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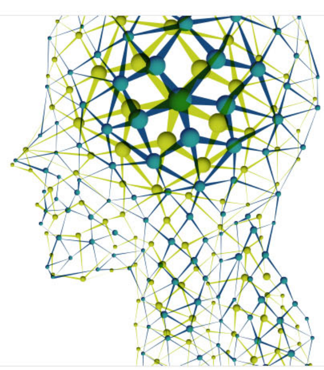

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On continuum dynamics

Giovanni Romano,[a)] Raffaele Barretta, and Marina Diaco
Department of Structural Engineering, University of Naples Federico II, via Claudio 21, 80125 Naples, Italy

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The theory of continuous dynamical systems is developed with an intrinsic geometric approach based on the action principle formulated in the velocity-time manifold. By endowing the finite dimensional Riemannian ambient manifold with a connection, an induced connection is naturally defined in the infinite dimensional configuration manifold of maps. The motion is shown to be governed, in the Banach configuration manifold, by a generalized Lagrange law and, in the ambient manifold, by a generalized Euler law which is independent of the Banach topology of the configuration manifold. Extended versions of Euler–Poincaré law, Euler classical laws and d'Alembert law are also derived as special cases. Stress fields in the body are introduced as Lagrange's multipliers of the rigidity constraint on virtual velocities, dual to the Lie derivative of the metric. No special assumptions are made so that any constitutive behaviors can be modeled. © *2009 American Institute of Physics.* [doi:10.1063/1.3215979]

I. INTRODUCTION

In recent times the interest for geometric formulations of dynamics has considerably grown up in the literature but, despite of this, most treatments still refer to Newtonian dynamics of a finite system of point-mass particles or to rigid body dynamics and are expressed in terms of coordinates or recourse is made to local coordinates to prove the main results. As a matter of fact, a satisfactory physical and intrinsic geometrical picture of the theory for continuous dynamical systems is still lacking. We contribute here the elements for a foundation of continuum dynamics by assuming, as starting point, a geometric action principle in the velocity-time state space. The abstract formulation is formally similar to that pertaining to discrete systems but, to cover dynamics of continuous systems, the configuration manifold is assumed to be an infinite dimensional manifold of maps. Dynamical systems in continuum mechanics enjoy a peculiar geometric feature in which three differentiable structures are the playmates: the *ambient* finite dimensional manifold without boundary [usually the flat Euclidean three dimensional (3D) space] in which motions take place, a finite dimensional manifold with boundary (called the *body*) which provides the geometrical picture of the body, and the infinite dimensional *configuration* manifold of maps whose elements are embeddings of the body into the ambient manifold. The special geometric feature of the configuration manifold of a continuous dynamical system permits to define a connection induced by a given connection in the finite dimensional ambient manifold. The procedure is conceived in terms of parallel transport. It consists in performing the parallel transport of vector fields, from one placement to another one, along the sheaf of curves tracked by the body particles in the ambient manifold, in correspondence to a given curve in the configuration manifold. The induced connection leads to a dynamical theory of continuous systems whose governing rules are independent of the Banach topology of the configuration manifold. This key property is in agreement with the physical expectation that continuum dynamics must depend only on the geometric structure of the ambient manifold.

[a)]Electronic mail: romano@unina.it.

The plan of the paper is the following. Preliminarily we collect some basic definitions and notions of differential geometry and calculus on manifolds, mainly to clarify the notations for subsequent reference. The reader is anyway assumed to be familiar with the fundamentals of the matter. A comprehensive exposition, to our purposes, may be found, e.g., in Refs. 1–3. Next we introduce, in an abstract setting, the definition of the geometric action principle and of the related Euler conditions with a formulation which includes discontinuities along the trajectory and is consistent with the paradigm of the calculus of variations discussed in Refs. 3–6. According to the paradigm, the rate of change of the action integral, when the path is dragged by a virtual flow, is equal to the boundary integral of the outward flux of the virtual velocity field, plus a linear term given by the integral along the trajectory of the virtual power of regular and singular force systems acting on the body. This definition gives up with the fixed end-point assumption which in previous treatments has lead to identify the extremality property of the action functional with a stationarity or minimum property.[2,7,8] The corresponding nongeometric action principle, i.e., dependent on the time parametrization of the trajectory in the configuration manifold, provides an extended version of the classical Hamilton's principle. The related Euler condition of extremality provides the general form of the law of motion in the configuration manifold in the wake of the guidelines and ideas exposed in Refs. 3–6. Then we show that the assignment of a connection in a configuration manifold provides a generalized statement of Lagrange's law of motion, as introduced in Ref. 5. If the connection in the configuration manifold is induced by a given connection in the ambient manifold, it is possible to translate Lagrange's law from the configuration to the ambient manifold and this leads to the formulation of the law of dynamics which generalizes to continuous systems and to Riemannian ambient manifolds the classical one due to Euler.[9] This translation is a crucial point which seems to have been dealt with here in full generality for the first time. The proof of this generalized Euler's law is not trivial. It is based on the result of Lemma VIII.5 concerning the torsion of the induced connection in the configuration manifold. To get the generalized Euler's law the ansatz of mass conservation along the variations is also needed. This assumption is tacitly or implicitly made in other treatments and in the analytical dynamics context. Conservation of mass along the dynamical trajectory leads to the generalization of d'Alembert's law. The expression of the law of dynamics in terms of a connection in the ambient manifold is of the utmost importance since it permits to infer that the law of dynamics is, in fact, expressed by a variational condition on a bounded linear functional in the Hilbert space of conforming virtual velocity fields at the current placement. This result has a twofold basic implication. On one hand, it opens the way to the introduction of the Cauchy stress field as Lagrange multiplier of the rigidity constraint on conforming virtual velocity fields, thus providing the variational formulation of the law of dynamics suitable to the analysis of deformable bodies. On the other hand, it reveals that no distinctions has to be made between holonomic and nonholonomic linear constraints, in formulating the law of dynamics. The issue is discussed in detail in Ref. 3 where also a critical review of relevant contributions in literature is made.[10–19] Under special assumptions, particular expressions of the law of motion are provided. In fact, assuming a distant parallel transport in the ambient manifold, the extension to continuous systems of the Poincaré law of dynamics, which we call the Euler-Poincaré law,[5,11] is given. By considering Riemannian ambient manifolds with the Levi–Civita connection and dynamical systems governed by standard Lagrangians, an extended version of the classical Euler's law of motion is derived and by conservation of mass, an extended version of the classical d'Alembert's law is recovered. An account of the formulation of dynamics of hyperelastic materials is provided as a last issue.

II. PRELIMINARIES

In the sequel a superscript star * denotes duality and the crochet $\langle , \rangle$ is the duality pairing. The dot $\cdot$ indicates a linear dependence on the forthcoming argument. Given a set X and a Banach space Y, the Banach space of bounded linear maps from X to Y is denoted by $BL(X;Y)$. A detailed exposition of calculus on manifolds may be found, e.g., in Refs. 1 and 2. The following summary of concepts and definitions of differential geometry is deduced from the treatment in Ref. 3. Let $\mathbb{M}$

be a differentiable manifold modeled on a Banach space and let $(\mathbb{TM}, \boldsymbol{\tau}_{\mathbb{M}}, \mathbb{M})$ and $(\mathbb{T}^*\mathbb{M}, \boldsymbol{\tau}^*_{\mathbb{M}}, \mathbb{M})$ be the dual pair of tangent and cotangent bundles over $\mathbb{M}$ with fibration maps $\boldsymbol{\tau}_{\mathbb{M}} \in C^1(\mathbb{TM};\mathbb{M})$ and $\boldsymbol{\tau}^*_{\mathbb{M}} \in C^1(\mathbb{T}^*\mathbb{M};\mathbb{M})$ surjective submersions. The *tangent map* $T\boldsymbol{\varphi} \in C^0(\mathbb{TM};\mathbb{TN})$ to a morphism $\boldsymbol{\varphi} \in C^1(\mathbb{M};\mathbb{N})$ between manifolds is the linear vector bundle homomorphism, i.e., the fiber preserving and fiber-linear map pointwise defined by the differential,

$$(T\boldsymbol{\varphi} \circ \mathbf{v})(\mathbf{x}) = T_{\mathbf{x}}\boldsymbol{\varphi} \cdot \mathbf{v}(\mathbf{x}) \in \mathbb{T}_{\boldsymbol{\varphi}(\mathbf{x})}\mathbb{N}, \quad \forall\, \mathbf{v}(\mathbf{x}) \in \mathbb{T}_{\mathbf{x}}\mathbb{M}.$$

A vector field $\mathbf{v}_{\mathbb{N}} \in C^1(\mathbb{N};\mathbb{TN})$, with $\boldsymbol{\tau}_{\mathbb{N}}\circ\mathbf{v}_{\mathbb{N}}=\mathbf{id}_{\mathbb{N}}$, is related to the vector field $\mathbf{v}_{\mathbb{M}} \in C^1(\mathbb{M};\mathbb{TM})$, with $\boldsymbol{\tau}_{\mathbb{M}}\circ\mathbf{v}_{\mathbb{M}}=\mathbf{id}_{\mathbb{M}}$, by a morphism $\boldsymbol{\varphi} \in C^1(\mathbb{M};\mathbb{N})$ (briefly $\boldsymbol{\varphi}$-related) if $T\boldsymbol{\varphi}\circ\mathbf{v}_{\mathbb{M}}=\mathbf{v}_{\mathbb{N}}\circ\boldsymbol{\varphi}$. If $\boldsymbol{\varphi} \in C^1(\mathbb{M};\mathbb{N})$ is a diffeomorphism (invertible and C^1, with the inverse) we have the push forward $\mathbf{v}_{\mathbb{N}}=\boldsymbol{\varphi}\uparrow\mathbf{v}_{\mathbb{M}}$, and the pull back $\mathbf{v}_{\mathbb{M}}=\boldsymbol{\varphi}\downarrow\mathbf{v}_{\mathbb{N}}$. The push forward (pull back) of a covector field is defined by evaluating the covector field on the pull back (push forward) of the vector field arguments. The push and pull of arbitrary tensor fields are defined in an analogous way.

The flow generated by a vector field $\mathbf{v} \in C^1(\mathbb{M};\mathbb{TM})$ is denoted by $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in C^1(\mathbb{M};\mathbb{M})$. In a fiber bundle $(\mathbb{E}, \boldsymbol{\pi}, \mathbb{M})$, with fibration map $\boldsymbol{\pi} \in C^1(\mathbb{E};\mathbb{M})$, for any given section $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$, the *natural derivative* along a vector $\mathbf{v} \in \mathbb{TM}$ is the vector field $T_{\mathbf{v}} \in C^1(\mathbf{s}(\mathbb{M});\mathbb{TE})$ defined by $T_{\mathbf{v}}\mathbf{s}=T\mathbf{s}\circ\mathbf{v}$. It fulfils the property $T\boldsymbol{\pi}\circ T_{\mathbf{v}}=\mathbf{v}\circ\pi$ on $\mathbf{s}(\mathbb{M})$. The *vertical subbundle* $(\mathbb{VE}, \boldsymbol{\tau}_{\mathbb{E}}, \mathbb{E})$ of the tangent bundle $(\mathbb{TE}, \boldsymbol{\tau}_{\mathbb{E}}, \mathbb{E})$ has linear fibers made of the kernels $\mathbf{ker}(T_{\mathbf{e}}\boldsymbol{\pi})$ of the tangent maps $T_{\mathbf{e}}\boldsymbol{\pi} \in BL(\mathbb{T}_{\mathbf{e}}\mathbb{E};\mathbb{T}_{\boldsymbol{\pi}(\mathbf{e})}\mathbb{M})$. The horizontal part of a vector $\mathbf{X}_{\mathbf{e}} \in \mathbb{T}_{\mathbf{e}}\mathbb{E}$ is the velocity of its base point in $\mathbb{M}$, i.e., $T_{\mathbf{e}}\boldsymbol{\pi}\cdot\mathbf{X}_{\mathbf{e}} \in \mathbb{T}_{\boldsymbol{\pi}(\mathbf{e})}\mathbb{M}$. Vertical vectors $\mathbf{V}_{\mathbf{e}} \in \mathbb{T}_{\mathbf{e}}\mathbb{E}$ are those with a vanishing horizontal part or equivalently those tangent to a fiber of the bundle. A 1-form $\boldsymbol{\omega} \in \mathbb{T}^*_{\mathbf{e}}\mathbb{E}$ is said to be horizontal if it vanishes on the vertical vectors $\mathbf{V}_{\mathbf{e}} \in \mathbb{T}_{\mathbf{e}}\mathbb{E}$. A *connection* on a fiber bundle is a linear vector bundle homomorphism $P_V \in C^1(\mathbb{TE};\mathbb{TE})$, which is fiberwise a projector on the vertical subbundle, i.e., $P_V(\mathbf{e}) \in BL(\mathbb{T}_{\mathbf{e}}\mathbb{E};\mathbb{T}_{\mathbf{e}}\mathbb{E})$ with $P_V(\mathbf{e})\circ P_V(\mathbf{e})=P_V(\mathbf{e})$ and $\mathbf{im}(P_V(\mathbf{e}))=\mathbf{ker}(T_{\mathbf{e}}\boldsymbol{\pi})$. The complementary projector $P_H=I-P_V$ defines the *horizontal subbundle* $\mathbb{HE}\subset\mathbb{TE}$. The *horizontal lift* $\mathbf{H}_{\mathbf{v}}\mathbf{s} \in C^1(\mathbb{M};\mathbb{HE})$ and the *covariant derivative* $\bar{\nabla}_{\mathbf{v}}\mathbf{s} \in C^1(\mathbb{M};\mathbb{VE})$ of a section $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$ along a vector field $\mathbf{v} \in C^1(\mathbb{M};\mathbb{TM})$ are, respectively, the horizontal and vertical components of the natural derivative,

$$\mathbf{H}_{\mathbf{v}}\mathbf{s} := P_H \circ T_{\mathbf{v}}\mathbf{s}, \quad \bar{\nabla}_{\mathbf{v}}\mathbf{s} := P_V \circ T_{\mathbf{v}}\mathbf{s},$$

so that $T_{\mathbf{v}}=\mathbf{H}_{\mathbf{v}}+\bar{\nabla}_{\mathbf{v}} \in C^1(\mathbf{s}(\mathbb{M});\mathbb{TE})$ with $\mathbf{H}_{\mathbf{v}}=P_H\circ T_{\mathbf{v}}$ and $\bar{\nabla}_{\mathbf{v}}=P_V\circ T_{\mathbf{v}}$. Then $T\boldsymbol{\pi}\circ\mathbf{H}_{\mathbf{v}}=\mathbf{v}\circ\boldsymbol{\pi}$ on $\mathbf{s}(\mathbb{M})$. The horizontal lift is a linear homomorphism from the tangent bundle $\boldsymbol{\tau}_{\mathbb{M}} \in C^1(\mathbb{TM};\mathbb{M})$ to $\boldsymbol{\tau}_{\mathbb{E}} \in C^1(\mathbb{HE};\mathbb{E})$ which is fiberwise invertible and tensorial in $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$. A connection in a vector bundle $\boldsymbol{\pi} \in C^1(\mathbb{E};\mathbb{M})$ is linear if the horizontal lift depends linearly on the point values of $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$. The *parallel transport* $\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Uparrow\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$ of a section $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$ along the flow $\mathbf{Fl}^{\mathbf{v}}_{\lambda} \in C^1(\mathbb{M};\mathbb{M})$ is defined by

$$\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Uparrow\mathbf{s} := \mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda}\circ\mathbf{s}.$$

We put $\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Downarrow := \mathbf{Fl}^{\mathbf{v}}_{-\lambda}\Uparrow$. The horizontal lift and the covariant derivative are expressed in terms of parallel transport by

$$\mathbf{H}_{\mathbf{v}}\mathbf{s} = \partial_{\lambda=0}\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{\lambda}\circ\mathbf{s} = \partial_{\lambda=0}\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Uparrow\mathbf{s},$$

$$\bar{\nabla}_{\mathbf{v}}\mathbf{s} = \partial_{\lambda=0}\mathbf{Fl}^{\mathbf{H}_{\mathbf{v}}}_{-\lambda}\circ\mathbf{s}\circ\mathbf{Fl}^{\mathbf{v}}_{\lambda} = \partial_{\lambda=0}\mathbf{Fl}^{\mathbf{v}}_{\lambda}\Downarrow(\mathbf{s}\circ\mathbf{Fl}^{\mathbf{v}}_{\lambda}).$$

The product bundle or Whitney product of two vector bundles $(\mathbb{E}, \boldsymbol{\pi}_{\mathbb{E}}, \mathbb{M})$ and $(\mathbb{H}, \boldsymbol{\pi}_{\mathbb{H}}, \mathbb{M})$, over the same base manifold $\mathbb{M}$, is the vector bundle defined by[20]

$$\mathbb{E}\times_{\mathbb{M}}\mathbb{H} := \{(\mathbf{e},\mathbf{h}) \in \mathbb{E}\times\mathbb{H} \,|\, \boldsymbol{\pi}_{\mathbb{E}}(\mathbf{e}) = \boldsymbol{\pi}_{\mathbb{H}}(\mathbf{h})\}.$$

The vector bundle isomorphism $\mathbf{vl}_{\mathbb{TM}}(\mathbf{u}_{\mathbf{x}},\mathbf{v}_{\mathbf{x}}) \in C^1(\mathbb{TM}\times_{\mathbb{M}}\mathbb{TM};\mathbb{VTM})$ defined by

$$\mathbf{vl}_{\mathbb{TM}}(\mathbf{u_x},\mathbf{v_x}) := \partial_{\lambda=0}(\mathbf{u_x}+\lambda\mathbf{v_x}),$$

is called the *vertical lift* of $\mathbf{v_x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$ at $\mathbf{u_x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$. The relation $\mathbf{vl}_{\mathbb{TM}}(\mathbf{u})\cdot\nabla_\mathbf{v}\mathbf{u} := \bar{\bar{\nabla}}_\mathbf{v}\mathbf{u}$ provides the definition of the covariant derivative $\nabla_\mathbf{v}\mathbf{u} \in \mathbb{TM}$ as a tangent vector. The fiber derivative $d_\mathrm{F}L \in C^1(\mathbb{TM};\mathbb{T}^*\mathbb{M})$ associated with a Lagrangian functional $L \in C^2(\mathbb{TM};\mathfrak{R})$ is defined by

$$d_\mathrm{F}L(\mathbf{u_x})\cdot\mathbf{v_x} := \partial_{\lambda=0}L(\mathbf{u_x}+\lambda\mathbf{v_x}) = \langle TL(\mathbf{u_x}),\mathbf{vl}_{\mathbb{TM}}(\mathbf{u_x},\mathbf{v_x})\rangle$$

for all $\mathbf{u_x}$, $\mathbf{v_x} \in \mathbb{T}_\mathbf{x}\mathbb{M}$. The correspondence between tangent and cotangent bundles induced by the fiber derivative is the Legendre transform. The Lie derivative of a section $\mathbf{s} \in C^1(\mathbb{M};\mathbb{E})$ of a fiber bundle $(\mathbb{E},\boldsymbol{\pi},\mathbb{M})$ along a vector field $\mathbf{v} \in C^1(\mathbb{M};\mathbb{TM})$ is defined by $\mathcal{L}_\mathbf{v}\mathbf{s} = \partial_{\lambda=0}\mathbf{Fl}^\mathbf{v}_\lambda\!\downarrow\mathbf{s} = \partial_{\lambda=0}T\mathbf{Fl}^\mathbf{v}_{-\lambda}\circ\mathbf{s}\circ\mathbf{Fl}^\mathbf{v}_\lambda$. In the tangent bundle, the equality between the Lie derivative $\mathcal{L}_\mathbf{v}\mathbf{u} = \partial_{\lambda=0}\mathbf{Fl}^\mathbf{v}_\lambda\!\downarrow\mathbf{u}$ for vector fields $\mathbf{u}$, $\mathbf{v} \in C^1(\mathbb{M};\mathbb{TM})$ and the Lie bracket, defined by $d_{[\mathbf{v},\mathbf{u}]}f = d_\mathbf{v}d_\mathbf{u}f - d_\mathbf{u}d_\mathbf{v}f$ for any $f \in C^2(\mathbb{M};\mathfrak{R})$, provides the antisymmetry property $\mathcal{L}_\mathbf{v}\mathbf{u} = [\mathbf{v},\mathbf{u}] = -[\mathbf{u},\mathbf{v}] = -\mathcal{L}_\mathbf{u}\mathbf{v}$. The torsion of a linear connection in a tangent bundle $(\mathbb{TM},\boldsymbol{\tau}_\mathbb{M},\mathbb{M})$ is the field of tangent-valued two-forms defined by

$$\mathrm{tors}(\mathbf{u_x},\mathbf{v_x}) := (\nabla_{\mathbf{u_x}}\mathbf{v} - \nabla_{\mathbf{v_x}}\mathbf{u} - [\mathbf{u},\mathbf{v}])(\mathbf{x}) \in \mathbb{T}_\mathbf{x}\mathbb{M}.$$

Main tools of calculus on manifolds are the domain displacement formula and Poincaré–Stokes' formula,

$$\int_{\mathbf{Fl}^\mathbf{v}_\lambda(\Sigma)}\boldsymbol{\omega}^k = \int_\Sigma \mathbf{Fl}^\mathbf{v}_\lambda\!\downarrow\boldsymbol{\omega}^k, \quad \int_\Sigma d\boldsymbol{\omega}^{k-1} = \oint_{\partial\Sigma}\boldsymbol{\omega}^{k-1},$$

providing a defining property of the *exterior derivative* d, the integral *extrusion formula*,[3]

$$\partial_{\lambda=0}\int_{\mathbf{Fl}^\mathbf{v}_\lambda(\Sigma)}\boldsymbol{\omega}^k = \int_\Sigma (d\boldsymbol{\omega}^k)\cdot\mathbf{v} + \oint_{\partial\Sigma}\boldsymbol{\omega}^k\cdot\mathbf{v},$$

and the related *magic formula* of Henri Cartan,[2,3,21–23]

$$\mathcal{L}_\mathbf{v}\boldsymbol{\omega}^k := \partial_{\lambda=0}\mathbf{Fl}^\mathbf{v}_\lambda\!\downarrow\boldsymbol{\omega}^k = (d\boldsymbol{\omega}^k)\cdot\mathbf{v} + d(\boldsymbol{\omega}^k\cdot\mathbf{v}),$$

where Σ, is a k-chain with boundary $\partial\Sigma$, $\boldsymbol{\omega}^k$, is a k-form, and $\boldsymbol{\omega}^k\cdot\mathbf{v}$ is the contraction performed by inserting $\mathbf{v}$, as first argument of $\boldsymbol{\omega}^k$. The last formula may be readily inverted to get Palais formula for the exterior derivative.[24] Indeed, by Leibniz rule for the Lie derivative, for any 1-form $\boldsymbol{\omega}^1$, and vector fields $\mathbf{v}$, $\mathbf{w} \in C^1(\mathbb{M};\mathbb{TM})$, we have

$$d\boldsymbol{\omega}^1\cdot\mathbf{v}\cdot\mathbf{w} = (\mathcal{L}_\mathbf{v}\boldsymbol{\omega}^1)\cdot\mathbf{w} - d(\boldsymbol{\omega}^1\cdot\mathbf{v})\cdot\mathbf{w} = d_\mathbf{v}(\boldsymbol{\omega}^1\cdot\mathbf{w}) - \boldsymbol{\omega}^1\cdot[\mathbf{v},\mathbf{w}] - d_\mathbf{w}(\boldsymbol{\omega}^1\cdot\mathbf{v}).$$

The exterior derivative of a differential 1-form is a two-form which is well defined by Palais formula because the expression at the right hand side fulfills the tensoriality criterion.[3,22,25] The value of the exterior derivative at a point is then independent of the extension of the argument vectors to vector fields, extension needed to compute the involved directional derivatives and the Lie derivative.

III. ABSTRACT ACTION PRINCIPLE AND EULER CONDITIONS

We assume that a status of the system is described by a point of a manifold $\mathbb{M}$ the *state space*. In dynamics, the state space is the velocity-time manifold. To allow for the velocity to undergo abrupt changes at singular time instants, a reasonably general treatment requires to consider trajectories which are piecewise smooth on $\mathbb{M}$, possibly with a finite number of jump discontinuities. The evolution of the system is governed by a variational condition on the integral, along a trajectory $\gamma: I \to \mathbb{M}$, of a piecewise smooth differential *one-form* $\boldsymbol{\omega}^1 : \mathbb{M} \to \mathbb{T}^*\mathbb{M}$. The geometric trajectory is the 1-chain $\boldsymbol{\gamma} := \gamma(I)$ and the set of singular points of $\boldsymbol{\gamma}$, where discontinuities of the trajectory occur, is the 0-chain $\mathrm{Sing}(\boldsymbol{\gamma})$. The vector bundle $\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{M}$ is the restriction of the tangent bundle $\mathbb{TM}$ to $\boldsymbol{\gamma}$.

Definition III.1: (Action integral) The action, associated with a geometrical path $\boldsymbol{\gamma}$ *in the state space* $\mathbb{M}$ *and with a differential one-form* $\boldsymbol{\omega}^1$*, on* $\mathbb{M}$*, is the integral*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega}^1 .$$

In formulating the action principle, we consider a dynamical system subject to linear constraints, the extension to affine constraints being straightforward. Linear constraints acting on the trajectory speed and on virtual velocities are described by vector subbundles, respectively $\mathrm{Trial}_{\boldsymbol{\gamma}}$ and $\mathrm{Test}_{\boldsymbol{\gamma}}$ of the vector bundle $\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{M}$.

Source terms are represented by a two-form $\boldsymbol{\alpha}^2$, on $\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{M}$, the *regular source form*, which provides an abstract description of a possibly nonpotential system of forces acting along the trajectory. The source form $\boldsymbol{\alpha}^2$ is *potential* if it is defined on a neighborhood $U(\boldsymbol{\gamma})\subset\mathbb{M}$ of the path and there is exact. This amounts to assume that $\boldsymbol{\alpha}^2=d\boldsymbol{\beta}^1$, where d is the exterior differentiation and $\boldsymbol{\beta}^1\in \mathrm{C}^1(U(\boldsymbol{\gamma});\mathbb{T}^*\mathbb{M})$ is a differential one-form. The *impulsive source form* $\boldsymbol{\alpha}^1$ is a differential one-form on $\mathbb{T}_{\mathrm{Sing}(\boldsymbol{\gamma})}\mathbb{M}$, acting at singular points of the trajectory.

The statement of an action principle, sufficiently general for application to dynamics, requires a suitable definition of the *virtual flows* along which the trajectory is assumed to be varied.

Definition III.2: (Virtual flows and velocities) Virtual flows $\boldsymbol{\chi}_\lambda:\boldsymbol{\gamma}\to\mathbb{M}$*, dragging the trajectory* $\boldsymbol{\gamma}$ *in the state space* $\mathbb{M}$*, are such that, at discontinuity points, the duality, between their initial velocity fields* $\mathbf{v}_{\boldsymbol{\chi}}=\partial_{\lambda=0}\boldsymbol{\chi}_\lambda:\boldsymbol{\gamma}\to\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{M}$ *and covectors source terms, is well defined. Initial velocities of virtual flows are named virtual velocities.*

The value taken by a source form, when acting on a virtual velocity, is called a *virtual power*. An instance of explicit formulation of conditions to be fulfilled at discontinuity points will be given in Definition V.1 with reference to virtual velocities on the tangent bundle to the configuration manifold.

Proposition III.1: (Abstract action principle) In a dynamical system governed by a piecewise smooth differential one-form $\boldsymbol{\omega}^1$*, on* $\mathbb{M}$*, a piecewise smooth trajectory* $\boldsymbol{\gamma}:I\to\mathbb{M}$ *has tangent vector field* $\mathbf{v}_{\boldsymbol{\gamma}}:\boldsymbol{\gamma}\to\mathbb{T}\boldsymbol{\gamma}$*, belonging to the subbundle* $\mathrm{Trial}_{\boldsymbol{\gamma}}$ *and such that*

$$\partial_{\lambda=0}\int_{\boldsymbol{\chi}_\lambda(\boldsymbol{\gamma})} \boldsymbol{\omega}^1 = \oint_{\partial\boldsymbol{\gamma}} \boldsymbol{\omega}^1\cdot\mathbf{v}_{\boldsymbol{\chi}} + \int_{\boldsymbol{\gamma}} \boldsymbol{\alpha}^2\cdot\mathbf{v}_{\boldsymbol{\chi}} + \int_{\mathrm{Sing}(\boldsymbol{\gamma})} \boldsymbol{\alpha}^1\cdot\mathbf{v}_{\boldsymbol{\chi}}$$

for all virtual flows $\boldsymbol{\chi}_\lambda:\boldsymbol{\gamma}\to\mathbb{M}$ *with virtual velocities* $\mathbf{v}_{\boldsymbol{\chi}}:=\partial_{\lambda=0}\boldsymbol{\chi}_\lambda$ *in the subbundle* $\mathrm{Test}_{\boldsymbol{\gamma}}$.

This means that the initial rate of increase in the action integral along any virtual flow is equal to the outward flux of virtual velocities at end points plus the integrals of the exterior forms providing the virtual power performed by regular and impulsive source forms. Denoting by $\mathbf{x}_1$ and $\mathbf{x}_2$ the initial and final end points of $\boldsymbol{\gamma}$, it is $\partial\boldsymbol{\gamma}=\mathbf{x}_2-\mathbf{x}_1$ (a 0-chain), and the boundary integral may be written as

$$\oint_{\partial\boldsymbol{\gamma}} \boldsymbol{\omega}^1\cdot\mathbf{v}_{\boldsymbol{\chi}} = (\boldsymbol{\omega}^1\cdot\mathbf{v}_{\boldsymbol{\chi}})(\mathbf{x}_2)-(\boldsymbol{\omega}^1\cdot\mathbf{v}_{\boldsymbol{\chi}})(\mathbf{x}_1).$$

Singular points $\mathbf{x}_i$, $i=1,n$, along the trajectory also form a 0-chain, so that

$$\int_{\mathrm{Sing}(\boldsymbol{\gamma})} \boldsymbol{\alpha}^1\cdot\mathbf{v}_{\boldsymbol{\chi}} = \sum_{i=1}^{n} (\boldsymbol{\alpha}^1\cdot\mathbf{v}_{\boldsymbol{\chi}})(\mathbf{x}_i).$$

The action principle is purely geometrical since the trajectory $\boldsymbol{\gamma}$ may be affected by an arbitrary reparametrization. Under the assumption that the vector subbundle $\mathrm{Test}_{\boldsymbol{\gamma}}$ is rich enough to allow for localization, the action principle is equivalent to Euler's extremality conditions provided by the next theorem.

Euler's classical result assumes regular paths with fixed end points and is formulated in coordinates. The statement below considers piecewise regular paths and variations with nonfixed end points, leading to extremality conditions expressed by differential and jump terms. Its proof is a direct consequence of the extrusion formula and of a localization argument.

Theorem III.1: *(Euler's extremality conditions) A path $\boldsymbol{\gamma} \subset \mathbb{M}$ fulfills the action Principle III.1 if and only if the tangent vector field $\mathbf{v}_{\boldsymbol{\gamma}}: \boldsymbol{\gamma} \to \mathbb{T}\boldsymbol{\gamma}$, belonging to the subbundle* $\mathrm{Trial}_{\boldsymbol{\gamma}}$ *, meets, at regular points, the differential condition*

$$(d\boldsymbol{\omega}^1 - \boldsymbol{\alpha}^2) \cdot \mathbf{v}_{\boldsymbol{\gamma}} \cdot \mathbf{v}_{\boldsymbol{\chi}} = 0, \quad \forall\, \mathbf{v}_{\boldsymbol{\chi}} \in \mathrm{Test}_{\boldsymbol{\gamma}},$$

and, at singular points, the jump conditions

$$([[\boldsymbol{\omega}^1]] - \boldsymbol{\alpha}^1) \cdot \mathbf{v}_{\boldsymbol{\chi}} = 0, \quad \forall\, \mathbf{v}_{\boldsymbol{\chi}} \in \mathrm{Test}_{\boldsymbol{\gamma}},$$

where $[[\boldsymbol{\omega}^1]]$ is the difference between the right and the left limit.

If $\mathrm{Test}_{\boldsymbol{\gamma}} = \mathrm{Trial}_{\boldsymbol{\gamma}}$, and the restriction of the two-form $d\boldsymbol{\omega}^1$, to this subbundle has a one dimensional kernel at each point, Euler's differential condition ensures local existence and uniqueness of the trajectory through a given point of the state space. As is well known,[8] this is the case in rigid body dynamics. On the contrary, in continuum dynamics only the strict inclusion $\mathrm{Test}_{\boldsymbol{\gamma}} \subset \mathrm{Trial}_{\boldsymbol{\gamma}}$ holds and further conditions specifying the constitutive behavior of the material body are required. The issue will be discussed in greater detail in the sequel.

IV. KINEMATICS

The peculiar geometric feature of continuous dynamical systems is that three differentiable structures are playmates: the *ambient* space, a finite dimensional Riemannian manifold without boundary $(\mathcal{S}, \mathbf{g})$ (usually the flat Euclidean 3D space) in which motions take place, the *body*, a finite dimensional manifold $\mathcal{B}$, with boundary describing the geometrical properties of the continuous body, and the *configuration space*, the infinite dimensional manifold $\mathbb{C}$, describing the kinematics of the body in the ambient space.

The metric tensor $\mathbf{g}_{\mathbf{x}} \in BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}, \mathbb{T}_{\mathbf{x}}\mathcal{S}; \Re)$ and the linear map $\mathbf{g}_{\mathbf{x}}^{\flat} \in BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}; \mathbb{T}_{\mathbf{x}}^*\mathcal{S})$ may be identified by virtue of the one-to-one correspondence induced by the identity $\mathbf{g}_{\mathbf{x}}(\mathbf{a}, \mathbf{b}) = \langle \mathbf{g}_{\mathbf{x}}^{\flat}\mathbf{a}, \mathbf{b} \rangle$ for any pair $\mathbf{a}$, $\mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathcal{S}$. Moreover, by the positive definiteness of the metric $\mathbf{g}_{\mathbf{x}}$, the map $\mathbf{g}_{\mathbf{x}}^{\flat}$ is an isomorphism whose inverse $\mathbf{g}_{\mathbf{x}}^{\sharp} = (\mathbf{g}_{\mathbf{x}}^{\flat})^{-1}$ provides a metric tensor $\mathbf{g}_{\mathbf{x}}^{\sharp} \in BL(\mathbb{T}_{\mathbf{x}}^*\mathcal{S}, \mathbb{T}_{\mathbf{x}}^*\mathcal{S}; \Re)$ in the dual space.

The configuration space is a manifold of maps[26] which are C^k-*embeddings* of the body manifold $\mathcal{B}$ into the ambient manifold $(\mathcal{S}, \mathbf{g})$, i.e., injective maps $\boldsymbol{\xi} \in \mathrm{C}^k(\mathcal{B}; \mathcal{S})$, such that the *placements* $\boldsymbol{\xi}(\mathcal{B})$ are submanifolds of $\mathcal{S}$ and the corestricted maps $\boldsymbol{\xi} \in \mathrm{C}^1(\mathcal{B}; \boldsymbol{\xi}(\mathcal{B}))$ are diffeomorphisms.[22]

The theory of continuous dynamical systems is a *field* theory and it is essential to express differential properties of the *configuration space* in terms of the ones of the *ambient space*.

When morphisms, flows, and tensor fields in the configuration space and the ambient space are to be distinguished, a superscript $(\cdot)^{\mathbb{C}}$ will be used to denote quantities pertaining to the former, when there are analogous quantities pertaining to the latter. Geometrical objects in the two manifolds will be labeled by the prefixes $\mathbb{C}$ and $\mathcal{S}$, respectively.

In continuum mechanics velocity fields which are infinitesimal isometries (or rigid body velocities) play a central role. In fact, material bodies are insensitive to isometric changes of their placement in the ambient space. It follows that the constitutive properties of the materials do not enter in the dynamical equilibrium condition, if rigid body test velocities are considered. The elements of the linear space $\mathrm{Rig}(\boldsymbol{\Omega})$ of infinitesimal isometries at the placement $\boldsymbol{\Omega} = \boldsymbol{\xi}(\mathcal{B})$, also denoted by $\mathrm{Rig}_{\boldsymbol{\xi}}$, are vector fields $\delta\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\Omega}; \mathbb{T}\mathcal{S})$ characterized by the condition $\mathcal{L}_{\delta\mathbf{v}}\mathbf{g} = 0$.

In the configuration manifold $\mathbb{C}$, the family of all linear spaces $\mathrm{Rig}_{\boldsymbol{\xi}}$ defines a distribution Rig in the tangent bundle $(\mathbb{TC}, \boldsymbol{\tau}_{\mathbb{C}}, \mathbb{C})$. The property[3,27] of the Lie derivative, that $\mathcal{L}_{[\mathbf{u},\mathbf{v}]} = [\mathcal{L}_{\mathbf{u}}, \mathcal{L}_{\mathbf{v}}]$ for any pair of tangent vector fields $\mathbf{u}$, $\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\Omega}; \mathbb{T}\mathcal{S})$, ensures that the distribution Rig is involutive, i.e., $\mathcal{L}_{\mathbf{u}}\mathbf{g} = \mathcal{L}_{\mathbf{v}}\mathbf{g} = 0 \Rightarrow \mathcal{L}_{[\mathbf{u},\mathbf{v}]}\mathbf{g} = 0$, and hence integrable by Frobenius theorem.[2,3,22] It follows that the configuration manifold is foliated in a family of disjoint leaves characterized by the property that in each leaf the body can be displaced from one placement to any other by an isometric transformation.

In the standard realm of analytical dynamics, dynamical processes are assumed to evolve in a leaf of the foliation induced by Rig. The peculiar task of continuum mechanics is to eliminate the rigidity constraint by means of appropriate Lagrange multipliers in duality with the stretching evaluated by Euler's classical formula $\frac{1}{2}(\mathcal{L}_{\mathbf{v}}\mathbf{g})=\mathbf{g}\circ(\operatorname{sym}\nabla\mathbf{v})$. We provide hereafter a generalized version of it which is valid in an ambient Riemannian manifold with an arbitrary linear connection.[3] The linear space of tangent-valued k-linear alternating forms on $\mathcal{S}$ is denoted by $\Lambda^k(\mathcal{S};\mathbb{T}\mathcal{S})$.

Lemma IV.1: Let $\{\mathcal{S},\mathbf{g}\}$ *be a Riemannian manifold* ∇*, a linear connection in* $\mathcal{S}$ *with torsion* $\operatorname{tors}\in\Lambda^2(\mathcal{S};\mathbb{T}\mathcal{S})$ *and* $\operatorname{Tors}(\mathbf{v})$*, the field of linear operators defined by*

$$\operatorname{Tors}(\mathbf{v})\cdot\mathbf{u}=\operatorname{tors}(\mathbf{v},\mathbf{u}),\quad \forall\,\mathbf{v},\mathbf{u}\in C^1(\mathcal{S};\mathbb{T}\mathcal{S}).$$

Then, for any vector field $\mathbf{v}\in C^1(\mathcal{S};\mathbb{T}\mathcal{S})$ *,*

$$\tfrac{1}{2}(\mathcal{L}_{\mathbf{v}}\mathbf{g})=\mathbf{g}\circ(\operatorname{sym}\ \nabla\mathbf{v})+\tfrac{1}{2}(\nabla_{\mathbf{v}}\mathbf{g})+\mathbf{g}\circ(\operatorname{sym}\operatorname{Tors}(\mathbf{v})).$$

If ∇ *is* Levi–Civita, *i.e., metric* $\nabla\mathbf{g}=0$ *and torsion-free* $\operatorname{Tors}=0$ *, Euler's formula for the stretching is recovered.*

Proof: Applying the Leibniz rule to the Lie derivative and to the covariant derivative, we have that, for any vector fields $\mathbf{v}$, $\mathbf{u}$, $\mathbf{w}\in C^1(\mathcal{S};\mathbb{T}\mathcal{S})$,

$$(\mathcal{L}_{\mathbf{v}}\mathbf{g})(\mathbf{u},\mathbf{w})=\mathcal{L}_{\mathbf{v}}(\mathbf{g}(\mathbf{u},\mathbf{w}))-\mathbf{g}(\mathcal{L}_{\mathbf{v}}\mathbf{u},\mathbf{w})-\mathbf{g}(\mathbf{u},\mathcal{L}_{\mathbf{v}}\mathbf{w}),$$

$$(\nabla_{\mathbf{v}}\mathbf{g})(\mathbf{u},\mathbf{w})=\nabla_{\mathbf{v}}(\mathbf{g}(\mathbf{u},\mathbf{w}))-\mathbf{g}(\nabla_{\mathbf{v}}\mathbf{u},\mathbf{w})-\mathbf{g}(\mathbf{u},\nabla_{\mathbf{v}}\mathbf{w}).$$

Since the Lie derivative and the covariant derivative of a scalar field coincide, we also have that $\mathcal{L}_{\mathbf{v}}(\mathbf{g}(\mathbf{u},\mathbf{w}))=\nabla_{\mathbf{v}}(\mathbf{g}(\mathbf{u},\mathbf{w}))$ and hence

$$(\mathcal{L}_{\mathbf{v}}\mathbf{g})(\mathbf{u},\mathbf{w})=(\nabla_{\mathbf{v}}\mathbf{g})(\mathbf{u},\mathbf{w})+\mathbf{g}(\nabla_{\mathbf{v}}\mathbf{u},\mathbf{w})+\mathbf{g}(\mathbf{u},\nabla_{\mathbf{v}}\mathbf{w})-\mathbf{g}(\mathcal{L}_{\mathbf{v}}\mathbf{u},\mathbf{w})-\mathbf{g}(\mathbf{u},\mathcal{L}_{\mathbf{v}}\mathbf{w}).$$

Moreover, since $\operatorname{tors}(\mathbf{v},\mathbf{u}):=\nabla_{\mathbf{v}}\mathbf{u}-\nabla_{\mathbf{u}}\mathbf{v}-[\mathbf{v},\mathbf{u}]$ we may write

$$(\mathcal{L}_{\mathbf{v}}\mathbf{g})(\mathbf{u},\mathbf{w})=(\nabla_{\mathbf{v}}\mathbf{g})(\mathbf{u},\mathbf{w})+\mathbf{g}(\operatorname{tors}(\mathbf{v},\mathbf{u}),\mathbf{w})+\mathbf{g}(\nabla_{\mathbf{u}}\mathbf{v},\mathbf{w})+\mathbf{g}(\operatorname{tors}(\mathbf{v},\mathbf{w}),\mathbf{u})+\mathbf{g}(\nabla_{\mathbf{w}}\mathbf{v},\mathbf{u}),$$

which gives the result. □

A. Trajectories and flows

A trajectory in the configuration manifold is a piecewise smooth time-parametrized path $\gamma\in C^0(I;\mathbb{C})$ defined on a compact time interval I. The speed along the trajectory is the piecewise smooth vector field $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}}\in C^0(I;\mathbb{T}\boldsymbol{\gamma})$, with $\boldsymbol{\tau}_{\mathbb{C}}\circ\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}}=\gamma$, defined by $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}}(t):=\partial_{\tau=t}\gamma_{\tau}$. Let us denote by $1:I\to\mathbb{T}I$ the unit section, so that $\boldsymbol{\tau}_I\circ 1=\mathbf{id}_I$. The lifted trajectory in the velocity phase space $\Gamma:I\to\mathbb{T}\mathbb{C}$ is given by $\Gamma:=\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}}$.

A virtual flow $\boldsymbol{\varphi}^{\mathbb{C}}_{\lambda}:\boldsymbol{\gamma}\to\mathbb{C}$ in the configuration manifold is such that its velocity field $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\varphi}}=\partial_{\lambda=0}\boldsymbol{\varphi}^{\mathbb{C}}_{\lambda}\in C^0(\boldsymbol{\gamma};\mathbb{T}\mathbb{C})$ is continuous at singular points of the trajectory. The virtual velocity field along the trajectory, as a function of time, is denoted by $\delta\mathbf{v}^{\mathbb{C}}=\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\varphi}}\circ\gamma\in C^1(I;\mathbb{T}\mathbb{C})$. A virtual flow $\mathbf{Fl}^{\Theta}_{\lambda}\in C^1(I;\mathfrak{R})$ along the time axis enters in the definition of an asynchronous flow $\boldsymbol{\varphi}^{\mathbb{C}}_{\lambda}\times\mathbf{Fl}^{\Theta}_{\lambda}:\boldsymbol{\gamma}\times I\to\mathbb{C}\times\mathfrak{R}$ in the configuration-time manifold. A vanishing time-velocity Θ of the virtual flow at every time $t\in I$ defines a synchronous flow $\boldsymbol{\varphi}^{\mathbb{C}}_{\lambda}\times\mathbf{id}_I:\boldsymbol{\gamma}\times I\to\mathbb{C}\times I$ in the configuration-time manifold. For short, we will set $\mathbf{v}^{\mathbb{C}}_t:=\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}}(t)$ and $\delta\mathbf{v}^{\mathbb{C}}_t:=\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\varphi}}(\gamma_t)=\partial_{\lambda=0}\boldsymbol{\varphi}^{\mathbb{C}}_{\lambda}(\gamma_t)$ emphasizing that $\delta\mathbf{v}^{\mathbb{C}}$ is a unique symbol so that δ, by itself is meaningless.

In the velocity-time state space the lifted trajectory is $\Gamma_I : I \to \mathbb{TC} \times I$, with $\Gamma_I(t) = (\Gamma(t), t)$, and the trajectory speed is given by $(\mathbf{X}(\mathbf{v}_t^{\mathbb{C}}), 1_t) := \partial_{\tau=t}\Gamma_I(\tau)$. Trajectory images will be denoted by $\boldsymbol{\gamma} := \gamma(I) \subset \mathbb{C}$, $\boldsymbol{\Gamma} := \Gamma(I) \subset \mathbb{TC}$, and $\boldsymbol{\Gamma}_I := \boldsymbol{\Gamma} \times I \subset \mathbb{TC} \times I$, so that $\boldsymbol{\gamma} = \boldsymbol{\tau}_{\mathbb{C}} \circ \boldsymbol{\Gamma}$.

We consider continuous dynamical systems subject to linear kinematical constraints, denoting by $\mathrm{Conf}_{\boldsymbol{\gamma}} \subset \mathbb{TC}$, the vector subbundle of velocity fields which are conforming to the constraints and by $\mathrm{Rig}_{\boldsymbol{\gamma}} \subset \mathbb{TC}$, the vector subbundle of velocity fields which are rigid (i.e., infinitesimal isometries) at every configuration of the trajectory in the configuration manifold. The proper definition of conforming vector fields for continuous systems is delayed to Sec. IX A.

V. FORCE SYSTEMS

A force acting at a configuration $\boldsymbol{\xi} \in \mathbb{C}$ at time $t \in I$ is a one-form $\mathbf{f}_t \in \mathbb{T}^*_{\boldsymbol{\xi}}\mathbb{C}$. To formulate the law of dynamics on the tangent bundle, we need to express forces as one-forms on that bundle. Physical consistency requires that force-forms be represented by horizontal one-forms on the tangent bundle since the virtual power at a configuration must vanish for a vanishing virtual velocity field on the corresponding placement. Between a force one-form $\mathbf{f}_t \in \mathbb{T}^*_{\gamma_t}\mathbb{C}$ and the horizontal one-form $\mathbf{F}_t \in \mathbb{T}^*_{\Gamma_t}\mathbb{TC}$ on the lifted trajectory in the tangent bundle, there is a linear isomorphism defined by

$$\langle \mathbf{F}_t(\mathbf{v}_t^{\mathbb{C}}), \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \rangle := \langle \mathbf{f}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}})), T\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}) \cdot \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \rangle, \quad \forall\, \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \in \mathbb{T}_{\mathbf{v}_t^{\mathbb{C}}}\mathbb{TC}.$$

In the velocity-time state space forces are represented by *force two-forms* defined by

$$\mathbf{F}^2 := dt \wedge \mathbf{F}.$$

From the definition it follows that

$$\begin{aligned}(\mathbf{F}^2 \cdot (\mathbf{X},1) \cdot (\mathbf{Y},\Theta))(\mathbf{v},t) &= (dt \wedge \mathbf{F}_t(\mathbf{v}_t^{\mathbb{C}})) \cdot (\mathbf{X}(\mathbf{v}_t^{\mathbb{C}}),1_t) \cdot (\mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}),\Theta_t) \\ &= \mathbf{F}_t(\mathbf{v}_t^{\mathbb{C}}) \cdot \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) - (\mathbf{F}_t(\mathbf{v}_t^{\mathbb{C}}) \cdot \mathbf{X}(\mathbf{v}_t^{\mathbb{C}}))\Theta_t,\end{aligned}$$

where $\mathbf{X}(\mathbf{v}_t^{\mathbb{C}})$, $\mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \in \mathbb{T}_{\mathbf{v}_t^{\mathbb{C}}}\mathbb{TC}$, and $\Theta_t \in \mathbb{T}_t I$. For synchronous virtual velocities $\Theta_t = 0$, we get

$$(\mathbf{F}^2 \cdot (\mathbf{X},1) \cdot (\mathbf{Y},0))(\mathbf{v}^{\mathbb{C}},t) = \mathbf{F}_t(\mathbf{v}_t^{\mathbb{C}}) \cdot \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}), \quad \forall\, \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \in \mathbb{T}_{\mathbf{v}_t^{\mathbb{C}}}\mathbb{TC}.$$

Impulsive forces at singular points $\gamma_t \in \boldsymbol{\gamma}$ are described by one-forms $\boldsymbol{\alpha}_t(\gamma_t) \in \mathbb{T}^*_{\gamma_t}\mathbb{C}$. The virtual power performed by impulsive forces is well defined by assuming that virtual velocity fields are continuous along the whole trajectory, i.e., $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^0(\boldsymbol{\gamma};\mathbb{TC})$. The lifted trajectory $\Gamma : I \to \mathbb{TC}$ in the tangent bundle is discontinuous at singular points of the base trajectory $\gamma : I \to \mathbb{C}$ since there the velocity field suffers a jump, say from $\mathbf{v}^-$ to $\mathbf{v}^+$.

Definition V.1: (Virtual velocity field on the tangent bundle) A virtual velocity of the trajectory $\boldsymbol{\Gamma} \subset \mathbb{TC}$ *is a vector field* $\mathbf{Y} : \boldsymbol{\Gamma} \to \mathbb{TTC}$, *which projects to a vector field* $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^0(\boldsymbol{\gamma};\mathbb{TC})$ *with* $\boldsymbol{\gamma} = \boldsymbol{\tau}_{\mathbb{C}} \circ \boldsymbol{\Gamma}$, *i.e.,*

$$T\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{Y} = \mathbf{v}_{\boldsymbol{\varphi}} \circ \boldsymbol{\tau}_{\mathbb{C}}.$$

A well-posed definition of impulsive forces on the lifted trajectory $\Gamma : I \to \mathbb{TC}$ is based on the following property.

Lemma V.1: A virtual velocity field $\mathbf{Y} : \boldsymbol{\Gamma} \to \mathbb{TTC}$ *is such that, in correspondence to jumps from* $\mathbf{v}^-$ *to* $\mathbf{v}^+$ *of the velocity field of the projected trajectory* $\boldsymbol{\gamma} = \boldsymbol{\tau}_{\mathbb{C}} \circ \boldsymbol{\Gamma}$, *the virtual velocities* $\mathbf{Y}^- \in \mathbb{T}_{\mathbf{v}^-}\mathbb{TC}$ *and* $\mathbf{Y}^+ \in \mathbb{T}_{\mathbf{v}^+}\mathbb{TC}$ *project to the same horizontal part,*

$$T_{\mathbf{v}^-}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}^- = T_{\mathbf{v}^+}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}^+.$$

Proof: Since $\mathbf{Y} : \boldsymbol{\Gamma} \to \mathbb{TTC}$ projects to a vector field $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^0(\boldsymbol{\gamma};\mathbb{TC})$, we have that

$$T_{\mathbf{v}^-}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}^- = \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^-)),$$

$$T_{\mathbf{v}^+}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}^+ = \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^+)).$$

The result follows from the equality $\mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^-)) = \mathbf{v}_{\boldsymbol{\varphi}}(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^+))$ due to the continuity of $\mathbf{v}_{\boldsymbol{\varphi}} \in \mathrm{C}^0(\boldsymbol{\gamma};\mathbb{TC})$ at $\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^-) = \boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^+) \in \mathbb{C}$. □

From Lemma V.1 we infer that impulsive forces at discontinuity points of the lifted trajectory $\Gamma: I \to \mathbb{TC}$ are horizontal one-forms $\mathbf{A}_t(\mathbf{v}^-,\mathbf{v}^+) \in (\mathbb{T}_{\mathbf{v}^-}\mathbb{TC} \times \mathbb{T}_{\mathbf{v}^+}\mathbb{TC})^*$ well defined by

$$\mathbf{A}_t(\mathbf{v}^-,\mathbf{v}^+) \cdot (\mathbf{Y}^-,\mathbf{Y}^+) = \langle \boldsymbol{\alpha}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^-)), T_{\mathbf{v}^-}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}^- \rangle = \langle \boldsymbol{\alpha}_t(\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}^+)), T_{\mathbf{v}^+}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}^+ \rangle.$$

For brevity, we will set $\mathbf{A}_t(\mathbf{v}_t^{\mathbb{C}}) \cdot \mathbf{Y}(\mathbf{v}_t^{\mathbb{C}}) \coloneqq \mathbf{A}_t(\mathbf{v}^-,\mathbf{v}^+) \cdot (\mathbf{Y}^-,\mathbf{Y}^+)$ at singular time instants $t \in I$.

Remark V.1: The definition of a force acting on a mechanical system given above is classical and differs from the one recently, given e.g., in Refs. 23 and 28, *where force fields are considered as fiber preserving maps* $\mathbf{f}_t \in \mathrm{C}^1(\mathbb{TC};\mathbb{T}^*\mathbb{C})$. *Classically, a force acting on a mechanical system at a configuration* $\boldsymbol{\xi} \in \mathbb{C}$ *is an element of the cotangent space* $\mathbb{T}^*_{\boldsymbol{\xi}}\mathbb{C}$. *The virtual power performed for a virtual velocity* $\delta\mathbf{v}_t^{\mathbb{C}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$ *is the scalar* $\langle \mathbf{f}_t, \delta\mathbf{v}_t^{\mathbb{C}} \rangle \in \mathfrak{R}$. *The force acting on a body at a given configuration may depend on relative velocity fields between the body and its surroundings but, in general, is not related to the velocity field of the body (with respect to some reference frame). The dependence of a force on parameters, such as relative velocity, friction coefficients, electric charges, electromagnetic fields, etc., is to be modeled as a constitutive property, for instance, a multivalued maximal monotone relation between dual fields of forces and velocities.*[3] *Moreover, a dependence of force on body's velocity would violate Galilei's principle of relativity.*

VI. THE LAW OF DYNAMICS

The laws of dynamics may be classically formulated in different geometrical settings by assuming, as primary variables describing the system, either the configuration or the velocity or the velocity-time pair (or the covelocity-time pair). Each formulation has its merits and drawbacks and one may choose one or another depending on whether the geometric description or the computational machinery is prevailing. Hereafter, we present in three separate subsections the main approaches.

A. The law of dynamics in the state space

In the geometric action principle of dynamics the *state space* is either the velocity-time bundle $\mathbb{TC} \times I$ or the covelocity-time bundle $\mathbb{T}^*\mathbb{C} \times I$, respectively, in the Lagrangian and the Hamiltonian description. The Liouville one-form[2] on the cotangent bundle $\boldsymbol{\theta} \in \mathbb{T}^*_{\mathbf{v}^*}\mathbb{T}^*\mathbb{C}$ whose variational definition is

$$\langle \boldsymbol{\theta}(\mathbf{v}^*), \mathbf{Y}(\mathbf{v}^*) \rangle = \langle \mathbf{v}^*, T_{\mathbf{v}^*}\boldsymbol{\tau}^*_{\mathbb{C}} \cdot \mathbf{Y}(\mathbf{v}^*) \rangle, \qquad \forall\, \mathbf{Y}(\mathbf{v}^*) \in \mathbb{T}_{\mathbf{v}^*}\mathbb{T}^*\mathbb{C}.$$

The exterior derivative $d\boldsymbol{\theta}(\mathbf{v}^*)$ is a weakly nondegenerate[27] two-form on $\mathbb{T}^*\mathbb{C}$,

$$\langle d\boldsymbol{\theta} \cdot \mathbf{X}, \mathbf{Y} \rangle(\mathbf{v}^*) = 0, \qquad \forall\, \mathbf{Y}(\mathbf{v}^*) \in \mathbb{T}_{\mathbf{v}^*}\mathbb{T}^*\mathbb{C} \Rightarrow \mathbf{X}(\mathbf{v}^*) = 0.$$

The counterpart in the tangent bundle is the Poincaré–Cartan one-form[8] $\boldsymbol{\theta}_{L_t^{\mathbb{C}}}(\mathbf{v}^{\mathbb{C}}) \in \mathbb{T}^*_{\mathbf{v}^{\mathbb{C}}}\mathbb{TC}$ defined by means of the Legendre transform as

$$\langle \boldsymbol{\theta}_{L_t^{\mathbb{C}}}, \mathbf{Y} \rangle(\mathbf{v}^{\mathbb{C}}) = \langle d_{\mathrm{F}} L_t^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}}), T_{\mathbf{v}^{\mathbb{C}}}\boldsymbol{\tau}_{\mathbb{C}} \cdot \mathbf{Y}(\mathbf{v}^{\mathbb{C}}) \rangle, \qquad \forall\, \mathbf{Y}(\mathbf{v}^{\mathbb{C}}) \in \mathbb{T}_{\mathbf{v}^{\mathbb{C}}}\mathbb{TC}.$$

The Hamiltonian action one-form is given by

$$\boldsymbol{\omega}^1(\mathbf{v}^*,t) \coloneqq \boldsymbol{\theta}(\mathbf{v}^*) - H_t^{\mathbb{C}}(\mathbf{v}^*)dt \in \mathbb{T}^*_{(\mathbf{v}^*,t)}(\mathbb{T}^*\mathbb{C} \times I),$$

where the Hamiltonian $H_t^{\mathbb{C}} \in \mathrm{C}^2(\mathbb{T}^*\mathbb{C};\mathfrak{R})$ is Legendre conjugate to the Lagrangian $L_t^{\mathbb{C}} \in \mathrm{C}^2(\mathbb{TC};\mathfrak{R})$. In the Lagrangian description, the action one-form is given by

$$\boldsymbol{\omega}^1_{L^{\mathbb{C}}} := \boldsymbol{\theta}_{L^{\mathbb{C}}} - E^{\mathbb{C}} dt = \boldsymbol{\theta}_{L^{\mathbb{C}}} - \boldsymbol{\eta}_{E^{\mathbb{C}}} \in \mathbb{T}^*_{(\mathbf{v}^{\mathbb{C}},t)}(\mathbb{T}\mathbb{C} \times I),$$

where $E^{\mathbb{C}}_t(\mathbf{v}^{\mathbb{C}}) := H^{\mathbb{C}}_t(d_f L^{\mathbb{C}}_t(\mathbf{v}^{\mathbb{C}}))$ is the energy functional and $\boldsymbol{\eta}_{E^{\mathbb{C}}} := E^{\mathbb{C}} dt$ is the energy one-form. Let us now state the geometric action principle for a continuous dynamical system subject to linear kinematical constraints, the extension to affine kinematical constraints being straightforward. We set $\langle \mathbf{A}_t, (\mathbf{Y},0)\rangle(\mathbf{v}^{\mathbb{C}}_t, t) := \langle \mathbf{A}_t, \mathbf{Y}\rangle(\mathbf{v}^{\mathbb{C}}_t)$ and will say that the speed $(\mathbf{X},1) : \boldsymbol{\Gamma}_I \to \mathbb{T}\mathbb{T}\mathbb{C} \times \mathbb{T}I$ of a trajectory $\boldsymbol{\Gamma}_I \subset \mathbb{T}\mathbb{C} \times I$ in the *velocity-time* state space is conforming to the constraints if it is such that $\mathbf{X} : \boldsymbol{\Gamma} \to \mathbb{T}\mathbb{T}\mathbb{C}$ projects to a conforming velocity $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}} \in \mathrm{C}^0(I; \mathbb{T}\boldsymbol{\gamma})$ in the configuration space, i.e., $T\boldsymbol{\tau}_{\mathbb{C}} \circ \mathbf{X} = \mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}} \circ \boldsymbol{\tau}_{\mathbb{C}}$ with $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\gamma}} \in \mathrm{Conf}_{\boldsymbol{\gamma}}$.

Proposition VI.1: (Geometric action principle) A trajectory $\boldsymbol{\Gamma}_I \subset \mathbb{T}\mathbb{C} \times I$ *in the velocity-time state space has a speed conforming to the constraints and fulfills the synchronous action principle,*

$$\partial_{\lambda=0} \int_{\mathbf{Fl}^{(\mathbf{Y},0)}_{\lambda}(\boldsymbol{\Gamma}_I)} \boldsymbol{\omega}^1_{L^{\mathbb{C}}} = \oint_{\partial \boldsymbol{\Gamma}_I} \boldsymbol{\omega}^1_{L^{\mathbb{C}}} \cdot (\mathbf{Y},0) - \int_{\boldsymbol{\Gamma}_I} \mathbf{F}^2 \cdot (\mathbf{Y},0) - \int_{\mathrm{Sing}(\boldsymbol{\Gamma}_I)} \mathbf{A}_t \cdot (\mathbf{Y},0),$$

is fulfilled for any virtual flow $\mathbf{Fl}^{(\mathbf{Y},0)}_{\lambda}$ *whose virtual velocity field* $\mathbf{Y} : \boldsymbol{\Gamma} \to \mathbb{T}\mathbb{T}\mathbb{C}$ *projects to a virtual velocity field* $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\varphi}} \in \mathrm{C}^0(\boldsymbol{\gamma}; \mathrm{Conf}_{\boldsymbol{\gamma}} \cap \mathrm{Rig}_{\boldsymbol{\gamma}})$. *A standard localization procedure shows that the variational condition is equivalent, at regular points* $\mathbf{v}^{\mathbb{C}}_t \in \boldsymbol{\Gamma}$, *to Euler's differential condition,*

$$(d\boldsymbol{\omega}^1_{L^{\mathbb{C}}} - \mathbf{F}^2) \cdot (\mathbf{X},1) \cdot (\mathbf{Y},0) = 0,$$

and, at discontinuity points, to the jump condition,

$$[[\boldsymbol{\omega}^1_{L^{\mathbb{C}}} \cdot (\mathbf{Y},0)]] = \mathbf{A}_t \cdot \mathbf{Y}.$$

If virtual velocities are assumed to fulfill the energy conservation law, i.e., $\langle \mathbf{F}_t - dE^{\mathbb{C}}_t, \mathbf{Y}\rangle(\mathbf{v}^{\mathbb{C}}_t) = 0$, the geometric action principle VI.1 yields an extended form of the classical Maupertuis least action principle.[6]

B. The law of dynamics in the phase space

The geometric action principle of Proposition VI.1 and the relevant Euler's differential and jump conditions may be equivalently formulated in the context of the velocity phase space $\mathbb{T}\mathbb{C}$. This leads to an extension to continuum dynamics of Hamilton's equation in the velocity phase space and of the relevant jump conditions.

Proposition VI.2: (Hamilton's equation) The differential and jump Euler conditions are equivalent to the Hamilton equations in the velocity phase space,

$$d\boldsymbol{\theta}_{L^{\mathbb{C}}_t} \cdot \mathbf{X} \cdot \mathbf{Y} = \langle \mathbf{F}_t - dE^{\mathbb{C}}_t, \mathbf{Y}\rangle,$$

$$[[\boldsymbol{\theta}_{L^{\mathbb{C}}_t} \cdot \mathbf{Y}]] = \mathbf{A}_t \cdot \mathbf{Y}.$$

Proof: Recalling that $(\mathbf{F}^2 \cdot (\mathbf{X},1) \cdot (\mathbf{Y},0))(\mathbf{v}^{\mathbb{C}}, t) = \mathbf{F}_t(\mathbf{v}^{\mathbb{C}}_t) \cdot \mathbf{Y}(\mathbf{v}^{\mathbb{C}}_t)$, Euler's differential condition may be written as

$$(d\boldsymbol{\omega}^1_{L^{\mathbb{C}}} \cdot (\mathbf{X},1) \cdot (\mathbf{Y},0))((\mathbf{v}^{\mathbb{C}}_t, t)) = \mathbf{F}_t(\mathbf{v}^{\mathbb{C}}_t) \cdot \mathbf{Y}(\mathbf{v}^{\mathbb{C}}_t), \quad \forall\, \mathbf{Y}(\mathbf{v}^{\mathbb{C}}_t).$$

Then, observing that $d\boldsymbol{\omega}^1_{L^{\mathbb{C}}} = d\boldsymbol{\theta}_{L^{\mathbb{C}}} - d\boldsymbol{\eta}_{E^{\mathbb{C}}}$, a direct evaluation of the term involving the energy one-form $\boldsymbol{\eta}_{E^{\mathbb{C}}}$ may be performed by means of Palais formula (see Sec. II) to get

$$d\boldsymbol{\eta}_{E^{\mathbb{C}}} \cdot (\mathbf{X},1) \cdot (\mathbf{Y},0) = d_{(\mathbf{X},1)}\langle \boldsymbol{\eta}_{E^{\mathbb{C}}}, (\mathbf{Y},0)\rangle - d_{(\mathbf{Y},0)}\langle \boldsymbol{\eta}_{E^{\mathbb{C}}}, (\mathbf{X},1)\rangle - \langle \boldsymbol{\eta}_{E^{\mathbb{C}}}, [(\mathbf{X},1),(\mathbf{Y},0)]\rangle.$$

The first term at the right hand side vanishes because $\langle \boldsymbol{\eta}_{E^{\mathbb{C}}}, (\mathbf{Y},0)\rangle = E^{\mathbb{C}}\langle dt, (\mathbf{Y},0)\rangle = 0$. Moreover, we may extend $(\mathbf{X},1)$, along the flow $\mathbf{Fl}^{(\mathbf{Y},0)}_{\lambda}$ so that also the third term vanishes. Hence, being $\langle \boldsymbol{\eta}_{E^{\mathbb{C}}}, (\mathbf{X},1)\rangle = E^{\mathbb{C}}\langle dt, (\mathbf{X},1)\rangle = E^{\mathbb{C}}$, we have that $d\boldsymbol{\eta}_{E^{\mathbb{C}}} \cdot (\mathbf{X},1) \cdot (\mathbf{Y},0) = -\langle dE^{\mathbb{C}}, \mathbf{Y}\rangle$ and Hamilton's differential condition follows. A direct evaluation provides the jump condition. □

Hamilton's equation of rigid body dynamics[8,21] is recovered by assuming that $\mathbf{X}(\mathbf{v}_t^{\mathbb{C}})$ is a rigid speed, i.e., that it projects on a rigid velocity according to the relation $T\boldsymbol{\tau}_{\mathbb{C}}\circ\mathbf{X}=\mathbf{v}_t^{\mathbb{C}}\circ\boldsymbol{\tau}_{\mathbb{C}}$ with $\mathbf{v}_t^{\mathbb{C}} \in \mathrm{Rig}(\boldsymbol{\Omega}_t)$. In this context, setting $\mathbf{Y}(\mathbf{v}_t^{\mathbb{C}})=\mathbf{X}(\mathbf{v}_t^{\mathbb{C}})$, the skew symmetry of the two form $d\boldsymbol{\theta}_{L_t^{\mathbb{C}}}$ yields the energy conservation law,

$$\langle \mathbf{F}_t - dE_t^{\mathbb{C}}, \mathbf{X}\rangle(\mathbf{v}_t^{\mathbb{C}}) = (d\boldsymbol{\theta}_{L_t^{\mathbb{C}}}\cdot \mathbf{X}\cdot \mathbf{X})(\mathbf{v}_t^{\mathbb{C}}) = 0.$$

In continuum dynamics energy conservation still holds but with an additional term expressing the power expended by the stress field in the body against the stretching field, as shown in Sec. IX C.

C. The law of dynamics in the configuration manifold

The geometric action principle VI.1 may be equivalently expressed (in a nongeometric form) in terms of the time-parametrized trajectory $\gamma \in \mathrm{C}^1(I;\mathbb{C})$ in the configuration manifold. Theorem VI.1 below is an extended version of the classical Hamilton stationarity principle for the Lagrangian. The main innovative features are that no connections are involved in the statement and no fixed end-point conditions are imposed on the virtual flows, so that the statement is more properly an extremality principle rather than a stationarity one. This leads to a more general statement, leaving the freedom of introducing any convenient connection at a later stage (see Sec. VII). The formulation is also more satisfactory from an epistemological point of view since, with the elimination of the fixed end-point condition, the action principle recovers the natural property that, for any given partition of a path, if all the pieces fulfill the extremality condition, then also the whole path is extremal.[3] Moreover Noether's theorem is readily verified to be a special case of the new statement.[4]

Theorem VI.1: *(Action principle and law of dynamics) A piecewise smooth trajectory $\gamma : I \to \mathbb{C}$ of a continuous dynamical system in the configuration manifold has a speed $\mathbf{v}_t^{\mathbb{C}} \in \mathrm{Conf}_{\gamma_t}$ and fulfills the extremality principle,*

$$\oint_{\partial I} \langle d_{\mathrm{F}} L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}), \delta \mathbf{v}_t^{\mathbb{C}}\rangle - \partial_{\lambda=0}\int_I L_t^{\mathbb{C}}(\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\uparrow \mathbf{v}_t^{\mathbb{C}})\circ \gamma dt = \int_I \langle \mathbf{f}_t, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}\rangle \circ \gamma dt + \int_{\mathrm{Sing}(I)} \langle \boldsymbol{\alpha}_t, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}\rangle \circ \gamma .$$

equivalent to the differential condition,

$$\partial_{\tau=t}\langle d_{\mathrm{F}} L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}), \delta \mathbf{v}_\tau^{\mathbb{C}}\rangle - \partial_{\lambda=0} L_t^{\mathbb{C}}(\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\uparrow \mathbf{v}_t^{\mathbb{C}}) = \langle \mathbf{f}_t(\gamma_t), \delta \mathbf{v}_t^{\mathbb{C}}\rangle,$$

and to the jump conditions,

$$\langle [[d_{\mathrm{F}} L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}})]], \delta \mathbf{v}_t^{\mathbb{C}}\rangle = \langle \boldsymbol{\alpha}_t(\gamma_t), \delta \mathbf{v}_t^{\mathbb{C}}\rangle,$$

for any virtual flow $\boldsymbol{\varphi}_\lambda^{\mathbb{C}} \in \mathrm{C}^1(\boldsymbol{\gamma};\mathbb{C})$ with virtual velocity $\delta\mathbf{v}_t^{\mathbb{C}}=\mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}(\gamma_t) \in \mathrm{Conf}_{\gamma_t}\cap \mathrm{Rig}_{\gamma_t}$.

Proof: Let us consider the projection $\boldsymbol{\varphi}_\lambda^{\mathbb{C}}:\boldsymbol{\gamma}\to\mathbb{C}$ of the flow $\mathbf{Fl}_\lambda^{\mathbf{Y}}:\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C}\to\mathbb{TC}$ on the configuration manifold according to the relation $T\boldsymbol{\tau}_{\mathbb{C}}\circ\mathbf{Fl}_\lambda^{\mathbf{Y}}=\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\circ\boldsymbol{\tau}_{\mathbb{C}}$ and let $T\boldsymbol{\varphi}_\lambda^{\mathbb{C}}:\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C}\to\mathbb{TC}$ be the lifted flow. Then the vector fields $\mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}=\partial_{\lambda=0}T\boldsymbol{\varphi}_\lambda^{\mathbb{C}}:\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C}\to\mathbb{TTC}$ and $\mathbf{Y}:\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C}\to\mathbb{TTC}$ have the same horizontal part. Hence $\mathbf{Y}=\mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}+\mathbf{V}$, with $\mathbf{V}:\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C}\to\mathbb{TTC}$, a vertical vector field. The fulfillment of Euler's condition $(d\boldsymbol{\theta}_{L_t^{\mathbb{C}}}\cdot\mathbf{X}\cdot\mathbf{V})(\mathbf{v}_t^{\mathbb{C}})+\langle dE_t^{\mathbb{C}},\mathbf{V}\rangle(\mathbf{v}_t^{\mathbb{C}})=0$ for any vertical virtual velocity $\mathbf{V}(\mathbf{v}_t^{\mathbb{C}}) \in \mathbb{T}_{\mathbf{v}_t^{\mathbb{C}}}\mathbb{T}_{\boldsymbol{\tau}_{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}})}\mathbb{C}$ is equivalent[3,6] to require that $T_{\mathbf{v}_t^{\mathbb{C}}}\boldsymbol{\tau}_{\mathbb{C}}\cdot\mathbf{X}(\mathbf{v}_t^{\mathbb{C}})=\mathbf{v}_t^{\mathbb{C}}$, which implies that $\mathbf{X}(\mathbf{v}_t^{\mathbb{C}})=\dot{\mathbf{v}}_t^{\mathbb{C}}$. Then, if the trajectory in the velocity phase space is lifted from the trajectory in the configuration manifold, i.e., $\Gamma=\mathbf{v}_{\gamma}^{\mathbb{C}}$, extremality with respect to vertical variations is trivially fulfilled and hence synchronous variations may be performed by the sole lifted virtual flows $T\boldsymbol{\varphi}_\lambda \in \mathrm{C}^1(\mathbb{T}_{\boldsymbol{\gamma}}\mathbb{C};\mathbb{TC})$. Hamilton's equation then writes

$$d\boldsymbol{\theta}_{L_t^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}}) \cdot \dot{\mathbf{v}}_t^{\mathbb{C}} \cdot \mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}) = (\langle \mathbf{f}_t, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}} \rangle \circ \boldsymbol{\tau}_{\mathbb{C}} - \langle dE_t^{\mathbb{C}}, \mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}} \rangle)(\mathbf{v}_t^{\mathbb{C}}).$$

To get the corresponding action principle, we may evaluate each term in the geometric action principle VI.1 setting $\mathbf{Y}=\mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}$. In this respect we observe that, to compute the Lagrangian on the paths drifted by the flow, one has to assume that it is defined on points outside the trajectory. Then the Lagrangian must be evaluated on the velocity of a synchronously varied trajectory which is equal to the push of the velocity of the trajectory. A direct computation gives[3]

$$\boldsymbol{\omega}_{L^{\mathbb{C}}}^1(\boldsymbol{\varphi}_\lambda^{\mathbb{C}} \uparrow \mathbf{v}_t^{\mathbb{C}}, t) \cdot (T\boldsymbol{\varphi}_\lambda^{\mathbb{C}} \uparrow \dot{\mathbf{v}}_t^{\mathbb{C}}, 1_t) = L_t^{\mathbb{C}}(\boldsymbol{\varphi}_\lambda^{\mathbb{C}} \uparrow \mathbf{v}_t^{\mathbb{C}}),$$

$$\boldsymbol{\omega}_{L^{\mathbb{C}}}^1(\mathbf{v}_t^{\mathbb{C}}, t) \cdot (\mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}), 0) = \langle d_{\mathrm{F}} L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}), \delta \mathbf{v}_t^{\mathbb{C}} \rangle,$$

and we infer that

$$\int_{(T\boldsymbol{\varphi}_\lambda^{\mathbb{C}} \times \mathbf{Fl}_\lambda^0)(\boldsymbol{\Gamma}_I)} \boldsymbol{\omega}_{L^{\mathbb{C}}}^1 = \int_I L_t^{\mathbb{C}}(\boldsymbol{\varphi}_\lambda^{\mathbb{C}} \uparrow \mathbf{v}_t^{\mathbb{C}}) \circ \gamma dt,$$

$$\oint_{\partial \boldsymbol{\Gamma}_I} \boldsymbol{\omega}_{L^{\mathbb{C}}}^1(\mathbf{v}_t^{\mathbb{C}}) \cdot (\mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}), 0) = \oint_{\partial I} \langle d_{\mathrm{F}} L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}), \delta \mathbf{v}_t^{\mathbb{C}} \rangle,$$

$$\int_{\boldsymbol{\Gamma}_I} \mathbf{F}_t^2 \cdot (\mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}, 0) = \int_I \langle \mathbf{f}_t, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}} \rangle \circ \gamma dt,$$

$$\int_{\mathrm{Sing}(\boldsymbol{\Gamma}_I)} \langle \mathbf{A}_t, (\mathbf{v}_{T\boldsymbol{\varphi}}^{\mathbb{C}}, 0) \rangle (\mathbf{v}_t^{\mathbb{C}}) = \int_{\mathrm{Sing}(I)} \langle \boldsymbol{\alpha}_t, \mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}} \rangle \circ \gamma_t.$$

Substituting we get the result.

VII. THE LAW OF DYNAMICS IN TERMS OF A CONNECTION

Our first goal is a generalized version of Lagrange's law of dynamics which proves that, at each point of the trajectory in the configuration manifold, the law of dynamics is tensorial and is expressed by the vanishing of a linear form on the linear subspace of test vectors. The proof of this tensoriality result, which is basic for the foundation of continuum dynamics, is provided in Theorem VII.1 and requires a linear connection to be fixed in the configuration manifold. Later, in Sec. VIII, we show that a connection is naturally induced in the configuration manifold by a given connection in the ambient manifold.

As a preliminary result we provide a split formula generalizing the usual partial differentiation formula valid in linear spaces adopted, e.g., in Refs. 10, 27, and 29. This decomposition was provided in Ref. 3 and later and independently introduced for vector bundles in Ref. 30, where the *base derivative* is called the *parallel derivative* and the *fiber-covariant derivative* is called the *fiber derivative*.

Lemma VII.1: (A split formula) Let $\mathbb{N}$ *be a manifold,* $\boldsymbol{\pi} \in \mathrm{C}^1(\mathbb{E};\mathbb{M})$ *a fiber bundle with a connection* ∇*, and* $\mathbf{f} \in \mathrm{C}^1(\mathbb{E};\mathbb{N})$ *a morphism. Then, for any section* $\mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{E})$ *of the fiber bundle, the map tangent to the composition* $\mathbf{f} \circ \mathbf{s} \in \mathrm{C}^1(\mathbb{M};\mathbb{N})$ *may be uniquely split as sum of the fiber-covariant derivative and the base derivative,*

$$T(\mathbf{f} \circ \mathbf{s}) = T\mathbf{f} \circ T\mathbf{s} = d_{\mathrm{F}}\mathbf{f}(\mathbf{s}) \cdot \nabla \mathbf{s} + d_{\mathrm{B}}\mathbf{f}(\mathbf{s}).$$

Proof: Denoting by $\mathbf{Fl}_\lambda^{\mathbf{v}} \Uparrow = \mathbf{Fl}_\lambda^{\mathbf{H}_{\mathbf{v}}} \in \mathrm{C}^1(\mathbb{E};\mathbb{E})$ the parallel transport along the flow associated with a vector field $\mathbf{v} \in \mathrm{C}^1(\mathbb{M};\mathbb{TM})$, by the definitions and the chain rule we have that

$$d_{\mathrm{F}}\mathbf{f}(\mathbf{s_x})\cdot\nabla_{\mathbf{v_x}}\mathbf{s}=T_{\mathbf{s_x}}\mathbf{f}\cdot\mathrm{P}_V\cdot T_{\mathbf{v_x}}\mathbf{s}=T_{\mathbf{s_x}}\mathbf{f}\cdot\nabla_{\mathbf{v_x}}\mathbf{s}=T_{\mathbf{s_x}}\mathbf{f}\cdot\partial_{\lambda=0}\mathbf{Fl}^{\mathbf{v}}_\lambda\Downarrow\mathbf{s}_{\mathbf{Fl}^{\mathbf{v}}_\lambda(\mathbf{x})}=\partial_{\lambda=0}\mathbf{f}(\mathbf{Fl}^{\mathbf{v}}_\lambda\Downarrow\mathbf{s}_{\mathbf{Fl}^{\mathbf{v}}_\lambda(\mathbf{x})}),$$

$$d_{\mathrm{B}}\mathbf{f}(\mathbf{s_x})\cdot\mathbf{v_x}=T_{\mathbf{s_x}}\mathbf{f}\cdot\mathrm{P}_H\cdot T_{\mathbf{v_x}}\mathbf{s}=T_{\mathbf{s_x}}\mathbf{f}\cdot\mathbf{H}_{\mathbf{v_x}}\mathbf{s}=T_{\mathbf{s_x}}\mathbf{f}\cdot\partial_{\lambda=0}\mathbf{Fl}^{\mathbf{v}}_\lambda\Uparrow\mathbf{s_x}=\partial_{\lambda=0}\mathbf{f}(\mathbf{Fl}^{\mathbf{v}}_\lambda\Uparrow\mathbf{s_x}),$$

so that $T_{\mathbf{s_x}}(\mathbf{f}\circ\mathbf{s})\cdot\mathbf{v_x}=d_{\mathrm{F}}\mathbf{f}(\mathbf{s_x})\cdot\nabla_{\mathbf{v_x}}\mathbf{s}+d_{\mathrm{B}}\mathbf{f}(\mathbf{s_x})\cdot\mathbf{v_x}$. □

Let $\mathbf{\Omega}_t=\gamma_t(\mathcal{B})$ be the placement of the body at time $t\in I$ along the trajectory $\gamma\in\mathrm{C}^0(I;\mathbb{C})$. The displacement along the trajectory is described by the diffeomorphism $\gamma_{\tau,t}:=\gamma_\tau\circ\gamma_t^{-1}\in\mathrm{C}^1(\mathbf{\Omega}_t;\mathbf{\Omega}_\tau)$.

Theorem VII.1: *(Generalized Lagrange's law of motion) Let* $\nabla^{\mathbb{C}}$ *be a linear connection in the configuration manifold* $\mathbb{C}$ *with parallel transport* $\Uparrow$ *and torsion* $\mathrm{tors}^{\mathbb{C}}$. *The law of motion for a trajectory with speed* $\mathbf{v}_t^{\mathbb{C}}\in\mathrm{Conf}_{\gamma_t}$ *is then expressed in terms of parallel transport by*

$$\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\gamma_{\tau,t}\Uparrow\delta\mathbf{v}_t^{\mathbb{C}}\rangle-\langle d_{\mathrm{B}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\delta\mathbf{v}_t^{\mathbb{C}}\rangle+\langle d_{\mathrm{F}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\mathrm{tors}^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}},\delta\mathbf{v}_t^{\mathbb{C}})\rangle=\langle\mathbf{f}_t(\gamma_t),\delta\mathbf{v}_t^{\mathbb{C}}\rangle$$

for any virtual velocity $\delta\mathbf{v}_t^{\mathbb{C}}\in\mathrm{Conf}_{\gamma_t}\cap\mathrm{Rig}_{\gamma_t}$.

Proof: The differential law of motion writes

$$\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\delta\mathbf{v}_\tau^{\mathbb{C}}\rangle-\partial_{\lambda=0}L_t^{\mathbb{C}}(T\boldsymbol{\varphi}_\lambda^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}))=\langle\mathbf{f}_t(\gamma_t),\delta\mathbf{v}_t^{\mathbb{C}}\rangle.$$

The invariance of the scalar product with respect to the pull back gives:

$$\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\delta\mathbf{v}_\tau^{\mathbb{C}}\rangle=\partial_{\tau=t}\langle\gamma_{\tau,t}\Downarrow d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\gamma_{\tau,t}\Downarrow\delta\mathbf{v}_\tau^{\mathbb{C}}\rangle,$$

so that, by Leibniz rule and the definition of covariant derivative, we get

$$\begin{aligned}\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\delta\mathbf{v}_\tau^{\mathbb{C}}\rangle&=\partial_{\tau=t}\langle\gamma_{\tau,t}\Downarrow d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\delta\mathbf{v}_t^{\mathbb{C}}\rangle+\partial_{\tau=t}\langle d_{\mathrm{F}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\gamma_{\tau,t}\Downarrow\delta\mathbf{v}_\tau^{\mathbb{C}}\rangle\\&=\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\gamma_{\tau,t}\Uparrow\delta\mathbf{v}_t^{\mathbb{C}}\rangle+\langle d_{\mathrm{F}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\nabla^{\mathbb{C}}_{\mathbf{v}_t^{\mathbb{C}}}\delta\mathbf{v}^{\mathbb{C}}\rangle.\end{aligned}$$

On the other hand, defining the vector field $\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})\in\mathrm{C}^1(\mathbb{C};\mathbb{TC})$, as the extension of the trajectory velocity by push along the virtual flow,

$$(\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})\circ\boldsymbol{\varphi}_\lambda^{\mathbb{C}})(\gamma_t)=(\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\uparrow\mathbf{v}_t^{\mathbb{C}}\circ\boldsymbol{\varphi}_\lambda^{\mathbb{C}})(\gamma_t)=T\boldsymbol{\varphi}_\lambda^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),$$

the second term at the left hand side of the law of motion writes

$$\partial_{\lambda=0}L_t^{\mathbb{C}}(T\boldsymbol{\varphi}_\lambda^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}))=\partial_{\lambda=0}(L_t^{\mathbb{C}}\circ\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})\circ\boldsymbol{\varphi}_\lambda^{\mathbb{C}})(\gamma_t)=\langle T(L_t^{\mathbb{C}}\circ\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})),\delta\mathbf{v}_t^{\mathbb{C}}\rangle,$$

and the split formula of Lemma VII.1 yields

$$\langle T(L_t^{\mathbb{C}}\circ\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})),\delta\mathbf{v}_t^{\mathbb{C}}\rangle=\langle d_{\mathrm{F}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\nabla^{\mathbb{C}}_{\delta\mathbf{v}_t^{\mathbb{C}}}\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})\rangle+\langle d_{\mathrm{B}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\delta\mathbf{v}_t^{\mathbb{C}}\rangle.$$

The left hand side of the law of motion of Theorem VI.1 may then be written as

$$\begin{aligned}&\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathbb{C}}\circ\mathbf{v}_\tau^{\mathbb{C}},\delta\mathbf{v}_\tau^{\mathbb{C}}\rangle-\partial_{\lambda=0}L_t^{\mathbb{C}}(T\boldsymbol{\varphi}_\lambda^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}))\\&\quad=\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathbb{C}}(\mathbf{v}_\tau^{\mathbb{C}}),\boldsymbol{\gamma}_{\tau,t}\Uparrow\delta\mathbf{v}_t^{\mathbb{C}}\rangle-\langle d_{\mathrm{B}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\delta\mathbf{v}_t^{\mathbb{C}}\rangle\\&\qquad+\langle d_{\mathrm{F}}L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}),\nabla^{\mathbb{C}}_{\mathbf{v}_t^{\mathbb{C}}}\delta\mathbf{v}^{\mathbb{C}}-\nabla^{\mathbb{C}}_{\delta\mathbf{v}_t^{\mathbb{C}}}\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})\rangle.\end{aligned}$$

Let us then consider a virtual flow $\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\in\mathrm{C}^1(\boldsymbol{\gamma};\mathbb{C})$ and its velocity field $\mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}\circ\boldsymbol{\varphi}_\lambda^{\mathbb{C}}:=\partial_{\mu=\lambda}\boldsymbol{\varphi}_\mu^{\mathbb{C}}$. Then the Lie bracket $[\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}}),\mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}]$ vanishes since

$$-[\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}}),\mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}]=[\mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}},\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})]=\mathcal{L}_{\mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}}\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})=\partial_{\lambda=0}\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\downarrow\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}})=\partial_{\lambda=0}\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\downarrow\boldsymbol{\varphi}_\lambda^{\mathbb{C}}\uparrow\mathbf{v}_t^{\mathbb{C}}=\partial_{\lambda=0}\mathbf{v}_t^{\mathbb{C}}=0.$$

Hence, by the tensoriality of the torsion of a connection, we have that

$$\mathrm{tors}^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}},\delta\mathbf{v}_t^{\mathbb{C}}) \coloneqq \nabla^{\mathbb{C}}_{\mathbf{v}_t^{\mathbb{C}}}\delta\mathbf{v}^{\mathbb{C}} - \nabla^{\mathbb{C}}_{\delta\mathbf{v}_t^{\mathbb{C}}}\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}}) - [\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}}),\mathbf{v}_{\boldsymbol{\varphi}}^{\mathbb{C}}] = \nabla^{\mathbb{C}}_{\mathbf{v}_t^{\mathbb{C}}}\delta\mathbf{v}^{\mathbb{C}} - \nabla^{\mathbb{C}}_{\delta\mathbf{v}_t^{\mathbb{C}}}\mathcal{F}_{\boldsymbol{\varphi}^{\mathbb{C}}}(\mathbf{v}_t^{\mathbb{C}}).$$

Substituting we get the result. □

VIII. POSITION FIBRATION AND INDUCED CONNECTION

Let us make reference to notions and definitions introduced in Sec. IV. The next result, which is plausible on an intuitive ground, is an essential tool for the theory developed in the sequel. It is quoted by Marsden and Hughes in Ref. 27, Box 4.2, property (*ii*), p. 170. Proofs are provided, in the context of the theory of manifolds of maps, by Eliasson in Ref. 31, Theorem 5.2, p. 186, and by Palais in Ref. 26, Theorem 13.6, p. 51.

Lemma VIII.1: (Identification) Let $\mathbb{C} \coloneqq \mathrm{C}^k(\mathcal{B};\mathcal{S})$. *Then there is a natural identification between the vectors* $\mathbf{v}_{\boldsymbol{\xi}}^{\mathbb{C}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$ *of the tangent space at a configuration* $\boldsymbol{\xi} \in \mathrm{C}^k(\mathcal{B};\mathcal{S})$ *and the tangent vector fields* $\mathbf{v}_{\boldsymbol{\xi}} \in \mathrm{C}^k(\boldsymbol{\xi}(\mathcal{B});\mathbb{T}\mathcal{S})$ *on the placement* $\boldsymbol{\xi}(\mathcal{B})$, *with* $\boldsymbol{\tau}_{\mathcal{S}} \circ \mathbf{v}_{\boldsymbol{\xi}} = \mathbf{id}_{\boldsymbol{\xi}(\mathcal{B})}$.

For our purposes it is convenient to provide an interpretation of this result in terms of a fibration map, which we call the *position map*. This map is a suitable analytical tool for the definition of a connection in a manifold of maps induced from a given connection in the codomain manifold, as illustrated hereafter.

Definition VIII.1: (Position map) The position map is a surjective submersion $\mathrm{pos}_{\mathbf{p}} \in \mathrm{C}^1(\mathbb{C};\mathcal{S})$ *which provides the position* $\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})$ *of particle* $\mathbf{p} \in \mathcal{B}$ *at the configuration* $\boldsymbol{\xi} \in \mathrm{C}^1(\mathcal{B};\mathcal{S})$,

$$\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi}) \coloneqq \boldsymbol{\xi}(\mathbf{p}) \in \boldsymbol{\xi}(\mathcal{B}).$$

In the configuration space of a continuous body to any particle $\mathbf{p} \in \mathcal{B}$ there corresponds a fiber bundle, denoted by $(\mathbb{C},\mathrm{pos}_{\mathbf{p}},\mathcal{S})$, whose fiber over the position $\boldsymbol{\xi}(\mathbf{p}) \in \mathcal{S}$ is the equivalence class of all configurations $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{B};\mathcal{S})$ mapping the particle into that position. The surjective tangent map $T_{\boldsymbol{\xi}}\,\mathrm{pos}_{\mathbf{p}} \in BL(\mathbb{T}_{\boldsymbol{\xi}}\mathbb{C};\mathbb{T}_{\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})}\mathcal{S})$ induces a fiber-linear correspondence between tangent spaces,

$$\mathbf{v}_{\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})} = T_{\boldsymbol{\xi}}\,\mathrm{pos}_{\mathbf{p}} \cdot \mathbf{v}_{\boldsymbol{\xi}}^{\mathbb{C}},$$

where $\mathbf{v}_{\boldsymbol{\xi}}^{\mathbb{C}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$ and $\mathbf{v}_{\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})} \in \mathbb{T}_{\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})}\mathcal{S}$.

Given a field of tangent vectors on a placement $\boldsymbol{\xi}(\mathcal{B})$ of the body, the tangent map $T_{\boldsymbol{\xi}}\,\mathrm{pos}_{\mathbf{p}}$ samples the vector tangent at the position of the particle $\mathbf{p} \in \mathcal{B}$. In geometric terms this relation is expressed by saying that the vector field $\mathbf{v} \in \mathrm{C}^1(\mathcal{S};\mathbb{T}\mathcal{S})$ is $\mathrm{pos}_{\mathbf{p}}$-related to the vector field $\mathbf{v}^{\mathbb{C}} \in \mathrm{C}^1(\mathbb{C};\mathbb{T}\mathbb{C})$ according to the commutative diagram,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathbf{v}^{\mathbb{C}}} & \mathbb{T}\mathbb{C} \\ \mathrm{pos}_{\mathbf{p}}\downarrow & & \downarrow T\,\mathrm{pos}_{\mathbf{p}} \\ \mathcal{S} & \xrightarrow{\mathbf{v}} & \mathbb{T}\mathcal{S} \end{array} \quad \Leftrightarrow \quad \mathbf{v} \circ \mathrm{pos}_{\mathbf{p}} = T\mathrm{pos}_{\mathbf{p}} \circ \mathbf{v}^{\mathbb{C}}.$$

By uniqueness of the solution of an ordinary differential equation, the $\mathrm{pos}_{\mathbf{p}}$-relatedness above is equivalent to the following commutative diagram for the respective flows:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathbf{Fl}_{\lambda}^{\mathbf{v}^{\mathbb{C}}}} & \mathbb{C} \\ \mathrm{pos}_{\mathbf{p}}\downarrow & & \downarrow \mathrm{pos}_{\mathbf{p}} \\ \mathcal{S} & \xrightarrow{\mathbf{Fl}_{\lambda}^{\mathbf{v}}} & \mathcal{S} \end{array} \quad \Leftrightarrow \quad \mathbf{Fl}_{\lambda}^{\mathbf{v}} \circ \mathrm{pos}_{\mathbf{p}} = \mathrm{pos}_{\mathbf{p}} \circ \mathbf{Fl}_{\lambda}^{\mathbf{v}^{\mathbb{C}}}.$$

For any fixed configuration $\boldsymbol{\xi} \in \mathrm{C}^1(\mathcal{B};\mathcal{S})$, by varying $\mathbf{p} \in \mathcal{B}$, the vector $\mathbf{v}_{\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})}$ spans the vector field which, according to Lemma VIII.1, can be identified with the tangent vector $\mathbf{v}_{\boldsymbol{\xi}}^{\mathbb{C}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$.

There is a natural way of endowing the configuration space $\mathbb{C}$, an infinite dimensional manifold of maps, with a connection induced by a given one in the finite dimensional ambient space $\mathcal{S}$, to which the codomains of the configuration embeddings belong.

Lemma VIII.2: (Induced connection) A connection in ambient space $\mathcal{S}$ induces a corresponding connection in the configuration space $\mathbb{C}$.

Proof: The correspondence in the statement is best described in terms of parallel transport of a tangent vector along a $\mathbb{C}$-curve $\gamma \in C^1(I;\mathbb{C})$ from a configuration γ_{t_0} to another γ_{t_1}. By Lemma VIII.1 a vector $\mathbf{v}_{t_0}^{\mathbb{C}} \in \mathbb{T}_{\gamma_{t_0}}\mathbb{C}$ is a vector field $\mathbf{v}_{t_0}^{\mathbb{C}} \in C^1(\gamma_{t_0}(\mathcal{B});\mathbb{T}\mathcal{S})$ with $\boldsymbol{\tau}_{\mathcal{S}} \circ \mathbf{v}_{t_0}^{\mathbb{C}} = \mathbf{id}_{\gamma_{t_0}(\mathcal{B})}$. A pointwise parallel transport of each vector $\mathbf{v}_{t_0}^{\mathbb{C}}(\mathbf{x})$, with $\mathbf{x} \in \gamma_{t_0}(\mathcal{B})$, along the $\mathcal{S}$-curve $\gamma(\mathbf{x}) \in C^1(I;\mathcal{S})$ in the ambient manifold yields a vector field $\mathbf{v}_{t_1}^{\mathbb{C}} \in C^1(\gamma_{t_1}(\mathcal{B});\mathbb{T}\mathcal{S})$, with $\boldsymbol{\tau}_{\mathcal{S}} \circ \mathbf{v}_{t_1}^{\mathbb{C}} = \mathbf{id}_{\gamma_{t_1}(\mathcal{B})}$. This is the vector $\mathbf{v}_{t_1}^{\mathbb{C}} \in \mathbb{T}_{\gamma_{t_1}}\mathbb{C}$ result of the parallel transport along the $\mathbb{C}$-curve $\gamma \in C^1(I;\mathbb{C})$. □

The construction in Lemma VIII.2 is equivalently described by the following statement. If the vector fields $\mathbf{u}$, $\mathbf{v} \in C^1(\mathcal{S};\mathbb{T}\mathcal{S})$, are $\mathrm{pos}_{\mathbf{p}}$-related to the vector fields $\mathbf{u}^{\mathbb{C}}$, $\mathbf{v}^{\mathbb{C}} \in C^1(\mathbb{C};\mathbb{T}\mathbb{C})$, then the parallel transport $\mathbf{Fl}_\lambda^{\mathbf{v}} \Uparrow \mathbf{u}$ is $\mathrm{pos}_{\mathbf{p}}$-related to the parallel transport $\mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \Uparrow \mathbf{u}^{\mathbb{C}}$ according to the commutative diagram,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \Uparrow \mathbf{u}^{\mathbb{C}}} & \mathbb{T}\mathbb{C} \\ \mathrm{pos}_{\mathbf{p}} \downarrow & & \downarrow T\mathrm{pos}_{\mathbf{p}} \\ \mathcal{S} & \xrightarrow{\mathbf{Fl}_\lambda^{\mathbf{v}} \Uparrow \mathbf{u}} & \mathbb{T}\mathcal{S} \end{array} \quad \Leftrightarrow \quad \mathbf{Fl}_\lambda^{\mathbf{v}} \Uparrow \mathbf{u} \circ \mathrm{pos}_{\mathbf{p}} = T\mathrm{pos}_{\mathbf{p}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \Uparrow \mathbf{u}^{\mathbb{C}}.$$

Lemma VIII.3: (Induced covariant derivative) Let ∇ be the covariant derivative associated with a linear connection in the ambient manifold $\mathcal{S}$ and $\nabla^{\mathbb{C}}$ be the covariant derivative according to the induced connection in the configuration manifold $\mathbb{C}$. Then the covariant derivatives of $\mathrm{pos}_{\mathbf{p}}$*-related vector fields are* $\mathrm{pos}_{\mathbf{p}}$*-related,*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\nabla_{\mathbf{v}^{\mathbb{C}}}^{\mathbb{C}} \mathbf{u}^{\mathbb{C}}} & \mathbb{T}\mathbb{C} \\ \mathrm{pos}_{\mathbf{p}} \downarrow & & \downarrow T\,\mathrm{pos}_{\mathbf{p}} \\ \mathcal{S} & \xrightarrow{\nabla_{\mathbf{v}} \mathbf{u}} & \mathbb{T}\mathcal{S} \end{array} \quad \Leftrightarrow \quad \nabla_{\mathbf{v}} \mathbf{u} \circ \mathrm{pos}_{\mathbf{p}} = T\,\mathrm{pos}_{\mathbf{p}} \circ \nabla_{\mathbf{v}^{\mathbb{C}}}^{\mathbb{C}} \mathbf{u}^{\mathbb{C}}.$$

Proof: By the property of the position map described above, we have that

$$T\,\mathrm{pos}_{\mathbf{p}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \Downarrow \mathbf{u}^{\mathbb{C}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} = \mathbf{Fl}_\lambda^{\mathbf{v}} \Downarrow \mathbf{u} \circ \mathrm{pos}_{\mathbf{p}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} = \mathbf{Fl}_\lambda^{\mathbf{v}} \Downarrow \mathbf{u} \circ \mathbf{Fl}_\lambda^{\mathbf{v}} \circ \mathrm{pos}_{\mathbf{p}},$$

$$\begin{aligned} T\,\mathrm{pos}_{\mathbf{p}} \cdot \nabla_{\mathbf{v}^{\mathbb{C}}}^{\mathbb{C}} \mathbf{u}^{\mathbb{C}} &= T\,\mathrm{pos}_{\mathbf{p}} \circ \partial_{\lambda=0} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \Downarrow \mathbf{u}^{\mathbb{C}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \\ &= \partial_{\lambda=0} \circ T\,\mathrm{pos}_{\mathbf{p}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \Downarrow \mathbf{u}^{\mathbb{C}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \\ &= \partial_{\lambda=0} \mathbf{Fl}_\lambda^{\mathbf{v}} \Downarrow \mathbf{u} \circ \mathbf{Fl}_\lambda^{\mathbf{v}} \circ \mathrm{pos}_{\mathbf{p}} \\ &= \nabla_{\mathbf{v}} \mathbf{u} \circ \mathrm{pos}_{\mathbf{p}}. \end{aligned}$$

The commutation property $T\,\mathrm{pos}_{\mathbf{p}} \circ \partial_{\lambda=0} = \partial_{\lambda=0} \circ T\,\mathrm{pos}_{\mathbf{p}}$ holds by linearity of the tangent map $T_{\boldsymbol{\xi}}\,\mathrm{pos}_{\mathbf{p}} \in BL(\mathbb{T}_{\boldsymbol{\xi}}\mathbb{C};\mathbb{T}_{\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})}\mathcal{S})$ since the curve $\lambda \to (\mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}} \Downarrow \mathbf{u}^{\mathbb{C}} \circ \mathbf{Fl}_\lambda^{\mathbf{v}^{\mathbb{C}}})(\boldsymbol{\xi})$ evolves in the linear space $\mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$ and its image through $T_{\boldsymbol{\xi}}\,\mathrm{pos}_{\mathbf{p}}$ is a curve in the linear space $\mathbb{T}_{\mathrm{pos}_{\mathbf{p}}(\boldsymbol{\xi})}\mathcal{S}$. □

Lemma VIII.4: (Lie brackets) The Lie brackets of $\mathrm{pos}_{\mathbf{p}}$*-related vector fields are* $\mathrm{POS}_{\mathbf{p}}$ *-related,*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{[\mathbf{v}^{\mathbb{C}},\mathbf{u}^{\mathbb{C}}]} & \mathbb{TC} \\ \text{pos}_{\mathbf{p}} \downarrow & & \downarrow T\,\text{pos}_{\mathbf{p}} \\ \mathcal{S} & \xrightarrow{[\mathbf{v},\mathbf{u}]} & \mathbb{T}\mathcal{S} \end{array} \quad \Leftrightarrow \quad [\mathbf{v},\mathbf{u}] \circ \text{pos}_{\mathbf{p}} = T\,\text{pos}_{\mathbf{p}} \circ [\mathbf{v}^{\mathbb{C}},\mathbf{u}^{\mathbb{C}}].$$

Proof: This is a basic property of Lie brackets.[1–3] □

The next result, based on the tensoriality property of the torsion, will be resorted to as an essential ingredient in the proof of Theorem IX.2. An analogous result holds for the curvature of the connection.

Lemma VIII.5: (Torsion of the induced connection) *Let* tors *be the torsion of a linear connection in the ambient manifold* $\mathcal{S}$ *and* $\text{tors}^{\mathbb{C}}$ *the torsion of the induced connection in the configuration manifold* $\mathbb{C}$. *Then*

$$T_{\boldsymbol{\xi}}\,\text{pos}_{\mathbf{p}} \cdot \text{tors}^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\xi}},\mathbf{u}^{\mathbb{C}}_{\boldsymbol{\xi}}) = \text{tors}(\mathbf{v}_{\text{pos}_{\mathbf{p}}(\boldsymbol{\xi})},\mathbf{u}_{\text{pos}_{\mathbf{p}}(\boldsymbol{\xi})}).$$

Proof: By tensoriality, to evaluate the torsion of the connection $\nabla^{\mathbb{C}}$ on any pair of $\mathbb{C}$-vectors $\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\xi}}$, $\mathbf{u}^{\mathbb{C}}_{\boldsymbol{\xi}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$, we may perform an extension of these vectors to vector fields $\mathbf{u}^{\mathbb{C}}$, $\mathbf{v}^{\mathbb{C}} \in \mathrm{C}^1(\mathbb{C};\mathbb{TC})$. Then, from Lemmata VIII.3 and VIII.4, we infer that $T\,\text{pos}_{\mathbf{p}} \circ (\nabla^{\mathbb{C}}_{\mathbf{v}^{\mathbb{C}}}\mathbf{u}^{\mathbb{C}} - \nabla^{\mathbb{C}}_{\mathbf{u}^{\mathbb{C}}}\mathbf{v}^{\mathbb{C}} - [\mathbf{v}^{\mathbb{C}},\mathbf{u}^{\mathbb{C}}]) = (\nabla_{\mathbf{v}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{v} - [\mathbf{v},\mathbf{u}]) \circ \text{pos}_{\mathbf{p}}$, and hence that the torsion vector fields, of $\text{pos}_{\mathbf{p}}$-related vector fields, are $\text{pos}_{\mathbf{p}}$-related

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{tors}^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}},\mathbf{u}^{\mathbb{C}})} & \mathbb{TC} \\ \text{pos}_{\mathbf{p}} \downarrow & & \downarrow T\,\text{pos}_{\mathbf{p}} \\ \mathcal{S} & \xrightarrow{\text{tors}(\mathbf{v},\mathbf{u})} & \mathbb{T}\mathcal{S} \end{array} \quad \Leftrightarrow \quad T\,\text{pos}_{\mathbf{p}} \circ \text{tors}^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}},\mathbf{u}^{\mathbb{C}}) = \text{tors}(\mathbf{v},\mathbf{u}) \circ \text{pos}_{\mathbf{p}}.$$

By tensoriality of the torsion, we have that

$$\text{tors}^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}},\mathbf{u}^{\mathbb{C}})(\boldsymbol{\xi}) = \text{tors}^{\mathbb{C}}(\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\xi}},\mathbf{u}^{\mathbb{C}}_{\boldsymbol{\xi}}),$$

$$\text{tors}(\mathbf{v},\mathbf{u}) \circ \text{pos}_{\mathbf{p}}(\boldsymbol{\xi}) = \text{tors}(\mathbf{v}_{\text{pos}_{\mathbf{p}}(\boldsymbol{\xi})},\mathbf{u}_{\text{pos}_{\mathbf{p}}(\boldsymbol{\xi})}).$$

Hence, by evaluating both members of the relatedness equality at a configuration $\boldsymbol{\xi} \in \mathbb{C}$, we get the result. □

IX. THE LAW OF MOTION IN THE AMBIENT MANIFOLD

Let us assume that the ambient space is an n-D Riemannian manifold $\{\mathcal{S},\mathbf{g}\}$ with volume n-form $\boldsymbol{\mu}$, induced by the metric tensor $\mathbf{g}$. By Lemma VIII.1 a virtual velocity $\delta\mathbf{v}^{\mathbb{C}} \in \mathbb{T}_{\boldsymbol{\xi}}\mathbb{C}$ can be identified with a vector field $\delta\mathbf{v} \in \mathrm{C}^1(\boldsymbol{\Omega}_{\boldsymbol{\xi}};\mathbb{T}\mathcal{S})$ with $\boldsymbol{\tau}_{\mathcal{S}} \circ \delta\mathbf{v} = \mathbf{id}_{\boldsymbol{\Omega}_{\boldsymbol{\xi}}}$ and $\boldsymbol{\Omega}_{\boldsymbol{\xi}} = \boldsymbol{\xi}(\mathcal{B})$.

Then each pair of covector fields $\mathbf{b}_t \in \mathrm{C}^1(\boldsymbol{\Omega}_{\boldsymbol{\xi}};\mathbb{T}^*\mathcal{S})$ (body forces), with $\boldsymbol{\tau}^*_{\mathbb{S}} \circ \mathbf{b}_t = \mathbf{id}_{\boldsymbol{\Omega}_{\boldsymbol{\xi}}}$ and $\mathbf{t}_t \in \mathrm{C}^1(\partial\boldsymbol{\Omega}_{\boldsymbol{\xi}};\mathbb{T}^*\mathcal{S})$ (boundary tractions), with $\boldsymbol{\tau}^*_{\mathbb{S}} \circ \mathbf{t}_t = \mathbf{id}_{\partial\boldsymbol{\Omega}_{\boldsymbol{\xi}}}$, defines a force one-form $\mathbf{f}_t \in \mathbb{T}^*_{\boldsymbol{\xi}}\mathbb{C}$ by

$$\langle \mathbf{f}_t, \delta\mathbf{v}^{\mathbb{C}} \rangle := \int_{\boldsymbol{\Omega}_{\boldsymbol{\xi}}} \langle \mathbf{b}_t, \delta\mathbf{v} \rangle \boldsymbol{\mu} + \oint_{\partial\boldsymbol{\Omega}_{\boldsymbol{\xi}}} \langle \mathbf{t}_t, \delta\mathbf{v} \rangle\, \partial\,\boldsymbol{\mu},$$

where $\delta\mathbf{v}_{\text{pos}_{\mathbf{p}}(\boldsymbol{\xi})} = T_{\boldsymbol{\xi}}\,\text{pos}_{\mathbf{p}} \cdot \delta\mathbf{v}^{\mathbb{C}}_{\boldsymbol{\xi}}$ and $\partial\boldsymbol{\mu} := \boldsymbol{\mu}\mathbf{n}$ is the volume $(n-1)$-form on the boundary $\partial\boldsymbol{\Omega}_{\boldsymbol{\xi}}$, $\mathbf{n}$, being the outward normal.

Let us denote by $\mathbf{m}_t = \rho_t\boldsymbol{\mu}$ the mass form, with $\rho_t \in \mathrm{C}^1(\boldsymbol{\Omega}_t;\mathfrak{R})$, the scalar mass density. In continuum dynamics, the Lagrangian per unit mass at the placement $\boldsymbol{\Omega}_t := \gamma_t(\mathcal{B})$ is a function $L_t \in \mathrm{C}^2(\mathbb{T}_{\boldsymbol{\Omega}_t}\mathcal{S};\mathfrak{R})$. The corresponding Lagrangian on the tangent bundle to the configuration manifold $L^{\mathbb{C}}_t \in \mathrm{C}^2(\mathbb{TC};\mathfrak{R})$ is then defined by the integral,

$$L_t^{\mathbb{C}}(\mathbf{v}_t^{\mathbb{C}}) := \int_{\mathbf{\Omega}_t} (L_t \circ \mathbf{v}_t)\mathbf{m}_t,$$

where $\mathbf{v}_t(\mathrm{pos}_{\mathbf{p}}(\gamma_t)) = T_{\gamma_t}\mathrm{pos}_{\mathbf{p}} \cdot \mathbf{v}_{\boldsymbol{\gamma}}^{\mathbb{C}}(\gamma_t)$. By Lemma VIII.1 the tangent vector field $\mathbf{v}_t \in \mathrm{C}^1(\mathbf{\Omega}_t; \mathbb{T}\mathcal{S})$, spanned by $\mathbf{v}_t(\mathrm{pos}_{\mathbf{p}}(\gamma_t))$ when $\mathbf{p}$ ranges over $\mathcal{B}$, is identified with the tangent vector $\mathbf{v}_{\boldsymbol{\gamma}}^{\mathbb{C}}(\gamma_t) \in \mathbb{T}_{\gamma_t}\mathbb{C}$.

A. Virtual velocity fields

A proper formulation of the law of motion for a continuous body, in an *ambient* finite dimensional Riemannian manifold $(\mathcal{S}, \mathbf{g})$, needs a sufficiently general definition of the linear space of spatial virtual velocity fields on the placement $\mathbf{\Omega}_t := \gamma_t(\mathcal{B})$, at time $t \in I$, along the trajectory in the ambient manifold. To this end, let us give the following definitions. A *patchwork* $\mathrm{Pat}(\mathbf{\Omega}_t)$ is a finite family of open connected, nonoverlapping subsets of $\mathbf{\Omega}_t$, called elements, such that the union of their closures is a covering for $\mathbf{\Omega}_t$. The set of all patchworks of $\mathbf{\Omega}_t$ is a directed set for the relation *finer than* and the coarsest patchwork finer than two given ones $\mathrm{Pat}_1(\mathbf{\Omega}_t)$ and $\mathrm{Pat}_2(\mathbf{\Omega}_t)$ is the *grid*[3] $\mathrm{Pat}_1(\mathbf{\Omega}_t) \wedge \mathrm{Pat}_2(\mathbf{\Omega}_t)$.

The kinematic space $\mathrm{Kin}(\mathbf{\Omega}_t)$ is made up of vector fields $\mathbf{v}_t : \mathbf{\Omega}_t \to \mathbb{T}\mathcal{S}$ which are square integrable with a distributional gradient $\nabla \mathbf{v}_t$ and square integrable in each element of a patchwork $\mathrm{Pat}_{\mathbf{v}_t}(\mathbf{\Omega}_t)$. This space is pre-Hilbert with the positive definite symmetric bilinear form,

$$\int_{\mathrm{Pat}_{(\mathbf{v}_t,\mathbf{w}_t)}(\mathbf{\Omega}_t)} (\mathbf{g}(\mathbf{v}_t, \mathbf{w}_t) + \langle \nabla \mathbf{v}_t, \nabla \mathbf{w}_t \rangle_{\mathbf{g}})\boldsymbol{\mu},$$

where $\mathrm{Pat}_{(\mathbf{v}_t,\mathbf{w}_t)}(\mathbf{\Omega}_t) = \mathrm{Pat}_{\mathbf{v}_t}(\mathbf{\Omega}_t) \wedge \mathrm{Pat}_{\mathbf{w}_t}(\mathbf{\Omega}_t)$ and $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ is the inner product between tensors induced by the metric $\mathbf{g}$. A continuous body at $\mathbf{\Omega}_t$ is defined by a fixed patchwork $\mathrm{Pat}(\mathbf{\Omega}_t)$ and by a closed linear subspace of conforming virtual velocities $\mathrm{Conf}(\mathbf{\Omega}_t) \subset \mathrm{Kin}(\mathbf{\Omega}_t)$, such that all of its vector fields have $\mathrm{Pat}(\mathbf{\Omega}_t)$ as a regularity patchwork. Then $\mathrm{Conf}(\mathbf{\Omega}_t)$ is a Hilbert space for the topology induced by $\mathrm{Kin}(\mathbf{\Omega}_t)$. Since $\mathrm{Conf}(\mathbf{\Omega}_t)$ is a linear space, this definition includes any linear or affine kinematical constraint.

Nonlinear constraints must instead be modeled by suitable constitutive laws described by fiberwise monotone maximal graphs in the Whitney bundle whose fiber is the product of tangent vector and covector spaces based at the same point.[3]

B. Law of motion

The virtual flow $\boldsymbol{\varphi}_{\lambda,t} \in \mathrm{C}^1(\mathbf{\Omega}_t; \mathcal{S})$, dragging a placement $\mathbf{\Omega}_t$ in the ambient space, is $\mathrm{pos}_{\mathbf{p}}$-related to the virtual flow $\boldsymbol{\varphi}_{\lambda,t}^{\mathbb{C}} \in \mathrm{C}^1(\gamma_t; \mathbb{C})$ of the configuration $\gamma_t \in \mathbb{C}$, according to the equality $\boldsymbol{\varphi}_{\lambda,t}(\gamma_t(\mathbf{p})) = \mathrm{pos}_{\mathbf{p}}(\boldsymbol{\varphi}_{\lambda}^{\mathbb{C}}(\gamma_t))$. The virtual velocity $\delta \mathbf{v}_t := \partial_{\lambda=0} \boldsymbol{\varphi}_{\lambda,t} \in \mathrm{C}^1(\mathbf{\Omega}_t; \mathbb{T}_{\mathbf{\Omega}_t}\mathcal{S})$, at the placement $\mathbf{\Omega}_t$, is assumed to fulfill the following condition.

Ansatz IX.1: (Virtual mass conservation) The mass form is dragged by virtual flows,

$$\mathbf{m}_t \circ \boldsymbol{\varphi}_{\lambda,t} := \boldsymbol{\varphi}_{\lambda,t} \uparrow \mathbf{m}_t.$$

Then, along virtual flows, the mass of any sub-body is preserved,

$$\mathcal{L}_{\delta \mathbf{v}_t} \mathbf{m}_t = 0 \Leftrightarrow \partial_{\lambda=0} \int_{\boldsymbol{\varphi}_{\lambda,t}(\mathcal{P})} \mathbf{m}_t = 0, \qquad \forall\, \mathcal{P} \subseteq \mathbf{\Omega}_t.$$

This assumption amounts in defining a proper way of extending the mass form to placements of the body outside the trajectory and mimics the one tacitly made in analytical mechanics in assuming that the material particles retain their mass measure along the variations.

Theorem IX.1: *(Law of motion in the ambient manifold) In a Riemannian manifold* $\{\mathcal{S}, \mathbf{g}\}$*, the law of motion of a continuous dynamical system imposes that the trajectory speed* $\mathbf{v}_t = \partial_{\tau=t} \gamma_{\tau,t} \in \mathrm{Conf}(\mathbf{\Omega}_t)$ *fulfills the variational condition,*

$$\partial_{\tau=t}\int_{\gamma_{\tau,t}(\mathbf{\Omega}_t)} \langle d_{\mathrm{F}}L_\tau \circ \mathbf{v}_\tau, \delta\mathbf{v}_\tau\rangle \mathbf{m}_\tau - \partial_{\lambda=0}\int_{\boldsymbol{\varphi}_{\lambda,t}(\mathbf{\Omega}_t)} L_t(\boldsymbol{\varphi}_{\lambda,t}\uparrow\mathbf{v}_t)\mathbf{m}_t = \int_{\mathbf{\Omega}_t}\langle \mathbf{b}_t, \delta\mathbf{v}_t\rangle \boldsymbol{\mu} + \oint_{\partial\mathbf{\Omega}_t}\langle \mathbf{t}_t, \delta\mathbf{v}_t\rangle\, \partial\,\boldsymbol{\mu},$$

for any virtual flow $\boldsymbol{\varphi}_{\lambda,t} \in \mathrm{C}^1(\mathbf{\Omega}_t;\mathcal{S})$, *at time* $t \in I$, *such that the virtual velocity field* $\delta\mathbf{v}_t = \partial_{\lambda=0}\boldsymbol{\varphi}_{\lambda,t}$ *is conforming and isometric, i.e.,* $\delta\mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t) \cap \mathrm{Rig}(\mathbf{\Omega}_t)$.

Proof: According to Theorem VI.1, the law of motion in the configuration manifold is expressed by the variational condition,

$$\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathrm{C}}(\mathbf{v}_\tau^{\mathrm{C}}), \delta\mathbf{v}_\tau^{\mathrm{C}}\rangle - \partial_{\lambda=0}L_t^{\mathrm{C}}(\boldsymbol{\varphi}_\lambda^{\mathrm{C}}\uparrow\mathbf{v}_t^{\mathrm{C}}) = \langle \mathbf{f}_t(\gamma_t), \delta\mathbf{v}_t^{\mathrm{C}}\rangle.$$

Setting $\mathbf{v}_{\mathrm{pos}_{\mathbf{p}}(\gamma_t)} = T_{\gamma_t}\,\mathrm{pos}_{\mathbf{p}} \cdot \mathbf{v}_{\gamma_t}^{\mathrm{C}}$ and $\delta\mathbf{v}_{\mathrm{pos}_{\mathbf{p}}(\gamma_t)} = T_{\gamma_t}\,\mathrm{pos}_{\mathbf{p}} \cdot \delta\mathbf{v}_{\gamma_t}^{\mathrm{C}}$, we have

$$\langle d_{\mathrm{F}}L_\tau^{\mathrm{C}}(\mathbf{v}_\tau^{\mathrm{C}}), \delta\mathbf{v}_\tau^{\mathrm{C}}\rangle = \int_{\gamma_{\tau,t}(\mathbf{\Omega}_t)} \langle d_{\mathrm{F}}L_\tau(\mathbf{v}_\tau), \delta\mathbf{v}_\tau\rangle \mathbf{m}_\tau.$$

On the other hand, recalling that by assumption $\mathbf{m}_t \circ \boldsymbol{\varphi}_{\lambda,t} := \boldsymbol{\varphi}_{\lambda,t}\uparrow\mathbf{m}_t$, we have

$$L_t^{\mathrm{C}}(\boldsymbol{\varphi}_\lambda^{\mathrm{C}}\uparrow\mathbf{v}_t^{\mathrm{C}}) = \partial_{\lambda=0}\int_{\boldsymbol{\varphi}_{\lambda,t}(\mathbf{\Omega}_t)} L_t(\boldsymbol{\varphi}_{\lambda,t}\uparrow\mathbf{v}_t)\mathbf{m}_t.$$

Substituting, we get the result. □

Each one of the two terms at the left hand side of the law of motion in Theorem IX.1 depends on the choice of the family of virtual flows $\boldsymbol{\varphi}_{\lambda,\tau} \in \mathrm{C}^1(\mathbf{\Omega}_\tau;\mathbb{T}\mathcal{S})$ with index $\tau \in I$. However, Theorem IX.2 proves that the sum of the two terms at time $t \in I$ depends (linearly) only on the virtual velocity at that time, thus defining a bounded linear functional on $\mathrm{Conf}(\mathbf{\Omega}_t) \cap \mathrm{Rig}(\mathbf{\Omega}_t)$. This result, which generalizes Euler's law of motion, makes an essential recourse to the notions of a connection in the ambient manifold and of the induced connection in the infinite dimensional configuration manifold.

Theorem IX.2: *(Generalized Euler's law of motion) Let* ∇ *be a linear connection in the ambient manifold* $\mathcal{S}$, *with parallel transport* $\Uparrow$ *and torsion tors. The law of motion for a trajectory with speed* $\mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t)$ *is expressed by the variational condition,*

$$\partial_{\tau=t}\int_{\gamma_{\tau,t}(\mathbf{\Omega}_t)} \langle d_{\mathrm{F}}L_\tau(\mathbf{v}_\tau), \gamma_{\tau,t}\Uparrow\delta\mathbf{v}_t\rangle \mathbf{m}_\tau - \int_{\mathbf{\Omega}_t}\langle d_{\mathrm{B}}L_t(\mathbf{v}_t), \delta\mathbf{v}_t\rangle\mathbf{m}_t + \int_{\mathbf{\Omega}_t}\langle d_{\mathrm{F}}L_t(\mathbf{v}_t), \mathrm{tors}(\mathbf{v}_t, \delta\mathbf{v}_t)\rangle\mathbf{m}_t = \int_{\mathbf{\Omega}_t}\langle \mathbf{b}_t, \delta\mathbf{v}_t\rangle\boldsymbol{\mu} + \oint_{\partial\mathbf{\Omega}_t}\langle \mathbf{t}_t, \delta\mathbf{v}_t\rangle\,\partial\,\boldsymbol{\mu}$$

for any virtual velocity field $\delta\mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t) \cap \mathrm{Rig}(\mathbf{\Omega}_t)$.

Proof: By Theorem VII.1, the left hand side of the Lagrange law of motion in the configuration manifold, according to the connection ∇^{C}, there induced by the connection ∇ in the ambient manifold, writes

$$\partial_{\tau=t}\langle d_{\mathrm{F}}L_\tau^{\mathrm{C}}(\mathbf{v}_\tau^{\mathrm{C}}), \gamma_{\tau,t}\Uparrow\delta\mathbf{v}_t^{\mathrm{C}}\rangle - \langle d_{\mathrm{B}}L_t^{\mathrm{C}}(\mathbf{v}_t^{\mathrm{C}}), \delta\mathbf{v}_t^{\mathrm{C}}\rangle + \langle d_{\mathrm{F}}L_t^{\mathrm{C}}(\mathbf{v}_t^{\mathrm{C}}), \mathrm{tors}^{\mathrm{C}}(\mathbf{v}_t^{\mathrm{C}}, \delta\mathbf{v}_t^{\mathrm{C}})\rangle.$$

Translating in terms of fields in the ambient manifold, by Lemmata VIII.2 and VIII.5, we have

$$\langle d_{\mathrm{F}}L_\tau^{\mathrm{C}}(\mathbf{v}_\tau^{\mathrm{C}}), \gamma_{\tau,t}\Uparrow\delta\mathbf{v}_t^{\mathrm{C}}\rangle = \int_{\gamma_{\tau,t}(\mathbf{\Omega}_t)}\langle d_{\mathrm{F}}L_\tau(\mathbf{v}_\tau), \gamma_{\tau,t}\Uparrow\delta\mathbf{v}_t\rangle\mathbf{m}_\tau,$$

$$\langle d_{\mathrm{F}}L_t^{\mathrm{C}}(\mathbf{v}_t^{\mathrm{C}}), \mathrm{tors}^{\mathrm{C}}(\mathbf{v}_t^{\mathrm{C}}, \delta\mathbf{v}_t^{\mathrm{C}})\rangle = \int_{\mathbf{\Omega}_t}\langle d_{\mathrm{F}}L_t(\mathbf{v}_t), \mathrm{tors}(\mathbf{v}_t, \delta\mathbf{v}_t)\rangle\mathbf{m}_t,$$

$$\langle d_{\mathrm{B}} L_t^{\mathrm{C}}(\mathbf{v}_t^{\mathrm{C}}), \delta \mathbf{v}_t^{\mathrm{C}} \rangle = \partial_{\lambda=0} L_t^{\mathrm{C}}(\boldsymbol{\varphi}_\lambda^{\mathrm{C}} \Uparrow \mathbf{v}_t^{\mathrm{C}})$$

$$= \partial_{\lambda=0} \int_{\boldsymbol{\varphi}_{\lambda,t}(\boldsymbol{\Omega}_t)} L_t(\boldsymbol{\varphi}_{\lambda,t} \Uparrow \mathbf{v}_t)\mathbf{m}_t$$

$$= \int_{\boldsymbol{\Omega}_t} \partial_{\lambda=0} \boldsymbol{\varphi}_{\lambda,t}\!\downarrow\!(L_t(\boldsymbol{\varphi}_{\lambda,t} \Uparrow \mathbf{v}_t)\mathbf{m}_t)$$

$$= \int_{\boldsymbol{\Omega}_t} \langle d_{\mathrm{B}} L_t(\mathbf{v}_t), \delta \mathbf{v}_t \rangle \mathbf{m}_t + \int_{\boldsymbol{\Omega}_t} L_t(\mathbf{v}_t) \mathcal{L}_{\delta \mathbf{v}_t} \mathbf{m}_t,$$

with the last equality inferred by Leibniz rule. Setting $\mathcal{L}_{\delta \mathbf{v}_t} \mathbf{m}_t = 0$, we get the result. □

C. Stress field

Let us preliminarily observe that, by the linear isomorphism between the spaces $\mathbb{T}_{\mathbf{x}}\mathcal{S}$ and $\mathbb{T}_{\mathbf{x}}^*\mathcal{S}$ provided by the metric tensor $\mathbf{g}$ and the natural identifications $BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}, \mathbb{T}_{\mathbf{x}}\mathcal{S}; \Re) = BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}; \mathbb{T}_{\mathbf{x}}^*\mathcal{S})$ and $BL(\mathbb{T}_{\mathbf{x}}^*\mathcal{S}, \mathbb{T}_{\mathbf{x}}^*\mathcal{S}; \Re) = BL(\mathbb{T}_{\mathbf{x}}^*\mathcal{S}; \mathbb{T}_{\mathbf{x}}\mathcal{S})$, the twice covariant tensor $\boldsymbol{\varepsilon}_t(\mathbf{x}) \in BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}; \mathbb{T}_{\mathbf{x}}^*\mathcal{S})$ and the dual twice contravariant tensor $\boldsymbol{\sigma}_t(\mathbf{x}) \in BL(\mathbb{T}_{\mathbf{x}}^*\mathcal{S}; \mathbb{T}_{\mathbf{x}}\mathcal{S})$ are associated with the operators $\mathbf{T}_t(\mathbf{x}) \in BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}; \mathbb{T}_{\mathbf{x}}\mathcal{S})$ and $\mathbf{D}_t(\mathbf{x}) \in BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}; \mathbb{T}_{\mathbf{x}}\mathcal{S})$ through the relations

$$\boldsymbol{\sigma}_t = \mathbf{T}_t \circ \mathbf{g}^{-1},$$

$$\boldsymbol{\varepsilon}_t = \mathbf{g} \circ \mathbf{D}_t,$$

The pairing between dual tensors and the inner product between operators are defined by linear invariant of the composition $(\boldsymbol{\sigma}_t \circ \boldsymbol{\varepsilon}_t)(\mathbf{x}) \in BL(\mathbb{T}_{\mathbf{x}}\mathcal{S}; \mathbb{T}_{\mathbf{x}}\mathcal{S})$,

$$\langle \boldsymbol{\sigma}_t, \boldsymbol{\varepsilon}_t \rangle = \langle \mathbf{T}_t, \mathbf{D}_t \rangle_{\mathbf{g}} := I_1(\boldsymbol{\sigma}_t \circ \boldsymbol{\varepsilon}_t),$$

whose integral over $\boldsymbol{\Omega}_t$ is the inner product in the Hilbert space $\mathrm{Sqit}(\boldsymbol{\Omega}_t)$ of square integrable tensor fields on $\boldsymbol{\Omega}_t$ which is identified with its dual by the Riesz–Fržchet theorem.[32] In turn, this property is inferred from Korn's second inequality.[33–36] The next theorem shows that the class of virtual velocities considered in Theorem IX.2 may be enlarged to the whole conformity subspace by eliminating the rigidity condition through the introduction of Lagrange's multipliers dual to the stretching. The proof is based on the property that the image, by the differential operator sym ∇, of any closed subspace of the Hilbert space $\mathrm{Conf}(\boldsymbol{\Omega}_t)$ is a closed subspace of $\mathrm{Sqit}(\boldsymbol{\Omega}_t)$.

Theorem IX.3: *(Law of motion in terms of a stress field) There exists at least a stress field* $\mathbf{T}_t \in \mathrm{Sqit}(\boldsymbol{\Omega}_t)$ *such that the law of motion of a continuous dynamical system in the ambient Riemannian manifold* $\{\mathcal{S}, \mathbf{g}\}$ *is equivalent to the variational condition,*

$$\partial_{\tau=t} \int_{\gamma_{\tau,t}(\boldsymbol{\Omega}_t)} \langle d_{\mathrm{F}} L_\tau(\mathbf{v}_\tau), \delta \mathbf{v}_\tau \rangle \mathbf{m}_\tau - \partial_{\lambda=0} \int_{\boldsymbol{\varphi}_{\lambda,t}(\boldsymbol{\Omega}_t)} L_t(\boldsymbol{\varphi}_{\lambda,t} \uparrow \mathbf{v}_t)\mathbf{m}_t$$

$$= \int_{\boldsymbol{\Omega}_t} \langle \mathbf{b}_t, \delta \mathbf{v}_t \rangle \boldsymbol{\mu} + \oint_{\partial \boldsymbol{\Omega}_t} \langle \mathbf{t}_t, \delta \mathbf{v}_t \rangle \, \partial \boldsymbol{\mu} - \int_{\mathrm{Pat}(\boldsymbol{\Omega}_t)} \langle \mathbf{T}_t, \mathrm{sym}\ \nabla\ \delta \mathbf{v}_t \rangle_{\mathbf{g}} \boldsymbol{\mu}$$

for any virtual flow $\boldsymbol{\varphi}_{\lambda,t} \in \mathrm{C}^1(\boldsymbol{\Omega}_t; \mathcal{S})$*, at time* $t \in I$ *whose virtual velocity field is conforming, i.e.,* $\delta \mathbf{v}_t \in \mathrm{Conf}(\boldsymbol{\Omega}_t)$.

Proof: Theorem IX.2 proves that the difference between the right hand side and the left hand side of the equation of motion defines, at each time instant along the trajectory, a bounded linear functional $\mathrm{Fun}_{(\gamma,t)} \in (\mathrm{Conf}(\boldsymbol{\Omega}_t) \cap \mathrm{Rig}(\boldsymbol{\Omega}_t))^*$, on the linear subspace of conforming rigid velocities. The law of motion may then be written as

$$\langle \mathrm{Fun}_{(\gamma,t)}, \delta \mathbf{v}_t \rangle = 0, \quad \forall\ \delta \mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t) \cap \mathrm{Rig}(\mathbf{\Omega}_t).$$

By Euler's formula for the stretching in Lemma IV.1, we have $\mathrm{Rig}(\mathbf{\Omega}_t)=\ker(\mathrm{sym}\ \nabla)$, where ∇ is the Levi–Civita covariant derivative in $\{\mathcal{S},\mathbf{g}\}$ acting on $\mathrm{Kin}(\mathbf{\Omega}_t)$ and $\ker(\bullet)$ stands for *kernel* of $\bullet$. Then, introducing the restriction ∇_{Conf} of ∇ to $\mathrm{Conf}(\mathbf{\Omega}_t)$, we have that $\ker(\nabla_{\mathrm{Conf}}) = \mathrm{Conf}(\mathbf{\Omega}_t) \cap \mathrm{Rig}(\mathbf{\Omega}_t)$. The dual operator $(\mathrm{sym}\ \nabla_{\mathrm{Conf}})^* \in BL(\mathrm{Sqit}(\mathbf{\Omega}_t);\mathrm{Conf}^*(\mathbf{\Omega}_t))$ of the operator $\mathrm{sym}\ \nabla_{\mathrm{Conf}} \in BL(\mathrm{Conf}(\mathbf{\Omega}_t);\mathrm{Sqit}(\mathbf{\Omega}_t))$ is defined by the identity

$$\langle (\mathrm{sym}\ \nabla_{\mathrm{Conf}})^* \delta \mathbf{T}_t, \delta \mathbf{v}_t \rangle := \int_{\mathrm{Pat}(\mathbf{\Omega}_t)} \langle \delta \mathbf{T}_t, \mathrm{sym}\ \nabla\ \delta \mathbf{v}_t \rangle_{\mathbf{g}} \boldsymbol{\mu}, \quad \forall\ \delta \mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t), \quad \forall\ \delta \mathbf{T}_t \in \mathrm{Sqit}(\mathbf{\Omega}_t).$$

Korn's inequality[35] implies that the range $(\mathrm{sym}\ \nabla)(\mathrm{Conf}(\mathbf{\Omega}_t))$ is closed in $\mathrm{Sqit}(\mathbf{\Omega}_t)$ and Banach's closed range theorem[32] assures that the range $(\mathrm{sym}\ \nabla_{\mathrm{Conf}})^*(\mathrm{Sqit}(\mathbf{\Omega}_t))$ of the dual operator is closed in the dual space $\mathrm{Conf}(\mathbf{\Omega}_t)^*$. The law of motion expressed by the variational condition in Theorem IX.1 may then be written as

$$\mathrm{Fun}_{(\gamma,t)} \in (\mathrm{Conf}(\mathbf{\Omega}_t) \cap \ker(\mathrm{sym}\nabla))^\circ = (\ker(\nabla_{\mathrm{Conf}}))^\circ = (\mathrm{sym}\nabla_{\mathrm{Conf}})^*(\mathrm{Sqit}(\mathbf{\Omega}_t)),$$

where $(\bullet)^\circ$ denotes the annihilator, i.e., the closed subspace of bounded linear functionals vanishing on $\bullet$.

This means that there exists a stress tensor field $\mathbf{T}_t \in \mathrm{Sqit}(\mathbf{\Omega}_t)$, such that $\mathrm{Fun}_{(\gamma,t)} = (\mathrm{sym}\ \nabla_{\mathrm{Conf}})^* \mathbf{T}_t$, that is, for all $\delta \mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t)$,

$$\langle \mathrm{Fun}_{(\gamma,t)}, \delta \mathbf{v}_t \rangle = \langle (\mathrm{sym}\ \nabla_{\mathrm{Conf}})^* \mathbf{T}_t, \delta \mathbf{v}_t \rangle = \int_{\mathrm{Pat}(\mathbf{\Omega}_t)} \langle \mathbf{T}_t, \mathrm{sym}\ \nabla\ \delta \mathbf{v}_t \rangle_{\mathbf{g}} \boldsymbol{\mu}.$$

The proof of the converse result is trivial since for conforming rigid virtual velocity fields $\delta \mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t) \cap \mathrm{Rig}(\mathbf{\Omega}_t)$ the variational condition above, being $\mathcal{L}_{\delta \mathbf{v}_t} \mathbf{g} = 0$, gives $\langle \mathrm{Fun}_{(\gamma,t)}, \delta \mathbf{v}_t \rangle = 0$. □

It is straightforward to see that the law of dynamics of Theorem IX.3 implies as a simple corollary a generalized statement of Noether's theorem for continuous dynamical systems.[4] Under the assumption that the mass is conserved and that mass form and Lagrangian are independent of time, the law of motion in Theorem IX.3, setting $\delta \mathbf{v}_t = \mathbf{v}_t$, leads to the following energy conservation law:

$$\partial_{\tau=t} \int_{\gamma_{\tau,t}(\mathbf{\Omega}_t)} E_\tau(\mathbf{v}_\tau) \mathbf{m}_t = \int_{\mathbf{\Omega}_t} \langle \mathbf{b}_t, \mathbf{v}_t \rangle \boldsymbol{\mu} + \int_{\partial \mathbf{\Omega}_t} \langle \mathbf{t}_t, \mathbf{v}_t \rangle\, \partial \boldsymbol{\mu} - \int_{\mathrm{Pat}(\mathbf{\Omega}_t)} \langle \mathbf{T}_t, \mathrm{sym}\ \nabla\ \mathbf{v}_t \rangle_{\mathbf{g}} \boldsymbol{\mu}.$$

The energy $E_t \in \mathrm{C}^2(\mathbb{T}_{\mathbf{\Omega}_t} \mathcal{S}; \Re)$ per unit mass is defined by Legendre transform, $E_t(\mathbf{v}_t) := \langle d_{\mathrm{F}} L_t(\mathbf{v}_t), \mathbf{v}_t \rangle - L_t(\mathbf{v}_t)$.

X. SPECIAL FORMS OF THE LAW OF MOTION

From the general law of motion provided in Theorem IX.3, other expressions valid under special assumptions may be derived. The following one is the extension to continuous systems of the law of dynamics formulated by Poincaré in the context of analytical dynamics for systems described in terms of vector components in a mobile reference frame.[3,5,11]

Theorem X.1: *(Euler–Poincaré law of motion) Let* ∇ *be a connection in the ambient manifold* $\mathcal{S}$*, with a distant parallel transport* $\Uparrow$ *and torsion tors. Let, moreover,* $\mathbf{S}(\mathbf{v}_{\mathbf{x}}) \in \mathrm{C}^1(U(\mathbf{x}); \mathbb{T}\mathcal{S})$ *be the vector field extension of the vector* $\mathbf{v}_{\mathbf{x}} \in \mathbb{T}_{\mathbf{x}} \mathcal{S}$ *in a neighborhood* $U(\mathbf{x}) \subset \mathcal{S}$ *by distant parallel transport. The law of motion is then expressed by the variational condition,*

$$\partial_{\tau=t}\int_{\gamma_{\tau,t}(\boldsymbol{\Omega}_t)}\langle d_{\mathrm{F}}L_\tau(\mathbf{v}_\tau),\gamma_{\tau,t}\Uparrow\delta\mathbf{v}_t\rangle\mathbf{m}_\tau$$

$$-\int_{\boldsymbol{\Omega}_t}\langle d_{\mathrm{B}}L_t(\mathbf{v}_t),\delta\mathbf{v}_t\rangle\mathbf{m}_t-\int_{\boldsymbol{\Omega}_t}\langle d_{\mathrm{F}}L_t(\mathbf{v}_t),[\mathbf{S}(\mathbf{v}_t),\mathbf{S}(\delta\mathbf{v}_t)]\rangle\mathbf{m}_t$$

$$=\int_{\boldsymbol{\Omega}_t}\langle\mathbf{b}_t,\delta\mathbf{v}_t\rangle\boldsymbol{\mu}+\oint_{\partial\boldsymbol{\Omega}_t}\langle\mathbf{t}_t,\delta\mathbf{v}_t\rangle\,\partial\,\boldsymbol{\mu}$$

$$-\int_{\mathrm{Pat}(\boldsymbol{\Omega}_t)}\langle\mathbf{T}_t,\mathrm{sym}\ \nabla\ \delta\mathbf{v}_t\rangle_{\mathbf{g}}\boldsymbol{\mu}$$

for any virtual velocity field $\delta\mathbf{v}_t\in\mathrm{Conf}(\boldsymbol{\Omega}_t)$.

Proof: To evaluate the torsion at a given pair of vectors $\mathbf{u}_\mathbf{x}$, $\mathbf{v}_\mathbf{x}\in\mathbb{T}_\mathbf{x}\mathcal{S}$ we may extend them in a neighborhood $U(\mathbf{x})\subset\mathcal{S}$ by distant parallel transport to a pair of vector fields $\mathbf{S}(\mathbf{u}_\mathbf{x})$, $\mathbf{S}(\mathbf{v}_\mathbf{x})\in\mathrm{C}^1(U(\mathbf{x});\mathbb{T}\mathcal{S})$ so that

$$\mathrm{tors}(\mathbf{u}_\mathbf{x},\mathbf{v}_\mathbf{x}):=\nabla_{\mathbf{u}_\mathbf{x}}\mathbf{S}(\mathbf{v}_\mathbf{x})-\nabla_{\mathbf{v}_\mathbf{x}}\mathbf{S}(\mathbf{u}_\mathbf{x})-[\mathbf{S}(\mathbf{u}_\mathbf{x}),\mathbf{S}(\mathbf{v}_\mathbf{x})]_\mathbf{x}=-[\mathbf{S}(\mathbf{u}_\mathbf{x}),\mathbf{S}(\mathbf{v}_\mathbf{x})]_\mathbf{x},$$

and the result follows from Theorem IX.2. □

The standard expression of Poincaré law[11] for rigid body dynamics is recovered by considering a mobile reference frame $\{\mathbf{e}_i\}$, with structure constants defined by $[\mathbf{e}_i,\mathbf{e}_j]=c_{i,j}^k\mathbf{e}_k$. Indeed, assuming the distant parallel transport $\mathbf{S}(\mathbf{u}_\mathbf{x}):=u_\mathbf{x}^k\mathbf{e}_k$ which keeps constant the components of the vector $\mathbf{u}_\mathbf{x}=u_\mathbf{x}^k\mathbf{e}_k(\mathbf{x})$ in the field of reference frames, the term $[\mathbf{S}(\mathbf{u}_\mathbf{x}),\mathbf{S}(\mathbf{v}_\mathbf{x})]_\mathbf{x}$ becomes $u_\mathbf{x}^k v_\mathbf{x}^j[\mathbf{e}_k,\mathbf{e}_j]_\mathbf{x}=u_\mathbf{x}^k v_\mathbf{x}^j c_{k,j}^i(\mathbf{x})\mathbf{e}_i(\mathbf{x})$.

The *standard* bulk Lagrangian per unit mass is $L_t=K_t+P_t\circ\boldsymbol{\tau}_\mathcal{S}\in\mathrm{C}^2(\mathbb{T}_{\boldsymbol{\Omega}_t}\mathcal{S};\Re)$, where $K_t=\frac{1}{2}\mathbf{g}\circ\mathrm{diag}\in\mathrm{C}^2(\mathbb{T}_{\boldsymbol{\Omega}_t}\mathcal{S};\Re)$ is the positive definite quadratic form of the bulk *kinetic energy* per unit mass, with $\mathrm{diag}(\mathbf{v}):=(\mathbf{v},\mathbf{v})$ so that $K_t(\mathbf{v}_t)=\frac{1}{2}\mathbf{g}(\mathbf{v}_t,\mathbf{v}_t)$, and $P_t\in\mathrm{C}^2(\boldsymbol{\Omega}_t;\Re)$ is the bulk *load potential* per unit mass.

Lemma X.1: Let the ambient manifold $\{\mathcal{S},\mathbf{g}\}$ *be a Riemannian manifold with the Levi–Civita connection* ∇. *Then the scalar fields* $K_t\in\mathrm{C}^2(\mathbb{T}_{\boldsymbol{\Omega}_t}\mathcal{S};\Re)$ *and* $P_t\in\mathrm{C}^2(\boldsymbol{\Omega}_t;\Re)$ *fulfill the relations*

$$\begin{cases}d_{\mathrm{F}}K_t=\mathbf{g}\\ d_{\mathrm{B}}K_t=\frac{1}{2}d_{\mathrm{B}}(\mathbf{g}\circ\mathrm{diag})=0,\end{cases}\qquad\begin{cases}d_{\mathrm{F}}(P_t\circ\boldsymbol{\tau}_\mathcal{S})=0\\ d_{\mathrm{B}}(P_t\circ\boldsymbol{\tau}_\mathcal{S})=TP_t\circ\boldsymbol{\tau}_\mathcal{S}.\end{cases}$$

Then, being $L_t:=K_t+P_t\circ\boldsymbol{\tau}_\mathcal{S}$, *with*

$$K_t(\mathbf{v}_t):=\tfrac{1}{2}\langle d_{\mathrm{F}}L_t(\mathbf{v}_t),\mathbf{v}_t\rangle,$$

$$E_t(\mathbf{v}_t):=\langle d_{\mathrm{F}}L_t(\mathbf{v}_t),\mathbf{v}_t\rangle-L_t(\mathbf{v}_t),$$

we have the relation $E_t=2K_t-L_t=K_t-P_t\circ\boldsymbol{\tau}_\mathcal{S}$.

Proof: Recalling that $\delta\mathbf{v}_t:=\partial_{\lambda=0}\boldsymbol{\varphi}_{\lambda,t}$, by definition of fiber and base derivative, for any $\mathbf{u}_t$, $\mathbf{v}_t$, $\delta\mathbf{v}_t\in\mathbb{T}_{\boldsymbol{\Omega}_t}\mathcal{S}$, with $\boldsymbol{\tau}_\mathcal{S}(\mathbf{u}_t)=\boldsymbol{\tau}_\mathcal{S}(\mathbf{v}_t)=\boldsymbol{\tau}_\mathcal{S}(\delta\mathbf{v}_t)$, we have that

$$\langle d_{\mathrm{F}}K_t(\mathbf{u}_t),\mathbf{v}_t\rangle=\partial_{\varepsilon=0}K_t(\mathbf{u}_t+\varepsilon\mathbf{v}_t)=\partial_{\varepsilon=0}\tfrac{1}{2}\mathbf{g}(\mathbf{u}_t+\varepsilon\mathbf{v}_t,\mathbf{u}_t+\varepsilon\mathbf{v}_t)=\mathbf{g}(\mathbf{u}_t,\mathbf{v}_t),$$

$$\langle d_{\mathrm{B}}K_t(\mathbf{v}_t),\delta\mathbf{v}_t\rangle=\partial_{\lambda=0}\boldsymbol{\varphi}_{\lambda,t}{\downarrow}K_t(\boldsymbol{\varphi}_{\lambda,t}\Uparrow\mathbf{v}_t)=\partial_{\lambda=0}K_t(\boldsymbol{\varphi}_{\lambda,t}\Uparrow\mathbf{v}_t)\circ\boldsymbol{\varphi}_{\lambda,t}=0,$$

$$d_{\mathrm{F}}(P_t\circ\boldsymbol{\tau}_\mathcal{S})(\mathbf{v}_t)\cdot\delta\mathbf{v}_t=TP_t(\boldsymbol{\tau}_\mathcal{S}(\mathbf{v}_t))\cdot T\boldsymbol{\tau}_\mathcal{S}(\mathbf{v}_t)\cdot\bar{\nabla}\mathbf{v}_t\cdot\delta\mathbf{v}_t=0,$$

$$d_{\mathrm{B}}(P_t \circ \boldsymbol{\tau}_{\mathcal{S}})(\mathbf{v}_t) \cdot \delta\mathbf{v}_t = TP_t(\boldsymbol{\tau}_{\mathcal{S}}(\mathbf{v}_t)) \cdot T\boldsymbol{\tau}_{\mathcal{S}}(\mathbf{v}_t) \cdot \mathbf{H}\mathbf{v}_t \cdot \delta\mathbf{v}_t = TP_t(\boldsymbol{\tau}_{\mathcal{S}}(\mathbf{v}_t)) \cdot \delta\mathbf{v}_t.$$

The second equality in the list holds since the Levi–Civita parallel transport in $\{\mathcal{S},\mathbf{g}\}$ preserves the metric, that is,

$$\mathbf{g}(\boldsymbol{\varphi}_{\lambda,t} \Uparrow \mathbf{v}_t, \boldsymbol{\varphi}_{\lambda,t} \Uparrow \mathbf{v}_t) \circ \boldsymbol{\varphi}_{\lambda,t} = \mathbf{g}(\mathbf{v}_t,\mathbf{v}_t).$$

The last two equalities follow from the verticality of the covariant derivative and the fact that the horizontal lift[3] is a right inverse to $T\boldsymbol{\tau}_{\mathcal{S}}$, the tangent map to the projection, so that $T\boldsymbol{\tau}_{\mathcal{S}}(\mathbf{v}_t)\cdot\mathbf{H}\mathbf{v}_t = \mathbf{id}_{\mathbb{T}_{\mathbf{\Omega}_t}\mathcal{S}}$. □

Theorem X.2: *(Euler's law of motion: Special form) Let the Lagrangian per unit mass have the standard form, $L_t = K_t + P_t \circ \boldsymbol{\tau}_{\mathcal{S}} \in \mathrm{C}^2(\mathbb{T}_{\mathbf{\Omega}_t}\mathcal{S};\Re)$, and ∇ be the Levi–Civita connection in the Riemannian ambient manifold $\{\mathcal{S},\mathbf{g}\}$. Then the law of motion writes*

$$\begin{aligned}\partial_{\tau=t}\int_{\gamma_{\tau,t}(\mathbf{\Omega}_t)} \mathbf{g}(\mathbf{v}_\tau, \gamma_{\tau,t} \Uparrow \delta\mathbf{v}_t)\mathbf{m}_\tau &= \int_{\mathbf{\Omega}_t} \langle TP_t(\boldsymbol{\tau}_{\mathcal{S}}(\mathbf{v}_t)), \delta\mathbf{v}_t\rangle \mathbf{m}_t + \int_{\mathbf{\Omega}_t} \langle \mathbf{b}_t, \delta\mathbf{v}_t\rangle \boldsymbol{\mu} + \oint_{\partial\mathbf{\Omega}_t} \langle \mathbf{t}_t, \delta\mathbf{v}_t\rangle\, \partial\boldsymbol{\mu} \\ &\quad - \int_{\mathrm{Pat}(\mathbf{\Omega}_t)} \langle \mathbf{T}_t, \mathrm{sym}\ \nabla\, \delta\mathbf{v}_t\rangle_{\mathbf{g}} \boldsymbol{\mu}\end{aligned}$$

for any virtual velocity field $\delta\mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t)$.

Proof: The result follows from Theorem IX.2 since the Levi–Civita connection is torsion-free, and by Lemma X.1 we have that $\langle d_{\mathrm{B}}L_t(\mathbf{v}_t), \delta\mathbf{v}_t\rangle = \langle TP_t(\boldsymbol{\tau}_{\mathcal{S}}(\mathbf{v}_t)), \delta\mathbf{v}_t\rangle$ and $\langle d_{\mathrm{F}}L_\tau(\mathbf{v}_\tau), \gamma_{\tau,t} \Uparrow \delta\mathbf{v}_t\rangle = \mathbf{g}(\mathbf{v}_\tau, \gamma_{\tau,t} \Uparrow \delta\mathbf{v}_t)$. □

In the Euclidean ambient space, a simple body is defined by the property that conforming isometric virtual velocity fields are simple infinitesimal isometries, expressible as the sum of a *speed of translation* and of an *angular velocity* around a pole. Then we recover the classical Euler's laws for the time rate of variation of momentum and of moment of momentum.

Theorem X.3: *(d'Alembert's law of motion) By conservation of mass the special Euler's law of motion translates into d'Alembert's law,*

$$\begin{aligned}\int_{\mathbf{\Omega}_t} \mathbf{g}(\mathbf{a}_t, \delta\mathbf{v}_t)\mathbf{m}_t &= \int_{\mathbf{\Omega}_t} \langle TP_t(\boldsymbol{\tau}_{\mathcal{S}}(\mathbf{v}_t)), \delta\mathbf{v}_t\rangle \mathbf{m}_t \\ &\quad + \int_{\mathbf{\Omega}_t} \langle \mathbf{b}_t, \delta\mathbf{v}_t\rangle \boldsymbol{\mu} + \oint_{\partial\mathbf{\Omega}_t} \langle \mathbf{t}_t, \delta\mathbf{v}_t\rangle\, \partial\boldsymbol{\mu} - \int_{\mathrm{Pat}(\mathbf{\Omega}_t)} \langle \mathbf{T}_t, \mathrm{sym}\ \nabla\, \delta\mathbf{v}_t\rangle_{\mathbf{g}} \boldsymbol{\mu},\end{aligned}$$

for any virtual velocity field $\delta\mathbf{v}_t \in \mathrm{Conf}(\mathbf{\Omega}_t)$.

Proof: Applying the transport formula and Leibniz rule we get the identity

$$\begin{aligned}\partial_{\tau=t}\int_{\gamma_{\tau,t}(\mathbf{\Omega}_t)} \mathbf{g}(\mathbf{v}_\tau, \gamma_{\tau,t} \Uparrow \delta\mathbf{v}_t)\mathbf{m}_\tau &= \int_{\mathbf{\Omega}_t} \partial_{\tau=t}\gamma_{\tau,t}\!\downarrow\!(\mathbf{g}(\mathbf{v}_\tau, \gamma_{\tau,t} \Uparrow \delta\mathbf{v}_t)\mathbf{m}_\tau) \\ &= \int_{\mathbf{\Omega}_t} \partial_{\tau=t}\mathbf{g}(\mathbf{v}_\tau, \gamma_{\tau,t} \Uparrow \delta\mathbf{v}_t) \circ \gamma_{\tau,t}\mathbf{m}_t + \int_{\mathbf{\Omega}_t} \mathbf{g}(\mathbf{v}_t, \delta\mathbf{v}_t)\partial_{\tau=t}\gamma_{\tau,t}\!\downarrow\!\mathbf{m}_\tau \\ &= \int_{\mathbf{\Omega}_t} \mathbf{g}(\partial_{\tau=t}\gamma_{\tau,t} \Downarrow \mathbf{v}_\tau, \delta\mathbf{v}_t)\mathbf{m}_t + \int_{\mathbf{\Omega}_t} \mathbf{g}(\mathbf{v}_t, \delta\mathbf{v}_t)\mathcal{L}_{t,\mathbf{v}_t}\mathbf{m} \\ &= \int_{\mathbf{\Omega}_t} \mathbf{g}(\mathbf{a}_t, \delta\mathbf{v}_t)\mathbf{m}_t + \int_{\mathbf{\Omega}_t} \mathbf{g}(\mathbf{v}_t, \delta\mathbf{v}_t)\mathcal{L}_{t,\mathbf{v}_t}\mathbf{m},\end{aligned}$$

where $\mathbf{g}(\mathbf{v}_\tau, \gamma_{\tau,t} \Uparrow \delta\mathbf{v}_t) \circ \gamma_{\tau,t} = \mathbf{g}(\gamma_{\tau,t} \Downarrow \mathbf{v}_\tau, \delta\mathbf{v}_t)$ since Levi–Civita connection is metric. Imposing conservation of mass,[3] $\mathcal{L}_{t,\mathbf{v}_t}\mathbf{m} := \partial_{\tau=t}\mathbf{m}_\tau + \mathcal{L}_{\mathbf{v}_t}\mathbf{m}_t = 0$, the result follows from Theorem X.2. □

A. Boundary value problems

The basic tool in boundary value problems governed by a linear partial differential operator diff, of order n, is Green's formula of integration by parts, which formally may be written as

$$\int_{\text{Pat}(\boldsymbol{\Omega}_t)} \langle \bullet, \text{Diff}\circ\rangle \boldsymbol{\mu} = \int_{\text{Pat}(\boldsymbol{\Omega}_t)} \langle \text{AdjDiff}\bullet, \circ\rangle \boldsymbol{\mu} + \oint_{\partial\text{Pat}(\boldsymbol{\Omega}_t)} \langle \text{Flux}\bullet, \text{Val}\circ\rangle\, \partial\boldsymbol{\mu},$$

where $\boldsymbol{\Omega}_t$ is a submanifold of a finite dimensional Riemannian space $\{\mathcal{S}, \mathbf{g}\}$, $\text{Pat}(\boldsymbol{\Omega}_t)$ is a fixed patchwork, $\partial\text{Pat}(\boldsymbol{\Omega}_t)$ is its boundary, $\partial\boldsymbol{\mu}$ is the volume form induced on the surfaces $\partial\text{Pat}(\boldsymbol{\Omega}_t)$ by the volume form in $\mathcal{S}$, and all the integrals are assumed to take a finite value. The differential operator adjdiff, of order n, is the *formal adjoint* of diff. The boundary integral acts on the duality pairing between the two fields Flux• and Val∘, with the differential operators flux and val, being n-tuples of normal derivatives of order from 0, to $n-1$, in inverse sequence, so that the duality pairing is the sum of n terms, whose kth term is the pairing of normal derivatives of two fields, respectively, of order k and $n-1-k$.

Boundary value problems are characterized by the property that the closed linear subspace $\text{Conf}(\boldsymbol{\Omega}_t)$ of conforming test fields includes the whole linear subspace ker(val) of test fields in $\text{Kin}(\boldsymbol{\Omega}_t)$, with vanishing boundary values on $\partial\text{Pat}(\boldsymbol{\Omega}_t)$, i.e.,

$$\ker(\text{Val}) \subseteq \text{Conf}(\boldsymbol{\Omega}_t).$$

Let us assume that the force virtual power $\langle \mathbf{f}_t, \delta\mathbf{v}_t\rangle$ is expressed in terms of forces per unit volume $\mathbf{b} \in \text{Sqiv}(\boldsymbol{\Omega}_t)$ (Sqiv := square integrable vector fields) and of forces per unit area (tractions) $\mathbf{t} \in \text{Sqiv}(\partial\text{Pat}(\boldsymbol{\Omega}_t))$, so that the force virtual power is given by

$$\int_{\boldsymbol{\Omega}_t} \langle \mathbf{f}_t, \delta\mathbf{v}_t\rangle \mathbf{m}_t := \int_{\boldsymbol{\Omega}_t} \mathbf{g}(\mathbf{b}_t, \delta\mathbf{v}_t)\boldsymbol{\mu} + \oint_{\partial\text{Pat}(\boldsymbol{\Omega}_t)} \mathbf{g}(\mathbf{t}_t, \delta\mathbf{v}_t)\, \partial\boldsymbol{\mu}.$$

d'Alembert's law may then be rewritten as

$$\int_{\boldsymbol{\Omega}_t} \mathbf{g}(\nabla_{\mathbf{v}_t}\mathbf{v}_t, \delta\mathbf{v}_t)\mathbf{m}_t + \int_{\text{Pat}(\boldsymbol{\Omega}_t)} \langle \mathbf{T}_t, \text{sym}\ \nabla\ \delta\mathbf{v}_t\rangle_{\mathbf{g}} \boldsymbol{\mu} = \int_{\boldsymbol{\Omega}_t} \mathbf{g}(\mathbf{b}_t, \delta\mathbf{v}_t)\boldsymbol{\mu} + \oint_{\partial\text{Pat}(\boldsymbol{\Omega}_t)} \mathbf{g}(\mathbf{t}_t, \delta\mathbf{v}_t)\, \partial\boldsymbol{\mu},$$

and a standard localization procedure[3] leads to the differential equation,

$$-\text{Div}\ \mathbf{T}_t = \mathbf{b}_t - \rho_t \cdot \mathbf{g} \circ \nabla_{\mathbf{v}_t}\mathbf{v}_t \quad \text{in}\ \ \text{Pat}_\infty(\boldsymbol{\Omega}_t),$$

and the boundary conditions on the jump $[[\mathbf{T}_t\mathbf{n}]]$ across the boundary of the domain $\boldsymbol{\Omega}_t$ and across the interfaces of the patchwork $\text{Pat}_\infty(\boldsymbol{\Omega}_t)$,fulfill the conditions

$$\mathbf{T}_t\mathbf{n} \in \mathbf{t} + \text{Conf}^\circ \quad \text{on}\ \ \boldsymbol{\Omega}_t,$$

$$[[\mathbf{T}_t\mathbf{n}]] \in \mathbf{t}^+ + \mathbf{t}^- + \text{Conf}^\circ \quad \text{on}\ \ \text{Sing}(\text{Pat}_\infty(\boldsymbol{\Omega}_t)),$$

where the fields $\mathbf{t}$ of surficial forces are taken to be zero outside their domain of definition and Pat_∞ denotes a patchwork sufficiently fine for the statement at hand.

XI. CONSTITUTIVE RELATIONS

As briefly pointed out here and there in the previous sections, the basic distinguishing feature of continuum dynamics in comparison with analytical dynamics is the need for a specification of the mechanical material behavior by means of a well-posed set of relations between suitable stress and strain measures and/or their rates which, in most engineering applications, are nonlinear and time dependent. Under these additional relations, uniqueness of the dynamical process is expected. We will not enter in a detailed discussion on the modeling of materials behavior, since it will not add by itself new light to notion and methods peculiar of dynamics. Anyway, for comparison sake, we show hereafter in some detail how an hyperelastic behavior in finite deformations may be implemented. Let us adopt as strain measure, associated with a displacement $\gamma_{\tau,t} \in C^1(\mathbf{\Omega}_t;\mathbf{\Omega}_\tau)$ from the placement $\mathbf{\Omega}_t$ to $\mathbf{\Omega}_\tau$, the Green deformation tensor field: $\mathbf{G}_{\gamma_{\tau,t}} := \frac{1}{2}(\gamma_{\tau,t}\downarrow \mathbf{g}-\mathbf{g})$, on $\mathbf{\Omega}_t$, pointwise defined at $\mathbf{x} \in \mathbf{\Omega}_t$ by

$$(2\mathbf{G}_{\gamma_{\tau,t}} + \mathbf{g})_{\mathbf{x}}(\mathbf{a},\mathbf{b}) = (\gamma_{\tau,t}\downarrow\mathbf{g})_{\mathbf{x}}(\mathbf{a},\mathbf{b}) := \mathbf{g}_{\gamma_{\tau,t}(\mathbf{x})}(T\gamma_{\tau,t}\cdot\mathbf{a}, T\gamma_{\tau,t}\cdot\mathbf{b}), \quad \forall\, \mathbf{a},\mathbf{b} \in \mathbb{T}_{\mathbf{x}}\mathcal{S}.$$

Setting $\mathbf{g}\circ\mathbf{D}_t = \frac{1}{2}\mathcal{L}_{\delta\mathbf{v}_t}\mathbf{g} = \text{sym}\,\nabla\,\delta\mathbf{v}_t$ and being $\partial_{\lambda=0}\frac{1}{2}(\boldsymbol{\varphi}_{\lambda,t}\circ\gamma_{t,s})\downarrow\mathbf{g} = \mathbf{g}\circ(d\mathbf{G}(\gamma_{t,s})\cdot\delta\mathbf{v}_t)$, the Lie derivative formula,

$$\partial_{\lambda=0}(\boldsymbol{\varphi}_{\lambda,t}\circ\gamma_{t,s})\downarrow\mathbf{g} = \gamma_{t,\sigma s}\downarrow\mathcal{L}_{\delta\mathbf{v}_t}\mathbf{g},$$

gives $d\mathbf{G}(\gamma_{t,s})\cdot\delta\mathbf{v}_{t,s} = \gamma_{t,s}\downarrow\mathbf{D}_t$, where $\delta\mathbf{v}_{t,s} := \delta\mathbf{v}_t\circ\gamma_{t,s}$. The global virtual power performed by the stress field $\mathbf{T}_t$, for the virtual stretching field $\mathbf{D}_t$ on the placement $\mathbf{\Omega}_t$, may be evaluated at a reference placement $\mathbf{\Omega}_s$ by the equality

$$\int_{\text{Pat}(\mathbf{\Omega}_s)} \langle \mathbf{S}_{t,s}, d\mathbf{G}(\gamma_{t,s})\cdot\delta\mathbf{v}_{t,s}\rangle_{\mathbf{g}}\boldsymbol{\mu} = \int_{\text{Pat}(\mathbf{\Omega}_t)} \langle \mathbf{T}_t,\mathbf{D}_t\rangle_{\mathbf{g}}\boldsymbol{\mu},$$

which holds for the symmetric Piola–Kirchhoff stress $\mathbf{S}_{t,s} = J(\gamma_{t,s}\downarrow\mathbf{T}_t)$, where J is the Jacobian defined by $J\boldsymbol{\mu} = \gamma_{t,s}\downarrow\boldsymbol{\mu}$. The hyperelastic behavior postulates the existence of a differentiable potential elastic energy per unit volume w, which is a convex function of the Green strain measured from an elastically underformed material local geometry. Then $\mathbf{S}_{t,s}(\mathbf{x}) = dw(\mathbf{G}(\gamma_{t,s}))_{\mathbf{x}}$, and hence

$$\langle \mathbf{S}_{t,s}, d\mathbf{G}(\gamma_{t,s})\cdot\delta\mathbf{v}_{t,s}\rangle_{\mathbf{g}} = \langle dw(\mathbf{G}(\gamma_{t,s})), d\mathbf{G}(\gamma_{t,s})\cdot\delta\mathbf{v}_{t,s}\rangle_{\mathbf{g}} = d(w\circ\mathbf{G})(\gamma_{t,s})\cdot\delta\mathbf{v}_{t,s} = \partial_{\lambda=0}(w\circ\mathbf{G})(\boldsymbol{\varphi}_{\lambda,t}\circ\gamma_{t,s}).$$

The global elastic stress virtual power is then given by the integral

$$\int_{\text{Pat}(\mathbf{\Omega}_s)} \langle \mathbf{S}_{t,s}, d\mathbf{G}(\gamma_{t,s})\cdot\delta\mathbf{v}_{t,s}\rangle_{\mathbf{g}}\boldsymbol{\mu} = \partial_{\lambda=0}\int_{\text{Pat}(\mathbf{\Omega}_s)} (w\circ\mathbf{G})(\boldsymbol{\varphi}_{\lambda,t}\circ\gamma_{t,s})\boldsymbol{\mu}.$$

More complex constitutive relations, for instance, those describing plastic or viscous behaviors, may be taken into account by implementing the relevant model of material response.

XII. CONCLUSIONS

The treatment of continuum dynamics developed in the paper extends notions and results of geometric analytical dynamics to dynamical systems whose configuration manifold is infinite dimensional. Up to now, most formulations of dynamics, including the ones based on a clear intrinsic geometric approach, have been strongly tied to a presentation of the matter which makes essential reference to Newtonian particle mechanics, according to the Lagrangian or Hamiltonian formulation (see, e.g., Refs. 8, 21, and 23). This fact witnesses that the extension to continua is not a trivial one. On the other hand there is no doubt that rigid body dynamics should be inferred from the dynamics of deformable bodies by imposing the rigidity constraint. It is also apparent that the dynamics of discrete models is to be deduced by discretization of a continuous one. In fact, in defining a discrete model without reference to a parent continuous one, a basic difficulty arises when describing the constitutive behavior, and this is because the continuous Cauchy model is the universally accepted standard for theoretical and experimental analyses of materials.

The treatment developed in this paper is based on a geometric definition of the action principle and is innovative under several aspects. It has been shown that, due to the special geometric feature of the configuration manifold for continuous dynamical systems, the law of dynamics is independent of the Banach topology there introduced. Moreover a connection in the configuration manifold is induced in a natural way by the one defined in the ambient manifold. These results are in accordance with the physical requirement that the dynamics of continuous systems should be affected only by the geometric structure of the finite dimensional ambient manifold. The relations between the descriptions of dynamics in the two manifolds have been illustrated in detail. The introduction of a linear connection in the ambient manifold, the assumption of mass conservation along virtual motions, and the result concerning the torsion of the induced connection in the configuration manifold lead eventually to a (very) generalized form of Euler's law of dynamics.

The ansatz of virtual mass conservation expresses a natural assumption which is tacitly made in classical analytical dynamics where, in performing the variations, the mass of the particles is taken as constant. Euler's law of dynamics shows that, at any time instant, the variational statement consists in the vanishing of a bounded linear functional on a closed subspace of the Hilbert space of conforming test velocities. This result opens the way to the introduction of a stress field as Lagrange's multiplier of the rigidity constraint and to the formulation of a variational statement suitable for the dynamics of deformable bodies.

The theory leads to the following main results. The classical Euler's and d'Alembert's laws of analytical dynamics, and the related Poincaré's law, are generalized to variational statements governing the dynamics of continuous bodies undergoing motions in a finite dimensional Riemannian ambient manifold. In the context of continuum dynamics, the generalized Lagrange's law of dynamics, formulated in Theorem VII.1, holds in the infinite dimensional configuration manifold. The same is true for Hamilton's law of dynamics.[3,5] Its counterpart in the finite dimensional ambient manifold is the generalized Euler's law, formulated in Theorem IX.2. Special forms are deduced by considering special Lagrangians or special connections in the ambient manifold. A peculiar feature of Continuum dynamics as a field theory is that the governing laws must be stated in variational form.

It is interesting to compare the approach developed in this paper with other treatments of continuum dynamics. The one developed in Ref. 27, Sec. 5.4, is based on the coordinate expression of Lagrange's law in terms of a Lagrangian functional defined, by means of a local chart, on an infinite dimensional Banach model space of the configuration manifold. This Lagrangian functional is the integral over a reference placement of a Lagrangian scalar field per unit volume which is assumed to be a differentiable function of position, speed, and deformation gradient. Since a referential description is adopted, the ansatz of virtual mass conservation is implictly and tacitly assumed. The formulation adopted in Ref. 27 is not intrinsic and is admittedly limited to nonlinear elasticity. Its extension, to include other constitutive behaviors and to consider connections other than the one induced by the local charts, is not at all straightforward. Our choice has been instead to start from the extremality principle for the action and to develop a completely intrinsic approach in the general context of Riemannian ambient manifolds. The Lagrangian scalar field is defined per unit mass in the actual placement so that the standard Lagrangian is given by the kinetic energy per unit mass, i.e., one-half the squared speed of motion (force potentials do not play a significant role here). The introduction of stress fields as Lagrange multipliers dual to the stretching of conforming virtual velocities, measured by one-half the Lie derivative of the metric tensor, opens the way for the implementation of constitutive laws according to well-developed procedures of continuum mechanics.[3] Our formulation of continuum dynamics has the generality sufficient to perform the analysis of continuous structural models of engineering interest, including nonlinear constitutive models which describe elastoplastic, elastoviscoplastic, phase transition, and dissipative behaviors.

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