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Well-posedness and numerical performances of the strain gap method

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SUMMARY

A mixed method of approximation is discussed starting from a suitably modified expression of the Hu-Washizu variational principle in which the independent fields are displacements, stresses and strain gaps defined as the difference between compatible strains and strain fields. The well-posedness of the discrete problem is discussed and necessary and sufficient conditions are provided. The analysis of the mixed method reveals that the discrete problem can be split into a reduced problem and in a stress recovery. Accordingly, the discrete stress solution is univocally determined once an interpolating stress subspace is chosen. The enhanced assumed strain method by Simo and Rifai is based on an orthogonality condition between stresses and enhanced strains and coincides with the reduced problem. It is shown that the mixed method is stable and converges. Computational issues in the context of the finite element method are discussed in detail and numerical performances and comparisons are carried out.

KEY WORDS: mixed methods; well-posedness; convergence; numerical performances

1. INTRODUCTION

In recent years mixed methods based on multi-field variational principles with enhanced strains have become popular in finite element method (FEM) literature since they provide the possibility to improve the performance of low-order FES and to overcome locking phenomena.

The enhanced assumed strain (EAS) method proposed in Reference [1] was originally developed to provide a variational basis to the incompatible mode element of Wilson *et al.* [2]. Enhanced strain methods have been widely adopted in the literature for both linear and non-linear elastic models as well as for elastoplastic problems [3–9].

The treatment developed in Reference [1] is based on the Hu-Washizu variational principle in which the independent fields are displacements, enhanced strains and stresses. The role of three conditions to be imposed on these fields is emphasized in References [1, 10] to provide

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well-posedness, convergence and stress independence of the FEM problem. The three assumptions are: (i) compatible and enhanced strain shape functions must be linearly independent; (ii) the shape functions of the stress fields and those of the enhanced strains must be mutually orthogonal; (iii) the space of stress fields must include at least piece-wise constant functions.

Condition (iii) was motivated in Reference [1] by the fulfilment of the patch test. Condition (i) ensures the uniqueness of the discrete solution in terms of displacements and enhanced strains. Condition (ii) is designed to eliminate the stress parameters from the mixed problem. An *a posteriori* stress recovery strategy must then be envisaged and in fact several proposal have been made in the literature [1, 7, 11–13].

This paper is motivated by the observation that variationally consistent stress recovery strategies can be derived from the formulation of the mixed method which fulfils the well-posedness conditions. It is then aimed to get a deeper understanding of the EAS method and of the related well-posedness and convergence properties. The difference between compatible strains and independent strain fields are called strain gaps. Accordingly, the new formulation of the discrete method is referred to as the strain gap method (SGM).

It is shown that necessary and sufficient conditions for well-posedness of the SGM, that is for existence and uniqueness of its solution, are the following (a) effective strain gaps (i.e. the ones orthogonal to the stress fields) and compatible strains must be linearly independent and (b) stresses must be controlled by strain gaps. Condition (a) ensures the uniqueness of the solution in terms of displacement and strain gap and condition (b) pertains to the uniqueness of the stress solution.

According to the general treatment of mixed methods [14, 15], the SGM can be split into a sequence of two steps: a reduced problem, formulated in terms of displacements and strain gaps, in which the orthogonality constraint between stresses and strain gaps is assumed to be fulfilled and a stress recovery problem which depends on the solution of the reduced problem.

An explicit comparison between the SGM and the EAS method clarifies the significant differences of the two formulations. According to the SGM, the subspaces of strain gap and stress fields are assigned so that the well-posedness requirements (a) and (b) are fulfilled. On the contrary, in the EAS method these two discrete subspaces are imposed to be mutually orthogonal so that the well-posedness requirement (b) is violated.

The troubles faced in envisaging an *a posteriori* stress recovery strategy are in fact due to the partially *ill-posedness* of the EAS method.

The reduced problem which is the first step of the SGM is equivalent to the whole EAS method. It can be reformulated as a modified displacement method with an enhanced flexibility. Once the reduced problem has been solved in terms of nodal displacements and strain gap parameters, the stress parameters can be univocally recovered at the element level by following the stress recovery strategy defined by the second step of the SGM. In this respect, we shall prove that the computation of the discrete stress according to the elastic constitutive relation is variationally consistent despite of the opposite opinion expressed in References [7, 11, 12]. This result is in accordance with the analogous statement in Reference [13] which was based on a more involved matricial arguments and limited to undistorted meshes.

The convergence analysis of the EAS method developed in References [10, 16] was based on the interpolation properties of the displacement shape functions and on a special orthogonality assumption between the enhanced strains and polynomials of suitable degree. This spourious requirement is in apparent contradiction with the observation that no interpolation properties are required to the strain gap shape functions since any strain gap subspace includes the null field, that is the exact solution.

In fact, the convergence analysis of the SGM shows that the error estimate depends only on the interpolation properties of the discrete subspaces of stress and displacement shape functions [15].

Finally, we develop a general formulation of the SGM, in which the orthogonality constraint is not satisfied *a priori* but enters as one of the equations of the discrete mixed problem. This formulation is a useful tool in detecting the computational performances relevant to different implementations of the discrete method.

Numerical examples of two-dimensional elastostatic problems, which are commonly adopted in the literature as significant benchmarks, are developed and discussed to get information about the comparative convergence properties, the distortion sensitivity and the reliability of the stress approximation.

2. THE STRUCTURAL MODEL

In this paper we will perform the analysis of mixed methods for a continuous linearly elastic structural model defined on a bounded domain Ω of an Euclidean space with boundary $\partial\Omega$ and closure $\bar{\Omega}=\Omega\cup\partial\Omega$.

The kinematic space $\mathscr{V}$ is endowed with a suitable Hilbert topology and its dual $\mathscr{F}$ is the space of external forces. To each displacement field $\mathbf{u}\in\mathscr{V}$ there corresponds a boundary field $\mathbf{\Gamma u}\in\partial\mathscr{V}$. Conforming displacement fields belong to the subspace $\mathscr{L}$. The dual space $\partial\mathscr{F}$ of $\partial\mathscr{V}$ is the space of boundary tractions $\bar{\mathbf{t}}$. Body forces $\bar{\mathbf{b}}$ belong to the Hilbert space W of square integrable vector fields on the structural model. External forces are collected in the set $\bar{\ell}=\{\bar{\mathbf{b}},\bar{\mathbf{t}}\}\in\mathscr{F}$. Strains ε and stresses $\boldsymbol{\sigma}$ belong to the Hilbert space $\mathscr{H}$ of square integrable vector fields on Ω [17, 18]. The strain space will be also denoted by $\mathscr{D}\equiv\mathscr{H}$.

The kinematic operator $\mathbf{B}\in\mathrm{Lin}\{\mathscr{V},\mathscr{D}\}$ associates the strain fields ε with the corresponding displacement fields $\mathbf{u}$ and it is a continuous linear map from $\mathscr{V}$ into $\mathscr{D}$ with closed range [19] and finite dimensional kernel [20].

The stress fields belong to the Hilbert space $\mathscr{S}=\{\boldsymbol{\sigma}\in\mathscr{H}\colon\ \mathbf{B}'_o\boldsymbol{\sigma}\in H\}$ where the differential operator $\mathbf{B}'_o\in\mathrm{Lin}\{\mathscr{S},W\}$ is the formal adjoint of $\mathbf{B}$ and provides the body forces corresponding to the stress field $\boldsymbol{\sigma}\in\mathscr{S}$.

The elastic strain energy $\phi:\mathscr{H}\mapsto\Re\cup\{+\infty\}$ is the convex quadratic functional given by

$$\phi(\boldsymbol{\varepsilon})=\tfrac{1}{2}((\mathbf{E}(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}))_{\mathscr{H}}$$

where $\mathbf{E}$ is the elastic stiffness of the material.

The structural problem can be written in an operator form as [21, 22]

$$\mathbf{Bu}=\boldsymbol{\varepsilon},\quad \mathbf{B}'\boldsymbol{\sigma}=\bar{\ell},\quad \boldsymbol{\sigma}=\mathrm{d}\phi(\boldsymbol{\varepsilon}) \tag{1}$$

where $\mathbf{B}'\in\mathrm{Lin}\{\mathscr{S},\mathscr{F}\}$ is the dual operator of $\mathbf{B}$ defined as $((\boldsymbol{\sigma},\mathbf{B}\boldsymbol{\sigma}))_{\mathscr{H}}=((\mathbf{B}'\boldsymbol{\sigma},\boldsymbol{\sigma}))_{\mathscr{H}}$ for any $\mathbf{u}\in\mathscr{V}$ and $\boldsymbol{\sigma}\in\mathscr{S}$.

Let us consider the Hu-Washizu functional [23] given by

$$H(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})=\phi(\boldsymbol{\varepsilon})-((\boldsymbol{\sigma},\boldsymbol{\varepsilon}))_{\mathscr{H}}+((\boldsymbol{\sigma},\mathbf{Bu}))_{\mathscr{H}}-\gamma_o(\mathbf{u}) \tag{2}$$

where the unknown displacement field is conforming, i.e. $\mathbf{u} \in \mathscr{L}$. The linear functional

$$\gamma_o(\mathbf{u}) = \langle \bar{\ell}, \mathbf{u} \rangle = (\bar{\mathbf{b}}, \mathbf{u})_W + \langle \bar{\mathbf{t}}, \mathbf{\Gamma u} \rangle$$

yields the virtual work of body forces and boundary tractions.

The stationarity conditions of the functional H provides the constitutive relation, the equilibrium equation and the constitutive relation given by (1).

Let us now introduce the SGM by defining the strain gap field $\mathbf{g} \in \mathscr{D}$ as the difference between the compatible strain $\mathbf{Bu} \in \mathscr{D}$ and the stress $\boldsymbol{\varepsilon} \in \mathscr{D}$ in the form $\mathbf{g} = \mathbf{Bu} - \boldsymbol{\varepsilon}$. Accordingly, the Hu-Washizu functional can be re-written as

$$\begin{aligned} \tilde{H}(\mathbf{u}, \mathbf{g}, \boldsymbol{\sigma}) &= \phi(\mathbf{Bu} - \mathbf{g}) + ((\boldsymbol{\sigma}, \mathbf{g}))_{\mathscr{H}} - \langle \bar{\ell}, \mathbf{u} \rangle \\ &= \tfrac{1}{2}((\mathbf{E}(\mathbf{Bu} - \mathbf{g}), \mathbf{Bu} - \mathbf{g}))_{\mathscr{H}} + ((\boldsymbol{\sigma}, \mathbf{g}))_{\mathscr{H}} - \langle \bar{\ell}, \mathbf{u} \rangle \end{aligned}$$

with $\mathbf{u} \in \mathscr{L}$, $\mathbf{g} \in \mathscr{D}$, $\boldsymbol{\sigma} \in \mathscr{S}$.

The stationarity of $\tilde{H}$ yields the mixed variational problem in the three fields $\{\mathbf{u}, \mathbf{g}, \boldsymbol{\sigma}\}$:

$$\begin{cases} ((\mathbf{EBu}, \mathbf{B}\delta\mathbf{u}))_{\mathscr{H}} - ((\mathbf{Eg}, \mathbf{B}\delta\mathbf{u}))_{\mathscr{H}} = \langle \bar{\ell}, \delta\mathbf{u} \rangle & \forall \delta\mathbf{u} \in \mathscr{L} \\ ((\mathbf{Eg}, \delta\mathbf{g}))_{\mathscr{H}} + ((\boldsymbol{\sigma}, \delta\mathbf{g}))_{\mathscr{H}} - ((\mathbf{EBu}, \delta\mathbf{g}))_{\mathscr{H}} = 0 & \forall \delta\mathbf{g} \in \mathscr{D} \\ ((\delta\boldsymbol{\sigma}, \mathbf{g}))_{\mathscr{H}} = 0 & \forall \delta\boldsymbol{\sigma} \in \mathscr{S} \end{cases}$$

Note that the last equation imposes the kinematic compatibility by requiring that the strain gap $\mathbf{g}$ must vanish in correspondence of a solution of the continuous problem.

2.1. *FEM interpolation*

With a standard notation in FE analysis [24] we consider, for each element, the interpolations

$$\mathbf{u}_h^e(\mathbf{x}) = \mathbf{N_u}(\mathbf{x})\mathbf{p_u}^e \in \mathscr{V}_h^e, \quad \mathbf{g}_h^e(\mathbf{x}) = \mathbf{N_g}(\mathbf{x})\mathbf{p_g}^e \in \mathscr{D}_h^e, \quad \boldsymbol{\sigma}_h^e(\mathbf{x}) = \mathbf{N}_{\boldsymbol{\sigma}}(\mathbf{x})\mathbf{p}_{\boldsymbol{\sigma}}^e \in \mathscr{S}_h^e \quad \mathbf{x} \in \bar{\Omega}_e$$

where $\bar{\Omega}_e \in \mathscr{T}_{\text{FEM}}(\Omega)$ is the domain decomposition induced by the meshing of Ω.

Let us set $n_{\mathbf{u}}^e = \dim \mathscr{V}_h^e$, $n_{\mathbf{g}}^e = \dim \mathscr{D}_h^e$, $n_{\boldsymbol{\sigma}}^e = \dim \mathscr{S}_h^e$. The interpolating spaces are collected in the following product spaces:

$$\mathscr{V}_h = \prod_{e=1}^{\mathscr{N}} \mathscr{V}_h^e, \quad \mathscr{D}_h = \prod_{e=1}^{\mathscr{N}} \mathscr{D}_h^e, \quad \mathscr{S}_h = \prod_{e=1}^{\mathscr{N}} \mathscr{S}_h^e \tag{3}$$

No interelement continuity condition is imposed on the strain gap and stress fields so that the corresponding global fields are simply the collection of the local ones:

$$\mathbf{g}_h = \left\{\mathbf{g}_h^1, \mathbf{g}_h^2, \ldots, \mathbf{g}_h^{\mathscr{N}}\right\} \in \mathscr{D}_h, \quad \boldsymbol{\sigma}_h = \left\{\boldsymbol{\sigma}_h^1, \boldsymbol{\sigma}_h^2, \ldots, \boldsymbol{\sigma}_h^{\mathscr{N}}\right\} \in \mathscr{S}_h$$

where $\mathscr{N}$ is the total number of elements pertaining to the FE discretization.

Accordingly, the virtual work performed by an interpolating stress field $\boldsymbol{\sigma}_h \in \mathscr{S}_h$ by an interpolating strain gap field $\mathbf{g}_h \in \mathscr{D}_h$ is defined as the sum of the contributions of each element

$$((\boldsymbol{\sigma}_h, \mathbf{g}_h)) = \sum_{e=1}^{\mathscr{N}} ((\boldsymbol{\sigma}_h^e, \mathbf{g}_h^e))_{\Omega_e} = \sum_{e=1}^{\mathscr{N}} \int_{\Omega_e} \mathbf{N}_{\boldsymbol{\sigma}}^e \mathbf{p}_{\boldsymbol{\sigma}}^e * \mathbf{N}_{\mathbf{g}}^e \mathbf{p}_{\mathbf{g}}^e$$

The local parameters $\mathbf{p}_{\mathbf{g}}^e \in \Re^{n_{\mathbf{g}}^e}$ and $\mathbf{p}_{\boldsymbol{\sigma}}^e \in \Re^{n_{\boldsymbol{\sigma}}^e}$ can be condensed at the element level.

We shall consider a conforming FE interpolation. The conforming displacement fields

$$\mathbf{u}_h = \{\mathbf{u}_h^1, \mathbf{u}_h^2, \ldots, \mathbf{u}_h^{\mathscr{N}}\} \in \mathscr{L}_h \subset \mathscr{V}_h$$

satisfy the homogeneous boundary constraints and the interelement continuity conditions. The dimension of the subspace $\mathscr{L}_h \subset \mathscr{V}_h$ will be denoted by $n_{\text{dof}} = \dim \mathscr{L}_h$.

As customary we assume that rigid-body displacements are ruled out by the conformity requirements so that $\mathscr{L} \cap \operatorname{Ker} \mathbf{B} = \{\mathbf{0}\}$ and the condition $\mathscr{L}_h \subset \mathscr{L}$ implies $\mathscr{L}_h \cap \operatorname{Ker} \mathbf{B} = \{\mathbf{0}\}$.

The parameters $\mathbf{p}_{\mathbf{u}}^e \in \Re^{n_{\mathbf{u}}^e}$ can be expressed in terms of the nodal parameters $\mathbf{q}_{\mathbf{u}} \in \Re^{n_{\text{dof}}}$ by means of the standard FE assembly operator $\mathscr{A}_{\mathbf{u}}^e$ according to the parametric representation $\mathbf{p}_{\mathbf{u}}^e = \mathscr{A}_{\mathbf{u}}^e \mathbf{q}_{\mathbf{u}}$. On the contrary, the strain gap and stress local parameters are simply collected in the global lists $\mathbf{q}_{\mathbf{g}}$ and $\mathbf{q}_{\boldsymbol{\sigma}}$ according to the expressions $\mathbf{p}_{\mathbf{g}}^e = \mathscr{J}_{\mathbf{g}}^e \mathbf{q}_{\mathbf{g}}$ and $\mathbf{p}_{\boldsymbol{\sigma}}^e = \mathscr{J}_{\boldsymbol{\sigma}}^e \mathbf{q}_{\boldsymbol{\sigma}}$ where the operators $\mathscr{J}_{\mathbf{g}}^e$ and $\mathscr{J}_{\boldsymbol{\sigma}}^e$ are the canonical extractors which pick up, from the global lists $\mathbf{q}_{\mathbf{g}}$ and $\mathbf{q}_{\boldsymbol{\sigma}}$, the local parameters $\mathbf{p}_{\mathbf{g}}^e$ and $\mathbf{p}_{\boldsymbol{\sigma}}^e$.

The interpolated counterpart of the Hu-Washizu functional $\tilde{H}_h(\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h)$ is obtained by adding-up the contributions of each non-assembled element and imposing that the interpolating displacement $\mathbf{u}_h$ satisfies the conformity requirement to get

$$\tilde{H}_h(\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h) = \tfrac{1}{2}((\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h), \mathbf{B}\mathbf{u}_h - \mathbf{g}_h)) + ((\boldsymbol{\sigma}_h, \mathbf{g}_h)) - \langle \bar{\ell}, \mathbf{u}_h \rangle \tag{4}$$

where $\{\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h\} \in \mathscr{L}_h \times \mathscr{D}_h \times \mathscr{S}_h$.

The matrix form of the discrete problem is obtained by imposing the stationarity of $\tilde{H}_h$ and is given by

$$(\mathbb{P}_h)\ \mathbf{M} \begin{vmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{vmatrix} = \sum_{e=1}^{\mathscr{N}} \begin{bmatrix} \mathscr{A}_{\mathbf{u}}^{e\mathrm{T}} \mathbf{K}^e \mathscr{A}_{\mathbf{u}}^e & -\mathscr{A}_{\mathbf{u}}^{e\mathrm{T}} \mathbf{G}^{e\mathrm{T}} \mathscr{J}_{\mathbf{g}}^e & \mathbf{0} \\ -\mathscr{J}_{\mathbf{g}}^{e\mathrm{T}} \mathbf{G}^e \mathscr{A}_{\mathbf{u}}^e & \mathscr{J}_{\mathbf{g}}^{e\mathrm{T}} \mathbf{H}^e \mathscr{J}_{\mathbf{g}}^e & \mathscr{J}_{\mathbf{g}}^{e\mathrm{T}} \mathbf{Q}^e \mathscr{J}_{\boldsymbol{\sigma}}^e \\ \mathbf{0} & \mathscr{J}_{\boldsymbol{\sigma}}^{e\mathrm{T}} \mathbf{Q}^{e\mathrm{T}} \mathscr{J}_{\mathbf{g}}^e & \mathbf{0} \end{bmatrix} \begin{vmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{vmatrix}$$

$$= \begin{bmatrix} \mathbf{K} & -\mathbf{G}^{\mathrm{T}} & \mathbf{0} \\ -\mathbf{G} & \mathbf{H} & \mathbf{Q} \\ \mathbf{0} & \mathbf{Q}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{vmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{vmatrix} = \sum_{e=1}^{\mathscr{N}} \begin{vmatrix} \mathscr{A}_{\mathbf{u}}^{e\mathrm{T}} \mathbf{f}_{\mathbf{u}}^e \\ \mathbf{0} \\ \mathbf{0} \end{vmatrix} = \begin{vmatrix} \mathbf{f}_{\mathbf{u}} \\ \mathbf{0} \\ \mathbf{0} \end{vmatrix} \tag{5}$$

The component submatrices and subvectors appearing in Equation (5) are defined by

$$
\begin{aligned}
\mathbf{H}^e &= \int_{\Omega_e} \mathbf{N}_{\mathbf{g}}^{e\mathrm{T}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{N}_{\mathbf{g}}^e(\mathbf{x}), \quad \mathbf{Q}^e = \int_{\Omega_e} \mathbf{N}_{\mathbf{g}}^{e\mathrm{T}}(\mathbf{x})\mathbf{N}_{\boldsymbol{\sigma}}^e(\mathbf{x}) \\
\mathbf{G}^e &= \int_{\Omega_e} \mathbf{N}_{\mathbf{g}}^{e\mathrm{T}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{B}_*\mathbf{N}_{\mathbf{u}}^e(\mathbf{x}), \quad \mathbf{K}^e = \int_{\Omega_e} (\mathbf{B}_*\mathbf{N}_{\mathbf{u}}^e)^{\mathrm{T}}(\mathbf{x})\mathbf{E}_*(\mathbf{x})\mathbf{B}_*\mathbf{N}_{\mathbf{u}}^e(\mathbf{x}) \\
\mathbf{f}_{\mathbf{u}}^e &= \int_{\Omega_e} \mathbf{N}_{\mathbf{u}}^{e\mathrm{T}}(\mathbf{x})\bar{\mathbf{b}}(\mathbf{x}) + \int_{\partial\Omega_e} (\boldsymbol{\Gamma}\mathbf{N}_{\mathbf{u}}^e)^{\mathrm{T}}(\mathbf{x})\,\bar{\mathbf{t}}(\mathbf{x})
\end{aligned}
\tag{6}
$$

Here $\mathbf{B}_*$ and $\mathbf{E}_*$ denote the matrix form of the operators $\mathbf{B}$ and $\mathbf{E}$. The elastic stiffness matrix $\mathbf{E}_*$ is positive definite and hence the matrix $\mathbf{H}$ turns out to be positive definite as well.

From the computational point of view it is more convenient to carry out the assembly operation after the condensation, at the element level, of the strain and stress parameters to put the global discrete problem in terms of the sole displacement parameters $\mathbf{q}_{\mathbf{u}}$. The procedure will be illustrated in Section 7. On the contrary, the well-posedness analysis discussed in the next two subsections is more conveniently developed in terms of the three-field problem $\mathbb{P}_h$.

3. WELL-POSEDNESS ANALYSIS

As customary in computational analysis we will assume that the structure cannot undergo conforming rigid displacements.

• *Definition of well-posedness*: The discrete mixed problem $\mathbb{P}_h$ is said to be *well-posed* if there exists a unique solution $\{\mathbf{u}_h, \mathbf{g}_h, \boldsymbol{\sigma}_h\} \in \mathscr{L}_h \times \mathscr{D}_h \times \mathscr{S}_h$ for any data $\mathbf{f}_{\mathbf{u}}$.

Well-posedness is often characterized in the literature by the requirements that the discrete problem admits a unique solution for any data and that the discrete solution tends to the solution of the continuous problem as the FE mesh is refined ever more. We prefer here to treat separately these two requirements since the conditions for their fulfilment can be proved following two different arguments.

A necessary and sufficient condition for well-posedness is thus provided by the next statement. The proof, in terms of the kernel of the matrix $\mathbf{M}$, is reported in the appendix.

Proposition 3.1 (*Well-posedness criterion*). If there are no rigid conforming displacements, that is $\operatorname{Ker}\mathbf{B} \cap \mathscr{L}_h = \{\mathbf{0}\}$, the conditions

$$
\begin{aligned}
\tilde{\mathscr{D}}_h \cap \mathbf{B}\mathscr{L}_h &= \{\mathbf{0}\} \\
\mathscr{S}_h \cap \mathscr{D}_h^{\perp} &= \{\mathbf{0}\}
\end{aligned}
\tag{7}
$$

are necessary and sufficient for the well-posedness of the discrete mixed problem $\mathbb{P}_h$.

The strain gaps $\mathbf{g}_h$ belonging to the subspace $\tilde{\mathscr{D}}_h = \mathscr{D}_h \cap \mathscr{S}_h^{\perp}$ are referred to as *effective* strain gaps since they effectively contribute to relax the compatibility condition. The orthogonality relation $\perp$ is intended according to the inner product in $L^2(\Omega)$.

The well-posedness condition $(7)_1$ requires that effective strain gaps $\mathbf{g}_h \in \tilde{\mathscr{D}}_h = \mathscr{D}_h \cap \mathscr{S}_h^{\perp}$ and compatible discrete strains $\mathbf{B}\mathbf{u}_h \in \mathbf{B}\mathscr{L}_h$ must be linearly independent. The well-posedness condition $(7)_2$ means that stresses $\boldsymbol{\sigma}_h \in \mathscr{S}_h$ must be controlled by strain gaps $\mathbf{g}_h \in \mathscr{D}_h$. Condition $(7)_1$ can be conveniently substituted by the local condition $\tilde{\mathscr{D}}_h \cap \mathbf{B}\mathscr{V}_h = \{\mathbf{0}\}$ which does not involve the unknown assembly operation. Condition $(7)_2$ can be imposed by chosing $\mathscr{D}_h$ such that $\mathscr{S}_h \subseteq \mathscr{D}_h$.

The conditions which guarantee that the convergence in the energy norm of the discrete solution to the continuous one will be analysed in Section 5 by resorting to the general treatment of mixed methods [14].

3.1. Element shape functions

The shape functions are defined in the reference element K and are evaluated in each element Ω_e of the mesh by performing the composition with the one-to-one isoparametric map $\boldsymbol{\chi}_e$: $K \mapsto \Omega_e$. We shall denote the gradient by $\mathbf{F}_e = \operatorname{grad} \boldsymbol{\chi}_e$ and the Jacobian determinant by $J^e = \det \mathbf{F}_e$, if the transformation is affine we have $J^e = V_e/V_K$.

The condition $\mathscr{D}_h \cap \mathscr{S}_h^{\perp} \neq \{\mathbf{0}\}$ can be effectively checked in terms of the subspaces $\mathscr{D}_K$ and $\mathscr{S}_K$ defined in the reference element by means of the change of co-ordinates described by the map $\boldsymbol{\chi}_e^{-1}$. The corresponding inner product in K is performed by an integration over the reference element which involves an unknown Jacobian determinant.

If we consider affine equivalent FE meshes, the Jacobian determinant is constant and no problem arises in imposing the orthogonality conditions.

On the contrary, in the case of general isoparametric maps, the Jacobian determinant is no more constant and as a consequence the integral of the product of two fields in the reference element is no more proportional to the corresponding integral in an actual element of the mesh.

A skilful trick was proposed in Reference [1] in order to overcome this difficulty. Following their proposal, the shape functions of the stresses and of the strain gaps are defined according to

$$\boldsymbol{\sigma}_h^e(\mathbf{x}) = \boldsymbol{\sigma}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right], \quad \mathbf{g}_h^e(\mathbf{x}) = \frac{J_o^e}{J^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right]}\mathbf{g}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right], \quad \mathbf{x} \in \bar{\Omega}_e \tag{8}$$

where J_o^e is obtained by evaluating $J^e(\boldsymbol{\xi})$ at $\boldsymbol{\xi} = \mathbf{0}$. Setting $\mathbf{x} = \boldsymbol{\chi}_e(\boldsymbol{\xi})$, we have

$$\int_{\Omega_e} \boldsymbol{\sigma}_h^e(\mathbf{x}) \cdot \mathbf{g}_h^e(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Omega_e} \boldsymbol{\sigma}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right] \cdot \frac{J_o^e}{J^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right]}\mathbf{g}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right]\,\mathrm{d}\mathbf{x} = J_o^e \int_K \boldsymbol{\sigma}(\boldsymbol{\xi}) \cdot \mathbf{g}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

and the orthogonality condition is preserved by general isoparametric mapping.

It is worthnoting that this procedure leads to a non-polynomial approximation of the strain gap since in Definition (8) the polynomials $g^e(\boldsymbol{\xi})$ are divided by the Jacobian $J^e(\boldsymbol{\xi})$. Nevertheless, it is admissible since the approximation properties of the strain gap subspace $\mathscr{D}_h$ do not play any role in the estimate of the asymptotic rate of convergence as proved in Reference [15] and discussed hereafter in Section 5.

Additional transformation rules which preserve the point-wise inner product between stress and strain tensors can be envisaged but it seems that they can only be motivated by an *a posteriori* evaluation of the quality of the numerical results. Examples are provided by the push/pull transformations of differential geometry [25]. In this respect, the following expressions for plane problems have been adopted, see e.g. References [1, 26]:

$$\boldsymbol{\sigma}_h^e(\mathbf{x}) = \mathbf{T}_o^e \boldsymbol{\sigma}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right], \quad \mathbf{g}_h^e(\mathbf{x}) = \frac{J_o^e}{J^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right]}\mathbf{T}_o^{e-T}\mathbf{g}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right] \tag{9}$$

where $\mathbf{T}_o^e$ is the value, at the origin of the reference element, of the matrix field

$$\mathbf{T}^e(\boldsymbol{\xi}) = \begin{bmatrix} F_{11}^2 & F_{12}^2 & 2F_{11}F_{12} \\ F_{21}^2 & F_{22}^2 & 2F_{21}F_{22} \\ F_{11}F_{21} & F_{12}F_{22} & F_{11}F_{22} + F_{12}F_{21} \end{bmatrix}^e (\boldsymbol{\xi}) \quad \text{with } \boldsymbol{\xi} \in K$$

Accordingly, the inner product in the real space is given by

$$\int_{\Omega_e} \boldsymbol{\sigma}_h^e(\mathbf{x}) \cdot \mathbf{g}_h^e(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Omega_e} \mathbf{T}_o^e \boldsymbol{\sigma}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right] \cdot \frac{J_o^e}{J^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right]}\mathbf{T}_o^{e-T}\mathbf{g}^e\left[\boldsymbol{\chi}_e^{-1}(\mathbf{x})\right]\mathrm{d}\mathbf{x} = J_o^e \int_K \boldsymbol{\sigma}(\boldsymbol{\xi}) \cdot \mathbf{g}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

4. REDUCED PROBLEM AND STRESS RECOVERY

The SGM can be cast in the theoretical framework of the mixed methods analysed in Reference [14]. In fact, the discrete mixed problem deriving from the stationarity of Equation (4) can be written in the form

$$\begin{cases} ((\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h), \mathbf{B}\delta\mathbf{u}_h)) = \langle \bar{\ell}, \delta\mathbf{u}_h \rangle \quad \forall \delta\mathbf{u}_h \in \mathscr{L}_h \\ ((\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h, \delta\mathbf{g}_h)) = 0 \quad \forall \delta\mathbf{g}_h \in \mathscr{D}_h \\ ((\delta\boldsymbol{\sigma}_h, \mathbf{g}_h)) = 0 \quad \forall \delta\boldsymbol{\sigma}_h \in \mathscr{S}_h \end{cases} \tag{10}$$

The relations (10) provide the discrete equilibrium, elastic equations and the compatibility conditions. It is convenient to consider $(10)_3$ as a constraint condition for the discrete problem in which the discrete stresses play the role of Lagrangian multipliers. This constraint amounts to require that $\mathbf{g}_h \in \tilde{\mathscr{D}}_h = \mathscr{D}_h \cap \mathscr{S}_h^{\perp}$. The $\{\mathbf{u}_h, \mathbf{g}_h\}$ solution of problem (10) can be obtained by solving the following reduced problem in which the strain gap variations meet the constraint condition:

$$\begin{cases} ((\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h),\ \mathbf{B}\delta\mathbf{u}_h)) = \langle \bar{\ell}, \delta\mathbf{u}_h \rangle \quad \forall \delta\mathbf{u}_h \in \mathscr{L}_h \\ ((\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h),\ \delta\mathbf{g}_h)) = 0 \quad \forall \delta\mathbf{g}_h \in \tilde{\mathscr{D}}_h \end{cases} \tag{11}$$

Once the solution $\{\mathbf{u}_h, \mathbf{g}_h\}$ of Equation (11) has been obtained, the discrete stresses $\boldsymbol{\sigma}_h \in \mathscr{S}_h$ can be evaluated by solving the stress recovery problem

$$((\mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h) - \boldsymbol{\sigma}_h, \delta\mathbf{g}_h)) = 0 \quad \forall \delta\mathbf{g}_h \in \mathscr{D}_h \tag{12}$$

This condition involves a number of equations which is larger than the number of unknown stress parameters. Nevertheless, the stress recovery problem (12) admits a unique solution if the well-posedness requirement $\mathscr{S}_h \cap \mathscr{D}_h^{\perp} = \{\mathbf{0}\}$ is fulfilled, since then the number of independent equations is equal to the number of unknowns.

The stress recovery (12) can be interpreted in geometrical terms as a projection procedure: to get the approximate stress $\boldsymbol{\sigma}_h \in \mathscr{S}_h$, the field $(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ must be projected on the subspace $\mathscr{S}_h$ with a projection orthogonal in elastic energy to the subspace $\mathscr{D}_h$.

4.1. *The elastic stress recovery*

From the computational standpoint, the most convenient stress recovery consists in computing the discrete stresses at the element level according to the elastic constitutive relation $\boldsymbol{\sigma}_h = \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ once the reduced problem has been solved.

Some authors [7, 11, 12] claimed that this simple computation is not variationally consistent but the following direct argument leads to the opposite conclusion.

Let us preliminarily observe that the well-posedness conditions require that any admissible choice of the subspaces $\mathscr{S}_h$ and $\mathscr{D}_h$ must fulfill the following three rules: (a) $\mathscr{S}_h \cap \mathscr{D}_h^{\perp} = \{\mathbf{0}\}$, (b) $\mathscr{D}_h \cap \mathscr{S}_h^{\perp} \neq \{\mathbf{0}\}$, (c) *constant stress fields must be included in* $\mathscr{S}_K$. Conditions (a) and (b) can always be satisfied by setting $\mathscr{D}_h = \mathscr{S}_h \oplus \tilde{\mathscr{D}}_h$ since the choice of $\mathscr{D}_h$ is not subjected to other conditions. The subspace $\tilde{\mathscr{D}}_h$ is defined as the linear span of shape functions with zero mean values. This choice is motivated by the orthogonality condition $\tilde{D}_h \subseteq \mathscr{S}_h^{\perp}$ since $\mathscr{S}_h$ must fulfil the condition (c).

Condition (a) ensures uniqueness and convergence of the approximate stress solution as will be discussed in the next section. Condition (b) is necessary in order to get an enhanced flexibility since otherwise the mixed method would collapse into the standard displacement method. Condition (c) is also motivated by convergence requirements (see next section).

To prove the variational consistency of the elastic stress recovery, we consider the stress subspace $\bar{\Sigma}_h$ composed by the stress fields $\bar{\boldsymbol{\sigma}}_h = \mathbf{E}(\mathbf{B}\bar{\mathbf{u}}_h - \bar{\mathbf{g}}_h)$ with $\bar{\mathbf{u}}_h \in \mathscr{V}_h$ and $\bar{\mathbf{g}}_h \in \tilde{\mathscr{D}}_h$. The elastic stress is $\boldsymbol{\sigma}_h^{\mathrm{E}} = \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ with $\{\mathbf{u}_h, \mathbf{g}_h\}$ solution of the reduced problem. It is then apparent that $\boldsymbol{\sigma}_h^{\mathrm{E}} \in \bar{\Sigma}_h$. Moreover, by condition $(4.2)_2$, we have also that $\boldsymbol{\sigma}_h^{\mathrm{E}} \in \tilde{\mathscr{D}}_h^{\perp}$. Then $\boldsymbol{\sigma}_h^{\mathrm{E}}$ belongs to the subspace

$$\Sigma_h = \{\bar{\boldsymbol{\sigma}}_h \in \bar{\Sigma}_h\colon\ ((\bar{\boldsymbol{\sigma}}_h, \delta\mathbf{g}_h)) = 0\ \forall \delta\mathbf{g}_h \in \tilde{\mathscr{D}}_h\} = \bar{\Sigma}_h \cap \tilde{\mathscr{D}}_h^{\perp}$$

We can then define the stress subspace to be $\mathscr{S}_h = \Sigma_h \oplus \mathscr{S}_h^{*}$ where $\mathscr{S}_h^{*}$ is any subspace included in $\tilde{\mathscr{D}}_h^{\perp}$. As a consequence $\Sigma_h \subseteq \mathscr{S}_h$ and $\mathscr{S}_h \subseteq \tilde{\mathscr{D}}_h^{\perp}$.

Being $\boldsymbol{\sigma}_h^{\mathrm{E}} \in \Sigma_h \subseteq \mathscr{S}_h$, the projection procedure (12) of the stress recovery problem yields trivially $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h^{\mathrm{E}}$.

The variationally consistency of the elastic stress recovery has been recently claimed in Reference [13] by an argument explicitly limited to undistorted meshes and based on a matricial formulation.

As a consequence of the choice $\mathscr{D}_h = \mathscr{S}_h \oplus \tilde{\mathscr{D}}_h$, the stress recovery strategy (12) reduces to an orthogonal projection of $\boldsymbol{\sigma}_h^{\mathrm{E}} = \mathbf{E}(\mathbf{B}\mathbf{u}_h - \mathbf{g}_h)$ on the subspace $\mathscr{S}_h$:

$$((\boldsymbol{\sigma}_h^{\mathrm{E}} - \boldsymbol{\sigma}_h, \delta\boldsymbol{\sigma}_h)) = 0 \quad \forall \delta\boldsymbol{\sigma}_h \in \mathscr{S}_h$$

In fact, the condition containing the variations $\delta\mathbf{g}_h \in \tilde{\mathscr{D}}_h$ are identically satisfied by the virtue of $(11)_2$.

Several non-variational stress recovery procedures have been envisaged in the literature to complement the enhanced strain method. The one proposed in Reference [1] performs the orthogonal projection in the elastic energy of the discrete field $\mathbf{E}\mathbf{B}\mathbf{u}_h$ on a subspace $\mathscr{S}_h$ fulfilling the condition $\mathscr{S}_h \subseteq \mathscr{D}_h^{\perp}$. This stress recovery is not variationally consistent since it cannot be deduced from a projection procedure of the type (12).

5. CONVERGENCE PROPERTIES

The convergence of the approximate solution provided by the SGM method to the exact one requires that the discrete problems must be *uniformly well-posed* with respect to the mesh size parameter h.

In geometrical terms, this requirement consists in assessing that the subspaces $\mathbf{B}\mathscr{V}_h$ and $\tilde{\mathscr{D}}_h$ must not tend to become parallel and that the subspaces $\mathscr{S}_h$ and $\mathscr{D}_h$ must not tend to become orthogonal as h goes to zero. In topological terms, these two conditions can be stated by requiring that the subspaces $\mathbf{B}\mathscr{L}_h + \tilde{\mathscr{D}}_h$ and $\mathscr{S}_h + \mathscr{D}_h^{\perp}$ must be uniformly closed in $\mathscr{H}$ with respect to the parameter h and, in turn, uniform closedness is equivalent to the following inequalities [15]:

$$\begin{aligned} \|\Pi_{\mathbf{B}\mathscr{L}_h}\mathbf{g}_h\|_{\mathscr{H}} &\leqslant \theta\|\mathbf{g}_h\|_{\mathscr{H}} \quad \forall \mathbf{g}_h \in \tilde{\mathscr{D}}_h \;\; \theta < 1 \\ \|\Pi_{\mathscr{D}_h}\boldsymbol{\sigma}_h\|_{\mathscr{H}} &\geqslant c\|\boldsymbol{\sigma}_h\|_{\mathscr{H}} \quad \forall \boldsymbol{\sigma}_h \in \mathscr{S}_h \;\; c > 0 \end{aligned} \tag{13}$$

with θ and c independent of h. The symbols $\Pi_{\mathbf{B}\mathscr{L}_h}$ and $\Pi_{\mathscr{D}_h}$ denote the orthogonal projectors on the subspaces $\mathbf{B}\mathscr{L}_h$ and $\mathscr{D}_h$ and $\|\cdot\|_{\mathscr{H}}$ is the norm in $\mathscr{H}$.

Being $\mathscr{S}_h \subset \mathscr{D}_h$, we have $\|\Pi_{\mathscr{D}_h}\boldsymbol{\sigma}_h\|_{\mathscr{H}} = \|\boldsymbol{\sigma}_h\|_{\mathscr{H}}$ so that the inequality $(5.1)_2$ is trivially fulfilled with $c = 1$. A sufficient local condition for Equation $(5.1)_1$ can be obtained by substituting $\mathscr{L}_h$ with the non-conforming displacement subspace $\mathscr{V}_h \supseteq \mathscr{L}_h$ to get:

$$\|\Pi_{\mathbf{B}\mathscr{V}_h}\mathbf{g}_h\|_{\mathscr{H}} \leqslant \theta\|\mathbf{g}_h\|_{\mathscr{H}} \quad \forall \mathbf{g}_h \in \tilde{\mathscr{D}}_h \;\; \theta < 1$$

If this condition is fulfilled, the following error estimate holds

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_1 \leqslant \beta \left(\inf_{\bar{\mathbf{u}}_h \in \mathscr{L}_h} \|\mathbf{u} - \bar{\mathbf{u}}_h\|_1 + \inf_{\bar{\boldsymbol{\sigma}}_h \in \mathscr{S}_h} \|\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_h\|_0 \right)$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ are the norms in the Sobolev spaces $H^0(\Omega) = L^2(\Omega)$ and $H^1(\Omega)$ [15].

A linear rate of convergence is ensured by polynomial interpolation theory [27] if the displacement shape functions can reproduce any polynomial of degree $\leqslant 1$ and the stress shape functions can reproduce any constant tensor field. More precisely, the subspace $\mathscr{V}_K$ generated by the displacement shape functions on the reference element K must fulfill the property

$\mathscr{V}_K \supseteq P_1(K)$ for simplicial elements or $\mathscr{V}_K \supseteq Q_1(K)$ for n–cubes. As usual $P_1(K)$ denotes the subspace of polynomials of degree $\leqslant 1$ and $Q_1(K)$ denotes the subspace of polynomials of degree $\leqslant 1$ separately in each variable.

The stress subspace $\mathscr{S}_K$ must fulfill the constant stress condition $\mathscr{S}_K \supseteq P_0(K) = Q_0(K)$. This condition was also recognized to be necessary for convergence in Reference [1] by appealing to the patch test. The effective strain gaps must then have a null mean value, a property also referred to in Reference [1]. Under these assumptions the following linear estimates for the rate of convergence hold [27]:

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leqslant c_{\mathbf{u}} h(|\mathbf{u}|_2 + |\boldsymbol{\sigma}|_1), \quad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leqslant c_{\boldsymbol{\sigma}} h(|\mathbf{u}|_2 + |\boldsymbol{\sigma}|_1)$$

provided that the solution is smooth in the sense that $\mathbf{u} \in H^2(\Omega)$ and $\boldsymbol{\sigma} \in H^1(\Omega)$. Here the symbol $|\cdot|_m$ is the seminorm in the Sobolev space $H^m(\Omega)$ involving only derivatives of total order m.

6. UNIFORM WELL-POSEDNESS

In the FE analysis the local sufficient conditions for the uniform well-posedness of the SGM

$$\tilde{\mathscr{D}}_h \cap \mathbf{B}\mathscr{V}_h = \{\mathbf{0}\}, \quad \tilde{\mathscr{D}}_h + \mathbf{B}\mathscr{V}_h \textit{ uniformly closed in } \mathscr{H} \tag{14}$$

have to be verified on the reference element K in terms of shape functions, that is

$$\tilde{\mathscr{D}}_K \cap \mathbf{B}^e_K\mathscr{V}_K = \{\mathbf{0}\}, \quad \tilde{\mathscr{D}}^e_K + \mathbf{B}^e_K\mathscr{V}_K \textit{ uniformly closed in } \mathscr{H} \tag{15}$$

The kinematic operator $\mathbf{B}^e_K$, which acts on the fields $\mathbf{u}_K$ in the reference element, is defined by $\mathbf{B}^e_K\mathbf{u}_K(\chi_e^{-1}(\mathbf{x})) = \mathbf{B}\mathbf{u}(\mathbf{x})$ for any $\mathbf{x} \in \Omega_e$.

In the case of undistorted elements $\mathbf{B}^e_K$ and $\mathbf{B}$ are proportional through the mesh size h, i.e. $\mathbf{B}^e_K = h\mathbf{B}$. This proportionality implies that the subspace $\mathbf{B}^e_K\mathscr{V}_K$ is equal to $\mathbf{B}\mathscr{V}_K$ and hence is independent of h. The condition $\tilde{\mathscr{D}}_K \cap \mathbf{B}^e_K\mathscr{V}_K = \tilde{\mathscr{D}}_K \cap \mathbf{B}\mathscr{V}_K = \{\mathbf{0}\}$ can be checked by evaluating the Gram determinant [28] of a set of shape functions spanning the subspace $\tilde{\mathscr{D}}_K \times \mathbf{B}\mathscr{V}_K$. The Gram determinant is in fact positive if and only if this set of shape functions is linearly independent [29]. Remarkably, the uniform closedness condition concerning $\tilde{\mathscr{D}}_K + \mathbf{B}^e_K\mathscr{V}_K$ is trivially fulfilled since $\mathbf{B}^e_K\mathscr{V}_K = \mathbf{B}\mathscr{V}_K$ is independent of h.

For general isoparametric maps, the condition (15) cannot be checked in the reference element, a drawback which seems to have been overridden in previous analyses [1, 11].

7. THE DISCRETE PROBLEM

Let us assume a strain gap interpolation in the reference element of the form

$$\mathscr{D}_K = \mathscr{S}_K \oplus \tilde{\mathscr{D}}_K \tag{16}$$

where $\tilde{\mathscr{D}}_K \subseteq \mathscr{S}_K^{\perp}$ is the subspace of effective strain gaps and the symbol $\oplus$ denotes the direct sum.

Then $\mathscr{D}_h = \mathscr{S}_h \oplus \tilde{\mathscr{D}}_h$ but in general we do not have $\tilde{\mathscr{D}}_h \subseteq \mathscr{S}_h^{\perp}$ unless the Jacobians of the isoparametric maps are constant. As a consequence, in the general case, it is not possible to split the variational problem into a sequence of a reduced problem and of a stress recovery projection.

Let us then develop hereafter a general formulation of the SGM in which the orthogonality constraint $\tilde{\mathscr{D}}_h \subseteq \mathscr{S}_h^{\perp}$ is not satisfied *a priori* but enters as one of the equations of the discrete problem. This formulation is a useful tool in detecting the computational performances relevant to different implementation of the method and will be referred to in the numerical examples discussed in Section 8.

The well-posedness condition $\mathscr{S}_h \cap \mathscr{D}_h^{\perp} = \{\mathbf{0}\}$ is equivalent to the non-singularity of the matrix $\mathbf{Q}$ as defined in Equation (6). The property $\operatorname{Ker} \mathbf{Q} = \{\mathbf{0}\}$ is crucial for the derivation of the element stiffness matrix. We can in fact eliminate the stress and the strain gap parameters, at the element level, according to the procedure

$$\begin{cases} \mathbf{H}^e \mathbf{p}_{\mathbf{g}}^e = \mathbf{G}^e \mathbf{p}_{\mathbf{u}}^e - \mathbf{Q}^e \mathbf{p}_{\boldsymbol{\sigma}}^e \\ (\mathbf{Q}^{e\mathrm{T}} \mathbf{H}^{e-1} \mathbf{Q}^e) \mathbf{p}_{\boldsymbol{\sigma}}^e = \mathbf{Q}^{e\mathrm{T}} \mathbf{H}^{e-1} \mathbf{G}^e \mathbf{p}_{\mathbf{u}}^e \\ [\mathbf{K}^e + \mathbf{G}^{e\mathrm{T}} \mathbf{H}^{e-1} (\mathbb{H}^e - \mathbf{H}^e) \mathbf{H}^{e-1} \mathbf{G}^e] \mathbf{p}_{\mathbf{u}}^e = \mathbf{f}_{\mathbf{u}}^e \end{cases}$$

where $\mathbb{H}^e = \mathbf{Q}^e (\mathbf{Q}^{e\mathrm{T}} \mathbf{H}^{e-1} \mathbf{Q}^e)^{-1} \mathbf{Q}^{e\mathrm{T}}$. Denoting the element stiffness matrix by

$$\mathbf{S}^e = \mathbf{K}^e + \mathbf{G}^{e\mathrm{T}} \mathbf{H}^{e-1} (\mathbb{H}^e - \mathbf{H}^e) \mathbf{H}^{e-1} \mathbf{G}^e \tag{17}$$

the mixed problem at the element level can be written as $\mathbf{S}^e \mathbf{p}_{\mathbf{u}}^e = \mathbf{f}_{\mathbf{u}}^e$. The global problem is then expressed in terms of nodal displacement parameters as

$$\mathbb{K} \mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathscr{N}} \left(\mathscr{A}_{\mathbf{u}}^{e\mathrm{T}} \mathbf{K}^e \mathscr{A}_{\mathbf{u}}^e \right), \quad \mathbf{q}_{\mathbf{u}} = \sum_{e=1}^{\mathscr{N}} \mathscr{A}_{\mathbf{u}}^{e\mathrm{T}} \mathbf{f}_{\mathbf{u}}^e = \mathbf{f}_{\mathbf{u}}$$

We underline that, due to the positive definiteness of $\mathbf{H}^e$ and the non-singularity of $\mathbf{Q}^e$, the matrix $\mathbf{Q}^{e\mathrm{T}} \mathbf{H}^{e-1} \mathbf{Q}^e$ is invertible. Once the global structural problem has been solved in terms of nodal displacements, the stress and the strain parameters can be evaluated at the element level by following the elimination procedure backwards:

$$\begin{aligned} \mathbf{p}_{\boldsymbol{\sigma}}^e &= \left(\mathbf{Q}^{e\mathrm{T}} \mathbf{H}^{e-1} \mathbf{Q}^e \right)^{-1} \mathbf{Q}^{e\mathrm{T}} \mathbf{H}^{e-1} \mathbf{G}^e \mathbf{p}_{\mathbf{u}}^e \\ \mathbf{p}_{\mathbf{g}}^e &= \mathbf{H}^{e-1} (\mathbf{G}^e \mathbf{p}_{\mathbf{u}}^e - \mathbf{Q}^e \mathbf{p}_{\boldsymbol{\sigma}}^e) \end{aligned} \tag{18}$$

If the Jacobian determinant is introduced in the definition of the strain gaps according to formula (8), the property $\tilde{\mathscr{D}}_h \subseteq \mathscr{S}_h^{\perp}$ is preserved by the isoparametric map. In this case, a more convenient computation strategy, based on the reduced problem (11) and the stress recovery projection (12), can be exploited.

7.1. Reduced problem and stress recovery

Let us now derive the expression of the element stiffness matrix and of the stress recovery for the SGM. To solve the reduced problem (11), we preliminary note that the discrete strain gaps belonging to the subspace $\tilde{\mathscr{D}}_h$ are such that the corresponding parameters belong to $\operatorname{Ker} \mathbf{Q}^{e\mathrm{T}}$, see also the equivalence (A.6) reported in the appendix.

The reduced problem (11) at the element level reduces to the following matrix form:

$$\begin{bmatrix} \mathbf{K}^e_{\mathbf{uu}} & -\mathbf{G}^{e\mathrm{T}}_{\alpha\mathbf{u}} \\ -\mathbf{G}^e_{\alpha\mathbf{u}} & \mathbf{H}^e_{\alpha\alpha} \end{bmatrix} \begin{vmatrix} \mathbf{p}^e_{\mathbf{u}} \\ \mathbf{p}^e_{\alpha} \end{vmatrix} = \begin{vmatrix} \mathbf{f}^e_{\mathbf{u}} \\ \mathbf{0} \end{vmatrix} \tag{19}$$

where $\mathbf{p}^e_{\alpha}$ is the element effective strain gap parameter.

The element stiffness matrix $\mathbf{S}^e$ is then

$$\mathbf{S}^e = \mathbf{K}^e_{\mathbf{uu}} - \mathbf{G}^{e\mathrm{T}}_{\alpha\mathbf{u}} \mathbf{H}^{e-1}_{\alpha\alpha} \mathbf{G}^e_{\alpha\mathbf{u}} \tag{20}$$

so that the problem at the element level can be written as $\mathbf{S}^e\mathbf{p}^e_{\mathbf{u}} = \mathbf{f}^e_{\mathbf{u}}$.

Once the global problem has been solved in terms of nodal parameters, the effective strain gap parameters can then be computed from (19) in the form:

$$\mathbf{p}^e_{\alpha} = \mathbf{H}^{e-1}_{\alpha\alpha} \mathbf{G}^e_{\alpha\mathbf{u}} \mathbf{p}^e_{\mathbf{u}} \tag{21}$$

At this point, the variationally consistent stress recovery (12) can be pursued to get

$$-\mathbf{G}^e_{\sigma\mathbf{u}}\mathbf{p}^e_{\mathbf{u}} + \mathbf{H}^e_{\sigma\alpha}\mathbf{p}^e_{\alpha} + \mathbf{Q}^e_{\sigma\sigma}\mathbf{p}^e_{\sigma} = \mathbf{0}$$

and recalling the expression (21) of $\mathbf{p}^e_{\alpha}$ we have

$$\mathbf{p}^e_{\sigma} = \mathbf{Q}^{e-1}_{\sigma\sigma}(\mathbf{G}^e_{\sigma\mathbf{u}} - \mathbf{H}^e_{\sigma\alpha}\mathbf{H}^{e-1}_{\alpha\alpha}\mathbf{G}^e_{\alpha\mathbf{u}})\mathbf{p}^e_{\mathbf{u}} \tag{22}$$

It is worthnoting that the reduced problem (19) coincides with the matrix formulation of the EAS method where the enhanced strains of the EAS method coincide with the effective strain gaps of the SGM to within an irrelevant change of sign. The well-posedness of the SGM provides the variationally consistent stress recovery (22).

8. COMPUTATIONAL ANALYSIS

Let us preliminarily consider some shape functions adopted in the literature in the context of the EAS method for plane problems with reference to a standard four-node bilinear isoparametric square element $\square = [-1,1]\times[-1,1]$.

A five-parameter interpolation for the strain gap field is provided in Reference [1] starting from the six-parameter strain interpolation of Wilson *et al*. incompatible element. The shape functions for the strain gaps are

$$\mathbf{N}^{\square}_{\mathbf{g}} = \begin{bmatrix} \xi & 0 & 0 & 0 & \xi\eta \\ 0 & \eta & 0 & 0 & -\xi\eta \\ 0 & 0 & \xi & \eta & \xi^2-\eta^2 \end{bmatrix} \tag{23}$$

Note that, deleting the last column of $\mathrm{N}^{\square}_{\mathbf{g}}$, we obtain the shape functions pertaining to the modified incompabile mode approximation of Taylor *et al.* [30]

$$\hat{\mathbf{N}}^{\square}_{\mathbf{g}} = \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix} \tag{24}$$

A seven-parameter strain gap interpolation has been assumed in References [7, 8] which is given by

$$\mathbf{N}_{\mathbf{g}}^{\square} = \begin{bmatrix} \xi & 0 & 0 & 0 & \xi\eta & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & \xi\eta & 0 \\ 0 & 0 & \xi & \eta & 0 & 0 & \xi\eta \end{bmatrix} \tag{25}$$

Over the standard four-node isoparametric element $\square$, the strain gap shape functions (23)–(25) and the Pian and Sumihara stress shape functions [31] given by

$$\mathbf{N}_{\boldsymbol{\sigma}}^{\square} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{26}$$

are apparently mutually orthogonal in the $L^2(\square)$ inner product.

According to the decomposition (16), the effective strain gap interpolation of the SGM is given by Equation (23). The strain gap shape functions are the collection of Equations (26) and (23).

The well-posedness condition requires that compatible and effective strain gap shape functions must be linear independent. Noting that the compatible strain subspace $\mathbf{B}\mathscr{V}_{\square}$ is given by

$$\mathbf{B}\mathscr{V}_{\square} = \operatorname{span} \begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & \xi & \eta \end{bmatrix}$$

well-posedness can be checked by considering the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{15}\}$, which represent the columns of the set $\{\mathbf{B}\mathscr{V}_{\square}, \mathbf{N}_{\mathbf{g}}^{\square}\}$ and imposing that the Gram matrix $G_{ij} = \int_{\square} \mathbf{a}_i \cdot \mathbf{a}_j$ is not singular.

8.1. Numerical examples

The numerical performances of the SGM are evaluated with reference to some discriminating examples selected from the literature and compared with the EAS, the Hellinger–Reissner (HR) and the standard displacement methods. All the examples are two-dimensional, linearly elastic isotropic and in plane stress state. A square four-node isoparametric element is adopted.

We begin by considering the results for the tapered cantilever, commonly known as the Cook membrane problem. The values $E = 250$ and $v = 0.4999$ for Young's modulus and Poisson's ratio are used such that a nearly incompressible response is effectively obtained, as reported in Reference [26]. A uniformly distributed in-plane shearing load with total value 100 is applied on the free end. Figure 1 shows a graph of the vertical tip deflection obtained by adopting the SGM, the EAS, the HR and the standard displacement methods. The superior coarse mesh accuracy achieved with the various enhanced strain or assumed stress elements is apparent.

The aim of this example is to illustrate the computational performances relevant to different implementations of the discrete method. In particular, the SGM0 plot shows the results obtained following the general formulation, according to formulae (17) and (18), in which

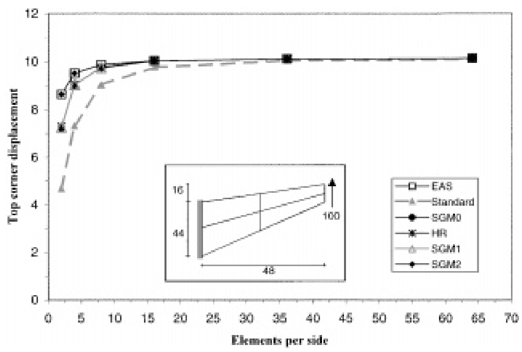


Figure 1. Vertical tip displacement for the Cook membrane problem.

the orthogonality constraint is considered as one of the equations of the discrete mixed problem. The SGM1 and SGM2 plots are obtained by adopting the formulation which include the Jacobian determinant (8) and the push-transformation (9), respectively.

The results provided by the SGM0 and SGM1 are similar showing that the variability of the Jacobian determinant (8) has only minor influence on the numerical performances.

The two-field HR method provides comparably good results for coarse meshes and the same convergence behaviour of the SGM and EAS method. The displacement method exhibits a rather poor performance for coarse meshes but no locking phenomenon is shown in contrast with the example reported in Reference [26].

In Figures 2(a)–2(c), the normal and tangential stresses σ_x, σ_y and τ_{xy} at the node A are plotted. The elastic stress recovery is reported in the SGM3 plot and turns out to be numerically similar to the EAS stress recovery. Figures 2(a) and 2(b) clearly shows a poor performance of the SGM and EAS method for coarse meshes.

The next test example of Figure 3 consists of a rectangular plate constrained at one end and subjected to a uniformly distributed shearing load, with intensity 100, at the other. The values of Young's modulus and Poisson's ratio are $E = 1500$ and $v = 0.25$. Since the mesh is not distorted, the Jacobian determinant and the push-transformation has no influence on the numerical performance of the SGM and of the EAS method which coincides with the HR method according to a limitation phenomenon. A detailed analysis can be found in References [11, 12, 32].

As a further example, we analyse a classical benchmark [16, 13] consisting in the bending problem of the rectangular plate reported in Figure 4 to address the issue of sensitivity to mesh distorsions. The values $E = 1500$ and $v = 0.25$ for Young's modulus and Poisson's ratio are used.

A two-element mesh is considered for the plate which is constrained at one end and is subjected to a linearly distributed axial load, equivalent to a couple with value 2000, at the other. The analytical solution in terms of displacements and stresses is

$$\begin{aligned} u(x,y) &= 2x(1-y) \\ v(x,y) &= x^2 + \tfrac{1}{4}(y^2 - 2y) \end{aligned} \quad \text{and} \quad \sigma_x(x,y) = 3000(1-y), \;\; \sigma_y(x,y) = \tau_{xy}(x,y) = 0$$

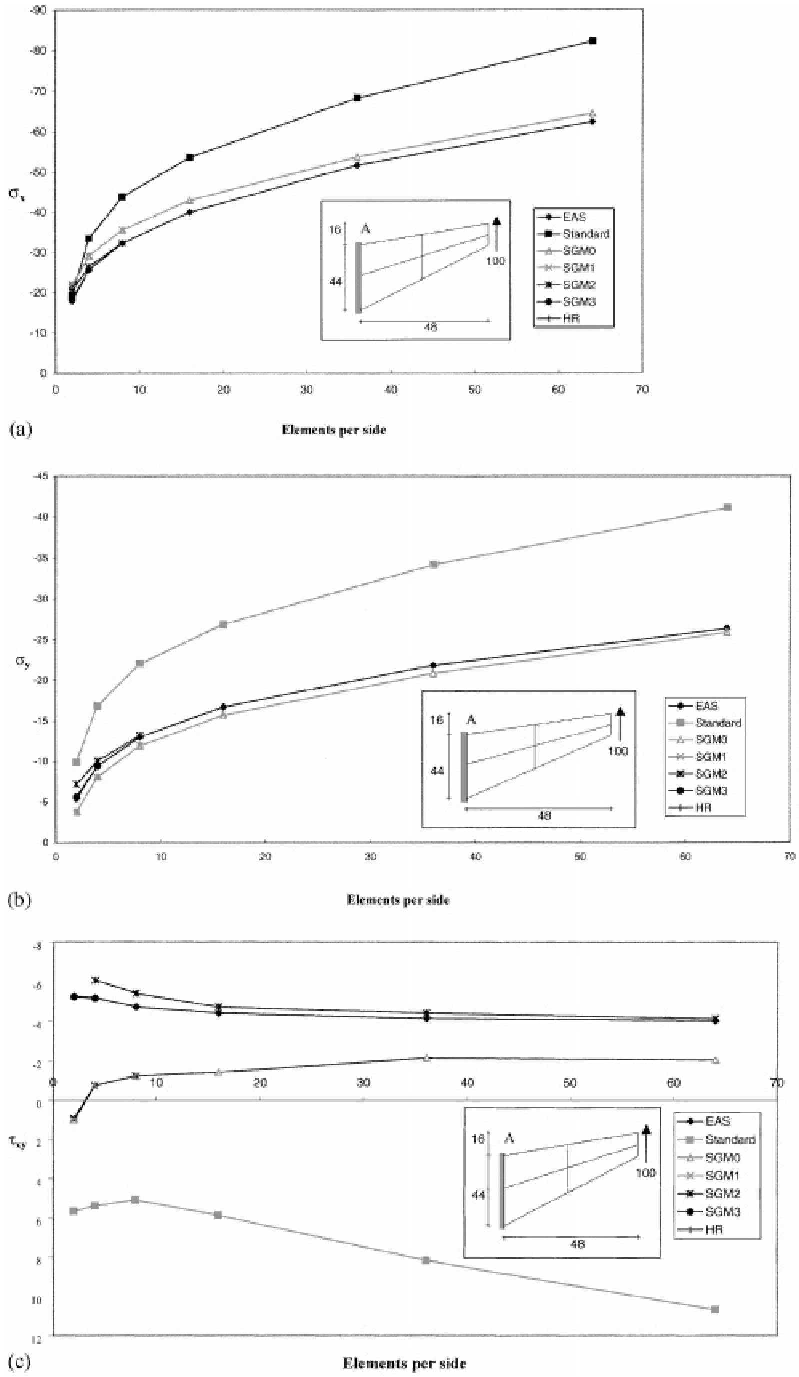


Figure 2. Stress results for the Cook membrane problem: (a) axial stress; (b) vertical stress; (c) tangential stress.

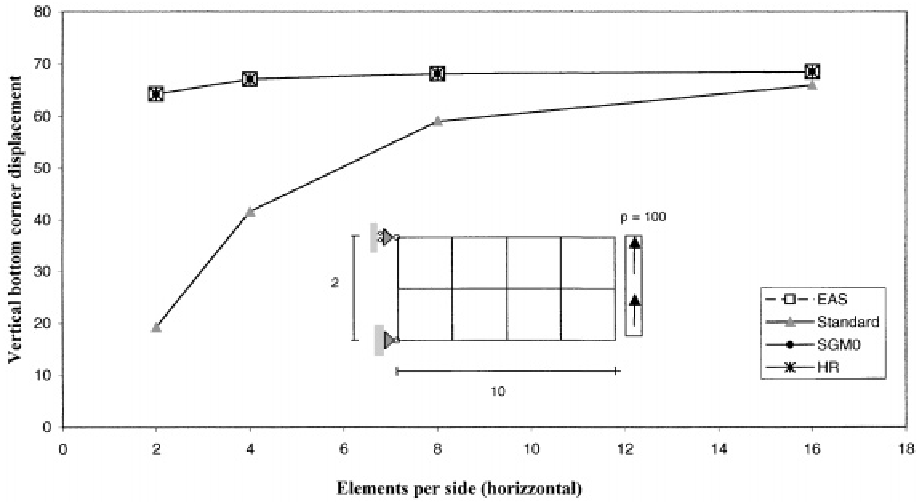


Figure 3. Vertical displacement of the bottom corner point for a cantilever rectangular plate subjected to a shearing force.

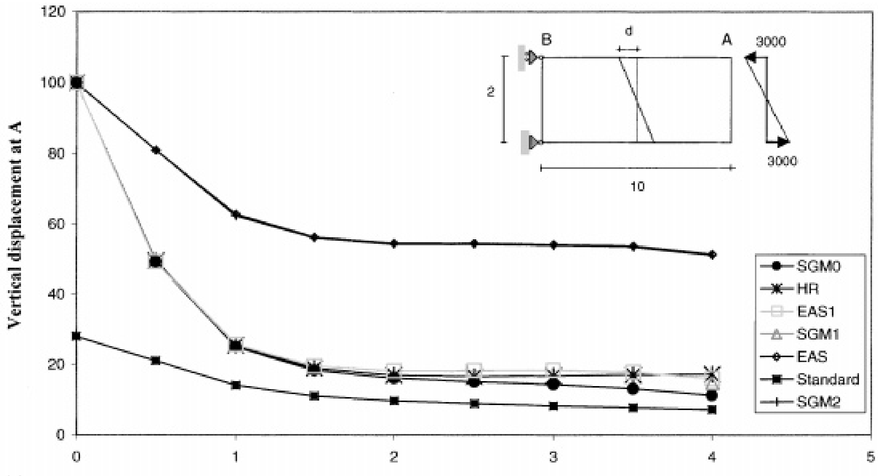
Figure 4(a) gives the results for the vertical displacement of the point A measured against the distorsion parameter d. It is evident that the push-transformation has a beneficial effect. In Figure 4(b), we report the vertical displacement of the point B measured against the distorsion parameter d. In this case, the EAS method with the push-transformation and the SGM2 have the worst performance since the difference between the discrete and the exact null solution increases with the distortion. On the contrary, the SGM0, SGM1 and the HR method provide more reliable results.

Further results concerning the normal and tangential stresses at the point B are reported in the Figures 5(a)–5(c). In terms of axial stress, the SGM2 and the EAS shows a less distortion sensitivity. In terms of vertical stress the SGM, the EAS and the HR methods give similar results. The SGM2 and the EAS method have the better performance in terms of the tangential stress.

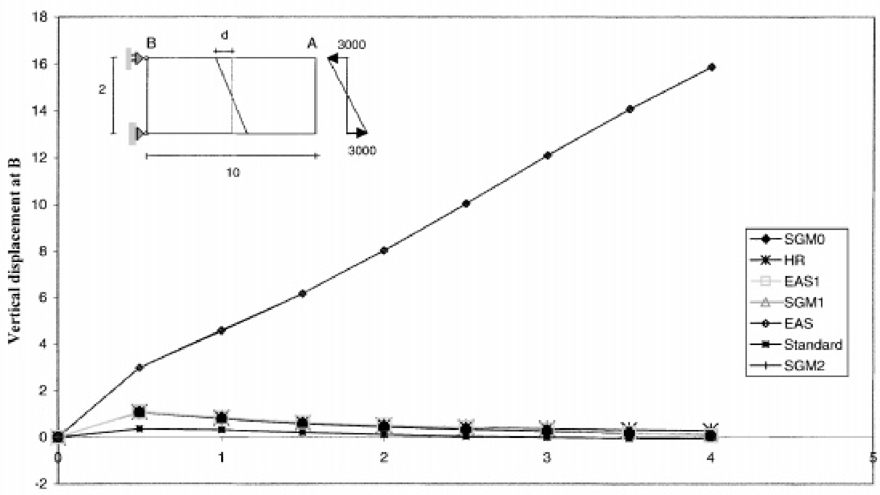
9. CONCLUDING REMARKS

Since the original formulation of the EAS method proposed by Simo and Rifai in Reference [1], the theoretical aspects and the computational performances of this method have been investigated in a number of subsequent papers (see e.g. References [6, 16, 13, 12]). All these treatments rely on the basic assumption, made in Reference [1], concerning the mutual orthogonality between stress and enhanced strain subspaces.

Seemingly, the merit of this orthogonality assumption is that the discrete stress fields are eliminated from the problem. A displacement-like formulation is thus obtained in which a modified stiffness operator provides an enhanced flexibility and better numerical performances with coarse meshes.



(a)



(b)

Figure 4. Vertical displacements for a cantilever rectangular plate subjected to a couple: (a) for the point A; (b) for the point B.

In reality, as a consequence of this assumption, two main troubles become apparent. The first one concerns with the evaluation of the discrete stress field, an issue which has been longly debated with several proposal [11–13].

The analysis carried out in this paper, which is based on the standard theoretical treatment of mixed methods [14], reveals that the three-field discrete problem can always be split into a sequence of two steps: (a) the resolution of a reduced problem in which the orthogonality constraint between stresses and strain gaps is imposed and (b) a stress recovery problem which is analogous to the evaluation of the Lagrangian multiplier relevant to the constraint. Such a decomposition requires no *a priori* orthogonality assumption.

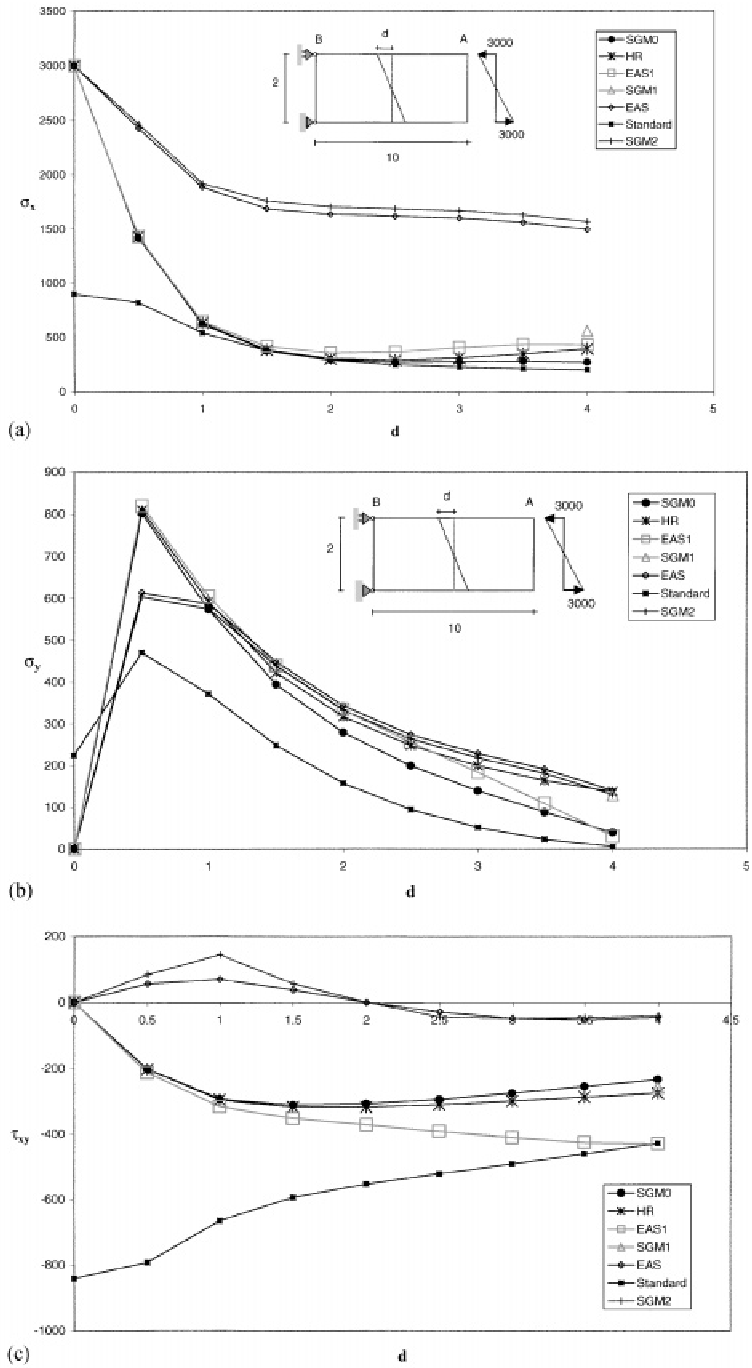


Figure 5. Stress results at the node A for a cantilever rectangular plate subjected to a couple: (a) axial stress; (b) vertical stress; (c) tangential stress.

In the EAS method only the reduced problem was discussed since the orthogonality constraint was assumed to be identically satisfied. Such an assumption is in contrast with the well-posedness requirement concerning the uniqueness of the solution in terms of discrete stress field. It is thus clear that the troubles faced with the theoretical assessment of the EAS method and in envisaging an *a posteriori* stress recovery strategy are motivated by the following consideration: the discrete variational problem of the EAS method is partially *ill-posed* due to the orthogonality assumption which leads to a complete indeterminancy of the discrete stress solution.

Another observation concerns with the peculiar convergence properties of the EAS method. In fact, since the exact constraint imposes the vanishing of the enhanced strain at the solution, it is apparent that no interpolation properties are required to the discrete space of enhanced strains.

The convergence analysis developed in References [10, 16] with reference to the EAS method was instead based on interpolation properties of the displacement shape functions and on a special orthogonality requirement of the enhanced strain subspace to polynomials of degree $k-1$ if k is the degree of the displacement polynomial interpolation.

This spourious requirement is in apparent contradiction with the previous observation that any discrete subspace of enhanced strain fields includes the null field (i.e. the exact strain gap solution).

In fact, there is no variational motivation for this orthogonality requirement and it seems that the authors of References [10, 16] were compelled to an *escamotage* in order to prove the convergence of the reduced problem without resorting to the interpolation properties of the stress fields.

Such interpolation properties were anyway invoked in References [10, 16] to prove the convergence of their stress recovery strategy.

No problem arises if all the variational well-posedness conditions are respected and specifically if the discrete strain gaps effectively control the discrete stresses so that a unique discrete stress solution is ensured. This feature seems to have been overlooked in previous treatments of the EAS method [10, 16].

We conclude this comparison between the EAS method and the SGM with the following considerations. The original idea contributed in Reference [1] was far reaching. In fact, the choice of the strain gap as a basic unknown of the three-field problem has some peculiar and important consequences. Due to the simplicity of the constraint, the reduced problem can be formulated as a modified displacement method with an enhanced flexibility. No interpolation requirements are requested on the discrete strain gaps and this fact allows to adopt a non-polynomial interpolation whose orthogonality properties can be controlled on the reference element of an isoparametric FE mesh.

On the other hand, the simplicity of the constraint instigates to jump to the conclusion that the convergence of the reduced discrete problem is independent of the choice of the discrete stress space which, in fact, does not explicitly appear in the reduced problem.

A full comprehension of the method requires however to recognize that the basic role played by the stress interpolation is hidden in the orthogonality condition.

The main difference between the EAS method and the SGM stems from the circumstance that in the EAS method the orthogonality condition $\mathscr{S}_h \subset \mathscr{D}_h^{\perp}$ was imposed as an essential requirement of the method and hence a well-posedness condition of the mixed method was violated. On the contrary, the SGM, which is fully respectful of the well-posedness conditions, allows for a consistent variational computation of the stress parameters and leads to a consistent method of approximation which permits to get a full convergence result.

The numerical tests reveal that there is no apparent motivation to a preferential adoption of the SGM or EAS method in comparison with the two-field HR method. Further, special procedures, such as the push-transformation, have no variational ground, show better results in terms of displacements for coarse meshes but poorer results in terms of stress fields. In this respect, the local lack of convergence of the stress field shown in the Cook membrane problem reveal that further improvements are needed and that scientific and commercial codes are still not completely reliable for engineering purposes.

APPENDIX

To discuss the well-posedness of the discrete problem $\mathbb{P}_h$ (5), it is essential to provide a representation formula for the kernel of the matrix $\mathbf{M}$ in terms of the component submatrices. To this end let us first define the reduced matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & -\mathbf{G}^{\mathrm{T}} \\ -\mathbf{G} & \mathbf{H} \end{bmatrix} \tag{A.1}$$

obtained from the global matrix $\mathbf{M}$ and the associated bilinear form

$$a(\{\mathbf{q_u}, \mathbf{q_g}\}, \{\delta\mathbf{q_u}, \delta\mathbf{q_g}\}) = \mathbf{H}\mathbf{q_g} \cdot \delta\mathbf{q_g} - \mathbf{G}\mathbf{q_u} \cdot \delta\mathbf{q_g} - \mathbf{G}^{\mathrm{T}}\mathbf{q_g} \cdot \delta\mathbf{q_u} + \mathbf{K}\mathbf{q_u} \cdot \delta\mathbf{q_u} \tag{A.2}$$

We can now prove a preparatory result.

Proposition A.1 (*The quadratic form of the reduced matrix*). The positive quadratic form $a(\{\mathbf{q_u}, \mathbf{q_g}\})$ associated with $\mathbf{A}$ is given by

$$\begin{aligned} a(\{\mathbf{q_u}, \mathbf{q_g}\}) &= \mathbf{H}\mathbf{q_g} \cdot \mathbf{q_g} - 2\mathbf{G}\mathbf{q_u} \cdot \mathbf{q_g} + \mathbf{K}\mathbf{q_u} \cdot \mathbf{q_u} \\ &= \sum_{e=1}^{\mathscr{N}} \int_{\Omega_e} \mathbf{E}_* \left(\mathbf{N}^e_{\mathbf{g}} \mathscr{J}^e_{\mathbf{g}} \mathbf{q_g} - \mathbf{B}_* \mathbf{N}^e_{\mathbf{u}} \mathscr{A}^e_{\mathbf{u}} \mathbf{q_u}\right) \cdot \left(\mathbf{N}^e_{\mathbf{g}} \mathscr{J}^e_{\mathbf{g}} \mathbf{q_g} - \mathbf{B}_* \mathbf{N}^e_{\mathbf{u}} \mathscr{A}^e_{\mathbf{u}} \mathbf{q_u}\right) \end{aligned} \tag{A.3}$$

with $a(\{\mathbf{q_u}, \mathbf{q_g}\}) \geqslant 0$ for any $\mathbf{q_u}$ and $\mathbf{q_g}$, and its kernel is

$$\operatorname{Ker} a = \left\{\{\mathbf{q_u}, \mathbf{q_g}\}:\ \mathbf{N}^e_{\mathbf{g}} \mathscr{J}^e_{\mathbf{g}} \mathbf{q_g} - \mathbf{B}_* \mathbf{N}^e_{\mathbf{u}} \mathscr{A}^e_{\mathbf{u}} \mathbf{q_u} = 0,\ e = 1, \ldots, \mathscr{N}\right\} \tag{A.4}$$

Proof. The proposition is a direct consequence of the definitions of $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{K}$ and of the positive definiteness of the elastic matrix $\mathbf{E}_*$. □

To provide a representation of the kernel of the matrix $\mathbf{M}$ we preliminarily recall that, due to the positivity of $a(\{\mathbf{q_u}, \mathbf{q_g}\})$, the matrix $\mathbf{A}$ and the associated quadratic form $a(\mathbf{q_u}, \mathbf{q_g})$ have the same kernel [28], that is

$$\operatorname{Ker} \mathbf{A} = \operatorname{Ker} a \tag{A.5}$$

We are now ready to prove the next result.

Proposition A.2 (*Representation of the kernel of the matrix* M). The interpolation parameters which annihilate the response of the global matrix $\mathbf{M}$, that is $\{\mathbf{q_u}, \mathbf{q_g}, \mathbf{q_\sigma}\} \in \operatorname{Ker} \mathbf{M}$, are characterized by the property

$$\begin{cases} \mathbf{q_g} \in \operatorname{Ker} \mathbf{Q}^{\mathrm{T}} \\ \mathbf{N_g} \mathscr{J}^e_{\mathbf{g}} \mathbf{q_g} - \mathbf{B}_* \mathbf{N}^e_{\mathbf{u}} \mathscr{A}^e_{\mathbf{u}} \mathbf{q_u} = \mathbf{0}, \quad e = 1, \ldots, \mathscr{N} \\ \mathbf{q_\sigma} \in \operatorname{Ker} \mathbf{Q}. \end{cases}$$

Proof. Let $\{\mathbf{q_u}, \mathbf{q_g}, \mathbf{q_\sigma}\} \in \operatorname{Ker} \mathrm{M}$ then

$$\begin{cases} \text{(i)} \ \mathbf{K}\mathbf{q_u} - \mathbf{G}^{\mathrm{T}}\mathbf{q_g} = \mathbf{0} \\ \text{(ii)} \ -\mathbf{G}\mathbf{q_u} + \mathbf{H}\mathbf{q_g} + \mathbf{Q}\mathbf{q_\sigma} = \mathbf{0} \\ \text{(iii)} \ +\mathbf{Q}^{\mathrm{T}}\mathbf{q_g} = \mathbf{0} \end{cases}$$

From Equation (iii) we can infer $\mathbf{q_g} \in \operatorname{Ker} \mathbf{Q}^{\mathrm{T}}$.

By taking the dot product of (ii) by $\mathbf{q_g}$ we get $\mathbf{H}\mathbf{q_g} \cdot \mathbf{q_g} + \mathbf{Q}\mathbf{q_\sigma} \cdot \mathbf{q_g} - \mathbf{G}\mathbf{q_u} \cdot \mathbf{q_g} = 0$ so that, by means of equation (iii), it turns out to be $\mathbf{H}\mathbf{q_g} \cdot \mathbf{q_g} - \mathbf{G}\mathbf{q_u} \cdot \mathbf{q_g} = 0$. Moreover, by summing up this equation and equation (i) multiplied by $\mathbf{q_u}$, given by $\mathbf{K}\mathbf{q_u} \cdot \mathbf{q_u} - \mathbf{G}\mathbf{q_u} \cdot \mathbf{q_g} = 0$, we get $\mathbf{H}\mathbf{q_g} \cdot \mathbf{q_g} - 2\mathbf{G}\mathbf{q_u} \cdot \mathbf{q_g} + \mathbf{K}\mathbf{q_u} \cdot \mathbf{q_u} = 0$. Then the displacement and the strain gap parameters $\{\mathbf{q_u}, \mathbf{q_g}\}$ belong to the kernel of the bilinear form $a(\mathbf{q_u}, \mathbf{q_g})$ so that, from Proposition A.1, we have $\mathbf{N_g} \mathscr{J}^e_{\mathbf{g}} \mathbf{q_g} - \mathbf{B}_* \mathbf{N}^e_{\mathbf{u}} \mathscr{A}^e_{\mathbf{u}} \mathbf{q_u} = \mathbf{0}$ for $e = 1, \ldots, \mathscr{N}$.

Moreover, the displacement and the strain gap parameters $\{\mathbf{q_u}, \mathbf{q_g}\}$ belong to the kernel of the reduced matrix $\mathbf{A}$ so that we have

$$\begin{cases} \mathbf{K}\mathbf{q_u} - \mathbf{G}^{\mathrm{T}}\mathbf{q_g} = \mathbf{0} \\ -\mathbf{G}\mathbf{q_u} + \mathbf{H}\mathbf{q_g} = \mathbf{0} \end{cases}$$

A comparison with equations (i)–(iii) shows that $\mathbf{Q}\mathbf{q_\sigma} = \mathbf{0}$ or equivalently $\mathbf{q_\sigma} \in \operatorname{Ker} \mathbf{Q}$. Conversely, the properties $\mathbf{q_\sigma} \in \operatorname{Ker} \mathbf{Q}$, $\mathbf{q_g} \in \operatorname{Ker} \mathbf{Q}^{\mathrm{T}}$, and $\mathbf{N_g} \mathscr{J}^e_{\mathbf{g}} \mathbf{q_g} - \mathbf{B}_* \mathbf{N}^e_{\mathbf{u}} \mathscr{A}^e_{\mathbf{u}} \mathbf{q_u} = \mathbf{0}$ for $e = 1, \ldots, \mathscr{N}$ ensure that $\{\mathbf{q_u}, \mathbf{q_g}, \mathbf{q_\sigma}\} \in \operatorname{Ker} \mathbf{M}$.

A better understanding of well-posedness can be got by means of an equivalent geometrical formulation in terms of interpolating subspaces. To this end we quote the following equivalences:

$$\begin{aligned} \mathbf{q_\sigma} \in \operatorname{Ker} \mathbf{Q} \ &\Leftrightarrow \ \boldsymbol{\sigma}_h \in S_h \cap \mathscr{D}_h^{\perp} \\ \mathbf{q_g} \in \operatorname{Ker} \mathbf{Q}^{\mathrm{T}} \ &\Leftrightarrow \ \mathbf{g}_h \in \mathscr{D}_h \cap \mathscr{S}_h^{\perp} \end{aligned} \tag{A.6}$$

The kernel of $\mathbf{M}$ can then be rewritten, by virtue of Proposition A.2, as

$$\begin{vmatrix} \mathbf{q}_{\mathbf{u}} \\ \mathbf{q}_{\mathbf{g}} \\ \mathbf{q}_{\boldsymbol{\sigma}} \end{vmatrix} \in \operatorname{Ker}\mathbf{M} \Leftrightarrow \begin{cases} \mathbf{g}_h \in \mathscr{D}_h \cap \mathscr{S}_h^{\perp} \\ \mathbf{u}_h \in \mathscr{L}_h\colon\ \mathbf{g}_h = \mathbf{B}\mathbf{u}_h \\ \boldsymbol{\sigma}_h \in \mathscr{S}_h \cap \mathscr{D}_h^{\perp} \end{cases} \tag{A.7}$$

The necessary and sufficient condition for well-posedness is hereafter proved.

- *Well-posedness criterion*: If there are no rigid conforming displacements, that is $\operatorname{Ker}\mathbf{B} \cap \mathscr{L}_h = \{\mathbf{0}\}$, the conditions

$$\mathrm{G}_1)\quad \tilde{\mathscr{D}}_h \cap \mathbf{B}\mathscr{L}_h = \{\mathbf{0}\}, \qquad \mathrm{G}_2)\quad \mathscr{S}_h \cap \mathscr{D}_h^{\perp} = \{\mathbf{0}\}$$

are necessary and sufficient for the well-posedness of the discrete mixed problem $\mathbb{P}_h$.

Proof. Conditions G_1 and G_2 are equivalent to assume that $\operatorname{Ker}\mathbf{M} = \{\mathbf{0}\}$. Due to the symmetry of the global matrix $\mathbf{M}$ we have $\operatorname{Im}\mathbf{M} = (\operatorname{Ker}\mathbf{M}^{\mathrm{T}})^{\perp} = (\operatorname{Ker}\mathbf{M})^{\perp} = \mathscr{D}_h \times \mathscr{S}_h \times \mathscr{L}_h$ so that there exists a unique solution for any data.

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