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AUTHORS' CLOSURE

First of all the authors wish to express their appreciation to Professor Giulio Maier and his co-workers for their careful reading of the paper.

The relevance of Maier's contributions to the development of a modern theory of plasticity are widely known and deeply appreciated by the international scientific community. Especially remarkable advances were performed several years ago by approaching the problems of engineering plasticity by means of a systematic recourse to the most updated results of Quadratic Programming (Q. P.). This branch of applied mathematics, first developed in the field of economics, was not in the luggage of most structural engineers at that time, and still now it is sufficiently familiar only to a small number of specialized researchers in structural mechanics.

Q. P. was revealed to be well suited for the treatment of the rate formulations of elasto-plastic problems; in fact only inequality and complementarity relations are included in the constitutive model since the flow rule is expressed by the condition that the plastic flow must be normal to a convex cone (the tangent cone to the elastic domain).

In the analysis of finite-step elasto-plastic problems, when adopting an implicit integration scheme, the step flow rule requires that the finite increment of plastic strain must be normal to the convex elastic domain at the final stress point; the *convex cone* is thus replaced by a *convex set*. The formulation of variational principles for these more general problems cannot be carried out in the framework of Q. P., the appropriate context being that of Convex Programming (C. P.).

The aim of our paper was to show how a systematic treatment of finite-step elastic-plastic problems could be performed by appealing to the potential theory for monotone multi-valued operators and to convex analysis. In this respect the authors are pleased to record the commendations of Professor Maier and his co-authors concerning the new facilities provided by this approach.

A point in the discussion needs, however, to receive an explicit reply. The statement that "Among different variational properties that hold true, which one is optimal appears to be a rather subjective, purpose-dependent judgement", and that "clandestine constraint conditions represent minor disadvantages" cannot be agreed with. There is no doubt that several equivalent variational principles can be associated with a given problem provided that the governing operator is conservative. The discussion should be rather focused on the following point. An extremum principle still remains valid if some natural conditions, to be satisfied by the solution, are added to the essential ones as side constraints. In fact any solution still meets the new extremality property and any new extremal point still turns out to be a solution. Such a modified principle must, however, be classified as ill-stated and refused.

A simple example can be given by stating the well known complementary energy principle of elasto-statics in the following form. A stress field is a solution of the elastostatic problem if, and only if, it minimizes the complementary energy functional in the class of the stresses which are in equilibrium with the applied load *and are elastically compatible with a displacement field.* The statement does hold true, but certainly no one would agree with the useless italicized part of the statement.

With specific reference to the minimum principle presented in our paper, it can be interesting to re-formulate it in the context of perfect plasticity defined by the Mises yield condition $\|\mathrm{dev}\sigma\| \leqslant Y$. The minimum principle then specializes to

$$\inf_{\mathbf{u},\mathbf{p},\lambda} \{\mathbf{\Phi}(\mathbf{Tu}-\mathbf{p})-\mathbf{\Gamma}(\mathbf{u})+\lambda Y\}$$

under the conditions

$$\lambda \geqslant 0, \quad \|\mathbf{p}-\mathbf{p}_0\| \leqslant \lambda, \quad \mathrm{sph}(\mathbf{p}-\mathbf{p}_0) = 0,$$

while the side conditions in the parallel principle by Maier *et al.* would be

$$\lambda \geqslant 0, \quad \mathbf{p}-\mathbf{p}_0 = \lambda \frac{\mathrm{dev}\,\sigma}{\|\mathrm{dev}\,\sigma\|}, \quad \sigma = \mathrm{d}\boldsymbol{\Phi}(\mathbf{Tu}-\mathbf{p}),$$

the second condition implying that

$$\|\mathbf{p}-\mathbf{p}_0\| = \lambda, \quad \mathrm{sph}(\mathbf{p}-\mathbf{p}_0) = 0.$$

While it is evident that this last set of conditions must be satisfied by the solution, it also appears that the feasible set turns out to be a non-convex subset of the convex cone provided by the direct approach followed in our paper.

In conclusion it must be underlined that there is no way to avoid the eventuality to get an ill-stated, though valid, variational principle when it is derived by direct inspection and suggested by *skillful intuition*. The only remedy is to invoke the suitable potential theory which automatically provides the well-stated variational principle. This was done in our paper.

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