that the new paradigm of variational calculus yields directly the differential condition equivalent to the extremality principle, without requiring neither partial differentiation nor integration by parts, which are not available unless a connection is defined on the configuration manifold.

Proposition 5.2 (*Action principle and general law of dynamics*). *The trajectory of a dynamical system in the configuration manifold is a piecewise regular path $\gamma\in\mathrm{C}^1(\mathscr{T}(I);\mathbb{C})$ fulfilling the extremality principle*:

$$\partial_{\lambda=0}\int_I L_t\circ T\boldsymbol{\varphi}_\lambda\circ\mathbf{v}_\gamma\circ\gamma dt=\oint_{\partial I}\langle d_{\mathrm{F}}L_t\circ\mathbf{v}_\gamma,\mathbf{v}_{\boldsymbol{\varphi}}\rangle\circ\gamma-\int_I\langle\mathbf{F}_t,\mathbf{v}_{\boldsymbol{\varphi}}\rangle\circ\gamma\,dt-\int_{\mathscr{I}(I)}\langle\mathbf{A}_t,\mathbf{v}_{\boldsymbol{\varphi}}\rangle\circ\gamma.$$

This non-geometric form of the action principle is equivalent to the differential condition

$$d_{\mathbf{v}_\gamma}\langle d_{\mathrm{F}}L_t\circ\mathbf{v}_\gamma,\mathbf{v}_{\boldsymbol{\varphi}}\rangle-\langle T(L_t\circ\mathscr{F}_{\boldsymbol{\varphi}}(\mathbf{v}_\gamma)),\mathbf{v}_{\boldsymbol{\varphi}}\rangle=\langle\mathbf{F}_t,\mathbf{v}_{\boldsymbol{\varphi}}\rangle-\langle\partial_{\tau=t}d_{\mathrm{F}}L_\tau\circ\mathbf{v}_\gamma,\mathbf{v}_{\boldsymbol{\varphi}}\rangle,$$

and the jump conditions $\langle [[d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma]], \mathbf{v}_\varphi \rangle = \langle \mathbf{A}_t, \mathbf{v}_\varphi \rangle$, *for any virtual flow* $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C};\mathbb{C})$ *whose virtual velocity* $\mathbf{v}_\varphi = \partial_{\lambda=0} \boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\boldsymbol{\Gamma}; \boldsymbol{\Delta}_{\boldsymbol{\Gamma}} \cap \mathrm{RIG})$ *is a conforming infinitesimal isometry at* $\boldsymbol{\Gamma}$.

Proof. For any regular element $\mathscr{P}$ of a patchwork of the time-interval I finer than the regularity patchwork $\mathscr{T}(I)$ we have that

$$\begin{aligned}
\partial_{\lambda=0} \int_{\mathscr{P}} L_t \circ T\boldsymbol{\varphi}_\lambda \circ \mathbf{v}_\gamma \circ \gamma \, dt &= \int_{\mathscr{P}} \partial_{\lambda=0} (L_t \circ T\boldsymbol{\varphi}_\lambda \circ \mathbf{v}_\gamma \circ \gamma) \, dt \\
&= \int_{\mathscr{P}} \partial_{\lambda=0} (L_t \circ \mathscr{F}_\varphi(\mathbf{v}_\gamma) \circ \boldsymbol{\varphi}_\lambda \circ \gamma) \, dt \\
&= \int_{\mathscr{P}} \langle T(L_t \circ \mathscr{F}_\varphi(\mathbf{v}_\gamma)), \mathbf{v}_\varphi \rangle \circ \gamma \, dt, \\
\oint_{\partial \mathscr{P}} \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle \circ \gamma &= \int_{\mathscr{P}} (\langle \partial_{\tau=t} d_{\mathrm{F}} L_\tau \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle \\
&\quad + d_{\mathbf{v}_\gamma} \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle) \circ \gamma \, dt.
\end{aligned}$$

The equivalence then follows by evaluating the variational condition in each element of the patchwork and summing up. $\square$

We remark that the differential law of dynamics is independent of the values of the virtual velocity field outside the trajectory. In fact the r.h.s. is tensorial in the virtual velocity field and the l.h.s is tensorial too, even if the two addends are not such. An alternative tensorial expression of the terms at the l.h.s will be provided in Proposition 5.3 by introducing a connection in the configuration manifold. Moreover, recalling that $\mathbf{v}_{T\varphi} = \partial_{\lambda=0} T\boldsymbol{\varphi}_\lambda = \mathbf{k} \circ \mathbf{T}\mathbf{v}_\varphi = \mathbf{k} \circ \mathbf{T} \partial_{\lambda=0} \boldsymbol{\varphi}_\lambda$, with $\mathbf{k} \in \mathrm{C}^1(\mathbb{TTC}; \mathbb{TTC})$ the canonical flip, we have that

$$\partial_{\lambda=0} (L_t \circ T\boldsymbol{\varphi}_\lambda \circ \mathbf{v}_\gamma \circ \gamma) = \langle TL_t \circ \mathbf{v}_\gamma, \mathbf{v}_{T\varphi} \circ \mathbf{v}_\gamma \rangle = \langle TL_t \circ \mathbf{v}_\gamma, \mathbf{k} \circ \mathbf{T}\mathbf{v}_\varphi \circ \mathbf{v}_\gamma \rangle,$$

and the differential condition in Proposition 5.2 may be rewritten as

$$d_{\mathbf{v}_\gamma} \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle - \langle TL_t \circ \mathbf{v}_\gamma, \mathbf{v}_{T\varphi} \circ \mathbf{v}_\gamma \rangle = \langle \mathbf{F}_t, \mathbf{v}_\varphi \rangle - \langle \partial_{\tau=t} d_{\mathrm{F}} L_\tau \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle.$$

The deduction of the law of dynamics from the action principle, in the non-geometric form of Proposition 5.2, is based on the following intrinsic result [6]. A somewhat ambiguous special expression in coordinates is reported in [20,21].

Lemma 5.1 (*A split formula*). *Let* $\mathbb{N}$ *be a manifold,* $\mathbf{p} \in \mathrm{C}^1(\mathbb{E};\mathbb{M})$ *a fiber bundle with a connection* ∇ *and* $\mathbf{f} \in \mathrm{C}^1(\mathbb{E};\mathbb{N})$ *a morphism. Then, for any section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ *of the fiber bundle, the map tangent to the composition* $\mathbf{f} \circ \mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{N})$ *may be uniquely split as the sum of the fiber-covariant derivative and the base derivative*:

$$T(\mathbf{f} \circ \mathbf{s}) = T\mathbf{f} \circ T\mathbf{s} = d_{\mathrm{F}} \mathbf{f}(\mathbf{s}) \cdot \nabla \mathbf{s} + d_{\mathrm{B}} \mathbf{f}(\mathbf{s}).$$

Proof. Denoting by $\mathbf{Fl}^{\mathbf{v}}_\lambda \Uparrow = \mathbf{Fl}^{\mathbf{H}_\mathbf{v}}_\lambda \in \mathrm{C}^1(\mathbb{E};\mathbb{E})$ the parallel transport along the flow associated with a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M};\mathbb{TM})$, by the definitions and the chain rule we get

$$\begin{aligned}
d_{\mathrm{F}} \mathbf{f}(\mathbf{s}(\mathbf{x})) \cdot \nabla_{\mathbf{v}(\mathbf{x})} \mathbf{s} &= (T\mathbf{f} \circ \nabla_{\mathbf{v}(\mathbf{x})} \mathbf{s})(\mathbf{x}) \\
&= (T\mathbf{f} \circ \mathrm{V} \circ T\mathbf{s} \circ \mathbf{v})(\mathbf{x}) \\
&= T_{\mathbf{s}(\mathbf{x})} \mathbf{f} \cdot \partial_{\lambda=0} (\mathbf{Fl}^{\mathbf{v}}_\lambda \Downarrow \mathbf{s})(\mathbf{Fl}^{\mathbf{v}}_\lambda(\mathbf{x})) \\
&= \partial_{\lambda=0} (\mathbf{f} \circ \mathbf{Fl}^{\mathbf{v}}_\lambda \Downarrow \mathbf{s} \circ \mathbf{Fl}^{\mathbf{v}}_\lambda)(\mathbf{x}), \\
d_{\mathrm{B}} \mathbf{f}(\mathbf{s}(\mathbf{x})) \cdot \mathbf{v}(\mathbf{x}) &= (T\mathbf{f} \circ \mathbf{H}_{\mathbf{v}(\mathbf{x})} \mathbf{s})(\mathbf{x}) \\
&= (T\mathbf{f} \circ \mathrm{H} \circ T\mathbf{s} \circ \mathbf{v})(\mathbf{x}) \\
&= (T\mathbf{f} \circ \partial_{\lambda=0} \mathbf{Fl}^{\mathbf{v}}_\lambda \Uparrow \mathbf{s})(\mathbf{x}) \\
&= \partial_{\lambda=0} (\mathbf{f} \circ \mathbf{Fl}^{\mathbf{v}}_\lambda \Uparrow \mathbf{s})(\mathbf{x}),
\end{aligned}$$

so that $T(\mathbf{f} \circ \mathbf{s}) \cdot \mathbf{v}(\mathbf{x}) = d_{\mathrm{F}} \mathbf{f}(\mathbf{s}(\mathbf{x})) \cdot \nabla_{\mathbf{v}(\mathbf{x})} \mathbf{s} + d_{\mathrm{B}} \mathbf{f}(\mathbf{s}(\mathbf{x})) \cdot \mathbf{v}(\mathbf{x})$. $\square$

Proposition 5.3 (*General law of dynamics in terms of a connection*). *The trajectory of a dynamical system in the configuration manifold is a piecewise regular path* $\gamma \in \mathrm{C}^1(\mathscr{T}(I);\mathbb{C})$ *which, for any given connection* ∇ *on* $\mathbb{C}$, *fulfills the differential condition*

$$\begin{aligned}
&\langle \nabla_{\mathbf{v}_\gamma} (d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma), \mathbf{v}_\varphi \rangle - \langle d_{\mathrm{B}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle - \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \mathrm{TORS}(\mathbf{v}_\gamma, \mathbf{v}_\varphi) \rangle \\
&\quad = \langle \mathbf{F}_t, \mathbf{v}_\varphi \rangle - \langle \partial_{\tau=t} d_{\mathrm{F}} L_\tau \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle,
\end{aligned}$$

for any virtual flow $\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{C};\mathbb{C})$ *with* $\mathbf{v}_\varphi = \partial_{\lambda=0} \boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\boldsymbol{\Gamma}; \boldsymbol{\Delta}_{\boldsymbol{\Gamma}} \cap \mathrm{RIG})$ *a conforming infinitesimal isometry at* $\boldsymbol{\Gamma}$.

Proof. The split formula in Lemma 5.1 yields

$$\langle T(L_t \circ \mathscr{F}_\varphi(\mathbf{v}_\gamma)), \mathbf{v}_\varphi \rangle = \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \nabla_{\mathbf{v}_\varphi} \mathscr{F}_\varphi(\mathbf{v}_\gamma) \rangle + \langle d_{\mathrm{B}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle.$$

Moreover, by Leibniz rule we have that

$$d_{\mathbf{v}_\gamma} \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle = \langle \nabla_{\mathbf{v}_\gamma} (d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma), \mathbf{v}_\varphi \rangle + \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \nabla_{\mathbf{v}_\gamma} \mathbf{v}_\varphi \rangle.$$

Then, setting $\mathbf{F}^{\mathrm{GEN}} := \mathbf{F}_t - \partial_{\tau=t} (d_{\mathrm{F}} L_\tau \circ \mathbf{v}_\gamma)$, the differential condition in Proposition 5.2 may be written as

$$\begin{aligned}
\langle \nabla_{\mathbf{v}_\gamma} (d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma), \mathbf{v}_\varphi \rangle &= \langle d_{\mathrm{B}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle + \langle \mathbf{F}^{\mathrm{GEN}}, \mathbf{v}_\varphi \rangle \\
&\quad + \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \nabla_{\mathbf{v}_\gamma} \mathbf{v}_\varphi - \nabla_{\mathbf{v}_\varphi} \mathscr{F}_\varphi(\mathbf{v}_\gamma) \rangle.
\end{aligned}$$

Since the bracket $[\mathscr{F}_\varphi(\mathbf{v}_\gamma), \mathbf{v}_\varphi]$ vanishes, we have that

$$\begin{aligned}
\mathrm{TORS}(\mathbf{v}_\gamma, \mathbf{v}_\varphi) &:= \nabla_{\mathbf{v}_\gamma} \mathbf{v}_\varphi - \nabla_{\mathbf{v}_\varphi} \mathscr{F}_\varphi(\mathbf{v}_\gamma) - [\mathscr{F}_\varphi(\mathbf{v}_\gamma), \mathbf{v}_\varphi] \\
&= \nabla_{\mathbf{v}_\gamma} \mathbf{v}_\varphi - \nabla_{\mathbf{v}_\varphi} \mathscr{F}_\varphi(\mathbf{v}_\gamma),
\end{aligned}$$

and the differential law takes the tensorial form in the statement. $\square$

Remark 5.1. Given a trajectory $\gamma \in \mathrm{C}^1(I;\mathbb{C})$, the Euler–Lagrange map associates the differential one-form $\mathscr{E}_{L_t} \circ \mathbf{v}_\gamma \in \mathbb{T}^*_{\boldsymbol{\Gamma}} \mathbb{C}$ to the field $\mathbf{v}_\gamma \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{T}\boldsymbol{\Gamma})$:

$$\begin{aligned}
\langle \mathscr{E}_{L_t} \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle &:= d_{\mathbf{v}_\gamma} \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle - \langle TL_t \circ \mathbf{v}_\gamma, \mathbf{v}_{T\varphi} \circ \mathbf{v}_\gamma \rangle \\
&= \langle \nabla_{\mathbf{v}_\gamma} (d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma), \mathbf{v}_\varphi \rangle - \langle d_{\mathrm{B}} L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle \\
&\quad - \langle d_{\mathrm{F}} L_t \circ \mathbf{v}_\gamma, \mathrm{TORS}(\mathbf{v}_\gamma, \mathbf{v}_\varphi) \rangle, \quad \forall \mathbf{v}_\varphi \in \mathrm{C}^1(\boldsymbol{\Gamma}; \mathbb{T}_{\boldsymbol{\Gamma}} \mathbb{C}).
\end{aligned}$$

The differential law of dynamics is then written $\langle \mathscr{E}_{L_t} \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle = \langle \mathbf{F}^{\mathrm{GEN}}, \mathbf{v}_\varphi \rangle$. In [22] the negative of the former expression is taken to be the Euler–Lagrange operator, but tensoriality in $\mathbf{v}_\varphi$ is not proved.

There are some special, but important, contexts in dynamics where the variational law of motion may be written as an equation. This situation occurs when the test virtual velocities are exactly the vector fields tangent to the configuration manifold at the trajectory. We shall refer to these contexts as *perfect dynamical systems*. Two main instances of perfect dynamical systems are rigid body dynamics and elastodynamics. In the former, the configuration manifold is a leaf of the foliation induced by the (integrable) rigidity constraint and the sections of the tangent bundle to such a leaf are exactly the infinitesimal isometries. In the latter, the bundle of test fields is enlarged to the whole tangent bundle to the configuration manifold by introducing a stress tensor field as Lagrange multiplier in duality with the Eulerian implicit description of the rigidity constraint, according to which the Lie derivative of the metric tensor is the field characterizing the lack of rigidity. This extension is at the very heart of continuum mechanics. The stress tensor field is then related to the strain tensor field by a pointwise elastic law in the body. By adding the negative elastic potential energy and the force potential to the kinetic energy, the Lagrangian for the field theory of elastodynamics is formulated.

6. Special forms of the law of dynamics

The original LAGRANGE's law of dynamics is immediately recovered from the general expression by endowing the configuration manifold with the local connection induced by a coordinate system. This connection is conveniently described by considering the distant parallel transport obtained by pushing the standard translation in the model linear space by the coordinate map. Both curvature and torsion tensor fields vanish for the standard connection in the model linear space and the diffeomorphic coordinate map simply pushes the curvature and torsion tensors to the ones of the induced connection in the configuration manifold which hence vanish too [6]. By the vanishing of the torsion of the induced connection, the law of dynamics specializes to the invariant form of LAGRANGE law

$$\langle \nabla_{\mathbf{v}_\gamma}(d_{\mathrm{F}}L_t \circ \mathbf{v}_\gamma), \mathbf{v}_\varphi \rangle - \langle d_{\mathrm{B}}L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle = \langle \mathbf{F}_t, \mathbf{v}_\varphi \rangle - \langle \partial_{\tau=t} d_{\mathrm{F}}L_\tau \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle,$$

and for perfect dynamical systems takes the standard aspect of LAGRANGE equation [23]

$$\nabla_{\mathbf{v}_\gamma}(d_{\mathrm{F}}L_t \circ \mathbf{v}_\gamma) - d_{\mathrm{B}}L_t \circ \mathbf{v}_\gamma = \mathbf{F}_t - \partial_{\tau=t} d_{\mathrm{F}}L_\tau \circ \mathbf{v}_\gamma.$$

The more general form of the laws of dynamics due to POINCARÉ holds when the reference system is a mobile frame whose base vectors are not necessarily the velocities along the coordinate lines. A common instance of this occurrence is provided by the so-called engineering reference systems which consist in curvilinear coordinate systems with velocity base vectors normalized to a unit length. In a mobile frame the induced connection in the configuration manifold is such that the related distant parallel transport $\mathbf{S} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{TC})$ is defined by the property that the components of a vector do not change when the frame base point is displaced. The torsion of this connection is evaluated on a given pair of vectors $\mathbf{u_x}, \mathbf{v_x} \in \mathbb{T}_{\mathbf{x}}\mathbb{C}$ by extending them to a pair of vector fields $\mathbf{S}(\mathbf{u_x}), \mathbf{S}(\mathbf{v_x}) \in \mathrm{C}^1(\mathbb{C}; \mathbb{TC})$ according to the distant parallel transport. Then, by tensoriality

$$\mathrm{TORS}(\mathbf{u_x}, \mathbf{v_x}) := \nabla_{\mathbf{u_x}}\mathbf{S}(\mathbf{v_x}) - \nabla_{\mathbf{v_x}}\mathbf{S}(\mathbf{u_x}) - [\mathbf{S}(\mathbf{u_x}), \mathbf{S}(\mathbf{v_x})] = -[\mathbf{S}(\mathbf{u_x}), \mathbf{S}(\mathbf{v_x})].$$

The law of dynamics then takes the form

$$\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle + \langle \nabla_{\mathbf{v}_\gamma}(d_{\mathrm{F}}L_t \circ \mathbf{v}_\gamma), \mathbf{v}_\varphi \rangle = \langle d_{\mathrm{B}}L_t \circ \mathbf{v}_\gamma, \mathbf{v}_\varphi \rangle + \langle \mathbf{F}_t, \mathbf{v}_\varphi \rangle - \langle d_{\mathrm{F}}L_t \circ \mathbf{v}_\gamma, [\mathbf{S} \circ \mathbf{v}_\gamma, \mathbf{S} \circ \mathbf{v}_\varphi] \rangle,$$

which for perfect systems gives POINCARÉ law [23] in invariant form.

Let us now consider the standard case in which the Lagrangian is the sum $L = K + P \circ \boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{TC}; \mathfrak{R})$ of the positive definite quadratic functional *kinetic energy* $K \in \mathrm{C}^1(\mathbb{TC}; \mathfrak{R})$ and of the load *potential* $P \circ \boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{TC}; \mathfrak{R})$. By polarization, a metric tensor $\mathbf{g} \in BL\,(\mathbb{TC}, \mathbb{TC}; \mathfrak{R})$ can then be associated with the kinetic energy, so that $K = 1/2\mathbf{g} \circ \mathrm{DIAG}$, where $\mathrm{DIAG}(\mathbf{v}) := (\mathbf{v}, \mathbf{v})$, so that $\mathbf{g} := d_{\mathrm{F}}K \in BL\,(\mathbb{TC}; \mathbb{T}^*\mathbb{C})$ and $\mathbf{g}^{-1} := d_{\mathrm{F}}K^{-1} \in BL\,(\mathbb{T}^*\mathbb{C}; \mathbb{TC})$. Adopting the LEVI-CIVITA connection ∇ in the Riemannian manifold $\{\mathbb{C}, \mathbf{g}\}$ we have that $\nabla d_{\mathrm{F}}K = \nabla \mathbf{g} = 0$ and $\mathrm{TORS} = 0$. Moreover, from the invariance of the norm of a vector field which is parallel transported according to a metric connection, we infer that $d_{\mathrm{B}}K = d_{\mathrm{B}}((1/2)\mathbf{g} \circ \mathrm{DIAG}) = 0$ and, by the definition of fiber and base derivative, we have that

$$d_{\mathrm{F}}(P \circ \boldsymbol{\pi})(\mathbf{v}) = TP(\boldsymbol{\pi}(\mathbf{v})) \cdot T\boldsymbol{\pi}(\mathbf{v}) \cdot \nabla \mathbf{v} = 0,$$
$$d_{\mathrm{B}}(P \circ \boldsymbol{\pi})(\mathbf{v}) = TP(\boldsymbol{\pi}(\mathbf{v})) \cdot T\boldsymbol{\pi}(\mathbf{v}) \cdot \mathbf{H}\mathbf{v} = dP(\boldsymbol{\pi}(\mathbf{v})).$$

Recalling that $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$ is the subbundle of $\mathbb{T}_{\boldsymbol{\Gamma}}\mathbb{C}$ described by the virtual velocities at $\boldsymbol{\Gamma}$ which are conforming to the (also non-holonomic) linear constraint, and setting $\mathbf{F}^{\mathrm{GEN}} := -\partial_{\tau=t}(d_{\mathrm{F}}L_\tau \circ \mathbf{v}_\gamma) + \mathbf{F}_t + dP$, the law of motion for a perfect dynamical system becomes

$$\nabla_{\mathbf{v}_\gamma}\mathbf{v}_\gamma - \mathbf{g}^{-1}\mathbf{F}^{\mathrm{GEN}} \in \boldsymbol{\Delta}_{\boldsymbol{\Gamma}}^{\perp}.$$

Denoting by $\boldsymbol{\Pi}, \boldsymbol{\Pi}^{\perp} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{TC})$ the fiberwise orthogonal projectors on $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$ and $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}^{\perp}$, the law of motion may be rewritten as $\boldsymbol{\Pi}(\nabla_{\mathbf{v}_\gamma}\mathbf{v}_\gamma - \mathbf{g}^{-1}\mathbf{F}^{\mathrm{GEN}}) = 0$. Introducing in the WHITNEY sum $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}} \oplus \boldsymbol{\Delta}_{\boldsymbol{\Gamma}}^{\perp}$ the tensorial WEINGARTEN map: $\mathbf{W}(\mathbf{u}, \mathbf{v}) := \boldsymbol{\Pi}^{\perp}(\nabla_{\mathbf{u}}\mathbf{v})$, we may write the law of motion for a perfect dynamical system as in [3,24]

$$\nabla_{\mathbf{v}_\gamma}\mathbf{v}_\gamma - \mathbf{W}(\mathbf{v}_\gamma, \mathbf{v}_\gamma) = \boldsymbol{\Pi}\mathbf{g}^{-1}\mathbf{F}^{\mathrm{GEN}}.$$

In the free dynamics of a perfect system with no mass-loss time-rate, we have that $\mathbf{F}^{\mathrm{GEN}} = 0$ and the law of dynamics becomes

$$\nabla_{\mathbf{v}_\gamma}\mathbf{v}_\gamma = \mathbf{W}(\mathbf{v}_\gamma, \mathbf{v}_\gamma).$$

In the absence of constraints it is $\mathbf{W}(\mathbf{v}_\gamma, \mathbf{v}_\gamma) = 0$ and the law of dynamics yields the differential equation of a geodesic $\nabla_{\mathbf{v}_\gamma}\mathbf{v}_\gamma = 0$.

The jump conditions then give $\langle [[d_{\mathrm{F}}L_t \circ \mathbf{v}_\gamma]], \mathbf{v}_\varphi \rangle = 0$, which, recalling that the virtual velocities are required to be tangent to the discontinuity interfaces in the configuration manifold, directly yield the conservation of the tangent component of the momentum at the singularity interfaces. We remark that the constrained dynamics formulated above is the classical one which is also called *d'Alembertian* [24] as opposed to the recently proposed *vakonomic* constrained dynamics [23].

7. Conclusions

The HEUN-HAMEL *central equation*, quoted in [20,21] as $\delta L = d/dt(p \cdot dq)$ (recall that in the standard notation in analytical dynamics q is the vector of Lagrangian variables and $p = dL/d\dot{q}$ the vector of momenta) is the coordinate expression, in the special context of discrete systems and time independent Lagrangian without non-potential forces, of the differential law of dynamics formulated in invariant terms in Proposition 5.2. In [20,21] the variation δL of the Lagrangian is evaluated as $\delta L = (dL/d\dot{q})\delta\dot{q} + (dL/dq)\delta q$ and the law is deduced from the LAGRANGE variational law of dynamics (see Section 6) relying on the key property $\delta\dot{q} = (\delta q)\dot{}$ which is also resorted to in standard treatments, to deduce LAGRANGE's equations from HAMILTON's principle [1]. Alas, the δ-$(\,)\dot{}$ notation, although adopted in most textbooks and articles, does not unambiguously clarify the operations to be performed especially when dealing with a non-linear configuration manifold. Let *speed* denote the time-rate. Then the rate of variation of the speed is defined by $\partial_{\lambda=0}\partial_{\tau=t}(\boldsymbol{\varphi}_\lambda \circ \gamma)(\tau)$ and the speed of the rate of variation is defined by $\partial_{\tau=t}\partial_{\lambda=0}(\boldsymbol{\varphi}_\lambda \circ \gamma)(\tau)$.They are, respectively, equal to the evaluations of the vector fields $\mathbf{v}_{T\boldsymbol{\varphi}} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{TTC})$ and $T\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^1(\mathbb{TC}; \mathbb{TTC})$ at the point $\mathbf{v}_\gamma(\gamma(t)) \in \mathbb{T}_{\gamma(t)}\mathbb{C}$:

$$\begin{aligned}\partial_{\lambda=0}\partial_{\tau=t}(\boldsymbol{\varphi}_\lambda \circ \gamma)(\tau) &= \partial_{\lambda=0}(T\boldsymbol{\varphi}_\lambda \circ \mathbf{v}_\gamma)(\gamma(t))\\ &= (\mathbf{v}_{T\boldsymbol{\varphi}} \circ \mathbf{v}_\gamma)(\gamma(t)) \in \mathbb{T}_{\mathbf{v}_\gamma}\mathbb{T}_{\gamma(t)}\mathbb{C},\\ \partial_{\tau=t}\partial_{\lambda=0}(\boldsymbol{\varphi}_\lambda \circ \gamma)(\tau) &= \partial_{\tau=t}(\mathbf{v}_{\boldsymbol{\varphi}} \circ \gamma)(\tau)\\ &= (T\mathbf{v}_{\boldsymbol{\varphi}} \circ \mathbf{v}_\gamma)(\gamma(t)) \in \mathbb{T}_{\mathbf{v}_\gamma}\mathbb{T}_{\gamma(t)}\mathbb{C}.\end{aligned}$$

These vectors, which belong to the second tangent bundle $\mathbb{TTC}$, are related to one another by the canonical involutive flip operation $\mathbf{k} \in \mathrm{C}^1(\mathbb{TTC}; \mathbb{TTC})$ according to the formula $\mathbf{v}_{T\boldsymbol{\varphi}} = \mathbf{k} \circ T\mathbf{v}_{\boldsymbol{\varphi}}$, see, e.g. [16,6]. If the manifold $\mathbb{C}$ is a linear space, each tangent space $\mathbb{T}_{\gamma(t)}\mathbb{C}$ is identified with the linear space itself. Hence $\mathbb{T}_{\mathbf{v}_\gamma}\mathbb{T}_{\gamma(t)}\mathbb{C}$ is identified with $\mathbb{C}$ and the flip involution collapses to the identity by the EULER–SCHWARZ theorem [6]. This is the case when working in coordinates, so that the terms $\delta\dot{q}$ and $(\delta q)\dot{}$ coincide. Anyway, as shown in Proposition 5.3 and in Section 6, in relating the general law of dynamics to LAGRANGE's law, the symmetry property to be invoked is $\nabla_{\mathbf{v}_\gamma}\mathbf{v}_\varphi = \nabla_{\mathbf{v}_\varphi}\mathscr{F}_\varphi(\mathbf{v}_\gamma)$. Here the covariant derivatives are performed according to the connection defined by pushing the standard connection in the linear model space. This property is a consequence of the vanishing of the torsion of the induced connection and of the vanishing of the LIE bracket $[\mathscr{F}_{\varphi}(\mathbf{v}_{\gamma}), \mathbf{v}_{\varphi}]$, due to the extension of the trajectory speed, which give

$$\begin{aligned}\mathrm{TORS}(\mathbf{v}_{\gamma}, \mathbf{v}_{\varphi}) &= \nabla_{\mathbf{v}_{\gamma}}\mathbf{v}_{\varphi} - \nabla_{\mathbf{v}_{\varphi}}\mathscr{F}_{\varphi}(\mathbf{v}_{\gamma}) - [\mathscr{F}_{\varphi}(\mathbf{v}_{\gamma}), \mathbf{v}_{\varphi}] \\ &= \nabla_{\mathbf{v}_{\gamma}}\mathbf{v}_{\varphi} - \nabla_{\mathbf{v}_{\varphi}}\mathscr{F}_{\varphi}(\mathbf{v}_{\gamma}) = 0.\end{aligned}$$

If the manifold $\mathbb{C}$ is a linear space, the identifications discussed above result in killing the horizontal subspaces in $\mathbb{TTC}$ and in the consequent identification of vertical subspaces with tangent spaces, so that the covariant derivative equals the natural derivative [6,11] and then

$$\nabla_{\mathbf{v}_{\gamma}}\mathbf{v}_{\varphi} = T\mathbf{v}_{\varphi} \circ \mathbf{v}_{\gamma}, \quad \nabla_{\mathbf{v}_{\varphi}}\mathscr{F}_{\varphi}(\mathbf{v}_{\gamma}) = T\mathscr{F}_{\varphi}(\mathbf{v}_{\gamma}) \circ \mathbf{v}_{\varphi},$$

with $T\mathscr{F}_{\varphi}(\mathbf{v}_{\gamma}) \circ \mathbf{v}_{\varphi} = \partial_{\lambda=0} T\boldsymbol{\varphi}_{\lambda} \circ \mathbf{v}_{\gamma} = \mathbf{v}_{T\varphi} \circ \mathbf{v}_{\gamma}$. We conclude that, far from being a question of points of view, as affirmed in [20,21], the equality $\delta\dot{q} = (\delta q)^{\cdot}$ holds in coordinates with the standard connection. In the general, non-linear case LAGRANGE's law must be substituted by the law of dynamics provided in Proposition 5.3, whose expression depends on the torsion of the adopted connection. These results confirm that geometric formulations of dynamics, powered by the tools of differential geometry and calculus on manifolds, are able to define and discuss in precise mathematical terms the relevant principles and variational conditions. As a consequence most long debated issues may be answered and clarified.

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