Geometry & Continuum Mechanics

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Dedicated to the memory of my twin brother Manfredi.

Contents

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Introductory Chapter

" Lie theory is in the process of becoming the most important part of modern mathematics. Little by little it became obvious that the most unexpected theories, from arithmetic to quantum physics, came to encircle this Lie field like a gigantic axis." [1]

– Jean Dieudonné

1.1 State of the art

In the general context of finite displacements, the state of the art presently referred to in literature is the one contributed in *The Non-Linear Field Theories of Mechanics* (NLFTM) by Truesdell and Noll (1965).

The finite strain constitutive theory of elasticity exposed in (Truesdell and Noll, 1965), Sect. 43, stipulates an abstract law relating the Cauchy stress tensor $\mathbf{T}$ to the deformation gradient $\mathbf{F}$:

$$\mathbf{T} = \mathbb{E}(\mathbf{F})\,. \tag{1.1}$$

A rate theory is also exposed in (Truesdell and Noll, 1965), Sect. 99, based on the original proposal by Truesdell (1955) under the name *hypo-elasticity*:

$$\dot{\mathbf{T}} = \mathbb{H}(\mathbf{T}, \mathbf{L}(\mathbf{v}))\,, \tag{1.2}$$

with $\dot{\mathbf{T}}$ material time derivative and $\mathbf{L}(\mathbf{v}) = \nabla\mathbf{v}$ velocity gradient.

The *reduction* argument adduced by Noll (1955), relying on previous work by Richter (1952), is usually applied to rewrite the finite strain elastic law in a reference local placement as

$$\mathbf{S} = \mathbb{E}_{\text{REF}}(\mathbf{U})\,, \tag{1.3}$$

[1] Quoted from (Arild Stubhaug, 2002).

and the hypo-elastic law as

$$\overset{\circ}{\mathbf{T}} = \mathbb{H}(\mathbf{T}) \cdot \mathbf{D}(\mathbf{v}) \,, \tag{1.4}$$

$\mathbf{S}$ is the symmetric PIOLA-KIRCHHOFF referential stress,

$\mathbf{U} = \sqrt{\mathbf{F}^T \mathbf{F}}$ is the referential right stretch,

$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{W}(\mathbf{v})\,\mathbf{T} - \mathbf{T}\,\mathbf{W}(\mathbf{v})$ is the JAUMANN co-rotational stress-rate,

$\mathbf{D}(\mathbf{v}) = \mathrm{sym}\nabla\mathbf{v}$ is the stretching,

$\mathbb{H}$ is the elastic tangent stiffness, nonlinearly dependent on the stress $\mathbf{T}$.

The *reduction* procedure is based on an appeal to the principle of *Material Frame Indifference* (MFI) enunciated by Noll (1958). A careful analysis reveals however that the formal expression of the principle of MFI is affected by a geometrically improper interpretation of the relation between points of view of distinct observers (Romano, Barretta, 2013b). The correct geometric formulation of frame invariance leads to the new principle of *Constitutive Frame Invariance* (CFI) as substitute of the MFI, so that *reduction* procedures are not feasible, as will be discussed in Sect. 3.7.

As explicitly observed in (Truesdell and Noll, 1965), Sect. 80, the elasticity map $\mathbb{E}$ in Eq. (1.1) depends on the choice of a reference local placement. Consequently the theory requires an assumption concerning invariance with respect to this choice. But this invariance eventually amounts to assume that the elasticity map $\mathbb{E}$ does not depend on the deformation gradient. Moreover, *not* discussed in (Truesdell and Noll, 1965) are the following issues.

1. The formula Eq. (1.1) states a relation between a tensor $\mathbf{T}$, based at an event on the trajectory, and a two point tensor $\mathbf{F}$, pertaining to a pair of events. This contradiction cannot be resolved just by imposing an invariance property, as observed above. To be more explicit about this comment, one should imagine to perform a thought experiment to evaluate the constitutive properties of an elastic material. Assuming that the stress-state and its time rate are evaluated by means of statical measurements and theoretical reasonings and that non-elastic phenomena are excluded by a careful testing procedure, the dual state variable, allowed to enter in the description of the material behavior at that time, is the time-rate of change of metric properties, the *stretching*. A finite strain or a deformation gradient do on the contrary refer to a start and to a target body placement. The latter is the current placement while the former is not detectable by laboratory tests.

2. The formula (1.1) should better describe the change in the elastic deformation in response to a given change of stress. In this way the

geometric change of deformation can be directly evaluated as the sum of various contributions described by different inelastic constitutive responses to changes of various state variables, such as stress, temperature, electromagnetic fields and internal structural parameters.

The assumption that the deformation gradient is a driving factor in describing the constitutive behavior of elastic materials, embodied in Eq. (1.1), contrasts with the physical evidence that materials do not react to isometric displacements. NOLL's *reduction* argument, intended to eliminate the incongruence, is a belated remedy based on an appeal to the geometrically incorrect statement of MFI, as evidenced above. The further remedy, adduced to include plasticity and other inelastic behaviors by means of a chain decomposition of the deformation gradient into inelastic and elastic parts (Lee and Liu, 1967; Lee, 1969), was worse than the disease. Indeed, concerning intermediate local placements and ordering of parts in the chain, troubles soon began and are still persistent after some fifty years. This clear indication of inadequacy of the proposal was not effective in dissuading many valuable researchers from perseverating and now the poisoning remedy has risen to the role of *deus ex machina* in formulating geometrically nonlinear constitutive behaviors (Lubarda, 2004).

The reason why it was, and is still commonly considered to be, difficult to give up with the untenable chain decomposition of the deformation gradient, is that a satisfactory rate theory of elasticity was not at hand.

The hypo-elastic law expressed by Eq. (1.4) is indeed affected by drawbacks concerning the following issues.

1. In formula Eq. (1.4) the stress rate $\overset{\circ}{\mathbf{T}}$ suffers of a longly debated intrinsic indeterminacy which cannot be resolved without a consistent geometric treatment.

2. In order to give to hypo-elasticity the physical role of satisfactory elastic model, applicable integrability and conservativeness conditions are required.

3. The formula Eq. (1.4) should rather describe the rate of change in the elastic response to a given rate of change of the stress. In fact rates of change in the response may well be due to causes other than stress rates of change, such, for instance, as temperature rates of change, and the geometric stretching will in general also include rate inelastic responses of the material.

Items 1 and 2 were already treated in (Truesdell and Noll, 1965), Sects. 99, 100, but the indeterminacy was not resolved, being rather accepted as an unavoidable feature of rate theories. Integrability was discussed in (Bernstein, 1960) by performing a comparison between the hypo-elastic law Eq.

(1.2) and the time derivative of the elastic law Eq. (1.1) leading to problematic conclusions. The same treatment of integrability was also adopted, but with a simpler exposition, in (Sansour, Bednarczyk, 1993). These unsuccessful investigations led researchers involved in computational issues to strive to abandon the rate elastic model (Simo and Pister, 1984; Simo and Ortiz, 1985; Simo, 1987, 1988).

1.2 Why geometry?

All the difficulties listed in the previous section, can in fact be overcome by undertaking a new, geometric line of attack to the problem, as is being evolving in a recent research activity (Romano, Barretta, Diaco, 2009a,b; Romano, Barretta, Barretta, 2009; Romano, Diaco, Barretta, 2010; Romano, Barretta, Diaco, 2010; Romano, G., 2011; Romano, Barretta, 2011, 2013a,b; Romano, Barretta, Diaco, 2014a,b,c).

The leading ideas are the following.

1. New physico-geometric notions of *material* and *spatial* fields, both defined on the trajectory manifold, are introduced to clarify basic issues and restore a proper nomenclature. Constitutive properties are described in terms of material fields. Comparisons of material tensors at different times are performed in a natural way by push along the motion.[2] Spatial fields are instead to be compared by a choosing a parallel transport along the motion in the trajectory manifold.

2. The *geometric stretching* is defined in a natural manner as the covariant tensor given by one-half the Lie derivative of the material metric tensor along the space-time motion.

3. The *stress* is described by a material contravariant tensor in duality with the *geometric stretching*. The duality interaction between stress and stretching provides the mechanical power per unit mass.

4. The elastic response is expressed in rate form by defining the *elastic stretching* as a covariant tensor depending nonlinearly on the *stress* and linearly on its time-rate, the *stressing*, which is the Lie derivative of the stress tensor along the space-time motion.

5. The *geometric stretching* is assumed to be the result of the (commutative) addition of various physical contributions such as *elastic stretching*, *thermal stretching*, *visco-plastic stretching*, *phase-change stretching*, *electromagnetic stretching*, *growth stretching*, etc. provided by specific models of constitutive response in function of current values

[2] The notion of *naturality* is illustrated in detail in Sect. 3.2.1.

of the state variables, such as stress, temperature and internal parameters, and of the relevant time-rates along the motion.

A geometrically consistent constitutive theory can then be developed with integrability, frame invariance and computational methods fully available, with a clear physical interpretation of the involved fields and with direct experimental strategies designable for testing material properties. These capabilities will be evidenced in the sequel with explicit reference to elasticity.

For the reader's convenience an essential background of geometric notions and properties is provided in Ch. 2.

2

Basic differential geometry

" I am certain, absolutely certain that... these theories will be recognized as fundamental at some point in the future." [1]

– Marius Sophus Lie, 1888

2.1 Manifolds and morphisms

A manifold $\mathbf{M}$ is a geometric object which generalizes the notion of a curve, surface or ball in the Euclid space. It is characterized by a family of local charts which are differentiable and invertible maps onto open sets in a model linear space, say $\Re^n$. Then n is the manifold dimension. The inverse maps are local coordinate systems. Velocities of parametrized curves through a point $\mathbf{e} \in \mathbf{M}$ on the manifold provide the tangent vectors at that point, describing the tangent linear space $T_{\mathbf{e}}\mathbf{M}$.

The dual space of real-valued linear maps on $T_{\mathbf{e}}\mathbf{M}$ is denoted by $T^*_{\mathbf{e}}\mathbf{M}$ or $(T_{\mathbf{e}}\mathbf{M})^*$ and its elements are called *covectors* at $\mathbf{e} \in \mathbf{M}$.

To a smooth transformation $\boldsymbol{\chi} : \mathbf{M} \mapsto \mathbf{N}$ there corresponds, at each point $\mathbf{e} \in \mathbf{M}$, a linear infinitesimal transformation $T_{\mathbf{e}}\boldsymbol{\chi} : T_{\mathbf{e}}\mathbf{M} \mapsto T_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N}$ between the tangent spaces, called the *differential*, whose action on the tangent vector $\mathbf{u}_{\mathbf{e}} := \partial_{s=0}\, \mathbf{c}(s) \in T_{\mathbf{e}}\mathbf{M}$ to a curve $\mathbf{c} : \Re \mapsto \mathbf{M}$, at the point $\mathbf{e} = \mathbf{c}(0)$, is defined by

$$T_{\mathbf{e}}\boldsymbol{\chi} \cdot \mathbf{u}_{\mathbf{e}} = \partial_{s=0}\, (\boldsymbol{\chi} \circ \mathbf{c})(s)\,. \tag{2.1}$$

A dot $\cdot$ denotes linear dependence on subsequent arguments belonging to linear spaces. A circle $\circ$ denotes composition of maps.

The chochét $\langle\, , \rangle$ denotes the bilinear, non-degenerate duality between pairs of dual linear spaces $(T_{\mathbf{e}}\mathbf{M}\,, T^*_{\mathbf{e}}\mathbf{M})$ or $(T_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N}\,, T^*_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N})$.

[1] Quoted from (Arild Stubhaug, 2002).

The dual linear map

$$(T_{\mathbf{e}}\boldsymbol{\chi})^* : T^*_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N} \mapsto T^*_{\mathbf{e}}\mathbf{M}\,, \tag{2.2}$$

is defined, for any $\mathbf{u}_{\mathbf{e}} \in T_{\mathbf{e}}\mathbf{M}$ and $\mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})} \in T_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N}$, by the identity

$$\langle T_{\mathbf{e}}\boldsymbol{\chi} \cdot \mathbf{u}_{\mathbf{e}}, \mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})} \rangle = \langle \mathbf{u}_{\mathbf{e}}, (T_{\mathbf{e}}\boldsymbol{\chi})^* \cdot \mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})} \rangle\,. \tag{2.3}$$

The *tangent* bundle $T\mathbf{M}$ and the *cotangent* bundle $T^*\mathbf{M}$ are disjoint unions respectively of the linear tangent spaces and of the dual spaces based at points of the manifold.

The global transformation between tangent bundles $T\boldsymbol{\chi} : T\mathbf{M} \mapsto T\mathbf{N}$ is called the *tangent transformation.* The operator T, acting on manifolds and on maps between them, is the *tangent functor.*

Zeroth order tensors are just real-valued functions. Second order tensors at $\mathbf{e} \in \mathbf{M}$ are bilinear maps on pairs of vectors or covectors based at that point.

Tensors of order two are named covariant, contravariant or mixed, depending on whether the arguments are both vectors, both covectors or a vector and a covector. The corresponding linear tensor spaces at $\mathbf{e} \in \mathbf{M}$ are denoted by $\mathrm{F{\small UN}}(T_{\mathbf{e}}\mathbf{M})$, $\mathrm{C{\small OV}}(T_{\mathbf{e}}\mathbf{M})$, $\mathrm{C{\small ON}}(T_{\mathbf{e}}\mathbf{M})$, $\mathrm{M{\small IX}}(T_{\mathbf{e}}\mathbf{M})$.

First order covariant tensors are covectors and first order contravariant tensors are tangent vectors. Second order tensors at $\mathbf{e} \in \mathbf{M}$ are equivalently defined as linear operators from a tangent or cotangent space to another such space at that point:

$$\begin{aligned}
\mathbf{s}_{\mathrm{COV}}(\mathbf{e}) &: T_{\mathbf{e}}\mathbf{M} \mapsto T^*_{\mathbf{e}}\mathbf{M} \in \mathrm{C{\small OV}}(T_{\mathbf{e}}\mathbf{M})\,,\\
\mathbf{s}_{\mathrm{CON}}(\mathbf{e}) &: T^*_{\mathbf{e}}\mathbf{M} \mapsto T_{\mathbf{e}}\mathbf{M} \in \mathrm{C{\small ON}}(T_{\mathbf{e}}\mathbf{M})\,,\\
\mathbf{s}_{\mathrm{MIX}}(\mathbf{e}) &: T_{\mathbf{e}}\mathbf{M} \mapsto T_{\mathbf{e}}\mathbf{M} \in \mathrm{M{\small IX}}(T_{\mathbf{e}}\mathbf{M})\,.
\end{aligned} \tag{2.4}$$

A covariant tensor $\mathbf{g}^{\mathbf{M}}_{\mathbf{e}} \in \mathrm{C{\small OV}}(T_{\mathbf{e}}\mathbf{M})$ is non-degenerate if

$$\mathbf{g}^{\mathbf{M}}_{\mathbf{e}}(\mathbf{u}_{\mathbf{e}}, \mathbf{w}_{\mathbf{e}}) = 0 \quad \forall\, \mathbf{w}_{\mathbf{e}} \in T_{\mathbf{e}}\mathbf{M} \implies \mathbf{u}_{\mathbf{e}} = \mathbf{0}_{\mathbf{e}}\,. \tag{2.5}$$

The corresponding linear operator $\mathbf{g}^{\mathbf{M}}_{\mathbf{e}} : T_{\mathbf{e}}\mathbf{M} \mapsto T^*_{\mathbf{e}}\mathbf{M}$ is then invertible and provides a tool to change tensorial type (alterations). The most important alterations are those which transform covariant or contravariant tensors into mixed ones and vice versa.

$$\begin{aligned}
(\mathbf{s}_{\mathrm{COV}})_{\mathbf{e}} \in \mathrm{C{\small OV}}(T_{\mathbf{e}}\mathbf{M}) &\implies (\mathbf{g}^{\mathbf{M}}_{\mathbf{e}})^{-1} \cdot (\mathbf{s}_{\mathrm{COV}})_{\mathbf{e}} \in \mathrm{M{\small IX}}(T_{\mathbf{e}}\mathbf{M})\,,\\
(\mathbf{s}_{\mathrm{CON}})_{\mathbf{e}} \in \mathrm{C{\small ON}}(T_{\mathbf{e}}\mathbf{M}) &\implies (\mathbf{s}_{\mathrm{CON}})_{\mathbf{e}} \cdot \mathbf{g}^{\mathbf{M}}_{\mathbf{e}} \qquad \in \mathrm{M{\small IX}}(T_{\mathbf{e}}\mathbf{M})\,.
\end{aligned} \tag{2.6}$$

Symmetry of covariant or contravariant tensors means invariance of their values under an exchange of the two arguments.

The adjoint $\mathbf{s}^{A}_{\mathrm{COV}}$ of a covariant tensor $\mathbf{s}_{\mathrm{COV}}$ is defined by

$$\mathbf{s}^{A}_{\mathrm{COV}}(\mathbf{u}\,, \mathbf{w}) = \mathbf{s}_{\mathrm{COV}}(\mathbf{u}\,, \mathbf{w})\,, \tag{2.7}$$

and hence symmetry amounts to the equality $\mathbf{s}^A_{\text{COV}} = \mathbf{s}_{\text{COV}}$.

A pseudo-metric tensor is a non-degenerate covariant tensor which is symmetric, i.e.

$$\mathbf{g}^{\mathbf{M}}_{\mathbf{e}}(\mathbf{u}_{\mathbf{e}}, \mathbf{w}_{\mathbf{e}}) = \mathbf{g}^{\mathbf{M}}_{\mathbf{e}}(\mathbf{w}_{\mathbf{e}}, \mathbf{u}_{\mathbf{e}}) \,. \tag{2.8}$$

A metric tensor $\mathbf{g}^{\mathbf{M}}_{\mathbf{e}} \in \text{Cov}(T_{\mathbf{e}}\mathbf{M})$ is symmetric and positive definite, i.e. such that

$$\mathbf{u}_{\mathbf{e}} \neq \mathbf{0} \implies \mathbf{g}^{\mathbf{M}}_{\mathbf{e}}(\mathbf{u}_{\mathbf{e}}, \mathbf{u}_{\mathbf{e}}) > 0 \,. \tag{2.9}$$

A tensor bundle $\text{Tens}(T\mathbf{M})$ is the disjoint union of tensor fibers which are linear tensor spaces based at points of the manifold.

A bundle is characterized by a projection $\boldsymbol{\pi}_{\mathbf{M}} : \text{Tens}(T\mathbf{M}) \mapsto \mathbf{M}$ which is an operator assigning to each element $\mathbf{s}_{\mathbf{e}} \in \text{Tens}(T_{\mathbf{e}}\mathbf{M})$ of the bundle the corresponding base point $\mathbf{e} \in \mathbf{M}$.

The fibers $\boldsymbol{\pi}^{-1}_{\mathbf{M}}(\mathbf{e})$ are the inverse images of the projection and are assumed to be related each-other by diffeomorphic transformations, so that they are all of the same dimension.

A *tensor field* is a map $\mathbf{s} : \mathbf{M} \mapsto \text{Tens}(T\mathbf{M})$ from a manifold $\mathbf{M}$ to a tensor bundle $\text{Tens}(T\mathbf{M})$ such that a point $\mathbf{e} \in \mathbf{M}$ is mapped to a tensor based at the same point, i.e. such that $\boldsymbol{\pi}_{\mathbf{M}} \circ \mathbf{s}$ is the identity map on $\mathbf{M}$. In geometrical terms it is said that a tensor field is a *section* of a tensor bundle.

A transformation $\boldsymbol{\chi} : \mathbf{M} \mapsto \mathbf{N}$ maps a curve on $\mathbf{M}$ into a curve in $\mathbf{N}$ and, under suitable assumptions, scalar, vector and covector fields from $\mathbf{M}$ onto $\boldsymbol{\chi}(\mathbf{M}) \subset \mathbf{N}$ (push forward $\uparrow$) and vice versa (pull back $\downarrow$).[2]

A synopsis is provided below. Assumptions of differentiability and invertibility of the differential, are claimed whenever needed by the formulae.

Push forward from $\mathbf{M}$ on $\boldsymbol{\chi}(\mathbf{M})$, $\boldsymbol{\chi} : \mathbf{M} \mapsto \mathbf{N}$ injective .

$$\begin{aligned}
&\psi : \mathbf{M} \mapsto \Re \,, && (\boldsymbol{\chi}{\uparrow}\psi)_{\boldsymbol{\chi}(\mathbf{e})} = \psi_{\mathbf{e}} \,, \\
&\mathbf{v} : \mathbf{M} \mapsto T\mathbf{M} \,, && (\boldsymbol{\chi}{\uparrow}\mathbf{v})_{\boldsymbol{\chi}(\mathbf{e})} = T_{\mathbf{e}}\boldsymbol{\chi} \cdot \mathbf{v}_{\mathbf{e}} \,, \\
&\mathbf{v}^* : \mathbf{M} \mapsto T^*\mathbf{M} \,, && \langle \boldsymbol{\chi}{\uparrow}\mathbf{v}^*, \mathbf{w} \rangle_{\boldsymbol{\chi}(\mathbf{e})} = \langle \mathbf{v}^*_{\mathbf{e}}, (T_{\mathbf{e}}\boldsymbol{\chi})^{-1} \cdot \mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})} \rangle \,.
\end{aligned} \tag{2.10}$$

Pull back from $\boldsymbol{\chi}(\mathbf{M})$ to $\mathbf{M}$.

$$\begin{aligned}
&\phi : \mathbf{N} \mapsto \Re \,, && (\boldsymbol{\chi}{\downarrow}\phi)_{\mathbf{e}} = \phi_{\boldsymbol{\chi}(\mathbf{e})} \,, \\
&\mathbf{w} : \mathbf{N} \mapsto T\mathbf{N} \,, && (\boldsymbol{\chi}{\downarrow}\mathbf{w})_{\mathbf{e}} = (T_{\mathbf{e}}\boldsymbol{\chi})^{-1} \cdot \mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})} \,, \\
&\mathbf{w}^* : \mathbf{N} \mapsto T^*\mathbf{N} \,, && \langle \boldsymbol{\chi}{\downarrow}\mathbf{w}^*, \mathbf{v} \rangle_{\mathbf{e}} = \langle \mathbf{w}^*_{\boldsymbol{\chi}(\mathbf{e})}, T_{\mathbf{e}}\boldsymbol{\chi} \cdot \mathbf{v}_{\mathbf{e}} \rangle \,.
\end{aligned} \tag{2.11}$$

[2] In differential geometry push and pull are respectively denoted by low and high asteriscs $_*;^*$ (Abraham, Marsden and Ratiu, 2002; Spivak, 1970). This standard notation leads however to consider too many similar stars in the geometric sky, i.e. push, pull, duality, Hodge operator.

Push-pull relations for second order covariant, contravariant and mixed tensors, are defined so that their scalar values be invariant and are given by the formulae

$$\begin{aligned} (\boldsymbol{\chi}{\downarrow}\mathbf{s}_{\mathrm{COV}})_{\mathbf{e}} &= (T_{\mathbf{e}}\boldsymbol{\chi})^* \cdot (\mathbf{s}_{\mathrm{COV}})_{\boldsymbol{\chi}(\mathbf{e})} \cdot T_{\mathbf{e}}\boldsymbol{\chi} \in \mathrm{C{\scriptsize OV}}(T_{\mathbf{e}}\mathbf{M})\,, \\ (\boldsymbol{\chi}{\uparrow}\mathbf{s}_{\mathrm{CON}})_{\boldsymbol{\chi}(\mathbf{e})} &= T_{\mathbf{e}}\boldsymbol{\chi} \cdot (\mathbf{s}_{\mathrm{CON}})_{\mathbf{e}} \cdot (T_{\mathbf{e}}\boldsymbol{\chi})^* \quad \in \mathrm{C{\scriptsize ON}}(T_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N})\,, \\ (\boldsymbol{\chi}{\uparrow}\mathbf{s}_{\mathrm{MIX}})_{\boldsymbol{\chi}(\mathbf{e})} &= T_{\mathbf{e}}\boldsymbol{\chi} \cdot (\mathbf{s}_{\mathrm{MIX}})_{\mathbf{e}} \cdot (T_{\mathbf{e}}\boldsymbol{\chi})^{-1} \quad \in \mathrm{M{\scriptsize IX}}(T_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N})\,. \end{aligned} \tag{2.12}$$

These transformation rules play an important role in Mechanics since the metric tensor is covariant and the dual stress tensor is contravariant. As the result of a push, a mixed tensor symmetric with respect to a metric tensor is transformed into a mixed tensor symmetric with respect to the pushed metric tensor.

A *morphism* $\boldsymbol{\chi}$ over $\boldsymbol{\phi}$ is made of a pair of maps $(\boldsymbol{\chi}\,,\boldsymbol{\phi})$ between tensor bundles and their base manifolds, that preserve the tensorial fibers, as expressed by the commutative diagram

$$\begin{tikzcd} \mathrm{T{\scriptsize ENS}}(T\mathbf{M}) \arrow[r, "\chi"] \arrow[d, "\pi_{\mathbf{M}}"'] & \mathrm{T{\scriptsize ENS}}(T\mathbf{N}) \arrow[d, "\pi_{\mathbf{N}}"] \\ \mathbf{M} \arrow[r, "\phi"] & \mathbf{N} \end{tikzcd} \iff \boldsymbol{\pi}_{\mathbf{N}} \circ \boldsymbol{\chi} = \boldsymbol{\phi} \circ \boldsymbol{\pi}_{\mathbf{M}}\,.$$

Morphisms that are invertible and differentiable with the inverse, are named *diffeomorphisms*. Important instances of diffeomorphisms are the displacements from a placement of a body to another one, changes of observer, and straightening out maps. On the other hand, differentiable maps which may be *not* diffeomorphisms are, for instance the following

1. *immersions* (maps with injective differentials)
2. *submersions* (maps with surjective differentials)
3. *projections* (surjective submersions).

The $(\mathbf{g}^{\mathbf{M}}, \mathbf{g}^{\mathbf{N}})$-adjoint tangent map

$$(T\boldsymbol{\chi})^A : T(\boldsymbol{\chi}(\mathbf{M})) \mapsto T\mathbf{M}\,, \tag{2.13}$$

is pointwise defined by

$$((T\boldsymbol{\chi})^A \circ \boldsymbol{\chi})(\mathbf{e}) := (\mathbf{g}^{\mathbf{M}}_{\mathbf{e}})^{-1} \cdot (T_{\mathbf{e}}\boldsymbol{\chi})^* \cdot \mathbf{g}^{\mathbf{N}}_{\boldsymbol{\chi}(\mathbf{e})}\,, \tag{2.14}$$

as expressed by the commutative diagram

$$\begin{tikzcd} (T\mathbf{M})^* & (T\mathbf{N})^* \arrow[l, "(T\chi)^*"'] \\ T\mathbf{M} \arrow[u, "\mathbf{g}^{\mathbf{M}}"] & T\mathbf{N} \arrow[l, "(T\chi)^A"'] \arrow[u, "\mathbf{g}^{\mathbf{N}}"'] \end{tikzcd} \iff \mathbf{g}^{\mathbf{M}} \cdot (T\boldsymbol{\chi})^A = (T\boldsymbol{\chi})^* \cdot \mathbf{g}^{\mathbf{N}}\,. \tag{2.15}$$

By Eq. (2.3), the relation Eq. fm: adj may be written as an identity

$$\mathbf{g}^{\mathbf{N}}_{\boldsymbol{\chi}(\mathbf{e})}(\mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})}\,, T_{\mathbf{e}}\boldsymbol{\chi}\cdot\mathbf{u}_{\mathbf{e}}) = \mathbf{g}^{\mathbf{M}}_{\mathbf{e}}((T_{\boldsymbol{\chi}(\mathbf{e})}\boldsymbol{\chi})^{A}\cdot\mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})}\,, \mathbf{u}_{\mathbf{e}})\,, \tag{2.16}$$

for any $\mathbf{u}_{\mathbf{e}} \in T_{\mathbf{e}}\mathbf{M}$ and $\mathbf{w}_{\boldsymbol{\chi}(\mathbf{e})} \in T_{\boldsymbol{\chi}(\mathbf{e})}\mathbf{N}$.

2.2 Lie derivatives

In a vector bundle $\boldsymbol{\pi} : \mathbf{E} \mapsto \mathbf{M}$ the velocity of a curve in a linear fiber belongs to the vertical subbundle $V\mathbf{E}$ of the tangent bundle $T\mathbf{E}$.

By linearity of the fibers, we may introduce the *vertical lift* as the fiber-wise linear, invertible correspondence $\textsc{vlift} : \mathbf{E} \times_{\mathbf{M}} \mathbf{E} \mapsto V\mathbf{E}$ defined, for any $\mathbf{v}, \mathbf{d} \in \mathbf{E}(\mathbf{x})$, by (Romano, G., 2007)

$$\textsc{vlift}(\mathbf{v}\,, \mathbf{d}) := \partial_{\lambda=0}\,(\mathbf{v} + \lambda\mathbf{d}) \in T_{\mathbf{v}}(\mathbf{E}(\mathbf{x}))\,. \tag{2.17}$$

To any vertical vector $\mathbf{V} \in V\mathbf{E}$ based at the vector $\mathbf{v} \in \mathbf{E}(\mathbf{x})$ there corresponds exactly one vector $\mathbf{d} \in \mathbf{E}(\mathbf{x})$ such that $\mathbf{V} = \textsc{vlift}(\mathbf{v}\,, \mathbf{d})$.

The Lie derivative [3] of a vector field $\mathbf{h} \in: \mathbf{M} \mapsto T\mathbf{M}$ according to a vector field $\mathbf{u} : \mathbf{M} \mapsto T\mathbf{M}$ is defined, by considering the flow $\mathbf{Fl}^{\mathbf{u}}_{\lambda}$ generated by solutions of the differential equation $\mathbf{u} = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{u}}_{\lambda}$, as the derivative of the pull-back along the flow

$$\textsc{vlift}(\mathbf{h}\,, \mathcal{L}_{\mathbf{u}}\,\mathbf{h}) := \partial_{\lambda=0}\,(\mathbf{Fl}^{\mathbf{u}}_{\lambda}{\downarrow}\mathbf{h}) = \partial_{\lambda=0}\,T\mathbf{Fl}^{\mathbf{u}}_{-\lambda}\cdot(\mathbf{h}\circ\mathbf{Fl}^{\mathbf{u}}_{\lambda})\,. \tag{2.18}$$

The Lie derivative of a tensor field is defined in an analogous way and the Lie derivative of scalar fields coincides with the directional derivative.

The commutator of tangent vector fields $\mathbf{u}, \mathbf{h} : \mathbf{M} \mapsto T\mathbf{M}$ is the skew-symmetric tangent-vector valued operator defined by

$$[\mathbf{u}\,, \mathbf{h}]f := (\mathcal{L}_{\mathbf{u}}\mathcal{L}_{\mathbf{h}} - \mathcal{L}_{\mathbf{h}}\mathcal{L}_{\mathbf{u}})f \tag{2.19}$$

with $f : \mathbf{M} \mapsto \textsc{Fun}(T\mathbf{M})$ a scalar field.

A basic theorem concerning Lie derivatives states that $\mathcal{L}_{\mathbf{u}}\mathbf{h} = [\mathbf{u}\,, \mathbf{h}]$ and hence the commutator of tangent vector fields is called the Lie bracket. For any injective morphism $\boldsymbol{\chi} : \mathbf{M} \mapsto \mathbf{N}$ the Lie bracket enjoys the push-naturality property (Romano, G., 2007)

$$\boldsymbol{\chi}{\uparrow}(\mathcal{L}_{\mathbf{u}}\,\mathbf{h}) = \boldsymbol{\chi}{\uparrow}[\mathbf{u}\,, \mathbf{h}] = [\boldsymbol{\chi}{\uparrow}\mathbf{u}\,, \boldsymbol{\chi}{\uparrow}\mathbf{h}] = \mathcal{L}_{\boldsymbol{\chi}{\uparrow}\mathbf{u}}\,\boldsymbol{\chi}{\uparrow}\mathbf{h}\,. \tag{2.20}$$

For any tensor field $\mathbf{s} : \mathbf{M} \mapsto \textsc{Tens}(T\mathbf{M})$ the Lie derivative is defined by

$$\textsc{vlift}(\mathbf{s}\,, \mathcal{L}_{\mathbf{u}}\,\mathbf{s}) := \partial_{\lambda=0}\,(\mathbf{Fl}^{\mathbf{u}}_{\lambda}{\downarrow}\mathbf{s}) = \partial_{\lambda=0}\,\mathbf{Fl}^{\mathbf{u}}_{\lambda}{\downarrow}(\mathbf{s}\circ\mathbf{Fl}^{\mathbf{u}}_{\lambda})\,, \tag{2.21}$$

[3] This basic notion was introduced by Marius Sophus Lie, Norwegian mathematician (Lie and Engel, 1888).

and the push-naturality property can be extended to

$$\boldsymbol{\chi}\!\uparrow\!(\mathcal{L}_{\mathbf{u}}\,\mathbf{s}) = \mathcal{L}_{\boldsymbol{\chi}\uparrow\mathbf{u}}\,\boldsymbol{\chi}\!\uparrow\!\mathbf{s}\,. \tag{2.22}$$

By commutativity between push and composition, LEIBNIZ rule for the $\partial_{\lambda=0}$ derivative yields the analogous LEIBNIZ rule for LIE derivatives of tensor fields

$$\mathcal{L}_{\mathbf{u}}\,(\mathbf{s}_{\mathrm{CON}}\cdot\mathbf{s}_{\mathrm{COV}}) = (\mathcal{L}_{\mathbf{u}}\,\mathbf{s}_{\mathrm{CON}})\cdot\mathbf{s}_{\mathrm{COV}} + \mathbf{s}_{\mathrm{CON}}\cdot(\mathcal{L}_{\mathbf{u}}\,\mathbf{s}_{\mathrm{COV}})\,. \tag{2.23}$$

Forms $\boldsymbol{\omega}^k : \mathbf{M} \mapsto \mathrm{ALT}^k(T\mathbf{M})$ are fields of alternating tensors of order $k \le n$, i.e. sign changes under exchange of any two argument vectors.

All forms of order greater than the dimension n of the manifold $\mathbf{M}$ vanish identically. A volume-form is a non null form of maximal order $\boldsymbol{\mu} : \mathbf{M} \mapsto \mathrm{MAX}(T\mathbf{M})$. The associated divergence operator div is defined by the equality

$$\mathcal{L}_{\mathbf{u}}\,\boldsymbol{\mu} = \mathrm{div}(\mathbf{u})\,\boldsymbol{\mu}\,. \tag{2.24}$$

A noteworthy property (Romano, G., 2007) is that for any scalar field $f : \mathbf{M} \mapsto \mathrm{FUN}(T\mathbf{M})$:

$$\mathcal{L}_{\mathbf{u}}\,(f\,\boldsymbol{\mu}) = \mathcal{L}_{(f\,\mathbf{u})}\,\boldsymbol{\mu}\,. \tag{2.25}$$

A volume-form induces a measure defined by $\mathrm{MEAS}(\boldsymbol{\mu}) := \mathrm{SIGNUM}(\boldsymbol{\mu})\,\boldsymbol{\mu}$. The *density* associated with a scalar field $\rho : \mathcal{T} \mapsto \Re$ and with a volume form $\boldsymbol{\mu} \in \mathrm{MAX}(V\mathcal{T})$ is the product $\rho\,\mathrm{MEAS}(\boldsymbol{\mu})$.

2.3 Connections

A linear connection ∇ in a manifold $\mathbf{M}$ fulfills the characteristic properties of a point derivation (Dieudonné, 1969) Vol.III, XVII-18,

$$\begin{aligned}
\nabla_{\mathbf{w}}(\mathbf{u}_1 + \mathbf{u}_2) &= \nabla_{\mathbf{w}}\mathbf{u}_1 + \nabla_{\mathbf{w}}\mathbf{u}_2\,,\\
\nabla_{(\mathbf{w}_1+\mathbf{w}_2)}\mathbf{u} &= \nabla_{\mathbf{w}_1}\mathbf{u} + \nabla_{\mathbf{w}_2}\mathbf{u}\,,\\
\nabla_{\mathbf{w}}(f\mathbf{u}) &= f\,\nabla_{\mathbf{w}}\mathbf{u} + (\nabla_{\mathbf{w}}f)\mathbf{u}\,,\\
\nabla_{(f\,\mathbf{w})}\mathbf{u} &= f\,\nabla_{\mathbf{w}}\mathbf{u}\,,
\end{aligned} \tag{2.26}$$

where $f : \mathbf{M} \mapsto \mathrm{FUN}(T\mathbf{M})$, $\mathbf{u}, \mathbf{u}_i : \mathbf{M} \mapsto T\mathbf{M}$ and $\mathbf{w}_i : \mathbf{M} \mapsto T\mathbf{M}$ for $i = 1, 2$. $\nabla_{\mathbf{w}}f$ is the standard derivative of scalar fields. In terms of parallel transport along a curve $\mathbf{c} : \Re \mapsto \mathbf{M}$, with $\mathbf{u} = \partial_{\lambda=0}\,\mathbf{c}(\lambda)$, the derivative according to the connection is the *parallel derivative*, given by

$$\mathrm{VLIFT}(\mathbf{w}\,, \nabla_{\mathbf{u}}\mathbf{w}) := \partial_{\lambda=0}\,\mathbf{c}(\lambda)\Downarrow(\mathbf{w}\circ\mathbf{c})(\lambda)\,. \tag{2.27}$$

Parallel transported vector fields $(\mathbf{w} \circ \mathbf{c})(\lambda) = \mathbf{c}(\lambda)\Uparrow\mathbf{w}_0$ have a null parallel derivative, since

$$\begin{aligned} \text{VLIFT}(\mathbf{w}\,, \nabla_{\mathbf{u}}\mathbf{w}) &:= \partial_{\lambda=0}\, \mathbf{c}(\lambda)\Downarrow(\mathbf{w} \circ \mathbf{c})(\lambda) \\ &= \partial_{\lambda=0}\, \mathbf{c}(\lambda)\Downarrow\mathbf{c}(\lambda)\Uparrow\mathbf{w}_0 = \partial_{\lambda=0}\, \mathbf{w}_0 = 0\,. \end{aligned} \tag{2.28}$$

The curvature of the connection is the *tensorial* [4] map $\mathbf{R}$, which acting on a vector field $\mathbf{s} : \mathbf{M} \mapsto T\mathbf{M}$ gives a tangent-vector valued two-form $\mathbf{R}(\mathbf{s})$ defined by [5]

$$\mathbf{R}(\mathbf{s})(\mathbf{u}\,, \mathbf{w}) := ([\nabla_{\mathbf{u}}\,, \nabla_{\mathbf{w}}] - \nabla_{[\mathbf{u}\,,\mathbf{w}]})(\mathbf{s})\,, \tag{2.29}$$

and the torsion $\mathbf{T}$ is the tangent-vector valued two-form defined by

$$\mathbf{T}(\mathbf{u}\,, \mathbf{w}) := \nabla_{\mathbf{u}}\mathbf{w} - \nabla_{\mathbf{w}}\mathbf{u} - [\mathbf{u}\,, \mathbf{w}]\,. \tag{2.30}$$

Mixed tensor fields $\mathbf{T}(\mathbf{u})$ and $\mathbf{R}(\mathbf{s}\,, \mathbf{u})$ are defined by the identities

$$\begin{aligned} \mathbf{T}(\mathbf{u}) \cdot \mathbf{w} &:= \mathbf{T}(\mathbf{u}\,, \mathbf{w}) = -\mathbf{T}(\mathbf{w}\,, \mathbf{u})\,, \\ \mathbf{R}(\mathbf{s}\,, \mathbf{u}) \cdot \mathbf{w} &:= \mathbf{R}(\mathbf{s})(\mathbf{u}\,, \mathbf{w}) = -\mathbf{R}(\mathbf{s})(\mathbf{w}\,, \mathbf{u})\,. \end{aligned} \tag{2.31}$$

A connection with vanishing torsion is named *torsion-free* or *symmetric*, and a connection with vanishing curvature is named *curvature-free* or *flat*.

Connections whose parallel transport is path independent are flat, as can be easily deduced by assuming that, in performing the derivative in Eq. (2.29) at a point $\mathbf{x} \in \mathbf{M}$, the vector field $\mathbf{w} : \mathbf{M} \mapsto T\mathbf{M}$ is generated by parallel transport from that point (a procedure allowed for by tensoriality and path independence). The same reasoning reveals that the torsion form of Eq. (2.30) reduces to the Lie bracket, i.e.

$$\mathbf{T}(\mathbf{u}\,, \mathbf{w}) = -[\mathbf{u}\,, \mathbf{w}] = [\mathbf{w}\,, \mathbf{u}]\,. \tag{2.32}$$

The expression of Lie derivatives in terms of parallel derivatives is given for vectors, covectors, covariant, contravariant and mixed tensors by

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}\, \mathbf{u} &= \nabla_{\mathbf{v}}\, \mathbf{u} - \mathbf{Y}(\mathbf{v}) \cdot \mathbf{u}\,, \\ \mathcal{L}_{\mathbf{v}}\, \mathbf{u}^* &= \nabla_{\mathbf{v}}\, \mathbf{u}^* + \mathbf{u}^* \cdot \mathbf{Y}(\mathbf{v})\,, \\ \mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} &= \nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Cov}} + \mathbf{s}_{\mathrm{Cov}} \cdot \mathbf{Y}(\mathbf{v}) + \mathbf{Y}(\mathbf{v})^* \cdot \mathbf{s}_{\mathrm{Cov}}\,, \\ \mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Con}} &= \nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Con}} - \mathbf{Y}(\mathbf{v}) \cdot \mathbf{s}_{\mathrm{Con}} - \mathbf{s}_{\mathrm{Con}} \cdot \mathbf{Y}(\mathbf{v})^*\,, \\ \mathcal{L}_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Mix}} &= \nabla_{\mathbf{v}}\, \mathbf{s}_{\mathrm{Mix}} - \mathbf{Y}(\mathbf{v}) \cdot \mathbf{s}_{\mathrm{Mix}} + \mathbf{s}_{\mathrm{Mix}} \cdot \mathbf{Y}(\mathbf{v})\,, \end{aligned} \tag{2.33}$$

[4] Tensoriality of a multilinear map, acting on vector fields and generating a vector field, means that point values of the image field depends only on the values of the source fields at the same point. An *exterior form*, or simply a *form*, is then a vector-valued, tensorial, alternating multilinear map.

[5] The curvature form of connection on a fiber bundle and the relevant expression in terms of parallel derivatives are treated in (Romano, G., 2007, 2011).

where $\mathbf{Y}(\mathbf{v}) := \nabla\mathbf{v} + \mathbf{T}(\mathbf{v})$. For an exhaustive presentation with proofs we refer the reader to (Romano, G., 2007).

A result due to the author is provided by next Lemma 1. An insight on the involved notions of differential geometry is provided in (Romano, G., 2007). The result will be resorted to in Eq. (3.23) of Sect. 3.2.1.

Lemma 1 *Let a time-parametrized family* $\boldsymbol{\varphi}_\alpha : \mathbf{M} \mapsto \mathbf{M}$ *of diffeomorphisms be acted upon by the tangent functor to give* $T\boldsymbol{\varphi}_\alpha : T\mathbf{M} \mapsto T\mathbf{M}$ *and define the velocity field* $\mathbf{v} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha : \mathbf{M} \mapsto T\mathbf{M}$ *and the following parallel time-derivative*

$$\mathbf{L}(\mathbf{v}) := \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \Downarrow T\boldsymbol{\varphi}_\alpha) : T\mathbf{M} \mapsto T\mathbf{M}\,. \tag{2.34}$$

Then the parallel time derivative of the spatial velocity field

$$\nabla\mathbf{v} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \Downarrow (\mathbf{v} \circ \boldsymbol{\varphi}_\alpha) : T\mathbf{M} \mapsto T\mathbf{M}\,, \tag{2.35}$$

and the tensor field $\mathbf{L}(\mathbf{v})$ *are related by the formula*

$$\mathbf{L}(\mathbf{v}) - \nabla\mathbf{v} = \mathbf{T}(\mathbf{v})\,. \tag{2.36}$$

Proof. Let us consider a curve $\mathbf{c} \in \mathrm{C}^1([-\varepsilon, \varepsilon]^{\mathbf{M}})$ with $\partial_{\lambda=0}\, \mathbf{c}(\lambda) = \mathbf{h} \in T\mathbf{M}$. The fiberwise linear *connector* $\mathcal{K} \in \mathrm{C}^1(T^2\mathbf{M}^T\mathbf{M})$ is related to the parallel derivative of the velocity vector field by the relation

$$\nabla\mathbf{v} \cdot \mathbf{h} := \mathcal{K} \cdot T\mathbf{v} \cdot \mathbf{h}\,. \tag{2.37}$$

Denoting by $T^2\mathbf{M}$ the second tangent bundle and by $\text{FLIP} : T^2\mathbf{M} \mapsto T^2\mathbf{M}$ the *canonical flip* defined by (Romano, G., 2007, 1.8.1)

$$\text{FLIP} \cdot (\partial_{\alpha=0}\, \partial_{\lambda=0}\, \boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda)) = \partial_{\lambda=0}\, \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda)\,, \tag{2.38}$$

we get the formula

$$\begin{aligned} \mathbf{L}(\mathbf{v}) \cdot \mathbf{h} &= \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \Downarrow (T\boldsymbol{\varphi}_\alpha \cdot \partial_{\lambda=0}\, \mathbf{c}_\lambda) = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \Downarrow \partial_{\lambda=0}\, \boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda) \\ &= \mathcal{K} \cdot (\partial_{\alpha=0}\, \partial_{\lambda=0}\, \boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda)) = (\mathcal{K} \circ \text{FLIP}) \cdot (\partial_{\lambda=0}\, \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha(\mathbf{c}_\lambda)) \\ &= (\mathcal{K} \circ \text{FLIP}) \cdot (\partial_{\lambda=0}\, \mathbf{v}(\mathbf{c}_\lambda)) = (\mathcal{K} \circ \text{FLIP}) \cdot (T\mathbf{v} \cdot \mathbf{h})\,. \end{aligned} \tag{2.39}$$

The conclusion follows from the expression of the torsion-form of a linear connection in terms of the *connector* (Romano, G., 2007, 1.8.12)

$$\mathbf{T}(\mathbf{v}, \mathbf{h}) = (\mathcal{K} \circ \text{FLIP} - \mathcal{K}) \cdot T\mathbf{v} \cdot \mathbf{h}\,, \tag{2.40}$$

and from the definition of the tensor field $\mathbf{T}(\mathbf{v})$. ■

2.4 Exterior calculus and Stokes' formula

The modern way to introduce integral transformations is to consider maximal-forms as geometric objects to be integrated over a (orientable) manifold and to resort to the notion of exterior differential of a form (Marsden and Hughes, 1983; Romano, G., 2007).

In a m-dimensional manifold $\mathbf{M}$, let $\mathbf{\Gamma}$ be any n-dimensional submanifold ($m \geq n$) with $(n-1)$-dimensional boundary manifold $\partial\mathbf{\Gamma}$.

The classical AMPÈRE-KELVIN-STOKES formula, in its modern formulation by VOLTERRA-POINCARÉ-BROUWER, characterizes the exterior derivative of a $(n-1)$-form $\boldsymbol{\omega} : \mathbf{M} \mapsto \mathrm{ALT}^{n-1}(T\mathbf{M})$, defined as the n-form $d\boldsymbol{\omega} : \mathbf{M} \mapsto \mathrm{ALT}^{n}(T\mathbf{M})$ fulfilling the identity

$$\int_{\mathbf{\Gamma}} d\boldsymbol{\omega} = \int_{\partial\mathbf{\Gamma}} \partial\mathbf{i}{\downarrow}\boldsymbol{\omega}\,, \tag{2.41}$$

where $\partial\mathbf{i} : \partial\mathbf{\Gamma} \mapsto \mathbf{\Gamma}$ is the injective immersion of the boundary manifold $\partial\mathbf{\Gamma}$ into the manifold $\mathbf{\Gamma}$. The pull-back $\partial\mathbf{i}{\downarrow}$ by the immersion is needed to transform exterior forms on $T\mathbf{\Gamma}$ to exterior forms on $T\partial\mathbf{\Gamma}$ but, for the sake of notational simplicity, it is often, and will be, abusively omitted in Eq. (2.41) briefly denoted as STOKES' formula.

The exterior derivative of the exterior product of a p-form times a k-form is given by the formula

$$d(\boldsymbol{\alpha}^p \wedge \boldsymbol{\omega}^k) = (d\boldsymbol{\alpha}^p) \wedge \boldsymbol{\omega}^k + (-1)^p\, \boldsymbol{\alpha}^p \wedge d\boldsymbol{\omega}^k\,. \tag{2.42}$$

Being $\partial\partial\mathbf{\Gamma} = 0$ for any manifold $\mathbf{\Gamma}$, it follows that also $dd\boldsymbol{\omega} = 0$ for any form $\boldsymbol{\omega}$ (Marsden and Hughes, 1983; Romano, G., 2007).

The exterior derivative of differential forms is characterized by the peculiar property of commutation with the pull-back by an injective immersion $\boldsymbol{\chi} : \mathbf{M} \mapsto \mathbf{N}$

$$d \circ \boldsymbol{\chi}{\downarrow} = \boldsymbol{\chi}{\downarrow} \circ d\,, \tag{2.43}$$

a result inferred, from STOKES and integral transformation formulae

$$\begin{aligned} \int_{\mathbf{\Gamma}} d(\boldsymbol{\chi}{\downarrow}\boldsymbol{\omega}) &= \oint_{\partial\mathbf{\Gamma}} \boldsymbol{\chi}{\downarrow}\boldsymbol{\omega} = \oint_{\boldsymbol{\chi}(\partial\mathbf{\Gamma})} \boldsymbol{\omega} \\ &= \oint_{\partial\boldsymbol{\chi}(\mathbf{\Gamma})} \boldsymbol{\omega} = \int_{\boldsymbol{\chi}(\mathbf{\Gamma})} d\boldsymbol{\omega} = \int_{\mathbf{\Gamma}} \boldsymbol{\chi}{\downarrow}(d\boldsymbol{\omega})\,. \end{aligned} \tag{2.44}$$

Then for $\mathbf{v} := \partial_{\lambda=0}\, \boldsymbol{\chi}_\lambda$ we infer that

$$\mathcal{L}_{\mathbf{v}}\,(d\boldsymbol{\omega}^1) = d\,(\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\omega}^1)\,. \tag{2.45}$$

The *geometric homotopy formula* relates the boundary chain generated by the extrusion of a manifold $\mathbf{\Gamma}$ and of its boundary $\partial\mathbf{\Gamma}$, as follows

$$\partial(J_{\boldsymbol{\chi}}(\mathbf{\Gamma}, \lambda)) = \boldsymbol{\chi}_\lambda(\mathbf{\Gamma}) - \mathbf{\Gamma} - J_{\boldsymbol{\chi}}(\partial\mathbf{\Gamma}, \lambda)\,,$$

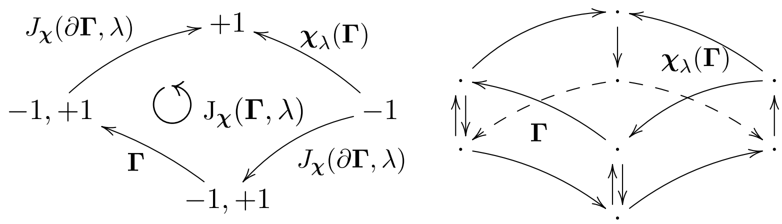
with $\lambda \in \Re$ extrusion parameter and $\boldsymbol{\chi} : \boldsymbol{\Gamma} \times \Re \mapsto \mathbf{M} \times \Re$ extrusion-map fulfilling the commutative diagram

$$\begin{tikzcd} \boldsymbol{\Gamma} \times \Re \arrow[r, "\boldsymbol{\chi}_\lambda"] \arrow[d, "\boldsymbol{\pi}_\Re"'] & \mathbf{M} \times \Re \arrow[d, "\boldsymbol{\pi}_\Re"] \\ \Re \arrow[r, "\boldsymbol{\theta}_\lambda"] & \Re \end{tikzcd} \iff \boldsymbol{\pi}_\Re \circ \boldsymbol{\chi}_\lambda = \boldsymbol{\theta}_\lambda \circ \boldsymbol{\pi}_\Re \,, \tag{2.46}$$

with $\boldsymbol{\theta}_\lambda : \Re \mapsto \Re$ the translation defined by $\boldsymbol{\theta}_\lambda(\alpha) := \alpha + \lambda$ for $\alpha, \lambda \in \Re$.

The signs in the formula are motivated as follows. The orientation of the $(n+1)$-dimensional flow tube $J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda)$ induces an orientation on its boundary $\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda))$.

In the boundary chain, composed by the manifolds $\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})$, $\boldsymbol{\Gamma}$ and $J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \lambda)$, each one with the induced orientation, the element $\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})$ has orientation opposed to the orientation of $\boldsymbol{\chi}_0(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma}$ and $J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma}, \lambda)$, as depicted in the diagrams Eq. (2.47), for $\dim \boldsymbol{\Gamma} = 1$ and $\dim \boldsymbol{\Gamma} = 2$.



(2.47)

Let $\boldsymbol{\omega}$ be an n-form defined on the $(n+1)$-manifold $J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma}, \lambda)$ spanned by extrusion of the n-manifold $\boldsymbol{\Gamma}$, so that the geometric homotopy formula gives

$$\int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \int_{\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\lambda))} \boldsymbol{\omega} + \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\lambda)} \boldsymbol{\omega} + \int_{\boldsymbol{\Gamma}} \boldsymbol{\omega} \,. \tag{2.48}$$

Differentiation with respect to the extrusion-time yields

$$\partial_{\lambda=0} \int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \partial_{\lambda=0} \Big(\int_{\partial(J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\lambda))} \boldsymbol{\omega} + \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\lambda)} \boldsymbol{\omega} \Big) \,. \tag{2.49}$$

Then, denoting by $\mathbf{v} := \partial_{\lambda=0}\, \boldsymbol{\chi}_\lambda$ the velocity field of the extrusion, applying Stokes formula and taking into account that by Fubini theorem (Abraham, Marsden and Ratiu, 2002)

$$\begin{aligned} \partial_{\lambda=0} \int_{J_{\boldsymbol{\chi}}(\boldsymbol{\Gamma},\lambda)} d\boldsymbol{\omega} &= \int_{\boldsymbol{\Gamma}} (d\boldsymbol{\omega}) \cdot \mathbf{v} \,, \\ \partial_{\lambda=0} \int_{J_{\boldsymbol{\chi}}(\partial\boldsymbol{\Gamma},\lambda)} \boldsymbol{\omega} &= \int_{\partial\boldsymbol{\Gamma}} \boldsymbol{\omega} \cdot \mathbf{v} \,, \end{aligned} \tag{2.50}$$

we get the integral *extrusion formula*

$$\boxed{\partial_{\lambda=0} \int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \int_{\boldsymbol{\Gamma}} (d\boldsymbol{\omega}) \cdot \mathbf{v} + \int_{\boldsymbol{\Gamma}} d(\boldsymbol{\omega} \cdot \mathbf{v}) \,.} \tag{2.51}$$

On the other hand, taking the time-rate of the integral transformation formula, leads to LIE-REYNOLDS formula

$$\partial_{\lambda=0} \int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\Gamma})} \boldsymbol{\omega} = \partial_{\lambda=0} \int_{\boldsymbol{\Gamma}} (\boldsymbol{\chi}_\lambda \downarrow \boldsymbol{\omega}) = \int_{\boldsymbol{\Gamma}} \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega}\,. \tag{2.52}$$

The comparison of the expressions in Eqs. (2.52) and (2.51) and localization, yield the *differential homotopy formula* (Cartan, 1951, 1967) expressing the LIE derivative $\mathcal{L}$ of a k-form in terms of exterior derivatives

$$\boxed{\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega} = d(\boldsymbol{\omega} \cdot \mathbf{v}) + (d\boldsymbol{\omega}) \cdot \mathbf{v}\,.} \tag{2.53}$$

From Eq. (2.53) and the recursive LEIBNIZ formula

$$\mathcal{L}_{\mathbf{v}}\, \boldsymbol{\omega} \cdot \mathbf{w} := \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega} \cdot \mathbf{w}) - \boldsymbol{\omega} \cdot \mathcal{L}_{\mathbf{v}}\mathbf{w} = \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega} \cdot \mathbf{w}) - \boldsymbol{\omega} \cdot [\mathbf{v}\,, \mathbf{w}]\,, \tag{2.54}$$

we get the recursive PALAIS formula for the exterior derivative of a n-form $\boldsymbol{\omega}$ in terms the exterior derivative of a $(n-1)$-form $\boldsymbol{\omega} \cdot \mathbf{v}$ and of LIE derivatives

$$d\boldsymbol{\omega} \cdot \mathbf{v} \cdot \mathbf{w} = -d(\boldsymbol{\omega} \cdot \mathbf{v}) \cdot \mathbf{w} + \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega} \cdot \mathbf{w}) - \boldsymbol{\omega} \cdot [\mathbf{v}\,, \mathbf{w}]\,. \tag{2.55}$$

Performing the recursion from the $(n{+}1)$-form $d\boldsymbol{\omega}$ till the exterior derivative of the 0-form $(\boldsymbol{\omega} \cdot \mathbf{w}_1 \ldots \cdot \mathbf{w}_n)$ and observing that $d(\boldsymbol{\omega} \cdot \mathbf{w}_1 \ldots \cdot \mathbf{w}_n) \cdot \mathbf{v} = \mathcal{L}_{\mathbf{v}}(\boldsymbol{\omega} \cdot \mathbf{w}_1 \ldots \cdot \mathbf{w}_n)$, yields the original result in (Palais, 1954).

The next result, contributed in (Romano, G., 2007), provides the expression of the exterior derivative of a form in terms of linear connections. The statement here refers to 1-forms, but can be recursively extended to forms of any degree.

Proposition 1 (Exterior derivative in terms of a connection) *The exterior derivative* $d\,\boldsymbol{\omega}^1$ *of a* 1*-form* $\boldsymbol{\omega}^1 \in \Lambda^1(T\mathbf{M})$ *is expressed in terms of a linear connection* ∇ *by the formula*

$$\boxed{d\boldsymbol{\omega}^1 = \nabla\boldsymbol{\omega}^1 - (\nabla\boldsymbol{\omega}^1)^A + \boldsymbol{\omega}^1 \cdot \mathbf{T}\,,} \tag{2.56}$$

where the 2*-forms at the r.h.s. are defined by*

$$\begin{aligned} (\nabla\boldsymbol{\omega}^1) \cdot \mathbf{v} \cdot \mathbf{w} &= (\nabla_{\mathbf{v}}\, \boldsymbol{\omega}^1) \cdot \mathbf{w}\,, \\ (\nabla\boldsymbol{\omega}^1)^A \cdot \mathbf{v} \cdot \mathbf{w} &= (\nabla_{\mathbf{w}}\, \boldsymbol{\omega}^1) \cdot \mathbf{v}\,, \\ (\boldsymbol{\omega}^1 \cdot \mathbf{T}) \cdot \mathbf{v} \cdot \mathbf{w} &= \boldsymbol{\omega}^1 \cdot \mathbf{T}(\mathbf{v}\,, \mathbf{w})\,, \quad \forall\, \mathbf{v}, \mathbf{w} \in T\mathbf{M}\,. \end{aligned} \tag{2.57}$$

Proof. On 0-forms exterior and parallel derivatives are identical, so that by LEIBNIZ rule and definition of torsion we have

$$\begin{aligned} d_{\mathbf{v}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{w}) &= \nabla_{\mathbf{v}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{w}) = (\nabla_{\mathbf{v}}\,\boldsymbol{\omega}^1) \cdot \mathbf{w} + \boldsymbol{\omega}^1 \cdot \nabla_{\mathbf{v}}\,\mathbf{w}\,, \\ d_{\mathbf{w}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v}) &= \nabla_{\mathbf{w}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v}) = (\nabla_{\mathbf{w}}\,\boldsymbol{\omega}^1) \cdot \mathbf{v} + \boldsymbol{\omega}^1 \cdot \nabla_{\mathbf{w}}\,\mathbf{v}\,, \\ \mathbf{T}(\mathbf{v}\,,\mathbf{w}) &= \nabla_{\mathbf{v}}\,\mathbf{w} - \nabla_{\mathbf{w}}\,\mathbf{v} - [\mathbf{v}\,,\mathbf{w}]\,. \end{aligned} \tag{2.58}$$

PALAIS formula Eq. (2.55) yields

$$(d\,\boldsymbol{\omega}^1) \cdot \mathbf{v} \cdot \mathbf{w} = d_{\mathbf{v}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{w}) - d_{\mathbf{w}}\,(\boldsymbol{\omega}^1 \cdot \mathbf{v}) - \boldsymbol{\omega}^1 \cdot [\mathbf{v}\,,\mathbf{w}]\,, \tag{2.59}$$

so that, substituting Eq. (2.58), we get the result. ■

2.5 Split formulae

Let $(\boldsymbol{\phi}\,, \mathrm{ID}_{\mathbf{M}})$ be a smooth non-linear morphism, between the tensor bundles $\mathrm{TENS}_1(T\mathbf{M})$ and $\mathrm{TENS}_2(T\mathbf{M})$, described by the commutative diagram

$$\begin{tikzcd} \mathrm{TENS}_1(T\mathbf{M}) \arrow[r, "\phi"] \arrow[d, "\boldsymbol{\pi}_{\mathrm{TENS}_1}"'] & \mathrm{TENS}_2(T\mathbf{M}) \arrow[d, "\boldsymbol{\pi}_{\mathrm{TENS}_2}"] \\ \mathbf{M} \arrow[r, "\mathrm{ID}_{\mathbf{M}}"] & \mathbf{M} \end{tikzcd} \quad \Longleftrightarrow \quad \boldsymbol{\pi}_{\mathrm{TENS}_2} \circ \boldsymbol{\phi} = \boldsymbol{\pi}_{\mathrm{TENS}_1}\,. \tag{2.60}$$

Lemma 2 (Differential split formulae) *Let the tensor field*

$$\boldsymbol{\phi} \circ \mathbf{s} : \mathbf{M} \mapsto \mathrm{TENS}_2(T\mathbf{M})\,, \tag{2.61}$$

be the composition of the morphism $\boldsymbol{\phi} : \mathrm{TENS}_1(T\mathbf{M}) \mapsto \mathrm{TENS}_2(T\mathbf{M})$ *with a tensor field* $\mathbf{s} : \mathbf{M} \mapsto \mathrm{TENS}_1(T\mathbf{M})$. *The* LIE *and parallel derivatives along the flow* $\boldsymbol{\varphi}_\alpha := \mathbf{Fl}^{\mathbf{v}}_\alpha$ *of a vector field* $\mathbf{v} : \mathbf{M} \mapsto T\mathbf{M}$, *may then be expressed by the split formulae*

$$\boxed{\begin{aligned} \mathcal{L}_{\mathbf{v}}(\boldsymbol{\phi} \circ \mathbf{s}) &= (\mathcal{L}_{\mathbf{v}}\boldsymbol{\phi})(\mathbf{s}) + d_F\boldsymbol{\phi}(\mathbf{s}) \cdot \mathcal{L}_{\mathbf{v}}\mathbf{s}\,, \\ \nabla_{\mathbf{v}}(\boldsymbol{\phi} \circ \mathbf{s}) &= (\nabla_{\mathbf{v}}\boldsymbol{\phi})(\mathbf{s}) + d_F\boldsymbol{\phi}(\mathbf{s}) \cdot \nabla_{\mathbf{v}}\mathbf{s}\,. \end{aligned}} \tag{2.62}$$

Proof. By definition $(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\phi}) \circ (\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{s}) = \boldsymbol{\varphi}_\alpha\!\downarrow\!(\boldsymbol{\phi} \circ \mathbf{s})$ and hence by LEIBNIZ rule

$$\begin{aligned} \mathcal{L}_{\mathbf{v}}(\boldsymbol{\phi} \circ \mathbf{s}) &= \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha\!\downarrow\!(\boldsymbol{\phi} \circ \mathbf{s} \circ \boldsymbol{\varphi}_\alpha) \\ &= \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\phi}) \circ \boldsymbol{\varphi}_\alpha\!\downarrow\!(\mathbf{s} \circ \boldsymbol{\varphi}_\alpha) \\ &= \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\phi})(\mathbf{s}) + \partial_{\alpha=0}\,\boldsymbol{\phi} \circ \boldsymbol{\varphi}_\alpha\!\downarrow\!(\mathbf{s} \circ \boldsymbol{\varphi}_\alpha) \\ &= \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\phi})(\mathbf{s}) + d_F\boldsymbol{\phi}(\mathbf{s}) \cdot \mathcal{L}_{\mathbf{v}}\mathbf{s}\,. \end{aligned} \tag{2.63}$$

The result in Eq. (2.62)$_1$ follows by observing that by definition

$$(\mathcal{L}_{\mathbf{v}}\boldsymbol{\phi})(\mathbf{s}) := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\phi})(\mathbf{s}) = \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha{\downarrow}(\boldsymbol{\phi}(\boldsymbol{\varphi}_\alpha{\uparrow}\mathbf{s}))\,. \tag{2.64}$$

By substituting the pull-back $\downarrow$ with the inverse parallel transport $\Downarrow$ and defining

$$(\boldsymbol{\varphi}_\alpha{\Downarrow}\boldsymbol{\phi}) \circ (\boldsymbol{\varphi}_\alpha{\Downarrow}\mathbf{s}) = \boldsymbol{\varphi}_\alpha{\Downarrow}(\boldsymbol{\phi} \circ \mathbf{s})\,, \tag{2.65}$$

so that

$$(\nabla_{\mathbf{v}}\boldsymbol{\phi})(\mathbf{s}) := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha{\Downarrow}\boldsymbol{\phi})(\mathbf{s}) = \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha{\Downarrow}(\boldsymbol{\phi}(\boldsymbol{\varphi}_\alpha{\Uparrow}\mathbf{s}))\,, \tag{2.66}$$

the result in Eq. (2.62)$_2$ is got. ■

2.6 A bit of matrix algebra

Let us denote by $\{\,\mathbf{d}_i\,, i = 1, 2, 3\,\}$ and $\{\,\mathbf{d}^j\,, j = 1, 2, 3\,\}$ dual bases in the space bundle, so that $[\langle\,\mathbf{d}^i, \mathbf{d}_j\,\rangle]$ is the identity matrix $[\delta^i_j]$.

The matrices associated with tensors $\boldsymbol{\varepsilon} \in \mathrm{Cov}(V\mathcal{E})$ and $\boldsymbol{\sigma} \in \mathrm{Con}(V\mathcal{E})$, considered as linear operators $\boldsymbol{\varepsilon} : V\mathcal{E} \mapsto (V\mathcal{E})^*$ and $\boldsymbol{\sigma} : (V\mathcal{E})^* \mapsto V\mathcal{E}$, are given by

$$\begin{aligned} \boldsymbol{\varepsilon} \cdot \mathbf{d}_i\, &= [\varepsilon]_{ki}\,\mathbf{d}^k\,, \\ \boldsymbol{\sigma} \cdot \mathbf{d}^i\, &= [\boldsymbol{\sigma}]^{ki}\,\mathbf{d}_k\,, \end{aligned} \tag{2.67}$$

and are the transpose of the corresponding Gram matrices

$$\begin{aligned} \boldsymbol{\varepsilon}(\mathbf{d}_i\,, \mathbf{d}_j)\, &= \langle\,\boldsymbol{\varepsilon} \cdot \mathbf{d}_i, \mathbf{d}_j\,\rangle = \langle\,[\varepsilon]_{ki}\,\mathbf{d}^k, \mathbf{d}_j\,\rangle = [\varepsilon]_{ji}\,, \\ \boldsymbol{\sigma}(\mathbf{d}^i\,, \mathbf{d}^j)\, &= \langle\,\boldsymbol{\sigma} \cdot \mathbf{d}^i, \mathbf{d}^j\,\rangle = \langle\,[\boldsymbol{\sigma}]^{ki}\,\mathbf{d}_k, \mathbf{d}^j\,\rangle = [\boldsymbol{\sigma}]^{ji}\,. \end{aligned} \tag{2.68}$$

The matrices of a linear operator $\mathbf{L} : V_{\mathbf{e}}\mathcal{E} \mapsto V_{\mathbf{e}}\mathcal{E}$ and of the dual operator $\mathbf{L}^* : (V_{\mathbf{e}}\mathcal{E})^* \mapsto (V_{\mathbf{e}}\mathcal{E})^*$ fulfill the relations

$$\mathbf{L} \cdot \mathbf{d}_k = [\mathbf{L}]^{j.}_{.k}\,\mathbf{d}_j \tag{2.69}$$

$$\mathbf{L}^* \cdot \mathbf{d}^i = [\mathbf{L}^*]^{.i}_{j.}\,\mathbf{d}^j \tag{2.70}$$

$$\langle\,\mathbf{L}^* \cdot \mathbf{d}^i, \mathbf{d}_k\,\rangle = \langle\,\mathbf{d}^i, \mathbf{L} \cdot \mathbf{d}_k\,\rangle \tag{2.71}$$

$$\langle\,\mathbf{d}^i, \mathbf{L} \cdot \mathbf{d}_k\,\rangle = \langle\,\mathbf{d}^i, [\mathbf{L}]^{j.}_{.k}\,\mathbf{d}_j\,\rangle = [\mathbf{L}]^{i.}_{.k} \tag{2.72}$$

$$\langle\,\mathbf{L}^* \cdot \mathbf{d}^i, \mathbf{d}_k\,\rangle = \langle\,[\mathbf{L}^*]^{.i}_{j.}\,\mathbf{d}^j, \mathbf{d}_k\,\rangle = [\mathbf{L}^*]^{.i}_{k.} \tag{2.73}$$

that is, the matrix of the dual is equal to the transpose of the matrix

$$[\mathbf{L}^*] = [\mathbf{L}]^T\,. \tag{2.74}$$

If the basis $\{\,\mathbf{d}_i\,, i = 1, 2, 3\,\}$ is $\mathbf{g}$-orthonormal, then $\mathbf{d}^i = \mathbf{g} \cdot \mathbf{d}_i$. The $\mathbf{g}$-adjoint operator $\mathbf{L}^A : V_{\mathbf{e}}\mathcal{E} \mapsto V_{\mathbf{e}}\mathcal{E}$, defined by

$$\langle\,\mathbf{L} \cdot \mathbf{d}_i, \mathbf{g} \cdot \mathbf{d}_k\,\rangle = \langle\,\mathbf{g} \cdot \mathbf{d}_i, \mathbf{L}^A \cdot \mathbf{d}_k\,\rangle\,, \tag{2.75}$$

is then represented by a matrix equal to the transpose of the matrix of the linear operator $\mathbf{L}$

$$[\mathbf{L}^A] = [\mathbf{L}]^T \,. \tag{2.76}$$

Let us consider two basis $\mathbf{h}_i \in T_{\mathbf{x}}\boldsymbol{\Omega}$ and $\mathbf{d}_j \in T_{\boldsymbol{\varphi}_\alpha(\mathbf{x})}(\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}))$. Then the deformation gradient $\mathbf{F}_\alpha$ and its adjoint $\mathbf{F}_\alpha^A$, fulfilling the characteristic relation

$$\mathbf{g}(\mathbf{F}_\alpha \cdot \mathbf{h} \,, \mathbf{d}) = \mathbf{g}(\mathbf{F}_\alpha^A \cdot \mathbf{d} \,, \mathbf{h}) \,, \quad \forall\, \mathbf{h} \in T\boldsymbol{\Omega} \,, \quad \forall\, \mathbf{d} \in T(\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})) \,. \tag{2.77}$$

are represented by the matrices

$$\begin{aligned} &\mathbf{F}_\alpha \cdot \mathbf{h}_i = F^k_{.i}\, \mathbf{d}_k \,, \\ &(\mathbf{F}_\alpha^A) \cdot \mathbf{d}_j = (F^A)^k_{.j}\, \mathbf{h}_k \,, \end{aligned} \tag{2.78}$$

so that

$$\begin{aligned} &\mathbf{g}(\mathbf{F}_\alpha \cdot \mathbf{h}_i \,, \mathbf{d}_j) = F^k_{.i}\, \mathbf{g}(\mathbf{d}_k \,, \mathbf{d}_j) \,, \\ &\mathbf{g}((\mathbf{F}_\alpha^A) \cdot \mathbf{d}_j \,, \mathbf{h}_i) = (F^A)^k_{.j}\, \mathbf{g}(\mathbf{h}_k \,, \mathbf{h}_i) \,. \end{aligned} \tag{2.79}$$

Under orthonormal bases, being $\mathbf{g}(\mathbf{d}_k \,, \mathbf{d}_j) = \delta_{kj}$ and $\mathbf{g}(\mathbf{h}_k \,, \mathbf{h}_i) = \delta_{ki}$, we infer that

$$F^j_{.i} = (F^A)^i_{.j} \,. \tag{2.80}$$

3

Continuum mechanics

> *" The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."*
>
> – Hermann Minkowski, 1908

3.1 Kinematics

Continuum Mechanics is best developed in the general framework of a 4D (four dimensional) manifold of events $\mathbf{e} \in \mathcal{E}$ and of the relevant tangent bundle $T\mathcal{E}$ with projection $\boldsymbol{\tau}_{\mathcal{E}} : T\mathcal{E} \mapsto \mathcal{E}$.[1]

3.1.1 Space-time splitting

Each observer performs a double foliation of the 4D events manifold $\mathcal{E}$ into complementary

- 3D *space-slices* $\mathcal{S}$ of *isochronous* events (with a same corresponding time instant) and
- 1D *time-lines* of *isotopic* events (with a same corresponding space location).

Time-lines do not intersect one another and each time-line intersects a space-slice just at one point. Analogously, space-slices do not intersect one another and each space-slice intersects a time-line just at one point.

[1] The tangent bundle $T\mathcal{E}$ is the manifold made of the disjoint union of all linear spaces tangent to the event manifold $\mathcal{E}$, with the property that the projection is a surjective submersion. A submersion (immersion) is a map between manifolds such that the tangent map at each point is surjective (injective).

The *time-lines* are parametrized in such a way that a real valued *time-projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ [2] assigns the same time instant $t_{\mathcal{E}}(\mathbf{e}) \in \mathcal{Z}$ to each event in a *space-slice*, that is

$$t_{\mathcal{E}}(\overline{\mathbf{e}}) = t_{\mathcal{E}}(\mathbf{e}) \,, \quad \forall \, \overline{\mathbf{e}} \in \mathcal{S} \,. \tag{3.1}$$

Velocities of *time-lines* define the field of *time arrows* $\mathbf{Z} : \mathcal{E} \mapsto T\mathcal{E}$.

The tangent space $T_{\mathbf{e}}\mathcal{E}$ at any event $\mathbf{e} \in \mathcal{E}$ is split into a complementary pair of a 3D time-vertical subspace $V_{\mathbf{e}}\mathcal{E}$ (tangent to a space-slice) and a 1D time-horizontal subspace $H_{\mathbf{e}}\mathcal{E}$ (tangent to a time-line) generated by the time arrow $\mathbf{Z}(\mathbf{e}) \in T_{\mathbf{e}}\mathcal{E}$.

The time-projection $t_{\mathcal{E}} : \mathcal{E} \mapsto \Re$ and the time arrow $\mathbf{Z}(\mathbf{e}) \in T_{\mathbf{e}}\mathcal{E}$ are *tuned* if they are such that

$$\langle dt_{\mathcal{E}}, \mathbf{Z} \rangle = 1 \circ t_{\mathcal{E}} \,. \tag{3.2}$$

Definition 1 (Space and time bundles) *The tangent bundle $T\mathcal{E}$ is split by the differential of time-projection into a* space *bundle and a* time *bundle. The former is the* time-vertical *subbundle $V\mathcal{E}$ while the latter is the* time-horizontal subbundle *$H\mathcal{E}$, respectively, disjoint unions of all time-vertical and time-horizontal subspaces.*

In the familiar EUCLID setting of classical Mechanics, the space slices and the *time-projection* $t_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{Z}$ are the same for all observers (universality of time).

A reference frame $\{ \mathbf{d}_i \,; \, i = 0, 1, 2, 3 \}$ for the event manifold is *adapted* if $\mathbf{d}_0 = \mathbf{Z}$ and $\mathbf{d}_i \in V\mathcal{E}$, $i = 1, 2, 3$.

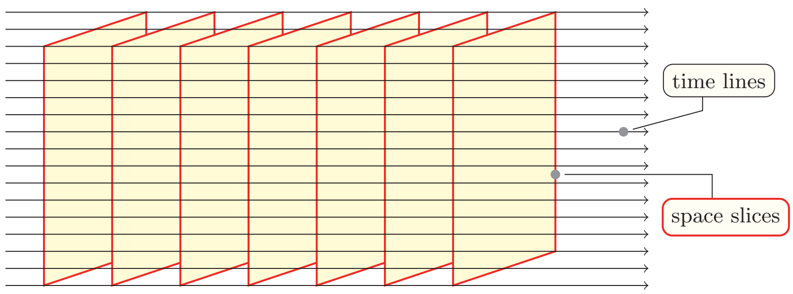


Figure 3.1: EUCLID space-time slicing.

[2] *Zeit* is the German word for *Time*.

3.1.2 Motion and material bundle

Definition 2 (Trajectory and motion) *The trajectory manifold is the geometric object investigated in Mechanics, characterized by an embedding* [3] $\mathbf{i} : \mathcal{T} \mapsto \mathcal{E}$ *into the event manifold* $\mathcal{E}$ *such that the image* $\mathcal{T}_{\mathcal{E}} := \mathbf{i}(\mathcal{T})$ *is a submanifold of* $\mathcal{E}$ *. The* motion *along the trajectory*

$$\{ \varphi^{\mathcal{T}}_{\alpha} : \mathcal{T} \mapsto \mathcal{T} \,, \; \alpha \in \Re \} \tag{3.3}$$

is a simultaneity preserving one-parameter family of maps, fulfilling the composition rule

$$\varphi^{\mathcal{T}}_{\alpha} \circ \varphi^{\mathcal{T}}_{\alpha} = \varphi^{\mathcal{T}}_{(\alpha+\beta)} \,, \tag{3.4}$$

for any pair of time-lapses $\alpha, \beta \in \mathcal{Z}$ *. Each* $\varphi^{\mathcal{T}}_{\alpha} : \mathcal{T} \mapsto \mathcal{T}$ *is a* displacement.

The trajectory can alternatively be considered as a manifold $\mathcal{T}$ by itself, with $\dim \mathcal{T} = 1 + n$, $1 \leq n \leq 3$, or as a submanifold $\mathcal{T}_{\mathcal{E}} = \mathbf{i}(\mathcal{T}) \subset \mathcal{E}$ of the event manifold.

Then, a $(1 + n)$ coordinate system describes $\mathcal{T}$, while an adapted 4D space-time coordinate system in $\mathcal{E}$ describes $\mathcal{T}_{\mathcal{E}}$.

The space-time displacement $\varphi_{\alpha} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ and the trajectory displacement $\varphi^{\mathcal{T}}_{\alpha} : \mathcal{T} \mapsto \mathcal{T}$ are related by the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\mathcal{E}} \arrow[r, "\varphi_{\alpha}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{T}_{\mathcal{E}} \arrow[dd, bend left=60, "t_{\mathcal{E}}"] \\ \mathcal{T} \arrow[u, "\mathbf{i}"] \arrow[r, "\varphi^{\mathcal{T}}_{\alpha}"] \arrow[d, "t_{\mathcal{T}}"] & \mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[d, "t_{\mathcal{T}}"'] \\ \mathcal{Z} \arrow[r, "t_{\alpha}"] & \mathcal{Z} \end{tikzcd} \quad \iff \quad \begin{cases} \varphi_{\alpha} \circ \mathbf{i} &= \mathbf{i} \circ \varphi^{\mathcal{T}}_{\alpha} \,, \\ t_{\mathcal{E}} \circ \varphi_{\alpha} &= t_{\alpha} \circ t_{\mathcal{E}} \,, \end{cases} \tag{3.5}$$

where the time translation $t_{\alpha} : \mathcal{Z} \mapsto \mathcal{Z}$ is defined by

$$t_{\alpha}(t) := t + \alpha \,, \quad t, \alpha \in \mathcal{Z} \,. \tag{3.6}$$

As sketched in fig. 3.2, to a space-time displacement $\varphi_{\alpha} : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ there corresponds a pair of maps:

1. a time-preserving *spatial displacement* $\varphi^{\mathcal{S}}_{\alpha} : \mathcal{E} \mapsto \mathcal{E}$,
2. a location-preserving *time step* $\varphi^{\mathcal{Z}}_{\alpha} : \mathcal{E} \mapsto \mathcal{E}$,

which fulfill the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\mathcal{E}} \arrow[r, "\varphi^{\mathcal{S}}_{\alpha}"] \arrow[d, "\varphi^{\mathcal{Z}}_{\alpha}"'] \arrow[dr, "\varphi_{\alpha}"] & \mathcal{E} \arrow[d, "\varphi^{\mathcal{Z}}_{\alpha}"] \\ \mathcal{E} \arrow[r, "\varphi^{\mathcal{S}}_{\alpha}"] & \mathcal{T}_{\mathcal{E}} \end{tikzcd} \quad \iff \quad \varphi_{\alpha} = \varphi^{\mathcal{S}}_{\alpha} \circ \varphi^{\mathcal{Z}}_{\alpha} = \varphi^{\mathcal{Z}}_{\alpha} \circ \varphi^{\mathcal{S}}_{\alpha} \,. \tag{3.7}$$

[3] An embedding is an injective immersion whose co-restriction is continuous with the inverse.

The spatial motion $\{\varphi_\alpha^{\mathcal{S}} : \mathcal{E} \mapsto \mathcal{E}\,,\ \alpha \in \mathcal{Z}\}$ is generated by intersecting each spatial-slice with the time-lines passing through the events of each material particle, as represented by thin red lines in fig. 3.2.

The space-time *velocity* of the motion is defined by the derivative

$$\mathbf{V} := \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha \in T\mathcal{T}_{\mathcal{E}}\,, \tag{3.8}$$

Taking the time derivative of (3.5) we have

$$\partial_{\alpha=0}\,(t_{\mathcal{E}} \circ \boldsymbol{\varphi}_\alpha) = \langle dt_{\mathcal{E}}, \mathbf{V}\rangle = (\partial_{\alpha=0}\, t_\alpha) \circ t_{\mathcal{E}} = 1 \circ t_{\mathcal{E}}\,, \tag{3.9}$$

and comparing with Eq. (3.2) we infer the decomposition into space and time components

$$\mathbf{V} = \mathbf{v} + \mathbf{Z}\,, \tag{3.10}$$

with $\langle dt_{\mathcal{E}}, \mathbf{v}\rangle = 0$.

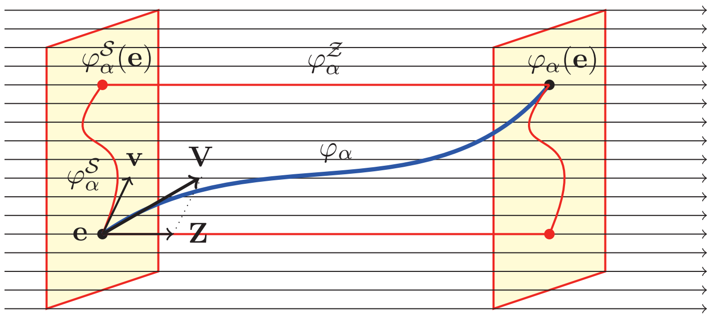


Figure 3.2: Displacement decomposition.

Definition 3 (Material particles and body manifold) *The physical notion of material particle corresponds in the geometric view to a time parametrized curve of events in the trajectory, related by the motion as described by the characteristic property*

$$\exists\, \alpha \in \mathcal{Z}\ :\ \mathbf{e}_2 = \boldsymbol{\varphi}_\alpha^{\mathcal{T}}(\mathbf{e}_1), \quad \mathbf{e}_1, \mathbf{e}_2 \in \mathcal{T}\,. \tag{3.11}$$

Accordingly, we will say that a geometrical object is defined along (not at) a material particle. Events belonging to a material particle form a class of equivalence and the quotient manifold so induced in the trajectory is the body manifold.

Definition 4 (Trajectory time bundle and body placements) *The trajectory inherits from the events manifold the time projection* $t_{\mathcal{T}} := t_{\mathcal{E}} \circ \mathbf{i} : \mathcal{T} \mapsto \mathcal{Z}$. *A fiber of simultaneous events, in the corresponding vertical time-bundle* $V\mathcal{T}$, *is a body placement* $\boldsymbol{\Omega} \subset \mathcal{T}_{\mathcal{E}}$, *with* $\dim \boldsymbol{\Omega} = n$.

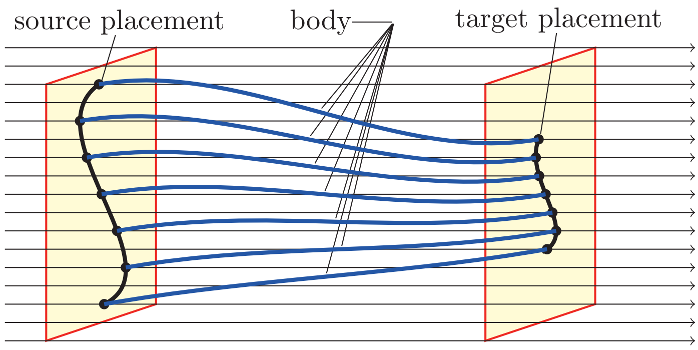


Figure 3.3: Particles, body and placements.

The motions of a body is characterized by the conservation property concerning the mass. This is a measure induced by a maximal material form $\mathbf{m} : \mathcal{T} \mapsto \mathrm{MAX}(V\mathcal{T})$, that is a form on the trajectory manifold whose values are n-covectors on the time vertical bundle (Romano, G., 2007).

Definition 5 (Mass conservation) *Mass conservation along the motion is expressed by the integral condition that, for all placements* $\boldsymbol{\Omega}$

$$\int_{\varphi_\alpha(\boldsymbol{\Omega})} \mathbf{m} = \int_{\boldsymbol{\Omega}} \varphi_\alpha{\downarrow}\mathbf{m} = \int_{\boldsymbol{\Omega}} \mathbf{m}\,. \tag{3.12}$$

Upon localisation, Eq. (3.12) *may be expressed by the equivalent pull-back and* LIE*-derivative conditions*

$$\varphi_\alpha{\downarrow}\mathbf{m} = \mathbf{m} \iff \mathcal{L}_{\mathbf{V}}\,\mathbf{m} = \mathbf{0}\,. \tag{3.13}$$

3.2 Spatial and material tensor fields

On the basis of the geometric framework set forth, the notions of spatial and material fields can be introduced in a natural way.

As a warning for the reader, we emphasize that these notions, which make no appeal to a local reference placement, do not comply with the homonymic nomenclature of usage in literature, in the wake of the one adopted in (Truesdell and Noll, 1965).

According to the nomenclature in (Truesdell and Noll, 1965) *material* tensor fields are based on a reference placement and are just pull-back of tensor fields defined on the current placement, there called *spatial* tensor fields.

In the new geometric theory *spatial* and *material* vector fields are instead both defined on body placements in the trajectory, the former being tangent to space slices, while the latter to body placements.

Definition 6 (Space-time bundle) *The space-time bundle* $(T\mathcal{E})_{\mathcal{T}}$ *is the restriction of the tangent bundle* $T\mathcal{E}$ *to vectors based on the trajectory.*

Definition 7 (Spatial bundle) *The* spatial bundle $(V\mathcal{E})_{\mathcal{T}}$ *is a sub-bundle of the space-time bundle* $(T\mathcal{E})_{\mathcal{T}}$ *made of time-vertical tangent vectors*

$$(V\mathcal{E})_{\mathcal{T}} := \{ \mathbf{v}_{\mathcal{E}} \in (T\mathcal{E})_{\mathcal{T}} \quad \textit{such that} \quad \langle dt_{\mathcal{E}}, \mathbf{v}_{\mathcal{E}} \rangle = 0 \} . \tag{3.14}$$

Definition 8 (Material bundle) *The* material bundle $V\mathcal{T}$ *is the sub-bundle of the tangent trajectory bundle* $T\mathcal{T}$ *made of time-vertical tangent vectors*

$$V\mathcal{T} := \{ \mathbf{v}_{\mathcal{T}} \in T\mathcal{T} \quad \textit{such that} \quad \langle dt, \mathbf{v}_{\mathcal{T}} \rangle = 0 \} , \tag{3.15}$$

and the immersed material bundle $V\mathcal{T}_{\mathcal{E}}$ *is defined by*

$$V\mathcal{T}_{\mathcal{E}} := \{ \mathbf{v}_{\mathcal{E}} \in T\mathcal{T}_{\mathcal{E}} \quad \textit{such that} \quad \langle dt_{\mathcal{E}}, \mathbf{v}_{\mathcal{E}} \rangle = 0 \} . \tag{3.16}$$

To simplify, the space-time immersion $V\mathcal{T}_{\mathcal{E}} = \mathbf{i}{\uparrow}(V\mathcal{T})$ *will also be referred to as the material bundle.*

Spatial and material tensors are multilinear maps acting respectively on spatial and material vectors (Romano, Barretta, Diaco, 2014a).

All tensor fields of interest in Continuum Mechanics are defined on the trajectory manifold and are therefore either *spatial* or *material* tensor fields, according to Defs. 8 and 7.

The only (and important) exception is the *metric tensor field* which is defined on the whole event manifold $\mathcal{E}$.

Acceleration, force and metric are *spatial* vector, covector and tensor fields. Stress, stressing, stretching, heat flux, temperature and thermodynamical potentials, are *material* tensor, vector and scalar fields. Only material fields are allowed to enter in constitutive relations. These involve in fact material tensors and their time rates along the motion.

Definition 9 (Covariant, contravariant & mixed tensors) *Covariant tensors act on pairs of tangent vectors and may be equivalently interpreted as linear maps from tangent to cotangent spaces. In the same way, contravariant tensors acting on pairs of cotangent vectors, may be interpreted as linear maps from cotangent to tangent spaces, and mixed tensors acting on pairs of tangent-cotangent vectors, may be interpreted as linear maps from tangent spaces to themselves.*

Definition 10 (Strain and stress bundles) *The linear bundle of covariant (contravariant) tensors on the material bundle* $V\mathcal{T}$*, respectively denoted by* $\textsc{Cov}(V\mathcal{T})$ *(*$\textsc{Con}(V\mathcal{T})$ *) will be called the strain (stress) bundle.*

3.2.1 Natural comparison of material tensors

Definition 11 (Naturality) *A notion concerning* material *tensors is said to be* natural *if it depends only on the* metric *properties of the event manifold and on the* motion*, no other arbitrary assumption (such as the choice of a parallel transport) being involved (Romano, Barretta, 2013b; Romano, Barretta, Diaco, 2014a).*

To perform the time-derivative of a material tensor field along the motion, a transportation tool must be employed to bring the base point to the event in the trajectory corresponding to the evaluation time, prior to taking the derivative with respect to the time lapse.

In this respect, we underline that two material tensor fields $\mathbf{s}_1$ and $\mathbf{s}_2$, based at a same event $\mathbf{e} \in \mathcal{T}$ on the trajectory, are naturally compared by taking the difference between their evaluation on any pair of argument vectors based at that event.

The question, of how to compare material tensors based at distinct events along a particle on the trajectory, requires a more careful geometric examination.

At its root there is the question concerning the comparison of material vectors tangent to distinct placements of the body along a particle. We call attention to the following items.

1. The comparison requires the availability of a map apt to transform, in a linear and invertible way, a tangent vector based at an event along a particle, into another one based at the evaluation event, where subtraction can be operated upon.

2. The temptation of defining equality by parallel transport is to be resisted because an unnatural choice is involved. The same comment holds if equality is defined by invariance of cartesian components.

3. Parallel transport is not feasible for lower dimensional bodies since parallel transported material vectors (tangent to a placement) will in general no more be material (tangent to the transformed placement), see fig. 3.5.

The natural way to compare the values of a material tensor field along a particle consists in pulling-back by the displacement map and leads to the following definition.

Definition 12 (Material time invariance) Time-invariance *of a* material *tensor* $\mathbf{s}_{\text{MAT}} \in \text{TENS}(V\mathcal{T})$ *along the motion, means fulfilment of the pull-back relation*

$$\mathbf{s}_{\text{MAT}} = \boldsymbol{\varphi}_{\alpha}^{\mathcal{T}}\!\downarrow\!\mathbf{s}_{\text{MAT}}\,. \tag{3.17}$$

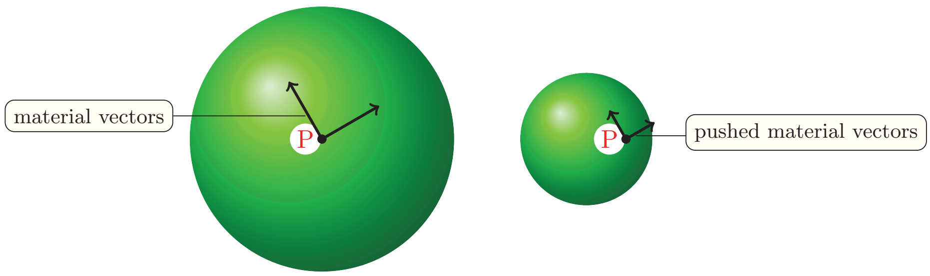


Figure 3.4: Push of vectors tangent to a material surface.

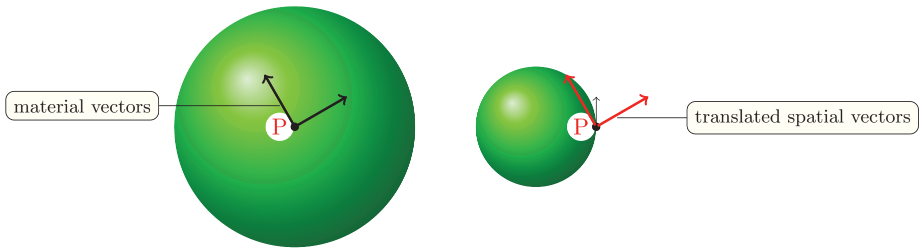


Figure 3.5: Parallel transport of vectors tangent to a material surface.

According to this definition, the time rate is evaluated as LIE *derivative* along the motion, still a material tensor field, see fig. 3.4 where two events belonging to the same particle P are considered.

Definition 13 (Material time derivative) *The* time derivative *of a* material *tensor field* $\mathbf{s}_{\text{MAT}} \in \text{TENS}(V\mathcal{T})$ *is naturally provided by the* LIE *derivative along the motion*

$$\dot{\mathbf{s}}_{\text{MAT}} := \mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\,\mathbf{s}_{\text{MAT}} := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_{\alpha}^{\mathcal{T}}\downarrow\mathbf{s}_{\text{MAT}})\,. \tag{3.18}$$

3.2.2 Comparison of spatial tensors by parallel transport

In general, the time-derivative of a spatial tensor field along the motion cannot be performed by pull-back along the motion, because, for lower dimensional bodies, the immersed material bundle is only a proper sub-bundle of the spatial tensor bundle and therefore the tangent displacement cannot operate on spatial vectors.

Accordingly, a *not natural* choice of spatial parallel transport in the event manifold $\mathcal{E}$ is needed. In the EUCLID framework the parallel transport by translation is tacitly assumed.

Anyway, even in the EUCLID framework, different choices of parallel transport are possible and may be more convenient.

An instance occurs when curvilinear coordinate systems are considered (Romano, Barretta, 2013a).

The choice of a linear parallel transport leads to the following notions.

Definition 14 (Spatial time invariance) Time-invariance *of a spatial tensor* $\mathbf{s}_{\text{SPA}} \in \text{TENS}((V\mathcal{E})_{\mathcal{T}})$ *along the motion, means fulfilment of the transport-back relation*

$$\mathbf{s}_{\text{SPA}} = \boldsymbol{\varphi}_\alpha \Downarrow \mathbf{s}_{\text{SPA}} \,. \tag{3.19}$$

Definition 15 (Spatial time derivative) *The definition of time derivative of a* spatial *tensor field* $\mathbf{s} \in \text{TENS}((V\mathcal{E})_{\mathcal{T}})$ *along the motion, is provided by the parallel derivative*

$$\dot{\mathbf{s}}_{\text{SPA}} := \nabla_{\mathbf{V}}\, \mathbf{s}_{\text{SPA}} := \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \Downarrow \mathbf{s}_{\text{SPA}}) \,. \tag{3.20}$$

The derivative defined by in (3.20) is usually split into spatial and time components by setting

$$\nabla_{\mathbf{V}}\, \mathbf{s}_{\text{SPA}} = \nabla_{\mathbf{v}}\, \mathbf{s}_{\text{SPA}} + \nabla_{\mathbf{Z}}\, \mathbf{s}_{\text{SPA}} \,, \tag{3.21}$$

The split form Eq. (3.21) of the parallel derivative Eq. (3.20) is commonly named *material time derivative* but this nomenclature is not appropriate because the field resulting from Eq. (3.20) is *not* a material tensor, but a spatial tensor, as evidenced by the sketch in fig. 3.5.

Moreover the split (3.21) is in general not performable because the vectors $\mathbf{v}$ and $\mathbf{Z}$ may be transversal to the trajectory, for lower dimensional bodies.

The spatial time derivative of the tangent displacement $\mathbf{F}_\alpha := T\boldsymbol{\varphi}_\alpha : V\mathcal{T}_{\mathcal{E}} \mapsto V\mathcal{T}_{\mathcal{E}}$ is the spatial tensor defined, according to Eq. (3.20), by [4]

$$\mathbf{L}(\mathbf{v}) := \dot{\mathbf{F}} = \partial_{\alpha=0}\, (\boldsymbol{\varphi}_\alpha \Downarrow T\boldsymbol{\varphi}_\alpha) \,. \tag{3.22}$$

Lemma 1 provides an expression in terms of the spatial velocity field

$$\mathbf{L}(\mathbf{v}) = \nabla\mathbf{v} + \mathbf{T}(\mathbf{v}) \,. \tag{3.23}$$

The usual formula $\mathbf{L}(\mathbf{v}) = \nabla\mathbf{v}$ is recovered when a linear torsion-free connection, such as EUCLID translation, is adopted.

The *spatial* tensor field $\mathbf{L}(\mathbf{v})$ is *not* a *natural* notion, being dependent on the choice of a linear connection in the event manifold. Therefore, being neither *material* nor *natural*, its appearance in constitutive relations must be carefully avoided (Romano, Barretta, Diaco, 2014a).

[4] The standard notations $\mathbf{F}_\alpha$ (the cryptic symbol $\mathbf{F}$ is commonly adopted) and $\dot{\mathbf{F}}$ are recalled here for direct comparison with treatments in literature.

This comment contributes to deprive of physical basis treatments of geometrically nonlinear continuum mechanics in which the tangent displacement $\mathbf{F}_\alpha := T\boldsymbol{\varphi}_\alpha : V\mathcal{T}_\mathcal{E} \mapsto V\mathcal{T}_\mathcal{E}$ and its spatial time derivative $\mathbf{L}(\mathbf{v}) := \dot{\mathbf{F}} : V\mathcal{T}_\mathcal{E} \mapsto (V\mathcal{E})_{\mathcal{T}_\mathcal{E}}$ play the role of state variables.

Although this inadequacy was pointed out in (Romano, Barretta, 2011, 2013b; Romano, Barretta, Diaco, 2014a) such unphysical treatments are still being proposed.

3.3 The geometry of metric measurements

At the very foundation of continuum mechanics is the way metric measurements are performed.

The axioms of Euclid geometry assure that the distance between space points can be measured in such a way that the associated vector space is endowed with a norm topology. This means that the map defining the length of vectors is positively homogeneous, positive definite and fulfils the triangle inequality.

A result in the theory of normed linear spaces, reported in Def. **??**, is of extraordinary importance in kinematics of continua.

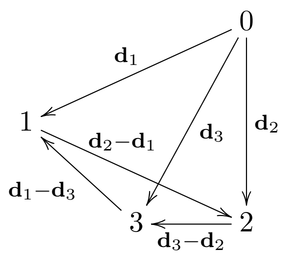
It states that, if the norm fulfils the parallelogram identity, then the polarization formula defines a symmetric and positive definite bilinear form on tangent spaces.

This bilinear form is the *metric*, a twice covariant tensor (its arguments are both tangent vectors) which provides the master way of investigating, point by point, the geometrical properties of a continuum and their variations along a motion.

The mechanical behavior of materials, under various conditions of interest, is investigated, in the last instance, by performing direct or indirect metric measurements of lengths.

To get a complete description of the metric properties at a material point, it is sufficient to measure the lengths $\ell_k\,, k = 1, \ldots, m$ of the edges $\mathbf{d}_k\,, k = 1, \ldots, m$ of an infinitesimal non degenerate simplex at that point.

Being $n+1$ the number of vertices of the simplex in a body placement of dimension n, the number of edges is given by $m = n(n+1)/2$.



(3.24)

The lengths $\ell_k \,, k = 1, \ldots, m$ of the edges are conveniently denoted by

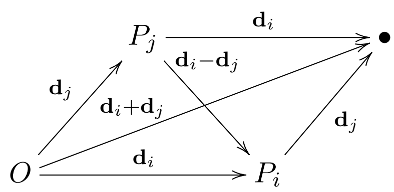
$$\|\mathbf{d}_k\| = \ell_k \,, \quad k = 1, \ldots, m \,. \tag{3.25}$$

In a 3D body the simplex is a tetrahedron and $m = 6$.

To evaluate the dilation rates along any direction, on the basis of the measurements in Eq. (3.25), it is expedient to assume $\mathbf{d}_i \,, i = 1, \ldots, 3$ as a basis and write

$$\mathbf{d}_4 = \mathbf{d}_2 - \mathbf{d}_1 \,, \quad \mathbf{d}_5 = \mathbf{d}_3 - \mathbf{d}_2 \,, \quad \mathbf{d}_6 = \mathbf{d}_1 - \mathbf{d}_3 \,. \tag{3.26}$$

Definition 16 (Parallelogram identity) *With reference to the diagram*



(3.27)

for $i, j = 1, 2, 3$*, the folling* parallelogram identity *holds*

$$||\mathbf{d}_i + \mathbf{d}_j||^2 + ||\mathbf{d}_i - \mathbf{d}_j||^2 = 2\,(||\mathbf{d}_i||^2 + ||\mathbf{d}_j||^2) \,. \tag{3.28}$$

The next result shows that the length of any tangent vector can be evaluated from the knowledge of the lengths of the sides of a non-degenerate tetrahedron as the one in the diagram Eq. (3.24).

Proposition 2 (Space metric) *In a normed linear space, in which the parallelogram identity is fulfilled, the polarization formula*

$$\mathbf{g}(\mathbf{d}_i \,, \mathbf{d}_j) := \tfrac{1}{4} \Big(||\mathbf{d}_i + \mathbf{d}_j||^2 - ||\mathbf{d}_i - \mathbf{d}_j||^2 \Big) \,, \tag{3.29}$$

and the equivalent one involving the sides of triangle (O, P_i, P_j) *in the diagram Eq.* (3.27)

$$\mathbf{g}(\mathbf{d}_i \,, \mathbf{d}_j) := \tfrac{1}{2} \Big((||\mathbf{d}_i||^2 + ||\mathbf{d}_j||^2) - ||\mathbf{d}_i - \mathbf{d}_j||^2 \Big) \,, \tag{3.30}$$

define a twice covariant symmetric and positive definite metric tensor.[5]

Proposition 2, applied to the space slices of Euclid space-time, in which the parallelogram identity is fulfilled, provides the key to introduce the space metric tensor $\mathbf{g}_{\mathrm{SPA}} : V\mathcal{E} \mapsto (V\mathcal{E})^*$.

[5] This result is a theorem due to Maurice Fréchet, John von Neumann, Pascual Jordan (Yosida, 1980).

To include the general case of possibly lower dimensional bodies, we consider the linear operator

$$\mathbf{\Pi_e} : T_{\mathbf{e}}\mathcal{S} \mapsto T_{\mathbf{e}}\mathbf{\Omega}\,, \tag{3.31}$$

that at $\mathbf{e} \in \mathbf{\Omega}$ projects, in an orthogonal way, the space $T_{\mathbf{e}}\mathcal{S}$ of tangent *space* vectors onto the subspace of vectors tangent to the body placement $\mathbf{\Omega}$, that is *material* vectors.

The adjoint linear operator $\mathbf{\Pi}_{\mathbf{e}}^{A} : T_{\mathbf{e}}\mathbf{\Omega} \mapsto T_{\mathbf{e}}\mathcal{S}$ is the injection of material vectors to generate the corresponding spatial vectors.

Definition 17 (Material metric) *The material metric* $\mathbf{g}_{\text{MAT}} \in \text{Cov}(V\mathcal{T})$ *is the restriction of the space metric tensor* $\mathbf{g}_{\text{SPA}} \in \text{Cov}(V\mathcal{E})$ *to material vectors, as expressed by the pull-back relation*

$$\mathbf{g}_{\text{MAT}} := \mathbf{i}{\downarrow}\mathbf{g}_{\text{SPA}}\,, \tag{3.32}$$

which, see Def. 2, is explicitly written as

$$\mathbf{g}_{\text{MAT}}(\mathbf{h}\,,\mathbf{d}) := \mathbf{g}(T\mathbf{i}\cdot\mathbf{h}\,,T\mathbf{i}\cdot\mathbf{d})\,,\quad \forall\,\mathbf{h},\mathbf{d} \in V\mathcal{T}\,. \tag{3.33}$$

For 3D bodies, the identification $\mathbf{g}_{\text{SPA}} = \mathbf{g}_{\text{MAT}}$ is feasible, even if not advisable for sake of conceptual clarity.

Definition 18 (Geometric stretching) *Setting* $\varphi_\alpha{\downarrow}\mathbf{g}_{\text{MAT}} = \mathbf{g}_{\text{MAT}}\cdot(\mathbf{F}^A\mathbf{F}) = \mathbf{g}_{\text{MAT}}\cdot\mathbf{U}^2$, *the geometric stretching is defined as* LIE*-derivative of the material metric tensor along the motion*

$$\boldsymbol{\varepsilon}_{\mathbf{V}} := \tfrac{1}{2}\dot{\mathbf{g}}_{\text{MAT}} = \tfrac{1}{2}\mathcal{L}_{\mathbf{V}}\,\mathbf{g}_{\text{MAT}} = \mathbf{g}_{\text{MAT}}\cdot\dot{\mathbf{U}}\,. \tag{3.34}$$

The geometric stretching isexpressed in terms of velocities by introducing the $\mathbf{g}_{\text{SPA}}$-symmetric mixed EULER stretching tensor (Euler, 1744, 1761)

$$\mathbf{D}(\mathbf{v}) = \tfrac{1}{2}{\mathbf{g}_{\text{SPA}}}^{-1}\cdot\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\text{SPA}} = \text{sym}_{\mathbf{g}}(\nabla + \mathbf{T})(\mathbf{v}) \in \text{Mix}(V\mathcal{E})\,, \tag{3.35}$$

whose kernel consists in spatial velocity fields that are infinitesimal isometries.[6] In Eq. (3.35) $\mathbf{T}$ is the torsion form of the connection, expressed in terms of parallel derivatives as in Eq. (2.30)

$$\mathbf{T}(\mathbf{v}\,,\mathbf{u}) := \nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v}\,,\mathbf{u}]\,. \tag{3.36}$$

We recall also that $[\mathbf{v}\,,\mathbf{u}] = \mathcal{L}_{\mathbf{v}}\,\mathbf{u}$ is the antisymmetric LIE bracket defined, setting $\mathbf{u}\,f := \nabla_{\mathbf{u}}\,f$ for any smooth scalar function $f : \mathbf{\Omega} \mapsto \Re$, by

$$[\mathbf{v}\,,\mathbf{u}]\,f = \mathbf{v}\mathbf{u}f - \mathbf{u}\mathbf{v}f\,, \tag{3.37}$$

Projecting, we get the material covariant geometric stretching tensor

$$\boldsymbol{\varepsilon}_{\mathbf{V}} = \mathbf{g}_{\text{MAT}}\cdot\mathbf{\Pi}\cdot\mathbf{D}(\mathbf{v})\cdot\mathbf{\Pi}^A \in \text{Cov}(V\mathcal{T})\,. \tag{3.38}$$

[6] By time invariance of the space metric tensor $\mathcal{L}_{\mathbf{Z}}\,\mathbf{g} = \mathbf{0}$ and hence $\mathcal{L}_{\mathbf{V}}\,\mathbf{g} = \mathcal{L}_{\mathbf{v}}\,\mathbf{g}$.

3.4 Comparison with standard treatments

The space metric tensor $\mathbf{g}_{\text{SPA}} \in \mathrm{Cov}(V\mathcal{E})$, being positive definite and hence non-singular in the space bundle, admits an inverse and provides the standard tool to perform alteration of tensors by changing their arguments from tangent to cotangent vectors and vice versa. Deceptively, most treatments of continuum mechanics do not even explicitly mention the metric tensor.

In standard treatments of classical mechanics, space slices of simultaneous events, in the 4D space-time manifold $\mathcal{E}$, are identified and cumulatively denoted by $\mathcal{S}$.

The change from a placement $\boldsymbol{\Omega}$, due to the motion along the trajectory in a time lapse $\alpha \in \Re$, is called a *deformation* $\boldsymbol{\varphi}_\alpha : \boldsymbol{\Omega} \mapsto \mathcal{E}$.

The corresponding tangent map $\mathbf{F}_\alpha := T\boldsymbol{\varphi}_\alpha : T\boldsymbol{\Omega} \mapsto T\mathcal{S}$ is called the *deformation gradient*.

The space metric tensor $\mathbf{g}_{\text{SPA}} \in \mathrm{Cov}(V\mathcal{E})$ enters implicitly and tacitly into the treatment by considering the $\mathbf{g}_{\text{SPA}}$-adjoint $\mathbf{F}_\alpha^A : T(\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})) \mapsto T\boldsymbol{\Omega}$ of $\mathbf{F}_\alpha$ uniquely defined by the identity [7]

$$\mathbf{g}(\mathbf{F}_\alpha \cdot \mathbf{h}\,,\mathbf{d}) = \mathbf{g}(\mathbf{F}_\alpha^A \cdot \mathbf{d}\,,\mathbf{h})\,, \quad \forall\,\mathbf{h} \in T\boldsymbol{\Omega}\,, \quad \forall\,\mathbf{d} \in T(\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}))\,. \tag{3.39}$$

The *polar decomposition*, an algebraic decomposition inferred from Cauchy eigenvalue theory for symmetric operators, occupies a central position in treatments of kinematics (Truesdell, 1977)

$$\begin{aligned} \mathbf{F}_\alpha &= \mathbf{R}_\alpha \mathbf{U}_\alpha = \mathbf{V}_\alpha \mathbf{R}_\alpha : T\boldsymbol{\Omega} \mapsto T\mathcal{S}\,, \\ \mathbf{U}_\alpha^2 &= \mathbf{F}_\alpha^A \mathbf{F}_\alpha : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}\,, \\ \mathbf{V}_\alpha^2 &= \mathbf{F}_\alpha \mathbf{F}_\alpha^A : T(\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})) \mapsto T(\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}))\,. \end{aligned} \tag{3.40}$$

Several observations, questions and comments may arise at this point. The answers to most of them can be now given on the ground of the geometric theory illustrated before.

1. The deformation $\boldsymbol{\varphi}_\alpha : \boldsymbol{\Omega} \mapsto \mathcal{E}$ maps $\boldsymbol{\Omega}$ into $\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})$ and hence the deformation gradient

   $$\mathbf{F}_\alpha := T\boldsymbol{\varphi}_\alpha : T\boldsymbol{\Omega} \mapsto T\mathcal{S}\,, \tag{3.41}$$

   maps $T\boldsymbol{\Omega}$ onto $T(\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega}))$. Then, at each $\mathbf{e} \in \boldsymbol{\Omega}$, $\mathbf{F}_\alpha(\mathbf{e})$ is not a linear transformation of a linear space into itself, but rather a linear transformation from a tangent space, based at an event, to another tangent space, based at the transformed event, i.e.

   $$\mathbf{h} \in T_{\mathbf{e}}\mathcal{S} \quad \mapsto \quad \mathbf{F}_\alpha(\mathbf{e}) \cdot \mathbf{h} \in T_{\boldsymbol{\varphi}_\alpha(\mathbf{e})}\mathcal{S}\,, \quad \forall\,\mathbf{e} \in \boldsymbol{\Omega}\,. \tag{3.42}$$

[7] In most treatments Eq. (3.39) defines the transpose $\mathbf{F}_\alpha^T$. We reserve the transpose for matrices with interchanged rows and columns, the matrix of the adjoint being the transpose only under orthonormal bases, see Sect. 2.6. Here and henceforth a dot (sometimes omitted) declares linear dependence on the subsequent argument.

This means that the isometry in Eq. (3.40) is such that $\mathbf{R}_\alpha(\mathbf{e}) \cdot \mathbf{h} \in T_{\varphi_\alpha(\mathbf{e})}\mathcal{S}$.

- **How to compare tangent vectors based at distinct events along the motion?**
  The standard (and often tacit) way is by translation in the Euclid space, a procedure which is feasible only for 3D bodies. It is in fact clear that for lower dimensional bodies the translation of a tangent vector at a placement will in general not yield a vector tangent to the new placement.
- **There are other ways to perform the comparison?**
  There is not a unique way to perform a parallel transport along a path. Henceforth the question is answered positively, so that an embarrassing choice is left to the questioner, with no guiding principle other than preference for tradition, if feasible.
- **There is a natural way to perform the comparison?**
  A unique natural way is provided by the tangent map $\mathbf{F}_\alpha := T\boldsymbol{\varphi}_\alpha : T\boldsymbol{\Omega} \mapsto T\mathcal{S}$ itself, through the notion of pull-back. Indeed the tangent map acts pointwise as a one-to-one linear transformation between the pairs of involved tangent spaces. The idea is almost bicentenary, going back till to George Green (1839) and to Bernhard Riemann (1854). It involves the notion of a metric tensor $\mathbf{g}_{\text{SPA}} \in \mathrm{Cov}(V\mathcal{E})$ in space and consists in considering the pull-back to the material metric $\mathbf{g}_{\text{MAT}} = \mathbf{i}{\downarrow}\mathbf{g}_{\text{SPA}} \in \mathrm{Cov}(V\mathcal{T})$ and in introducing a fictitious metric on $\boldsymbol{\Omega}$ pulled back from a placement $\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})$, and defined for all $\mathbf{h}, \mathbf{d} \in T\boldsymbol{\Omega}$ by the identity
  $$(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g}_{\text{MAT}})(\mathbf{h}\,,\mathbf{d}) := \mathbf{g}_{\text{MAT}}(T\boldsymbol{\varphi}_\alpha \cdot \mathbf{h}\,, T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d})\,. \tag{3.43}$$
  This means that the pulled-back material metric tensor field, when evaluated on a pair of *material vectors*, that is vectors tangent to the current placement $\boldsymbol{\Omega}$, yields the value of the material metric evaluated on the corresponding pair of transformed material vectors tangent to the transformed placement $\boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})$. By bilinearity, comparing the metric tensor and its pull-back is equivalent to comparing the length of any tangent vector with the length of the transformed one.
- **Is this procedure equivalent to polar decomposition?**
  To answer, let us rewrite the pull-back as
  $$(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g})(\mathbf{h}\,,\mathbf{d}) = \mathbf{g}((T\boldsymbol{\varphi}_\alpha)^A\, T\boldsymbol{\varphi}_\alpha \cdot \mathbf{h}\,, \mathbf{d})\,, \tag{3.44}$$
  which can be rewritten as
  $$\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g} = \mathbf{g} \cdot (\mathbf{F}_\alpha^A \mathbf{F}_\alpha) = \mathbf{g} \cdot \mathbf{U}_\alpha^2\,. \tag{3.45}$$

The answer is then essentially positive only for 3D bodies. However Eq. (3.43) is more general, being applicable to bodies of any dimensionality. On the contrary, the polar decomposition does not, since the isometry $\mathbf{R}_\alpha$ is uniquely defined only for 3D bodies. Moreover, by definition, it plays no role in metric comparisons. The introduction of the pull-back metric avoids the needless polar decomposition and provides instead a direct definition of the stretch tensor $\mathbf{U}^2_\alpha = \mathbf{F}^A_\alpha \mathbf{F}_\alpha$ as mixed alteration of $\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g}$, according to Eq. (3.45).

2. A more subtle but decisive point concerns the definition of time-rate of the deformation gradient. Taking the derivative $\partial_{\alpha=0}$ in Eq. (3.45), recalling by Eq. (3.8) that $\mathbf{V} := \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha$ is the space-time motion velocity, and observing that $\mathbf{U}_0 = \mathbf{I}$, we get

$$
\begin{aligned}
\dot{\mathbf{g}} := \mathcal{L}_{\mathbf{V}}\mathbf{g} &= \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g} \\
&= \partial_{\alpha=0}\,\mathbf{g}\cdot(\mathbf{F}^A_\alpha \mathbf{F}_\alpha) \\
&= \partial_{\alpha=0}\,\mathbf{g}\cdot\mathbf{U}^2_\alpha \\
&= 2\,\mathbf{g}\cdot\partial_{\alpha=0}\,\mathbf{U}_\alpha\,.
\end{aligned}
\tag{3.46}
$$

The evaluations $\dot{\mathbf{g}} := \mathcal{L}_{\mathbf{V}}\mathbf{g} = \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g}$ and $\dot{\mathbf{U}} := \partial_{\alpha=0}\,\mathbf{U}_\alpha$ are legitimate because the tensors $(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g})(\mathbf{e})$ and $\mathbf{U}_\alpha(\mathbf{e})$ refer to the same tangent space $T_{\mathbf{e}}\mathcal{S}$ for all $\alpha \in \Re$.

The geometric stretching is then defined on the basis of Eq. (3.46) by

$$
\boldsymbol{\varepsilon}_{\mathbf{V}} := \tfrac{1}{2}\dot{\mathbf{g}} = \tfrac{1}{2}\mathcal{L}_{\mathbf{V}}\mathbf{g} = \mathbf{g}\cdot\dot{\mathbf{U}}\,.
\tag{3.47}
$$

On the contrary the derivative $\dot{\mathbf{F}} := \partial_{\alpha=0}\,\mathbf{F}_\alpha$ is feasible only for 3D bodies and is *not* a natural operation, since the evaluation of $\partial_{\alpha=0}\,\mathbf{F}_\alpha(\mathbf{e})$ requires the choice of a parallel transport along the motion, in order to bring the base point $\boldsymbol{\varphi}_\alpha(\mathbf{e}) \in \boldsymbol{\varphi}_\alpha(\boldsymbol{\Omega})$ back to $\mathbf{e} \in \boldsymbol{\Omega}$.

The practice, of setting $\mathbf{L} := \dot{\mathbf{F}}_0$, with $\dot{\mathbf{F}}_0 \cdot \mathbf{h} := \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha{\Downarrow}(\mathbf{F}_\alpha\cdot\mathbf{h})$ for any material vector $\mathbf{h} \in V\mathcal{T}_{\mathcal{E}}$, and of performing the additive splitting $\mathbf{L} = \mathbf{D} + \mathbf{W}$ of the tensor $\mathbf{L}$ into symmetry and skew-symmetric parts, is therefore not applicable to the formulation of constitutive relations. The difficulty is evident when bodies of any dimensionality are included in the analysis. The mixed tensor $\mathbf{L}$, whose expression is given in Lemma 1, is *not* a *material* tensor and *not* a *natural* notion. In fact it is a *space* tensor and its definition depends on the choice of a space parallel transport along the motion (a *not natural* operation).

3. A further question concerns the formulation of constitutive equations in terms of strain or deformation gradient. Both these notions are in fact pertinent to a change of placement of a body and cannot be

defined on the current placement unless some other reference (local) placement is fixed. Moreover deformations due to phenomena other than the ones under investigation may occur and so the geometric strain cannot be a significant constitutive parameter. For instance, in investigating an elastic behavior, the body may deform due to temperature changes, viscosity, plastic flows or phase transformations. To exclude these extraneous deformations one is compelled, in the last instance, to *define* the elastic stretching starting from the stress change and excluding other action rates. The controlling role is therefore to be attributed to stress, temperature, electric and magnetic fields etc., i.e. to state variables, and to their time variations along the motion, while the geometric stretching is to be determined as the global output of the combination of constitutive relations describing different kinds of the material behavior.

- **How to fix the reference placement?** Effective rules or criteria are usually not enunciated and in fact can hardly be conceived. All efforts are directed towards the goal of ensuring that any reasonable choice should be equally acceptable. At the end of this route the conclusion seems however to be in the implicit admission that the choice of a reference placement must be avoided to get a meaningful theoretical framework for constitutive relations (Noll, 2004).
- **How to write constitutive relations?** Constitutive laws cannot connect the finite strain with the stress state, since the strain variable depends on the comparison of two body placements, while the stress state variable pertains only to the current placement. This point is usually hidden by a synthetic but obscure notation which does not display, in an explicit way, the two placements to be put in comparison. Once that reference placements have been banned from constitutive relations, the master choice consists in a rate formulation, by defining the instantaneous response of the material to current values of state variables and their rates along the motion. Experimental tests are in fact to be designed and interpreted according to this paradigm. As will be discussed in Sect. 4.6, a class of isometric local placements, suitable to perform laboratory tests, is considered to provide an experimental basis to rate constitutive parameters. Invariance along the motion is imposed to define the elastic constitutive behavior in other placements.

The next and last question deals with a longly debated issue in mechanics about the way the rate of state variables in constitutive relations are to be evaluated.

- **How to evaluate stress rates in constitutive relations?** A definite answer to this question is possible only after a strict analysis has been carried out and its mathematical formulation again requires the recourse to differential geometric notions. The issue will be detailedly investigated in the sequel but we may here set a guiding principle. Material tensors, the ones entering in constitutive relations, based at different placements of the body, must be compared in a natural way, by push along the motion.

3.5 Trajectory straightening

To apply the tools of calculus in linear spaces to the nonlinear context at hand, the key procedure consists in making recourse to a diffeomorphism [8]

$$\boldsymbol{\xi} : \boldsymbol{\Omega}_{\text{REF}} \times I \mapsto \mathcal{T}_{\mathcal{E}} \,, \tag{3.48}$$

whose inverse transforms the $(n+1)$D trajectory segment $\mathcal{T}_{\mathcal{E}}$ into a straightened one representable as a direct product $\boldsymbol{\Omega}_{\text{REF}} \times I$ of a nD manifold $\boldsymbol{\Omega}_{\text{REF}}$ times an interval of time I .

By the transformation, the motion is reduced to a time translation $\text{tr}_\alpha : \boldsymbol{\Omega}_{\text{REF}} \times I \mapsto \boldsymbol{\Omega}_{\text{REF}} \times I$, defined by

$$\text{tr}_\alpha(\mathbf{x}\,, t) := (\mathbf{x}\,, t+\alpha)\,, \tag{3.49}$$

for all $\mathbf{x} \in \boldsymbol{\Omega}_{\text{REF}}$ and all $t \in I$, as described by the commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\varphi_\alpha"] & \mathcal{T} \\ \boldsymbol{\Omega}_{\text{REF}} \times I \arrow[u, "\boldsymbol{\xi}"] \arrow[r, "\text{tr}_\alpha"] & \boldsymbol{\Omega}_{\text{REF}} \times I \arrow[u, "\boldsymbol{\xi}"'] \end{tikzcd} \quad \Longleftrightarrow \quad \varphi_\alpha \circ \boldsymbol{\xi} = \boldsymbol{\xi} \circ \text{tr}_\alpha \,. \tag{3.50}$$

By the standard identification $\{\boldsymbol{\Omega}_{\text{REF}}\,, t\} = \boldsymbol{\Omega}_{\text{REF}}$ for all $t \in I$, the straightened trajectory may be interpreted as a reference manifold $\boldsymbol{\Omega}_{\text{REF}}$ and the motion as a dependence on a time-parameter.

A straightening map $\boldsymbol{\xi} : \boldsymbol{\Omega}_{\text{REF}} \times I \mapsto \mathcal{T}_{\mathcal{E}}$, with a finite dimensional reference manifold $\boldsymbol{\Omega}_{\text{REF}}$, is adopted in computational methods of continuum mechanics, to perform linear operations.

The co-restricted map $\boldsymbol{\xi} : \boldsymbol{\Omega}_{\text{REF}} \times I \mapsto \boldsymbol{\xi}(\boldsymbol{\Omega}_{\text{REF}} \times I)$ is invertible but an explicit expression of the inverse map is usually not available.

The relevant tangent map is thus evaluated as the inverse of the co-restricted tangent map $T\boldsymbol{\xi} : T\boldsymbol{\Omega}_{\text{REF}} \times TI \mapsto T\boldsymbol{\xi}(T\boldsymbol{\Omega}_{\text{REF}} \times TI)$.

[8] A *morphism* is a fiber respecting map between fiber bundles. A *diffeomorphism* is an invertible morphism which is continuously differentiable with the inverse.

A most useful property follows from the naturality of the Lie derivative with respect to push-pull transformations by diffeomorphisms.

Hence a Lie derivative along the motion is transformed by the straightening map into a partial time derivative, at a fixed point of $\mathbf{\Omega}_{\mathrm{REF}}$, as explicated by the equality

$$\boldsymbol{\xi}{\downarrow}(\mathcal{L}_{\mathbf{V}}\, \mathbf{s}) = \mathcal{L}_{\boldsymbol{\xi}{\downarrow}\mathbf{V}}\, (\boldsymbol{\xi}{\downarrow}\mathbf{s}) = \mathcal{L}_{\mathbf{Z}}\, \mathbf{s}_{\mathrm{REF}} = \partial_{\alpha=0}\, (\mathbf{s}_{\mathrm{REF}} \circ \mathrm{tr}_{\alpha})\,, \tag{3.51}$$

where $\mathbf{s}$ is any material tensor field and $\mathbf{s}_{\mathrm{REF}} := \boldsymbol{\xi}{\downarrow}\mathbf{s}$. Instances of applications in theoretical and computational contexts will respectively be illustrated in Th. 1 of Sect. 4.4 and in Sect. 4.5.

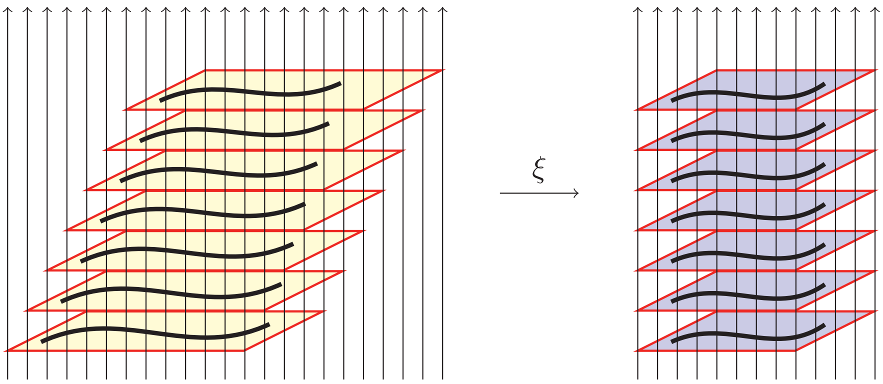


Figure 3.6: Straightening of the trajectory.

3.6 Dynamical equilibrium

Definition 19 (Kirchhoff stress) *To the infinitesimal isometricity constraint* $\boldsymbol{\varepsilon}_{\delta\mathbf{v}} = \mathbf{0}$ *on virtual velacity fields* $\delta\mathbf{v} : \mathbf{\Omega} \mapsto T\mathcal{S}$ *, there corresponds a field* $\boldsymbol{\sigma} : \mathbf{\Omega} \mapsto \mathrm{Con}(T\mathbf{\Omega})$ *of* Kirchhoff *stresses, twice contravariant and symmetric tensors playing the role of* Lagrange *multipliers dual to the virtual geometric stretching* $\boldsymbol{\varepsilon}_{\delta\mathbf{v}} \in \mathrm{Cov}(T\mathbf{\Omega})$ *, defined by Eq.* (3.38)*.* [9]

The composition $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_{\delta\mathbf{v}}$ at $\mathbf{e} \in \mathcal{E}$ is a linear operator on the tangent space $T_{\mathbf{e}}\mathbf{\Omega}$, and the relevant linear invariant defines the duality pairing

$$\langle\, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\delta\mathbf{v}} \,\rangle := J_1(\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}_{\delta\mathbf{v}})\,, \tag{3.52}$$

which provides the internal virtual mechanical power expended per unit mass.

[9] The idea, of introducing the stress as Lagrange multiplier for the virtual geometric stretching, dates back to Gabrio Piola in (Piola, 1833).

Definition 20 (Equilibrium) *The principle of dynamical equilibrium states that the virtual power performed by external forces* $\mathbf{f}_{\text{EXT}}$ *acting on a virtual velocity field, minus the internal virtual power performed by the stress field acting on the stretching* $\boldsymbol{\varepsilon}_{\delta\mathbf{v}} \in \text{COV}(T\boldsymbol{\Omega})$ *, must be equal to the rate of variation of the kinetic momentum along the motion. By conservation of mass Eq.* (3.13)*,* D'ALEMBERT *formulation of the principle writes*

$$\langle \mathbf{f}_{\text{EXT}}, \delta\mathbf{v} \rangle - \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\delta\mathbf{v}} \rangle \, \mathbf{m} = \int_{\boldsymbol{\Omega}} \mathbf{g}_{\text{SPA}}(\mathbf{a} \, , \delta\mathbf{v}) \, \mathbf{m} \, , \quad \forall \, \delta\mathbf{v} : \boldsymbol{\Omega} \mapsto T\mathcal{S} \, , \tag{3.53}$$

where $\mathbf{a} : \boldsymbol{\Omega} \mapsto T\mathcal{S}$ *is the acceleration field, i.e. the spatial vector field defined, according to Eq.* (3.20)*, as the time-rate of variation of the spatial velocity along the motion*

$$\mathbf{a} := \nabla_{\mathbf{V}}(\mathbf{v}) \, . \tag{3.54}$$

If test fields in Eq. (3.53) are restricted to be infinitesimal isometries, that is such that $\boldsymbol{\varepsilon}_{\delta\mathbf{v}} = \mathbf{0}$, the condition of dynamical equilibrium implies that

$$\langle \mathbf{f}_{\text{EXT}}, \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} \mathbf{g}_{\text{SPA}}(\mathbf{a} \, , \delta\mathbf{v}) \, \mathbf{m} \, , \quad \forall \, \delta\mathbf{v} : \boldsymbol{\Omega} \mapsto T\mathcal{S} : \boldsymbol{\varepsilon}_{\delta\mathbf{v}} = \mathbf{0} \, . \tag{3.55}$$

Vice versa, fulfillment of the variational condition in Eq. (3.55) assures that there exists at least a field of KIRCHHOFF stresses $\boldsymbol{\sigma} : \boldsymbol{\Omega} \mapsto \text{CON}(T\boldsymbol{\Omega})$ verifying the equilibrium condition Eq. (3.53). This implication holds as a proven theorem for 3D bodies (Romano and Diaco, 2004). The proof follows from KORN's second inequality, by resorting to BANACH's closed range theorem (Yosida, 1980).

Definition 21 (Stressing) *The rate of variation of the stress field along the motion is evaluated in a natural way by the* LIE*-derivative*

$$\mathcal{L}_{\mathbf{V}} \, \boldsymbol{\sigma} = \partial_{\alpha=0} \, (\boldsymbol{\varphi}_{\alpha} \!\downarrow\! \boldsymbol{\sigma}) \, . \tag{3.56}$$

In performing LIE derivatives, it should be recalled that the pull back of a material scalar field $f : \boldsymbol{\Omega} \mapsto \text{FUN}(T\boldsymbol{\Omega})$ and of a material vector field $\mathbf{h} : \boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}$ along the motion, are respectively given by

$$\begin{aligned} \boldsymbol{\varphi}_{\alpha} \!\downarrow\! f &:= f \circ \boldsymbol{\varphi}_{\alpha} \, , \\ \boldsymbol{\varphi}_{\alpha} \!\downarrow\! \mathbf{h} &:= T\boldsymbol{\varphi}_{-\alpha} \cdot (\mathbf{h} \circ \boldsymbol{\varphi}_{\alpha}) \, . \end{aligned} \tag{3.57}$$

The pull back $\boldsymbol{\varphi}_{\alpha} \!\downarrow\! \boldsymbol{\omega}$ of a material covector field $\boldsymbol{\omega}$ is defined in a natural way as

$$\langle \boldsymbol{\varphi}_{\alpha} \!\downarrow\! \boldsymbol{\omega}, \boldsymbol{\varphi}_{\alpha} \!\downarrow\! \mathbf{h} \rangle = \boldsymbol{\varphi}_{\alpha} \!\downarrow\! \langle \boldsymbol{\omega}, \mathbf{h} \rangle = \langle \boldsymbol{\omega}, \mathbf{h} \rangle \circ \boldsymbol{\varphi}_{\alpha} \, . \tag{3.58}$$

The pull back of other tensor fields is introduced in an analogous way.

We also recall that, denoting, as usual, dual operators by a star $()^*$, a direct evaluation yields for covariant, contravariant and mixed tensors,[10] the pull-back formulae

$$\begin{aligned} \boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{s}_{\mathrm{COV}} &= (T\boldsymbol{\varphi}_\alpha)^* \cdot \mathbf{s}_{\mathrm{COV}} \cdot T\boldsymbol{\varphi}_\alpha\,, \\ \boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{s}_{\mathrm{CON}} &= T\boldsymbol{\varphi}_{-\alpha} \cdot \mathbf{s}_{\mathrm{CON}} \cdot (T\boldsymbol{\varphi}_{-\alpha})^*\,, \\ \boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{s}_{\mathrm{MIX}} &= T\boldsymbol{\varphi}_{-\alpha} \cdot \mathbf{s}_{\mathrm{MIX}} \cdot T\boldsymbol{\varphi}_\alpha\,. \end{aligned} \tag{3.59}$$

The pull-back of an operator is defined by the property that, acting on pulled-back arguments, it provides the pull-back of the original result, a notion that will be resorted to in Eq. (4.8).

A metric is induced on the bundle of mixed material tensors $\mathbf{s}_{\mathrm{MIX}} \in \mathrm{MIX}(V\mathcal{T})$ by the material metric tensor $\mathbf{g}_{\mathrm{MAT}} \in \mathrm{COV}(V\mathcal{T})$, according to the definition

$$\langle \mathbf{s}_{\mathrm{MIX}}, \bar{\mathbf{s}}_{\mathrm{MIX}} \rangle = J_1(\mathbf{s}^A_{\mathrm{MIX}} \cdot \bar{\mathbf{s}}_{\mathrm{MIX}})\,, \tag{3.60}$$

where the $\mathbf{g}_{\mathrm{MAT}}$-adjoint mixed tensor $\mathbf{s}^A_{\mathrm{MIX}} \in \mathrm{MIX}(V\mathcal{T})$ is defined by the identity

$$\mathbf{g}_{\mathrm{MAT}}\,(\mathbf{s}_{\mathrm{MIX}} \cdot \mathbf{h}, \mathbf{d}) = \mathbf{g}_{\mathrm{MAT}}\,(\mathbf{h}, \mathbf{s}^A_{\mathrm{MIX}} \cdot \mathbf{d})\,, \quad \forall\, \mathbf{h}, \mathbf{d} \in T\boldsymbol{\Omega}\,. \tag{3.61}$$

Since $\mathbf{s}_{\mathrm{CON}} : T\boldsymbol{\Omega}^* \mapsto T\boldsymbol{\Omega}$ and $\mathbf{s}_{\mathrm{COV}} : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}^*$, the composition $\mathbf{s}_{\mathrm{CON}} \cdot \mathbf{s}_{\mathrm{COV}} : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}$ is a mixed tensor.

Covariant and contravariant tensors are then put in separating duality by the pairing given by [11]

$$\langle \mathbf{s}_{\mathrm{CON}}, \mathbf{s}_{\mathrm{COV}} \rangle := J_1(\mathbf{s}_{\mathrm{CON}} \cdot \mathbf{s}^A_{\mathrm{COV}})\,, \tag{3.62}$$

where J_1 denotes the linear invariant induced by the material metric $\mathbf{g}_{\mathrm{MAT}} \in \mathrm{COV}(V\mathcal{T})$ and the adjoint tensor $\mathbf{s}^A_{\mathrm{COV}} \in \mathrm{CON}(V\mathcal{T})$ is defined by the identity in Eq. (2.7)

$$\mathbf{s}^A_{\mathrm{COV}}(\mathbf{h}\,, \mathbf{d}) := \mathbf{s}_{\mathrm{COV}}(\mathbf{d}\,, \mathbf{h})\,, \qquad \forall\, \mathbf{a}, \mathbf{b} \in T\boldsymbol{\Omega}\,. \tag{3.63}$$

Remark 1 (Symmetry) *Lack of attention for the metric tensor has often led to vain discussions concerning the question whether a given mixed tensor is symmetric or not. Strictly speaking, such a question is ill-posed because symmetry is a property of bilinear maps acting on a pair of vectors (or covectors), while a mixed tensor is an operator from a vector space onto itself and hence as bilinear map it acts on a vector-covector pair. Symmetry can be detected only after an alteration is performed to get a covariant*

[10] Twice covariant tensors are bilinear forms on pairs of tangent vectors, contravariant tensors on pairs of cotangent vectors, and mixed tensors on pairs of tangent-cotangent vectors. They are respectively equivalent to linear operators from tangent to cotangent vectors, from cotangent to tangent vectors and from tangent to tangent vectors.

[11] A *separating* duality pairing between linear spaces is a bilinear form such that vanishing for any value of one of its arguments implies vanishing of the other argument.

or a contravariant tensor. A correct question should ask whether a given mixed tensor is symmetrizable or not, that is, if there exists a metric tensor performing the alteration to a symmetric tensor. Therefore the metric with respect to which symmetry does occur, should explicitly be made mention of. The nice properties of symmetric operators, i.e. real eigenvalues and basis of eigenvectors, are in fact properties of symmetrizable operators. A discussion, concerning the lack of commutativity between pull-back and alteration, will be performed in Rem. 6.

3.7 Frame-invariance

A *change of frame* in the EUCLID space-time is an isometric automorphism $\boldsymbol{\zeta}_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{E}$ in the event time-bundle, characterized by the property of invariance of the spatial metric under pull-back

$$\boldsymbol{\zeta}_{\mathcal{E}}{\downarrow}\mathbf{g}_{\text{SPA}} = \mathbf{g}_{\text{SPA}} \,. \tag{3.64}$$

Definition 22 (Trajectory transformation) *A* trajectory transformation $\boldsymbol{\zeta}_{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T}_{\boldsymbol{\zeta}}$ *is induced by a* change of frame $\boldsymbol{\zeta}_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{E}$ *, as described by the commutative diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\zeta_{\mathcal{E}}"] \arrow[dd, bend right=60, "t_{\mathcal{E}}"'] & \mathcal{E} \arrow[dd, bend left=60, "t_{\mathcal{E}_\zeta}"] \\
\mathcal{T} \arrow[u, "\mathbf{i}"'] \arrow[r, "\zeta=\zeta_{\mathcal{T}}"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T}_\zeta \arrow[u, "\mathbf{i}_\zeta"] \arrow[d, "t_{\mathcal{T}_\zeta}"] \\
\mathcal{Z} \arrow[r, leftrightarrow, "\mathrm{ID}"] & \mathcal{Z}
\end{tikzcd} \tag{3.65}$$

The material metric tensors $\mathbf{g}_{\text{MAT}} \in \mathrm{Cov}(V\mathcal{T})$ *and* $(\mathbf{g}_{\text{MAT}})_{\boldsymbol{\zeta}} \in \mathrm{Cov}(V\mathcal{T}_{\boldsymbol{\zeta}})$ *, in the source and the target trajectory time-bundle, are defined by pull-back according to the relevant immersions*

$$\begin{aligned} \mathbf{g}_{\text{MAT}} &:= \mathbf{i}{\downarrow}\mathbf{g}_{\text{SPA}} \,, \\ (\mathbf{g}_{\text{MAT}})_{\boldsymbol{\zeta}} &:= \mathbf{i}_{\boldsymbol{\zeta}}{\downarrow}\mathbf{g}_{\text{SPA}} \,. \end{aligned} \tag{3.66}$$

From (3.64) and (3.65) it follows that $\boldsymbol{\zeta} : \mathcal{T} \mapsto \mathcal{T}_{\boldsymbol{\zeta}}$ *is an* isometric isomorphism *between the two trajectory time-bundles, as seen by distinct observers, according to the property*

$$\boldsymbol{\zeta}{\downarrow}(\mathbf{g}_{\text{MAT}})_{\boldsymbol{\zeta}} = \mathbf{g}_{\text{MAT}} \,. \tag{3.67}$$

By the trajectory transformation $\boldsymbol{\zeta} : \mathcal{T} \mapsto \mathcal{T}_{\zeta}$ the motion $\varphi_{\alpha}^{\mathcal{T}} : \mathcal{T} \mapsto \mathcal{T}$ and the pushed motion $\boldsymbol{\zeta}\!\uparrow\!\varphi_{\alpha}^{\mathcal{T}} : \mathcal{T}_{\zeta} \mapsto \mathcal{T}_{\zeta}$ are related according to the commutative diagram

$$\begin{array}{ccc} \mathcal{T}_{\zeta} & \xrightarrow{\boldsymbol{\zeta}\uparrow\varphi_{\alpha}^{\mathcal{T}}} & \mathcal{T}_{\zeta} \\ \uparrow{\scriptstyle\zeta} & & \uparrow{\scriptstyle\zeta} \\ \mathcal{T} & \xrightarrow{\varphi_{\alpha}^{\mathcal{T}}} & \mathcal{T} \end{array} \iff (\boldsymbol{\zeta}\!\uparrow\!\varphi_{\alpha}^{\mathcal{T}}) \circ \boldsymbol{\zeta} = \boldsymbol{\zeta} \circ \varphi_{\alpha}^{\mathcal{T}} . \tag{3.68}$$

The space-time velocity $\mathbf{V} := \partial_{\alpha=0}\, \varphi_{\alpha}^{\mathcal{T}}$ is then also transformed by push

$$\mathbf{V}_{\zeta} := \partial_{\alpha=0}\, \boldsymbol{\zeta}\!\uparrow\!\varphi_{\alpha}^{\mathcal{T}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{V} . \tag{3.69}$$

Remark 2 *The transformation law for the space velocity, due to a relative rigid motion between observers, is recovered by considering the explicit expression of an isometric frame-transformation in the* Euclid *space which, in terms of time-dependent rotation* $\mathbf{Q}$ *and translation* $\mathbf{c}$, *is given by*

$$\boldsymbol{\zeta}_{\mathcal{E}} : \begin{cases} \mathbf{x} \mapsto \mathbf{Q}(t) \cdot \mathbf{x} + \mathbf{c}(t) \\ t \mapsto t \end{cases} \tag{3.70}$$

The associated Jacobi *space-time block matrix writes*

$$[T\boldsymbol{\zeta}_{\mathcal{E}}] = \begin{bmatrix} \mathbf{Q} & (\dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}) \\ \mathbf{0} & 1 \end{bmatrix} \tag{3.71}$$

and the pushed velocity $\boldsymbol{\zeta}\!\uparrow\!\mathbf{V}$ *has the space-time block expression*

$$[T\boldsymbol{\zeta}_{\mathcal{E}}] \cdot [\mathbf{V}] = \begin{bmatrix} \mathbf{Q} & (\dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}}) \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}} \\ 1 \end{bmatrix} \tag{3.72}$$

which yields the transformation rule for the space velocity. The space-time formulation of frame invariance reveals that the statement that velocity is not objective *(Truesdell and Noll, 1965) is in fact an instance of the importance of taking note that, in detecting transformation rules, it is not convenient to look only at space velocity and space transformations.*

By definition frame-transformations in Euclid space are isometric and hence by Eq. (3.67) the metric tensor is frame-invariant

$$(\mathbf{g}_{\mathrm{MAT}})_{\zeta} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{g}_{\mathrm{MAT}} . \tag{3.73}$$

A natural axiom of Continuum Mechanics, which formalizes the physical requirement that material behavior must be independent of the special observer performing measurements, is enunciated below.

Axiom 1 (Material Frame Invariance) *All material tensors are* EUCLID *frame invariant.*

Denoting by a subscript $()_\zeta$ transformed material tensor fields, frame-invariance of stress and elastic stretching is expressed by

$$\boldsymbol{\sigma}_\zeta = \boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma}\,, \quad \mathbf{es}_\zeta = \boldsymbol{\zeta}\!\uparrow\!\mathbf{es}\,. \tag{3.74}$$

It follows that the elastic power $\langle \boldsymbol{\sigma}, \mathbf{es} \rangle$ is frame-invariant

$$\langle \boldsymbol{\sigma}_\zeta, \mathbf{es}_\zeta \rangle = \langle \boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma}, \boldsymbol{\zeta}\!\uparrow\!\mathbf{es} \rangle = \boldsymbol{\zeta}\!\uparrow\!\langle \boldsymbol{\sigma}, \mathbf{es} \rangle\,. \tag{3.75}$$

Proposition 3 (Frame invariance of time rates) *Frame-invariance of a material tensor field implies frame-invariance of its time-rate along the motion*

$$\mathbf{s}_\zeta = \boldsymbol{\zeta}\!\uparrow\!\mathbf{s} \implies \mathcal{L}_{\mathbf{V}_\zeta}\, \mathbf{s}_\zeta = \boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{V}}\, \mathbf{s})\,.$$

Proof. The push transformation law for the space-time velocity Eq. (3.69) and the naturality property of the LIE derivative with respect to push, expressed for any tensor field $\mathbf{s}$ on the trajectory, by

$$\mathcal{L}_{\boldsymbol{\zeta}\uparrow\mathbf{V}}(\boldsymbol{\zeta}\!\uparrow\!\mathbf{s}) = \boldsymbol{\zeta}\!\uparrow\!(\mathcal{L}_{\mathbf{V}}\, \mathbf{s})\,, \tag{3.76}$$

lead to assess the result. ■

By Eq. (3.76) it follows that invariance of stress implies invariance of stressing

$$\boldsymbol{\sigma}_\zeta = \boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\sigma} \implies \dot{\boldsymbol{\sigma}}_\zeta = \boldsymbol{\zeta}\!\uparrow\!\dot{\boldsymbol{\sigma}}\,. \tag{3.77}$$

3.8 Constitutive frame-invariance

In providing a mathematical model of mechanical properties of materials, it is natural to require that a principle of Constitutive Frame Invariance (CFI) is fulfilled.

To illustrate the issue, let the rate constitutive behavior be expressed by means of a *constitutive operator* $\mathbf{C}$ which depends, in each material fiber, in a possibly nonlinear way on the *stress* and on the *stressing*.

Definition 23 (Rate constitutive operator) *The rate constitutive law defines the* geometric stretching $\boldsymbol{\varepsilon}_{\mathbf{V}} \in \mathrm{COV}(V\mathcal{T})$ *as a symmetric covariant material tensor expressed by*

$$\boldsymbol{\varepsilon}_{\mathbf{V}} := \mathbf{C}(\boldsymbol{\sigma}\,, \dot{\boldsymbol{\sigma}})\,, \tag{3.78}$$

where $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{(\mathbf{i}\downarrow\mathbf{V})}\, \boldsymbol{\sigma} \in \mathrm{CON}(V\mathcal{T})\,.$

Definition 24 (Constitutive Frame Invariance) *Frame invariance requires that constitutive operators described by distinct observers must fulfil the relation*

$$\mathbf{C}_\zeta = \zeta{\uparrow}\mathbf{C}\,. \tag{3.79}$$

The pushed constitutive operator $\zeta{\uparrow}\mathbf{C}$ is defined by the property that the transformed triplet $(\zeta{\uparrow}\boldsymbol{\varepsilon}_\mathbf{v}, \zeta{\uparrow}\boldsymbol{\sigma}, \zeta{\uparrow}\dot{\boldsymbol{\sigma}})$ will fulfills the transformed rate constitutive law

$$\zeta{\uparrow}\boldsymbol{\varepsilon}_\mathbf{v} = (\zeta{\uparrow}\mathbf{C})(\zeta{\uparrow}\boldsymbol{\sigma}\,, \zeta{\uparrow}\dot{\boldsymbol{\sigma}})\,, \tag{3.80}$$

if and only if the triplet $(\boldsymbol{\varepsilon}_\mathbf{v}, \boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}})$ fulfills the rate-elastic law (4.2). The requirement (3.79) of CFI may then be expressed by the equivalence

$$\boldsymbol{\varepsilon}_\mathbf{v} = \mathbf{C}(\boldsymbol{\sigma}\,, \dot{\boldsymbol{\sigma}}) \iff \zeta{\uparrow}\boldsymbol{\varepsilon}_\mathbf{v} = \mathbf{C}_\zeta(\zeta{\uparrow}\boldsymbol{\sigma}\,, \zeta{\uparrow}\dot{\boldsymbol{\sigma}})\,. \tag{3.81}$$

The physical requirement expressed by CFI is that the results of laboratory experiments performed by two observers should be comparable in a *natural* way, that is on the basis of the knowledge of their relative motion.

If the first observer detects a constitutive operator $\mathbf{C}$, relating state variables and constitutive responses, he is able to foretell the result of the same experiment performed by the second observer, who detects a constitutive operator $\mathbf{C}_\zeta$ relating the transformed state variables and the transformed constitutive responses.

Remark 3 *The principle of* Material Frame Indifference *(MFI), as enunciated in (Truesdell and Noll, 1965), p. 403, when translated in geometric notations, consists in the requirement that*

$$\mathbf{C} = \zeta{\uparrow}\mathbf{C}\,, \tag{3.82}$$

where the isometry $\zeta_\mathcal{T} : \mathcal{T} \mapsto \mathcal{T}_\zeta$, fulfilling the following sub-diagram of Eq. (3.65)*, describes the relative motion between observers*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\zeta_\mathcal{T}"] \arrow[d, "t_\mathcal{T}"'] & \mathcal{T}_\zeta \arrow[d, "\mathcal{T}_\zeta"] \\ \mathcal{Z} \arrow[r, "\mathrm{ID}"] & \mathcal{Z} \end{tikzcd} \tag{3.83}$$

The constitutive maps $\mathbf{C}$ and $\zeta{\uparrow}\mathbf{C}$ involved in the equality Eq. (3.82) *refer to constitutive descriptions made by distinct observers, and have therefore domains and codomains which are material tensor bundles based on distinct trajectory manifolds $\mathcal{T}$ and $\mathcal{T}_\zeta$. By definition Eq.* (3.80)*, the equality in Eq.* (3.82) *expressing* MFI *amounts to require that*

$$\boldsymbol{\varepsilon}_\mathbf{v} = \mathbf{C}(\boldsymbol{\sigma}\,, \dot{\boldsymbol{\sigma}}) \iff \zeta{\uparrow}\boldsymbol{\varepsilon}_\mathbf{v} = \mathbf{C}(\zeta{\uparrow}\boldsymbol{\sigma}\,, \zeta{\uparrow}\dot{\boldsymbol{\sigma}})\,. \tag{3.84}$$

The r.h.s. of this equivalence is however geometrically incorrect because the constitutive operator $\mathbf{C}$ *acts on material tensors based on the trajectory* $\mathcal{T}$*, while the argument material tensors at the r.h.s. Eq. (3.84) are based on the transformed trajectory* $\mathcal{T}_\zeta$*. In place of the incorrect equality Eq. (3.82) expressing the MFI, the geometrically consistent requirement Eq. (3.79) of the CFI, must be adopted. The constitutive maps* $\mathbf{C}_\zeta$ *and* $\boldsymbol{\zeta}\!\uparrow\!\mathbf{C}$ *involved in the equality Eq. (3.79) are in fact based on the same transformed trajectory manifold* $\mathcal{T}_\zeta$ *and their equality may well be imposed and checked by a single observer.*

3.9 Material isotropy

Definition 25 (Material isotropy) *The condition of* isotropy *of a constitutive operator* $\mathbf{C}$ *consists in the requirement that, for any simultaneity preserving isometric transformation* $\boldsymbol{\xi} : \mathcal{T} \mapsto \mathcal{T}$*, the following equality holds*

$$\mathbf{C} = \boldsymbol{\xi}\!\uparrow\!\mathbf{C}\,, \tag{3.85}$$

a condition explicitly expressed by the equivalence

$$\boldsymbol{\varepsilon}_{\mathbf{v}} = \mathbf{C}(\boldsymbol{\sigma}\,,\dot{\boldsymbol{\sigma}}) \iff \boldsymbol{\xi}\!\uparrow\!\boldsymbol{\varepsilon}_{\mathbf{v}} = \mathbf{C}(\boldsymbol{\xi}\!\uparrow\!\boldsymbol{\sigma}\,,\boldsymbol{\xi}\!\uparrow\!\dot{\boldsymbol{\sigma}})\,. \tag{3.86}$$

The isometry $\boldsymbol{\xi} : \mathcal{T} \mapsto \mathcal{T}$ *is defined by the condition* $\mathbf{g}_{\mathrm{MAT}} = \boldsymbol{\xi}\!\downarrow\!\mathbf{g}_{\mathrm{MAT}}$ *and by the commutative diagram*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\xi"] \arrow[d, "t_{\mathcal{T}}"'] & \mathcal{T} \arrow[d, "t_{\mathcal{T}}"] \\ \mathcal{Z} \arrow[r, "\mathrm{ID}"] & \mathcal{Z} \end{tikzcd} \tag{3.87}$$

The physical interpretation of the condition (3.85)-(3.86) is that a single observer gets the same constitutive response by testing two mutually rotated material specimens.[12]

The similarity, between the statement (3.82)-(3.84) of the MFI and the condition of isotropy (3.85)-(3.86), has been the source of confusion that led to sustain the unphysical conclusion that MFI implies isotropy of material behavior (Truesdell and Noll, 1965) fn.1, Sect. 47, p.140 and fn.2, Sect. 99, p.403.

The geometric treatment reveals instead a basic difference because, while $\boldsymbol{\zeta} : \mathcal{T} \mapsto \mathcal{T}_\zeta$ is an isometric transformation between two distinct trajectories, induced by a change of observer according to Def. 22, the map $\boldsymbol{\xi} : \mathcal{T} \mapsto \mathcal{T}$ is

[12] The group of isometries such that (3.85) holds characterises the symmetry properties of the material response.

an isometric transformation of a trajectory into itself, evaluated by a single observer.

As a consequence the statement Eq. (3.82)-(3.84) of the MFI is untenable, as observed above, while the requirement Eq. (3.85)-(3.86) of isotropy is correctly formulated.

The geometric treatment reveals that the requirement of *Constitutive Frame Invariance* formulated in Eq. (3.79) has nothing to do with material isotropy because two distinct constitutive operators, evaluated by distinct observers, are involved. [13]

3.10 Material homogeneity

Let us leave to the reader the challenging answer to the following question: How to define material homogeneity?

The solution should fulfill the requirement of being equally applicable to bodies of any dimension. A comparison with the definition proposed by the author can be made by consulting Sect. 15 of (Romano, Barretta, Diaco, 2014a).

[13] The stress independent isotropic rate elastic law fulfills the CFI principle (Romano, Barretta, 2013b).

4

Elasticity

“ CEIIINOSSSTTVV... VT TENSIO SIC VIS ” [1]

– Robert Hooke, 1675...1679

What can reasonably be measured in a laboratory test is the incremental mechanical stretching response of an elastic material to an increment of stress state, starting from a known stress state. A general expression of Hooke’s law, for a uniaxially solicited specimen, should then declare direct proportionality between stressing (vis) and stretching (ex-tensio).

4.1 Elastic constitutive relation

The mechanical response of an elastic material is expressed by an elastic stretching $\boldsymbol{\varepsilon}_{\mathrm{EL}}$ that is a symmetric twice covariant tensor field which, in a *purely elastic* process, equals the geometric stretching

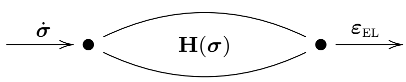
$$\boldsymbol{\varepsilon}_{\mathrm{EL}} = \boldsymbol{\varepsilon}_{\mathbf{v}} \,. \tag{4.1}$$

A natural, and experimentally tested, assumption is the linear dependence of the elastic stretching $\boldsymbol{\varepsilon}_{\mathrm{EL}}$ on the twice contravariant Kirchhoff stressing $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{v}}\, \boldsymbol{\sigma}$.

Definition 26 (Elastic law) *The elastic stretching, corresponding to given stress and stressing states, is expressed by the rate constitutive law*

$$\boldsymbol{\varepsilon}_{\mathrm{EL}} := \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \,, \tag{4.2}$$

schematically depicted by



(4.3)

[1] Lectiones Cutlerianæ (Robert Hooke, 1679).

The tangent elastic compliance $\mathbf{H}(\boldsymbol{\sigma}) : \text{Con}(V\mathcal{T}) \mapsto \text{Cov}(V\mathcal{T})$ is a linear map from contravariant to covariant tensors, nonlinearly dependent on stress state, fulfilling the following properties.

1. Time-invariance along the motion
$$\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{H} = \mathbf{H}\,, \tag{4.4}$$

2. Fiberwise integrability to a time invariant stress potential
$$\Xi : \text{Con}(V\mathcal{E}) \mapsto \text{Fun}(V\mathcal{E})\,, \tag{4.5}$$
such that
$$\mathbf{H} = d_F^2\Xi\,. \tag{4.6}$$

3. Positive definiteness
$$\delta\boldsymbol{\sigma} \neq \mathbf{0} \implies \langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma}, \delta\boldsymbol{\sigma} \rangle > 0\,. \tag{4.7}$$

Let us discuss in detail the three items above.

1. The pull-back $\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{H}$, of the constitutive operator $\mathbf{H}$ appearing in Eq. (4.4), is defined by the requirement that, if the pair $\{\boldsymbol{\varepsilon}_{\text{EL}}\,, \dot{\boldsymbol{\sigma}}\}$ is related by the tangent compliance $\mathbf{H}(\boldsymbol{\sigma})$, then the pulled back tangent compliance $\boldsymbol{\varphi}_\alpha\!\downarrow\!(\mathbf{H}(\boldsymbol{\sigma})) = (\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{H})(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma})$ relates the pair $\{\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\varepsilon}_{\text{EL}}\,, \boldsymbol{\varphi}_\alpha\!\downarrow\!\dot{\boldsymbol{\sigma}}\}$ and vice versa, i.e.
$$\boldsymbol{\varepsilon}_{\text{EL}} = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \iff \boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\varepsilon}_{\text{EL}} = (\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{H})(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma}) \cdot \boldsymbol{\varphi}_\alpha\!\downarrow\!\dot{\boldsymbol{\sigma}}\,. \tag{4.8}$$
The time invariance condition, expressed by Eq. (4.4), may then be also explicitly stated as
$$\boldsymbol{\varepsilon}_{\text{EL}} = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \implies \boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\varepsilon}_{\text{EL}} = \mathbf{H}(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma}) \cdot \boldsymbol{\varphi}_\alpha\!\downarrow\!\dot{\boldsymbol{\sigma}}\,. \tag{4.9}$$
Recalling that material tensors at different times are to be compared by push along the motion, the invariance property in Eq. (4.9) can be enunciated by the statement:
   - Time invariance of the constitutive operator of the rate elastic law ($\boldsymbol{\varphi}_\alpha\!\downarrow\!\mathbf{H} = \mathbf{H}$) means that time invariance of the stress field ($\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma} = \boldsymbol{\sigma}$) and of the stressing ($\boldsymbol{\varphi}_\alpha\!\downarrow\!\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}$) assure time invariance of the elastic stretching response ($\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\varepsilon}_{\text{EL}} = \boldsymbol{\varepsilon}_{\text{EL}}$).

2. Fiberwise integrability, discussed in (Romano, Barretta, 2011; Romano, Barretta, Diaco, 2014c), is illustrated below. If the symmetry condition
$$\langle d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma} \cdot \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma} \rangle = \langle d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma} \cdot \delta_2\boldsymbol{\sigma}, \delta_1\boldsymbol{\sigma} \rangle\,, \tag{4.10}$$

is fulfilled for all test fields $\delta\boldsymbol{\sigma}, \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma}$, the constitutive operator $\mathbf{H}$ of rate-elasticity is CAUCHY integrable. Denoting by $\boldsymbol{\pi}_{\mathcal{E}} : T\mathcal{E} \mapsto \mathcal{E}$ the tangent bundle projection, this means that there exists a fiber-differentiable morphism $\boldsymbol{\Psi}$ between the stress and the strain bundles, described by the commutative diagram

$$\begin{tikzcd} \mathrm{CON}(V\mathcal{E}) \arrow[r, "\boldsymbol{\Psi}"] \arrow[d, "\pi_{\mathcal{E}}"'] & \mathrm{COV}(V\mathcal{E}) \arrow[d, "\pi_{\mathcal{E}}"] \\ \mathcal{E} \arrow[r, "\mathrm{ID}_{\mathcal{E}}"] & \mathcal{E} \end{tikzcd} \iff \quad \boldsymbol{\pi}_{\mathcal{E}} \circ \boldsymbol{\Psi} = \boldsymbol{\pi}_{\mathcal{E}} \,, \tag{4.11}$$

and such that

$$\mathbf{H} = d_F \boldsymbol{\Psi} \,. \tag{4.12}$$

The morphism $\boldsymbol{\Psi}$ is called the *elastic response*.

The symbol d_F denotes the *fiber derivative*, that is the derivative taken while leaving the base material point fixed and letting the involved material tensor to vary along a curve of tensors all based at the same point and therefore belonging to the same linear tensor space.

GREEN-integrability is ensured by the further symmetry condition

$$\langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma} \rangle = \langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta_2\boldsymbol{\sigma}, \delta_1\boldsymbol{\sigma} \rangle \,. \tag{4.13}$$

This means that there exists a fiberwise differentiable morphism Ξ between the stress and the scalar bundles, described by the commutative diagram

$$\begin{tikzcd} \mathrm{CON}(V\mathcal{E}) \arrow[r, "\Xi"] \arrow[d, "\pi_{\mathcal{E}}"'] & \mathrm{FUN}(V\mathcal{E}) \arrow[d, "\pi_{\mathcal{E}}"] \\ \mathcal{E} \arrow[r, "\mathrm{ID}_{\mathcal{E}}"] & \mathcal{E} \end{tikzcd} \iff \quad \boldsymbol{\pi}_{\mathcal{E}} \circ \Xi = \boldsymbol{\pi}_{\mathcal{E}} \,, \tag{4.14}$$

and such that

$$\boldsymbol{\Psi} = d_F \Xi \,. \tag{4.15}$$

The morphism Ξ is called the *elastic stress potential*.

Hence, combining with Eq. (4.12), we get Eq. (4.6).

3. By the assumed positive definiteness of $\mathbf{H}(\boldsymbol{\sigma})$, the elastic response $\boldsymbol{\Psi}$ fulfills the monotonicity property

$$\langle \boldsymbol{\Psi}(\boldsymbol{\sigma}_2) - \boldsymbol{\Psi}(\boldsymbol{\sigma}_1), \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1 \rangle > 0 \,. \qquad \boldsymbol{\sigma}_2 \neq \boldsymbol{\sigma}_1 \,. \tag{4.16}$$

The monotonicity property Eq. (4.16), implies that the potential fulfils the convexity property

$$\lambda\Xi(\boldsymbol{\sigma}_2) + (1-\lambda)\Xi(\boldsymbol{\sigma}_1) > \Xi(\lambda\boldsymbol{\sigma}_2 + (1-\lambda)\boldsymbol{\sigma}_1) \,, \quad 0 < \lambda < 1 \,. \tag{4.17}$$

As we will see in Th. 1, existence of a Green potential ensures that the mechanical work expended in a stress cycle is vanishing.

In this respect, we underline that the notion of stress cycles involves the comparison of stress tensors based at the same material particle but at different times along the trajectory, a comparison that is to be performed in the natural way, by push along the motion.

This comparison becomes an equality by means of a pull-back to a straightened trajectory and identification of the relevant material slices with a reference manifold.

In accord with the general rule, pull-back of the Cauchy elastic response is defined by setting

$$(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\Psi})(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma}) := \boldsymbol{\varphi}_\alpha\!\downarrow\!(\boldsymbol{\Psi}(\boldsymbol{\sigma}))\,. \tag{4.18}$$

A noteworthy property is commutativity of fiber-derivatives with pull-back

$$\boldsymbol{\varphi}_\alpha\!\downarrow\!(d_F\boldsymbol{\Psi}) = d_F(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\Psi})\,, \tag{4.19}$$

which follows from fiber-linearity of the pull-back operation.

By taking the derivative $\partial_{\alpha=0}$ we infer that fiber and Lie-derivatives also commute

$$\begin{aligned} \mathcal{L}_{\mathbf{V}}(d_F\boldsymbol{\Psi}) &= \partial_{\alpha=0}\,\boldsymbol{\varphi}_\alpha\!\downarrow\!(d_F\boldsymbol{\Psi}) \\ &= \partial_{\alpha=0}\,d_F(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\Psi}) = d_F(\mathcal{L}_{\mathbf{V}}\boldsymbol{\Psi})\,. \end{aligned} \tag{4.20}$$

Definitions and properties Eqs. (4.18), (4.19), (4.20) apply as well to the *elastic stress potential* Ξ and to any morphism between tensor bundles.

Let us now come to the central point of our presentation.

Definition 27 (Elastic state) *The elastic state* $\mathbf{es} \in \mathrm{Cov}(V\mathcal{T})$ *is a material symmetric twice covariant tensor, output of the (invertible) constitutive relation*

$$\mathbf{es} := \boldsymbol{\Psi}(\boldsymbol{\sigma}) = d_F\Xi(\boldsymbol{\sigma})\,, \tag{4.21}$$

schematically depicted as

$$\xrightarrow{\ \boldsymbol{\sigma}\ } \bullet \quad \boldsymbol{\Psi}{=}d_F\Xi \quad \bullet \xrightarrow{\ \mathbf{es}\ } \tag{4.22}$$

Taking the Lie-derivative of Eq. (4.21) and applying the split formula Eq. (2.62) we get the relation [2]

$$\begin{aligned} \dot{\mathbf{es}} := \mathcal{L}_{\mathbf{V}}\mathbf{es} = \mathcal{L}_{\mathbf{V}}(\boldsymbol{\Psi}\circ\boldsymbol{\sigma}) &= \mathcal{L}_{\mathbf{V}}\boldsymbol{\Psi}(\boldsymbol{\sigma}) + d_F\boldsymbol{\Psi}(\boldsymbol{\sigma})\cdot\dot{\boldsymbol{\sigma}} \\ &= \dot{\boldsymbol{\Psi}}(\boldsymbol{\sigma}) + \mathbf{H}(\boldsymbol{\sigma})\cdot\dot{\boldsymbol{\sigma}} \\ &= d_F\dot{\Xi}(\boldsymbol{\sigma}) + d_F^2\Xi(\boldsymbol{\sigma})\cdot\dot{\boldsymbol{\sigma}} = d_F\mathcal{L}_{\mathbf{V}}(\Xi\circ\boldsymbol{\sigma})\,. \end{aligned} \tag{4.23}$$

[2] The two letter symbol $\mathbf{es}$ denotes an elastic state and $\dot{\mathbf{es}}$ is a simplified writing for its time rate $(\mathbf{es})^{\cdot} := \mathcal{L}_{\mathbf{V}}\mathbf{es}$.

By time-invariance along the motion $\dot{\Xi}(\boldsymbol{\sigma}) = 0$ and hence Eq. (4.23) implies that the elastic stretching is equal to the rate of the elastic state

$$\boldsymbol{\varepsilon}_{\mathrm{EL}} = \dot{\mathbf{es}}\,. \tag{4.24}$$

By Eq. (4.24) and property Eq. (3.51) we may introduce the following notion.

Proposition 4 (Accumulated elastic strain) *The elastic strain* $\mathbf{es}_{\mathrm{REF}}(\alpha\,,\mathbf{x})$ *accumulated in the time lapse* α *, at a point* $\mathbf{x} \in \boldsymbol{\Omega}_{\mathrm{REF}}$ *of a fixed reference placement* $\boldsymbol{\Omega}_{\mathrm{REF}}$ *, is defined by the integral formula*

$$\mathbf{es}_{\mathrm{REF}}(\alpha\,,\mathbf{x}) := \int_0^\alpha (\boldsymbol{\varphi}_\tau{\downarrow}\boldsymbol{\varepsilon}_{\mathrm{EL}})_{\mathbf{x}}\,d\tau = \int_0^\alpha (\boldsymbol{\varphi}_\tau{\downarrow}(\mathcal{L}_{\mathbf{V}}\mathbf{es}))_{\mathbf{x}}\,d\tau = (\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{es} - \mathbf{es})_{\mathbf{x}}\,. \tag{4.25}$$

Proof. Time-integration in Eq. (4.25) is legitimate, since the tensors $\boldsymbol{\varphi}_\tau{\downarrow}\boldsymbol{\varepsilon}_{\mathrm{EL}}$ are all based at a same point in the fixed placement. The last equality in Eq. (4.25) follows from the definition

$$\mathcal{L}_{\mathbf{V}}\mathbf{es} := \partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{es})\,, \tag{4.26}$$

and the evaluation

$$\begin{aligned} \boldsymbol{\varphi}_\tau{\downarrow}(\mathcal{L}_{\mathbf{V}}\mathbf{es}) &= \boldsymbol{\varphi}_\tau{\downarrow}(\partial_{\alpha=0}\,(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{es})) \\ &= \partial_{\alpha=0}\,(\boldsymbol{\varphi}_{\tau+\alpha}{\downarrow}\mathbf{es}) \\ &= \partial_{t=\tau}\,(\boldsymbol{\varphi}_t{\downarrow}\mathbf{es})\,. \end{aligned} \tag{4.27}$$

An analogous result holds if the pull-back to a reference manifold $\boldsymbol{\Omega}_{\mathrm{REF}}$ is considered. ■

The time-rate of the elastic state $\dot{\mathbf{es}}$ at a given placement is equal to the time-derivative of the elastic strain $\mathbf{es}_{\mathrm{REF}}(\alpha)$ accumulated at that placement along the motion

$$\mathcal{L}_{\mathbf{V}}\mathbf{es} = \partial_{\alpha=0}\,\mathbf{es}_{\mathrm{REF}}(\alpha)\,. \tag{4.28}$$

In defining the notion of accumulated elastic strain, Cauchy integrability of the rate elastic operator and the introduction of the notion of elastic state play a basic role.

Integrability is a peculiar feature of elasticity, and therefore no notion of anelastic state or of accumulated anelastic strain can be introduced in continuum mechanics, due to the inherent lack of integrability of the relevant rate constitutive law.

In spite of this, most constitutive models of ultra-elastic behavior formulated in literature make still explicit recourse to the notion of a *plastic strain*, leaving however the evaluation procedure, and the involved reference state, completely undetermined.

In the geometric theory, elasticity is well characterized, both from the physical and the mathematical point of view, by the rate formulation of Eq. (4.2), governed by a tangent elastic compliance, with a constitutive operator $\mathbf{H}$ which is CAUCHY and GREEN integrable, with a monotone response $\mathbf{\Psi}$ and a convex, time invariant, elastic stress potential Ξ.

4.2 Legendre transform

The relation between the *elastic stress potential* Ξ and the conjugate *elastic state potential* Ξ^* is provided by the LEGENDRE transform

$$\begin{cases} \mathbf{es} := d_F \Xi(\boldsymbol{\sigma}) \,, \\ \Xi^*(\mathbf{es}) := \langle \boldsymbol{\sigma}, \mathbf{es} \rangle - \Xi(\boldsymbol{\sigma}) \,, \\ \boldsymbol{\sigma} = d_F \Xi^*(\mathbf{es}) \,. \end{cases} \tag{4.29}$$

Nomenclature and notation here adopted are just the converse of the standard ones in the linearized theory of elasticity.

There $W^* = \Xi$ is called the conjugate or complementary (stress) potential while the elastic potential W is assumed to be a function of an elastic strain measure. But this last is a quantity depending on an undetermined reference placement.

Our change of nomenclature emphasises that the primary state variable is the stress state, while the elastic state is defined in terms of stress state by means of the elastic constitutive law Eq. (4.21), reproduced above in Eq. $(4.29)_1$.

Time invariance of the *elastic stress potential* Ξ along the motion entails time invariance of the conjugate *elastic state potential* Ξ^*. Indeed, assuming $\boldsymbol{\varphi}_\alpha{\downarrow}\Xi = \Xi$, we have that

$$\begin{aligned} \boldsymbol{\varphi}_\alpha{\downarrow}\langle \boldsymbol{\sigma}, \mathbf{es} \rangle &= \langle \boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\sigma}, \boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{es} \rangle \\ &= (\boldsymbol{\varphi}_\alpha{\downarrow}\Xi)(\boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\sigma}) + (\boldsymbol{\varphi}_\alpha{\downarrow}\Xi^*)(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{es}) \\ &= \Xi(\boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\sigma}) + (\boldsymbol{\varphi}_\alpha{\downarrow}\Xi^*)(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{es}) \,, \end{aligned} \tag{4.30}$$

for any $\mathbf{es} \in \mathrm{COV}(V\mathcal{E})$, which implies $\boldsymbol{\varphi}_\alpha{\downarrow}\Xi^* = \Xi^*$.

4.2.1 Rate potentials

From Eq. (2.62) we get the split formulae

$$\begin{cases} \mathcal{L}_{\mathbf{V}}(\Xi \circ \boldsymbol{\sigma}) = \mathcal{L}_{\mathbf{V}}\Xi(\boldsymbol{\sigma}) + d_F\Xi(\boldsymbol{\sigma}) \cdot \mathcal{L}_{\mathbf{V}}\boldsymbol{\sigma} \,, \\ \mathcal{L}_{\mathbf{V}}(\Xi^* \circ \mathbf{es}) = \mathcal{L}_{\mathbf{V}}\Xi^*(\mathbf{es}) + d_F\Xi^*(\mathbf{es}) \cdot \mathcal{L}_{\mathbf{V}}\mathbf{es} \,, \end{cases} \tag{4.31}$$

and from LEGENDRE transform Eq. (4.29) we infer that

$$\begin{cases} \mathcal{L}_{\mathbf{V}}(\Xi \circ \boldsymbol{\sigma}) + \mathcal{L}_{\mathbf{V}}(\Xi^* \circ \mathbf{es}) = \mathcal{L}_{\mathbf{V}}(\Xi \circ \boldsymbol{\sigma} + \Xi^* \circ \mathbf{es}) \\ \qquad\qquad = \langle \boldsymbol{\sigma}, \mathbf{es} \rangle^{\cdot}, \\ \langle (d_F\Xi)(\boldsymbol{\sigma}), \dot{\boldsymbol{\sigma}} \rangle + \langle (d_F\Xi^*)(\mathbf{es}), \dot{\mathbf{es}} \rangle = \langle \mathbf{es}, \dot{\boldsymbol{\sigma}} \rangle + \langle \boldsymbol{\sigma}, \dot{\mathbf{es}} \rangle \\ \qquad\qquad = \langle \boldsymbol{\sigma}, \mathbf{es} \rangle^{\cdot}. \end{cases} \tag{4.32}$$

Then, summing up the two equalities in Eq. (4.32) and setting

$$\begin{cases} \dot{\Xi} \ := \mathcal{L}_{\mathbf{V}}\Xi\,, \\ \dot{\Xi}^* := \mathcal{L}_{\mathbf{V}}\Xi^*\,, \end{cases} \tag{4.33}$$

we infer the relation between the rates of conjugate potentials

$$\begin{cases} \mathbf{es} = d_F\Xi(\boldsymbol{\sigma})\,, \\ \boldsymbol{\sigma} = d_F\Xi^*(\mathbf{es})\,, \\ \dot{\Xi}(\boldsymbol{\sigma}) + \dot{\Xi}^*(\mathbf{es}) = 0\,. \end{cases} \tag{4.34}$$

Eq. (4.34) is formally analogous to Eq. (13) in (Hill and Rice, 1973), where a ultra elastic constitutive theory, with finite strains measured with respect to a reference configuration, was considered. In elasticity Eq. $(4.34)_3$ is trivially fulfilled by time-invariance of the elastic potentials.

4.3 Variational formulation

Let us assume that along the virtual motion the elastic state field, the conjugate elastic potential and the mass form are prolonged in such a way that

$$\begin{cases} \mathcal{L}_{\delta\mathbf{v}}\,\mathbf{es} \ = \boldsymbol{\varepsilon}(\delta\mathbf{v}) = \frac{1}{2}\,\mathcal{L}_{\delta\mathbf{v}}\,\mathbf{g}_{\mathrm{MAT}}\,, \\ \mathcal{L}_{\delta\mathbf{v}}\,\Xi^*(\mathbf{es}) \ = 0\,, \\ \mathcal{L}_{\delta\mathbf{v}}\,\mathbf{m} \ = \mathbf{0}\,. \end{cases} \tag{4.35}$$

Eq. $(4.35)_1$ states that along the virtual motion the constitutive behavior is assumed to be purely elastic. Eqs. $(4.35)_{2,3}$ express invariance of the conjugate elastic potential and of the mass form along the virtual motion.

Proposition 5 (Variational principle of elastodynamics) *Assuming prolongations according to Eq.* (4.35)*, the elastic state solution of the elastodynamic problem is the tensor field fulfilling the variational condition*

$$\partial_{\lambda=0} \int_{\delta\boldsymbol{\varphi}_\lambda(\boldsymbol{\Omega})} \Xi^*(\mathbf{es})\,\mathbf{m} = \langle \mathbf{f}_{\mathrm{EXT}}, \delta\mathbf{v} \rangle - \int_{\boldsymbol{\Omega}} \mathbf{g}_{\mathrm{SPA}}(\mathbf{a}\,, \delta\mathbf{v})\,\mathbf{m}\,, \tag{4.36}$$

for all virtual velocity field $\delta\mathbf{v} := \partial_{\lambda=0}\,\delta\boldsymbol{\varphi}_\lambda$.

Proof. From Lemma 2 and Eqs. (4.35) we infer that

$$\begin{aligned}
&\partial_{\lambda=0} \int_{\delta\varphi_\lambda(\mathbf{\Omega})} \Xi^*(\mathbf{es})\,\mathbf{m} = \int_{\mathbf{\Omega}} \mathcal{L}_{\delta\mathbf{v}}\,(\Xi^*(\mathbf{es})\,\mathbf{m}) \\
&= \int_{\mathbf{\Omega}} (\mathcal{L}_{\delta\mathbf{v}}\,(\Xi^* \circ \mathbf{es}))\,\mathbf{m} = \int_{\mathbf{\Omega}} (\mathcal{L}_{\delta\mathbf{v}}\,\Xi^*(\mathbf{es}) + d_F\Xi^*(\mathbf{es}) \cdot \mathcal{L}_{\delta\mathbf{v}}\,\mathbf{es})\,\mathbf{m} \\
&= \int_{\mathbf{\Omega}} d_F\Xi^*(\mathbf{es}) \cdot \boldsymbol{\varepsilon}(\delta\mathbf{v})\,\mathbf{m}\,,
\end{aligned} \tag{4.37}$$

which, substituted in Eq. 4.36 gives the dynamical equilibrium condition Eq. (3.53) for the stress field $\boldsymbol{\sigma} = d_F\Xi^*(\mathbf{es})$. ■

In elastostatics the acceleration field is assumed to vanish. Assuming in addition that there are no external forces, we may set $\mathbf{a} = \mathbf{0}$ and $\mathbf{f}_{\text{EXT}} = \mathbf{0}$ so that Eq. 4.36 yields the stationarity condition for the global potential elastic energy along virtual motions

$$\partial_{\lambda=0} \int_{\delta\varphi_\lambda(\mathbf{\Omega})} \Xi^*(\mathbf{es})\,\mathbf{m} = 0\,, \tag{4.38}$$

which, by convexity of the integrand $\Xi^* : \mathrm{Cov}(T\mathbf{\Omega}) \mapsto \mathrm{Fun}(T\mathbf{\Omega})$, is in fact a minimum property for the elastic state solution $\mathbf{es} : \mathbf{\Omega} \mapsto \mathrm{Cov}(T\mathbf{\Omega})$.

4.4 Conservativeness

Lemma 3 (Time invariance of stress and elastic state) *Time invariance of the stress state along the motion implies a corresponding time invariance of the elastic state and vice versa, as expressed by the equivalence*

$$\boldsymbol{\sigma} = \varphi_\alpha{\downarrow}\boldsymbol{\sigma} \iff \mathbf{es} = \varphi_\alpha{\downarrow}\mathbf{es}\,. \tag{4.39}$$

Proof. Time invariance of the elastic response Eq. (4.4) implies that

$$\varphi_\alpha{\downarrow}\mathbf{\Psi} = \mathbf{\Psi}\,. \tag{4.40}$$

Then time invariance of stress $\boldsymbol{\sigma} = \varphi_\alpha{\downarrow}\boldsymbol{\sigma}$ implies that

$$\varphi_\alpha{\downarrow}\mathbf{es} = \varphi_\alpha{\downarrow}(\mathbf{\Psi}(\boldsymbol{\sigma})) = (\varphi_\alpha{\downarrow}\mathbf{\Psi})(\varphi_\alpha{\downarrow}\boldsymbol{\sigma}) = \mathbf{\Psi}(\boldsymbol{\sigma}) = \mathbf{es}\,, \tag{4.41}$$

and vice versa by invertibility of $\mathbf{\Psi}$. ■

Definition 28 (Stress cycles) *A path* $\mathbf{p} : [0, \Delta t] \mapsto \mathrm{Con}(V\mathcal{T})$ *in the stress bundle is a cycle if*

$$\mathbf{p}_{\Delta t} = \varphi_{\Delta t}{\uparrow}\mathbf{p}_0\,. \tag{4.42}$$

An analogous definition holds for cycles of elastic states in the strain bundle. By Lemma 3 any stress cycle is also a cycle of elastic states and vice versa.

The next result shows that the two properties, of mass conservation Eq. (3.13) and of GREEN integrability of the elastic operator Eq. (4.15), assure vanishing of the elastic work performed in cycles of elastic states along the motion.

Theorem 1 (Conservativeness of elastic response) *The constitutive operator of an elastic material is conservative, that is, in any cycle of stress states (or elastic states) along the motion, no mechanical work is performed, as expressed by the implication*

$$\left.\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\varphi}_{\Delta t}\!\downarrow\!\boldsymbol{\sigma}\\ \mathbf{es} &= \boldsymbol{\varphi}_{\Delta t}\!\downarrow\!\mathbf{es}\end{aligned}\right\} \implies \int_0^{\Delta t}\int_{\boldsymbol{\varphi}_t(\boldsymbol{\Omega})} \langle\, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\text{EL}} \,\rangle \, \mathbf{m}\, dt = 0\,. \tag{4.43}$$

Proof. By the relation Eq. (4.24) the scalar integrand in Eq. (4.43) may be written as

$$\langle\, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\text{EL}} \,\rangle = \langle\, \boldsymbol{\sigma}, \dot{\mathbf{es}} \,\rangle\,. \tag{4.44}$$

From the split formula Eq. $(4.31)_2$ and time invariance, we then get

$$\langle\, \boldsymbol{\sigma}, \dot{\mathbf{es}} \,\rangle = \langle\, d_F\Xi^*(\mathbf{es}), \dot{\mathbf{es}} \,\rangle = \mathcal{L}_{\mathbf{V}}(\Xi^* \circ \mathbf{es}) - \mathcal{L}_{\mathbf{V}}\Xi^*(\mathbf{es}) = \mathcal{L}_{\mathbf{V}}(\Xi^* \circ \mathbf{es})\,. \tag{4.45}$$

The restriction $\boldsymbol{\xi}_t : \boldsymbol{\Omega}_{\text{REF}} \times \{\, t \,\} \mapsto \boldsymbol{\varphi}_t(\boldsymbol{\Omega}) \subset \mathcal{T}_{\mathcal{E}}$ of a straightening map $\boldsymbol{\xi} : \boldsymbol{\Omega}_{\text{REF}} \times I \mapsto \mathcal{T}_{\mathcal{E}}$ transforms by pull-back the integral on $\boldsymbol{\varphi}_t(\boldsymbol{\Omega})$ in Eq. (4.43) to an integral on the reference manifold $\boldsymbol{\Omega}_{\text{REF}}$. Observing that the pull-back of a LIE derivative along the motion becomes a partial time derivative at a fixed position in $\boldsymbol{\Omega}_{\text{REF}}$, and setting $\boldsymbol{\xi}_\tau = \boldsymbol{\varphi}_{\tau,t} \circ \boldsymbol{\xi}_t$, the integral writes

$$\begin{aligned}&\int_0^{\Delta t}\int_{\boldsymbol{\varphi}_t(\boldsymbol{\Omega})} \mathcal{L}_{\mathbf{V}}(\Xi^* \circ \mathbf{es})\, \mathbf{m}\, dt = \int_0^{\Delta t}\int_{\boldsymbol{\Omega}_{\text{REF}}} \boldsymbol{\xi}_t\!\downarrow\!(\mathcal{L}_{\mathbf{V}}(\Xi^* \circ \mathbf{es}))\, \boldsymbol{\xi}_t\!\downarrow\!\mathbf{m}\, dt\\ &= \int_{\boldsymbol{\Omega}_{\text{REF}}} \Big(\int_0^{\Delta t} \partial_{\tau=t}\, (\boldsymbol{\xi}_\tau\!\downarrow\!\Xi^*) \circ (\boldsymbol{\xi}_\tau\!\downarrow\!\mathbf{es})\;\, dt \Big)\, \mathbf{m}_{\text{REF}}\\ &= \int_{\boldsymbol{\Omega}_{\text{REF}}} \Big((\boldsymbol{\xi}_{\Delta t}\!\downarrow\!\Xi^*) \circ (\boldsymbol{\xi}_{\Delta t}\!\downarrow\!\mathbf{es}) - (\boldsymbol{\xi}_0\!\downarrow\!\Xi^*) \circ (\boldsymbol{\xi}_0\!\downarrow\!\mathbf{es}) \Big)\, \mathbf{m}_{\text{REF}} = 0\,,\end{aligned} \tag{4.46}$$

where resort was made to the following properties:

1. $\mathbf{m}_{\text{REF}} := \boldsymbol{\xi}\!\downarrow\!\mathbf{m}$ is time independent by time-invariance of the mass form along the motion,

2. $\boldsymbol{\xi}_{\Delta t}\!\downarrow\!\Xi^* = \boldsymbol{\xi}_0\!\downarrow\!\Xi^*$, by the assumption of time-invariance of the elastic potential Ξ^*,

3. $\boldsymbol{\xi}_{\Delta t}\!\downarrow\!\mathbf{es} = \boldsymbol{\xi}_0\!\downarrow\!\mathbf{es}$, since by assumption the process is a cycle of elastic states.

This concludes the proof. ■

From Eq. (4.46) we infer that the pull-back $\Xi^*_{\text{REF}} := \boldsymbol{\xi}\!\downarrow\!\Xi^*$ of the elastic state potential to a reference manifold, defined by

$$\Xi^*_{\text{REF}}(\boldsymbol{\xi}\!\downarrow\!\mathbf{es}) := \boldsymbol{\xi}\!\downarrow\!(\Xi^*(\mathbf{es})) = \Xi^*(\mathbf{es}) \circ \boldsymbol{\xi}\,, \tag{4.47}$$

provides, with its variation in a time interval, the mechanical work performed, per unit reference mass, in deforming the elastic body along the motion.

Definition 29 (Referential elastic potential energy) *The pull-back of the elastic state potential $\Xi^*_{\text{REF}} := \boldsymbol{\xi}\!\downarrow\!\Xi^*$ to a straightened trajectory will be named* referential elastic potential energy*. Its variation, due to a variation of the elastic state, provides the mechanical work, per unit referential mass, performed in changing the elastic state of the body, as described by Eq. (4.46).*

Remark 4 *We underline that the* referential elastic potential energy *just introduced, and the specific* elastic energy *usually considered in literature, are distinct notions, pertaining to distinct theories. The latter is in fact a scalar field based in a reference placement, and function of an elastic strain measured from the reference placement to the current one. The definition of specific* elastic energy *requires then that a class of privileged local placements is assumed in the constitutive theory and that a procedure apt to measure the elastic strain, between such placements and the current placement, is detected. Physical consistency of these assumptions will be critically discussed in Sect. 6.2. The referential elastic potential energy is instead a brand new notion, consisting in a scalar field, based in an arbitrary straightened trajectory, which is function of the pull-back of the elastic state to that straightened trajectory. The notion of finite elastic strain, in fact absent in the new theory, is not referred to. The elastic strain accumulated in a straightened trajectory, as defined by Eq. (4.25), and later appearing in Eqs. (4.56)-(4.57) of Sect. 4.5, is just a computational item which lives only in a straightened trajectory but cannot be interpreted as a material field because the time parameter, entering in the relevant push forward to the current placement, was lost in the integration procedure. The transformation to a straightened trajectory is an effective mathematical tool to perform linear operations, not feasible in the nonlinear trajectory manifold.*

4.5 Computational algorithm

Let us here describe in synthesis the iterative scheme leading to the solution of an elastostatic problem in a finite time step, according to the new theory.

The finite deformation elastostatic evolution is defined by considering a control manifold C and a time parametrized curve $\mathbf{c} : I \mapsto \mathrm{C}$, the *control process.*

The loading acting on a body at time $\tau \in I$ in the time interval $I \subset \mathcal{Z}$, and similarly all other relevant data, are assumed to depend on the control point $\mathbf{c}(\tau)$ and on the displacement $\boldsymbol{\varphi}_{\tau,t}$ from the placement $\boldsymbol{\Omega}_t$ to the placement $\boldsymbol{\Omega}_\tau = \boldsymbol{\varphi}_{\tau,t}(\boldsymbol{\Omega}_t)$, as expressed by

$$\mathbf{f}_{\text{EXT}}(\tau) = \mathbf{A}(\mathbf{c}(\tau), \boldsymbol{\Omega}_\tau) : \boldsymbol{\Omega}_\tau \mapsto T^*\mathcal{S} \,. \tag{4.48}$$

The finite time step process is analyzed by an iterative trial and error procedure, as follows.

1. The start point is an equilibrium solution at time $t_1 \in I$ under the loading
$$\mathbf{f}_{\text{EXT}}(t_1) = \mathbf{F}(\mathbf{c}(t_1), \boldsymbol{\Omega}_{t_1}) : \boldsymbol{\Omega}_{t_1} \mapsto T^*\mathcal{S} \,, \tag{4.49}$$
fulfilling in conjunction with the stress field $\boldsymbol{\sigma}(t_1)$ the virtual power variational equality
$$\langle \mathbf{f}_{\text{EXT}}(t_1), \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}(t_1)} \langle \boldsymbol{\sigma}(t_1), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \,. \tag{4.50}$$

2. An initial guess of the displacement $\boldsymbol{\varphi}_{t_2,t_1}$ corresponding to the update of the input control point from $\mathbf{c}(t_1)$ to $\mathbf{c}(t_2)$ may be obtained by updating the loading to
$$\mathbf{f}_{\text{EXT}} = \mathbf{A}(\mathbf{c}(t_2), \boldsymbol{\Omega}_{t_1}) : \boldsymbol{\Omega}_{t_1} \mapsto T^*\mathcal{S} \,. \tag{4.51}$$
The placement $\boldsymbol{\Omega}_{t_1}$ is assumed as reference placement. Let the vector field $\mathbf{u} : \boldsymbol{\Omega}_{t_1} \mapsto T\mathcal{S}$ be the incremental displacement from $\boldsymbol{\Omega}_{t_1}$ and $\boldsymbol{\varepsilon}(\mathbf{u})$ be the corresponding stretching (linearized strain increment). The constitutive stress response is then expressed by
$$\boldsymbol{\sigma}_{\text{REF}} = \boldsymbol{\Psi}^{-1}_{\text{REF}}(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}(t_1)) + \boldsymbol{\varepsilon}(\mathbf{u})) \,. \tag{4.52}$$
The solution of the geometrically linearized elastostatic problem
$$\begin{aligned} \langle \mathbf{f}_{\text{EXT}}, \delta\mathbf{v} \rangle &= \langle \mathbf{A}(\mathbf{c}(t_2), \boldsymbol{\Omega}_{t_1}), \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}_{t_1}} \\ &= \int_{\boldsymbol{\Omega}_{t_1}} \langle \boldsymbol{\Psi}^{-1}_{\text{REF}}(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}(t_1)) + \boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m} \,, \end{aligned} \tag{4.53}$$
yields the spatial vector field $\mathbf{u} : \boldsymbol{\Omega}_{t_1} \mapsto T\mathcal{S}$ as first trial incremental displacement from $\boldsymbol{\Omega}_{t_1}$.

3. Setting
$$\boldsymbol{\varphi}_{t_2,t_1}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{x} \,, \tag{4.54}$$
for $\mathbf{x} \in \boldsymbol{\Omega}_{t_1}$, a first trial placement of the body at time t_2 is evaluated by the assigment $\boldsymbol{\Omega} = \boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}_{t_1})$.

4. The control algorithm provides the loading update

$$\mathbf{f}_{\text{EXT}} = \mathbf{A}(\mathbf{c}(t_2), \boldsymbol{\Omega}) : \boldsymbol{\Omega} \mapsto T^*\mathcal{S}\,, \tag{4.55}$$

and the trial referential finite-step elastic strain on $\boldsymbol{\Omega}_{t_1}$ is evaluated by

$$\mathbf{es}_{\text{REF}}(t_2, t_1) = \tfrac{1}{2}(\boldsymbol{\varphi}_{t_2,t_1}\!\downarrow\!\mathbf{g}_{\text{MAT}} - \mathbf{g}_{\text{MAT}})\,. \tag{4.56}$$

5. The updated stress on $\boldsymbol{\Omega}$ is thus given by

$$\boldsymbol{\sigma} = \boldsymbol{\varphi}_{t_2,t_1}\!\uparrow\!(\boldsymbol{\Psi}_{\text{REF}}^{-1}(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}(t_1)) + \mathbf{es}_{\text{REF}}(t_2, t_1)))\,, \tag{4.57}$$

and the related elastic response on the trial placement $\boldsymbol{\Omega}$ is evaluated by the virtual power variational expression

$$\langle \mathbf{r}, \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m}\,. \tag{4.58}$$

6. If the ratio, between a suitable norm of the force gap

$$\mathbf{f}_{\text{EXT}} - \mathbf{r} : \boldsymbol{\Omega} \mapsto T^*\mathcal{S}\,, \tag{4.59}$$

and the norm of the loading $\mathbf{f}_{\text{EXT}}$, is less than a prescribed tolerance, an approximated fixed point of the algorithm is deemed to be reached and the iterations stop.

7. Otherwise the force gap $\mathbf{f}_{\text{EXT}} - \mathbf{r}$ is applied to perform a correction of the previous guess concerning the displacement $\boldsymbol{\varphi}_{t_2,t_1}$ from $\boldsymbol{\Omega}_{t_1}$. This task is accomplished by assuming the previous guess $\boldsymbol{\Omega}$ as reference placement. The constitutive stress response, to the previous stress trial $\boldsymbol{\sigma}$ and to a linearized strain increment $\boldsymbol{\varepsilon}(\mathbf{u})$, is expressed by

$$\boldsymbol{\sigma}_{\text{REF}} = \boldsymbol{\Psi}_{\text{REF}}^{-1}(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}) + \boldsymbol{\varepsilon}(\mathbf{u}))\,, \tag{4.60}$$

and the solution of the geometrically linearized elastostatic problem

$$\langle \mathbf{f}_{\text{EXT}} - \mathbf{r}, \delta\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} \langle \boldsymbol{\Psi}_{\text{REF}}^{-1}(\boldsymbol{\Psi}_{\text{REF}}(\boldsymbol{\sigma}) + \boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\delta\mathbf{v}) \rangle \, \mathbf{m}\,, \tag{4.61}$$

yields the spatial vector field $\mathbf{u} : \boldsymbol{\Omega} \mapsto T\mathcal{S}$ as trial incremental displacement from $\boldsymbol{\Omega}$. The update of the displacement map from $\boldsymbol{\Omega}_{t_1}$ is given by the assignement

$$\boldsymbol{\varphi}_{t_2,t_1}(\mathbf{x}) = \boldsymbol{\varphi}_{t_2,t_1}(\mathbf{x}) + \mathbf{u}(\boldsymbol{\varphi}_{t_2,t_1}(\mathbf{x}))\,, \tag{4.62}$$

for $\mathbf{x} \in \boldsymbol{\Omega}_{t_1}$. Setting $\boldsymbol{\Omega} = \boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}_{t_1})$, the iteration loop proceeds from item 4.

After convergence, the next time-step is performed starting from placement $\boldsymbol{\Omega}_{t_2} = \boldsymbol{\varphi}_{t_2,t_1}(\boldsymbol{\Omega}_{t_1})$ under the force $\mathbf{f}_{\text{EXT}}(t_2) = \mathbf{A}(\mathbf{c}(t_2), \boldsymbol{\Omega}_{t_2}) : \boldsymbol{\Omega}_{t_2} \mapsto T^*\mathcal{S}$.

4.6 Rate elasticity in terms of mixed tensors

Alteration of material tensors may be performed by making recourse to a linear isomorphism between the tangent and the cotangent space at the base material point. To any non-degenerate symmetric twice covariant material tensor $\mathbf{g}_{\text{ALT}}(\mathbf{e}) : T_{\mathbf{e}}\boldsymbol{\Omega} \times T_{\mathbf{e}}\boldsymbol{\Omega} \mapsto \Re$, there corresponds a linear, self-dual isomorphism

$$\mathbf{g}_{\text{ALT}}(\mathbf{e}) : T_{\mathbf{e}}\boldsymbol{\Omega} \mapsto T^*_{\mathbf{e}}\boldsymbol{\Omega}\,, \tag{4.63}$$

that can do the job.

Denoting by $\mathbf{M} : \text{Con}(V\mathcal{E}) \mapsto \text{Mix}(V\mathcal{E})$ the alteration from contravariant to mixed tensors, the dual operator $\mathbf{M}^* : \text{Mix}(V\mathcal{E}) \mapsto \text{Cov}(V\mathcal{E})$, performing the alteration from mixed to covariant tensors, is defined by

$$\langle \mathbf{M}\boldsymbol{\sigma}, \mathbf{D} \rangle = \langle \boldsymbol{\sigma}, \mathbf{M}^*\mathbf{D} \rangle\,, \qquad \boldsymbol{\sigma} \in \text{Con}(V\mathcal{E})\,, \quad \mathbf{D} \in \text{Mix}(V\mathcal{E})\,. \tag{4.64}$$

We may then set

$$\begin{aligned} \mathbf{K} &= \mathbf{M} \cdot \boldsymbol{\sigma} := \boldsymbol{\sigma} \cdot \mathbf{g}_{\text{ALT}}\,, && \mathbf{K} \text{ mixed alteration of } \boldsymbol{\sigma}\,, \\ \mathbf{es} &= \mathbf{M}^* \cdot \mathbf{es}_{\text{MIX}} := \mathbf{g}_{\text{ALT}} \cdot \mathbf{es}_{\text{MIX}}\,, && \mathbf{es}_{\text{MIX}} \text{ mixed alteration of } \mathbf{es}\,, \\ \boldsymbol{\varepsilon}_{\text{EL}} &= \mathbf{M}^* \cdot \mathbf{D}_{\text{EL}} := \mathbf{g}_{\text{ALT}} \cdot \mathbf{D}_{\text{EL}}\,, && \mathbf{D}_{\text{EL}} \text{ mixed alteration of } \boldsymbol{\varepsilon}_{\text{EL}}\,. \end{aligned} \tag{4.65}$$

Symmetry of the contravariant Kirchhoff stress $\boldsymbol{\sigma} \in \text{Con}(V\mathcal{E})$ entails $\mathbf{g}_{\text{ALT}}$-symmetry of the mixed alteration $\mathbf{K} := \boldsymbol{\sigma} \cdot \mathbf{g}_{\text{ALT}} \in \text{Mix}(V\mathcal{E})$, since by definition

$$\begin{aligned} \mathbf{g}_{\text{ALT}}(\mathbf{K} \cdot \mathbf{h}\,, \mathbf{d}) &= \mathbf{g}_{\text{ALT}}(\boldsymbol{\sigma} \cdot \mathbf{g}_{\text{ALT}} \cdot \mathbf{h}\,, \mathbf{d}) \\ &= \mathbf{g}_{\text{ALT}}(\mathbf{d}\,, \boldsymbol{\sigma} \cdot \mathbf{g}_{\text{ALT}} \cdot \mathbf{h}) \\ &= \boldsymbol{\sigma}(\mathbf{g}_{\text{ALT}} \cdot \mathbf{h}\,, \mathbf{g}_{\text{ALT}} \cdot \mathbf{d})\,. \end{aligned} \tag{4.66}$$

Analogously, symmetry of $\boldsymbol{\varepsilon}_{\text{EL}} \in \text{Cov}(V\mathcal{E})$ entails $\mathbf{g}_{\text{ALT}}$-symmetry of $\mathbf{D}_{\text{EL}} \in \text{Mix}(V\mathcal{E})$.

The elastic power expended per unit mass is expressed, according to the pairings Eq. (3.60) by

$$\begin{aligned} \langle \mathbf{K}, \mathbf{D}_{\text{EL}} \rangle &= \langle \mathbf{M} \cdot \boldsymbol{\sigma}, \mathbf{M}^{-*} \cdot \boldsymbol{\varepsilon}_{\text{EL}} \rangle \\ &= \langle \mathbf{M}^{-1}\mathbf{M} \cdot \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\text{EL}} \rangle \\ &= \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}_{\text{EL}} \rangle\,. \end{aligned} \tag{4.67}$$

From Eq. (4.67) it follows that, as expected on a physical ground, the elastic power per unit mass is alteration-invariant, being a notion independent of the chosen representation of tensors.

Definition 30 (Rate elastic law in terms of mixed tensors) *The rate-elastic law expressed by $\boldsymbol{\varepsilon}_{\text{EL}} := \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}$, Eq. (4.2), is translated in terms of mixed material tensors as*

$$\mathbf{D}_{\text{EL}} := \mathbf{H}_{\text{MIX}}(\mathbf{K}) \cdot \dot{\mathbf{K}}\,, \tag{4.68}$$

with the definitions

$$\begin{aligned} \dot{\mathbf{K}} \quad &:= \mathbf{M} \cdot \dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\mathrm{ALT}} \,, && \textit{mixed alteration of stressing}\,, \\ \mathbf{D}_{\mathrm{EL}} &:= \mathbf{M}^* \cdot \dot{\mathbf{e}}\mathbf{s} = \mathbf{g}_{\mathrm{ALT}}^{-1} \cdot \dot{\mathbf{e}}\mathbf{s} \,, && \textit{mixed alteration of elastic stretching}\,. \end{aligned} \tag{4.69}$$

The constitutive operators $\mathbf{H}$ and $\mathbf{H}_{\mathrm{MIX}}$ are related by

$$\mathbf{H}(\boldsymbol{\sigma}) = \mathbf{M}^* \left(\mathbf{H}_{\mathrm{MIX}}(\mathbf{M} \cdot \boldsymbol{\sigma})\right)\mathbf{M} \,. \tag{4.70}$$

Then, from the expression

$$d_F\mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma} = \mathbf{M}^* \cdot (d_F\mathbf{H}_{\mathrm{MIX}}(\mathbf{M} \cdot \boldsymbol{\sigma}) \cdot \mathbf{M} \cdot \delta\boldsymbol{\sigma}) \cdot \mathbf{M} \,, \tag{4.71}$$

we infer that integrability of $\mathbf{H}_{\mathrm{MIX}}$, expressed by the symmetry conditions

$$\begin{aligned} \langle d_F\mathbf{H}_{\mathrm{MIX}}(\mathbf{K}) \cdot \delta\mathbf{K} \cdot \delta_1\mathbf{K}, \delta_2\mathbf{K} \rangle &= \langle d_F\mathbf{H}_{\mathrm{MIX}}(\mathbf{K}) \cdot \delta\mathbf{K} \cdot \delta_2\mathbf{K}, \delta_1\mathbf{K} \rangle \,, \\ \langle \mathbf{H}_{\mathrm{MIX}}(\mathbf{K}) \cdot \delta_1\mathbf{K}, \delta_2\mathbf{K} \rangle &= \langle \mathbf{H}_{\mathrm{MIX}}(\mathbf{K}) \cdot \delta_2\mathbf{K}, \delta_1\mathbf{K} \rangle \,, \end{aligned} \tag{4.72}$$

is equivalent to integrability of $\mathbf{H}$, expressed by Eqs. (4.12) and (4.15).

Setting $\mathbf{H}_{\mathrm{MIX}} = d_F\boldsymbol{\Psi}_{\mathrm{MIX}}$ and $\boldsymbol{\Psi}_{\mathrm{MIX}} = d_F\Xi_{\mathrm{MIX}}$, the integrated elastic law is expressed in terms of the mixed elastic state by

$$\begin{aligned} \mathbf{es}_{\mathrm{MIX}} = \boldsymbol{\Psi}_{\mathrm{MIX}}(\mathbf{K}) &= d_F\Xi_{\mathrm{MIX}}(\mathbf{K}) \,, \\ \mathbf{K} &= d_F\Xi^*_{\mathrm{MIX}}(\mathbf{es}_{\mathrm{MIX}}) \,, \end{aligned} \tag{4.73}$$

so that $\mathbf{D}_{\mathrm{EL}} = \dot{\mathbf{e}}\mathbf{s}_{\mathrm{MIX}}$.

To get time invariance along the motion, if the elastic operator $\mathbf{H}_{\mathrm{MIX}}$ is time-invariant, then the alterating tensor $\mathbf{M} : \mathrm{Con}(V\mathcal{E}) \mapsto \mathrm{Mix}(V\mathcal{E})$ introduced by Eq. (4.65), must also be assumed to be time-invariant along the motion. This amounts to assume that the material tensor field $\mathbf{g}_{\mathrm{ALT}} : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}^*$ is invariant along the motion.

The formulation in terms of mixed tensors is the one adopted in applications since the powerful representation in terms of a spectral decomposition is then available.

Remark 5 (Lie-derivatives and alterations) *The natural candidate as linear isomorphism for alteration of tensors at each event on the trajectory manifold, is the material metric. If this choice is made, one must be aware of the fact that pull-back along the motion (hence* Lie *derivative) and alteration by the material metric tensor field on the trajectory, are non-commutative operations. In particular, vanishing of the contravariant* Kirchhoff *stressing* $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{V}}\boldsymbol{\sigma}$ *does not imply vanishing of the mixed* Kirchhoff *stressing* $\mathcal{L}_{\mathbf{V}}\,\mathbf{K}$ *with* $\mathbf{K} = \boldsymbol{\sigma} \cdot \mathbf{g}_{\mathrm{MAT}}$, *because*

$$\mathcal{L}_{\mathbf{V}}\mathbf{K} = \mathcal{L}_{\mathbf{V}}(\boldsymbol{\sigma} \cdot \mathbf{g}_{\mathrm{MAT}}) = (\mathcal{L}_{\mathbf{V}}\boldsymbol{\sigma}) \cdot \mathbf{g}_{\mathrm{MAT}} + \boldsymbol{\sigma} \cdot (\mathcal{L}_{\mathbf{V}}\mathbf{g}_{\mathrm{MAT}}) \,. \tag{4.74}$$

Therefore, the LIE*-derivative* $\mathcal{L}_{\mathbf{V}}\mathbf{K}$ *is not* $\mathbf{g}_{\text{MAT}}$*-symmetric, unless the body is experiencing a rigid act of motion, so that* $\dot{\mathbf{g}}_{\text{MAT}} := \mathcal{L}_{\mathbf{V}}\mathbf{g}_{\text{MAT}} = \mathbf{0}$ *and hence* $\mathcal{L}_{\mathbf{V}}\mathbf{K} = \dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{MAT}}$ *. Lack of* $\mathbf{g}_{\text{MAT}}$*-symmetry of the convective derivative of mixed stress tensors was already put into evidence by Sedov (1960) and remarked in (Marsden and Hughes, 1983).*

Remark 6 *A standard procedure to preserve* $\mathbf{g}_{\text{MAT}}$*-symmetry of a mixed stress tensor* $\mathbf{K}$ *under pull-back to a straightened trajectory, consists in considering the contravariant alteration* $\boldsymbol{\sigma} = \mathbf{K} \cdot \mathbf{g}_{\text{MAT}}^{-1}$ *, in pulling it back and then in altering it back to the mixed form by post composition with the material metric tensor, to get* $\mathbf{K}_{\text{REF}} = \boldsymbol{\varphi}_{\alpha}\!\downarrow\!(\mathbf{K} \cdot \mathbf{g}_{\text{MAT}}^{-1}) \cdot \mathbf{g}_{\text{MAT}}$ *. Denoting by a superscript* $()^A$ *the* $\mathbf{g}_{\text{MAT}}$*-adjoint, recalling the pull-back law Eq. (3.59) and the defining relation*

$$\mathbf{g}_{\text{MAT}} \cdot (T\boldsymbol{\varphi}_{\alpha})^A = (T\boldsymbol{\varphi}_{\alpha})^* \cdot \mathbf{g}_{\text{MAT}} \,, \tag{4.75}$$

the procedure outlined above yields the well-known referential mixed PIOLA-KIRCHHOFF *stress (Piola, 1833; Kirchhoff, 1852), see (Truesdell and Noll, 1965), Sect. 43A:*

$$\begin{aligned} \mathbf{K}_{\text{REF}} &= T\boldsymbol{\varphi}_{-\alpha} \cdot (\mathbf{K} \cdot \mathbf{g}_{\text{MAT}}^{-1}) \cdot (T\boldsymbol{\varphi}_{-\alpha})^* \cdot \mathbf{g}_{\text{MAT}} \\ &= T\boldsymbol{\varphi}_{-\alpha} \cdot \mathbf{K} \cdot \mathbf{g}_{\text{MAT}}^{-1} \cdot \mathbf{g}_{\text{MAT}} \cdot (T\boldsymbol{\varphi}_{-\alpha})^A \\ &= T\boldsymbol{\varphi}_{-\alpha} \cdot \mathbf{K} \cdot (T\boldsymbol{\varphi}_{-\alpha})^A \,. \end{aligned} \tag{4.76}$$

If after the pull back of $\boldsymbol{\sigma} = \mathbf{K} \cdot \mathbf{g}_{\text{ALT}}^{-1}$ *to a reference placement, the final alteration were performed by means of the pull-back of the altering tensor, the result would be*

$$\boldsymbol{\varphi}_{\alpha}\!\downarrow\!\mathbf{g}_{\text{ALT}} = (T\boldsymbol{\varphi}_{\alpha})^* \cdot \mathbf{g}_{\text{ALT}} \cdot T\boldsymbol{\varphi}_{\alpha} \,. \tag{4.77}$$

This procedure yields in fact the pull-back of the mixed tensor $\mathbf{K}$ *Eq. (3.59)*

$$\begin{aligned} \boldsymbol{\varphi}_{\alpha}\!\downarrow\!\mathbf{K} &= T\boldsymbol{\varphi}_{-\alpha} \cdot (\mathbf{K} \cdot \mathbf{g}_{\text{ALT}}^{-1}) \cdot (T\boldsymbol{\varphi}_{-\alpha})^* \cdot (T\boldsymbol{\varphi}_{\alpha})^* \cdot \mathbf{g}_{\text{ALT}} \cdot T\boldsymbol{\varphi}_{\alpha} \\ &= T\boldsymbol{\varphi}_{-\alpha} \cdot \mathbf{K} \cdot T\boldsymbol{\varphi}_{\alpha} \,, \end{aligned} \tag{4.78}$$

revealing that the pull-back $\boldsymbol{\varphi}_{\alpha}\!\downarrow\!\mathbf{K}$ *is not* $\mathbf{g}_{\text{ALT}}$*-symmetric but is in fact* $(\boldsymbol{\varphi}_{\alpha}\!\downarrow\!\mathbf{g}_{\text{ALT}})$*-symmetric, with the property of sharing, with the original* $\mathbf{g}_{\text{ALT}}$*-symmetric* $\mathbf{K}$ *, the eigenvalue spectrum.*

4.7 Isotropic, stress independent rate elasticity

Let E and G be EULER and LAMÉ moduli of linearised elasticity.

It is convenient to introduce the modified EULER modulus $E_m = E/\rho$ and LAMÉ modulus $G_m = G/\rho$ per unit mass, measured in a local placement which will be called the *local testing placement.*[3]

[3] A local placement is a tangent space to a placement.

The simplest 3D rate-elastic isotropic constitutive operator $\mathbf{H}$, denoting by J_1 the linear invariant, is defined as follows

$$\boldsymbol{\varepsilon}_{\text{EL}} = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} := \frac{1}{2\,G_m}\,\mathbf{M}^* \cdot \mathbf{M} \cdot \dot{\boldsymbol{\sigma}} - \frac{\nu_m}{E_m}\,J_1(\mathbf{M} \cdot \dot{\boldsymbol{\sigma}})\,\mathbf{g}_{\text{ALT}}\,, \tag{4.79}$$

with

$$\frac{1}{2\,G_m} = \frac{1+\nu_m}{E_m}\,, \tag{4.80}$$

or explicitly

$$\boldsymbol{\varepsilon}_{\text{EL}} = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} := \frac{1+\nu_m}{E_m}\,\mathbf{g}_{\text{ALT}} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{ALT}} - \frac{\nu_m}{E_m}\,J_1(\dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{ALT}})\,\mathbf{g}_{\text{ALT}}\,. \tag{4.81}$$

Premultiplying by $\mathbf{g}_{\text{ALT}}^{-1}$ and taking the linear invariant we get

$$J_1(\mathbf{g}_{\text{ALT}}^{-1} \cdot \boldsymbol{\varepsilon}_{\text{EL}}) = \frac{1-2\,\nu_m}{E_m}\,J_1(\dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{ALT}})\,, \tag{4.82}$$

that is

$$J_1(\mathbf{D}_{\text{EL}}) = \frac{1-2\,\nu_m}{E_m}\,J_1(\dot{\mathbf{K}})\,. \tag{4.83}$$

Under the condition that $2\,\nu_m < 1$, the LAMÉ modulus per unit mass may be defined by

$$\lambda_m = \frac{\nu_m\,E_m}{(1+\nu_m)(1-2\,\nu_m)} = 2\,G_m\,\frac{\nu_m}{(1-2\,\nu_m)}\,, \tag{4.84}$$

and the inverse rate-elastic law may be written as

$$\dot{\boldsymbol{\sigma}} = 2\,G_m\,\mathbf{g}_{\text{ALT}}^{-1} \cdot \boldsymbol{\varepsilon}_{\text{EL}} \cdot \mathbf{g}_{\text{ALT}}^{-1} + \lambda_m\,J_1(\mathbf{g}_{\text{ALT}}^{-1} \cdot \boldsymbol{\varepsilon}_{\text{EL}}) \cdot \mathbf{g}_{\text{ALT}}^{-1}\,. \tag{4.85}$$

The alterating tensor $\mathbf{M} : \mathrm{CON}(V\mathcal{E}) \mapsto \mathrm{MIX}(V\mathcal{E})$ is defined by setting $\mathbf{g}_{\text{ALT}} = \mathbf{g}_{\text{MAT}}$ in the *local testing placement* [4] and assuming that $\mathbf{g}_{\text{ALT}}$ is pushed by the motion from that placement, i.e. that

$$\mathbf{g}_{\text{ALT}} = \varphi_\alpha{\uparrow}\mathbf{g}_{\text{MAT}}\,. \tag{4.86}$$

The tangent elastic compliance $\mathbf{H}(\boldsymbol{\sigma})$, given by Eq. (4.79), is then time-invariant along the motion (and stress independent). We will refer to the expression

$$\mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} := \frac{1}{2\,G_m}\,\mathbf{g}_{\text{MAT}} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{MAT}} - \frac{\nu_m}{E_m}\,J_1(\dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{MAT}})\,\mathbf{g}_{\text{MAT}}\,, \tag{4.87}$$

as the *simplest model* of elastic behavior.

[4] It should be underlined that, although introduced here in a special context, this notion has a character of generality, being the placement where experimental tests are performed, This is feasible only in the context of a rate theory.

Let a laboratory test be performed on the same material after a time lapse $\alpha \in \mathcal{Z}$. By time invariance, the outcome will be

$$\begin{aligned}\mathbf{H}(\boldsymbol{\varphi}_\alpha\!\uparrow\!\boldsymbol{\sigma}) \cdot \boldsymbol{\varphi}_\alpha\!\uparrow\!\dot{\boldsymbol{\sigma}} &= \frac{1+\nu_m}{E_m}\, \boldsymbol{\varphi}_\alpha\!\uparrow\!(\mathbf{g}_{\text{ALT}} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{ALT}}) - \frac{\nu_m}{E_m}\, J_1(\boldsymbol{\varphi}_\alpha\!\uparrow\!(\dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{ALT}}))\, \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{g}_{\text{ALT}} \\ &= \frac{1+\nu_m}{E_m}\, \mathbf{g}_{\text{ALT}} \cdot (\boldsymbol{\varphi}_\alpha\!\uparrow\!\dot{\boldsymbol{\sigma}}) \cdot \mathbf{g}_{\text{ALT}} - \frac{\nu_m}{E_m}\, J_1((\boldsymbol{\varphi}_\alpha\!\uparrow\!\dot{\boldsymbol{\sigma}}) \cdot \mathbf{g}_{\text{ALT}})\, \mathbf{g}_{\text{ALT}} \,.\end{aligned} \tag{4.88}$$

The elastic law will then be expressed by the simplest constitutive model Eq. (4.87) in all local placements related by a local isometry because there $\mathbf{g}_{\text{ALT}} = \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{g}_{\text{MAT}} = \mathbf{g}_{\text{MAT}}$. In non isometric local placements the law will take the expression in Eq. (4.81) with $\mathbf{g}_{\text{ALT}} = \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{g}_{\text{MAT}}$.

In terms of the mixed KIRCHHOFF stress $\mathbf{K} = \boldsymbol{\sigma} \cdot \mathbf{g}_{\text{MAT}}$ the simplest model takes the expression

$$\mathbf{H}_{\text{MIX}}(\mathbf{K}) := \frac{1+\nu_m}{E_m}\, \mathbb{I} - \frac{\nu_m}{E_m}\, \mathbf{I} \otimes_{\mathbf{g}_{\text{ALT}}} \mathbf{I} \,, \tag{4.89}$$

where $\mathbf{I}$ is the identity in the tangent space to the placement, $\mathbb{I}$ is the identity in the linear space of mixed tensors and ν is the POISSON ratio.

The simplest rate-elastic constitutive law, in the formulation corresponding to Eq. (4.68), expressed in terms of the mixed alteration of KIRCHHOFF stressing $\dot{\mathbf{K}} := \dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\text{ALT}}$ and of the mixed alteration of elastic stretching $\mathbf{D}_{\text{EL}} = \mathbf{g}_{\text{ALT}}^{-1} \cdot \boldsymbol{\varepsilon}_{\text{EL}}$, writes

$$\mathbf{D}_{\text{EL}} := \frac{1+\nu_m}{E_m}\, \dot{\mathbf{K}} - \frac{\nu_m}{E_m}\, J_1(\dot{\mathbf{K}})\, \mathbf{I} \,. \tag{4.90}$$

Upon geometric linearization around the testing placement, the law in Eq. (4.87) collapses into the familiar linear isotropic elastic compliance.

Stress independence of the constitutive expression Eq. (4.87) assures CAUCHY integrability, with the tensor potential

$$\boldsymbol{\Psi}_{\text{MIX}}(\mathbf{K}) = \frac{1+\nu_m}{E_m}\, \mathbf{K} - \frac{\nu_m}{E_m}\, J_1(\mathbf{K})\, \mathbf{I} \,, \tag{4.91}$$

and, by symmetry of Eq. (4.87), the scalar GREEN elastic potential is given by

$$\Xi_{\text{MIX}}(\mathbf{K}) = \frac{1+\nu_m}{2\,E_m}\, J_1(\mathbf{K}^2) - \frac{\nu_m}{2\,E_m}\, J_1(\mathbf{K})^2 \,. \tag{4.92}$$

Mass conservation, time invariance of the elastic moduli per unit mass and of the alteration operator $\mathbf{M} : \mathrm{CON}(V\mathcal{E}) \mapsto \mathrm{MIX}(V\mathcal{E})$, assure that the elastic law expressed by Eq. (4.79) is conservative, as explicated by Eq. (4.43).

It is worth noting that, if the rate elastic law in Eqs. (4.87)-(4.90) is expressed in terms of the CAUCHY stress $\mathbf{T} = \rho\,\mathbf{K}$, time independence of the elastic moduli does not ensure conservativeness, since the mass density is not time invariant, unless the motion is isochoric.

4.8 Pure elasticity

Let us consider a purely elastic behavior, so that the elastic stretching is given by

$$\boldsymbol{\varepsilon}_{\mathrm{EL}} = \boldsymbol{\varepsilon}_{\mathbf{v}} = \mathcal{L}_{\mathbf{V}} \mathbf{g}_{\mathrm{MAT}} = \partial_{\theta=0}\, \boldsymbol{\varphi}_{\theta}\!\downarrow\! \mathbf{g}_{\mathrm{MAT}}\,. \tag{4.93}$$

Preliminarily, observing that $\mathbf{g}_{\mathrm{MAT}}^{-1} \in \mathrm{Con}(V\mathcal{E})$, from Eq. $(3.59)_2$ we infer that

$$\begin{aligned}(\boldsymbol{\varphi}_{\alpha}\!\uparrow\! \mathbf{g}_{\mathrm{MAT}})^{-1} &= \Big((T\boldsymbol{\varphi}_{-\alpha})^{*} \cdot \mathbf{g}_{\mathrm{MAT}} \cdot T\boldsymbol{\varphi}_{-\alpha}\Big)^{-1} = T\boldsymbol{\varphi}_{\alpha} \cdot \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot (T\boldsymbol{\varphi}_{\alpha})^{*} \\ &= \boldsymbol{\varphi}_{\alpha}\!\uparrow\! \mathbf{g}_{\mathrm{MAT}}^{-1}\,.\end{aligned} \tag{4.94}$$

Then, pulling back Eq. (4.81) to a *local testing placement*, we get

$$\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \boldsymbol{\varepsilon}_{\mathrm{EL}} = \frac{1+\nu_m}{E_m}\, \mathbf{g}_{\mathrm{MAT}} \cdot (\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \dot{\boldsymbol{\sigma}}) \cdot \mathbf{g}_{\mathrm{MAT}} - \frac{\nu_m}{E_m}\, J_1((\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \dot{\boldsymbol{\sigma}}) \cdot \mathbf{g}_{\mathrm{MAT}})\, \mathbf{g}_{\mathrm{MAT}}\,, \tag{4.95}$$

where, by Eq. $(3.59)_1$

$$\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \boldsymbol{\varepsilon}_{\mathrm{EL}} = (T\boldsymbol{\varphi}_{\alpha})^{*} \cdot \boldsymbol{\varepsilon}_{\mathrm{EL}} \cdot T\boldsymbol{\varphi}_{\alpha}\,. \tag{4.96}$$

The increment of referential stress in a time lapse $\Delta t \in I$ is evaluated by substituting the expression Eq. (4.93) of $\boldsymbol{\varepsilon}_{\mathrm{EL}}$ into Eq. (4.96) and integrating in time Eq. (4.95). The push forward of the upgraded referential stress to the current placement yields the upgraded current stress.

The pull-back of the geometric stretching is given by

$$\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \boldsymbol{\varepsilon}_{\mathbf{v}} := \boldsymbol{\varphi}_{\alpha}\!\downarrow\!(\mathcal{L}_{\mathbf{V}} \mathbf{g}_{\mathrm{MAT}}) = \boldsymbol{\varphi}_{\alpha}\!\downarrow\! \partial_{\theta=0}\,(\boldsymbol{\varphi}_{\theta}\!\downarrow\! \mathbf{g}_{\mathrm{MAT}}) = \partial_{\theta=\alpha}\,(\boldsymbol{\varphi}_{\theta}\!\downarrow\! \mathbf{g}_{\mathrm{MAT}})\,. \tag{4.97}$$

Introducing the Piola-Kirchhoff tensor (Piola, 1833)

$$\mathbf{S}_{\alpha} := (\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \boldsymbol{\sigma}) \cdot \mathbf{g}_{\mathrm{MAT}} \in \mathrm{Mix}(V\mathcal{E})\,, \tag{4.98}$$

and the Green-St.Venant strain (George Green, 1839; St. Venant, 1844)

$$\mathbf{E}_{\alpha} := \tfrac{1}{2}\mathbf{g}_{\mathrm{MAT}}^{-1} \cdot (\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \mathbf{g}_{\mathrm{MAT}} - \mathbf{g}_{\mathrm{MAT}}) = \tfrac{1}{2}\Big((T\boldsymbol{\varphi}_{\alpha})^{A}\, T\boldsymbol{\varphi}_{\alpha} - \mathbf{I}\Big) \in \mathrm{Mix}(V\mathcal{E})\,, \tag{4.99}$$

with the last equality following from Eq. (4.75), we have that

$$\dot{\mathbf{S}}_{\alpha} = \partial_{\theta=\alpha}\, \mathbf{S}_{\theta} = \partial_{\theta=\alpha}\,(\boldsymbol{\varphi}_{\theta}\!\downarrow\! \boldsymbol{\sigma}) \cdot \mathbf{g}_{\mathrm{MAT}} = (\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \dot{\boldsymbol{\sigma}}) \cdot \mathbf{g}_{\mathrm{MAT}}\,, \tag{4.100}$$

$$\dot{\mathbf{E}}_{\alpha} = \partial_{\theta=\alpha}\, \mathbf{E}_{\theta} = \partial_{\theta=\alpha}\, \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot (\boldsymbol{\varphi}_{\theta}\!\downarrow\! \mathbf{g}_{\mathrm{MAT}}) = \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot (\boldsymbol{\varphi}_{\alpha}\!\downarrow\! \boldsymbol{\varepsilon}_{\mathbf{v}})\,. \tag{4.101}$$

The mixed alteration of Eq. (4.95) writes

$$\dot{\mathbf{E}}_{\alpha} = \frac{1+\nu_m}{E_m}\, \dot{\mathbf{S}}_{\alpha} - \frac{\nu_m}{E_m}\, J_1(\dot{\mathbf{S}}_{\alpha})\, \mathbf{I}\,. \tag{4.102}$$

Under the condition that $2\,\nu_m < 1$, the inverse law reads

$$\dot{\mathbf{S}}_{\alpha} = 2\, G_m \left(\dot{\mathbf{E}}_{\alpha} + \frac{\nu_m}{1 - 2\,\nu_m}\, J_1(\dot{\mathbf{E}}_{\alpha})\, \mathbf{I}\right). \tag{4.103}$$

5

Stretching of a rubber bar

“ The power of a theory can be better popularized by prediction of experimental results.”

– Anonymous

To illustrate the formulation of an elastic constitutive model in the range of finite strains, we consider the response of a rubber bar whose elastic law is expressed by Eq. (4.68), the tangent compliance being given by the simplest, stress independent, law in Eq. (4.89). Assuming $\nu_m = 0.5$, the elastic stretching has a vanishing linear invariant, so that the elastic process is isochoric. Invariance of the volumetric stretching implies a singular elastic response so that the elastic law cannot be inverted to provide the stressing field induced by a given elastic stretching. The elastic response of a rubber-like material subject to a uniaxial stress field, is discussed by considering a simple isochoric motion.

Conservation of volume and mass implies that the scalar mass density ρ is constant, i.e. $\dot{\rho} = 0$. Being E_m constant, the modulus $E = \rho\, E_m$ will also be constant.

5.1 Homogeneous extension

Let $\{\, \mathbf{d}_i\,, i = 1,2,3\,\}$ and $\{\, \mathbf{d}^j\,, j = 1,2,3\,\}$ be dual bases in the space bundle $V\mathcal{E}$ and in the space cobundle $(V\mathcal{E})^*$ respectively. The former base is assumed to be $\mathbf{g}$-orthonormal so that the latter is $\mathbf{g}^{-1}$-orthonormal.

Setting $\mathbf{x} = x\,\mathbf{d_x} + y\,\mathbf{d_y} + z\,\mathbf{d_z}$, the motion is expressed by the assignment

$$\begin{aligned} \boldsymbol{\varphi}_\alpha(\mathbf{x}) &= \alpha(x\,\varepsilon_x\,\mathbf{d_x} + y\,\varepsilon_y\,\mathbf{d_y} + z\,\varepsilon_z\,\mathbf{d_z}) + x\,\mathbf{d_x} + y\,\mathbf{d_y} + z\,\mathbf{d_z} \\ &= x\,(\alpha\,\varepsilon_x + 1)\,\mathbf{d_x} + y\,(\alpha\,\varepsilon_y + 1)\,\mathbf{d_y} + z\,(\alpha\,\varepsilon_z + 1)\,\mathbf{d_z}\,, \end{aligned} \tag{5.1}$$

with the parameters $\varepsilon_x, \varepsilon_y, \varepsilon_z$ assumed to be function of the time lapse.

The corresponding space velocity is then given by

$$\begin{aligned}(\mathbf{v} \circ \boldsymbol{\varphi}_\alpha)(\mathbf{x}) &= \partial_{\theta=\alpha}\, \boldsymbol{\varphi}_\theta(\mathbf{x}) \\ &= x\,(\varepsilon_x + \alpha\,\varepsilon'_x)\,\mathbf{d_x} + y\,(\varepsilon_y + \alpha\,\varepsilon'_y)\,\mathbf{d_y} + z\,(\varepsilon_z + \alpha\,\varepsilon'_z)\,\mathbf{d_z}\,.\end{aligned} \tag{5.2}$$

Then $\boldsymbol{\varphi}_0(\mathbf{x}) = \mathbf{x}$ and $\mathbf{v}(\mathbf{x}) = x\,\varepsilon_x\,\mathbf{d_x} + y\,\varepsilon_y\,\mathbf{d_y} + z\,\varepsilon_z\,\mathbf{d_z}$.

The matrices associated with the tangent displacement $T\boldsymbol{\varphi}_\alpha$ and with the Green-St.Venant strain $\mathbf{E}_\alpha := \frac{1}{2}\Big((T\boldsymbol{\varphi}_\alpha)^A\,T\boldsymbol{\varphi}_\alpha - \mathbf{I}\Big)$, are given by

$$[T\boldsymbol{\varphi}_\alpha] = \begin{bmatrix} \alpha\,\varepsilon_x + 1 & 0 & 0 \\ 0 & \alpha\,\varepsilon_y + 1 & 0 \\ 0 & 0 & \alpha\,\varepsilon_z + 1 \end{bmatrix} \tag{5.3}$$

$$[\mathbf{E}_\alpha] = \tfrac{1}{2} \begin{bmatrix} (\alpha\,\varepsilon_x + 1)^2 - 1 & 0 & 0 \\ 0 & (\alpha\,\varepsilon_y + 1)^2 - 1 & 0 \\ 0 & 0 & (\alpha\,\varepsilon_z + 1)^2 - 1 \end{bmatrix} \tag{5.4}$$

The matrix associated with the mixed Euler stretching

$$\mathbf{D}(\mathbf{v}) = \mathbf{g}_{\mathrm{SPA}}{}^{-1} \cdot \tfrac{1}{2}\mathcal{L}_{\mathbf{v}}\,\mathbf{g}_{\mathrm{SPA}} = \mathrm{sym}_{\mathbf{g}}(\nabla\mathbf{v})\,, \tag{5.5}$$

is evaluated by time differentiation of Eq. (5.2) to be

$$[\mathbf{D}(\mathbf{v})] = \begin{bmatrix} \frac{(\varepsilon_x + \alpha\,\varepsilon'_x)}{(\alpha\,\varepsilon_x + 1)} & 0 & 0 \\ 0 & \frac{(\varepsilon_y + \alpha\,\varepsilon'_y)}{(\alpha\,\varepsilon_y + 1)} & 0 \\ 0 & 0 & \frac{(\varepsilon_z + \alpha\,\varepsilon'_z)}{(\alpha\,\varepsilon_z + 1)} \end{bmatrix} \tag{5.6}$$

The matrix associated with the time rate $\dot{\mathbf{E}}_\alpha$ of the Green-St.Venant strain can be evaluated by time differentiation of Eq. (5.4) to be

$$[\dot{\mathbf{E}}_\alpha] = \begin{bmatrix} (\alpha\,\varepsilon_x + 1)\,(\varepsilon_x + \alpha\,\varepsilon'_x) & 0 & 0 \\ 0 & (\alpha\,\varepsilon_y + 1)\,(\varepsilon_y + \alpha\,\varepsilon'_y) & 0 \\ 0 & 0 & (\alpha\,\varepsilon_z + 1)\,(\varepsilon_z + \alpha\,\varepsilon'_z) \end{bmatrix} \tag{5.7}$$

Then $\dot{\mathbf{E}}_0 = \mathbf{D}(\mathbf{v})$. Alternatively from Eq. (4.75) and Eq. (4.96) we infer that

$$\begin{aligned}\dot{\mathbf{E}}_\alpha = \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot (\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\varepsilon}_{\mathbf{v}}) &= (T\boldsymbol{\varphi}_\alpha)^A \cdot (\mathbf{g}_{\mathrm{MAT}}^{-1} \cdot \boldsymbol{\varepsilon}) \cdot T\boldsymbol{\varphi}_\alpha \\ &= (T\boldsymbol{\varphi}_\alpha)^A \cdot \mathbf{D}(\mathbf{v}) \cdot T\boldsymbol{\varphi}_\alpha\,,\end{aligned} \tag{5.8}$$

which by Eq. (5.3) and Eq. (5.6) yields Eq. (5.7).

5.2 Incompressibility

Assuming transversal symmetry, we set $\varepsilon_y = \varepsilon_z = \varepsilon_t$. Then infinitesimal isochoricity at any time α holds if

$$\begin{aligned} J_1(\mathbf{D}(\mathbf{v})) &= \frac{\varepsilon_x}{\alpha\,\varepsilon_x + 1} + \frac{2\,(\varepsilon_t + \alpha\,\varepsilon_t')}{\alpha\,\varepsilon_t + 1} \\ &= \frac{\varepsilon_x + 2\,\varepsilon_t + 3\,\alpha\,\varepsilon_x\,\varepsilon_t + 2\,\alpha\,\varepsilon_t'\,(\alpha\,\varepsilon_x + 1)}{(\alpha\,\varepsilon_x + 1)(\alpha\,\varepsilon_t + 1)} = 0\,. \end{aligned} \tag{5.9}$$

Fixing the time scale so that $\varepsilon_x = 1$, the infinitesimal isochoricity Eq. (5.9) is fulfilled by imposing the condition

$$2\,\alpha\,(\alpha + 1)\,\varepsilon_t' + (3\,\alpha + 2)\,\varepsilon_t + 1 = 0\,, \tag{5.10}$$

whose general solution is

$$\varepsilon_t(\alpha) = -\frac{1}{\alpha} + \frac{C}{\alpha\,\sqrt{1+\alpha}}\,. \tag{5.11}$$

The constant, evaluated to give $\lim_{\alpha\to 0^+} \varepsilon_t(\alpha) = -0.5$, is $C = 1$ and we get the expression plotted in fig. 5.1

$$\varepsilon_t(\alpha) = \frac{1}{\alpha}\Big(\frac{1}{\sqrt{1+\alpha}} - 1\Big) \iff \alpha\,\varepsilon_t(\alpha) + 1 = \frac{1}{\sqrt{1+\alpha}}\,. \tag{5.12}$$

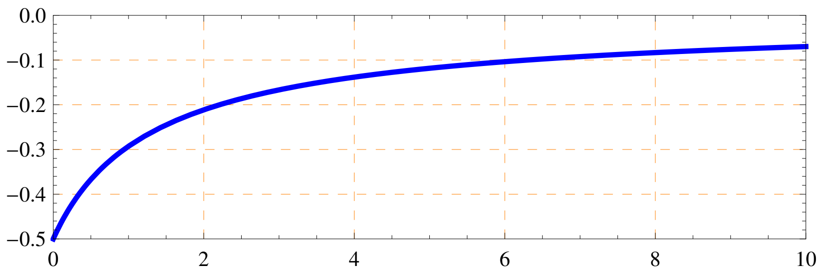


Figure 5.1: Transversal stretching $\varepsilon_t(\alpha)$ needed for isochoricity

On the other hand, finite isochoricity in the time lapse α holds if

$$\det(T\boldsymbol{\varphi}_\alpha) = (\alpha\,\varepsilon_x + 1)(\alpha\,\varepsilon_t(\alpha) + 1)^2 = 1\,. \tag{5.13}$$

From Eq. (5.13), being $\varepsilon_x = 1$, we infer that finite isochoricity in the time lapse α, expressed by the condition $\det(T\boldsymbol{\varphi}_\alpha) = 1$, leads to the same requirement Eq. (5.12), in accord with the kinematic relation

$$(\det(T\boldsymbol{\varphi}_\alpha))^{\cdot} = J_1(\mathbf{D}(\mathbf{v}))\,\det(T\boldsymbol{\varphi}_\alpha)\,. \tag{5.14}$$

Substituting Eq. (5.12) into Eq. (5.3), we get

$$[T\boldsymbol{\varphi}_\alpha] = \begin{bmatrix} 1+\alpha & 0 & 0 \\ 0 & 1/\sqrt{1+\alpha} & 0 \\ 0 & 0 & 1/\sqrt{1+\alpha} \end{bmatrix} \tag{5.15}$$

and substitution into Eq. (5.4) gives

$$[\mathbf{E}_\alpha] = \tfrac{1}{2}\begin{bmatrix} \alpha\,(2+\alpha) & 0 & 0 \\ 0 & -\alpha/(1+\alpha) & 0 \\ 0 & 0 & -\alpha/(1+\alpha) \end{bmatrix} \tag{5.16}$$

Fig. 5.2 yields the plot of the linear invariant $J_1(\mathbf{E}_\alpha)$ which, evaluated from Eq. (5.16), is given by

$$\begin{aligned} J_1(\mathbf{E}_\alpha) &= \frac{1}{2}\Big(\alpha\,(2+\alpha) - \frac{2\,\alpha}{1+\alpha}\Big) \\ &= \frac{\alpha^2(3+\alpha)}{2(1+\alpha)}\,. \end{aligned} \tag{5.17}$$

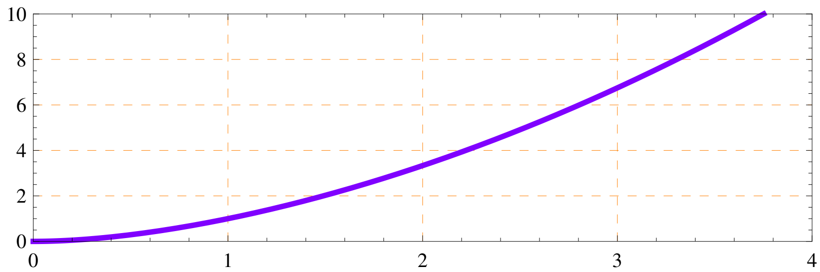


Figure 5.2: Plot of $J_1(\mathbf{E}_\alpha)$

Moreover substitution of Eq. (5.12) into Eq. (5.6) gives

$$[\mathbf{D}(\mathbf{v})] = \begin{bmatrix} \frac{1}{1+\alpha} & 0 & 0 \\ 0 & \frac{-1}{2(1+\alpha)} & 0 \\ 0 & 0 & \frac{-1}{2(1+\alpha)} \end{bmatrix} \tag{5.18}$$

and substitution into Eq. (5.7) , gives

$$\left[\dot{\mathbf{E}}_\alpha\right] = \tfrac{1}{2}\begin{bmatrix} 2\,(1+\alpha) & 0 & 0 \\ 0 & -1/(1+\alpha)^2 & 0 \\ 0 & 0 & -1/(1+\alpha)^2 \end{bmatrix} \tag{5.19}$$

which can be also got by evaluating the time derivative of Eq. (5.16).

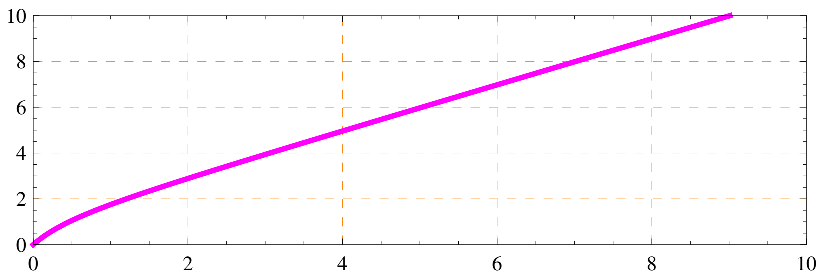


Figure 5.3: Plot of $J_1(\dot{\mathbf{E}}_\alpha)$

Fig. 5.3 yields the plot of the linear invariant $J_1(\dot{\mathbf{E}}_\alpha)$ which, evaluated from Eq. (5.19), is given by

$$J_1(\dot{\mathbf{E}}_\alpha) = \frac{1}{2}\Big(2\,(1+\alpha) - \frac{2}{(1+\alpha)^2}\Big) = \frac{\alpha\,(\alpha^2+3\alpha+3)}{(1+\alpha)^2}\,. \tag{5.20}$$

From Eqs. (5.15) and (5.16) we get

$$\Big[T\boldsymbol{\varphi}_\alpha \cdot (T\boldsymbol{\varphi}_\alpha)^A\Big] = \begin{bmatrix} (1+\alpha)^2 & 0 & 0 \\ 0 & 1/(1+\alpha) & 0 \\ 0 & 0 & 1/(1+\alpha) \end{bmatrix} \tag{5.21}$$

$$\Big[(T\boldsymbol{\varphi}_\alpha) \cdot \mathbf{E}_\alpha \cdot (T\boldsymbol{\varphi}_\alpha)^A\Big] = \tfrac{1}{2} \begin{bmatrix} \alpha\,(2+\alpha)\,(1+\alpha)^2 & 0 & 0 \\ 0 & -\frac{\alpha}{(1+\alpha)^2} & 0 \\ 0 & 0 & -\frac{\alpha}{(1+\alpha)^2} \end{bmatrix} \tag{5.22}$$

From Eq. (5.15) we infer that the longitudinal elongation ratio is given by

$$\lambda(\alpha) := \ell(\alpha)/\ell_0 = 1+\alpha \tag{5.23}$$

5.3 Elastic stress-elongation response

At a testing placement, considered as a straightened trajectory, the uniaxial elastic law gives the mixed stressing response

$$\mathbf{g}_{\mathrm{MAT}}(\dot{\mathbf{K}}\mathbf{d}_{\mathbf{x}}\,,\mathbf{d}_{\mathbf{x}}) = E_m\,\mathbf{g}_{\mathrm{MAT}}(\mathbf{D}(\mathbf{v})\cdot\mathbf{d}_{\mathbf{x}}\,,\mathbf{d}_{\mathbf{x}})\,, \tag{5.24}$$

$$\dot{\boldsymbol{\sigma}}(\mathbf{g}_{\mathrm{MAT}}\cdot\mathbf{d}_{\mathbf{x}}\,,\mathbf{g}_{\mathrm{MAT}}\cdot\mathbf{d}_{\mathbf{x}}) = E_m\,\boldsymbol{\varepsilon}_{\mathbf{v}}(\mathbf{d}_{\mathbf{x}}\,,\mathbf{d}_{\mathbf{x}})\,. \tag{5.25}$$

which, by time invariance, is equivalent, in the current placement, to the stressing response

$$\dot{\boldsymbol{\sigma}}(\mathbf{g}_{\mathrm{ALT}}\cdot\mathbf{d}_{\mathbf{x}}\,,\mathbf{g}_{\mathrm{ALT}}\cdot\mathbf{d}_{\mathbf{x}}) = E_m\,\boldsymbol{\varepsilon}_{\mathbf{v}}(\mathbf{d}_{\mathbf{x}}\,,\mathbf{d}_{\mathbf{x}})\,. \tag{5.26}$$

To simplify the notations, we avoid to be explicitly involved with the projection in the longitudinal direction $\mathbf{d_x}$, in the formulae listed below.

The subsequent steps to be followed in the evaluation of the elastic response of the incompressible elastic material under uniform axial traction, according to Eq. (4.86), are listed hereafter.

$$
\begin{cases}
\dot{\boldsymbol{\sigma}} = E_m\, \mathbf{g}_{\mathrm{ALT}}^{-1} \cdot \boldsymbol{\varepsilon}_{\mathbf{v}} \cdot \mathbf{g}_{\mathrm{ALT}}^{-1} & \text{natural law} \\
\boldsymbol{\varphi}_\alpha{\downarrow}\dot{\boldsymbol{\sigma}} = E_m\, \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot \boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\varepsilon}_{\mathbf{v}} \cdot \mathbf{g}_{\mathrm{MAT}}^{-1} & \text{pull back} \\
\dot{\mathbf{S}}_\alpha = (\boldsymbol{\varphi}_\alpha{\downarrow}\dot{\boldsymbol{\sigma}}) \cdot \mathbf{g}_{\mathrm{MAT}} & \text{mixed alteration} \\
\dot{\mathbf{E}}_\alpha = \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot (\boldsymbol{\varphi}_\alpha{\downarrow}\boldsymbol{\varepsilon}_{\mathbf{v}}) & \\
\dot{\mathbf{S}}_\alpha = E_m\, \dot{\mathbf{E}}_\alpha & \text{ref. mixed law} \\
\mathbf{S}_\alpha = E_m\, \mathbf{E}_\alpha & \text{time integration} \\
\mathbf{S}_\alpha \cdot \mathbf{g}_{\mathrm{MAT}}^{-1} = E_m\, \mathbf{E}_\alpha \cdot \mathbf{g}_{\mathrm{MAT}}^{-1} & \text{end alteration} \\
\boldsymbol{\sigma} = \boldsymbol{\varphi}_\alpha{\uparrow}(\mathbf{S}_\alpha \cdot \mathbf{g}_{\mathrm{MAT}}^{-1}) & \text{pushed stress} \\
\boldsymbol{\sigma} = E_m\, \boldsymbol{\varphi}_\alpha{\uparrow}(\mathbf{E}_\alpha \cdot \mathbf{g}_{\mathrm{MAT}}^{-1}) & \\
\quad = E_m\, (T\boldsymbol{\varphi}_\alpha) \cdot \mathbf{E}_\alpha \cdot \mathbf{g}_{\mathrm{MAT}}^{-1} \cdot (T\boldsymbol{\varphi}_\alpha)^* & \\
\quad = E_m\, (T\boldsymbol{\varphi}_\alpha) \cdot \mathbf{E}_\alpha \cdot (T\boldsymbol{\varphi}_\alpha)^A \cdot \mathbf{g}_{\mathrm{MAT}}^{-1} & \\
\mathbf{K} = \boldsymbol{\sigma} \cdot \mathbf{g}_{\mathrm{MAT}} & \text{mixed alteration} \\
\mathbf{K} = E_m\, (T\boldsymbol{\varphi}_\alpha) \cdot \mathbf{E}_\alpha \cdot (T\boldsymbol{\varphi}_\alpha)^A & \text{result}
\end{cases}
\tag{5.27}
$$

Being

$$A_0/A(\alpha) = \ell(\alpha)/\ell_0 = 1 + \alpha \tag{5.28}$$

and $\mathbf{K}\mathbf{d_x} = \mathbf{K}_0\mathbf{d_x}\,(1+\alpha)$, the stress $K_0 := \mathbf{g}(\mathbf{K}_0\mathbf{d_x}\,,\mathbf{d_x})$ per unit initial transversal area is given by

$$\frac{K_0}{E_m} = \frac{\alpha\,(1+\alpha)\,(2+\alpha)}{2}\,. \tag{5.29}$$

The initial slope in Eq. (5.29) is equal to 1.

To compare the outcome of the analysis carried out with the new geometric theory, we observe that the plot of the nominal stress vs the elongation ratio is given in (Treloar, 1987, (5.3) p. 81)

$$K_0(\lambda) = \hat{G}\,(\lambda - \frac{1}{\lambda^2}) \tag{5.30}$$

with the modulus $\hat{G}$ introduced on the basis of a statistical mechanics argument.

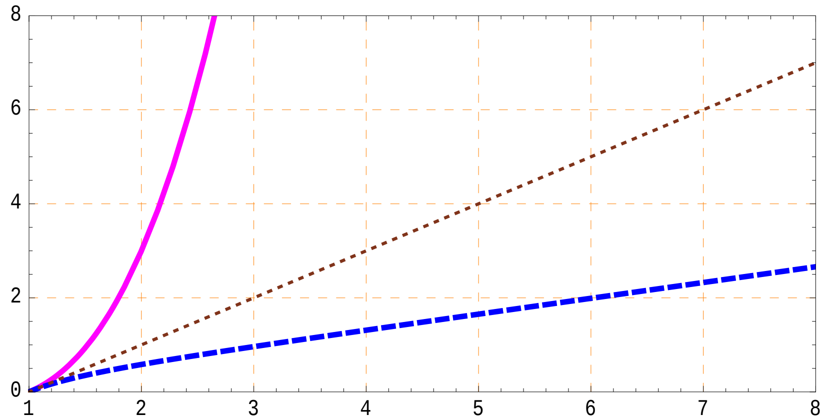


Figure 5.4: Comparison of laws K_0/E_m vs elongation ratio $\ell(\alpha)/\ell_0$

The initial slope of Eq. (5.30) normalised by E_m is equal to $3\,(\hat{G}/E_m)$ and hence coincides with the slope of Eq. (5.29) by setting $\hat{G} = E_m/3$.

TRELOAR's formula Eq. (5.30) may then be rewritten as

$$\frac{K_0(\lambda)}{E_m} = \frac{1}{3}\,(\lambda - \frac{1}{\lambda^2})\,. \tag{5.31}$$

The resulting curves, with the common slope given by the dotted line, are plotted in fig. 5.4 where

1. the solid line is the plot of the new Eq. (5.29),
2. the dashed line is is the plot of TRELOAR's formula Eq. (5.31).

The qualitative behavior of the elastic response represented by Eq. (5.29) is in agreement with the experimental results in (Treloar, 1987) reporting a progressive locking under increasing elongation. This characteristic behavior is not shared by Eq. (5.31).

It might be interesting to observe that, if the procedure in Eq. (5.27) is modified by performing the push forward directly on mixed alteration of the referential stress response, the final result, plotted in fig. 5.5, would be

$$\frac{K_0}{E_m} = \frac{\alpha\,(2+\alpha)}{2\,(1+\alpha)}\,, \tag{5.32}$$

and the progressive locking under increasing elongation would no more hold.

The theoretical treatment outlined on the basis of the new formulation of elasticity, may be compared to standard treatments of rubber-like materials by a statistical mechanics approach or on the base of the MOONEY-RIVLIN model, see (Rivlin and Saunders, 1951), and on its modifications (Ogden, 1972).

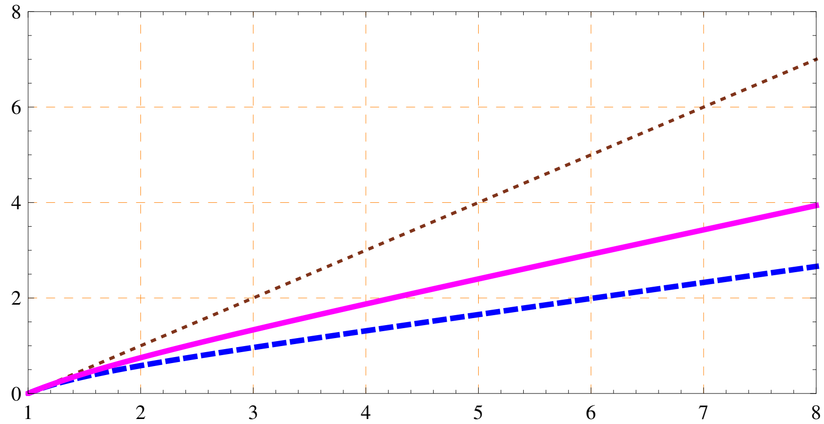


Figure 5.5: Modified comparison of K_0/E_m vs elongation ratio $\ell(\alpha)/\ell_0$

An account of the state of art is provided in the review (Boyce and Arruda, 2000) and in (Müller and Strehlow, 2004).

We conclude this case study with the following proviso. Material stress-strain relationship described by functions of strain invariants, depending on empirically evaluated constants, must rely upon reference placements entering into the constitutive theory, for the evaluation of the finite strain.

While the standard approach to finite elasticity leaves these reference placements substantially undetermined, the geometric theory illustrated in Ch. 4 makes explicit reference to a testing placement where experimental tests are carried out and measurements are performed according to a rate constitutive theory.

Significant comparisons of standard treatments with the present rate constitutive theory are thus problematic because elasticity models, in which the finite strain is taken as primary control variable, are in the viewfinder of the critical comments exposed in Sect. 3.4.

6

Epilogue

6.1 Standard stress-rate formulations

In literature, since the beginning of the twentieth century, a large number of proposals concerning the notion of stress-rate have been made (Zaremba, 1903; Jaumann, 1906, 1911; Oldroyd, 1950a,b; Thomas, 1955; Cotter and Rivlin, 1955; Green and Rivlin, 1955; Truesdell, 1955; Green, 1956).

An interpretation of these proposals in terms of convective derivatives of various tensor alterations along was provided in (Sedov, 1960) and (Marsden and Hughes, 1983, p.100).

The proposed time derivatives were evaluated by taking into account either the space motion or an artificial homomorphism connecting subsequent spaces tangent to rotated body placements, by dropping the stretching component of velocity gradient, as described by Eq. (6.8) expressing the JAUMANN (co-rotational) rate.

In all these treatments the time derivative of the stress tensor along the motion was split into the sum of partial time and spatial derivatives, a procedure which, while giving a deceptive feeling of computational convenience, is geometrically incorrect.

To motivate our assertion, we observe that the splitting of the space-time displacement $\boldsymbol{\varphi}_\alpha : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$, into a time-step at a fixed space position and a spatial displacement at a fixed time instant, cannot be invoked to get a corresponding decomposition of the time-derivative of material fields.

To illustrate this point, let us consider a formal procedure leading to the space-time splitting formulae, as exposed for instance in (Sedov, 1960; Marsden and Hughes, 1983).

1. The LIE derivative and the parallel derivative of the stress tensor along the space-time motion are split in time and space components

$$\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\sigma} = \mathcal{L}_{\mathbf{v}}\,\boldsymbol{\sigma} + \mathcal{L}_{\mathbf{Z}}\,\boldsymbol{\sigma}\,, \tag{6.1}$$

$$\nabla_{\mathbf{V}}\,\boldsymbol{\sigma} = \nabla_{\mathbf{v}}\,\boldsymbol{\sigma} + \nabla_{\mathbf{Z}}\,\boldsymbol{\sigma}\,. \tag{6.2}$$

2. The LIE derivative $\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\sigma}$ along a space motion, is expressed in terms of parallel (covariant) derivatives according to a LEVI-CIVITA (metric and torsion-free) connection in the space slice, see Eq. $(2.33)_4$, by means of the formula

$$\mathcal{L}_{\mathbf{v}}\,\boldsymbol{\sigma} = \nabla_{\mathbf{v}}\,\boldsymbol{\sigma} - \nabla\mathbf{v}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\nabla\mathbf{v})^* \,. \tag{6.3}$$

3. Taking into account the equality

$$\mathcal{L}_{\mathbf{Z}}\,\boldsymbol{\sigma} = \partial_{\alpha=0}\,\boldsymbol{\varphi}^{\mathcal{Z}}_{\alpha}\!\downarrow\!\boldsymbol{\sigma} = \partial_{\alpha=0}\,\boldsymbol{\varphi}^{\mathcal{Z}}_{\alpha}\!\Downarrow\!\boldsymbol{\sigma} = \nabla_{\mathbf{Z}}\,\boldsymbol{\sigma}\,, \tag{6.4}$$

valid in the EUCLID space-time, the formula Eq. fm: Lienablaplit may be written as

$$\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\sigma} = \nabla_{\mathbf{Z}}\,\boldsymbol{\sigma} + \nabla_{\mathbf{v}}\,\boldsymbol{\sigma} - \nabla\mathbf{v}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\nabla\mathbf{v})^* \,. \tag{6.5}$$

By the relation Eq. (3.39) between dual and adjoint operators

$$(\nabla\mathbf{v})^*\cdot\mathbf{g} = \mathbf{g}\cdot(\nabla\mathbf{v})^A \,, \tag{6.6}$$

and by the formula $\mathbf{K} := \boldsymbol{\sigma}\cdot\mathbf{g}_{\text{MAT}}$, the mixed counterpart of Eq. (6.5) is given by

$$\overset{\circ}{\mathbf{K}} = (\mathcal{L}_{\mathbf{V}}\,\boldsymbol{\sigma})\cdot\mathbf{g}_{\text{MAT}} = \dot{\mathbf{K}} - \nabla\mathbf{v}\cdot\mathbf{K} - \mathbf{K}\cdot(\nabla\mathbf{v})^A \,. \tag{6.7}$$

The term $\dot{\mathbf{K}} := \nabla_{\mathbf{V}}\,\mathbf{K}$ is usually called the *material time derivative.*[1] If the motion is an isometry, then $\text{sym}\nabla\mathbf{v} = \mathbf{0}$ and hence $\nabla\mathbf{v} = \mathbf{W}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^A)$. Consequently, being $\mathbf{W}(\mathbf{v})^A = -\mathbf{W}(\mathbf{v})$, Eq. (6.7) yields the JAUMANN (co-rotational) rate, defined by

$$\overset{\circ}{\mathbf{K}} := \dot{\mathbf{K}} - \mathbf{W}(\mathbf{v})\cdot\mathbf{K} + \mathbf{K}\cdot\mathbf{W}(\mathbf{v})\,. \tag{6.8}$$

The splitting in Eq. (6.1) and Eq. (6.2) and the ensuing formulae Eq. (6.5) or Eqs. (6.7), (6.8), although spread throughout the literature on nonlinear continuum mechanics, are affected by the following flaws.

1. The additive decompositions of the LIE derivative Eq. (6.1) and of the parallel derivative Eq. (6.2), along the space-time motion, have no general validity being subject to the special assumption that spatial and temporal components $\mathbf{v}, \mathbf{Z}$ of the velocity $\mathbf{V}$ are *not transversal* to the trajectory, see fig. 3.2. Consequently Eq. (6.3) and Eq. (6.4) cannot be applied to get Eq. (6.5).

[1] The nomenclature is improper because the parallel derivative will, as a rule, yield a spatial tensor.

2. The parallel (covariant) derivative Eq. (6.2) along the space-time velocity $\mathbf{V}$ requires the evaluation of the backward parallel transport of the stress field along the material particle but the resulting tensor field will not in general still belong to the material bundle but rather to the spatial bundle. As a result the parallel derivative cannot appear in constitutive relations Eq. (1.4). This comment applies to the formulation of hypo-elasticity given in (Truesdell and Noll, 1965).

The comments above point out again the peculiar position of Euler formula expressing the mixed material stretching in terms of parallel derivatives, a formula whose general validity relies on the fact that the material metric tensor field is the pull-back to the material bundle of the space metric tensor field which is defined on the whole event manifold. No other material tensor in Continuum Mechanics has this property and hence no general formula in terms of parallel derivatives can be found to express its time derivative along the motion.

These basic difficulties with the representation Eq. (6.5) are however of no concern because theoretical and numerical computations are conveniently performed in full generality by pushing the stress field $\boldsymbol{\sigma}$ and the stressing field $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{V}}\boldsymbol{\sigma}$ to a local straightened trajectory, where Lie derivatives along the motion reduce to partial time derivatives, as shown in Sect. 3.5.

6.2 Concluding remarks

The preliminary discussion in Sect. 3.4 puts into evidence that, a geometrically biased analysis of standard issues in constitutive theory, leads to conclude that a modification of basic ingredients is compelling.

Two main intertwined points are decisive.

The first point is mainly of epistemological character, requiring that reference states should not be invoked in the formulation of constitutive relations. Tracks of previous actions on the material are to be recorded by suitable internal variables and related evolution laws.

Despite the unavoidable indeterminacy intrinsic in constructions of constitutive equations involving reference states, most formulations in literature, following the wake of the treatment conceived in (Truesdell and Noll, 1965), are based on this untenable notion. However, motivations against the adoption of referential finite plastic strains in constitutive relations have been clearly expressed in literature, see e.g. (Rubin, 2001), and the adoption of reference states was eventually critically commented also by (Noll, 2004).

To envisage a consistent remedy to these problematics, a drastic geometric revision of fundamental concepts in continuum mechanics is an unavoidable task, undertaken in a systematic way by the first author and his associates, in recent contributions propaedeutic to the present treatment (Romano, Barretta, 2011, 2013a,b; Romano, Barretta, Diaco, 2014a,b,c).

A significant and clarifying innovation concerns the distinction between newly defined *material* and *space* tensor fields and the statements of the pertinent rules for their comparison along the space-time motion.

As a matter of fact, the denomination of *material* and *space* fields is still commonly adopted to denote tensor fields in a reference placement and on the actual placement, in one-to-one correspondence by means of the diffeomorphic transformation between placements.

But for these fields the attributes *current* and *referential* are instead appropriate. Both current and referential fields represent then the same physical entity that we classify as a *material* field.

The new definition makes a clear physical classification and provides the geometric definition of two quite distinct kinds of fields, which cannot be transformed one into the another.

Material tensors act on material vectors which are tangent to body placements. Space tensors act on space vectors which are based on body placements but tangent to space slices.

The former must be compared, in a natural manner, by pull-back along the motion, while the latter, which do not live in body placements, can only be compared, in a not natural manner, according to an arbitrarily chosen parallel transport in space, along the motion.

From the physical point of view, material fields pertain to the description of material behavior, such as stretching, stress, mass and temperature, and are deputed to enter in constitutive relations, while spatial fields describe the kinetics of a material body, such as velocity and acceleration.

A related point deals with the longly debated question about the proper definition of stress rate.

The geometric theory provides the answer in a natural and univocal way since the values attained by material fields in a time lapse along the motion must be compared by pull-back through the corresponding displacement map. It then follows that time-derivatives of material fields must be performed as LIE-derivatives along the motion.

This conception leads to the conclusion that mechanical constitutive laws must be formulated in rate form as relations involving current values of material state variables and their time rates along the motion, as control parameters. The output is the geometrical stretching, build up by the sum of various contributions, providing interpretations of distinct physical behaviors of materials of interest in mechanics.

In this framework, results of the present contribution may be summarised as follows.

1. The geometric revision of elasticity theory shows that, contrary to still repeatedly negative claims in literature, rate elasticity is a well founded constitutive model. Even better, elasticity *must* indeed be formulated as a rate model.

2. From the mathematical point of view, integrability of the rate constitutive relation, required by thermodynamical motivations, can be imposed by simple and operative symmetry conditions on the constitutive operator.

3. In the new theory, time invariance of elastic behavior is given a new geometric definition, consistent with a large displacement analysis, in which invariance along the motion means variance according to push along the motion.

4. The theory leads to the notion of *elastic states*, which are related to stress states by the integrated constitutive law, according to a one-to-one correspondence. Elastic states take in the new elastic constitutive law the position previously improperly occupied by *elastic strains*. Troubles concerning the role played by reference placements in constitutive relations are thus bypassed in a natural way.

5. Stress states are the primary variables and elastic energy is therefore expressed in terms of stress fields. The usual denomination of *elastic strain energy* ought accordingly to be changed into *elastic stress energy* and the complementary functional should be named *elastic state energy*.

6. It is clarified that, due to lack of integrability of the rate constitutive law, anelastic constitutive responses cannot be formulated as time rates of change of state variables along the motion. Thus, for instance, no accumulated plastic strain can be associated with the evolution of plastic stretching.

7. A careful analysis of contravariant-covariant tensors describing stress-stretching pairs in duality, and of their mixed alterations, dictates the rules for the proper formulation of a rate elastic law. This issue of mixed alterations is of the utmost importance in applications, since mixed tensors are adopted in engineering treatments to take advantage of the powerful spectral representation of symmetric linear operators. However, in performing time derivatives along the motion, invariance of the altering material metric must be considered.

Innovations adduced by the theory are outlined in the synoptic table.

The new theory is intended to provide a consistent framework for the formulation of elastic laws in the field of large displacements and is expected to be especially effective in applications involving thin elastic bodies, such as wires and membranes, or very soft matter, as in biomechanics.

Synoptic table			
new		***previous***	
stressing	$\dot{\boldsymbol{\sigma}} = \mathcal{L}_{\mathbf{V}}\, \boldsymbol{\sigma}$	stress rates	no univocal definition
elastic stretching	$\boldsymbol{\varepsilon}_{\mathrm{EL}} := \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}$		
mixed stretching	$\mathbf{D}_{\mathrm{EL}} := \mathbf{H}_{\mathrm{MIX}}(\mathbf{K}) \cdot \dot{\mathbf{K}}$	hypo elastic law	$\mathring{\mathbf{T}} = \mathbf{H}(\mathbf{T}) \cdot \mathbf{D}$
elastic stress potential	$\Xi(\boldsymbol{\sigma})$	referential elastic potential	$W(\mathbf{F})$
elastic state	$\mathbf{es} := \boldsymbol{\Psi}(\boldsymbol{\sigma}) = d_F \Xi(\boldsymbol{\sigma})$	referential elastic response	$\mathbf{P} = dW(\mathbf{F})$
rate elastic state	$\boldsymbol{\varepsilon}_{\mathrm{EL}} = \mathcal{L}_{\mathbf{V}} \mathbf{es}$	mixed stretching	$\mathbf{D}$
elastic state potential	$\Xi^*(\mathbf{es})$		
accumulated elastic strain	$\mathbf{es}_{\mathrm{REF}}(\alpha) := \int_0^{\alpha} (\boldsymbol{\varphi}_\tau \!\downarrow\! \boldsymbol{\varepsilon}_{\mathrm{EL}})\, d\tau = \boldsymbol{\varphi}_\alpha \!\downarrow\! \mathbf{es} - \mathbf{es}$	finite elastic strain	special reference and intermediate placements required

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