Manfredi Romano

SCRITTI SCELTI

ISTITUTO DI SCIENZA DELLE COSTRUZIONI
UNIVERSITA' DEGLI STUDI DI CATANIA

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Le doti umane e la personalità scientifica del prof. Manfredi Romano lasceranno un ricordo indelebile in quanti ebbero occasione di avere con Lui un quotidiano rapporto di collaborazione.

L'improvvisa scomparsa ha interrotto il Suo impegno appassionato quando, forse, più fecondo ed incisivo poteva essere il Suo contributo.

La vivacità e l'apertura nel confronto delle idee ed il profondo spirito critico che Egli coltivò nella quotidiana attività accademica, resterà d'esempio per tutti.

Questo volume, che raccoglie una collezione di Suoi scritti, vuole essere una concreta testimonianza della stima e del prestigio che Egli seppe dare alla Istituzione Universitaria con la Sua attività scientifica e con la Sua opera di Maestro.

INDICE

RICORDO DI MANFREDI

In un pomeriggio di settembre del 1973 arrivavo, un po' stanco, a Zakopane in Polonia, dopo un lungo viaggio in macchina da Varsavia, con l'amico Henryk Żorski e sua moglie Maria.

Il giorno dopo avrebbe avuto inizio il Convegno della Società di Meccanica Polacca.

Avevo appena completato le formalità del *check in* al *bureau* dell'elegante albergo, dove erano stati ospitati i congressisti, e mi avviavo verso la stanza che mi era stata destinata, quando una bella ragazza bionda mi si fece incontro ed in un italiano quasi perfetto mi disse: ''Ben arrivato professore, permetta mi presenti: sono sua cugina Silvia Romano''. Rimasi attonito! Incontrare una mia cugina a Zakopane non era un'eventualità che avevo previsto. Risposi cortesemente che la lista delle mie cugine, di sangue o acquisite, era cospicua, ma la tenevo ben aggiornata: non mi constava che lei ne facesse parte; non mi sarebbe certo sfuggita! Un giovane aveva assistito con aria divertita alla scena. Era certamente un bel ragazzo, bruno e dagli occhi espressivi ed intelligenti. Con inequivocabile inflessione partenopea mi disse: ''Professore, posso spiegarle tutto, sono Manfredi Romano, primo cugino di Bruna Romano, moglie di Pippo Carbonaro, suo primo cugino. E questa è Silvia, mia moglie''. Improvvisamente compresi quel qualcosa di familiare nella sua fisionomia che mi aveva colpito non appena lo avevo visto: la straordinaria rassomiglianza con Bruna, sposa di Pippo, figlio di una mia zia materna.

Parlando, Manfredi aveva cinto con il braccio le spalle di Silvia. Erano molto diversi fra loro e si vedeva che provenivano da due paesi diversi, ma formavano un'assai bella coppia. Mi piacquero subito.

Cominciò così un'esperienza umana che per me è stata di grande valore. Si stabilì fra loro e me un'amicizia sincera, cui, poche settimane dopo, si associò, con entusiasmo pari al mio, Matelda, mia moglie, alla quale volli far conoscere, appena possibile, i nostri nuovi ''cugini''.

Ma quello che rese saldo il mio legame con Manfredi fu l'accorgermi che, oltre a straordinarie doti morali, di gentilezza e di bontà, possedeva un talento scientifico fuori dal comune. Già nei giorni di Zakopane, durante i quali fummo inseparabili, constatai di trovarmi accanto ad uno studioso di classe.

Aveva una cultura eccezionale per un giovane della sua età e non solo nel campo della Scienza delle Costruzioni e della Meccanica, dove erano rivolti i suoi interessi, ma anche in quello dell'Analisi Matematica, che sapeva adoperare con una maestria inconsueta in un ''ingegnere''.

A quel tempo io lavoravo molto nella teoria del calcolo degli autovalori e Manfredi comprese subito che i miei risultati ed i miei metodi potevano tornargli utili in problemi di stabilità delle strutture, cui egli era interessato. Doveva, però, prima estendere la mia teoria, in modo da coprire certi casi che essa, così come io l'avevo elaborata, lasciava scoperti. Riuscì in questa impresa magnificamente e le sue ricerche dettero luogo al bel lavoro "*On a class of buckling problems in the theory of elastic structures*", uno dei contributi più importanti che siano stati portati alla teoria degli "invarianti ortogonali" per il calcolo degli autovalori. Ma la nostra collaborazione non si limitò a questo solo campo. Anche la teoria dei "problemi unilaterali" interessò Manfredi ed anche in questo difficile settore seppe portare apprezzabili contributi.

Sono vivi nella mia memoria i tanti incontri avuti con lui. In Convegni in Italia o in Polonia, nei quali, spesso, disertavamo le sedute ufficiali per parlare delle "nostre" cose. E le vacanze trascorse insieme a San Martino di Castrozza o sulla splendida costiera amalfitana. E le sue frequenti visite a Roma. Ma con particolare emozione ricordo un bel pomeriggio di sole a Napoli, allorché Matelda ed io reggevamo a turno, assai emozionati, un piccolo, delizioso esserino: Alessandra, la primogenita di Manfredi e Silvia, che essi avevano voluto tenessimo a battesimo.

Poi la gioia di tutti noi per la meritata conquista, da parte di Manfredi, della cattedra universitaria. La sua nomina nella Università di Catania. La scelta di questa sede suscitò in me *mixed feelings*. Da un lato, il dispiacere che il caro Manfredi si allontanasse vieppiù da Roma. Dall'altro, la soddisfazione di saperlo nella mia amata città d'origine e professore in quel glorioso *Siculorum Gymnasium* dove, tantissimi anni prima, avevo, adolescente, iniziato i miei studi universitari.

Inevitabilmente, i nostri rapporti un po' si allentarono. Ma non vennero meno le lunghe telefonate che di tanto in tanto ci scambiavamo.

Poi, improvvisa, la tragedia.

La drammatica telefonata di Bruna. Lo stordimento delle prime ore, che quasi non lasciava sentire il dolore. Il continuo iterarsi nella mente della frase: "Manfredi è morto!", talché si finiva quasi per perderne il senso.

In un piovoso mattino d'autunno l'estremo saluto in una Chiesa di Napoli. Poi il cordoglio. La pena. Il silenzio.

Amici, Colleghi ed il fratello Giovanni hanno voluto, raccogliendo in questo volume alcuni dei suoi più significativi lavori scientifici, onorare la Memoria di Manfredi. Chi leggerà questi scritti avrà certamente chiara l'idea delle sue non comuni qualità di studioso e di ricercatore.

Ma da essi certo non trasparirà la grande bontà e gentilezza del suo animo, la disponibilità verso gli altri, la sua gioia di vivere, la sua generosità.

Sono cose, queste, che quanti lo hanno conosciuto ed amato, custodiranno, con cura gelosa e con infinito rimpianto, nel profondo dei loro cuori.

Gaetano Fichera

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PUBBLICAZIONI

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FACOLTÀ DI INGEGNERIA
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MANFREDI ROMANO

ON LEON'S CRITERION

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ON LEON'S CRITERION

Manfredi Romano *

SOMMARIO: Si richiama la teoria delle superfici limiti di resistenza per materiali isotropi nello spazio delle componenti principali di tensione. Tale resistenza è intesa nei riguardi dei fenomeni di rottura duttile o fragile che possono intervenire in alternativa per uno stesso materiale in dipendenza dei diversi tipi di stato tensionale. Si introduce una trasformazione di coordinate che risulta particolarmente utile per il confronto delle varie condizioni limiti con quella derivante dalla teoria della curva intrinseca. In particolare si fa riferimento al criterio di Stassi, certo il più attendibile fra quelli che prendono in considerazione la dipendenza della condizione limite dal valore della tensione principale intermedia. La teoria della curva intrinseca viene generalizzata attraverso la modifica di Leon, che interpreta il fenomeno della crisi per distacco. Si mostra come la teoria di Griffith della rottura fragile conduca ad una condizione limite che si può considerare un caso particolare di quella derivante dalla teoria di Leon. Per confronto con alcuni dati sperimentali si verifica che questa ultima teoria bene si presta allo studio dei fenomeni di rottura fragile. Si propone una generalizzazione che, tenendo conto dell'influenza della tensione principale intermedia, consente di ottenere un buon accordo con l'esperienza anche nel campo di rottura duttile, pur conservando le caratteristiche della teoria di Leon in quello della rottura fragile.

SUMMARY: The theory of limit surfaces of resistance for isotropic materials in the space of the principal stress components is recalled. This resistance is viewed in terms of the phenomena of ductile or brittle failure that may arise in alternation for a given material, depending on the type of stress state. A transformation of coordinates is introduced that proves to be very useful for comparing the various limit conditions with that deriving from intrinsic curve theory. Reference is made to the criterion of Stassi, certainly the most reliable of the criteria that allow for dependence of the limit condition on the value of the principal intermediate stress. The theory of the intrinsic curve is generalised by means of Leon's modification, which interprets the phenomenon of tensile fracture. It is shown how Griffith's theory of brittle fracture leads to a limit condition that may be regarded as a special case of the one deriving from Leon's theory. Comparison with some experimental data shows that Leon's theory is well suitable for the study of brittle fracture. A generalisation that takes account of the influence of the principal intermediate stress and so permits good agreement with the experimental data in the field of ductile failure too, whilst conserving the characteristics of Leon's theory in brittle fracture, is proposed.

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1. General considerations.

For isotropic materials and assuming independence of the loading path, that is for ever-increasing statically applied loads, the attainment of the limit state of the material depends solely on the principal stresses (σ_ξ, σ_η, σ_ζ), which will be denoted in general terms as σ_p. The limit condition is therefore expressed analytically in the form:

$$f(\sigma_\xi, \sigma_\eta, \sigma_\zeta) = 0 . \tag{1}$$

Let f be continuous and uniform. The assumption of isotropy also implies that f must be invariant with respect to any permutations of its variables. With regard to the geometric representation of $f = 0$ in the system of coordinates $S_p(\sigma_\xi, \sigma_\eta, \sigma_\zeta)$, it should be noted that, as it has to be symmetric, for the reason stated, with respect to the planes $\sigma_\xi = \sigma_\eta$, $\sigma_\eta = \sigma_\zeta$, $\sigma_\zeta = \sigma_\xi$, it will be endowed with symmetry at 120° with respect to the trisector t of the positive octant. Because of this property it can be more easily represented in the S_p^* defined by the transformation:

	σ_ξ	σ_η	σ_ζ
σ_ξ^*	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
σ_η^*	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$-\sqrt{\frac{2}{3}}$
σ_ζ^*	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$.

2)

The axis of the σ_ζ^* values is thus coincident with t (see Fig. 1).

The representation of $f = 0$ in this system proves necessary when we wish to study its intersections with planes of equation $t =$ const. (Meldahl sections). But it has the snag of being an indirect evaluation of σ_p and hence of the stress state.

In S_p the three planes of symmetry of $f = 0$, all belonging to the set of planes containing the straight line t, divide the space into six regions (dihedral angles of 60°), in each of which is valid one of the six relations obtained by sub-

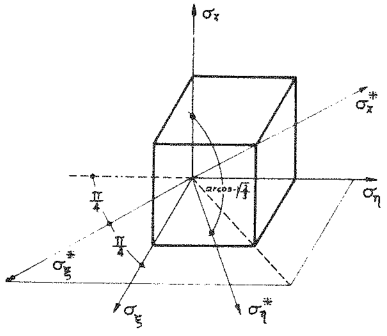

Fig. 1.

stituting the σ_p values, according to the possible arrangements, in:

$$\sigma_M \geqslant \sigma_i \geqslant \sigma_m \tag{3}$$

where σ_M, σ_i, σ_m denote the maximum, intermediate and minimum values of σ_p (Fig. 2). Because of its properties

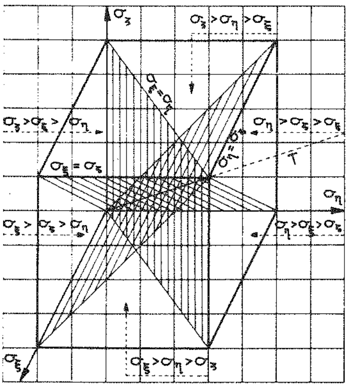

Fig. 2.

of symmetry, $f = 0$ is completely determinable when it is known in only one of the six regions defined by (3), that is when the equation:

$$f(\sigma_M , \sigma_i , \sigma_m) = 0 \tag{4}$$

is assigned in the corresponding part of S_p (Fig. 3).

It is interesting to study the limit condition (4) in a different frame, because that enables us, as we shall see, to relate the various criteria of strength of materials to the theory of the intrinsic curve and to evidence the role of the variability of σ_i in the determination of the limit state.

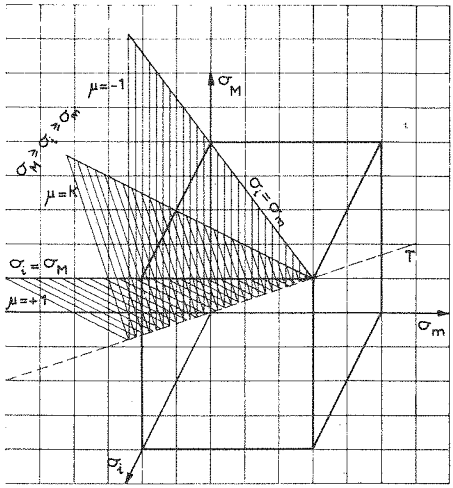

Fig. 3.

The representation of (4) in the new frame S_c defined by the transformation:

$$\begin{aligned} p &= \frac{\sigma_M + \sigma_m}{2} \\ r &= \frac{\sigma_M - \sigma_m}{2} \\ \mu &= \frac{2\sigma_i - \sigma_M - \sigma_m}{\sigma_M - \sigma_m} \end{aligned} \tag{5}$$

will be used, with the inverse

$$\begin{aligned} \sigma_M &= p + r \\ \sigma_i &= p + \mu r \\ \sigma_m &= p - r \,. \end{aligned} \tag{6}$$

By means of (6), (4) is transformed into

$$F(p, r, \mu) = 0 \,. \tag{7}$$

In (5) and (6) p and r represent respectively the abscissa of the center and the radius of the maximum, principal circle of Mohr, and μ is the parameter of Lode, which supplies σ_i once σ_M and σ_m have been assigned. From (3) and (5) we obtain:

$$-1 \leqslant \mu \leqslant +1 \,. \tag{8}$$

From the second of (5) we get:

$$r \geqslant 0 \,. \tag{9}$$

The valid region of the frame $(\sigma_M, \sigma_i, \sigma_m)$ is transformed by means of (6) into the region of space p, r, μ bound by the two planes $\mu = 1$ and $\mu = -1$ and the plane $r = 0$ (the zone lying in the half-space $r \geqslant 0$ must obviously be chosen). As shown by figures 3 and 4, to each half-plane

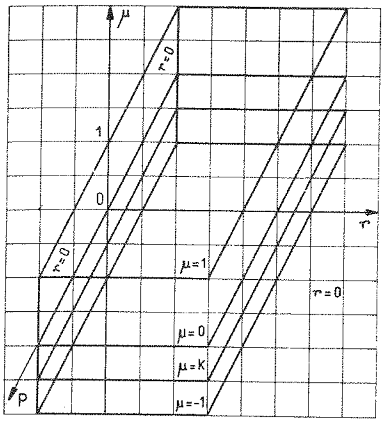

Fig. 4.

$\mu = K$ in S_c there corresponds in the frame $(\sigma_M, \sigma_i, \sigma_m)$ a half-plane of the set of planes which contain the straight line t of equation:

$$(1 + \mu)\sigma_M - 2\sigma_i + (1 - \mu)\sigma_m = 0 \qquad (10)$$

(10) is obtained directly from the third of (5).

As regards the loci $\mu = \text{const.}$ in any given plane of Meldahl (of equation $t = \text{const.}$), assuming $\sigma_\xi = \sigma_M$, $\sigma_\eta = \sigma_i$, $\sigma_\zeta = \sigma_m$ and transforming (10) by means of (2), we have:

$$(1 + \mu)\left(\frac{\sigma_\xi^*}{\sqrt{2}} + \frac{\sigma_\eta^*}{\sqrt{6}} + \frac{\sigma_\zeta^*}{\sqrt{3}}\right) - 2\left(-\frac{\sigma_\xi^*}{\sqrt{2}} + \frac{\sigma_\eta^*}{\sqrt{6}} + \frac{\sigma_\zeta^*}{\sqrt{3}}\right) +$$

$$+ (1 - \mu)\left(-\sqrt{\frac{2}{3}}\,\sigma_\eta^* + \frac{\sigma_\zeta^*}{\sqrt{3}}\right) = 0$$

and hence

$$\sigma_\xi^* \frac{\mu + 3}{\sqrt{2}} + \sigma_\eta^* \sqrt{\frac{2}{3}}(\mu - 1) = 0 \qquad (11)$$

and

$$\tan \vartheta = \frac{1}{\sqrt{3}} \frac{\mu + 3}{1 - \mu} = \frac{\sigma_\eta^*}{\sigma_\xi^*}. \qquad (12)$$

To every value of μ there thus corresponds a half-straight-line issuing from the origin and inclined at an angle of ϑ to the axis of σ_ξ^*.

Eqs. (10) and (11) hold good in sector *I* of Fig. 5.

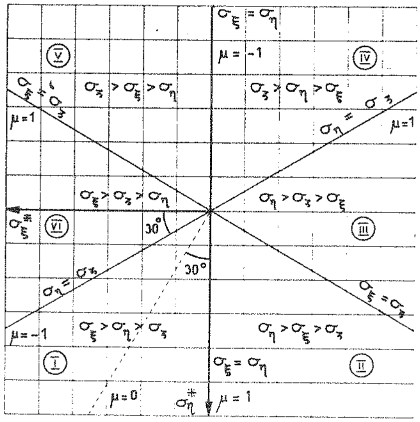

Fig. 5.

The law of variation of μ throughout the plane is obtained by symmetry.

For the plane stress states, assuming for example $\sigma_\zeta = 0$, the law of variation of μ in the stress plane $(\sigma_\xi, \sigma_\eta)$ may be obtained in the following way:

for $\sigma_\eta = \sigma_M$ and $\sigma_\xi = \sigma_i$ $(\sigma_m = 0)$ (zone I in Fig. 6) is obtained from the third of (5):

$$\frac{\sigma_\eta}{\sigma_\xi} = \frac{2}{1 + \mu}$$

and

$$\mu = 2\frac{\sigma_\xi}{\sigma_\eta} - 1$$

for $\sigma_\eta = \sigma_M$ and $\sigma_\xi = \sigma_m$ $(\sigma_i = 0)$ (zone II in Fig. 6) we have

$$\frac{\sigma_\eta}{\sigma_\xi} = \frac{\mu - 1}{\mu + 1}$$

and

$$\mu = \frac{\sigma_\xi + \sigma_\eta}{\sigma_\xi - \sigma_\eta}$$

for $\sigma_\eta = \sigma_i$ and $\sigma_\xi = \sigma_m$ $(\sigma_M = 0)$ (zone III in Fig. 6) we have

$$\frac{\sigma_\eta}{\sigma_\xi} = \frac{1 - \mu}{2}$$

and

$$\mu = 1 - 2\,\frac{\sigma_\eta}{\sigma_\xi}\,.$$

In the half-plane $\sigma_\eta \leqslant \sigma_\xi$ (zones IV, V and VI in Fig. 6) analogous relations are obtained by symmetry.

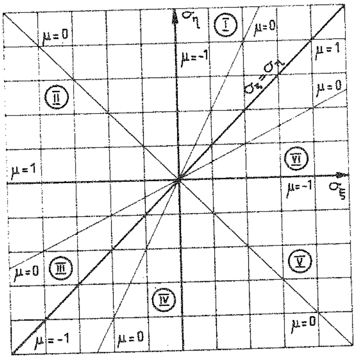

Fig. 6.

A particularly interesting example of representation in S_c and one which will be used in the sequel is that of the limit surface according to Stassi. As is known, in S_p it consists of a paraboloid of rotation about axis t of equation (1):

$$f = \sigma_\xi^2 + \sigma_\eta^2 + \sigma_\zeta^2 - (\sigma_\xi\sigma_\eta + \sigma_\eta\sigma_\zeta + \sigma_\zeta\sigma_\xi) - \\ - \Sigma(\sigma_\xi + \sigma_\eta + \sigma_\zeta) + \pi = 0 \qquad (13)$$

where it is assumed that (2): $\Sigma = \sigma_t + \sigma_c$ and $\pi = \sigma_t\sigma_c$. In the frame $(\sigma_M, \sigma_i, \sigma_m)$ we have:

$$f = \sigma_M^2 + \sigma_i^2 + \sigma_m^2 - (\sigma_M\sigma_i + \sigma_i\sigma_m + \sigma_m\sigma_M) - \\ - \Sigma(\sigma_M + \sigma_i + \sigma_m) + \pi = 0\,. \qquad (14)$$

From (14) by substituting relations (6) we obtain the equation of the limit surface of Stassi in S_c:

$$F = (3 + \mu^2)r^2 - 3\Sigma p - \Sigma\mu r + \pi = 0\,. \qquad (15)$$

The projection of (15) on plane (p, r) supplies a family of parabolas having a horizontal axis and all passing through point K of axis p, of abscissa $p_k = \pi/3\Sigma > 0$.

Given μ_1 and μ_2 two different values of parameter μ, the relative parabolas intersect with one another not only at K but also in the point of coordinates:

$$r = \frac{\Sigma}{\mu_1 + \mu_2}$$

$$p = \frac{\pi}{3\Sigma} + \frac{\Sigma}{3}\,\frac{3 - \mu_1\mu_2}{(\mu_1 + \mu_2)^2}$$

belonging to the valid half-plane $r \geqslant 0$ if $\mu_1 + \mu_2 < 0$.

The parabolas for which $\mu_1 = -\mu_2$ intersect only at K; indeed, they differ through a translation of the axis in the direction of r equal to $\Sigma\mu/(3 + \mu^2)$.

Eq. (15) may be written in the nondimensional form:

$$(3 + \mu^2)\bar{r}^2 - 3(1 + \varrho)\bar{p} - (1 - \varrho)\mu\bar{r} + \varrho = 0 \qquad (16)$$

where

$$\bar{p} = \frac{p}{\sigma_t}\,, \qquad \bar{r} = \frac{r}{\sigma_t}\,, \qquad \varrho = \frac{\sigma_c}{\sigma_t}\,.$$

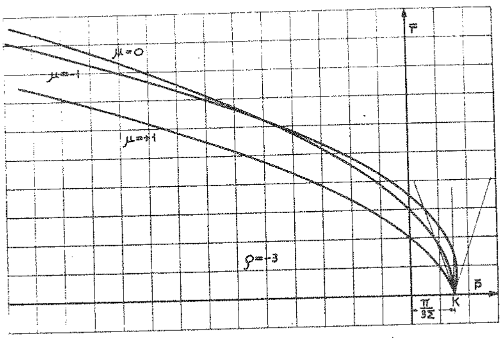

Fig. 7.

Fig. 7 represents in the plane $(\bar{p}, \bar{r})$ the parabolas of the family (16) corresponding to the values 1, 0, -1 of the parameter μ and for $\varrho = -3$.

2. The theory of the intrinsic curve.

The hypotheses of intrinsic curve theory for isotropic materials may be expressed as the condition that the maximum principal circles of Mohr for the limit stress states admit an envelope in the plane (σ, τ). Caquot called this envelope the intrinsic curve of the material. It generally has an equation of the type:

$$E(\sigma, \tau) = 0 \qquad (17)$$

and, being the envelope of a family of circles having their center on the axis of σ, by analytical definition it must be

(1) Eq. (13) has also been studied by C. Torre in a series of papers quoted in the references.
(2) σ_t and σ_c denote the limit values of σ in uniaxial traction and in uniaxial compression respectively.

symmetric with respect to that axis, continuous and regular in every one of its symmetric parts.

Further, the radius of curvature R at every point must be greater (³) than the distance δ of the σ values calculated according to the normal to the curve at that point.

Experiments on this show that the envelope is closed in the tension zone ($\sigma > 0$), is convex and, for high values of the normal compressive stresses ($\sigma \ll 0$), tends to become parallel to the axis of σ with an asymptotic trend (Mohr, von Karman).

It is as well to write (17) in explicit form by means of the function

$$\sigma = \sigma(\tau) \tag{18}$$

or its inverse

$$\tau = \tau(\sigma) \tag{19}$$

which describes only one of the two symmetric parts of the envelope. Form (18) will be used from now on as it leads to simpler analytical developments.

Because of what was said earlier about the shape of the envelope curve, (18) must satisfy the following relations:

$$\begin{gathered} \sigma(\tau) = \sigma(-\tau) \\ R > \delta \\ \sigma'' < 0 \end{gathered} \tag{20}$$

and that is:

$$0 > \sigma'' > \frac{\sigma'}{\tau}(1 + \sigma'^2) \tag{21}$$

with $\sigma(\tau)$ as even function. It is assumed that $\sigma' = d\sigma/d\tau$ and $\sigma'' = d^2\sigma/d\tau^2$.

In intrinsic curve theory a particular physical meaning attaches to the point of contact N between any given circle relating to a limit stress state and the envelope curve. This point, with its coordinates σ_n and τ_n, supplies the normal and shear stresses for the slip-plane and, as shown in Fig. 8, angle φ which the normal to the slip-plane forms with the direction of σ_M in plane σ_M, σ_m and for which $\tan 2\varphi = \sigma'$. In this approach the question of whether curve E has a radius of curvature of zero or different from zero in vertex R_v is of considerable importance. Indeed, Mohr (and others with him), by drawing this curve with a cusp at the vertex (or anyhow with $R_v = 0$), excluded the possibility of tensile fracture ($\varphi = 0$).

Experience contradicts this assumption, however, especially for the more brittle materials.

(³) Greater than, never equal to, because the osculating circle or circle of curvature of a plane curve crosses it at the point of tangency, having a second order contact with it. Stationary points of the curvature in which the curve and the osculating circle have a third order contact and do not cross are exceptions.

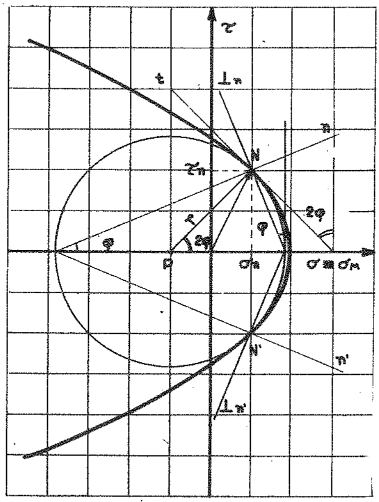

Fig. 8.

Alfons Leon proposed to reconcile Mohr's theory with the experimental evidence by considering only envelope curves with $R_v \neq 0$ for which the family of circles, with $r \leqslant R_v$ tangent to E at the vertex, supplies the set of limit stress states that cause tensile fracture ($\varphi = 0$).

3. The limit condition in intrinsic curve theory.

The limit condition, being imposed only on the maximum principal circles of Mohr, is of the type:

$$f(\sigma_M, \sigma_m) = 0 . \tag{22}$$

It is thus independent of σ_t.

Eq. (22) in the frame (σ_M, σ_t, σ_m) is represented by a cylinder with generatrices parallel to the axis of σ_t and directrix curve lying in the plane (σ_m, σ_M).

Thus in S_p we have a surface formed in general by six cylindrical portions in twos with generatrices that are parallel to the axes σ_ξ, σ_η, σ_ζ and that, at the limit of the respective zones of validity, intersect with one another according to lines lying in the planes of symmetry.

Eq. (22) in S_c becomes

$$F(p, r) = 0 \tag{23}$$

and is represented by a cylindrical surface with generatrices parallel to the axis μ and directrix curve lying in the plane (p, r).

Thus in the case of intrinsic curve theory independence on parameter μ makes it possible to study the limit condition in the plane (p, r).

In plane (p, r) (23) may be written in explicit form by means of the function:

$$p = p(r) . \tag{24}$$

It is interesting to study the relations between the expressions of the limit condition in plane (p, r) and in plane (σ, τ).

These are supplied in the two cases respectively by F and by E or, in explicit form, by $p = p(r)$ and by $\sigma = \sigma(\tau)$.

The equation for any given one of Mohr's maximum principal circles in the plane σ, τ is:

$$C(\sigma, \tau, p, r) = (\sigma - p)^2 + \tau^2 - r^2 = 0 \,. \tag{25}$$

Once the envelope is assigned in the form (17), the corresponding limit state can be obtained in the form (23) by eliminating the parameters σ and τ from the system:

$$\begin{aligned} &E(\sigma, \tau) = 0 \\ &C(\sigma, \tau, p, r) = 0 \\ &J(E, C, \sigma, \tau) = 0 \end{aligned} \tag{26}$$

where J denotes the functional determinant or Jacobian of the E and C values with respect to σ and τ. The vanishing of J expresses the condition that any given circle (25) is tangent to the envelope E. By rendering E explicit through (18), system (26) can be written in the form:

$$\begin{aligned} &\sigma = \sigma(\tau) \\ &(\sigma - p)^2 + \tau^2 - r^2 = 0 \\ &\tau + \sigma'(\sigma - p) = 0 \end{aligned} \tag{27}$$

where it is assumed that $\sigma' = d\sigma/d\tau$.

Conversely, if (23) is assigned, (17) is obtained by eliminating parameters p and r from the system:

$$\begin{aligned} &F(p, r) = 0 \\ &C(\sigma, \tau, p, r) = 0 \\ &J(F, C, p, r) = 0 \end{aligned} \tag{28}$$

where the vanishing of the Jacobian expresses the condition that the family of circles described by the F and C values admits of an envelope, system (28), if F is rendered explicit through (24), is written:

$$\begin{aligned} &p = p(r) \\ &(\sigma - p)^2 + \tau^2 - r^2 = 0 \\ &r + \dot{p}(\sigma - p) = 0 \end{aligned} \tag{29}$$

where it is assumed that $\dot{p} = dp/dr$.

Systems (27) and (29) define a one-one correspondence between the points of curves E and F. Through these equations we can establish two important relations between the functions $\sigma(\tau)$ and $p(r)$:

$$r\sigma' = \tau\dot{p} \tag{30}$$

$$1 + \sigma'^2 = \dot{p}^2. \tag{31}$$

They permit some interesting deductions.

For angle φ (Fig. 8) we obtain, via the third equation of (29), the relation:

$$\cos 2\varphi = \frac{\sigma - p}{r} = -\frac{1}{\dot{p}} \tag{32}$$

which for (31) agrees perfectly with the expression already quoted $\tan 2\varphi = \sigma'$.

From (21) we get $\sigma'/\tau < 0$. Since $r \geqslant 0$, (30) means that $\dot{p} \leqslant 0$. Bearing in mind (31), we then obtain the relation:

$$\dot{p} \leqslant -1 \,. \tag{33}$$

Eq. (33) sets a precise limit to $p(r)$ when it has to be considered as corresponding to a given envelope E.

By means of (30) and (31) we can study the various cases that arise out of the different assumptions made regarding the trend of curve E at the vertex ($\tau = 0$).

If for $\tau = 0$, σ' is different from zero, it is also discontinuous at that point, being an odd function of τ (as derivative of $\sigma(\tau)$ which is even for the symmetry with respect to the axis of σ). Hence the quantities $\sigma'' = d^2\sigma/d\tau^2$ and $R = -(1 + \sigma'^2)^{3/2}/\sigma''$ (radius of curvature of $\sigma(\tau)$) are not defined at the vertex. From (30), for $\tau = 0$, $\sigma' \neq 0$, $\dot{p} = -\sqrt{1 + \sigma'^2} < -1$, we get $r = 0$.

For $\tau = 0$ we have $R_v = -1/\sigma''(0)$, $\dot{p} = -1$ and from (30):

$$r = \lim_{\tau \to 0} -\frac{\tau}{\sigma'} = -\frac{1}{\sigma''(0)} = R_v \,.$$

The value of R_v depends on that of $\sigma''(0)$ and may be zero or less according to the different geometric forms chosen for the envelope. The curves for which $\sigma''(0) = 0$ and hence $R_v = \infty$ are not of interest here because they cannot be the envelope of a family of curves.

The foregoing may be summarised as follows:

$$\tau = 0 \begin{cases} \sigma' \neq 0 \text{ and discontinuous, } \sigma'' \text{ and } R_v \text{ not defined,} \\ \qquad \dot{p} < -1, \quad p = \sigma = h, \quad r = 0\,. \\ \sigma' = 0, \quad R_v = -\dfrac{1}{\sigma''}, \quad \dot{p} = -1, \\ \qquad p = h - R_v, \quad r = -\dfrac{1}{\sigma''} = R_v\,. \end{cases} \tag{34}$$

In the light of the above considerations we can examine some classic cases, representing curves E and F in the same plane by superposing planes (σ, τ) and (p, r). In this way we highlight the close geometric relationship that exists between them.

For Coulomb's theory $\sigma = \sigma(\tau)$ consists, as is known, of two lines of equation:

$$\sigma = \pm a\tau + h \tag{35}$$

where the minus sign is relative to the half-plane $\tau \geqslant 0$ and the plus sign to the half-plane $\tau \leqslant 0$. In this case, for $\tau = 0$ and $\sigma' = \pm a \neq 0$, $\dot{p} = -\sqrt{1 + a^2}$, $r = 0$. $p(r)$ is easily obtained on the basis of (27) and is:

$$p = -\sqrt{1 + a^2}\, r + h \tag{36}$$

where the minus sign is chosen because $\dot{p}$ must be $\leqslant -1$. As stated, (36) is valid in the half-plane $r \geqslant 0$ (Fig. 9).

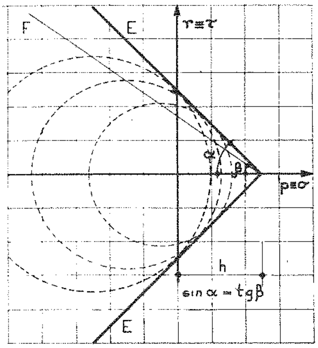

Fig. 9.

The same considerations can be repeated qualitatively for the envelope curve of Fig. 10 drawn according to Mohr.

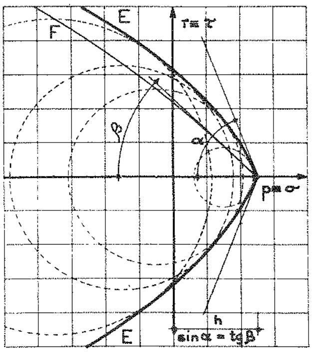

Fig. 10.

The envelope curve proposed by Caquot for conglomerates is a special case because, whilst crossing the axis of σ orthogonally, it has $R_v = 0$ and so cannot allow for tensile fracture phenomena, notwithstanding the assertions current in the literature. In this case the analytical expression is:

$$\sigma = -a\tau^{3/2} + h \tag{37}$$

which for $\sigma < h$ gives one of the two symmetric branches of Caquot's envelope curve, namely the one for the half-plane $\tau \geqslant 0$.

From (37) we get:

$$\sigma' = -\frac{3}{2} a\sqrt{\tau^{1/2}} \qquad \sigma'' = -\frac{3a}{4\sqrt{\tau}}$$

and so for $\tau = 0$

$$\sigma' = 0 \qquad \sigma'' \to \infty \qquad r = R_v = 0 \qquad \dot{p} = -1\,.$$

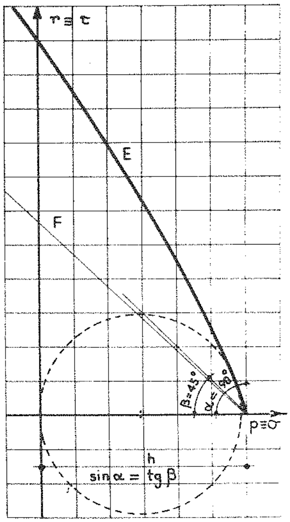

Fig. 11.

Fig. 11 is the drawing for a conglomerate of (37) and the corresponding $F = 0$, calculated numerically by means of (27) because it cannot be written in explicit form. The uniaxial tension failure circle is shown in dash-line.

4. Leon's theory.

As already stated, Alfons Leon always represents the envelope curve with a finite radius of curvature at the vertex. The circles of the family that determines the envelope curve are relative to shear fracture ($90° > \varphi > 0$) and for them $r > R_v$; the family of circles tangent at the vertex, for which $R_v \geqslant r \geqslant 0$, expresses tensile fracture ($\varphi = 0$). Some circles belonging to these two classes are drawn in Fig. 12. The circle of curvature in the vertex represents the element of separation.

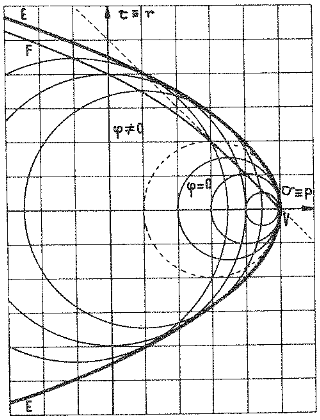

Fig. 12.

Leon's theory is valid with reference to envelope curves of any shape provided that $R_v \neq 0$ and (21) is satisfied.

When E has been assigned and the corresponding F obtained through (27), a few considerations must be made. For $r = R_v$ and $\dot{p} = -1$ (see schema 34), since $\ddot{p} = d\dot{p}/dr = \sigma'/\tau < 0$ (from Eq. 30) for $R_v > r$, then $\dot{p}$ would be > -1, against condition (33).

The F obtained via (27) is therefore valid only for $r \geqslant R_v$.

For $R_v \geqslant r \geqslant 0$ the expression $\sigma' = 0$ holds good and hence $\dot{p} = -1$. The corresponding limit condition is obtained by integrating and assuming for $r = 0$, $p = h$. We then have:

$$p = -r + h\,. \tag{38}$$

In the last analysis the limit condition in S_c is supplied by the following relations taken together:

$$\begin{aligned} &p = p(r) \\ &r > R_v \end{aligned} \tag{39}$$

$$\begin{aligned} &p = -r + h \\ &R_v \geqslant r \geqslant 0\,. \end{aligned} \tag{40}$$

This is represented in Fig. 12 together with the corresponding envelope curve by superposition of the planes (p, r) and (σ, τ).

Eq. (39) constitute the limit condition for shear failure stress states and the form of $p(r)$ naturally depends on the material under consideration. Eq. (40), which supply the limit condition for the stress states of tensile fracture ($\varphi = 0$), are represented by the straight part of the F in Fig. 12 (from the point V to the circle of curvature drawn in dash-line), inclined at an angle of 45° to the axis of p, irrespective of the material. The material is responsible only for the amplitude of this straight segment, which is greater for brittle materials, the latter having a greater R_v than ductile materials (which are rarely subject to tensile fracture).

The limit condition in the frame (σ_M, σ_i, σ_m) is easy to deduce by means of the first and second equations of (5) from (39) and (40), which supply respectively:

$$\begin{aligned} &f(\sigma_M\,,\ \sigma_m) = 0 \\ &\sigma_M > \sigma_m + 2R_v \end{aligned} \tag{41}$$

$$\begin{aligned} &\sigma_M = h \\ &\sigma_M \leqslant \sigma_m + 2R_v\,. \end{aligned} \tag{42}$$

Expressions (41)-(42) represent in space a cylindrical surface with generatrices parallel to the axis σ_i and, in the plane (σ_M, σ_m), the directrix curve (Fig. 13) that includes one

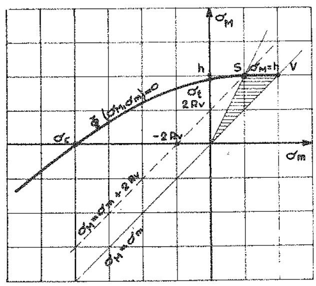

Fig. 13.

straight segment SV (Eq. 42) parallel to the axis of σ_m and one generally curvilinear (Eq. 41) for which we can repeat the considerations previously made for (39)-(40). Here again there is no angular discontinuity in S. Indeed f (σ_M, σ_m) = 0 is obtained from $F(p, r) = 0$ via the first and second equations of (5) and so we get:

$$\frac{d\sigma_M}{d\sigma_m} = -\frac{\dfrac{\partial f}{\partial \sigma_m}}{\dfrac{\partial f}{\partial \sigma_M}} = \frac{\dfrac{1}{2}\dfrac{\partial F}{\partial p} - \dfrac{1}{2}\dfrac{\partial F}{\partial r}}{\dfrac{1}{2}\dfrac{\partial F}{\partial p} + \dfrac{1}{2}\dfrac{\partial F}{\partial r}} = \frac{\dot{p}+1}{\dot{p}-1}\,. \tag{43}$$

For $\sigma_M = \sigma_m + 2R_v$ the relation $r = R_v$ holds good and so $\dot{p} = -1$ and, from (43), $d\sigma_M/d\sigma_m = 0$.

The region of plane (σ_M, σ_m) in dash-line in the figure relates to the stress states that give rise, through the proportional increase of σ_p, to tensile rupture and is bounded by the segment SV and, for (42), by the straight-line of equation:

$$\sigma_M = \frac{h}{h - 2R_v}\,\sigma_m\,. \tag{44}$$

The limit surface in S_p is obtained immediately from (41)-(42) because of the well known properties of symmetry. It consists of three sides of a cube (Eq. 42) whose common vertex lies on axis t, and of six portions of cylindrical surfaces (Eq. 41) joined smoothly with them. Its geometric

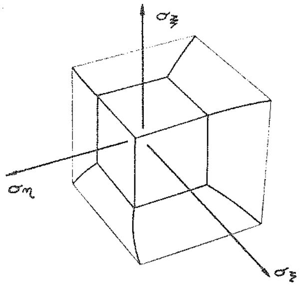

Fig. 14.

representation is given in Fig. 14 for the case of a brittle material (concrete). Figure 15 shows the three types of intersection with Meldahl planes.

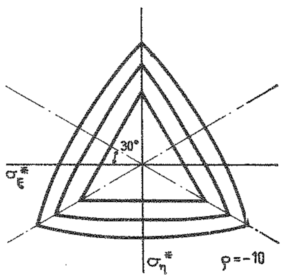

Fig. 15.

5. The parabolic envelope curve.

Leon dealt particularly with the case of a parabolic envelope curve. In that case E has the form:

$$E(\sigma, \tau) = \tau^2 + 2R_v(\sigma - h) = 0 \tag{45}$$

which satisfies (21). Writing (45) in explicit form, we get for $\sigma = \sigma(\tau)$ the expression:

$$\sigma = -\frac{\tau^2}{2R_v} + h\,. \tag{46}$$

From (46) by means of (27) we easily obtain $p = p(r)$. Indeed, since $\sigma' = -\tau/R_v$, the third equation of (27) is written:

$$\tau\left[1 - \frac{\sigma - p}{R_v}\right] = 0$$

from which for $\tau \neq 0$ we obtain

$$\sigma - p = R_v \tag{47}$$

by substituting (47) in the second of (27) we get:

$$R_v^2 + \tau^2 - r^2 = 0\,. \tag{48}$$

Having obtained σ and τ^2 from (47) and (48) and substituting in (46) we have

$$p = -\frac{r^2}{2R_v} + h - \frac{R_v}{2}\,. \tag{49}$$

The expression (49) of $p(r)$ is valid for $\tau \neq 0$, that is for $r > R_v$. For $R_v \geqslant r \geqslant 0$ we get, as generally, the limit condition:

$$p = -r + h\,. \tag{50}$$

In frame $(\sigma_M, \sigma_i, \sigma_m)$ Eq. (49) and (50) with their respective limitations are written:

$$\begin{aligned} &\sigma_M > \sigma_m + 2R_v \\ &(\sigma_M - \sigma_m)^2 + 4R_v(\sigma_M + \sigma_m) + 4R_v(R_v - 2h) = 0 \end{aligned} \tag{51}$$

$$\begin{aligned} &\sigma_M \leqslant \sigma_m + 2R_v \\ &\sigma_M = h \end{aligned} \tag{52}$$

6. Determination of parameters R_v and h by means of two experimental tests.

Form (45) for E and, consequently, (49) for F and (51) for f have the virtue of analytical simplicity. They depend on only two parameters R_v and h with a clear geometric and physical meaning and are therefore determinable by means of only two experimental tests. Since two limit circles of equations:

$$C_1 = (\sigma - p_1)^2 + \tau^2 - r_1^2 = 0$$

$$C_2 = (\sigma - p_2)^2 + \tau^2 - r_2^2 = 0$$

are known, envelope E is unequivocally determined when at least one of them is not tangent in the vertex V.

Supposing $r_1 < r_2$ the above condition demands that

C_2 be not tangent in V, that is that the two equivalent conditions:

$$\begin{aligned} R_v &< p_2 + r_2 \\ R_v &< h - p_2 \end{aligned} \qquad (53)$$

are satisfied. There are thus two possibilities:

1) C_1 is not tangent to E in the vertex. This circumstance is expressed by the relations

$$\begin{aligned} R_v &< p_1 + r_1 \\ R_v &< h - p_1 \end{aligned} \qquad (54)$$

2) C_2 is tangent to E in the vertex. We then have:

$$\begin{aligned} R_v &\geqslant p_1 + r_1 \\ R_v &\geqslant h - p_1 , \end{aligned} \qquad (55)$$

In the first case the relations linking R_v and h to p_1 and r_1 or to p_2 and r_2 are in the form (49) and that is may be written:

$$R_v^2 + 2(p_1 - h)R_v + r_1^2 = 0 \qquad (56)$$

$$R_v^2 + 2(p_2 - h)R_v + r_2^2 = 0 \, . \qquad (57)$$

Subtracting each side from the corresponding one, we have:

$$R_v = \frac{r_2^2 - r_1^2}{2(p_1 - p_2)} \qquad (58)$$

and substituting (58) in (56) or in (57) we obtain

$$h = \frac{r_2^2 p_1 - r_1^2 p_2}{r_2^2 - r_1^2} + \frac{r_2^2 - r_1^2}{4(p_1 - p_2)} \, . \qquad (59)$$

In the second case we have for (50)

$$p_1 + r_1 = h \, . \qquad (60)$$

Having thus obtained h, by solving (57) with respect to R_v, we obtain:

$$R_v = h - p_2 - \sqrt{(h - p_2)^2 - r_2^2} \, . \qquad (61)$$

where the minus sign is chosen before the root because the following must hold good for (53): $R_v < h - p_2$.

If we consider in particular the two limit circles for the plane tension state and compression state respectively, the previous expressions for R_v and h are simplified. Having chosen, for example, the uniaxial tension and compression failure circles C_t and C_c, assuming that $C_1 \equiv C_t$ and $C_2 \equiv C_c$, we have:

$$p_1 = r_1 = \frac{\sigma_t}{2} \, , \qquad p_2 = - r_2 = \frac{\sigma_c}{2} \, .$$

Eqs. (58), (59) and (60), (61) with their respective limitations (54) and (55) are written:

for

$$R_v < \frac{\sigma_t}{2} \qquad \begin{aligned} R_v &= - \frac{\sigma_t + \sigma_c}{4} \\ h &= - \frac{(\sigma_t - \sigma_c)^2}{8(\sigma_t + \sigma_c)} \end{aligned} \qquad (62)$$

for

$$R_v \geqslant \frac{\sigma_t}{2} \qquad \begin{aligned} R_v &= \sigma_t - \frac{\sigma_c}{2} - \sqrt{\sigma_t^2 - \sigma_c \sigma_t} \\ h &= \sigma_t \, . \end{aligned} \qquad (63)$$

In the limit case $R_v = \sigma_t/2$, from (62) or from (63) we obtain:

$$3\sigma_t + \sigma_c = 0 \qquad \varrho = \frac{\sigma_c}{\sigma_t} = -3 \, .$$

In the case in point (51) thus assumes the form:

$$\begin{aligned} &\varrho > -3 \\ &(\sigma_M - \sigma_m)^2 - \Sigma(\sigma_M + \sigma_m) + \pi = 0 \end{aligned} \qquad (64)$$

$$\begin{aligned} &\varrho \leqslant -3 \\ &(\sigma_M - \sigma_m)^2 + 2\Delta(\sigma_M + \sigma_m) + \Delta^2 - 4\Delta\sigma_t = 0 \end{aligned} \qquad (65)$$

where $\Sigma = \sigma_t + \sigma_c$, $\pi = \sigma_t \sigma_c$ and $\Delta = 2R_v = 2\sigma_t - \sigma_c - 2\sqrt{\sigma_t^2 - \sigma_c \sigma_t}$. Dividing by σ_t^2 (64) and (65) are reduced to nondimensional form depending solely on the parameter ϱ, whilst the tensile stresses are evaluated at less than σ_t.

7. The K line.

Leon also proposed a form for the envelope curve that satisfied the requirement of Mohr and von Karman of asymptotes parallel to the axis of σ in the compression zones. For this purpose he chose the curve of equation:

$$(\sigma - h)^2 = \frac{\tau^4}{a^2 - \tau^2} \, . \qquad (66)$$

Of this we need only consider the branch for the half-plane $\sigma < h$. It presents two horizontal asymptotes distant from the axis of σ. This property is obtained at the cost of greater complexity of function $\sigma(\tau)$, which, however, always remains determinable by two experimental tests.

By means of Eq. (27) it is no longer possible to obtain from (66) an explicit expression of $p(r)$, which must be calculated numerically through

$$p = \sigma + \frac{\tau}{\sigma'} \qquad\qquad r = \tau\sqrt{1 + \frac{1}{\sigma'^2}}$$

obtained from (27) by expressing p and r in explicit form as functions of σ, τ and σ'. This drawback makes (66) of little practical interest.

8. The domains of Leon for constant μ.

By intersecting Leon's space domain in frame (σ_M, σ_i, σ_m) with halfplanes of Eq. (10) (see Fig. 3), we obtain limit curves of the domain for stress states all characterised by the same μ value. It is worth representing these curves by projecting them on plane (σ_M, σ_m) so as to be able to read the value of σ_M and of σ_m directly on the reference axes.

Knowing μ we can obtain σ_i. These projections coincide, whatever be the value of μ, with the director curve (41)-(42). Figs. 16, 17 and 18 represent in the plane (σ_ξ, σ_η) of S_p the domains of Leon for $\mu = \pm 1$, assuming $\sigma_\eta = \sigma_\zeta$.

The envelope curve is supposed to be parabolic and three cases are considered, in which $\varrho = -1.5$, -3 and -10.

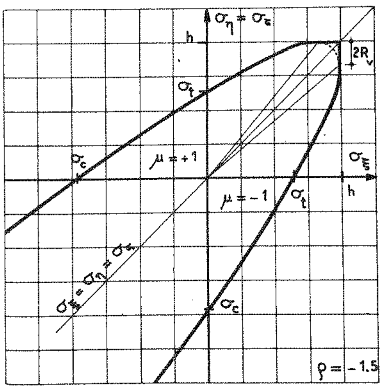

Fig. 16.

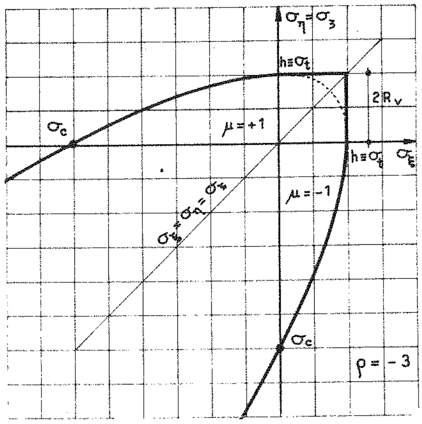

Fig. 17.

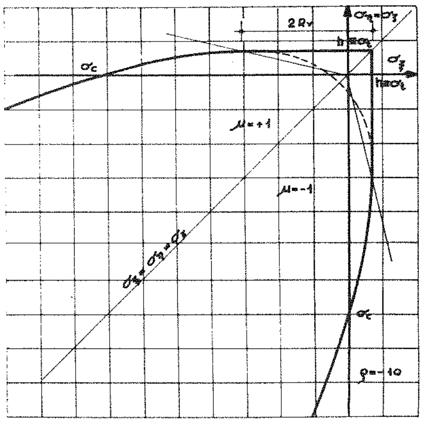

Fig. 18.

9. Leon's domains for plane stress states.

The analytical relations representing the frontier of these domains in S_p are obtained from (41)-(42) or, for an intrinsic parabolic curve, from (51)-(52). Assuming, for example, that $\sigma_\zeta = 0$; we obtain these relations in plane (σ_ξ, σ_η) indicating in square brackets the expressions valid only for the parabolic envelope.

In S_p the plane (σ_ξ, σ_η) is divided by the planes of symmetry of the domain into 6 zones (Fig. 6). The outlines of these planes are obviously the two axes $\sigma_\xi = 0$ and $\sigma_\eta = 0$ and the line bisecting the odd quadrants $\sigma_\xi = \sigma_\eta$. This line, belonging to the plane of symmetry orthogonal to plane (σ_ξ, σ_η), will be an axis of symmetry for the plane domain.

In zone I (Fig. 6) $\sigma_\eta = \sigma_M > 0$, $\sigma_\xi = \sigma_i > 0$, $\sigma_m = 0$. Therefore, assuming that $\sigma_m = 0$ in (41)-(42) [or (51)-(52)], we have:

$$\begin{aligned} &\sigma_\eta = \sigma_M > 2R_v \\ &\sigma_\eta = \sigma_M = \sigma_t = [-2R_v + 2\sqrt{2hR_v}] \end{aligned} \tag{67}$$

$$\begin{aligned} &\sigma_\eta = \sigma_M \leqslant 2R_v \\ &\sigma_\eta = \sigma_M = h = \sigma_t . \end{aligned} \tag{68}$$

Eq. (67) is thus valid for $\sigma_t > 2R_v$ (ductile materials) and (68) for $\sigma_t \leqslant 2R_v$ (brittle materials).

These limitations can be inferred immediately from Figs. 19 and 20, from which it is clear that only for $\sigma_t \leqslant 2R_v$ there are plane stress states that cause tensile fracture.

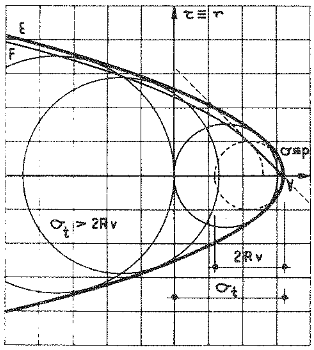

Fig. 19.

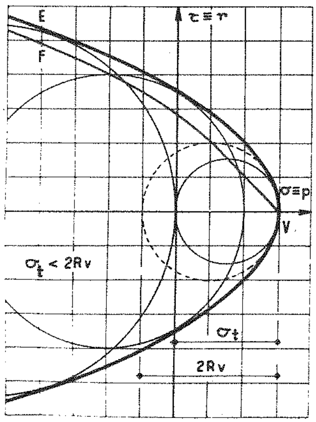

Fig. 20.

In zone II $\sigma_\eta = \sigma_M > 0$, $\sigma_i = 0$, $\sigma_\xi = \sigma_m < 0$. The limit condition is thus represented by (41)-(42) [or (51)-(52)]

In zone III $\sigma_M = 0$, $\sigma_\xi = \sigma_i < 0$, $\sigma_\eta = \sigma_m < 0$. Thus, assuming that $\sigma_M = 0$ in (41) [or (51)] we have:

$$\sigma_\eta = \sigma_m = \sigma_c = [-2R_v - 2\sqrt{2R_v h}] . \tag{69}$$

Figs. 21, 22 and 23 represent Leon's domains for plane stress states, in the case of a parabolic intrinsic curve, for the values -1.5, -3 and -10 respectively of ϱ.

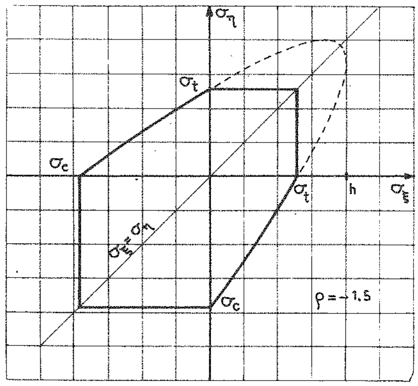

Fig. 21.

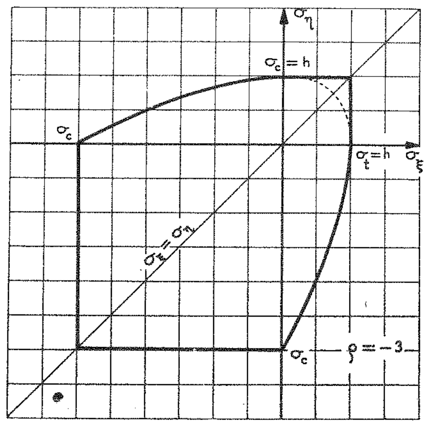

Fig. 22.

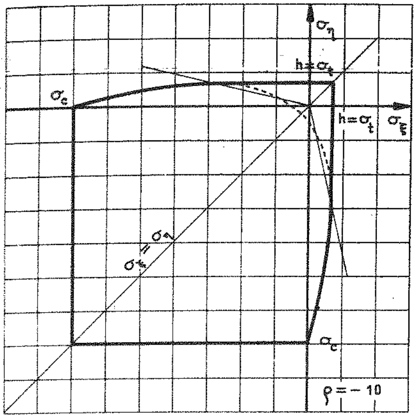

Fig. 23.

10. The criterion of Griffith.

It is interesting to compare the results of Leon's theory with those deduced from Griffith's theory of brittle fracture. The theoretical evaluations of the limit stress of uniaxial tensile failure of a perfectly homogeneous solid, i. e. free from flaws, made by some workers (Polanyi 1921, Zwicky 1923, Orowan 1934) supply values of the order of 10^2 or 10^3 times the experimental values. In justification of this gap it is admitted that real solids always contain flaws that act as focal points of the internal stresses, which reach values far above the average at some points. Griffith's theory, in assuming a uniform distribution of small internal fractures in a continuous isotropic and elastic material, traces failure to the attainment of a critical value for the tensile stresses (equal to the molecular cohesion of the material) at the boundary of these fractures. For plane stress states, on the assumption that the internal fractures are of very elongated elliptical form, all equal in length at the major axis and with equal radius of curvature at the ends of this axis and uniformly distributed throughout the material with all possible orientations [4], Griffith obtained a condition of failure using Inglis' solution for the distribution of stresses in a plate with an elliptical hole. This supplies the maximum tensile stress at the vertices of the ellipse of most dangerous orientation with respect to the principal stress axes, as function of the principal stresses σ_1 and σ_2. By equating this maximum tensile stress with the condition of failure and expressing the latter in terms of the mean stress in uniaxial tensile failure σ_t, we obtain equations representing the criterion of strength for biaxial stress states in which σ_t is the only physical constant characteristic of the material. Supposing that $\sigma_1 > \sigma_2$ and that, as usual, the tensile stresses are positive, the analytical expression of Griffith's criterion is supplied by the relations:

(4) To get closer to physical reality a probabilistic approach would be necessary.

$$3\sigma_1 + \sigma_2 \geqslant 0 \qquad \sigma_1 = \sigma_t \qquad \varphi = 0 \tag{70}$$

$$3\sigma_1 + \sigma_2 < 0 \qquad (\sigma_1 - \sigma_2)^2 + 8\sigma_t(\sigma_1 + \sigma_2) = 0$$

$$\cos 2\varphi = \frac{\sigma_2 - \sigma_1}{2(\sigma_1 + \sigma_2)} \tag{71}$$

where φ is the angle that the major axis of the critical elliptical fracture forms with the axis of σ_2 (Fig. 24).

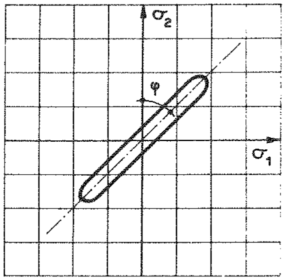

Fig. 24.

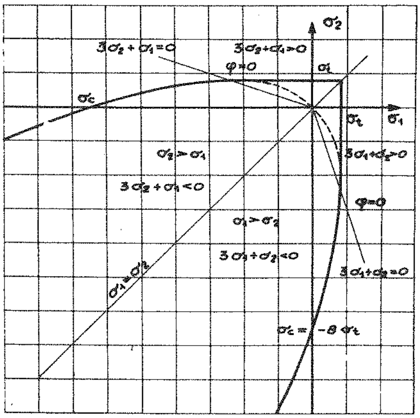

Fig. 25.

Fig. 25 is a graphic representation of the relations (70)-(71) (Orowan 1949). The frontier of the domain consists of a rectilinear segment parallel to the axis of σ_2 (Eq. 70) and of a parabolic segment (Eq. 71). From (71) we get:

$$\sigma_c = -8\sigma_t$$

$$\varrho = \frac{\sigma_c}{\sigma_t} = -8\,.$$

The constancy of the ratio ϱ for all materials is obviously not very realistic and is the result of over-schematisation of the model underlying the theory. Bearing in mind that in deriving Griffith's criterion it was supposed that the material behaves elastically until failure, the results obtained must be assumed to be applicable only to "brittle" fracture (5).

This being so, the value -8 for ϱ supplied by the condition under consideration is in good qualitative agreement with the experimental values for materials that present brittle failure at least in the tension-tension and tension-compression quadrants of plane (σ_1, σ_2) (concretes, rocks, cast iron and so on). Griffith's condition, formulated for biaxial tension states, may be regarded as applicable to triaxial states also. Indeed, within the range of approximation underlying the theory, the normal stress and shear stress acting in a plane perpendicular to the boundary of any given internal fracture have no appreciable influence. An experimental confirmation of this criterion was supplied by Orowan on the basis of the results of Bridgman's experiments on the strength of brittle materials under strong hydrostatic pressure. By introducing into (71) the value of σ_1 and σ_2 at failure, he obtained the σ_t values of the materials under consideration (pyrex, carboloy, beryllium), which were in satisfactory agreement with experience.

We now wish to show the perfect coincidence of the relations (70-71) with those representing Leon's criterion for a parabolic envelope curve and for $\varrho = -8$. Indeed, bearing in mind that in the case in point $b = \sigma_t$ and $\Delta = 2R_v = 4\sigma_t$, from (44) and (49) respectively we have:

$$\sigma_M = \frac{b}{b - 2R_v}\,\sigma_m = \frac{\sigma_t\sigma_m}{\sigma_t - \Delta} = -\frac{\sigma_m}{3} \qquad (72)$$

$$p = \frac{r^2}{4\sigma_t} \qquad\qquad \dot{p} = \frac{r}{2\sigma_t}\,. \qquad (73)$$

For (72-73) Eq. (32) may be written:

$$\cos 2\varphi = \frac{\sigma - p}{r} = -\frac{1}{\dot{p}} = -\frac{2\sigma_t}{r} = -\frac{r}{2p} =$$

$$= \frac{\sigma_m - \sigma_M}{2(\sigma_M + \sigma_m)}\,. \qquad (74)$$

Eq. (72) and (74) enable us to put (51)-(52) in the form:

$$3\sigma_M + \sigma_m \geqslant 0 \qquad \sigma_M = \sigma_t \qquad \varphi = 0$$

$$3\sigma_M + \sigma_m < 0 \qquad (\sigma_M - \sigma_m)^2 + 8\sigma_t(\sigma_M + \sigma_m) = 0$$

$$\cos 2\varphi = \frac{\sigma_m - \sigma_M}{2(\sigma_M + \sigma_m)}$$

and so there is exact coincidence with (70) (71).

(5) The adjective "brittle" is not applied to the process of fracture (the act of fracture itself can be characterized as tensile, shearing, etc.) but to what precedes it; it is understood, that is to say, that plastic slips and hence permanent distortions of the material before failure are negligible. In the same way we speak of "ductile" or "plastic" fracture when the reverse holds good.

11. Range of validity of Leon's theory.

Thus Griffith's theory yields a criterion of strength of materials that may be regarded as a special case of Leon's theory. The two theories, although deriving from such different sources, lead (for $\varrho = -8$ and parabolic envelope curve) to the determination of the same classes of stress states at fracture, whether tensile or shearing, and, for the latter, supply the same values of angle φ. Leon's theory, unlike Griffith's, does not assume that the material behaves elastically until fracture. It does, however, assume that the limit condition is independent of the value of σ_t (and hence of parameter μ), an assumption that the experimental data show is acceptable when fracture is not preceded by appreciable plastic deformations. So even Leon's theory must be regarded as confined to cases of brittle fracture. Experience indicates that markedly hetero-resistant materials ($\sigma_t \ll -\sigma_c$) have a wider range of tensile fracture (and this agrees with Leon's theory), which extends from the tension zone to the tension-compression zone of S_p.

Within that range, and in an adjacent zone, fracture is of the brittle type, whereas the nearer we get to the compression region the greater are the plastic slips and the influence of σ_t. It is for such materials that Leon's theory has a wider range of applicability.

With reference to the plane stress states, it is interesting to compare in plane $(\sigma_\xi, \sigma_\eta)$ Leon's theoretical limit domain (for the parabolic envelope curve) with the experimental data obtained by R. C. Grassi and I. Cornet, who did tests on thin-walled gray cast-iron tubes, investigating the tension zone and the tension-compression zone. For this type of cast iron $\sigma_t = 2.8 \times 10^4\ psi$, $\sigma_c = -9 \times 10^4\ psi$ and so $\varrho = -3.2$.

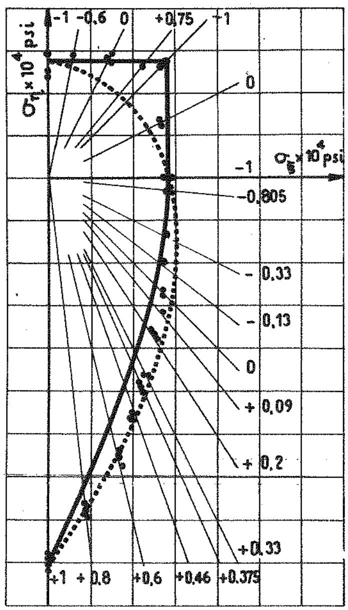

Fig. 26.

Fig. 26 shows the experimental points of Grassi and Cornet and (continuous-line curve) the limit domain of Leon. It will be noted that there is perfect agreement between theory and experience in the tension zone and in the tension-compression zone up to the value — 0.805 of μ, which is in theory the limit of the tensile fracture zone. Later Leon's parabola tends, especially from the value 0.2 of μ, to be inside the experimental points.

In line with the previous considerations, we can explain this behaviour because the first appreciable plastic deformations at failure begin to be observed from such value of μ. Fig. 27 represents in plane (σ, τ) the theoretical E and F

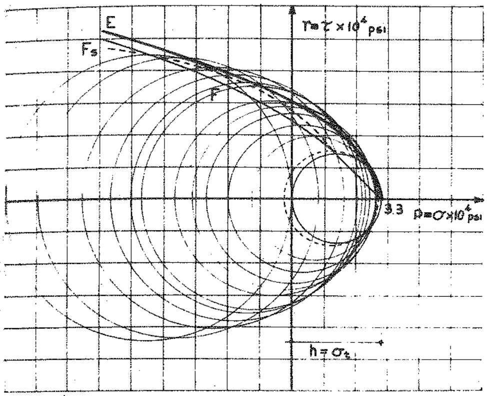

Fig. 27.

curves, the experimental circles of Mohr and curve F_S relating to them. It will also be noted that in the tension-compression zone the theoretical curves are inside, though close to, the experimental curves.

Fig. 26 shows also Stassi's curve for the same values of σ_t and σ_c (dash-line curve).

It approximates well to the experimental points in the tension-compression zone for $1 \geqslant \mu \geqslant 0$, is slightly outside it for $0 \geqslant \mu \geqslant -1$, and then passes decidedly inside in the tension zone, where it supplies for σ_{tt} [6] a value equal to about 3/5 of the experimental value.

Stassi's theory, undoubtedly the most reliable of those that take into account the influence of σ_i on the attainment of the limit state, does not appear to be valid in the brittle failure range.

12. Influence of σ_i.

In cases in which the influence of σ_i on the determination of the limit condition is not negligible, one can think in terms of an extension of Leon's theory based on the hypothesis that the principal maximum circles of Mohr for all the yield stress states characterised by a given value for parameter μ continue to admit an envelope curve in plane (σ, τ).

In that plane we shall then have a family of curves E, each relating to a given value of μ and all obviously passing through the representative point of the hydrostatic tension limit stress. Experience shows that these curves open out as μ decreases.

Similarly in the plane (p, r) we shall obtain a family of curves F issuing from successive points of the straight segment representing the stress states for tensile fracture.

In Fig. 28, by superposing planes (p, r) and (σ, τ), are represented these families of curves in the case of parabolic envelopes and with reference to a material having $\sigma_t = = 3.3 \times 10^4$ *psi*, $\sigma_c = 10.5 \times 10^4$ *psi*, $\sigma_{c1} = 12 \times 10^4$ *psi*, $\sigma_{c2} = 14 \times 10^4$ *psi* and $\sigma_{cc} = 16.5 \times 10^4$ *psi*, where σ_{c1} and σ_{c2} denote the values for biaxial compression failure for $\mu = 0.55$ and for $\mu = 0$. These values were deduced from experiments carried out by L. F. Coffin on thin-walled gray cast-iron tubes for plane stress states.

He explored the tension zone, the tension-compression zone and the compression zone, thus arriving at a full description of the limit domain in plane $(\sigma_\xi, \sigma_\eta)$.

Fig. 28 shows circles C_t, C_c, C_{c2}, C_{cc} in continuous line and the three circles of curvature in the vertex of the individual envelopes in broken line. They meet the segment for tensile fracture at the points in which the corresponding F curves start.

(6) In the sequel σ_{tt} and σ_{cc} will denote the σ values for biaxial tension and compression failure.

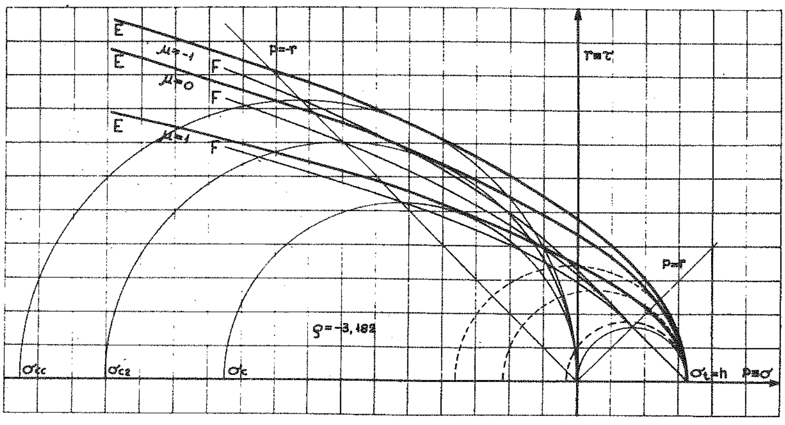

Fig. 28.

In frame $(\sigma_M, \sigma_i, \sigma_m)$ the limit domain, according to this generalised theory of Leon, is described by the curves obtained by intersecting the domain valid for any given value k of μ with the half-plane $\mu = k$.

The domain in S_p is obtained by symmetry. Fig. 29 shows the experimental data of Coffin and the limit domains of Leon corresponding to the values 1; 0.55; 0 and -1 of μ in the plane $(\sigma_\xi, \sigma_\eta)$.

The definition of symbol ϱ is generalised so that it means a ratio between the σ values for compression and tension failure for plane stress states characterised by the same μ values. In the case in point is always $\sigma_t < 2R_r$ and hence $\sigma = b$. We thus have:

$$\varrho_c = \frac{\sigma_c}{\sigma_t} = -3.18 \qquad \varrho_{c1} = \frac{\sigma_{c1}}{\sigma_t} = -3.64$$

$$\varrho_{c2} = \frac{\sigma_{c2}}{\sigma_t} = -4.24 \qquad \varrho_{cc} = \frac{\sigma_{cc}}{\sigma_t} = -5\,.$$

Since in any case $p < -3$, the different domains of Leon all coincide in the tension zone.

In Fig. 29 two points of the theoretical domain in the tension-compression zone are obtained by intersecting the relative domains of Leon with the lines $\mu = 0.55$ and $\mu = 0$. They are in excellent agreement with the experimental data.

Fig. 30, still with reference to Coffin's data, gives the values of ϱ and R_r as functions of μ for the four domains of Fig. 29. They vary appreciably according to a linear law and this, since $\sigma_t = b = \text{const.}$, enables us to assume with good approximation (7):

$$\begin{aligned} \sigma_c(\mu) &= a\mu + b \\ R_r(\mu) &= c\mu + d\,. \end{aligned} \tag{75}$$

(7) (75) could not be simultaneously correct because $\sigma_c = -2R_r - 2\sqrt{2R_r b}$.

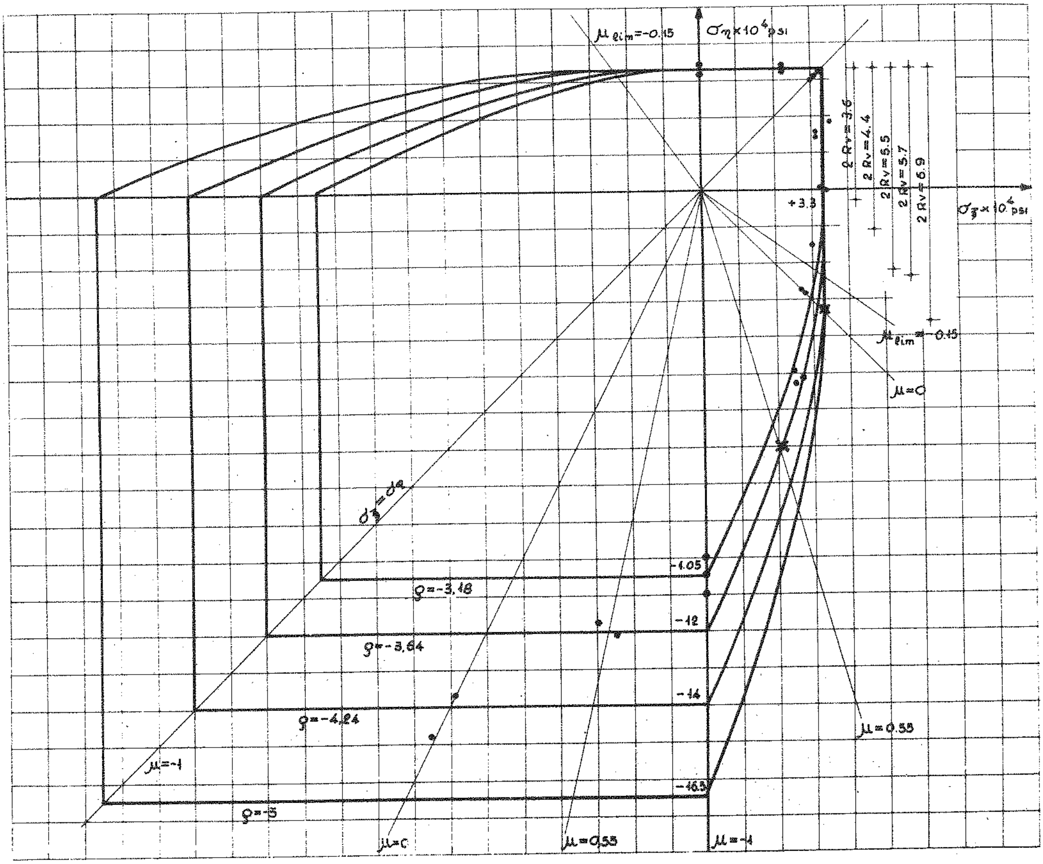

Fig. 29.

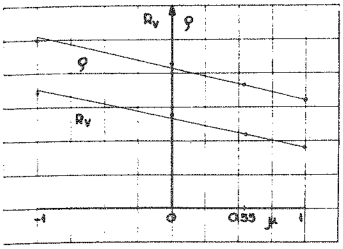

Fig. 30.

From the first of (75) it is easy to obtain an analytical expression for the limit condition in the compression zone. Indeed, since $\sigma_M = 0$, we have:

$$\sigma_m = \sigma_c(\mu) = a\mu + b = a\left(1 - 2\,\frac{\sigma_i}{\sigma_m}\right) + b$$

and hence

$$\sigma_m^2 - (b + a)\sigma_m + 2a\sigma_i = 0 \tag{76}$$

where we may assume:

$$a = \frac{\sigma_{cc} - \sigma_c}{2}$$

and

$$b = \frac{\sigma_{cc} + \sigma_c}{2}.$$

From the second of (75) we deduce the analytical expression for the limit condition in the tension-compression zone. In fact, substituting in (49) we get:

$$2c\mu p + 2dp + r^2 + (2cd - 2ab)\mu - 2db + d^2 - c^2\mu^2 = 0\,. \tag{77}$$

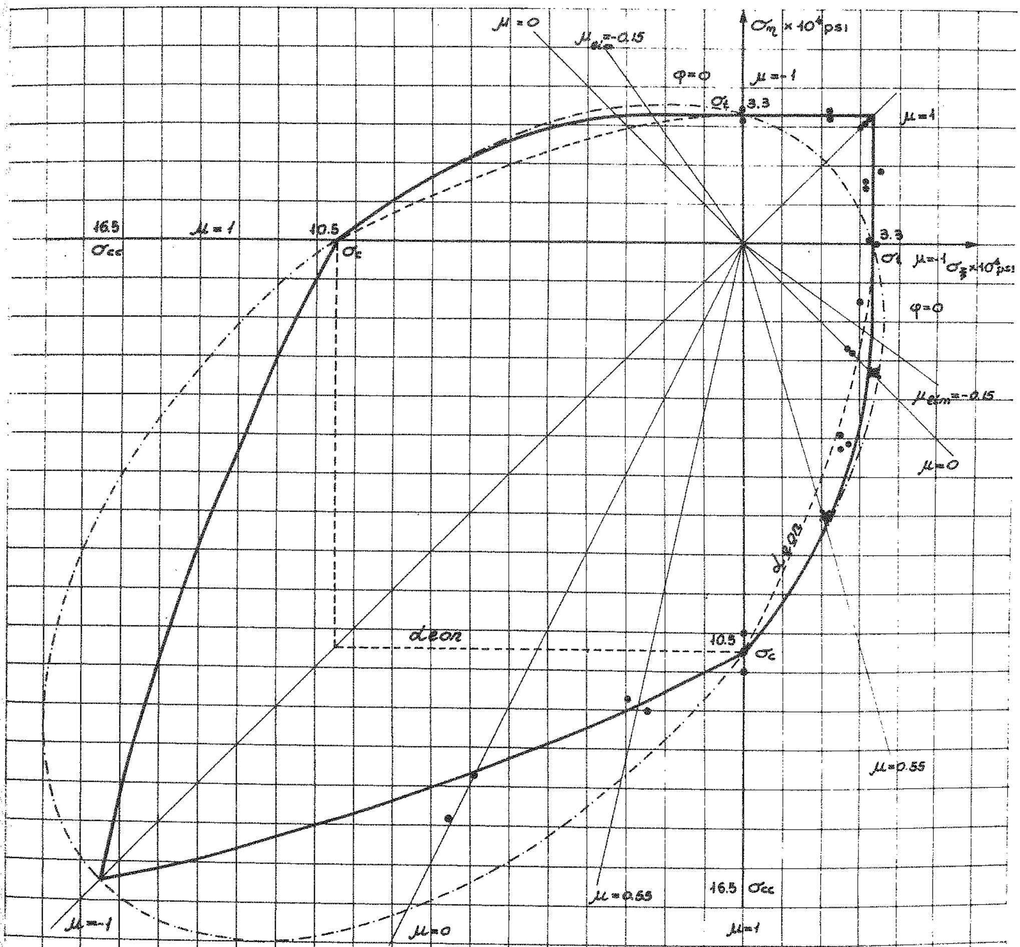

Fig. 31.

Remembering that $\sigma_i = 0$, we have:

$$\mu p = -\frac{(\sigma_M + \sigma_m)^2}{(\sigma_M - \sigma_m)} \qquad \mu^2 = \frac{(\sigma_M + \sigma_m)^2}{(\sigma_M - \sigma_m)}$$

$$p = \frac{\sigma_M + \sigma_m}{2} \qquad r^2 = \frac{(\sigma_M - \sigma_m)^2}{4}$$

$$\mu = -\frac{\sigma_M + \sigma_m}{\sigma_M - \sigma_m}$$

and substituting in (77):

$$(\sigma_M - \sigma_m)^4 + 4(\sigma_M^2 - \sigma_m^2)[d(\sigma_M - \sigma_m) - c(\sigma_M + \sigma_m)] - \\ - 8c(d - b)(\sigma_M^2 - \sigma_m^2) - 4c^2(\sigma_M + \sigma_m)^2 + \\ + 4(b^2 - 2bb)(\sigma_M - \sigma_m)^2 = 0 \qquad (78)$$

where we may assume:

$$c = \frac{R_v(1) - R_v(-1)}{2}$$

and

$$d = \frac{R_v(1) + R_v(-1)}{2}.$$

Parabola (76) and Eq. (78) are represented in Fig. 31, where, in addition to Coffin's experimental points, Leon's domain and Stassi's domain are shown.

The actual value of μ that marks the limit of the tensile fracture range in the tension-compression zone may be determined as follows according to the theory under consideration.

Considering, for example, the half-plane $\sigma_\xi > \sigma_\eta$ we have $\sigma_\eta/\sigma_\xi = \mu + 1/\mu - 1$ and substituting in (44) together with the second of (75) we obtain:

$$\mu_{lim} = \frac{c - d - \sqrt{(c - d)^2 - 4c(b - d)}}{2c}.$$

In the case of Fig. 31 $\mu_{lim} = 0.15$.

From Fig. 31 it is clear that, for the material under consideration, the domain according to the generalised theory coincides with that of Leon in the tension zone, whereas it goes outside in the tension-compression zone, presenting a wider tensile fracture range and good agreement with the experimental values. In the compression zone too approximation is satisfactory and so in this case the hypothesis of linearity of $\sigma_c(\mu)$ is acceptable. In the cases in which this did not occur the domain may be determined by points.

Whilst Stassi's domain virtually coincides with it to some extent in the tension-compression zone, it differs appreciably, above all in the tension zone and in the compression zone, where it does not agree with the experimental data.

The profound difference between the two theories in those zones can be gauged from Figs. 28 and 32, which show the E and F curves corresponding to Coffin's experimental values for $\mu = -1, 0, +1$ respectively according to the generalised theory of Leon and that of Stassi.

The circles of curvature in the vertices of curves E, corresponding to the points of curves F for which $p = -1$, are drawn for the latter also.

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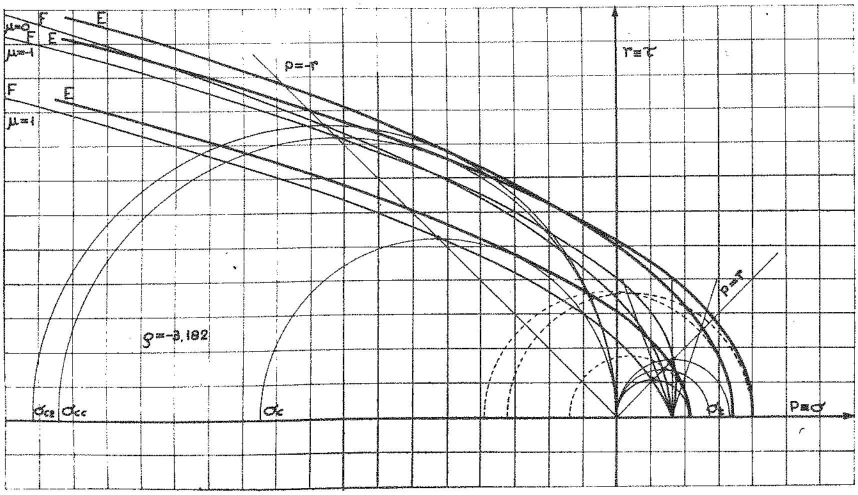

Fig. 32.

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AIMETA

ASSOCIAZIONE ITALIANA DI MECCANICA TEORICA ED APPLICATA

ESTRATTO DAGLI ATTI DEL

1° CONGRESSO NAZIONALE
DI
MECCANICA TEORICA ED APPLICATA

UDINE, 26-30 GIUGNO 1971

M. ROMANO (*)

EQUAZIONI COSTITUTIVE DIFFERENZIALI PER DEFORMAZIONI FINITE DEI MEZZI GRANULARI (**)

Riassunto

Si ricavano le equazioni costitutive per deformazioni finite di mezzi granulari con riferimento ad un semplice modello del comportamento di tali materiali, che non assume l'ipotesi classica dell'esistenza di una superficie limite di plasticità. La densità viene assunta come parametro fondamentale dello stato del materiale e ne influenza fortemente le proprietà meccaniche. Con riferimento alla teoria esposta, sono ricavate le relazioni tensione-deformazione per alcuni casi semplici di moti omogenei e privi di accellerazione. In questo lavoro si è inteso presentare le idee fondamentali della derivazione delle equazioni costitutive proposte, facendo riferimento ad un modello molto semplificato, che però può essere ampliato per interpretare meglio il comportamento reale dei mezzi granulari.

Ciò sarà oggetto di un prossimo lavoro.

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(**) L'argomento di questa nota fa parte di studi condotti dall'autore durante un periodo di ricerca presso l'Istituto per i problemi Fondamentali della Tecnica dell'Accademia della Scienza di Polonia in Varsavia, nell'anno 1970.

Indice di alcuni simboli usati nel testo

$\underset{\sim}{T}$	tensore degli sforzi di Cauchy
$\bar{\underset{\sim}{T}}$	parte deviatorica di $\underset{\sim}{T}$
$\dot{\underset{\sim}{T}}$	derivata temporale di $\underset{\sim}{T}$
$\overset{\circ}{\underset{\sim}{T}}$	derivata temporale invariante o corotazionale di $\underset{\sim}{T}$
p	pressione
$I = \mathrm{tr}\underset{\sim}{T} = -3p$	
$\bar{H} = \frac{1}{2}\mathrm{tr}(\bar{T}^2)$	
$\underset{\sim}{F}$	gradiente di deformazione
$\underset{\sim}{L}$	gradiente spaziale delle velocità
$\underset{\sim}{D}$	gradiente spaziale delle velocità di deformazione, parte simmetrica di $\underset{\sim}{L}$
$\underset{\sim}{W}$	gradiente spaziale delle velocità di rotazione, parte emisimmetrica di $\underset{\sim}{L}$
ϱ	densità
$E = \mathrm{tr}\underset{\sim}{D} = -\frac{\dot{\varrho}}{\varrho}$	
$M = \mathrm{tr}(\underset{\sim}{T}\underset{\sim}{D})$	
$\bar{M} = \mathrm{tr}(\bar{\underset{\sim}{T}}\bar{\underset{\sim}{D}})$	
$M = \mathrm{tr}(\underset{\sim}{T}^2\underset{\sim}{D})$	
λ , μ	costanti di Lamé
Γ	modulo di rigidità volumetrica elastica.

I – Derivazione delle equazioni costitutive

Per costruire un modello teorico che descriva il comportamento dei mezzi granulari, si considerano equazioni costitutive del tipo:

$$\overset{(p)}{\underset{\sim}{T}} = \underset{\sim}{g}(\underset{\sim}{T}\dot{\underset{\sim}{T}}, \dots \overset{(p-1)}{\underset{\sim}{T}}; \underset{\sim}{F}, \underset{\sim}{F}, \dots \overset{(r)}{\underset{\sim}{F}}; \underset{\sim}{\varrho}) \tag{1}$$

dove $\underset{\sim}{g}$ è una funzione tensoriale.

Come risulta dalla (1) la densità $\underset{\sim}{\varrho}$ è assunta come parametro di stato e compare esplicitamente nelle relazioni costitutive. Infatti, come si precisa nel seguito, le variazioni di densità influenzano profondamente il comportamento di un mezzo granulare in un generico processo di deformazione.

Un'equazione del tipo (1), in opportune ipotesi di regolarità per $\underset{\sim}{g}$ ed $\underset{\sim}{F}$, per ogni assegnata storia di deformazione $\underset{\sim}{F}(t)$ a partire dal tempo t_0, e per un sistema di valori iniziali

$$\underset{\sim}{T}(t_0), \dot{\underset{\sim}{T}}(t_0), \dots \overset{(p-1)}{\underset{\sim}{T}}(t_0) \tag{2}$$

ammette un'unica soluzione $\underset{\sim}{T}(t)$.

Si consideri il caso particolare in cui l'equazione (1) è del tipo:

$$\dot{\underset{\sim}{T}} = \underset{\sim}{g}(\underset{\sim}{T};\underset{\sim}{F},\dot{\underset{\sim}{F}};\varrho) . \tag{3}$$

Imponendo che sia soddisfatto il principio di indipendenza dal sistema di riferimento e supponendo che non esista per il materiale in esame una configurazione di riferimento preferenziale, la (3) si può ridurre nella forma:(°)

$$\overset{\circ}{\underset{\sim}{T}} = \underset{\sim}{h}(\underset{\sim}{T},\underset{\sim}{D},\varrho) \tag{4}$$

dove $\overset{\circ}{\underset{\sim}{T}} = \dot{\underset{\sim}{T}} - \underset{\sim}{W}\underset{\sim}{T} + \underset{\sim}{T}\underset{\sim}{W}$ è la derivata invariante corotazionale del tensore degli sforzi, ed $\underset{\sim}{h}$ è una generica funzione tensoriale isotropa dei suoi argomenti $\underset{\sim}{T}$ e $\underset{\sim}{D}$.

L'isotropia della funzione $\underset{\sim}{h}$ deriva direttamente dall'ipotesi di assenza di una configurazione di riferimento preferenziale.

In effetti non è stato dimostrato che la condizione che definisce l'isotropia del materiale, legata al concetto di gruppo d'isotropia, sia soddisfatta dai materiali aventi equazioni costitutive del tipo (4), anche se ciò appare probabile (°°).

Nell'ipotesi che le proprietà meccaniche del materiale in considerazione siano indipendenti dal tempo, si dimostra che se la $\underset{\sim}{h}$ è differenziabile in un intorno di $\underset{\sim}{D} = 0$, deve essere lineare in $\underset{\sim}{D}$. Tale ipotesi si può anche esprimere dicendo che

(°) Equazioni costitutive del tipo (4) sono state considerate da W. Noll (1)

(°°) Per una discussione su questo punto si veda C Truesdell e W. Noll (2), 99.

due storie di deformazione che differiscono solo per un cambiamento della scala dei tempi, determinano lo stesso stato tensionale, e dal punto di vista fisico, significa escludere la presenza di fenomeni viscosi.

Per un teorema di rappresentazione delle funzioni tensoriali isotrope, dovuto a Rivlin e generalizzato da Wang (°) la (4) si può porre allora nella forma:

$$\overset{\circ}{\underset{\sim}{T}} = (a_0 E + a_3 M + a_7 N)\underset{\sim}{1} + (a_2 E + a_6 M + a_{10} N)\underset{\sim}{T} + \\ + (a_5 E + a_9 M + a_{11} N)\underset{\sim}{T}^2 + a_1 \underset{\sim}{D} + \frac{1}{2} a_4 (\underset{\sim}{T}\underset{\sim}{D} + \underset{\sim}{D}\underset{\sim}{T}) + \\ + \frac{1}{2} a_8 (\underset{\sim}{T}^2 \underset{\sim}{D} + \underset{\sim}{D}\underset{\sim}{T}^2) \tag{5}$$

dove i dodici scalari a_i sono funzioni degli invarianti fondamentali del tensore degli sforzi $\underset{\sim}{T}$ e della densità ϱ.

Se si escludono i termini della (5) non lineari in $\underset{\sim}{T}$ e si pone per semplicità, $a_4 = 0$, si ottiene la relazione costitutiva che sarà argomento delle successive considerazioni:

$$\overset{\circ}{\underset{\sim}{T}} = (a_0 E + a_3 M)\underset{\sim}{1} + (a_2 E + a_6 M)\underset{\sim}{T} + a_1 \underset{\sim}{D} . \tag{6}$$

Se si scompone $\overset{\circ}{\underset{\sim}{T}}$ nelle sue parti idrostatica e deviatorica:

$$\overset{\circ}{\underset{\sim}{T}} = \overset{\circ}{\underset{\sim}{\bar{T}}} + \frac{I}{3}\underset{\sim}{1} \tag{7}$$

(°) - R.S. Rivlin (3), C.C. Wang (4).

dove è $\overset{\circ}{\underset{\sim}{\bar{T}}} = \dot{\underset{\sim}{\bar{T}}} - \underset{\sim}{W}\underset{\sim}{\bar{T}} + \underset{\sim}{\bar{T}}\underset{\sim}{W}$, alla (6) si può sostituire il sistema:

$$
(8)\qquad
\begin{aligned}
\dot{I} &= (3a_0 + a_1 + a_2 I)E + (3a_3 + a_6 I)M \\
\overset{\circ}{\underset{\sim}{\bar{T}}} &= a_1\underset{\sim}{\bar{D}} + \left[\left(a_6 - \frac{I}{3} + a_2\right)E + a_6\bar{M}\right]\underset{\sim}{\bar{T}} .
\end{aligned}
$$

Nell'ipotesi che $\dot{I}$ non dipenda dalla parte deviatorica $\underset{\sim}{\bar{D}}$ del gradiente spaziale della velocità di deformazione e che $\overset{\circ}{\underset{\sim}{\bar{T}}}$ non dipenda dalla parte idrostatica $E = \mathrm{tr}\,\underset{\sim}{D}$ dello stesso gradiente, le (8) assumono la forma:

$$
(9)\qquad
\begin{aligned}
\dot{I} &= \left(3a_0 + a_1 - a_6\frac{I^2}{3}\right)E \\
\overset{\circ}{\underset{\sim}{\bar{T}}} &= a_1\underset{\sim}{\bar{D}} + a_6\bar{M}\underset{\sim}{\bar{T}}
\end{aligned}
$$

dove ora a_0, a_1 ed a_6 sono funzioni solo di I e ϱ.
Per ottenere le (9) dalle (8) si è posto, al fine di soddisfare le ipotesi suddette:

$$
a_2 = a_3 = -a_6\frac{I}{3} .
$$

Al sistema (9) si può sostituire l'unica equazione:

$$
(10)\qquad \overset{\circ}{\underset{\sim}{T}} = \left(a_0 + \frac{a_1}{3} - a_6\frac{I^2}{9}\right)E\underset{\sim}{1} + a_1\underset{\sim}{\bar{D}} + a_6 M\underset{\sim}{\bar{T}} .
$$

Al fine di precisare la dipendenza dei coefficienti a_0, a_1 ed a_6 da ϱ, si premettono ora alcune considerazioni sull'influenza che le variazioni di densità hanno sul comportamento di un mezzo granulare soggetto ad un generico processo di deformazione.

Durante il consolidamento primario del materiale vergine, viene continuamente alterata la struttura interna e le conseguenti deformazioni irreversibili, da un punto di vista fenomenologico, ne modificano il comportamento in quello di un materiale di sempre minore deformabilità.

La risposta in tensione ad un generico processo di deformazione è pertanto funzione della densità del mezzo.

Di che tipo di funzione si tratti si può precisare nel modo seguente. Dal punto di vista fisico la densità di un mezzo granulare deve essere maggiore di un valore minimo ϱ_m che rappresenta lo stato in cui il materiale diviene incoerente ed ha resistenza nulla, e minore di un valore massimo ϱ_L che rappresenta lo stato di consolidazione limite in cui il materiale si comporterebbe come un corpo rigido.

Questi due valori ϱ_m e ϱ_M sono dei parametri caratteristici del materiale. E' chiaro che la risposta in tensione ad un generico processo di deformazione deve essere una funzione crescente di ϱ che si annulla in ϱ_m e tende all'infinito per ϱ che tende a ϱ_L .

Se nella prima delle (9) si pone

$$3a_0(\varrho\,,I) \;=\; 3\alpha_0(\varrho\,,I) + a_6(\varrho\,,I)\frac{I^2}{3}$$

si ha:

$$
(11) \qquad
\begin{aligned}
\dot{I} &= (3\alpha_0 + a_1)E \\
\dot{\bar{\underset{\sim}{T}}} &= a_1\bar{\underset{\sim}{D}} + a_6\overline{M}\bar{\underset{\sim}{T}} .
\end{aligned}
$$

Considerando che l'equazione costitutiva di un mezzo elastico isotropo per deformazioni infinitesime, in forma differenziale è:

$$
\dot{\underset{\sim}{T}} = \lambda E \underset{\sim}{1} + 2\mu\underset{\sim}{D}
$$

la corrispondente per deformazioni finite, in forma invariante sarà:

$$
\overset{\circ}{\underset{\sim}{T}} = \lambda E \underset{\sim}{1} + 2\mu\underset{\sim}{D}
$$

oppure:

$$
(12) \qquad
\begin{aligned}
\dot{I} &= (3\lambda + 2\mu)E = 3\Gamma E \\
\overset{\circ}{\bar{\underset{\sim}{T}}} &= 2\mu\bar{\underset{\sim}{D}} .
\end{aligned}
$$

Supponendo che nelle (11) α_0 ed a_1 siano funzioni solo di $\underset{\sim}{\rho}$ e confrontandole con le (12) poniamo:

$$
(13) \qquad \alpha_0 = \lambda_1 G(\underset{\sim}{\rho}) \qquad a_1 = 2\mu_1 G(\underset{\sim}{\rho}) \qquad 3\alpha_0 + a_1 = \Gamma_1 G(\underset{\sim}{\rho}) .
$$

Una conveniente espressione per la $G(\underset{\sim}{\rho})$, in accordo con le considerazioni precedenti è

$$
(14) \qquad G(\underset{\sim}{\rho}) = \frac{\underset{\sim}{\rho} - \underset{\sim}{\rho}_m}{\underset{\sim}{\rho}_L - \underset{\sim}{\rho}} .
$$

Sostituendo le (13) nelle (11) si ha:

$$\dot{I} = 3\Gamma_1 G(\varrho)E$$

$$\overset{\circ}{\bar{\tilde{T}}} = 2\mu_1 G(\varrho)\bar{\tilde{D}} + a_6 \bar{M}\bar{\tilde{T}} . \tag{15}$$

Dove Γ_1 e μ_1 sono costanti caratteristiche del materiale. In tal modo le (15) per deformazioni infinitesime in un intorno di $\varrho = \varrho_0$ e $\bar{\tilde{T}} = 0$, si riducono alle classiche equazioni dell'elasticità infinitesimale, essendo infatti:

$$\overset{\circ}{\bar{\tilde{T}}} = \dot{\bar{\tilde{T}}} - \tilde{W}\bar{\tilde{T}} + \bar{\tilde{T}}\tilde{W} = \dot{\bar{\tilde{T}}} , \quad \text{per} \quad \bar{\tilde{T}} = 0$$

$$\Gamma_1 G(\varrho_0) = \Gamma_a = \text{cost} \qquad \mu_1 G(\varrho_0) = \mu_a = \text{cost} .$$

In tal caso il materiale presenta quindi un comportamento elastico lineare reversibile. Si caratterizza ora il comportamento globale considerando dapprima la risposta a deformazioni idrostatiche. E' chiaro, dalla prima delle (15), che durante un processo di consolidamento primario del materiale vergine, questo si comporta da un punto di vista fenomenologico, come un mezzo elastico con modulo di deformabilità volumetrica funzione decrescente della densità. Nel corso di un tale processo risulta $E\dot{I} \geq 0$ (processo di carico). Se si arresta il consolidamento e si fa diminuire la densità durante questo successivo processo risulta $E\dot{I} < 0$ (processo di scarico), ed il comportamento del materiale è diverso perchè durante la fase precedente hanno avuto luogo alcune alterazioni irreversibili della struttura interna.

Un'ipotesi abbastanza vicina alla realtà fisica dei fenomeni in esame, è che il comportamento del materiale sia ora funzione della densità massima ϱ_M raggiunta nel processo di consolidamento primario e che per processi di deformazione, sia di carico che di scarico, che hanno luogo in un campo di densità minori di ϱ_M, il materiale si comporta come un mezzo elastico, nel senso che presenta solo deformazioni reversibili. In questo caso la risposta del materiale alle deformazioni è funzione non solo della densità variabile, ma anche della massima densità raggiunta precedentemente nel processo di consolidamento primario, ϱ_M, della quale il materiale ha quindi memoria.

Tenendo presenti anche considerazioni di carattere termodinamico, si assume come legame costitutivo relativo a questa seconda categoria di fenomeni sempre la prima delle (15), dove però la funzione $G(\varrho)$ ha ora la determinazione:

$$G(\varrho) = \frac{\varrho - \varrho_m}{\varrho_L - \varrho_M} . \tag{16}$$

Infine è ammissibile supporre che se il materiale nello stato vergine, alla densità ϱ_m è sottoposto ad una pressione, p_m, tale valore si può assumere come un limite in feriore per le pressioni, al di sotto del quale avviene la rottura. Deve quindi risultare sempre:

$$p > p_m \, . \tag{17}$$

Per quanto riguarda la risposta a processi di deformazione deviatorici, le cose sono notevolmente diverse. In realtà tali processi, più che vere e proprie alterazioni della struttura interna del materiale, provocano scorrimenti in parte irreversibili che però lasciano praticamente inalterate le caratteristiche di deformabilità del materiale. Si suppone che il valore delle tensioni che accompagnano i processi di deformazione deviatorici per i quali risulta $\bar{M} \geq 0$ (processi di carico) siano limitate da una condizione imposta all'invariante quadratico $\bar{H} = \frac{1}{2} \mathrm{tr}\, \bar{\underset{\sim}{T}}^2$, e precisamente:

$$(\bar{H} = B_1^2 G^2(\underset{\sim}{\varrho})) \Longrightarrow (\dot{\bar{H}} = 0) \, . \tag{18}$$

La costante B_1 è una caratteristica del materiale definita positiva.

Durante i processi di deformazione per i quali è $\bar{M} < 0$ (processi di scarico), la risposta del materiale si suppone di tipo elastico ponendo nella (15) $a_6 = 0$.

Per quanto riguarda la condizione espressa dalla (18), si osserva che, dalla seconda delle (15), moltiplicando primo e secondo membro per $\bar{\underset{\sim}{T}}$ si ha:

$$\dot{\bar{H}} = (2\mu_1 G(\underset{\sim}{\varrho}) + 2\bar{H} a_6)\bar{M} \, . \tag{19}$$

Perchè sia verificata la (18) deve risultare

$$(\bar{H} = B_1^2 G^2(\underset{\sim}{\varrho})) \Longrightarrow (2\mu_1 G(\underset{\sim}{\varrho}) + 2\bar{H}a_6 = 0)$$

cioè deve essere $2B_1^2 G^2(\underset{\sim}{\varrho})a_6 = -2\mu_1 G(\underset{\sim}{\varrho})$, da cui

$$a_6 = -\frac{\mu_1}{B_1^2 G(\underset{\sim}{\varrho})} \qquad (20)$$

In definitiva allora le (15) si possono scrivere nella forma:

$$\begin{aligned} \dot{I} &= 3\Gamma_1 G(\underset{\sim}{\varrho})E \\ \overset{\circ}{\bar{\underset{\sim}{T}}} &= 2\mu_1 G(\underset{\sim}{\varrho})\left(\bar{\underset{\sim}{D}} - \frac{\bar{M}\bar{\underset{\sim}{T}}}{2B_1^2 G^2(\underset{\sim}{\varrho})}\right) \end{aligned} \qquad (21)$$

per $\bar{M} \geq 0$,

$$\begin{aligned} \dot{I} &= 3\Gamma_1 G(\underset{\sim}{\varrho})E \\ \overset{\circ}{\bar{\underset{\sim}{T}}} &= 2\mu_1 G(\underset{\sim}{\varrho})\bar{\underset{\sim}{D}} \end{aligned} \qquad (22)$$

per $\bar{M} < 0$.

E' da notare che sia nelle (21) che nelle (22) la $G(\underset{\sim}{\varrho})$ è espressa dalla (14) se la deformazione idrostatica è un processo di consolidamento primario del materiale vergine, dalla (16) in tutti gli altri casi.

E' interessante osservare che le equazioni (21), e così anche le (22), possono essere risolte indipendentemente, fornendo rispettivamente la parte idrostatica e quella deviatorica di $\underset{\sim}{T}$.

Sostituendo la (20) nella (19), questa ultima si scrive per $\bar{M} \geq 0$

$$\dot{\bar{H}} = 2\mu_1 G(\varrho)\left(1 - \frac{\bar{H}}{B_1^2 G^2(\varrho)}\right)\bar{M} . \tag{23}$$

L'equazione differenziale (23) ammette assegnato $\bar{\underset{\sim}{T}}(t_0)$, una sola soluzione $\bar{H}(t)$, che tende asintoticamente al valore $B_1^2 G^2(\varrho)$, perchè, essendo sempre $\bar{H} \geq 0$, per $\bar{H} = B_1^2 G^2(\varrho)$ è $\dot{\bar{H}} > 0$ e per $\bar{H} > B_1^2 G^2(\varrho)$ è $\dot{\bar{H}} < 0$. Si esamina ora in maggiore dettaglio l'andamento della $\bar{H}(t)$ nei tre casi fondamentali in cui per $t > t_0$ la ϱ è costante, aumenta o diminuisce rispettivamente. Per $t = t_0$ si suppone in ogni caso $\bar{H} < B_1^2 G^2(\varrho)$.

a) Se $\varrho = \varrho_0$ la $\bar{H}(t)$ è sempre crescente e tende asintoticamente al valore costante $B_1^2 G^2(\varrho_0)$, senza mai raggiungerlo. Infatti in tal caso lo assumerebbe definitivamente e, per l'unicità della soluzione delle (23), ciò sarebbe possibile solo se risultasse identicamente $\bar{H}(t) = B_1^2 G^2(\varrho_0)$, contro l'ipotesi $\bar{H}(t_0) < B_1^2 G^2(\varrho_0)$. Pertanto deve essere sempre $\bar{H}(t) < B_1^2 G^2(\varrho_0)$.

b) Se è $\dot{\varrho} > 0$ si mostra che è sempre $\bar{H}(t) < B_1^2 G^2(\varrho)$. Infatti se per $t = t_1$ e $\varrho(t_1) = \varrho_1$ fosse $\bar{H}(t_1) = B_1^2 G^2(\varrho_1)$ si avrebbe anche $\dot{\bar{H}}(t_1) = 0$. Inoltre derivando la (23), poichè è $\dot{G} > 0$, e per ipotesi $\bar{M} > 0$ si ha:

$$\ddot{\bar{H}}(t_1) = 4\mu_1 \bar{M}\bar{H} \frac{\dot{G}}{B_1^2 G^2(\varrho_1)} > 0 .$$

Quindi la $\bar{H}(t)$ dovrebbe avere in t_1 un punto di minimo relativo e ciò è assurdo perchè essa è crescente per ogni $t < t_1$.

c) Se è $\dot{\varrho} < 0$ è invece possibile che risulti $\bar{H}(t_1) = B_1^2 G^2(\varrho_1)$ poichè in tal caso, essendo $\dot{\bar{H}}(t_1) = 0$ e $\ddot{\bar{H}}(t_1) < 0$, la $\bar{H}(t)$ ha in t_1 un punto di massimo relativo e ciò è in accordo col fatto che è crescente per $t < t_1$.

In questo caso anche almeno una delle componenti di $\bar{\underset{\sim}{T}}$ ha un punto di massimo relativo in t_1, perchè tutte le componenti di $\bar{\underset{\sim}{T}}$ figurano al quadrato nell'espressione di $\bar{H}$.

Nelle figure 1, 2 e 3 si mostra l'andamento della $\bar{H}(t)$ nei tre casi discussi e per $\bar{\underset{\sim}{T}}(t_0) = 0$.

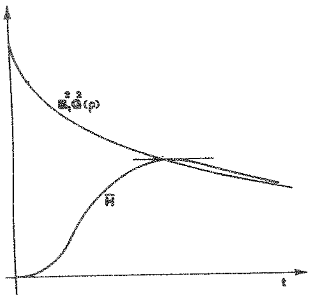

Fig. 1

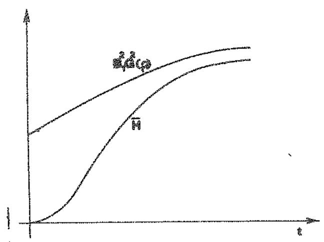

Fig. 2

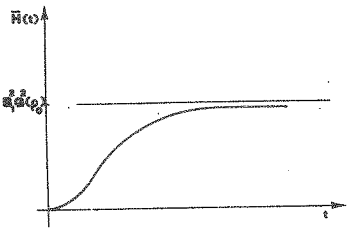

Fig. 3

II – Stabilità

E' importante studiare la stabilità di un materiale le cui proprietà meccaniche sono descritte dalle equazioni (21) e (22). A tal fine è necessario precisare il concetto di stabilità per deformazioni finite. La definizione che sembra più naturale è quella che considera stabile la risposta ad un processo di deformazione quando si ha $tr(\overset{\circ}{\underset{\sim}{T}}\underset{\sim}{D}) \geq 0$ ed instabile quando invece risulta $tr(\overset{\circ}{\underset{\sim}{T}}\underset{\sim}{D}) < 0$.

E' da notare che tale definizione coincide con quella che usualmente si dà nella teoria incrementale della plasticità, per deformazioni infinitesime.

In generale si ha:

$$tr(\overset{\circ}{\underset{\sim}{T}}\underset{\sim}{D}) = tr(\overset{\circ}{\underset{\sim}{\bar{T}}}\underset{\sim}{\bar{D}}) + \frac{E\dot{I}}{3} . \qquad (24)$$

Dalla prima delle (21) e delle (22), si ha:

$$E\dot{I} = 3\Gamma_1 G(\varrho)E^2 \geq 0 . \qquad (25)$$

Essendo Γ_1 e $G(\varrho)$ definite positive. La risposta del materiale a processi di deformazione idrostatica è pertanto sempre stabile.

Dalla seconda delle (21) si ha, per i processi di carico:

$$tr(\overset{\circ}{\underset{\sim}{\bar{T}}}\underset{\sim}{\bar{D}}) = 2\mu_1 G(\varrho)\left(tr\underset{\sim}{\bar{D}}^2 - \frac{\bar{M}^2}{2B_1^2 G^2(\varrho)}\right) . \qquad (26)$$

Dalla (26) si deduce che i processi di carico non danno sempre luogo a risposte stabili, anche se si riconosce facilmente che in un conveniente intorno di $\underset{\sim}{\bar{T}} = 0$ la risposta è sempre stabile perchè $\bar{M}$ si può rendere piccolo quanto si vuole. Dalla seconda delle (22), per i processi di scarico, si ha:

$$tr(\overset{\circ}{\underset{\sim}{\bar{T}}}\underset{\sim}{\bar{D}}) = 2\mu_1 G(\varrho) tr\underset{\sim}{\bar{D}}^2 \geq 0 . \qquad (27)$$

La (27) mostra che la risposta a processi di deformazione deviatorici di scarico è sempre stabile.

III – Le relazioni tensione-deformazione

Le relazioni tensione-deformazione non sono assegnate a priori ma sono il risultato della teoria sviluppata.

Per ottenere tali relazioni, caso per caso, si deve assegnare la storia di deformazione, a partire da un istante iniziale t_0, tramite la $\underset{\sim}{F}(t)$ che rappresenta il gradiente di deformazione relativo alla configurazione al tempo t_0 e specificare i valori di $\underset{\sim}{T}(t_0)$ e $\varrho(t_0)$.

La funzione $\varrho(t)$ è allora determinata dalla relazione $|\det \underset{\sim}{F}(t)| = \dfrac{\varrho(t_0)}{\varrho(t)}$.

La risposta in tensione $\underset{\sim}{T}(t)$ si ottiene risolvendo il sistema di equazioni costitutive. E' importante a questo riguardo osservare che insieme alle equazioni costitutive deve essere soddisfatta anche la prima delle equazioni fondamentali del moto, di Cauchy

$$\mathrm{div}\,\underset{\sim}{T} + \varrho\underset{\sim}{b} = \varrho\ddot{\underset{\sim}{x}} \tag{28}$$

dove $\underset{\sim}{b}$ è la forza di massa e $\ddot{\underset{\sim}{x}}$ l'accelerazione.

La determinazione di una funzione $\underset{\sim}{T}(t)$ che sia contemporaneamente soluzione della (28) e dalle equazioni costitutive è molto ardua.

In materiali omogenei, con equazioni costitutive del tipo (1), moti omogenei, descritti da equazioni del moto nella forma:

$$\underset{\sim}{x}(t) = \underset{\sim}{F}(t)\underset{\sim}{X} + \underset{\sim}{c}(t) \tag{29}$$

dove $\underset{\sim}{F}$ e $\underset{\sim}{c}$ dipendono solo dal tempo t, determinano stati tensionali anch'essi omogenei. Si ha quindi in tal caso $\operatorname{div}\underset{\sim}{T} = 0$ e la (28) si riduce a:

$$\underset{\sim}{b} = \ddot{\underset{\sim}{x}} \tag{30}$$

Nei casi di interesse pratico la forza di massa è rappresentata da un campo gravitazionale praticamente uniforme e costante, cioè risulta $\underset{\sim}{b}=\underset{\sim}{g}=\text{cost}$. Ciò implica che sia $\ddot{\underset{\sim}{c}}=\underset{\sim}{g}$, cioè $\underset{\sim}{c}=\frac{1}{2}t^2\underset{\sim}{g}+ t\underset{\sim}{e}+\underset{\sim}{f}$, dove $\underset{\sim}{e}$ ed $\underset{\sim}{f}$ sono dei vettori costanti, e che risulti:

$$\underset{\sim}{F}(t) = \underset{\sim}{F}_0(\underset{\sim}{1} + t\underset{\sim}{F}_1) \tag{31}$$

dove $\underset{\sim}{F}_0$ ed $\underset{\sim}{F}_1$ sono dei tensori costanti. Cioè il moto (29) è somma di un moto di traslazione con accelerazione costante, $\ddot{\underset{\sim}{x}} = \underset{\sim}{g}$ di equazione

$$\underset{\sim}{x}(t) = \frac{1}{2}t^2\underset{\sim}{g} + t\underset{\sim}{e} + \underset{\sim}{f} \tag{32}$$

e di un moto di deformazione omogeneo e privo di accelerazione

$$\underset{\sim}{x}(t) = \underset{\sim}{F}_0(\underset{\sim}{1} + t\underset{\sim}{F}_1)\underset{\sim}{X}\,. \tag{33}$$

Il corrispondente gradiente di velocità è allora fornito da:

$$\underset{\sim}{L}(t) = \underset{\sim}{D}(t) + \underset{\sim}{W}(t) = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} = \underset{\sim}{F}_0\underset{\sim}{F}_1(\underset{\sim}{1} + t\underset{\sim}{F}_1)^{-1}\underset{\sim}{F}_0^{-1}\,. \tag{34}$$

In molti casi è possibile trascurare l'influenza delle forze di massa e quindi porre $\underset{\sim}{b} = \ddot{\underset{\sim}{c}} = \ddot{\underset{\sim}{x}} = 0$, ed anche $\underset{\sim}{c} = 0$.

Il moto (33) è regolare ed $\underset{\sim}{F}$ invertibile, se e solo se $|\det \underset{\sim}{F}| \neq 0$, cioè se e solo se $\underset{\sim}{F}_1$ non ha autovalori reali.

Si ricavano ora le relazioni tensione-deformazione o, il che è lo stesso, le risposte in tensione $\underset{\sim}{T}(t)$ per alcuni tipi di moti di deformazione privi di accelerazione.

1) Moto idrostatico.

L'equazione del moto è:

$$\underset{\sim}{x}(t) = (1 + k_i t)\underset{\sim}{X} \tag{35}$$

dove k_i è una costante, positiva se il moto è di espansione, negativa se di contrazione. Dalla (35) si ha:

$$\underset{\sim}{F}(t) = (1 + K_i t)\underset{\sim}{1} \qquad |\det \underset{\sim}{F}| = \frac{\varrho(0)}{\varrho(t)} = (1 + k_i t)^3 .$$

Il moto non ha quindi singolarità se di espansione $(k_i > 0)$, ed ha una singolarità per $t = \frac{-1}{K_i}$ dove è $\varrho = \infty$, se di contrazione $(K_i < 0)$. Si ha

$$\underset{\sim}{L} = \underset{\sim}{D} = \frac{K_i}{1 + K_i t}\underset{\sim}{1} = -\frac{1}{3}\dot{\overline{\log \varrho}}\,\underset{\sim}{1} \qquad \underset{\sim}{W} = 0$$

Essendo $\overline{\underset{\sim}{D}} = 0$, per la seconda delle (21) o delle (22) è $\overset{\circ}{\overline{\underset{\sim}{T}}} = 0$.
L'equazione costitutiva si scrive allora:

$$\dot{p} = \Gamma_1 G(\varrho)\frac{\dot{\varrho}}{\varrho} . \tag{36}$$

Poichè sia che si consideri la (14), che la (16), risulta $G(\varrho) \geq 0$, dalla (36) si deduce:

$$\frac{dp}{d\varrho} = \Gamma_1 \frac{G(\varrho)}{\varrho} \geq 0 . \tag{37}$$

Inoltre è facile verificare che si ha sempre $\frac{d^2 p}{d\varrho^2} > 0$.

La (37), integrata in corrispondenza delle due determinazioni (14) e (16) di $G(\varrho)$, rispettivamente fornisce le relazioni pressione-densità:

$$p = p_m + \Gamma_1\left[\left(\frac{\varrho_m}{\varrho_L} - 1\right)\log\frac{\varrho_L - \varrho}{\varrho_L - \varrho_m} - \frac{\varrho_m}{\varrho_L}\log\frac{\varrho}{\varrho_m}\right] \tag{38}$$

per il processo di consolidamento del materiale vergine, e

$$p = p_M + \frac{\Gamma_1}{\varrho_L - \varrho_M}\left(\varrho - \varrho_M - \varrho_m\log\frac{\varrho}{\varrho_M}\right) \tag{39}$$

negli altri casi. I Diagrammi delle (38) e (39) sono riportati in fig. 4.

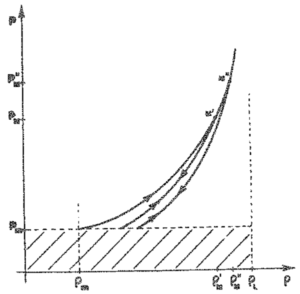

Fig. 4

2) Taglio semplice.

Le equazioni del moto sono:

$$\begin{aligned} x_1 &= X_1 + 2k_t t X_2 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \tag{40}$$

dove è $2k_t t = \mathrm{tg}\vartheta$, e $\vartheta = \mathrm{arctg}\,2k_t t$ è l'angolo di cui è ruotato al tempo t un piano inizialmente normale all'asse delle x_1. Dalle (40) si ha:

$$\underset{\sim}{F}(t) = \begin{vmatrix} 1 & 2k_t t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad |\det \underset{\sim}{F}| = \frac{\varrho(0)}{\varrho(t)} = 1 .$$

Il moto non ha quindi singolarità. Si ha inoltre

$$\underset{\sim}{L} = \begin{vmatrix} 0 & 2k_t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \underset{\sim}{D} = \underset{\sim}{\bar{D}} = \begin{vmatrix} 0 & k_t & 0 \\ k_t & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \underset{\sim}{W} = \begin{vmatrix} 0 & k_t & 0 \\ -k_t & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} .$$

Essendo $E = \mathrm{tr}\underset{\sim}{D} = \frac{-\dot{\varrho}}{\varrho} = 0$ si ha $\varrho = \varrho_0 = \mathrm{cost.}$ e per la prima delle (21) e delle (22), $\dot{I} = -3\dot{p} = 0$, cioè $p = \mathrm{cost.}$

L'equazione costitutiva si scrive quindi, posto $G(\varrho_0) = G_a$,

$$\dot{\underset{\sim}{\bar{T}}} = \underset{\sim}{W}\underset{\sim}{\bar{T}} - \underset{\sim}{\bar{T}}\underset{\sim}{W} + 2\mu_1 G_a\left(\underset{\sim}{\bar{D}} - \frac{\bar{M}}{2B_1^2 G^2}\underset{\sim}{\bar{T}}\right) \tag{41}$$

se $\bar{M} \geq 0$,

$$\dot{\underset{\sim}{\bar{T}}} = \underset{\sim}{W}\underset{\sim}{\bar{T}} - \underset{\sim}{\bar{T}}\underset{\sim}{W} + 2\mu_1 G_a \underset{\sim}{\bar{D}} \tag{42}$$

se $\bar{M} < 0$.

Supponiamo che sia:

$$\underset{\sim}{\bar{T}} = \begin{vmatrix} Q & S & 0 \\ S & -Q & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Ciò è lecito perchè se le altre componenti di $\bar{\underset{\sim}{T}}$ sono nulle al tempo $t = 0$, per la (41) o (42), tali sono anche le loro derivate, e quindi risultano definitivamente nulle.

Si ha quindi:

$$\bar{M} = tr(\underset{\sim}{\bar{T}}\underset{\sim}{\bar{D}}) = 2k_t S \qquad \bar{H} = \frac{1}{2}tr(\underset{\sim}{\bar{T}}^2) = S^2 + Q^2 .$$

Le (41) e (42) si possono scrivere in termini di componenti

$$\begin{aligned} \dot{Q} &= 2k_t S\left(1 - \frac{\mu_1}{B_1^2 G_a} Q\right) \\ \dot{S} &= k_t\left[-2Q + 2\mu_1 G_a\left(1 - \frac{S^2}{B_1^2 G_a^2}\right)\right] \end{aligned} \tag{43}$$

e

$$\begin{aligned} \dot{Q} &= 2k_t S \\ \dot{S} &= k_t(-2Q + 2\mu_1 G_a) . \end{aligned} \tag{44}$$

Per $\bar{M} > 0$ si ha inoltre:

$$tr(\dot{\bar{\underset{\sim}{T}}}\underset{\sim}{\bar{D}}) = 4\mu_1 k_t^2 G_a\left(1 - \frac{S^2}{B_1^2 G_a^2}\right) . \tag{45}$$

Dalla (45) si riconosce che la risposta a processi di deformazione di questo tipo è sempre stabile. Infatti essendo: $\varrho = \varrho_0$ si ha $\bar{H} = S^2 + Q^2 \leq B_1^2 G_a^2$ e quindi $S^2 < B_1^2 G_a^2$ e $tr(\dot{\bar{\underset{\sim}{T}}}\underset{\sim}{\bar{D}}) > 0$. Ciò nonostante la S(t) può avere un diagramma di tipo instabile. Un esempio è rappresentato in fig. 5. (Si suppone $B_1 < \mu_1$).

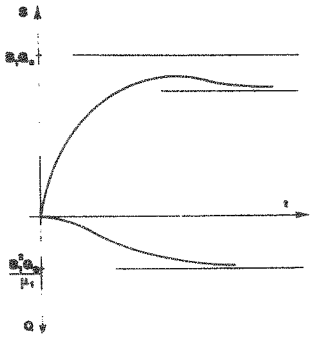

Fig. 5

3) Estensione o contrazione semplice.

Le equazioni del moto sono:

$$
\begin{aligned}
x_1 &= X_1 - k_0\sigma_0 t X_1 \\
x_2 &= X_2 - k_0\sigma_0 t X_2 \\
x_3 &= X_3 + k_0 t X_3
\end{aligned}
\tag{46}
$$

dove è $\sigma_0 \geq 0$ e $k_0 > 0$ se il moto è di estensione semplice, $k_0 < 0$ se di contrazione semplice. Dalle (46) si ha:

$$
\underset{\sim}{F} = \begin{vmatrix} 1 - k_0\sigma_0 t & 0 & 0 \\ 0 & 1 - k_0\sigma_0 t & 0 \\ 0 & 0 & 1 + k_0 t \end{vmatrix}
$$

$$
|\det \underset{\sim}{F}| = \frac{\varrho(0)}{\varrho(t)} = (1 - k_0\sigma_0 t)^2 (1 + k_0 t) \, .
$$

Il moto ha una singolarità per $t = \dfrac{1}{k_0\sigma_0}$ se di estensione, e per $t = -\dfrac{1}{k_0}$ se di contrazione.

In corrispondenza di tali singolarità si ha $\varrho = \infty$.

Si ha inoltre:

$$\underset{\sim}{L} = \underset{\sim}{D} = k \begin{vmatrix} -\sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \underset{\sim}{W} = 0$$

dove è $k = \dfrac{k_0}{1 + k_0 t}$ e $\sigma = \sigma_0 \dfrac{1 + k_0 t}{1 - k_0 \sigma_0 t}$.

Risulta $E = \det \underset{\sim}{D} = k(1 - 2\sigma) = -\dfrac{\dot{\varrho}}{\varrho}$ e

$$\overline{\underset{\sim}{D}} = k \begin{vmatrix} -\dfrac{1+\sigma}{3} & 0 & 0 \\ 0 & -\dfrac{1+\sigma}{3} & 0 \\ 0 & 0 & 2\dfrac{1+\sigma}{3} \end{vmatrix} .$$

Supponiamo che sia

$$\overline{\underset{\sim}{T}} = \begin{vmatrix} -S & 0 & 0 \\ 0 & -S & 0 \\ 0 & 0 & 2S \end{vmatrix} .$$

Ciò è lecito perchè se le altre componenti di $\overline{\underset{\sim}{T}}$ sono nulle al tempo $t = 0$, per la seconda delle (21) o delle (22), tali sono anche le loro derivate, e quindi risultano definitivamente nulle.

Si ha allora $\overline{M} = 2Sk(1 + \sigma)$.

La seconda delle (21) e (22), in termini di componenti si scrivono rispettivamente:

$$\dot{S} = 2\mu_1 G(\varrho)k(1 + \sigma)\frac{1}{3}\left(1 - \frac{3S^2}{B_1^2 G^2(\varrho)}\right) \tag{47}$$

e

$$\dot{S} = 2\mu_1 G(\varrho)k\frac{1 + \sigma}{3} \; . \tag{48}$$

E' interessante osservare che essendo $\bar{H} = 3\,S^2$ e

$$tr(\mathring{\bar{\underset{\sim}{T}}}\bar{\underset{\sim}{D}}) = 2\mu_1 G(\varrho)\frac{2}{3}k^2(1 + \sigma)^2\left(1 - \frac{3S^2}{B_1^2 G^2(\varrho)}\right) \tag{49}$$

la risposta del materiale alla parte deviatorica della deformazione diviene instabile quando è $\bar{H} > B_1^2 G^2(\varrho)$.

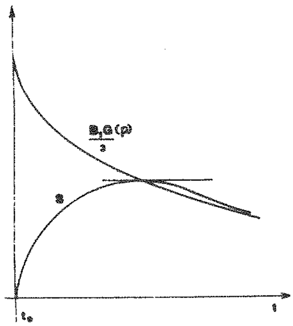

Fig. 6

Siccome $\dot{\varrho}(0)$ è negativo per il moto di estensione ($k_0 > 0$) se $\sigma_0 < \frac{1}{2}$, e per il moto di contrazione se $\sigma_0 > \frac{1}{2}$, mentre in ogni caso per $t > \frac{1 - 2\sigma_0^2}{3k_0\sigma_0}$ è $\dot{\varrho} > 0$, solo per tali valori di σ_0 rispettivamente nei due moti è possibile che si abbia $\bar{H} > B_1^2 G^2(\varrho)$ e quindi un comportamento instabile della S(t). Il diagramma di fig. 6 si riferisce a tale circostanza.

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A continuum theory for granular media with a critical state

M. ROMANO (NAPLES)

IN THIS work an attempt is presented to formulate a theory of the mechanical behaviour of granular media for finite deformations. The critical state assumption is fundamental, while no idea of plastic limit condition is presented. No true elastic range is exibited. Dilatancy effects are taken into account. An explicit form of the constitutive equations has been obtained and stress-strain relations are presented both for loading and unloading processes.

W pracy przedstawiono próbę sformułowania teorii mechanicznego zachowania się materiałów ziarnistych przy odkształceniach skończonych. Podstawowym założeniem pracy jest przyjęcie hipotezy stanu krytycznego przy równoczesnym odrzuceniu idei warunku plastyczności. W rozważaniach pominięto zakres sprężysty. Uwzględniono efekty dylatancji. Otrzymano jawną postać równań konstytutywnych, w których związki między naprężeniami i odkształceniami obowiązują zarówno w procesie obciążenia jak i odciążenia.

В работе представлена попытка формулировки теории механического поведения зернистых материалов при конечных деформациях. Основным предположением является принятие гипотезы критического состояния при одновременном непринятие идеи условия пластичности. В рассуждениях пренебрегается упругой областью. Учтены эффекты дилатации. Получен явный вид определяющих уравнений, в которых соотношения между напряжениями и деформациями обязывают так в процессе нагрузки, как и разгрузки.

1. Introduction

THE GRANULAR media this paper will be dealing with are isotropic, one-phase, non viscous, cohesionless solids. They are conceived as the continuum idealization of real materials as e.g. dry sand with uniform grain size. Many soil materials as well as some organic or artificial granular materials can show, at least under convenient assumptions, similar mechanical properties. In this context, the aim of the present paper is to develop, by a phenomenological approach, a theoretical model for the mechanical behaviour of granular media that can give an appropriate description of some basic properties of these materials. Let us now introduce the subject by some brief considerations that by no means intend to be a review of the overwhelming literature in this field. Only very few specific works will be listed, as being more strictly connected with the ideas that will be presented in this paper.

Classical soil mechanics theories were conceived to predict failure of soils, rather than to obtain a good description of real deformation processes. Such a point of view is evident in the assumption of a rigid or elastic behaviour up to rupture, and a Coulomb type failure condition. More attention to the deformation process brought about the formulations of new, more sophisticated models. Best developed and most satisfactory are the theories considering granular media as elasto-plastic materials admitting plastic potential with associated flow rule. The first attempts in this direction failed because, e.g. assuming a Coulomb type limit surface as yield surface, too large volume increase during plastic flow was predicted [1]. But these shortcomings are absent in the theories that consider the yield surface as a function of the density, giving account of the softening and hardening due to density variations. Worth mentioning are two main approaches. The Cambridge theory [2] developed the critical state conception in the framework of an elasto-plastic model, obtaining the plastic potential function, i.e. the yield surface, by the assumption of a special simple form for the specific power dissipation during plastic flow, which is supposed to be of "frictional" nature. On a similar line is Mróz theory of density hardening media [3] that can be considered, to a certain extent, a generalization of the previous one. This approach has also more sound foundations in general plasticity theory. The plastic potential is in general a function of the stress invariants and of the density, and can be chosen in the most convenient form. This theory has been generalized to allow for non-linear behaviour in the elastic range [16]. Many serious objections have been raised against classical elasto-plastic models with regard to the description of plastic deformations of metals and much more can be said thinking of granular media. In general, no elastic range is exhibited by these materials, in the sense that also small deformations are partially irreversible. For this reason ROSCOE and BURLAND presented a generalization of Cambridge theory for "wet" clays [4], giving account of irreversible shear deformations that take place beneath the yield surface, confirming so the non-existence of a true elastic region. Moreover, the stress strain relations are very smooth, so that also if irreversible deformations up to a certain amount are disregarded, the definition of conventional yield points is arbitrary. Different definitions give rise to different experimental determination of the initial yield surface. These differences can be important and any way become non-negligible for the subsequent yield surfaces, e.g. during hardening of the material, giving rise to big discrepancies between the deformation histories associated. It must be said that while experimental evidence shows that for granular media the relevant processes involve very large deformations, no one of the previously mentioned theories is properly formulated to take account of finite deformations. So there are good reasons to reject the previous approaches and to try new ones.

T. Y. THOMAS first attempted to develop a theory of the plastic behaviour of non-viscous metals without a yield surface [5]. Assuming a special form of Truesdell's Hypo-elastic [6, 7] constitutive relations, he described continuous transition from elastic to perfectly plastic behaviour during loading processes. The form of the constitutive relations range from the incremental version for infinitesimal elasticity (for zero stresses), to a correctly invariant form for finite deformations of the Prandtl-Reuss equations for isochoric perfectly plastic flow (when von Mises condition is satisfied). This is obtained by an appropriate choice of the constitutive functions. It is clear that von Mises condition in this case looses the meaning of yield limit, conceived in the classical sense, but is a limit condition, never definitively satisfied but asymptotically approached when deviatoric deformations increase. This interpretation is in perfect agreement with Truesdell's observation that during simple shear of a particular hypoelastic material of grade 2 (that can be considered a special case of Thomas material for purely deviatoric deformations) von Mises yield is never reached, but only asymptotically approached when shear deformations increase. T. Y. THOMAS developed also a similar theory for von Mises plasticity and with a reference to a generalized von Mises yield condition [8]. The Hypoelastic yield observed by TRUESDELL is of purely "mathematical" nature. A more general approach was presented by A. E. GREEN [9, 10], always in connection with the formulation of a theory for plastic flow of metals. By means of a representation theorem he reduced the general Hypoelastic constitutive relations to a tensorial polynomial form. Then he assumed a definition of loading and unloading processes in terms of the sign of the deviatoric stress power and determined the constitutive coefficients for processes of each kind, by means of some axiomatic assumptions on the material properties. The different determination of the constitutive coefficients for loading and unloading processes gives account of the irreversible deformations. Green requires that constitutive equations for loading and unloading must coincide when the stress power is zero, i.e. for neutral states, to assure a smooth transition from one process to the other. This hypothesis does not seem necessary and is not supported by the experimental results that show discontinuity of the derivatives at the transition points in the stress-strain relations. Green also tried to reconcile this new approach with the classical conception of a yield surface. He assumed that when the yield condition was satisfied, the constitutive coefficients had to be of such a form as to assure the condition to be satisfied further until unloading occurred. In this conception the yield condition has again the classical meaning and for the non-yield states elastic behaviour is hypothesized. We obtain in such a way a true generalization of Prandtl-Reuss theory, now formulated in a correctly invariant form for finite deformations. The yield condition can be any smooth function of the stress invariants, and "elastic" compressibility during plastic flow is taken into account. There is no idea here of continuous, smooth transition from elastic to perfectly plastic states.

All these theories refer to metallic materials, but it is natural to think that Green's general approach that describes irreversible behaviour assuming different constitutive laws for loading and unloading processes, when these are defined in a convenient way, can suggest a procedure to built up a model for the mechanical behaviour of granular media.

2. Choice of the model

Let us list some well established experimental facts about granular materials as constitutive assumptions:

1. Relevant processes involve finite deformations (so that a properly invariant theory is needed).

2. Density variations play a fundamental role and strongly influence the mechanical response (softening and hardening).

3. No elastic range is observed in general.

4. When undergoing increasing deviatoric deformations these materials may tend to reach "critical states" in which they flow as frictional "fluids". In these states some limit condition on stresses must be satisfied. Later on this behaviour will be discussed in detail.

The peculiarities in the mechanical behaviour of granular materials summarized above justify the choice of the model that will be developed in this paper.

The assumed definition for a granular material and the constitutive assumptions 1) and 2) suggest constitutive equations of the form[1]

$$\overset{\circ}{T} = H(T, \varrho)\,[D], \tag{2.1}$$

where T is the Cauchy stress tensor, D is the stretching tensor, ϱ the density, $\overset{\circ}{T}$ the corotational stress rate and the tensor function H is isotropic in its tensor arguments and linear in D. The linearity of H in D assures time scale independent mechanical properties. By a special case of a general representation theorem of C. C. Wang, we have [14]:

$$\begin{aligned} H(T, \varrho)\,[D] = {} & [\Box_1 \operatorname{tr} D + \Box_2 \operatorname{tr}(TD) + \Box_3 \operatorname{tr}(T^2 D)]1 \\ & + [\Box_4 \operatorname{tr} D + \Box_5 \operatorname{tr}(TD) + \Box_6 \operatorname{tr}(T^2 D)]T \\ & + [\Box_7 \operatorname{tr} D + \Box_8 \operatorname{tr}(TD) + \Box_9 \operatorname{tr}(T^2 D)]T^2 \\ & + \Box_{10} D + \Box_{11}(DT + TD) + \Box_{12}(DT^2 + T^2 D), \end{aligned} \tag{2.2}$$

where the $\Box_i$ $i = 1, \ldots, 12$ are scalar functions of the fundamental stress invariants and of the density ϱ. In this paper we will be dealing with a special simple form of the general representation (2.2) Indeed it will be assumed:

$$\overset{\circ}{T} = [\Box_1 \operatorname{tr} D + \Box_2 \operatorname{tr}(TD)]1 + [\Box_4 \operatorname{tr} D + \Box_5 \operatorname{tr}(TD)]T + \Box_{10} D. \tag{2.3}$$

The irreversible behaviour will be described following Green's approach, assuming the same form of the general constitutive equations for loading and unloading processes, but with different choice of the constitutive coefficients, let us say $\Box_i$ and $\Box_i'$, respectively. Loading states are characterized by positive stress power, i.e. $\operatorname{tr}(TD) > 0$, neutral states by zero stress power, i.e. $\operatorname{tr}(TD) = 0$, and unloading states by negative stress power, i.e. $\operatorname{tr}(TD) < 0$. This definition, also due to A .E. Green, seems to be the most appropriate for our purposes. We will reject Green's hypothesis that the constitutive coefficients $\Box_i$ and $\Box_i'$ must coincide for neutral states to assure smooth transition from loading to unloading processes and conversely because, as previously stated in the introduction, it seems not at all justified by the experimental results. Stress-strain relations show discontinuity in the derivatives at the transition points. In what follows we will distinguish only between loading states ($\operatorname{tr}(TD) \geqslant 0$) and unloading states ($\operatorname{tr}(TD) < 0$), assuming that the same constitutive equations are valid for neutral and loading processes.

On the basis of convenient constitutive assumptions it is possible to obtain an explicit form of the coefficients $\Box_i \Box_i'$. The procedure that will be followed is general but will be illustrated with reference to a special case, chosen due to the physical reliability of the constitutive assumptions and the clear meaning of the state parameters introduced.

3. The state space

Most of present knowledge about constitutive properties in soil mechanics come from laboratory triaxial tests on cylindrical specimens. It is evident that in these tests only stress states in which two principal stresses are equal are feasible. Then only two independent

(1) Constitutive equations of the same form are assumed in Noll's theory of hygrosteric materials [11], and have been assumed also in [12 and 13] with reference to granular media.

stress parameters are needed e.g. the axial and the radial principal stresses t_1 and t_2(²) or the mean pressure $p = -(t_1+2t_2)/3$ and the parameter $q' = t_1 - t_2$ that can be considered a "measure" of deviatoric stresses. In such a situation, to obtain an explicit form of the constitutive relations in terms of well established experimental facts, in this paper a point of view has been adopted similar to that of the Cambridge school, assuming only two constitutive stress parameters that can be considered an appropriate generalization of p and q'. Namely, we shall define:

$$p = -\frac{\operatorname{tr} T}{3}, \qquad q = \sqrt{\operatorname{tr}(T^*)^2}, \text{(}^3\text{)} \tag{3.1}$$

where $T^* = T + pI$ is the stress deviator $\left(\text{observe that for triaxial tests it is } q = \sqrt{\frac{2}{3}}\, q'\right)$. Such a choice is quite natural because all the constitutive assumptions, if they are to be founded on experimental evidence, at present can be expressed only in terms of these stress parameters. Anyway the procedure that will allow the determination of the constitutive coefficients is completely general. Therefore, as state parameters in what follows will be considered the pressure p, the density ϱ and the non-negative "measure" of the deviatoric stresses q. The constitutive coefficients $\square_1$, $\square_2$, $\square_4$, $\square_5$, $\square_{10}$ in the Eq. (2.3) will be assumed to be functions of p, q, ϱ only. The three-dimensional space with coordi-

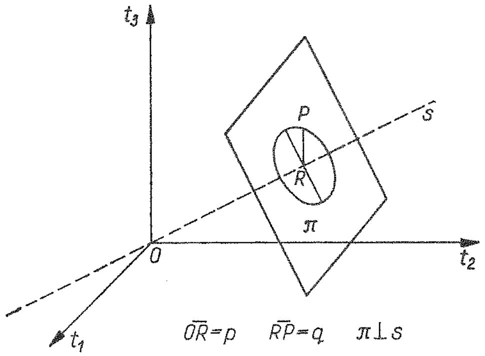

FIG. 1.

nates p, q, ϱ, will be called the state space. For a fixed value of the density, the points representative of the "state" characterized by the pair (p, q) are situated in the Haigh-Westergaard principal stress space, on a circumference of radius q contained in the plane orthogonal to the space diagonal at the point of abscissa p and with center on the space diagonal (Fig. 1).

4. Some experimental results

Let us now recall some basic features of the mechanical behaviour of granular media in the most indicative available experimental tests.

(²) Principal stresses are assumed positive if they correspond to tension, while in soil mechanics usually the opposite convention is adopted.

(³) The non-negative parameter q can be considered as the norm of the stress deviator.

4.1. Purely spherical motion under hydrostatic pressure

It is well known that the set of admissible states (pairs (p, ϱ)) in the p, ϱ plane is bounded by the so called virgin compression line, that represents the set of the "loosest states" of the material. It means that for every value of the pressure p, the point on the virgin compression line corresponds to the least admissible value of the density ϱ and is actually reached only when the material never before experienced greater densities, i.e. when it is yet

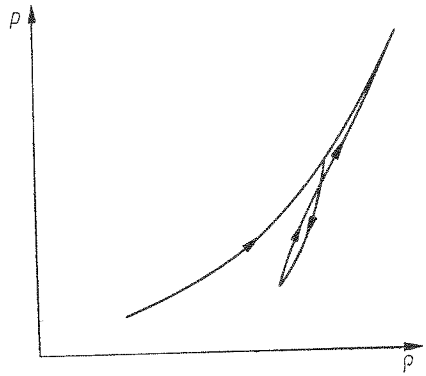

FIG. 2.

"virgin". Figure 2 shows a schematic picture of typical pressure-density paths for cohesionless materials.

4.2. Constant p tests

When deviatoric deformations increase, the density and the stress parameter q tend to reach limit values that depend only on the fixed value of p. If the initial value ϱ_i of the density is greater, equal or less than the limit "critical" one ϱ_c, the responses are of a differ-

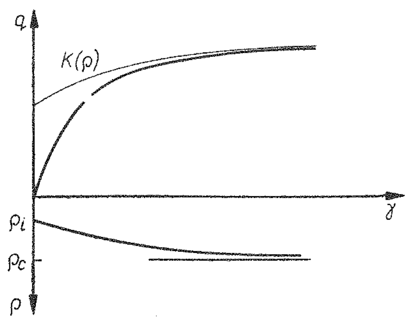

FIG. 3.

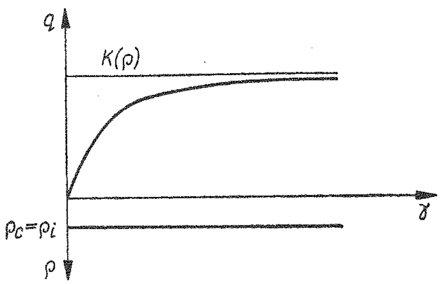

FIG. 4.

ent kind as is shown in Figs. 3, 4 and 5, where q and ϱ are plotted as functions of the parameter γ that indicates a "measure" of the deviatoric deformation.

If $\varrho_i < \varrho_c$ (Fig. 3) (loose states), both ϱ and q tend asymptotically to the final values.

If $\varrho_i = \varrho_c$ (Fig. 4) (critical states), there is no density variation and q behaves as before.

If $\varrho_i > \varrho_c$ (Fig. 5) (dense states), both ϱ and q first increase reaching a maximum, and subsequently decrease tending to reach the critical values.

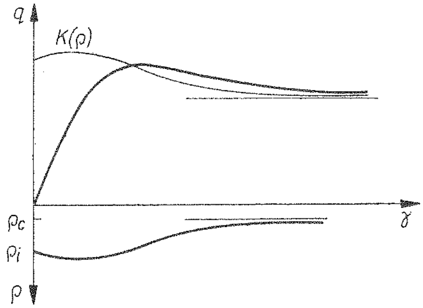

FIG. 5.

4.3. Constant ϱ tests

The deformation process is of course purely deviatoric. When deformation increases, the mean pressure and the stress parameter q tend to reach a limit value that depends only on the fixed value of ϱ. The responses are different if the initial value p_i of the mean pressure is greater, equal or less than the limit "critical" p_c as is shown in Figs. 6, 7 and 8 where p and q are plotted as functions of γ.

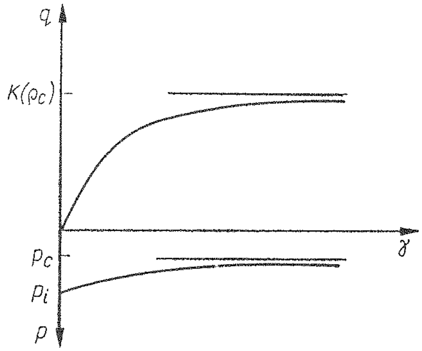

FIG. 6.

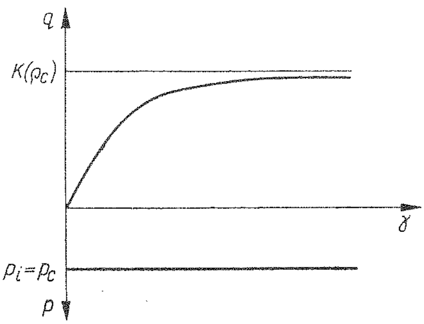

FIG. 7.

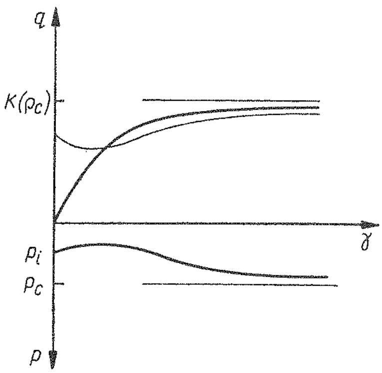

FIG. 8.

If $p_i > p_c$ (Fig. 6) (loose states), the mean pressure decreases monotonically to the critical value, q increases monotonically to the final value.

If $p_i = p_c$ (Fig. 7) (critical states), the mean pressure remains constant while q behaves as before.

If $p_i < p_c$ (Fig. 8) (dense states), the mean pressure reaches a minimum and after increases tending to the critical value, the parameter q behaves like before.

These results will be of great importance in the following formulation of the critical state assumption.

5. The critical state

ROSCOE, SCHOFIELD and WROTH [15] suggested that granular materials when undergoing increasing deviatoric deformations tend to reach a critical state in which they continue to distort without further change of p, q and ϱ. According to the experimental facts previously exposed it will be assumed that in the critical states the following relations must hold:

$$q = k(\varrho), \quad q = \psi p \quad \text{and then} \quad k(\varrho) = \psi p, \tag{5.1}$$

where k is a strictly increasing function of the density ϱ and ψ is a dimensionless positive constant. Relations (5.1) define in the state space two surfaces and a plane that intersect

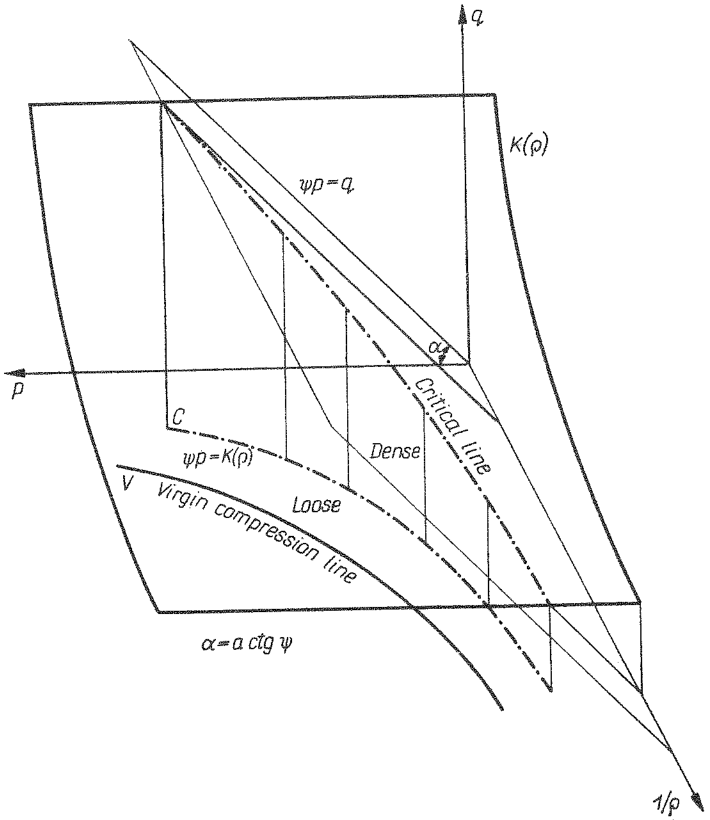

FIG. 9.

in a curve, called the critical state line, that is the set of all the points representative of critical states. In general we will call "dense" these states in which is $k(\varrho) > \psi p$ and "loose" those where $k(\varrho) < \psi p$ (Fig. 9). The critical state assumption can be enounced in the following way:

(5.2) $$\left.\begin{aligned} q &\to k(\varrho) \\ q &\to \psi p \\ \psi p &\to k(\varrho) \end{aligned}\right\} \Rightarrow \dot{p}, \dot{q}, \dot{\varrho} \to 0.$$

6. The constitutive equations

To obtain an explicit determination of the constitutive coefficients, it is convenient to split the Eq. (2.3) in two that describe the deviatoric and the spherical stress-responses, respectively. Because $\overset{\circ}{T} = \dot{T} - WT + TW = \dot{T}^* - WT^* + T^*W - \dot{p}1 = \overset{\circ}{T}{}^* - \dot{p}1$, where W is the spin tensor, and by the local form of the continuity equation

(6.1) $$\operatorname{tr} D = -\frac{\dot{\varrho}}{\varrho}$$

it can easily be shown that the following system:

(6.2) $$\begin{aligned} \dot{p} &= \left(\square_1 + \frac{\square_{10}}{3} - \square_4 p - \square_2 p + \square_5 p^2\right)\frac{\dot{\varrho}}{\varrho} + (\square_5 p - \square_2)\operatorname{tr}(T^*D^*), \\ \overset{\circ}{T}{}^* &= \square_{10} D^* + \left[(\square_5 p - \square_4)\frac{\dot{\varrho}}{\varrho} + \square_5 \operatorname{tr}(T^*D^*)\right] T^*, \end{aligned}$$

is equivalent to the Eq. (2.3).

A new important constitutive assumption is that purely spherical motions does not affect the deviatoric part of the stress response.

The second equation of the system (6.2) for purely spherical motions ($D^* = 0$) reduces to

(6.3) $$\overset{\circ}{T}{}^* = (\square_5 p - \square_4)\frac{\dot{\varrho}}{\varrho}$$

and then, by the stated assumption, necessarily must be

(6.4) $$\square_5 p - \square_4 = 0.$$

From (6.4) it results $\square_4 = \square_5 p$, and substituting in (6.2), we obtain the system

(6.5) $$\begin{aligned} \dot{p} &= \left(\square_1 + \frac{\square_{10}}{3} - \square_2 p\right)\frac{\dot{\varrho}}{\varrho} + (\square_5 p - \square_2)\operatorname{tr}(T^*D^*), \\ \overset{\circ}{T}{}^* &= \square_{10} D^* - \square_5 \operatorname{tr}(T^*D^*) T^*, \end{aligned}$$

that is equivalent to the unique equation

(6.6) $$\overset{\circ}{T} = [\square_1 \operatorname{tr} D + \square_2 \operatorname{tr}(TD)]1 + \square_5\left[\operatorname{tr}(TD) - \frac{(\operatorname{tr} T)(\operatorname{tr} D)}{3}\right] T + \square_{10} D.$$

The system (6.5) and the Eq. (6.6), by the arbitrariness of the functions $\Box_1$, $\Box_{10}$, $\Box_2$ and $\Box_5$, can, of course, be written in the form:

$$(6.7)\qquad \begin{aligned} \dot{p} &= \Gamma(p,q,\varrho)\frac{\dot{\varrho}}{\varrho} + C(p,q,\varrho)\operatorname{tr}(T^*D^*), \\ \mathring{T}^* &= 2\mu(p,q,\varrho)D^* - B(p,q,\varrho)\operatorname{tr}(T^*D^*)T^*, \end{aligned}$$

where

$$(6.8)\qquad \begin{aligned} \Gamma &= \frac{\Box_{10}}{3} + \Box_1 - \Box_2 p, \\ C &= \Box_5 p - \Box_2, \\ 2\mu &= \Box_{10}, \\ B &= \Box_5; \end{aligned}$$

Γ and 2μ usually denote the bulk and shear moduli of isotropic linear elastic materials, respectively. The analogy is clear if we consider that the constitutive equations of isotropic infinitesimal elasticity in incremental form is

$$(6.9)\qquad \dot{T} = \lambda(\operatorname{tr} T)1 + 2\mu D,$$

where λ and 2μ are the two Lamé constants.

By decomposing (6.9) in the spherical and deviatoric part we obtain

$$(6.10)\qquad \dot{p} = \Gamma\frac{\dot{\varrho}}{\varrho}, \qquad \dot{T}^* = 2\mu D^*,$$

where $\Gamma = \lambda + \frac{2}{3}\mu$ is the bulk modulus. Because $\mathring{T}^* = \dot{T}^* - WT^* + T^*W$, for small values of T^* it results $\mathring{T}^* = \dot{T}^*$ and for $T^* = 0$ is $\mathring{T}^* = \dot{T}^*$.

When $T^* \approx 0$, the (6.10) become

$$(6.11)\qquad \begin{aligned} \dot{p} &= \Gamma(p,0,\varrho)\frac{\dot{\varrho}}{\varrho}, \\ \dot{T}^* &= 2\mu(p,0,\varrho)D^*, \end{aligned}$$

so that the analogy is evident and we conclude that the materials whose mechanical properties are described by the constitutive Eqs. (6.7) behave in the neighbourhoods of states in which the deviatoric stresses are zero as isotropic elastic materials. System (6.7) is equivalent to the single equation

$$(6.12)\qquad \mathring{T} = [(\lambda + Bp^2 - Cp)\operatorname{tr} D + (Bp - C)\operatorname{tr}(TD)]1 + B[\operatorname{tr}(TD) + p(\operatorname{tr} D)]T + 2\mu D,$$

which on the basis of the inverses of relations (6.8), namely

$$(6.13)\qquad \begin{aligned} \Box_{10} &= 2\mu, \\ \Box_5 &= B, \\ \Box_2 &= Bp - C, \\ \Box_1 &= \Gamma - \frac{2}{3}\mu + Bp^2 - Cp = \lambda + Bp^2 - Cp, \end{aligned}$$

can be written again in the form

$$\overset{\circ}{T} = (\square_1 \operatorname{tr} D + \square_2 \operatorname{tr}(TD))1 + \square_5 \left(\operatorname{tr}(TD) - \frac{(\operatorname{tr} T)(\operatorname{tr} D)}{3}\right) T + \square_{10} D. \tag{6.14}$$

Let us now look for an explicit form of the constitutive coefficients for loading processes $(\operatorname{tr}(TD) \geqslant 0)$. A differential equation for q can be obtained remembering that $q^2 = \operatorname{tr}(T^*)^2$ and then $\dot{q}q = \operatorname{tr}(\overset{\circ}{T}{}^* T^*)$. Multiplying the second term in (6.10) by T^* and taking the trace, we have

$$\dot{q}q = (2\mu + Bq^2)\operatorname{tr}(T^* D^*). \tag{6.15}$$

From (6.15) it is clear that the deviatoric stress response in terms of q is described by the constitutive function B. Figure 5 shows that in dense states at least, this response depends more strictly on the density changes than on the mean pressure. A simple interpretation in agreement with the various typical stress results schematically illustrated in Figs. 3–8 comes out naturally in terms of the following constitutive hypothesis:

$$q \to k(\varrho), \qquad \dot{q} \to 0. \tag{6.16}$$

By (6.15) this implies $2\mu + Bq^2 \to 0$ when $q \to k(\varrho)$, so that by continuity it must be $2\mu + Bk^2(\varrho) = 0$, from which

$$B = -\frac{2\mu}{k^2(\varrho)}. \tag{6.17}$$

We can now substitute (6.17) in (6.15) to obtain

$$\dot{q}q = 2\mu\left(1 - \frac{q^2}{k^2(\varrho)}\right)\operatorname{tr}(T^* D^*). \tag{6.18}$$

From (6.18) it results that for loading processes in which the deviatoric part of the stress power $\operatorname{tr}(T^* D^*) = \operatorname{tr}(TD) - \frac{p\dot{\varrho}}{\varrho}$ is positive, $\dot{q}$ has the same sign of the difference $k(\varrho) - q$. The previous considerations make clear the interpretation of the test results of Figs. 3–8, on the basis of the Eq. (6.18).

Substituting (6.17), the second of (6.7) can be written in the form:

$$\overset{\circ}{T}{}^* = 2\mu\left[D^* - \frac{\operatorname{tr}(T^* D^*)}{k^2(\varrho)} T^*\right]. \tag{6.19}$$

The first of (6.7) for constant p tests assumes the form

$$0 = \Gamma \frac{\dot{\varrho}}{\varrho} + C \operatorname{tr}(T^* D^*) \tag{6.20}$$

and for constant ϱ tests

$$\dot{p} = C \operatorname{tr}(T^* D^*). \tag{6.21}$$

The typical stress results shown in Figs. 3–8 refer to deformation process in which $\operatorname{tr}(T^* D^*) > 0$. Because Γ is always positive, it is possible to deduce some implications concerning the constitutive function C that gives account of the coupling between deviatoric and spherical parts of the stress response. If C is zero, the first of (6.7) could be

6*

solved separately to give the spherical stresses. This is the case in [12]. Figs. 3 and 5, when compared with the Eq. (6.20), respectively suggest that

$$k(\varrho) \to \psi p_0, \quad C \to 0,$$

and

$$q \to \psi p_0, \quad C \to 0.$$

Moreover, Fig. 5 shows that ϱ has always the same sign of the difference $\psi p_0 - q$ and that $\psi p_0 = 0 \Rightarrow \dot{\varrho} = 0$.

Because C is a dimensionless function, we propose the following explicit form:

$$C(p, q, \varrho) = \frac{(q-\psi p)|k(\varrho)-\psi p|}{b}, \tag{6.22}$$

where b is a constant with dimension of stress. Of course, the smaller is b, the greater are the dilatancy effects during shear processes of dense materials (Fig. 5).

Expression (6.22) is in agreement with the behaviour illustrated in Fig. 4 and for constant ϱ tests, with that of Figs. 6, 7, 8 when compared with the Eq. (6.21).

We can now summarize the previous results writing (6.7) in the form

$$\begin{aligned} \dot{p} &= \Gamma\frac{\dot{\varrho}}{\varrho} + \frac{(q-\psi p)|k(\varrho)-\psi p|}{b}\,\mathrm{tr}(T^*D^*), \\ \mathring{T}^* &= 2\mu\left[D^* - \frac{\mathrm{tr}(T^*D^*)}{k^2(\varrho)}\,T^*\right]. \end{aligned} \tag{6.23}$$

The constitutive functions Γ, μ and k remain now to be given an explicit form. Their determination is really difficult, mainly because of the lack of suitable experimental data. By the same reason, the choice of a suitable form of the constitutive equations for unloading processes is troublesome. Anyway a major step toward the verification of the validity of the proposed model is the evaluation of the reliability, at least from a qualitative point of view, of the solutions obtainable for boundary-value problems that simulate some real processes. With these considerations in mind, the next step will be to conceive an explicit form of the constitutive coefficients that, if strongly simplified, can be realistic.

To this aim, for the unloading processes will be assumed the absence of dilatancy effects, i.e. $C = 0$, and an "elastic" behaviour with variable moduli Γ and μ, i.e. $B = 0$, so that the constitutive Eqs. (6.7) assume now the special form

$$\dot{p} = \Gamma(p, q, \varrho)\frac{\dot{\varrho}}{\varrho}, \qquad \mathring{T}^* = 2\mu(p, q, \varrho)D^*. \tag{6.24}$$

A leading hypothesis in the determination of the constitutive coefficients for loading and unloading processes is that every granular material has a limited range of admissible densities, with an upper bound ϱ_L that is asymptotically approached when the pressure increases, and a lower limit ϱ_M, beyond which the continuity of the body is lost. The deformability of the material, of course, decreases with the density and will be assumed that it tends to zero for $\varrho \to \varrho_L$ and to infinity as $\varrho \to \varrho_M$ (Fig. 2).

Let us suppose that, during purely hydrostatic compression of the virgin material, Γ is a function of the density only. The constitutive equations for such processes can then be written in the form

$$\dot{p} = \frac{\Gamma}{\varrho}\dot{\varrho}, \qquad \overset{\circ}{T}{}^{*} = 0. \tag{6.25}$$

The first of (6.25) gives

$$\frac{dp}{d\varrho} = \frac{\Gamma}{\varrho} \tag{6.26}$$

and by integration we obtain the following expression for the pressure on the virgin compression line:

$$p(\varrho) = \int_{\varrho_M}^{\varrho} \frac{\Gamma(\eta)}{\eta}\, d\eta + p_M, \tag{6.27}$$

where $p_M = p(\varrho_M)$, while the previous considerations and the diagrams of Fig. (11) suggest the following explicit simple form

$$p(\varrho) = \Gamma_v \frac{\varrho - \varrho_M}{\varrho_L - \varrho}, \tag{6.28}$$

where Γ_v is a constant with dimension of stress.

From (6.27) and (6.28) we have

$$\frac{\Gamma}{\varrho} = \Gamma_v \frac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2} \tag{6.29}$$

for virgin compression processes.

Experimental pressure-density relations (Fig. 2) show that it is reasonable to assume for unloading processes a similar expression of the rate Γ/ϱ but with a greater value of the constant, so that we take

$$\frac{\Gamma}{\varrho} = \Gamma_u \frac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2} \tag{6.30}$$

with $\Gamma_u > \Gamma_v$.

For general loading processes, observing that the rate $dp/d\varrho = \Gamma/\varrho$ decreases with the distance from the virgin compression line tending to reach the value (6.29), we will assume

$$\frac{\Gamma}{\varrho} = \Gamma_v \left[\left(\frac{\varrho - \varrho_M}{\varrho_L - \varrho}\right)\frac{\Gamma_v}{p}\right]^r \frac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2}, \tag{6.31}$$

where r is a dimensionless constant. For virgin processes by (6.29) it is $\dfrac{\varrho - \varrho_M}{\varrho_L - \varrho}\dfrac{\Gamma_v}{p} = 1$ and (6.31) reduces to (6.29). Thus we can summarize:

$$\Gamma(p, \varrho) = \begin{cases} \Gamma_v \left[\left(\dfrac{\varrho - \varrho_M}{\varrho_L - \varrho}\right)\dfrac{\Gamma_v}{p}\right]^r \dfrac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2}\varrho, & \text{if} \quad \mathrm{tr}(TD) \geqslant 0. \\ \Gamma_u \dfrac{\varrho(\varrho_L - \varrho_M)}{(\varrho_L - \varrho)^2}, & \text{if} \quad \mathrm{tr}(TD) \leqslant 0. \end{cases} \tag{6.32}$$

Let us now analyze the response to purely deviatoric loading deformation processes, characterized by the same stretch history but different values of the density.

Equation (6.19) can be written in the form

$$\dot{q} = 2\mu\left(1 - \frac{q^2}{k^2}\right)\frac{\operatorname{tr}(T^*D^*)}{(\operatorname{tr}T^{*2})^{1/2}}. \tag{6.33}$$

With the initial condition $T^*(0) = 0$, we have $q(0) = 0$ and

$$\dot{q}(0) = 2\mu\delta(D^*), \tag{6.34}$$

where

$$\lim_{T^*\to 0}\frac{\operatorname{tr}(T^*D^*)}{(\operatorname{tr}T^{*2})^{1/2}} = \delta(D^*) \geqslant 0. \tag{6.35}$$

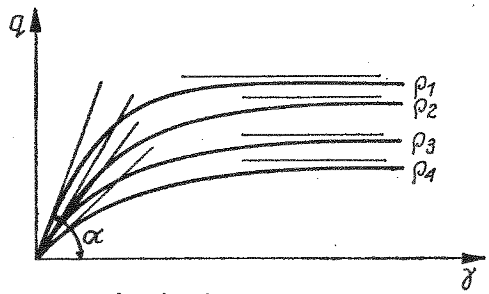

FIG. 10.

The non-negative function δ is linear and such that $D^* = 0 \Rightarrow \delta(D^*) = 0$. Typical test results are plotted in Fig. (10) and when interpreted in terms of (6.34) and (6.35), justify the assumption of the following expression for k and μ:

$$k = k_c\frac{\varrho - \varrho_M}{\varrho_L - \varrho}, \tag{6.36}$$

$$\mu = \mu_T\frac{\varrho - \varrho_M}{\varrho_L - \varrho}, \tag{6.37}$$

where k_c and μ_T are constants with dimension of stress. The constitutive parameters whose value must be determined experimentally for each material are therefore Γ_v, Γ_u, r, ϱ_M, ϱ_L, k_c, μ_T, ψ, b. The design of suitable experimental procedures for such determination will be discussed in a next paper.

Comparison with experimental data shows that as possible values for the constitutive parameters can be assumed

$$\varrho_M = 1.2, \quad \varrho_L = 2.2, \quad \mu_T = 600, \quad k_c = 90,$$

$$\Gamma_v = 400, \quad \Gamma_u = 4000, \quad r = 5, \quad b = 10000, \quad \psi = 1.$$

With such a choice, the constitutive equations can be integrated to give local stress-strain relations. The procedure is illustrated in detail in [17]. Here, only some results for spherical compression and simple contraction under constant pressure or constant density are reported. Initial states are always spherical. For spherical and constant pressure processes, also unloading-reloading paths are shown. The results are illustrated in Figs. 11–16.

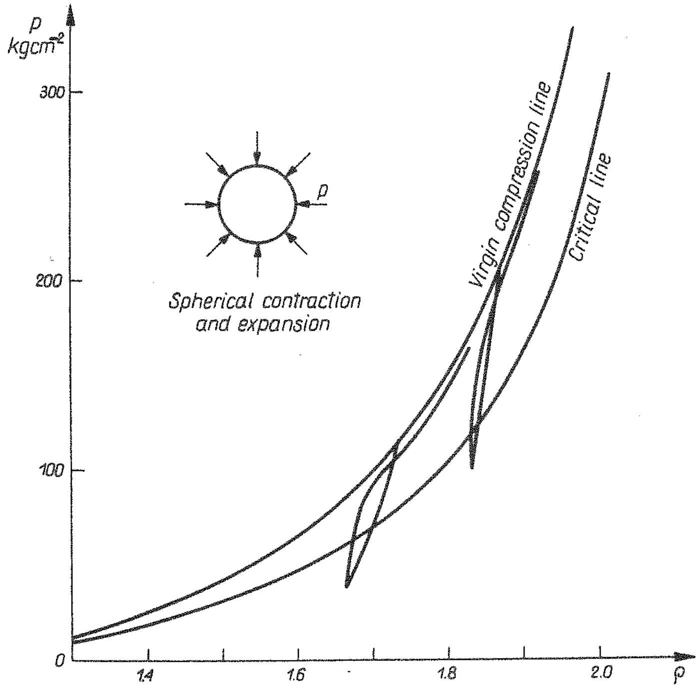

FIG. 11.

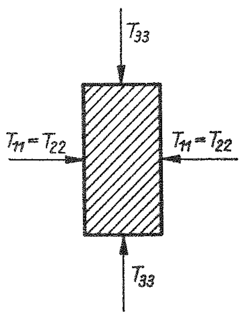

FIG. 12.

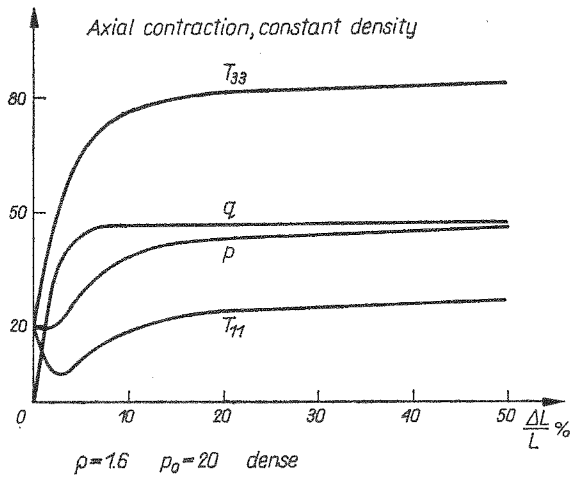

FIG. 13.

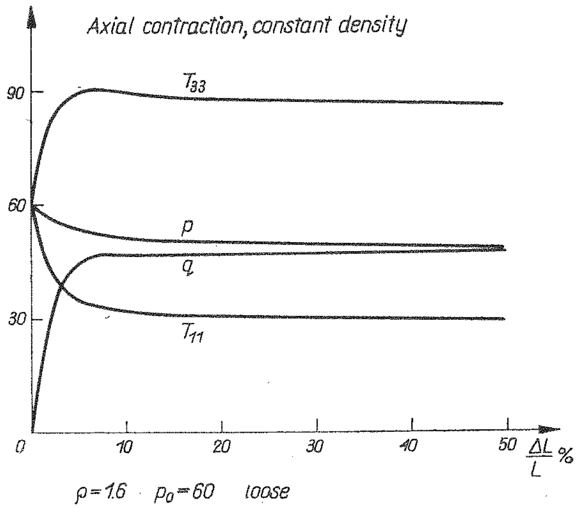

FIG. 14.

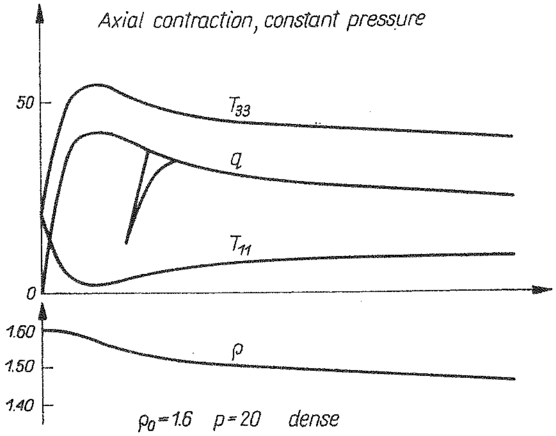

FIG. 15.

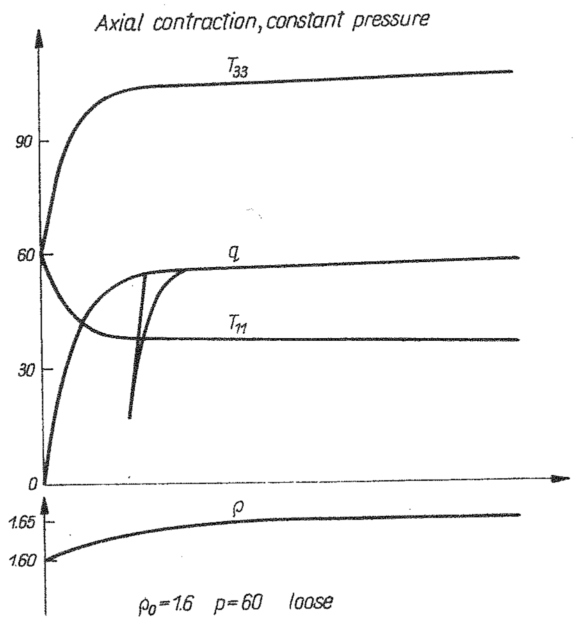

FIG. 16.

[1026]

7. Conclusions

The present theory is characterized by the inclusion of the powerfully simplifying, but realistic, critical state conception in the general framework of the theory of rate type materials. It is an attempt to provide a model of the mechanical behaviour of granular media for finite deformations whose main features will be listed below:

1. No true elastic range is exibited.
2. Account is given of the dilatancy effects under constant mean pressure, through the coupling of deviatoric and spherical parts of stress response.
3. Purely spherical processes do not affect deviatoric stresses.
4. Linear elastic behaviour in the neighbourhoods of states with zero deviatoric stresses, if the density is constant.
5. Continuous transition from elastic behaviour (when $T^* \approx 0$) to perfectly plastic behaviour as deviatoric deformations increase under constant pressure or constant density.

The determination of stress-strain relations for relevant homogeneous deformation processes, to be compared with experimental tests, the discussion of the procedures as well as the solution of boundary value problems for non-uniform motions, will be the subject of next works.

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UPPER AND LOWER BOUNDS TO THE EIGENFREQUENCIES OF ELASTIC FRAMES

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SOMMARIO: Si presenta un metodo per il calcolo di limiti superiori e inferiori delle frequenze proprie di vibrazione dei telai elastici. I limiti superiori sono ottenuti tramite il metodo di Rayleigh-Ritz, con riferimento alla formulazione variazionale del problema di autovalori. La formulazione di un problema equivalente di autovalori per un operatore positivo compatto, consente di applicare la teoria degli Invarianti Ortogonali per ottenere i limiti inferiori. Un esempio numerico mostra l'efficacia del metodo.

SUMMARY: A method for the computation of upper and lower bounds to the eigenfrequencies of elastic frames is presented. The upper bounds are obtained by the Rayleigh-Ritz method, with reference to the variational formulation of the eigenvalue problem. The formulation of an equivalent eigenvalue problem for a positive compact operator allows to apply the theory of Orthogonal Invariants to obtain the lower bounds. A numerical example shows the effectiveness of the method.

1. Introduction.

In this paper we consider the problem of the computation of upper and lower bounds to the eigenfrequencies of elastic frames. The eigenvalue boundary value problem for the classical differential equations governing the stationary flexural vibrations of elastic beams is formulated. The eigenvalues appear also in the boundary conditions. The upper bounds are computed by the Rayleigh-Ritz method, with reference to the variational formulation of the eigenvalue problem. It is shown that the original problem is equivalent to an eigenvalue problem for a positive integral operator that has finite trace in the sense of Hilbert-Schmidt. By the theory of Orthogonal Invariants of positive compact operators we obtain the lower bounds to the eigenfrequencies. The determination of lower bounds is of fundamental importance for a rigorous analysis of the eigenvalue problems. In fact, it is evident that only the knowledge of both upper and lower bounds gives a meaningful estimate of the approximation.

2. Adimensional form of the equations.

The differential equations governing the stationary flexural vibrations of the t members of the frame, are

$$k_i \frac{d^4 v_i}{d z_i} = \omega^2 m_i v_i \qquad (i = 1, \dots t) \tag{1}$$

where, for the i-th member:

- k_i elastic bending stiffness
- m_i mass per unit lenght
- l_i lenght
- v_i transversal displacement function
- z_i abscissa of the centroidal line, $z_i \in [0, l_i]$
- ω angular frequency

Introducing the adimensional abscissa $x = \dfrac{z_i}{l_i}$, $x \in [0,1]$ and the functions $u_i(x) = \dfrac{v_i(xl_i)}{l_i}$, eqs. (1) may be written in the form

$$\frac{k_i}{l_i} \frac{d^4 u_i}{dx^4} = \omega^2 m_i l_i^3 u_i \tag{2}$$

Assuming k_0, m_0, l_0 as reference quantities, we get the adimensional form of the equations

$$\chi_i \frac{d^4 u_i}{dx^4} = \lambda \bar{m}_i u_i \tag{3}$$

where

$$\chi_i = \frac{k_i l_0}{k_0 l_i}$$
$$\bar{m}_i = \frac{m_i l_i^3}{m_0 l_0^3} \tag{4}$$
$$\lambda = \omega^2 \frac{l_0^4 m_0}{k_0}$$

If we introduce the two-diagonal matrices

$$T = \text{diag}(\chi_1, \chi_2, \dots, \chi_t)$$
$$P = \text{diag}(\bar{m}_1, \bar{m}_2, \dots, \bar{m}_t),$$

the notation D^4 for d/dx^4 and the operator $L = TD^4$, eq. (3) may be written in the compact form

$$Lu = \lambda Pu \tag{5}$$

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3. The boundary conditions.

We shall consider the case of frames in which, neglecting the axial deformability of the members, both joint rotations and displacements may occur. While the method that will be presented is fairly general, it is convenient to treat here in detail the case, very common in practice, of frames in which all the members ending in the joints whose displacements are not independent, are parallel. In fact, in this case it is easier to assign the kinematic parameters that describe the rotations and the independent displacements of the joints. As a consequence, the whole procedure is made simpler, putting into evidence the main ideas, with a minimum of formal difficulty. An explicit formulation of the boundary conditions may be obtained in the following way. Let us suppose that to each member has been given a sequential number and a versus (the same for all parallel members), and let be

- t the number of the members of the frame
- m the number of the joint rotation parameters $\alpha_l (l = 1, \dots m)$
- s_l the number of the members that have at one end the rotation α_l
- m_j^l $(j = 1, \dots s_l)$ the indexes of the members that have at one end the rotation α_l
- τ the number of members that have zero rotation at one end
- d the number of independent joint displacement parameters $\delta_h (b = 1, \dots d)$
- r_h the number of the members that have at one end the displacement δ_h
- n_k^h $(k = 1, \dots r_h)$ the indexes of the members that have at one end the displacement δ_h
- ϱ the number of members that have zero displacement at one end.

Let us remark that

$$\begin{aligned} \varrho + \sum_{h=1}^{d} r_h &= 2t \\ \tau + \sum_{l=1}^{m} s_l &= 2t \end{aligned} \tag{6}$$

If we denote by η the value of the abscissa x at the ends of each member (so that η is 0 or 1), we may write in adimensional form the geometric compatibility conditions

$$\begin{aligned} &u_j(\eta) = 0 && (j = 1, \dots \varrho) \\ &Du_j(\eta) = 0 && (j = 1, \dots \tau) \\ &\bar{l}_{n_k^h} u_{n_k^h}(\eta) = \delta_h && (k = 1, \dots r_h)\ (b = 1, \dots d) \\ &Du_{m_j^l}(\eta) = \alpha_l && (j = 1, \dots s_l)\ (l = 1, \dots m) \end{aligned} \tag{7}$$

and the equilibrium conditions

$$\begin{aligned} &(-1)^{\eta} \sum_{i=1}^{s_l} \chi_{m_j^l} D^2 u_{m_j^l}(\eta) = 0 && (l = 1, \dots m) \\ &(-1)^{\eta+1} \sum_{k=1}^{r_h} \chi_{n_k^h} D^3 u_{n_k^h}(\eta)\, \bar{l}_{n_k^h}^{-1} = -\lambda \bar{M}_h \bar{\delta}_h && (b = 1, \dots d) \end{aligned} \tag{8}$$

where

$$\bar{l} = l/l_0, \qquad \bar{\delta}_h = \delta_h/l_0, \qquad \bar{M}_h = M_h/m_0 l_0,$$

and M_h is the mass associated with the members that translate, without bending, when only the displacement parameter δ_h is different from zero. The term $\lambda \bar{M}_h \bar{\delta}_h$ gives account of the inertia force that equilibrates the r_h shear forces acting on these members.

From (6) we see that the eqs. (7) are $4t$ and the eqs. (8) are $m + d$. Of course, we could eliminate the $m + d$ parameters α_l and δ_h from eqs. (7) obtaining a set of $4t - m - d$ homogeneous geometric boundary conditions. But the form (7) is more convenient for the subsequent analysis.

Let be $n = m + d$. We define s as the n-vector whose components are, in the order, the d parameters d_h and the m parameters α_l and $\bar{M}$ as the $n \times n$ diagonal matrix diag $(\bar{M}_1, \dots, \bar{M}_d, 0, \dots 0)$.

4. Variational formulation of the problem.

Let us denote by V the subspace of $H_{2[0,1]}$ determined by the geometric boundary conditions (7), and by U the subspace of V of the functions that satisfy also the equilibrium boundary conditions (8). If $w \in V$, we have, multiplying both sides of (5) by w and integrating by parts,

$$\begin{aligned} \int_0^1 Lu \cdot w\, dx = \int_0^1 TD^2u \cdot D^2w\, dx + |TD^3u \cdot w|_0^1 - \\ - |TD^2u \cdot Dw|_0^1 = \lambda \int_0^1 Pu \cdot w\, dx \end{aligned} \tag{9}$$

Since $w \in V$, from (7) we have

$$\begin{aligned} &|TD^3u \cdot w|_0^1 - |TD^2u \cdot Dw|_0^1 = \\ &= \sum_{l=1}^{m} \alpha_l \sum_{j=1}^{s_l} (-1)^{\eta} \chi_{m_j^2} D^2 u_{m_j^l}(\eta) + \\ &+ \sum_{h=1}^{d} \bar{\delta}_h \sum_{k=1}^{r_h} (-1)^{\eta+1} \chi_{n_k^h} \bar{l}_{n_k^h}^{-1} D^3 u_{n_k^h}(\eta) \end{aligned} \tag{10}$$

It follows from (10) and (8) that $u \in U$ iff

$$
\left| TD^3 u \cdot w \right|_0^1 - \left| TD^2 u \cdot Dw \right|_0^1 =
$$

$$
= -\lambda \sum_{h=1}^{d} \overline{M}_h \overline{\delta}_h \overline{\delta}'_h \qquad \forall\, w \in V \tag{11}
$$

where the $\overline{\delta}'_h(\overline{\delta}_h)$ are the displacement parameters corresponding to the function $u(w)$.

The eigenvalue problem (5), (7), (8) has then the following variational formulation

$$
B(u, w) = \lambda\, Q(u, w) \qquad u \in V, \quad \forall\, w \in V \tag{12}
$$

where $B(u, w)$ and $Q(u, w)$ are the hermitian bilinear forms given by

$$
B(u, w) = \int_0^1 TD^2 u \cdot D^2 w \, dx \tag{13}
$$

$$
Q(u, w) = \int_0^1 Pu \cdot w \, dx + \sum_{h=1}^{d} \overline{M}_h \overline{\delta}_h \overline{\delta}'_h \tag{14}
$$

It is easy to show that the quadratic form $B(u, u)$ is coercive on V and that $Q(u, u)$ is positive on V. It follows that the eigenvalue problem (12) (and then also (5), (7), (8)) has a countable set of real positive eigenvalues, each with finite multiplicity, which form a divergent sequence. In particular $\lambda = 0$ cannot be an eigenvalue of problem (12), (and then of (5), (7), (8)). The geometric boundary conditions (7) are stable and the equilibrium boundary conditions (8) are natural for the variational problem (12).

5. The Rayleigh-Ritz method.

Upper bounds to the eigenvalues of (12) may be computed by the Rayleigh-Ritz method. Let $\{w_i\}$ be a system of linearly independent functions in V; then the upper bounds $\tilde{\lambda}_i^{(\nu)}$ to the first ν eigenvalues λ_i are given by the roots of the following determinant equation

$$
\det \left\{ B(w_i, w_j) - \lambda\, Q(w_i, w_j) \right\} = 0 \qquad (i, j = 1, \ldots \nu) \tag{15}
$$

We may write eq. (15) in explicit form. From (13) and (14) we get

$$
\det \left\{ \int_0^1 TD^2 w_i \cdot D^2 w_j \, dx - \lambda \left(\int_0^1 P w_i \cdot w_j \, dx + \sum_{h=1}^{d} \overline{M}_h (w_i)_{n_1^h}(\eta)\, (w_j)_{n_1^h}(\eta) \right) \right\} = 0 \tag{16}
$$

We shall not discuss here the problem of the choice of the approximating functions $\{w_i\}$ and the various techniques that can be used to determine the roots of (16).

6. Equivalent eigenvalue problem for an integral operator.

The eigenvalue problem (5), (7), (8), in which the eigenvalues appear in the boundary conditions, may be reduced to an equivalent problem for an integral operator [8]. Let us substitute in (5) q for λPu and in (8) a for $-\lambda \overline{M}s$. Then we may split the original problem in the following two

$$
TD^4 \phi = q \tag{17}
$$

with the geometric boundary conditions

$$
\begin{aligned}
&\phi_j(\eta) = 0 \\
&D\phi_j(\eta) = 0 \\
&\overline{l}_{n_k^h}\, \phi_{n_k^h}(\eta) = \overline{\delta}'_h \\
&D\phi_{m_j^l}(\eta) = \alpha'_l
\end{aligned} \tag{18}
$$

and the equilibrium boundary conditions

$$
\begin{aligned}
&(-1)^{\eta} \sum_{j=1}^{s_l} \chi_{m_j^l}\, D^2 \phi_{m_j^l}(\eta) = 0 \\
&(-1)^{\eta+1} \sum_{k=1}^{r_h} \chi_{n_k^h}\, D^3 \phi_{n_k^h}(\eta)\, \overline{l}_{n_k^h}^{-1} = 0
\end{aligned} \tag{19}
$$

$$
D^4 \psi = 0 \tag{20}
$$

$$
\begin{aligned}
&\psi_j(\eta) = 0 \\
&D\psi_j(\eta) = 0 \\
&\overline{l}_{n_k^h}\, \psi_{n_k^h}(\eta) = \overline{\delta}''_h \\
&D\psi_{m_j^l}(\eta) = \alpha''_l
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&(-1)^{\eta} \sum_{j=1}^{s_l} \chi_{m_j^l}\, D^2 \psi_{m_j^l}(\eta) = 0 \\
&(-1)^{\eta+1} \sum_{k=1}^{r_h} \chi_{n_k^h}\, D^3 \psi_{n_k^h}(\eta)\, \overline{l}_{n_k^h}^{-1} = a_h
\end{aligned} \tag{22}
$$

where $\phi + \psi = u$ on $[0, 1]$ and $s' + s'' = s$, with $s' = (\overline{\delta}'_1, \ldots \overline{\delta}'_d, \alpha'_1, \ldots \alpha'_m)$ and $s'' = (\overline{\delta}''_1, \ldots \overline{\delta}''_d, \alpha''_1, \ldots \alpha''_m)$.

The problem (17), (18), (19), with homogeneous boundary conditions, may be solved in the following way. Let us consider the general solution of (17) in the form

$$
\phi_i(x) = r_i \cdot \sigma(x) + \chi_i^{-1} \int_0^1 \overline{g}(x, \xi)\, q_i(\xi)\, d\xi \qquad (i = 1, \ldots t) \tag{23}
$$

where

$$r_i(x) = (\phi(0), D\phi(0), \phi(1), D\phi(1)) \qquad (i = 1, \ldots t) \tag{24}$$

$$\sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x), \sigma_4(x)) \tag{25}$$

$$\begin{aligned} \sigma_1(x) &= 2x^3 - 3x^2 + 1 \\ \sigma_2(x) &= x^3 - 2x^2 + x \\ \sigma_3(x) &= -2x^3 + 3x^2 \\ \sigma_4(x) &= x^3 - x^2 \end{aligned} \tag{26}$$

$$\bar{g}(x, \xi) = \begin{cases} (1/6)\, x^2(\xi - 1)^2 (3\xi - 2x\xi - x) & x \leqslant \xi \\ (1/6)\, \xi^2 (x - 1)^2 (3x - 2\xi x - \xi) & \xi \leqslant x \end{cases} \tag{27}$$

$\bar{g}(x, \xi)$ is the Green function of the clamped beam.

The $4t$ constant components of the vectors $r_i (i = 1, \ldots t)$ can be determined by means of the boundary conditions (18), (19), but it is more convenient to satisfy only the stable (geometric) boundary condition and to obtain the solution from the variational formulation of the problem. So we are led to a minimum problem for the quadratic functional

$$I(\phi) = \frac{1}{2} B(\phi, \phi) - (q, \phi)_0 .^2 \qquad \phi \in V \tag{28}$$

If we consider the problem in the subspace W of the solutions of the eq. (17), we get a minimum problem for the functional (28) on a finite dimensional subspace (dim $W = 4t$). If $\phi \in W$ we may set $\phi = f + h$ where

$$\begin{aligned} f_i(x) &= r_i \cdot \sigma(x) \\ h_i(x) &= \chi_i^{-1} \int_0^1 g(x, \xi)\, q_i(\xi)\, d\xi \qquad (i = 1, \ldots t) \end{aligned} \tag{29}$$

and $D^4 f = 0$, $D^4 h = q$ with $h(0) = Dh(0) = h(1) = Dh(1) = 0$.

From the Green formula we have

$$B(f, h) = (Lf, h)_0 + |TD^3 f \cdot h - TD^2 f \cdot Dh|_0^1 = 0 \tag{30}$$

so that

$$\begin{aligned} B(\phi, \phi) &= B(f, f) + 2B(f, h) + B(h, h) = \\ &= B(f, f) + B(h, h) \end{aligned} \tag{31}$$

Moreover, $(q, \phi)_0 = (q, f)_0 + (q, h)_0$. Then

$$I(\phi) = I(f) + I(h) \tag{32}$$

[2] By $(.\,,.)_0$ we denote the scalar product in the space $L_{2[0,1]}$ and by a dot the scalar product in finite dimensional spaces. C^* indicates the transpose of C.

Since $I(h)$ is constant, we may seek the minimum of $I(f)$ on W. From (18) we obtain for each member the $4xu$ Boolean matrix C_i such that

$$r_i = C_i s' \qquad (i = 1, \ldots t) \tag{33}$$

C_i may have on each row only one nonzero entry, whose value is l_i^{-1} on odd rows and 1 on even rows.

The relations (33) give account of the geometric boundary conditions (18). Now, we have

$$\begin{aligned} B(f, f) &= \int_0^1 TD^2 f \cdot D^2 f \, dx = \\ &= \sum_{i=1}^{t} \chi_i \int_0^1 (r_i \cdot D^2\sigma(x))^2 \, dx = \\ &= \sum_{i=1}^{t} \chi_i K r_i \cdot r_i = \sum_{i=1}^{t} \chi_i C_i^* K C_i s' \cdot s' = E s' \cdot s' \end{aligned}$$

where

$$E = \sum_{i=1}^{t} \chi_i C_i^* K C_i \tag{34}$$

and

$$K_{ij} = \int_0^1 D^2\sigma_i(x)\, D^2\sigma_j(x)\, dx \qquad (i, j = 1, \ldots 4)$$

so that, by (26)

$$K_i = \begin{vmatrix} 12 & 6 & -12 & 6 \\ . & 4 & -6 & 2 \\ . & . & 12 & -6 \\ . & . & . & 4 \end{vmatrix}$$

Moreover,

$$\begin{aligned} (q, f)_0 &= \int_0^1 f(\xi) \cdot q(\xi)\, d\xi = \sum_{i=1}^{t} \int_0^1 r_j \cdot \sigma(\xi)\, q_j(\xi)\, d\xi = \\ &= \sum_{i=1}^{t} C_j^* \int_0^1 \sigma(\xi)\, q_j(\xi)\, d\xi \cdot s' \end{aligned}$$

Then

$$\operatorname{grad} I(f) = E s' - \sum_{j=1}^{t} C_j^* \int_0^1 \sigma(\xi)\, q_j(\xi)\, d\xi$$

so that

$$s' = E^{-1} \sum_{j=1}^{t} C_j^* \int_0^1 \sigma(\xi)\, q_j(\xi)\, d\xi \tag{35}$$

Since

$$\phi_i = f_i + h_i = r_i \cdot \sigma(x) + h_i = C_i s' \cdot \sigma(x) + h_i$$

we have

$$\phi_i(x) = C_i E^{-1} \sum_{j=1}^{t} C_j^* \int_0^1 \sigma(\xi) \cdot \sigma(x)\, q_j(\xi)\, d\xi +$$

$$+ \chi_i^{-1} \int_0^1 \bar{g}(x, \xi)\, q_i(\xi)\, d\xi$$

i.e.,

$$\phi = Gq$$

and

$$Gq = \int_0^1 g(x, \xi)\, q(\xi)\, d\xi \tag{36}$$

where the $t \times t$ matrix $g(x, \xi)$ has the entries

$$g_{ij}(x, \xi) = C_i E^{-1} C_j^* \sigma(\xi) \cdot \sigma(x) + \delta_{ij} \bar{\chi}_j^{-j} \bar{g}(x, \xi) \tag{37}$$

The variational formulation of problem (20)-(22) on the subspace of the solutions of eq. (20), where $\psi_i = r_i \cdot \sigma(x)$, leads to the minimum problem for the quadratic functional

$$I(s'') = \frac{1}{2} Es'' \cdot s'' + a \cdot s'' \tag{38}$$

From grad $I(s'') = Es'' + a = 0$ we get

$$s'' = - E^{-1} a \tag{39}$$

and then

$$\psi_i(x) = r_i \cdot \sigma(x) = C_i s'' \cdot \sigma(x) = - C_i E^{-1} a \cdot \sigma(x) =$$

$$= - E^{-1} C_i^* \sigma(x) \cdot a \tag{40}$$

Let us remember that $\lambda \neq 0$ and

$$q = \lambda P u \tag{41}$$

$$a = - \lambda \bar{M} s \tag{42}$$

From (41) and (42), since $u = \phi + \psi$ and $s = s' + s''$, by means of (35), (36), (39), and (40) and substituting μ^{-1} for λ, we get

$$\begin{cases} \mu q_i(x) = \bar{m}_i \int_0^1 g_{ij}(x, \xi)\, q_j(\xi)\, d\xi + \bar{m}_i (E^{-1} C_i^* \sigma(x))_h\, a_h \\ \mu a_k = \bar{M}_k \int_0^1 (E^{-1} C_j^* \sigma(\xi))_k\, q_j(\xi)\, d\xi + \bar{M}_k E_{kh}^{-1} a_h \end{cases} \tag{43}$$

where $(i, j = 1, \ldots t)$ and $(h, k = 1, \ldots d)$.

The eigenvalue problems (43) and (5), (7), (8) are equivalent in the sense that their eigenvalues are in the relation $\mu_i = \lambda_i^{-1}$.

If we set $y_i = \bar{m}_i^{-\frac{1}{2}} q_i$ $(i = 1, \ldots t)$ and $b_h = \bar{M}_h^{-\frac{1}{2}} a_h$ $(h = 1, \ldots d)$, we obtain the following equivalent symmetric form of system (43)

$$\begin{cases} \mu y_i(x) = \sqrt{\bar{m}_i} \int_0^1 g_{ij}(x, \xi) \sqrt{\bar{m}_j}\, y_j(\xi)\, d\xi + \\ \qquad + \sqrt{\bar{m}_i}\, (E^{-1} C_i^* \sigma(x))_h \sqrt{\bar{M}_h} b_h \\ \mu b_h = \sqrt{\bar{M}_k} \int_0^1 (E^{-1} C_j^* \sigma(\xi))_k \sqrt{\bar{m}_j}\, y_j(\xi)\, d\xi + \\ \qquad + \sqrt{\bar{M}_k}\, E_{kh}^{-1} \sqrt{\bar{M}_h} b_h \end{cases} \tag{44}$$

We intend to show that the right-hand side of (44) defines an integral operator. To this end let us consider the two sets $A_1 = [0, 1]$ and $A_2 = \{c\}$ where c is a real number. Let us define for $x_1, \xi_1 \in A_1$, and $x_2, \xi_2 \in A_2$ (i.e., $x_2 = \xi_2 = c$)

$$\begin{aligned} \gamma_{11}(x_1, \xi_1) &= \left\{ \sqrt{\bar{m}_i}\, g_{ij}(x_1, \xi_1) \sqrt{\bar{m}_j} \right\} \\ & \qquad (i, j = 1, \ldots t) \\ \gamma_{12}(x_1, c) &= \left\{ \sqrt{\bar{m}_i}\, (E^{-1} C_i^* \sigma(x))_h \sqrt{\bar{M}_h} \right\} \\ & \qquad (i = 1, \ldots t) \quad (h = 1, \ldots d) \\ \gamma_{21}(c, \xi_1) &= \left\{ \sqrt{\bar{M}_k}\, (E^{-1} C_j^* \sigma(\xi))_k \sqrt{\bar{m}_j} \right\} \\ & \qquad (k = 1, \ldots d) \quad (j = 1, \ldots t) \\ \gamma_{22}(c, c) &= \left\{ \sqrt{\bar{M}_k}\, E_{kh}^{-1} \sqrt{\bar{M}_h} \right\} \\ & \qquad (k, h = 1, \ldots d) \end{aligned} \tag{45}$$

$$b(x_2) = b(\xi_2) = b(c) = b \tag{46}$$

On A_1 we have the Lebesgue measure $d\xi_1 = dx_1$, on A_2 the Dirac measure $d\delta_c = dx_2 = d\xi_2$.

We see that for $x_2, \xi_2 \in A_2$, by (45) and (46), we may write (44) in the form

$$\begin{cases} \int_{A_1} \gamma_{11}(x_1, \xi_1)\, y(\xi_1)\, d\xi_1 + \int_{A_2} \gamma_{12}(x_1, \xi_2)\, b(\xi_2)\, d\xi_2 = \mu y(x_1) \\ \int_{A_1} \gamma_{21}(x_2, \xi_1)\, y(\xi_1)\, d\xi_1 + \int_{A_2} \gamma_{22}(x_2, \xi_2)\, b(\xi_2)\, d\xi_2 = \mu b(x_2) \end{cases} \tag{47}$$

Let us now consider the set $A = A_1 \cup A_2$, the function

$$z(x) = \begin{cases} y(x) & x \in A_1 \\ b(x) & x \in A_2 \end{cases}$$

and the measure

$$d\xi = \begin{cases} d\xi_1 & \text{on} \quad A_1 \\ d\xi_2 & \text{on} \quad A_2 \end{cases}$$

We have $A \times A = (A_1 \times A_1) \cup (A_1 \times A_2) \cup (A_2 \times A_1) \cup (A_2 \times A_2)$; hence, if we define the kernel $\gamma(x, \xi)$ on $A \times A$ in the following way

$$\gamma(x, \xi) = \begin{cases} \gamma_{11}(x_1, \xi_1) & \text{on} \quad A_1 \times A_1 \\ \gamma_{12}(x_1, \xi_2) & \text{on} \quad A_1 \times A_2 \\ \gamma_{21}(x_2, \xi_1) & \text{on} \quad A_2 \times A_1 \\ \gamma_{22}(x_2, \xi_2) & \text{on} \quad A_2 \times A_2 \end{cases} \tag{48}$$

we see that (47) may be written in the form

$$\Gamma z(x) = \int_A \gamma(x, \xi)\, z(\xi)\, d\xi = \mu z(x) \tag{49}$$

The eigenvalue problem (49) for the integral operator Γ is equivalent, in the above specified sense, to the problem (5), (7), (8). Then, since it is $\mu_i = \lambda_i^{-1} \geqslant 0$ (for every i), the operator Γ is a positive compact operator.

7. The computation of lower bounds.

We may use the theory of Orthogonal Invariants of positive compact operators [1] to obtain lower bounds for the eigenvalues λ_i. Let us first show that the operator Γ has finite trace in the sense of Hilbert-Schmidt. We recall that

$$tr\, \Gamma = \sum_{i=1}^{\infty} \mu_i = I_1^1(\Gamma) = \int_A tr\, \gamma(x, x)\, dx \tag{50}$$

where $I_1^1(\Gamma)$ is the orthogonal invariant of order 1 and degree 1 of Γ.

Then from (47), (44), and (37) we have

$$\begin{aligned} I_1^1(\Gamma) &= \int_A tr\, \gamma(x, x)\, dx = \int_{A_1} tr\, \gamma_{11}(x_1, x_1)\, dx_1 + \\ &+ \int_{A_2} tr\, \gamma_{22}(x_2, x_2)\, dx_2 = \\ &= \sum_{i=1}^{t} \bar{m}_i \int_0^1 g_{ii}(x, x)\, dx + \sum_{k=1}^{d} \overline{M}_k E_{kk}^{-1} = \\ &= \sum_{i=1}^{t} \bar{m}_i \left[tr\, C_i E^{-1} C_i^* M + \frac{1}{420 \chi_i} \right] + \sum_{k=1}^{d} \overline{M}_k E_{kk}^{-1} \end{aligned} \tag{51}$$

where

$M_{ij} = \int_0^1 \sigma_i(x)\, \sigma_j(x)\, dx \quad (i, j = 1, \ldots 4)$, so that by (26)

$$M = \begin{vmatrix} \frac{13}{35} & \frac{11}{210} & \frac{9}{70} & -\frac{13}{420} \\ \cdot & \frac{1}{105} & \frac{13}{420} & -\frac{1}{140} \\ \cdot & \cdot & \frac{13}{35} & -\frac{11}{210} \\ \cdot & \cdot & \cdot & \frac{1}{105} \end{vmatrix} \tag{52}$$

Let us point out that the practical computation of $I_1^1(\Gamma)$ is very simple. As formula (51) shows, it requires mainly the knowledge of the compliance matrix E^{-1}, that must be computed even for the simple static analysis of the frame. Moreover, both the matrix products $C_i E^{-1} C_i^* M$ (in (51)) and $C_i^* K C_i$ (in (34)) in practice will not be performed, because typical techniques of computer's programing allow to get the same result with great saving of time and memory requirements.

We may now get the lower bounds $\tilde{\tilde{\lambda}}_i^{(\nu)}$ $(i = 1, \ldots \nu)$ to the first ν eigenvalues of the original problem (5), (7), (8) by the following formula [1]

$$\tilde{\tilde{\lambda}}_i = \left[I_1^1(\Gamma) - \sum_{h=1}^{\nu} \tilde{\lambda}_h^{(\nu)-1} + \tilde{\lambda}_i^{(\nu)-1} \right]^{-1} \tag{53}$$

where the $\tilde{\lambda}_h^{(\nu)}$ $(h = 1, \ldots \nu)$ are the upper bounds computed by the Rayleigh-Ritz method.

8. An example.

Let us consider the steel frame of fig. 1, whose data

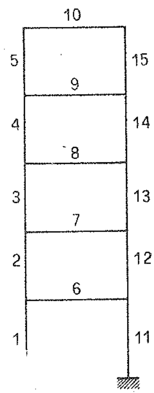

Fig. 1

are given in table 1. For this frame upper and lower bounds to the first 45 eigenvalues ($\nu = 45$) have been computed

Table 1

$t = 15$. $n = 15$. $\bar{k}_i = k_i/k_o$ $(i = 1, \ldots, 15)$			
ADIMENSIONAL VALUES			
i	$\bar{l}_i$	$\bar{k}_i$	$\bar{m}_i$
1	14	420.7	72
2	12	382.5	66
3	12	272.9	54
4	12	157.3	29
5	12	106.3	21
6	18	515	40
7	18	515	40
8	18	515	40
9	18	204.1	27
10	18	204.1	27
11	14	420.7	72
12	12	382.5	66
13	12	272.9	54
14	12	157.3	29
15	12	106.3	21

by the method presented in the paper. In table 2 are shown the results for the first nine eigenfrequencies, which are related to the adimensional eigenvalues λ_i by the formula

$$f_i = \frac{\sqrt{\lambda_i}}{2\pi l_0^2} \sqrt{\frac{k_0}{m_0}} \tag{54}$$

where

$l_0 = 1$ *ft.*, $k_0 = 30 \cdot 10^6$ *lb* $\cdot$ inch^2, $m_0 = 2.16 \cdot 10^{-4}$ sec^2 $\cdot$ $\cdot$ *lb*/inch^2

Table 2

$$\frac{1}{l_o^2} \sqrt{\frac{K_o}{m_o}} = 2588.12315$$

$I_1^1(\Gamma) = 12067.5190906$					$\nu = 45$
upper bounds		f		lower bounds	
4.287	≥	f1	≥	4.284	
10.260	≥	f2	≥	10.216	
18.779	≥	f3	≥	18.513	
28.560	≥	f4	≥	27.649	
38.356	≥	f5	≥	36.230	
44.002	≥	f6	≥	41.191	
54.248	≥	f7	≥	48.683	
59.409	≥	f8	≥	52.309	
67.082	≥	f9	≥	57.321	

9. Conclusive remarks.

We want to stress that the main purpose of this method is to separate the eigenvalues, i.e., to determine intervals in which only one eigenvalue is contained. In the above example (see table 2) the first 6 eigenvalues have been separated. When the approximation is not good enough, *ad hoc* techniques may be employed to restrict the bounds. These methods require anyway the previous rigorous separation of the eigenvalues.

2nd National Congress, October 1974

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BOUNDS TO THE CRITICAL LOADS FOR A CLASS OF ELASTIC BUCKLING PROBLEMS

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BOUNDS TO THE CRITICAL LOADS FOR A CLASS OF ELASTIC BUCKLING PROBLEMS

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Introduction

In this paper we consider a class of eigenvalue problems that arise in the buckling theory of elastic beams and plates. Such problems are formulated in a natural way in terms of differential operators. An upper bound to the critical load may be computed by means of the Rayleigh - Ritz method, in the classical form or in a more refined version with reference to the variational formulation of the eigenvalue problem. In latter case, as is well known, the test functions must satisfy only the geometric (stable) boundary conditions. In general a good choice of the test functions allows to get a good upper bound. The main goal of this paper is the determination of a lower bound to the critical load. Such problem is of great practical interest, because only the knowledge of rigorous upper and lower bounds to the first eigenvalue gives a meaningfull evaluation of the approximation. Furthermore from the technical point of view we are more interested in getting lower bounds, which give a safe estimation of the critical load. Also the problem of the approximation of the buckling modes, which is fundamental for the post-buckling analysis, requires the knowledge of lower bounds. We show how the original problem may be reduced to an equivalent eigenvalue problem for a compact operator. The two main results follow. All the relevant questions of existence and convergence can be answered satisfactorily by the spectral theory of compact operators. The theory of orthogonal invariants of compact operators (1)(2) may be used to solve the problem of the computation of a lower bound to the buckling load. For the sake of simplicity the method will be presented with reference

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to special (Dirichlet) boundary conditions, that for beams and plates correspond to the case of a clamped boundary. The extension to more general conditions will be considered in a next work. We shall begin with the study of the associated static problem, which is a boundary value problem for the iterated L^2 of an elliptic operator L. A fundamental step is the determination of the Green operator G of this problem. To this aim the representation of G in terms of an operator P, that is the projector on the kernel of L, and of a fundamental solution operator T for L, is of basic importance. In fact it is always possible to give T in a closed form if L has constant coefficients (3), while only in very special cases the Green operator may be found directly. We point out that T must be choosen in such a form to satisfy the boundary condition. Then we find, in terms of P and T, an hermitian compact operator K, whose eigenvalues are the inverses of those of the original problem. The method is presented in an abstract context, which allows to get a maximum of generality with a clear formalism. The whole procedure is illustrated by the application to the classical buckling problem for the clamped beam. We note that this problem could be solved in a simpler way (4), because the Green operator for the associated static problem may be easily found by direct integration. This example has been choosen to show, in the simplest way, how the abstract formalism specializes in a particular case. With this consideration in mind, we limit ourselves to the computation of the simplest orthogonal invariant of the operator K. The application to this particular problem of the Rayleigh - Ritz procedure, that allows to get upper bounds to the eigenvalues, is classical and will not be presented in this paper. The effectiveness of the method may be shown e.g. with respect to the buckling problem of elastic plates. Applications to such problems will be presented in connection with the extension of the method to more general boundary conditions and the discussion of the more complex cases in which there are positive and negative eigenvalues.

Analysis of the Eigenvalue Problem

Let S be a complex Hilbert space and $(\cdot,\cdot)$ the inner product in S. We shall consider eigenvalue problems of the form

$$Q L^2 v - \lambda N L v = 0 \qquad v \in V, \quad \lambda \in C \tag{1}$$

where :

L is a linear operator defined on the subspace D_L of S, dense in S (i.e. $\overline{D_L}=S$)

V is the space of admissible vectors ($V \subset S$)

Q is a strictly positive (and then hermitian and bounded) operator on S, such that $Q(D_L) \subset D_L$, $QL = LQ$ on D_L

N is an hermitian operator defined on S, such that $N(D_L) \subset D_L$, $NL = LN$ on D_L .

Suppose the following hypotheses be satisfied :

Hp. I. There exists an operator T defined on S, such that: i) T is hermitian and compact on S, ii) $T(S) \subset D_L$, iii) $L\,T\,u = u$ for every $u \in S$.

Hp. II. $\Omega_\circ = \ker L$ is closed on S, i.e. $\Omega_\circ = \overline{\Omega}_\circ$. Of course $\Omega_\circ \subset D_L$

Let us denote by P and Π the projectors of S on $\Omega_\circ$ and on $\Omega_\circ^\perp$ respectively.

We have the following Lemmas:

<u>Lemma</u> I. $T\,L\,u = u + \omega_\circ \quad \forall u \in D_L,\ \omega_\circ \in \Omega_\circ$ since $L(\,T\,L\,u - u\,) = Lu - Lu = 0$.

<u>Lemma</u> II. $u \in D_L \Rightarrow \Pi\,u \in D_L$ since $\Pi\,u = (I - P)\,u = u - P\,u$ and $P\,u \in \Omega_\circ \subset D_L$.

<u>Lemma</u> III. $L\,\Pi\,u = L\,u \quad \forall u \in D_L$ since $L\,\Pi\,u = L\,u - L\,P\,u = L\,u$.

<u>Lemma</u> IV. The subspaces $\Omega_\circ$ and $\Omega_\circ^\perp$ are invariant under Q and N. In fact

$Q(\Omega_\circ) \subset \Omega_\circ$, since $L\,Q\,\omega_\circ = Q\,L\,\omega_\circ = 0$

$Q(\Omega_\circ^\perp) \subset \Omega_\circ^\perp$, since $(Q\,w,\ \omega_\circ) = (w\ ,\ Q\,\omega_\circ) = 0$ if $w \in \Omega_\circ^\perp$

Identical proofs hold for N. Moreover it is easy to see that because Q^{-1} exists, it results $Q(\Omega_\circ) = \Omega_\circ$, $Q(\Omega_\circ^\perp) = \Omega_\circ^\perp$, $Q^{-1}(\Omega_\circ) = \Omega_\circ$, $Q^{-1}(\Omega_\circ^\perp) = \Omega_\circ^\perp$.

Let us define

$$H_L = \{\ v \in D_L \mid L\,v \in D_L\ \}$$

H_L is a subspace of D_L on which the iterated operator L^2 is defined. Let

$$V = \{\ v \in H_L \mid v = T\,w,\ w \in \Omega_\circ^\perp\}$$

be the subspace of H_L that we defined as the space of the admissible vectors. V is therefore determined by the choice of T. Remark : $v \in H_L \Rightarrow w \in D_L$.

<u>Theorem</u> I. The following problem : given $f \in S$, find a vector $v \in V$ such that

$$L^2\,v = f$$

has one and only one solution given by

$$v = G\,f$$

where
$$G = T\,\Pi\,T$$

<u>Proof</u>.

Existence. It results $L^2 v = L^2\,T\,w = L\,\Pi\,u = L\,u$. Hence if we assume $w = T\,f$ we get the thesis. $L^2\,v = L\,u = L\,T\,f = f$.

Uniqueness. We have to prove that $L^2 v = 0$ implies $v = 0$. In fact $L^2 v = 0$

both sides by $Q^{-\frac{1}{2}}$, we get the following equivalent eigenvalue problem

$$Q^{-\frac{1}{2}}\, N\, \Pi\, T\, \Pi\, Q^{-\frac{1}{2}}\, z = \mu\, z \qquad z \in S, \quad \mu \in C \tag{4}$$

The operator $K = Q^{-\frac{1}{2}}\, N\, \Pi\, T\, \Pi\, Q^{-\frac{1}{2}}$ is hermitian and compact on S. Then (4) may be written as follows

$$K\, z = \mu\, z \qquad z \in S, \quad \mu \in R \tag{5}$$

An example

Let us see, in a simple case, how the abstract formalism specializes. We shall consider the buckling problem for the clamped beam. The eigenvalue problem is

$$k\, D^4\, v = \lambda\, n\, D^2\, v \qquad v \in V, \quad \lambda \in C \tag{6}$$

where k and n are positive constants and

$$V = \{\, v \in H_{4(0,1)} \mid v(o)=v'(o)=v(1)=v'(1)=0 \,\} \tag{7}$$

It results $L = D^2$, $D_L = H_{2(0,1)}$, $H_L = H_{4(0,1)}$, $Q = k$, $N = n$.
$\Omega_o = \ker L$ is the subspace of the linear functions $ax + b$ on $(0,1)$(closed).
In Ω_o we have the orthonormal basis $\{u_1 , u_2\} = \{1 , \sqrt{3}(2x - 1)\}$, then

$$P\, u(x) = (u,u_i)\, u_i = \int_0^1 p(x,\xi)\, u(\xi)\, d\xi \tag{8}$$

where

$$p(x,\xi) = 1 + 3(2x - 1)\,(2\xi - 1) \tag{9}$$

The operator T is a fundamental solution operator for the differential operator D^2. We will show that to satisfy the boundary conditions (7), we may assume

$$v(x) = T\, w(x) = \int_0^1 g(x,\xi)\, w(\xi)\, d\xi \tag{10}$$

where $w \in \Omega_o^{\perp}$, and $g(x,\xi)$ is the kernel of the Green operator of the following boundary value problem :

$$D^2\, u = w \qquad u(\alpha) = u(\beta) = 0 \tag{11}$$

with $[\alpha,\beta] \supset [0,1]$. In fact since the function $v(x)$ is of class C_1 on $[\alpha,\beta]$ the boundary conditions (7) are satisfied if v vanishes on the intervals $[\alpha,0]$ and $[1,\beta]$. Such condition is equivalent to the following

$$\int_\alpha^0 x^k\, v(x)\, dx = 0 \qquad \int_1^\beta x^k\, v(x)\, dx = 0 \tag{12}$$

for every natural number k ($\{x_k\}$ is a complete system in $L_{2(\alpha,\beta)}$). Hence we have, from the first one of (12) and from (10)

$$\int_\alpha^0 x^k \int_0^1 g(x,\xi)\ w(\xi)\ d\xi\ dx = \int_0^1 w(\xi) \int_\alpha^0 g(x,\xi)\ x^k\ dx\ d\xi = 0 \tag{13}$$

The function

$$\phi(\xi) = \int_\alpha^0 g(x,\xi)\ x^k\ dx \tag{14}$$

may be written in the form

$$\phi(\xi) = \int_\alpha^\beta g(x,\xi)\ f(x)\ dx \tag{15}$$

where

$$f(x) = \begin{cases} x^k & \alpha \leqslant x \leqslant 0 \\ 0 & 0 \leqslant x \leqslant \beta \end{cases}$$

Hence $\phi(\xi)$ is the solution of the following boundary value problem (see (10))

$$D^2\ \phi(\xi) = f(\xi) \qquad\qquad \phi(\alpha) = \phi(\beta) = 0$$

On the interval $[0,1]$ we have $D^2\ \phi(\xi) = f\ (\xi) = 0$ and then $\phi(\xi) = a_k\xi + b_k$.

From (13) we get

$$\int_0^1 w(\xi)\ (\ a_k\xi + b_k\ )\ d\xi = 0 \tag{16}$$

If we set $\alpha = -1$, $\beta = 2$, it follows

$$g(x,\xi) = \begin{cases} \frac{1}{3}\ (x\xi + x - 2\xi - 2) & x \geqslant \xi \\ \frac{1}{3}\ (\xi x + \xi - 2x - 2) & \xi \geqslant x \end{cases} \tag{17}$$

From (7) - (8) and (9) we get

$$\Pi\ T\ \Pi\ u(x) = \int_0^1 \chi(x,\xi)\ u(\xi)\ d\xi \tag{18}$$

where

$$\chi(x,\xi) = \gamma(x,\xi) - \int_0^1 p(x,t)\ \gamma(t,\xi)\ dt \tag{19}$$

and

$$\gamma(x,\xi) = g(x,\xi) - \int_0^1 g(x,t)\ p(t,\xi)\ dt \tag{20}$$

From (19) and (20), because $g(t,\xi) = g(\xi,t)$ and $p(t,x) = p(x,t)$ it follows

$$\chi(x,\xi) = g(x,\xi) - \int_0^1 g(x,t)\ p(t,\xi)\ dt - \int_0^1 g(\xi,t)\ p(t,x)\ dt +$$

$$+ \int_0^1 \int_0^1 p(x,t)\ g(t,\eta)\ p(\eta,\xi)\ d\eta\ dt$$

Making use of (9) and (17) we obtain

$$\chi(x,\xi) = g(x,\xi) - 2(\xi x^3 + x\xi^3) + (x^3 + \xi^3) + 3(x^2\xi + \xi^2 x) - 2(x^2 + \xi^2) +$$

$$- \frac{23}{15} x\xi + \frac{23}{30}(x + \xi) + \frac{8}{15}$$

We may now evaluate the orthogonal invariant of degree 1 and order 1 [1] of the compact operator $K = k^{-1}\ n\ \Pi\ T\ \Pi$, which is given by the formula

$$I_1^1 = k^{-1}\ n \int_0^1 \chi(x,x)\ dx \tag{21}$$

Since

$$\chi(x,x) = -4\ x^4 + 8\ x^3 - \frac{26}{5} x^2 + \frac{6}{5} x - \frac{2}{15}$$

from (21) we get

$$I_1^1 = -k^{-1}\ n \frac{1}{15} \tag{22}$$

Let us suppose that the lower bounds $\bar{\bar{\mu}}_i$ $(i = 1,..h)$ to the first h eigenvalues μ_i $(i = 1,..h)$ have been evaluated e.g. by the Rayleigh - Ritz method. Recalling that for the orthogonal invariant I_1^1 we have the relation [2]

$$I_1^1 = \sum_{i=1}^{\infty} \mu_i = \sum_{i=1}^{\infty} \lambda_i^{-1} \tag{23}$$

upper bounds $\bar{\mu}_k$ $(k = 1,..h)$ to first h eigenvalues μ_i may be obtained by the formula

$$\bar{\mu}_k = I_1^1 - \sum_{i=1}^{h} \bar{\bar{\mu}}_i + \bar{\bar{\mu}}_k \tag{24}$$

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On a Class of Buckling Problems in the Theory of Elastic Structures

di **MANFREDI ROMANO** (a Napoli)

RIASSUNTO - *Viene presentata una teoria astratta per i problemi di autovalori che si incontrano nello studio dello svergolamento (buckling) di talune strutture elastiche. Vengono discussi metodi che consentono di ottenere valori per eccesso e per difetto dei singoli autovalori, non escludendo che questi possano essere sia positivi che negativi. La teoria viene illustrata da due esempi relativi a sistemi ellittici di ordine* 2m.

I. Introduction.

The analysis of many buckling problems in the linear theory of elastic structures leads to eigenvalue problems of the form

$$(1) \qquad Q\,L^2 v - \lambda\, N\, L\, v = 0 \qquad v \in V \quad \lambda \in C$$

where L is a differential operator defined on a linear variety $\mathcal{D}_L$ of a functional space S (usually a Sobolev space), Q is a positive definite operator on S, N an hermitian operator on S and V is the subspace of the admissible functions in S which is determined by the boundary conditions. In this paper we shall develope the analysis of such problems in an abstract framework, viewing L, Q and N as operators considered in a Hilbert space S.

Such approach allows to obtain the maximum of generality in the formulation, pointing out the essential hypotheses that are necessary to get the desidered results. Moreover the proofs of the theorems are simpler and the whole procedure is easier to follow in such abstract formalism than in the specific notations of each particular case. Under convenient assumptions, that are verified in the cases that arise in the applications, it will be shown that problem (1) is equivalent to a classical eigenvalue problem for an hermitian compact operator on S, for which the relevant questions of existence and convergence can be answered satisfactorily. A structural representation of this operator, in terms of a resolvent operator T for L and a projector P, is provided. The operators T and P determine the class V of the admissible functions, and their appropriate choice allows to cover a large class of boundary conditions. It is then shown how to apply the Rayleigh-Ritz method and the theory of orthogonal invariants of positive compact operators [1] to get a rigorous computation of the eigenvalues.

II. The Existence and Uniqueness Theorem.

We shall postpone the rigorous statement and analysis of the eigenvalue problem to the next section, since it is necessary first to introduce some definitions and state the basic hypotheses concerning the abstract operator L. In the present section we shall prove an existence and uniqueness theorem that plays a fundamental role in the subsequent analysis.

Let S be a complex Hilbert space and $(\cdot,\cdot)$ the inner product in S. We consider the linear operator

$$L:\ v \in \mathcal{D}_L \rightarrow L\,v \in S$$

defined on the linear variety $\mathcal{D}_L$ of S, dense in S (i. e. $\overline{\mathcal{D}_L} = S$). Suppose the following hypotheses be satisfied:

HYPOTHESIS I.

There exists an operator T, defined on S, such that.

i) T is hermitian and compact on S

ii) $T(S) \subset \mathcal{D}_L$

iii) $LTu = u \quad \forall u \in S$ i. e. $LT = I$ on S.

HYPOTHESIS II.

$\Omega_0 = \ker L$ is closed on S, i. e. $\overline{\Omega_0} = \Omega_0$. Of course $\Omega_0 \subset \mathcal{D}_L$.

LEMMA I. *For every* $u \in \mathcal{D}_L$ $\ TLu = u + \omega_0$ *with* $\omega_0 \in \Omega_0$.

In fact $L(TLu - u) = Lu - Lu = 0$.

Let Ω be a subspace (closed linear variety) of the kernel Ω_0 of L, and $\Omega^{\perp}$ its orthogonal complement in S. Denote by P and $\Pi = I - P$ the orthogonal projectors on Ω and on $\Omega^{\perp}$ respectively.

LEMMA II. $u \in \mathcal{D}_L \Rightarrow \Pi u \in \mathcal{D}_L$ since $\Pi u = (I-P)\, u = u - Pu$ and $Pu \in \Omega \subset \mathcal{D}_L$.

LEMMA III. $L\Pi u = Lu \quad \forall u \in \mathcal{D}_L$ since $L\Pi u = Lu - LPu = Lu$.

Let us define

$$\mathcal{H}_L = \{ v \in \mathcal{D}_L \mid L v \in \mathcal{D}_L \}$$

$\mathcal{H}_L$ is a subvariety of $\mathcal{D}_L$ on which the iterated operator L^2 is defined.

Let V be the linear subvariety of $\mathcal{H}_L$ such that for every $v \in V$

$$v = Tw, \quad w \in \Omega^{\perp}$$

and the following Hypothesis is satisfied:

HYPOTHESIS III.

On V the following formula holds

$$(L^2 v, z) = (L v, Lz) \qquad \forall z \in V, v \in V.$$

We define V as the space of the admissible functions.

LEMMA IV. $TLv = v \quad \forall v \in V$ since $TLv = TLTw = Tw = v$.

THEOREM I. *The following equation*

$$TLw = w + \omega \quad \forall w \in \Omega^{\perp} \cap \mathcal{D}_L, \ \omega \in \Omega \tag{2}$$

is equivalent to Hyp. III.

PROOF. Assume that equation (2) holds. If $v, u \in V$, we have $v = Tw$ and $u = Tw'$ with $w, w' \in \Omega^{\perp} \cap \mathcal{D}_L$, so that $(L^2 v, u) = (L^2 Tw, Tw') = (Lw, Tw') = (TLw, w') = (w + \omega, w') = (w, w') = (Lv, Lu)$. If Hyp. III holds, assuming v and u defined as above, since $Lv = w$ and $Lu = w'$, we have: $(TLw.\, w') = (Lw, Tw') = (L^2 v, u) = (L v, Lu) = (w, w')$; hence $(TLw - w, w') = 0 \ \forall w' \in \Omega^{\perp} \cap \mathcal{D}_L$. Since $\overline{\mathcal{D}_L} = S$, we have $\overline{\Omega^{\perp} \cap \mathcal{D}_L} = \Omega^{\perp}$. In fact $\overline{\Omega^{\perp} \cap \mathcal{D}_L} \subset \overline{\Omega^{\perp}} = \Omega^{\perp}$. On the other hand if we suppose that $w \in \Omega^{\perp}$, since $\overline{\mathcal{D}}_L = S$, we must have $w = \lim_n w_n$ with $w_n \in \mathcal{D}_L$. Moreover $w_n = Pw_n + \Pi w_n$ and $\lim_n Pw_n = Pw = 0$, $\lim_n \Pi w_n = \Pi w = w$. Since $Pw_n \in \Omega \subset \mathcal{D}_L$, we have $\Pi w_n = w_n - Pw_n \in \mathcal{D}_L$. Hence $w \in \overline{\Omega^{\perp} \cap \mathcal{D}_L}$. By the continuity of the inner product in S, we obtain that $(TLw - w, w') = 0 \ \forall w' \in \Omega^{\perp}$, i. e. $TLw - w = \omega \in \Omega$.

LEMMA V. *When we assume* $\Omega = \Omega_0$, Hyp. III *is automatically satisfied.*

In fact for Theorem I Hyp. III is equivalent to Eq. (2), which in this particular case is satisfied because of Lemma I. Alternatively we may verify that in this case $(L^2 v, u) = (L^2 Tw, Tw') = (Lw, Tw') = (TLw, w') = (w + \omega_0, w') = (w, w') = (L v, Lu)$ where $w, w' \in \Omega_0^{\perp} \cap \mathcal{D}_L$.

Let us now consider the following problem:

(*P*) Given $f \in S$ find a vector $v \in V$ such that

$$L^2 v = f.$$

THEOREM II. *Problem* (*P*) *has one and only one solution given by*

$$v = Gf \tag{3}$$

where $G = T\Pi T$.

PROOF. *Existence.* It is obvious that the vector v given by (3) belongs to V, since $v = Tw$, with $w = \Pi Tf \subset \Omega^{\perp} \cap \mathcal{D}_L$ (see Hyp. I, ii) and Lemma II). Moreover $L^2 v = L^2 T\Pi Tf = L\Pi Tf$ (see Hyp. I, iii)). For Lemma III we have $L\Pi Tf = LTf = f$. *Uniqueness.* Suppose that $v \in V$, $L^2 v = 0$. From Hyp. III we deduce $0 = (L^2 v, v) = (Lv, Lv)$. Hence $Lv = 0$. From Lemma IV we deduce $TLv = v = 0$.

III. The Eigenvalue Problem.

Let us now consider the eigenvalue problem

$$QL^2 v - \lambda NLv = 0 \qquad v \in V \qquad \lambda \in C \tag{4}$$

where the two operators Q and N, defined on S, verify the following hypotheses.

HYPOTHESIS IV.

i) Q is a strictly positive (and then hermitian and bounded) operator on S.

ii) $Q(\Omega)\subset\Omega$, i. e. the subspace Ω is invariant under Q.

LEMMA VI. $Q(\Omega^{\perp})\subset\Omega^{\perp}$.

In fact if $\omega\in\Omega$, $w\in\Omega^{\perp}$ we have $(Qw, \omega)=(w, Q\omega)=0$.

LEMMA VII. *There exists* Q^{-1} *which is strictly positive and hermitian on* S *and we have* $Q(\Omega)=\Omega$, $Q(\Omega^{\perp})=\Omega^{\perp}$, $Q^{-1}(\Omega)=\Omega$, $Q^{-1}(\Omega^{\perp})=\Omega^{\perp}$.

LEMMA VIII. *There exists* $Q^{1/2}$ *which is strictly positive and hermitian on* S.

HYPOTHESIS V.

i) N is hermitian (and then bounded) on S.

ii) $N(\mathcal{D}_L)\subset\mathcal{D}_L$

iii) $N(\Omega)\subset\Omega$

iv) $N(V)\subset V$ *and* $NLv=LNv\ \forall\ v\in V$.

LEMMA IX. $N(\Omega^{\perp})\subset\Omega^{\perp}$.

See Lemma VI.

LEMMA X. $N\Pi u=\Pi Nu\ \ \forall u\in S$.

In fact, because of Hyp. V iii) and Lemma IX, we have $u=Pu+\Pi u$ and $\Pi Nu=\Pi N(\Pi u+Pu)=\Pi N\Pi u=N\Pi u$.

THEOREM III. *Under the assumed hypotheses on* T *and under the Hyp. V* i), ii), iii) *on* N *the operators* N *and* T *commute on* $\Omega^{\perp}$, *i. e.*

$$NTw=TNw \qquad \forall w\in\Omega^{\perp}, \tag{5}$$

if and only if Hyp. iv) *holds.*

PROOF. Let us first show that Hyp. V iv) implies (5). $\forall w\in\Omega^{\perp}\cap\mathcal{D}_L$ by Lemma IV and Hyp. V iv) we have: $NTw=NTLv=Nv=TLNv=$ $=TNLTw=TNw$, i. e.

$$TNw=NTw \qquad \forall w\in\Omega^{\perp}\cap\mathcal{D}_L.$$

Since the operators TN and NT are bounded on S and $\Omega^1 \cap \mathcal{D}_L = \Omega^1$, Eq. (5) follows. For getting the inverse implication, let us observe that for every $v \in V$, because of (5), we have $Nv = NTw = TNw \in V$, i. e. $N(V) \subset V$. Moreover $NLv = NLTw = Nw = LTNw = LNTw = LNv$.

THEOREM IV. *The problem* (4) *has no zero eigenvalue.*

PROOF. Recalling Hyp. IV i) and Theorem I, we see that $\lambda = 0 \Rightarrow \Rightarrow QL^2 v = 0 \Rightarrow L^2 v = 0 \Rightarrow v = 0$.

Let us substitute in Eq. (4) (where we assume $\lambda \neq 0$), for $L^2 v$ the vector f of S and for v the vector $T\Pi T f$. We get

$$Qf - \lambda\, N\Pi T f = 0 \qquad f \in S, \qquad \lambda \in C - \{0\}. \tag{6}$$

The eigenvalue problem (6), because of Theorems I and III is equivalent to the eigenvalue problem (4). The operator $N\Pi T$ is compact but not hermitian on S.

THEOREM V. *The eigenvalue problem* (6) *is equivalent to the following one*

$$Qf - \lambda\, N\Pi T\Pi f = 0 \qquad f \in S, \qquad \lambda \in C - \{0\}. \tag{7}$$

PROOF. If λ and f are eigenvalue and eigenvector of problem (6), then $f = Q^{-1} \lambda\, N\Pi\, Tf$, hence by the Lemmas VII and IX, we have $f = \Pi f$. Substituting for f the vector Πf in Eq. (6) we get Eq. (7). Let us now suppose that λ and f are eigenvalue and eigenvector of (7). We have $f = Q^{-1} \lambda\, N\Pi\, T\Pi f$, so that again $f = \Pi f$ and substituting for Πf the vector f in Eq. (7) we get Eq. (6).

THEOREM VI. *The operator $N\Pi\, T\Pi$ is hermitian on S, i. e. the hermitian operators N and $\Pi\, T\, \Pi$ commute on S.*

PROOF. $\forall u, v \in S$ we have, making use of Lemma X and Theorem III, $(N\Pi\, T\Pi u, v) = (u, \Pi T\, \Pi N v) = (u, \Pi T\, N\Pi\, v) = (u, \Pi N\, T\Pi\, v) = = (u, N\Pi\, T\Pi\, v)$.

If we pose $z = Q^{1/2} f$ and $\mu = \lambda^{-1}$, then substituting in (7) for the vector f, $Q^{-1/2} z$ and for λ, μ^{-1} and multiplying at left both sides by

$Q^{-1/2}$, we get the following equivalent eigenvalue problem

$$Q^{-1/2} N\Pi\, T\Pi\, Q^{-1/2}\, z = \mu z \qquad z \in S, \qquad \mu \in C. \tag{8}$$

The operator $K = Q^{-1/2} N\Pi\, T\Pi\, Q^{-1/2}$ is hermitian and compact on S. Then we may write (8) as follows

$$Kz = \mu z \qquad z \in S, \qquad \mu \in R. \tag{9}$$

REMARK. The operator K has the eigenvalue $\mu = 0$ and the corresponding eigenspace ker K is such that ker $K \supset Q^{1/2}(\Omega)$.

Let us recall that, since K is hermitian and compact, the characteristic set (the set of the eigenvalues) of K is a countable subset of R and, if infinite, has zero as unique limit point. The eigenvalues of problem (4) are the inverses of the non zero eigenvalues of K. Let us denote by $\{\mu_h\}$ the countable sequence of such eigenvalues ($\mu_h \neq 0$), ordered e. g. so that their modulus is non increasing [1], and each repeated according to its multiplicity. The eigenvalues of problem (4) are given by the countable sequence $\{\lambda_h \in R \mid \lambda_h = \mu_h^{-1}\}$ that we shall briefly denote by $\{\lambda_h\}$. From $\{\mu_h\}$ we may construct the two (void, finite or infinite) non-increasing and non-decreasing sequences $\{\mu_i^+\}$ and $\{\mu_j^-\}$ of the positive and negative eigenvalues of K. From $\{\lambda_h\}$, in the same way, we obtain the non-decreasing sequence $\{\lambda_i^+\}$, and the non-increasing sequence $\{\lambda_j^-\}$.

IV. The Rayleigh-Ritz Method.

By the Rayleigh - Ritz method it is possible to get lower bounds to the positive eigenvalues and upper bounds to the negative eigenvalues of K. To prove the basic theorem let us first introduce some definitions and prove some preparatory results. If W is a subspace of S and $\tilde{P}$ the orthogonal projector of S on W, the hermitian compact operator $\tilde{P} K \tilde{P}$ (definite on S) is called the W-component of K. Let us consider the eigenvalue problem

$$\tilde{P} K \tilde{P} u - \tilde{\mu}\, u = 0 \qquad u \in S, \;\; \tilde{\mu} \in R. \tag{10}$$

(1) If two distinct eigenvalues have the same modulus we may assume that the negative follows the positive.

It is easily seen that $\widetilde{P}K\widetilde{P}$ has the zero eigenvalue and that the corresponding eigenspace $\ker \widetilde{P}K\widetilde{P}$ contains $S \ominus W$. Then non zero eigenvalues [2] of $\widetilde{P}K\widetilde{P}$ (each repeated according to its multiplicity) can be ordered in the positive (non-increasing) and negative (non-decreasing) sequences $\{\widetilde{\mu}_i^+\}$ and $\{\widetilde{\mu}_j^-\}$. Let us now prove the following two theorems.

THEOREM VII. *If K is an hermitian and compact operator and $\widetilde{P}K\widetilde{P}$ its W-component, we have*

$$\mu_i^+ \geq \widetilde{\mu}_i^+ \tag{11}$$

$$\mu_j^- \leq \widetilde{\mu}_j^-. \tag{12}$$

PROOF. Let H be a finite dimensional subspace of S and $H^\perp$ its orthogonal complement. Let us define $\widetilde{H}=\widetilde{P}(H)$ and $H^\perp=(\widetilde{P}(H))^\perp$. We have that $\widetilde{P}u \in H^\perp \Leftrightarrow u \in \widetilde{H}^\perp$ since $(u, \widetilde{P}h)=(\widetilde{P}u, h)$ where $h \in H$. Let us denote by $H_0^\perp$ and $\widetilde{H}_0^\perp$ the two sets $H^\perp - \{0\}$ and $\widetilde{H}^\perp - \{0\}$. By the H. Weyl theorem on the independent characterization of the eigenvalues of hermitian compact operators [2], if $\dim H < i \leq \dim W$, we have

$$\widetilde{\mu}_i^+ = \min_H \max_{u \in H_0^\perp} \frac{(\widetilde{P}K\widetilde{P}u, u)}{\|u\|^2} \leq \min_H \max_{u \in \widetilde{H}_0^\perp} \frac{(\widetilde{P}K\widetilde{P}u, u)}{\|u\|^2} \leq$$

$$\leq \min_H \max_{u \in \widetilde{H}_0^\perp} \frac{(K\widetilde{P}u, \widetilde{P}u)}{\|\widetilde{P}u\|^2} \leq \min_H \max_{\widetilde{P}u \in H_0^\perp} \frac{(K\widetilde{P}u, \widetilde{P}u)}{\|\widetilde{P}u\|^2} \leq$$

$$\leq \min_H \max_{u \in H_0^\perp} \frac{(Ku, u)}{\|u\|^2} = \mu_i^+$$

[2] In what follows we shall refer to the general case in which both K and $\widetilde{P}K\widetilde{P}$ have at least a finite number of positive and negative eigenvalues. If this is not the case, our results may be modified in an obvious way.

and if $\dim H < j \leq \dim W$

$$\tilde{\mu}_j^- = \max_H \min_{u \in H_0^1} \frac{(\tilde{P} K \tilde{P} u, u)}{\| u \|^2} \geq \max_H \min_{u \in \tilde{H}_0^1} \frac{(\tilde{P} K \tilde{P} u, u)}{\| u \|^2} \geq$$

$$\geq \max_H \min_{u \in \tilde{H}_0^1} \frac{(K \tilde{P} u, \tilde{P} u)}{\| \tilde{P} u \|^2} \geq \max_H \min_{\tilde{P} u \in H_0^1} \frac{(K \tilde{P} u, \tilde{P} u)}{\| \tilde{P} u \|^2} \geq$$

$$\geq \max_H \min_{u \in H_0^1} \frac{(K u, u)}{\| u \|^2} = \mu_j^- .$$

THEOREM VIII. *Let K and $\tilde{K}$ be two hermitian compact operators defined on S and denote by μ_i^+ (μ_j^-) the i-th (j-th) positive (negative) eigenvalue of K and by $\tilde{\mu}_i^+$ ($\tilde{\mu}_j^-$) the corresponding eigenvalue of $\tilde{K}$. Then we have*

$$|\mu_i^+ - \tilde{\mu}_i^+| \leq \|K - \tilde{K}\| \tag{13}$$

$$|\mu_j^- - \tilde{\mu}_j^-| \leq \|K - \tilde{K}\|. \tag{14}$$

PROOF. Let H and H_0^1 be the same subspaces of S as in Theorem VII. By the above mentioned theorem of H. Weyl we have

$$\mu_i^+ = \min_H \max_{u \in H_0^1} \frac{(K u, u)}{\| u \|^2} \leq \min_H \left\{ \max_{u \in H_0^1} \frac{((K - \tilde{K}) u, u)}{\| u \|^2} + \max_{u \in H_0^1} \frac{(\tilde{K} u, u)}{\| u \|^2} \right\} \leq \| K - \tilde{K} \| + \tilde{\mu}_i^+$$

i. e. $\mu_i^+ - \tilde{\mu}_i^+ \leq \|K - \tilde{K}\|$. Inverting the roles of μ_i^+ and of $\tilde{\mu}_i^+$ we get $\tilde{\mu}_i^+ - \mu_i^+ \leq \|K - \tilde{K}\|$, so that $|\mu_i^+ - \tilde{\mu}_i^+| \leq \|K - \tilde{K}\|$.

Moreover we have

$$\mu_j^- = \max_H \min_{u \in H_0^1} \frac{(K u, u)}{\| u \|^2} \geq \max_H \left\{ \min_{u \in H_0^1} \frac{((K - \tilde{K}) u, u)}{\| u \|^2} + \min_{u \in H_0^1} \frac{(\tilde{K} u, u)}{\| u \|^2} \right\} \geq - \| K - \tilde{K} \| + \tilde{\mu}_j^-$$

i. e. $\tilde{\mu}_j^- - \mu_j^- \leq \|K - \tilde{K}\|$. Analogously we get $\mu_j^- - \tilde{\mu}_j^- \leq \|K - \tilde{K}\|$ so that $|\mu_j^- - \tilde{\mu}_j^-| \leq \|K - \tilde{K}\|$.

Let $\{u_h\}$ be a complete system of linearly independent vectors in S. Let W_n be the n-dimensional subspace of S spanned by $u_1, u_2, \ldots, u_n$ and denote by $\tilde{P}_n$ *the* orthogonal projector of S on W_n. The non zero eigenvalues of the problem

$$\tilde{P}_n K \tilde{P}_n u - \mu u = 0 \qquad u \in S,\ \mu \in R \tag{15}$$

are given by the non zero roots of the following determinant equation [1]

$$\det\{(K u_h, u_k) - \mu (u_h, u_k)\} = 0 \qquad (h, k = 1, \ldots, n). \tag{16}$$

If we refer to the general case in which $\tilde{P}_n K \tilde{P}_n$ has positive eigenvalues $\tilde{\mu}_{i,n}^+ \, (i \geq 1)$ and negative eigenvalues $\tilde{\mu}_{j,n}^- (j \geq 1)$ (each repeated according to its multiplicity), the following theorem fournishes the Rayleigh - Ritz method for the lower (upper) approximation of the positive (negative) eigenvalues of K.

Theorem IX. *The non zero eigenvalues of the W_n-component $\tilde{P}_n K \tilde{P}_n$ of K have the following properties*

$$\tilde{\mu}_{i,n}^+ \leq \tilde{\mu}_{i,n+1}^+ \leq \mu_i^+ , \qquad \tilde{\mu}_{j,n}^- \geq \tilde{\mu}_{j,n+1}^- \geq \tilde{\mu}_j^- \tag{17}$$

$$\lim_n \tilde{\mu}_{i,n}^+ = \mu_i^+ \qquad , \qquad \lim_n \mu_{j,n}^- = \mu_j^- . \tag{18}$$

Proof. By the completeness of $\{u_h\}$ the projector $\tilde{P}_n$ converges strongly to the identity operator I, i. e. $\lim_n \tilde{P}_n u = u \ \forall u \in S$. Then we know [1] (Aronszajn) that the operator $\tilde{P}_n K \tilde{P}_n$ converges uniformly to K, i. e. $\lim_n \|\tilde{P}_n K \tilde{P}_n - K\| = 0$. Now if we substitute $\tilde{P}_n K \tilde{P}_n$ for K in Theorem VIII we see that (18) follow. Right inequalities in (17) are easily obtained if we substitute in Theorem VII $\tilde{P}_{n+1} K \tilde{P}_{n+1}$ for $\check{P} K \check{P}$. Left inequalities follow if we substitute $\tilde{P}_{n+1} K \tilde{P}_{n+1}$ for K and $\tilde{P}_n K \tilde{P}_n$ for $\tilde{P} K \tilde{P}$.

It is possible to formulate the Rayleigh - Ritz method directly for the problem (4). In fact we have the following theorem.

THEOREM X. *Let $\{v_h\}$ be a system of linearly independent functions of V, such that $\{Lv_h\}$ is complete in Ω^1. Consider the determinant equation*

$$\text{(19)} \qquad \det\{(QLv_h\,,\ Lv_k)-\lambda\,(NLv_h\,,v_k)\}=0 \qquad (h,k=1,\ldots,n)$$

and denote by $\tilde{\lambda}^+_{i,n}$ and $\tilde{\lambda}^-_{j,n}$ the positive and negative roots of (19), each repeated according to its multiplicity, and ordered as the λ_i^+ and λ_j^-. Then we have

$$\text{(20)} \qquad \tilde{\lambda}^+_{i,n}\geq\tilde{\lambda}^+_{i,n+1}\geq\lambda^+_i, \qquad \tilde{\lambda}^-_{j,n}\leq\tilde{\lambda}^-_{j,n+1}\leq\lambda^-_j$$

$$\text{(21)} \qquad \lim_n \tilde{\lambda}^+_{i,n}=\lambda^+_i \quad , \qquad \lim_n \tilde{\lambda}^-_{j,n}=\lambda^-_j\,.$$

PROOF. We will show that the non zero roots of Eq. (19) are the inverses of the non zero roots of Eq. (16) Then the thesis follows from Theorem IX. Set $u_h=Q^{1/2}\,Lv_h$. Then we have:

$$(Ku_h,u_k)=(Q^{1/2}N\Pi\, T\Pi Q^{1/2}\,u_h,u_k)=(Q^{1/2}\,N\Pi\,T\Pi\,L\,v_h\,,\,Q^{1/2}\,L\,v_k)=$$

$$(N\,\Pi\,T\,\Pi\,L\,v_h,L\,v_k)=(TLv_h,NLv_k)=(v_h,NLv_k)$$

and

$$(u_h,u_k)=(Q^{1/2}\,L\,v_h,Q^{1/2}\,L\,v_k)=(L\,v_h,QLv_k).$$

If we pose $\mu=\lambda^{-1}$ we see that (14) can be written as (17), and the proof is complete.

V. Upper and Lower Bounds to the Eigenvalues.

We intend to show how to compute upper (lower) bounds τ_i^+ (τ_j^-) to the eigenvalues μ_i^+ (μ_j^-) of K, by means of the Theory of Orthogonal Invariants of positive compact operators [1]. We refer to the general case in which K has both positive and negative eigenvalues, so that K is not positive (or negative). Then we have to consider the iterated operator K^2, which is obviously positive since its eigenvalues are the squares μ_h^2 of the eigenvalues of K. We have at our disposal the n Rayleigh - Ritz approximations $\tilde{\mu}^+_{i,n}$ and $\tilde{\mu}^-_{j,n}$. Let us denote by $\tilde{\mu}_{h,n}$ the finite sequence of these eigenvalues ordered so that their modulus $|\tilde{\mu}_{h,n}|$ is non-increasing. Then the $|\tilde{\mu}_{h,n}|$ $(h=1, \dots, n)$ are lower bounds to the first n $|\mu_h|$ (3).

Let us suppose that the operator K^2 has finite trace in the sense of Hilbert - Schmidt. Then we may compute e. g. the orthogonal invariant $I_1^1(K^2)$ of order one and degree one of K^2 [1], which has the following meaning

$$I_1^1(K^2) = \sum_{i=1}^{\infty} \mu_i^2.$$

We get n upper bounds $|\tau_{h,n}|$ to the $|\mu_h|$ $(h=1, \dots, n)$ by the following formula [1]

$$|\tau_{h,n}| = \left[I_1^1(K^2) - \sum_{i=1}^{n} \tilde{\mu}^2_{i,n} + \tilde{\mu}^2_{h,n}\right]^{\frac{1}{2}}$$

so that

$$|\tilde{\mu}_{h,n}| \leq |\mu_h| \leq |\tau_{h,n}|.$$

Let us suppose that each of the first ν eigenvalues $|\mu_h|$ $(h=1, \dots, n)$ $(\nu \leq n)$ has been separated i. e. that

$$|\tau_{h,n}| \leq |\tilde{\mu}_{h+1,n}| \qquad (h=1, \dots, \nu). \tag{22}$$

Then if $\tilde{\mu}_{h,n} > 0$ $(h < \nu)$, we have

$$\tilde{\mu}_{h,n} \leq \mu_h \leq |\tau_{h,n}| \tag{23}$$

and if we denote by $\mu^+_{i,n}$ the i-th positive eigenvalue of $\tilde{P}_n K \tilde{P}_n$ that coincides with $\tilde{\mu}_{h,n}$, and by τ_i^+ the bound $|\tau_{h,n}|$, we have $\mu_h = \mu_i^+$ and

(3) The $|\mu_h|$ may be regarded as the eigenvalues of the operator $(K^2)^{1/2}$

(23) may be written in the form

$$\tilde{\mu}_{i,n}^{+} \leq \mu_i^{+} \leq \tau_i^{+}.$$

If $\tilde{\mu}_{h,n}<0\,(h<\nu)$, we have

$$-|\tau_{h,n}| \leq \mu_h \leq \tilde{\mu}_{h,n} \tag{24}$$

and if we denote by $\tilde{\mu}_{j,n}^{-}$ the j-th negative eigenvalue of $\tilde{P}_n K \tilde{P}_n$ that coincides with $\tilde{\mu}_{h,n}$, and by τ_j^- the bound $-|\tau_{h,n}|$, we have $\mu_h=\mu_j^-$ and (24) may be written in the form

$$\tau_j^- \leq \mu_j^- \leq \tilde{\mu}_{j,n}^{-}.$$

In such a way we may separate each of the first ν μ_h.

If some of the $|\mu_h|$ are very near, we may not succeed to separate these eigenvalues from the others. This is also the case if some of the $|\mu_h|$ have multiplicity greater than one (4). Let us suppose that $\rho\,(1\leq\rho\leq\nu-1)$ eigenvalues $|\mu_k|, |\mu_{k+1}|, \dots, |\mu_{k+\rho-1}|$ $(k+\rho-1<\nu)$ have been separated from the others. Then we have (5)

$$|\tau_{k-1}| < |\mu_k| \text{ and } |\tau_{k+\rho-1}| < |\mu_{k+\rho}|. \tag{25}$$

In what follows we assume that the index h belongs to the interval $[k, k+\rho-1]$.

The number of positive (negative) eigenvalues in the interval $[|\mu_k|, |\tau_{k+\rho-1}|]$ is given by the number of positive (negative) eigenvalues $\tilde{\mu}_{h,n}$. Then if $\tilde{\mu}_{i,n}^{+}$ $(\tilde{\mu}_{j,n}^{-})$ is the greatest (the least) of the positive (negative) $\tilde{\mu}_{h,n}$, the positive (negative) eigenvalues μ_h belong to the interval

$$[\tilde{\mu}_{i,n}^{+}, |\tau_{k+\rho-1}|] \qquad ([-|\tau_{k+\rho-1}|, \tilde{\mu}_{j,n}^{-}]).$$

(4) This may happen even if K has only simple eigenvalues, if $|\mu_i^+| = |\mu_j^-|$. Geometric and algebraic multiplicities coincide for the hermitian operators that we consider.

(5) The first of inequalities (25) must be omitted if $k=1$.

Let us now consider the special situation in which we know that in the interval $[|\mu_k|, |\tau_{k+\varrho-1}|]$ there is only one eigenvalue $|\mu_h|$ of multiplycity ρ. Let us denote by $|\tilde{\mu}_{l,n}|$ the greatest of the $|\tilde{\mu}_{h,n}|$ and by $|\tau_{l,n}|$ the least of the $|\tau_{h,n}|$. Then if the $\tilde{\mu}_{h,n}$ are all positive (negative), also the μ_h are all positive (negative) and we have

$$|\tilde{\mu}_{l,n}| \leq \mu_h \leq |\tau_{l,n}| \tag{26}$$

$$(-|\tau_{l,n}| \leq \mu_h \leq -|\tilde{\mu}_{l,n}|). \tag{27}$$

If there are α positive and β negative eigenvalues $\tilde{\mu}_{h,n}$ $(\alpha+\beta=\rho)$, K has a positive eigenvalue of multiplicity α and a negative eigenvalue of multiplicity β. For these eigenvalues the inequalities (26) and (27) hold respectively. We see that the knowledge of the multiplicity of $|\mu_h|$ allows to choose the best bounds.

VI. Particular cases.

a) General remarks.

Let us consider the following linear differential matrix operator of order m

$$L(x, D)\, v = \sum_{0 \leq |s| \leq m} a_s(x)\, D^s v.$$

The $a_s(x)$ are $l \times l$ matrices of class $C^\infty(X^r)$ with complex entries. We suppose that L is elliptic, i. e. $\det \sum_{|s|=m} a_s(x)\, \xi^s \neq 0$ $\forall x \in X^r$, $\forall \xi \in X^r - \{0\}$, and that L is formally self adjoint, i. e. that $L(x, D) = L^*(x, D)$, where $L^*(x, D) = \sum_{0 \leq |s| \leq m} (-1)^{|s|} D^s \overline{a_s}(x)$ is the formal adjoint of $L(x, D)$. Moreover we shall suppose that there exists a fundamental solution operator T for L [1] defined for any $u \in L_2(X^r)$ having a bounded support, such that $LTu = u$. Let A be a properly regular domain (see [3]) of X^r. The elements of the Sobolev spaces H_m (6)

(6) If A is an open set of X^r, let us denote by $C^m(A)$ the space of the complex vector valued functions that have continuous derivatives up to the order m in A, and by $\overset{\circ}{C}{}^m(A)$ the subset of $C^m(A)$ formed by the functions

we shall consider are vector functions with l complex components. Let Q and N be two $l \times l$ matrices whose entries are functions of class $C^{\infty}(X^r)$ and suppose that Q and N satisfy the hypotheses IV and V of section III. In particular such hypotheses are satisfied if L is arbitrary but the two matrices Q and N have constant entries and N commutes with L (e. g. this is the case if N and L are both diagonal operators).

We shall consider two boundary value problems for the operator L^2 to show how they fit in the abstract scheme.

b) FIRST EXAMPLE.

As space S we shall assume the complex $L_2(A)$ space of l-vectors with complex components. Since the operator L has been defined, we shall assume as $\mathcal{D}_L$ the subvariety of $L_2(A)$ formed by all the functions which belong to any space $H_m(B)$ where B is such that $\overline{B} \subset A$. We have $\overline{\mathcal{D}}_L = S$. This assertion is an obvious consequence of the cicumstance that any function of $L_2(A)$ is the limit in the $L_2(A)$ norm of a sequence of polynomials. Let us now consider the fundamental solution operator T, we have assumed to exist

$$Tw = \int_A g(x, y)\, w(y)\, dx. \tag{28}$$

Because of the hypotheses on the coefficients $a_s(x)$ of the operator L, the $l \times l$ matrix $g(x, y)$, which is defined for $(x, y) \in (X^r \times X^r) - \delta$, where δ is the diagonal $x = y$ of the cartesian product $(X^r \times X^r)$, enjoys the following properties (see [4])

1) $g(x, y)$ is C^{∞} in $(X^r \times X^r) - \delta$.
2) $g(x, y) = \overline{g(y, x)}$.
3) $D_x^p g(x, y) = O(|x-y|^{m-r-|p|} \log |x-y|)$ (for any multiindex p).

with bounded support contained in A. The Sobolev spaces $H_m(A)$ and $\mathring{H}_m(A)$ are respectively obtained as functional completion of $C^m(A)$ and of $\mathring{C}^m(A)$ with respect to the scalar product

$$(u, v)_m = \sum_{0 \leq |s| \leq m} \int_A D^s u \cdot D^s v \, dx.$$

Form $m = 0$ we have $H_0(A) = \mathring{H}_0(A) = L_2(A)$.

T is a fundamental solution operator for the differential operator L in the sense that for any $w \in L_2(X^r)$ with a bounded support, the function

$$v(x) = \int_{X^r} g(x, y)\, w(y)\, dy$$

is a solution of the differential equation $Lv = w$ belonging to $H_m(E)$ for any bounded domain E.

Hyp. I of section II is satisfied. In fact, while condition i) follows from 2) and 3), the conditions ii) and iii) follow from the fact that T is a fundamental solution operator.

Since the subspace $\Omega_0 = \ker L$ is constitued by all the functions ω of $\mathcal{D}_L$ which are solutions in A of the differential equation $L\omega = 0$, we know from the theory of elliptic partial differential equations that Ω_0 is closed in the $L_2(A)$ norm. We assume $\Omega = \Omega_0$.

The space $\mathcal{H}_L$ in this particular case is defined by the conditions

$$v \in L_2(A) \cap H_m(B), \quad Lv \in L_2(A) \cap H_m(B)$$

for every B such that $\overline{B} \subset A$.

Because of Lemma V, Hyp. III is automatically satisfied.

THEOREM XI. *Under the above assumptions for S, L, T and Ω we have that problem (P) of section II is equivalent to the following boundary value problem* (7)

$$L^2 v = f \qquad f \in L_2(A) \qquad \text{in } A \tag{29}$$

$$D^p v = 0 \qquad 0 \leq |p| \leq m-1 \qquad \text{on } \partial A \tag{30}$$

$$v \in H_m(A) \cap H_{2m}(B) \qquad (\forall B\colon \overline{B} \subset A). \tag{31}$$

PROOF. In fact the solution of problem (P) is given by the formula

$$v(x) = T\Pi\, Tf.$$

Since the function (28) is continuous across ∂A with its derivatives up to the order $m-1$ (in the sense of the functions of $H_m(A)$), to have

(7) See [1] section 9.

$D^h v=0$ $0\leq |h| \leq m-1$ on ∂A it is sufficient to impose $v(x)=0$ in the exterior of A. If M is any domain at a positive distance from A and φ a continuous function such that $\operatorname{supp}\varphi\subset M$, the condition $v=0$ for $x\notin \bar{A}$ is equivalent to the following one

$$\int_M \varphi\cdot v\, dx=0 \qquad \forall\varphi, \qquad \forall M. \tag{32}$$

We take $v=Tw$. By (28) then we have

$$\begin{aligned}\int_M \varphi\cdot v\, dx &= \int_M \varphi\int_A g(x,y)\, w(y)\, dy\, dx= \\ &= \int_A w(y)\int_M g(x,y)\,\varphi(x)\, dx\, dy=\int_A w(y)\,\omega(y)\, dy\end{aligned} \tag{33}$$

where $\omega(y)=\int_M g(x,y)\,\varphi(x)\,dx$ is a function of $\Omega_0=\ker L$. Hence

the function

$$v(x)=\int_A g(x,y)\, w(y)\, dy$$

with $w(y)\in\Omega_0^{\perp}$ is a solution of the problem (29) - (30) - (31). The fact that a solution of problem (29) - (30) - (31) is a solution of problem (P), is a consequence of the uniqueness of the solution of the problem (29) - (30) - (31). This uniqueness can be proved in the following way (8).

For $v(x)\in H_m(A)\cap H_{2m}(B)$ (for every B such that $\bar{B}\subset A$) and for every $w\in\overset{\circ}{C}{}^{\infty}(A)$, the Green formula holds

$$\int_A Lv\cdot Lw\, dx=\int_A L^2 v\cdot w\, dx.$$

Hence if $L^2 v=0$ and $v\in\overset{\circ}{H}_m(A)$.

(8) See [1] section 13.

$$\int_A (L\, v)^2\, dx = 0$$

for every $\varphi \in L_2(A)$ we have that $z = T\varphi$ belongs to $H_m(A)$. Then since $v \in \overset{\circ}{H}_m(A)$ and $L = L^*$

$$\int_A v \cdot \varphi\, dx = \int_A v \cdot Lz\, dx = \int_A L\, v \cdot z\, dx = 0.$$

Hence, for the arbitrariness of φ, $v = 0$.

c) SECOND EXAMPLE.

As space S we assume the L_2 space of l-vectors with complex components. The operator L is defined as before. We suppose that the boundary ∂A of A can be decomposed in two open ipersurfaces Σ_1 and Σ_2 which have in common their border $\partial\Sigma_1 = \partial\Sigma_2$ and no other point. We shall consider Σ_i $(i = 1, 2)$ as open sets with respect to ∂A. Let us suppose that there exists a domain A' with a smooth C^∞ boundary and such that $A' \supset A$, $\partial A \cap \partial A' = \bar{\Sigma}_1$. We shall assume that $m = 2\nu$ and that in the domain A' the hermitian bilinear form $B(u, v)$ associated with the operator L is such that for every $v \in \overset{\circ}{H}_\nu(A')$

$$(-1)^\nu B(v, v) \geq c \, \| v \|_\nu^2$$

where c is a positive constant (coerciveness hypothesis). From this hypothesis we derive that the Dirichlet problem for the operator L in the domain A' has a unique solution. From the theory of strongly elliptic partial differential equations, if we denote by Gf the solution of the Dirichlet problem with homogeneous boundary conditions for the equation $Lu = f$, we know that G enjoys the following properties:

1) G is a bounded operator from $L_2(A)$ into $H_2(A)$.

2) When considered as an operator from $L_2(A)$ into $L_2(A)$, G is hermitian and compact.

3) If $f \in C^\infty(\bar{A}')$ then $Gf \in C^\infty(\bar{A}')$.

4) The operator G admits the following representation

$$Gf = \int_{A'} g(x, y)\, f(y)\, dy$$

where $g(x, y)$ enjoys the same properties (1) - (2) - (3) of the kernel of T in the last example, if X^r is replaced by A'. In the present case we shall assume as operator T of the abstract theory the operator G. The space $\mathcal{D}_L$ will be defined as follows. Let us denote by E any domain (open connected set), which enjoys the following properties:

1) E is contained in A.

2) $\partial E \cap \partial A \subset \Sigma_1$.

Then we pose, for every B such that $\overline{B} \subset A$ and every E which enjoys the properties 1) and 2),

$$\mathcal{D}_L = \{ v \in L_2(A) \cap H_m(B) \cap H_\nu(E) \mid D^p v = 0 \text{ on } \Sigma_1,\ 0 \leq |p| \leq \nu - 1 \}$$

i. e. $\mathcal{D}_L$ is the subspace of the space $L_2(A) \cap H_m(B) \cap H_\nu(E)$ formed by the functions which satisfy homogeneous Dirichlet boundary conditions on Σ_1. As it is easily seen $\overline{\mathcal{D}}_L = S$. Hypothesis I is satisfied because of the hypotheses on G. Let us remark that in any set E enjoying the above properties any function $\omega \in \Omega_0 = \ker L$ satisfies the following inequality (9)

$$\|\omega\|^2_{\nu, E} \leq c_0 \|\omega\|^2_{0, A} \qquad c_0 > 0.$$

Let φ be a C^∞ function on X^r, such that

$$\operatorname{supp} \varphi \cap \partial E \subset \Sigma_1$$

and belonging to $\overset{\circ}{H}_\nu(A)$. Let $\{\omega_n\}$ be a sequence of functions of Ω_0 converging in the $L_2(A)$ norm. We have

$$\int_A \omega_n L\varphi \, dx = B(\omega_n, \varphi) = 0.$$

If ω is the limit function of $\{\omega_n\}$, we have $B(\omega, \varphi) = 0$. From that the closedness of Ω_0 follows. Hypothesis III is satisfied because of Lemma V. Then the following theorem holds.

(9) This inequality can be proved by methods that are very similar to those of reference [5]. By $\|\omega\|^2_{0,A}$ we denote the norm of ω in the space $L_2(A)$.

THEOREM XII. *Under the above assumptions for S, L and T, we have that problem (P) of section II is equivalent to the following boundary value problem*:

$$(34)\qquad L^2 v = f \qquad f \in L_2(A) \qquad \text{in } A$$

$$D^h v = 0 \qquad 0 \leq |h| \leq m-1 \qquad \text{on } \Sigma_2$$

$$(35)\qquad D^q v = 0 \qquad 0 \leq |q| \leq \nu - 1 \qquad \text{on } \Sigma_1$$

$$D^q L v = 0 \qquad 0 \leq |q| \leq \nu - 1 \qquad \text{on } \Sigma_1$$

$$v \in H_m(A) \cap H_{2m}(B) \cap H_{m+\nu}(E) \qquad (\forall B\colon \overline{B} \subset A,\ \forall E \text{ satisfying } 1), 2)).$$

PROOF. The proof of this theorem parallels that of Theorem IX. Only on checking the boundary conditions on Σ_2 we must take the domain M in $A' - A$.

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SULLA TORSIONE NON UNIFORME NELLE TRAVI A SEZIONE VARIABILE

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Sommario - Si trae l'equazione differenziale lineare del quarto ordine autoaggiunta che regola il fenomeno della torsione non uniforme in alcuni tipi di travi a sezione sottile aperta variabile, e si mostra in quale caso l'equazione coincide con quella ottenuta sfruttando l'analogia con la trave inflessa, valida nel caso della sezione costante.

Abstract - The forth order self-adjoint linear differential equation which governs the non-uniform torsion of thin walled beams with variable open cross section is derived. It is shown that in a special case this equation coincides with that obtained by the analogy with the bending theory, which is valid for beams of constant cross-section.

1) Ci si limita a considerare travi a sezione retta sottile aperta, costituita da uno o più tratti (e cioè senza o con punti di diramazione della linea media).

La trave sia generata, seguendo la definizione classica, da una figura piana (sezione retta) che si muove, variando con continuità, in modo che il suo baricentro G appartenga ad una retta z (asse della trave), ed il suo piano si conservi ortogonale a tale retta. Una trave così definita è ad asse rettilineo.

Si suppone inoltre che i centri di taglio C appartengano ad una retta parallela al l'asse z (asse di taglio). Tale ipotesi, non strettamente necessaria in presenza di sole coppie torcenti applicate, lo diviene invece se l'effetto torcente deriva da forze; essa, e solo essa, permette di sostituire l'insieme di forze applicate con due altri insiemi, all'uno dei quali corrisponde sola flessione ($\varphi_z = 0$), ed all'altro (costituito da sole coppie torcenti applicate) sola torsione ($\varphi_x = \varphi_y = 0$).

Il vettore della generica coppia applicata sia parallelo all'asse z; sulla trave agiscano quindi solo coppie, di tipo torcente.

Ponendo $\varphi_z = \vartheta$, ed indicando con apice la derivata rispetto a z, l'ingobbimento w è ancora espresso [1] [2] dalla relazione valida per travi a sezione costante

$$w = -2\vartheta' f \tag{1}$$

dove f è l'area settoriale riferita ad un polo O coincidente con il centro C di taglio. Tale relazione infatti si trae dall'esame di un concio di lunghezza elementare dz e di sezione costante, e può accettarsi, limitatamente però all'ambito delle sezioni debolmente variabili, in cui cioè le variazioni di dimensioni della linea media tra due sezioni a distanza Δz siano trascurabili rispetto a Δz.

Essa è invece pienamente accettabile se la linea media è costante, e cioè se la variazione di sezione lungo z è da imputarsi alla sola variazione degli spessori δ

La scelta O = C (e ciò impone che i vincoli siano tali da consentire la rotazione intorno ad O) implica, come può dimostrarsi attraverso il teorema di Betti,

$$\begin{aligned} \int_m w\, x\, \delta\, ds &= 0 \\ \int_m w\, y\, \delta\, ds &= 0 \end{aligned} \tag{2}$$

variando il polo O le w variano di una quantità lineare in x ed y [3], e quindi le (2) non sono più rispettate.

Inoltre, in ogni sezione l'anomalia iniziale delle aree settoriali è tale da rispettare la condizione

$$\int_m w \delta \, ds = 0 \tag{3}$$

Dalla (1) si trae

$$\begin{aligned} \varepsilon &= - 2(\vartheta' f)' \\ \sigma &= - 2E(\vartheta' f)' . \end{aligned} \tag{4}$$

Lungo lo spessore generico la τ è costituita da una parte τ_1 variabile alla De Saint Venant, e cioè con legge lineare (diagramma delle τ_1 bitriangolare, con punto di nullo sulla linea media), e da una τ_2 costante. Poichè si suppone che valga ancora la relazione

$$(\text{rot } \tau)_z = 2G\vartheta'$$

(essa è ricavata in generale nella sola ipotesi di invariabilità di forma) l'inclinazione delle τ_1 sullo spessore è la stessa per tutti gli spessori di una data sezione. Quindi le τ_1 danno luogo ad un momento M_1 legato a ϑ' dalla relazione del De Saint Venant

$$M_1 = C_1 \vartheta' \quad , \tag{5}$$

dove C_1 è la rigidità torsionale.

La residua parte M_2 ($M_1 + M_2 = M_t$ è la caratteristica momento torcente) è data dalle τ_2, che sorgono per equilibrio, data la variabilità lungo z delle σ dettate dalla (4).

La soluzione è così equilibrata, ma non congruente, attesi i γ_2 connessi con τ_2, che si trascurano. Infatti i γ_2 alterano le w, (tratte nell'ipotesi di γ nulle sulla linea media) e quindi generano altre σ, ed altre τ. Di ciò [4] può tenersi conto, senza grosse complicazioni formali.

La condizione di equilibrio dell'elemento dz ds (fig.1) alla traslazione lungo z fornisce

$$\frac{\partial}{\partial s}(\tau_2 \delta) + (\sigma \delta)' = 0 \tag{6}$$

da cui, per la (4),

$$\frac{\partial}{\partial s}(\tau_2 \delta) = 2E\left[(\vartheta' f)' \delta\right]'$$

Dato un segmento AP della linea media lungo il quale non esistano punti di diramazione, è quindi

$$(\tau_2 \delta)_P = (\tau_2 \delta)_A + 2E\int_A^P \left[(\vartheta' f)' \delta\right]' ds \tag{7}$$

2) Se la sezione è costituita da un solo tratto AB (se cioè la linea media non presenta punti di diramazione), dando ad s il verso A e B, poichè $\tau_A = 0$ si ha

$$(\tau_2 \delta)_P = 2E\int_A^P \left[(\vartheta' f)' \delta\right] ds \tag{8}$$

Il momento M_2 è dato (fig.2) da

$$M_2 = \int_A^B \tau_2 \delta\, h\, ds \tag{9}$$

Poichè è pure

$$2\, df = 2\frac{\partial f}{\partial s} ds = h\, ds \tag{10}$$

dalle (9) e (10) si trae

$$M_2 = 2\int_A^B \tau_2 \delta \frac{\partial f}{\partial s} ds$$

e, per la (8),

$$
\begin{aligned}
M_2 &= 4E \int_A^B \left(\int_A^{P(s)} \left[(\vartheta' f)' \delta \right]' dt \right) \frac{\partial f}{\partial s}\, ds = \\
&= 4E \int_A^B \left(\int_A^{P(s)} (\vartheta' f)' \delta \, dt \right)' \frac{\partial f}{\partial s}\, ds = \\
&= 4E \int_A^B g' \frac{\partial f}{\partial s}\, ds
\end{aligned}
\tag{11}
$$

avendo posto

$$
g(s, z) = \int_A^{P(s)} (\vartheta' f)' \delta \, dt . \tag{12}
$$

Si può scrivere

$$
\int_A^B g' \frac{\partial f}{\partial s}\, ds = \left(\int_A^B g \frac{\partial f}{\partial s}\, ds \right)' - \int_A^B g \left(\frac{\partial f}{\partial s} \right)' ds . \tag{13}
$$

Si ha poi

$$
\begin{aligned}
\int_A^B g \frac{\partial f}{\partial s}\, ds &= \int_A^B \left(\int_A^{P(s)} (\vartheta' f)' \delta \, dt \right) \frac{\partial f}{\partial s}\, ds = \\
&= \left[f \int_A^{P(s)} (\vartheta' f)' \delta \, dt \right]_A^B - \int_A^B f (\vartheta' f)' \delta \, ds = \\
&= f_B \vartheta'' \int_A^B f \delta \, ds + f_B \vartheta' \int_A^B f' \delta \, ds - \vartheta'' \int_A^B f^2 \delta \, ds - \\
&- \vartheta' \int_A^B f f' \delta \, ds .
\end{aligned}
\tag{14}
$$

Così pure è

$$\int_A^B q \left(\frac{\partial f}{\partial s}\right)' ds = f'_B \vartheta'' \int_A^B f \delta \, ds + f'_B \vartheta' \int_A^B f' \delta \, ds - \vartheta'' \int_A^B f f' \delta \, ds - \vartheta' \int_A^B f'^2 \delta \, ds \tag{15}$$

Ricordando la (3), e ponendo

$$\begin{aligned} B_1 &= 4E \int_m f' \delta \, ds \\ C_2 &= 4E \int_m f^2 \delta \, ds \\ B_2 &= 4E \int_m f'^2 \delta \, ds \\ B_3 &= 4E \int_m f f' \delta \, ds \end{aligned} \tag{16}$$

(C_2 è la nota espressione del <u>bimomento</u>) dalle (11) e (13), per le (14) e (15), si ottiene

$$\begin{aligned} M_2 &= (f_B B_1 \vartheta' - C_2 \vartheta'' - B_3 \vartheta')' - \\ &- f'_B B_1 \vartheta' + B_3 \vartheta'' + B_2 \vartheta' = \\ &= f_B (B_1 \vartheta')' + B_2 \vartheta' - B'_3 \vartheta' - (C_2 \vartheta'')' . \end{aligned} \tag{17}$$

La (3) porge, poichè si verifica in ogni sezione,

$$\left(\int_m f \delta \, ds \right)' = 0 . \tag{18}$$

Da questa si trae

$$\int_m f' \delta \, ds + \int_m f \delta' \, ds = 0 . \tag{19}$$

Nella maggioranza dei casi di interesse tecnico, ambedue gli integrali della (19) sono nulli; se infatti la variazione della sezione è dovuta alla sola variazione degli spessori δ, mentre la linea media è inalterata, è $f' = 0$; se la variazione è dovuta alla sola variazione della linea media, mentre gli spessori sono inalterati, è $\delta' = 0$; se invece variano sia la linea media che gli spessori, e per questi ultimi vale la legge di proporzionalità

$$d\delta = \delta' dz = k(z) \delta \, dz$$

risulta

$$\int_m f \delta' \, ds = k(z) \int_m f \delta \, ds = 0$$

In tutti questi casi è $B_1 = 0$ in stretto rigore; ciò può assumersi però in generale (par.4) se la sezione è debolmente variabile.

Ponendo $B_1 = 0$, per la (5) e la (17) può scriversi

$$M_t = M_1 + M_2 = -(C_2 \vartheta'')' + (C_1 + B_2 - B'_3)\vartheta' . \qquad (20)$$

Poichè è $M'_t = -m_t$, dalla (20) si trae

$$(C_2 \vartheta'')'' - (C_1^* \vartheta')' = m_t \qquad (21)$$

con la posizione

$$C_1^* = C_1 + B_2 - B'_3 . \qquad (22)$$

L'equazione di una trave inflessa di sezione variabile, di rigidità EI, soggetta ad un carico trasversale q e ad un carico assiale cui si associa uno sforzo normale N (z), è

$$(EI\, v'')'' - (N\, v')' = q . \qquad (23)$$

Dal confronto tra le (21) e (23) emerge l'analogia tra i due fenomeni, e la possibilità di studiare la trave soggetta a coppie torcenti con gli stessi metodi elaborati per la trave inflessa.

Si osservi che se la linea media è invariabile ($f' = 0$, $\delta' \neq 0$) sono nulli, oltre che B_1, anche B_2 e B_3; è allora $C_1^* = C_1$, e la (21) si scrive

$$(C_2 \vartheta'')'' - (C_1 \vartheta')' = m_t . \qquad (24)$$

E' questa l'equazione fornita dal Kollbrünner [5] per le travi a sezione variabile; si riconosce che essa, come la (23), è del tipo autoaggiunto.

Se non esistono coppie distribuite applicate, le (23) e (24) diventano omogenee; esse regolano la torsione non uniforme per la sola variazione di sezione.

3) La trattazione del paragrafo 2 è stata eseguita nell'ipotesi che la sezione sia costituita da un solo tratto. Se la sezione presenta punti di diramazione, i risultati non variano. Basta infatti (fig. 3) scomporre la sezione in parti costituite ciascuna da un solò tratto; nei punti P_i corrispondenti allo stesso punto P è $\sum_i \delta(P_i) = \delta(P)$. La condizione sotto cui deve essere fatta tale scomposizione è che per ogni parte valga ancora la (3), in relazione allo stesso centro C originario. Poichè le σ sono sempre fornite dalla (4), in cui lo spessore non gioca, i prodotti $(\tau\delta)_{P_i}$ sono dati per ciascun tratto ancora dalla (8) e, poichè in essi δ compare alla prima potenza, può porsi

$$(\tau_2 \delta)_P = \sum_i (\tau_2 \delta)_{P_i} . \tag{25}$$

Il momento M_2 è somma delle espressioni (9) calcolate per le singole parti, e le quantità B_2 B_3 e C_2 della (21) vanno costruite come somma delle quantità (16) ottenute dallo studio delle parti stesse.

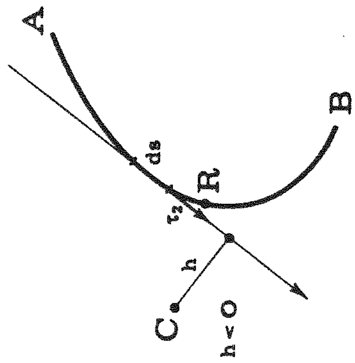
4) L'ordine di approssimazione in cui $B_1 = 0$ può essere valutato come segue. Considerando (fig.4) un tronco di lunghezza elementare dz, si ha, per la formula di Gauss,

$$\int_m f' \delta\, ds = \frac{1}{dz} \int_m f' \delta\, ds\, dz = \frac{1}{dz} \int_V f'\, dV =$$

$$= \frac{1}{dz} \int_S f \cos \widehat{nz}\, dS = \frac{1}{dz} \left[\int_{A_1} f \delta\, ds + \int_{A_2} f \delta\, ds + \int_{A_L} f \cos \widehat{nz}\, dS \right] =$$

$$= \frac{1}{dz} \int_{A_L} f \cos \widehat{nz}\, dS,$$

dove A_1 ed A_2 sono le due basi, A_L la superficie laterale. Se la sezione varia in modo che in corrispondenza dello spessore generico le normali n_1 ed n_2 ai due estremi possono autorizzare la posizione $\cos \widehat{n_1 z} = -\cos \widehat{n_2 z}$, è $B_1 = 0$. In rigore tale condizione si verifica se lo spessore non varia.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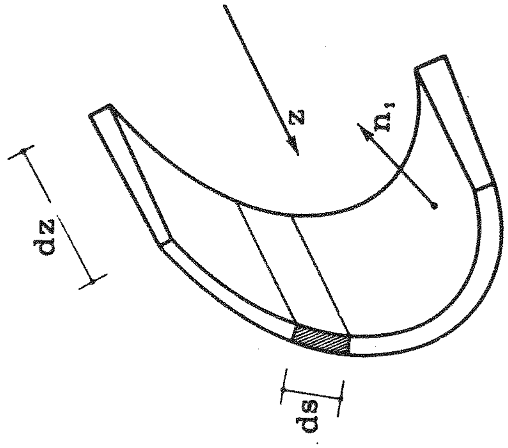
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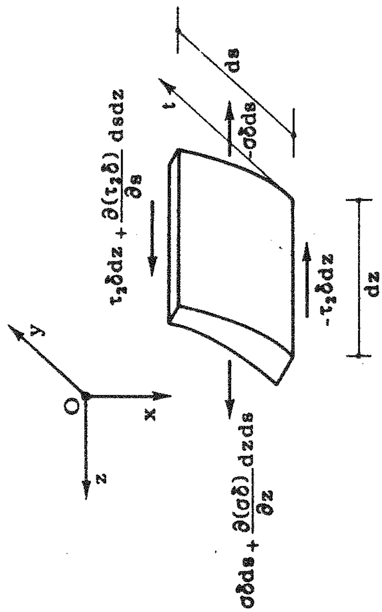
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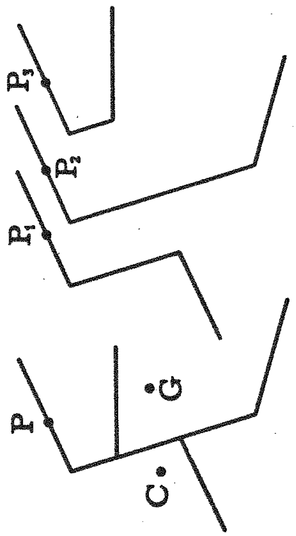
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UNIVERSITÀ DI NAPOLI
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VINCENZO FRANCIOSI - MANFREDI ROMANO

SEISMIC LOADS ON SPATIAL FRAMES

Pubblicazione N. 271

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Second national congress on theoretical and applied mechanics
Варна, 8 - 14.X. 1973, Varna ж Книга /vol./ 2, 1976

SEISMIC LOADS ON SPATIAL FRAMES

V. Franciosi, M. Romano /Italia/

Introduction. This work is founded on the assumption that the effects of an earthquake on spatial frames can be considered from the safety point of view, equivalent to the effect of a set of horizontal parallel forces, proportional to the seismic masses[1] and lying in the floor planes - Such assumption in suggested by the present Italian Standard.

Since the direction of the earthquake is unknown and variable, and the calculus must be performed in the most dangerous conditions, the set of seismic forces is assumed as rotating, with constant intensity, around the centroids of seismic masses in each floor.

The elastic solution for rotating seismic loads can be obtained by the theory of torsional ellipse of elasticity [1]. This effective procedure is illustrated in [2]. It is shown how to calculate the elliptical work domains that the seismic bending moments M_s describe in the M_a, M_b plane /fig. 1/ / a and b are the principal directions of inertia of the columns cross section/.

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[1] According to the present Italian Standard, the seismic masses are assumed proportional to the masses and to a fraction of the dead loads.

The study of the plastic behaviour of the structura requires shakedown analysis for the seismic cyclic load program. A procedure will be shown, that allows to obtain upper and lower bounds for incremental collapse and alternating plasticity multipliers.

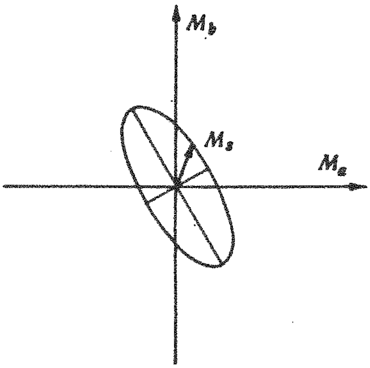

Fig. 1

The analysis is based on the assumption of elastic-perfectly plastic constitutive properties of the structure, when expressed in terms of generalized forces and deformation parameters.

Moreover, as seems reasonable, it is assumed that plastic deformations, may occur only in the terminal cross sections of the columns, between each couple of successive floors.

In full plastic bending there is a one to one correspondence between the position of the neutral axis and the limit bending moment vector, in each cross-section.

This correspondence is established when the plastic domain and the neutral line of the cross-section are known. Let us recall that the neutral line is the envelope of all the possible positions of the neutral axis under a given axial load N. At least for cross-sections with two axes of symmetry and for small values of the axial load this line almost degenerates in the centroid. Such degenerate situation can be assumed to hold true with good approximation in such cases, and this assumption strongly simplifies the analysis.

By the associate flow rule, when the limit bending moment is given, the direction of the neutral axis is known because coincides with the direction of the plastic rotation vector φ, i.e. is parallel to the normal at the plastic domain in the limit point. The position of the neutral axis is then determinate choosing between the two tangents at the neutral line, parallel to the φ direction, that which is in agreement with the sign of the plastic rotation /fig. 2/. The inverse relation is obtained reversing the above arguments.

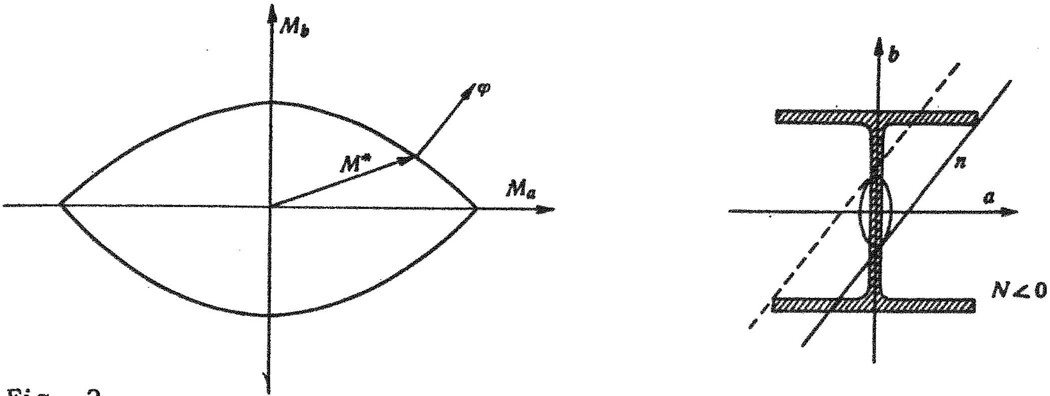

Fig. 2

The following considerations are relative to each set of n columns between two successive floors and to the set of the $2n$ lower and upper terminal cross-sections of these columns. Let us call M_{gi} $(i=1,\ldots,2n)$ the bending moments due to non-seismic, constant loads in these sections.

A l t e r n a t i n g p l a s t i c i t y. The crisis for alternating plasticity in full plastic bending may occur only in the sections in which plastic rotations of opposite signs around two parallel neutral axes take place. Indeed in such situation there is no cumulating of plastic rotations of the same sign, but low-cycle plastic fatigue without appreciable changes in the geometry of the structure[2].

Then the seismic ellipse of the cross-section must touch the plastic domain in two points in which the common tangents are parallel. To fit such requirement each ellipse can be translated and amplified by a factor γ_i $(i=1,\ldots,2n)$. The translation represents the residual moment M_{ri} in the cross-section. Let γ_k be the smallest of the γ_i, i.e. the k-th cross-section be the most dangerous for alternating plasticity, then $\gamma_A=\gamma_k$ is an upper bound for this kind of collapse, because it is a kinematically sufficient factor.

If there exist $2n-1$ residual moments that together with M_{rk} are in equilibrium under zero external loads, i.e. that

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[2]Permanent axial deformations are neglected.

form a compatible set, and moreover such that all the seismic ellipses, when translated of the corresponding amounts, don't go outside the limit plastic domains, the upper bound γ_A is also a lower bound ψ_A because it is a statically admissible factor. Then it is the safety factor for alternating plasticity.

Usually it is difficult to verify if such conditions are satisfied or not.

I n c r e m e n t a l c o l l a p s e. The collapse mechanism is assumed to be a rigid rotation between two successive floors, due to plastic rotations that take place one after the other at both the terminal cross-sections of the columns connecting these floors, under rotating seismic loads.

For each position of the center of rotation C in a floor plane, is possible to determine an upper bound to the incremental collapse multiplier. Indeed the neutral axes /i.e. the axes of the plastic rotations/ of all the terminal cross-sections in full plastic bending are known; they must contain C and be tangent to the neutral lines, on the side corresponding to the sign of the plastic rotations, that is, to the sign of the rigid rotation between the two floors.

By the correspondence previously mentioned, also the limit bending moments M_i^* in the terminal cross-sections, are known. It is sufficient to choose one of the two points, on each plastic domain, in which the normal is parallel to the corresponding neutral axis, depending on the sign of the plastic rotation vector, that must be directed outward.

Because the ellipses must translate and be amplified by a common factor γ it must be

$$/1/ \quad M_i^* = M_{gi} + M_{ri} + \gamma M_{si}, \qquad i = 1, \dots, 2n.$$

The residual moments M_{ri} must be in equilibrium under zero external forces; by the principle of virtual work, it must be

$$/2/ \quad \sum M_{ri}\varphi_i = 0,$$

where the φ_i are the plastic rotation vectors.
From /1/ and /2/ we have

$$/3/ \quad \gamma_I = \frac{\sum (M_i^* - M_{gi})\cdot\varphi_i}{\sum M_{si}\cdot\varphi_i}$$

γ_I is an upper bound to the incremental collapse multiplier because it is a kinematically sufficient factor.

Substituting γ_I in /1/ we can evaluate the residual moments M_{ri} by the relations

$$M_{ri} = M_i^* - M_{gi} - \gamma_I M_{si} \; .$$

If the seismic ellipses, translated of the quantities M_{ri} and amplified by the factor γ_I in some sections, go out of the plastic domain, let ρ be the smallest reduction factor that makes then be contained in the respective domains. The number $\psi_I = \gamma_I / \rho$ is a lower bound to the incremental collapse multiplier because it is a statically admissible factor.

A n e x a m p l e. Let us consider the three floor framed structure of fig. 3. All the bvertical columns are realized with

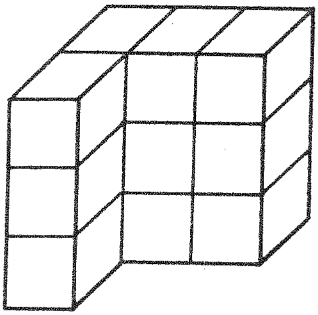

Fig. 3

I beams HE260 whose geometrical and inertia characteristics are given in fig. 4. The floors are considered as infinitely rigid

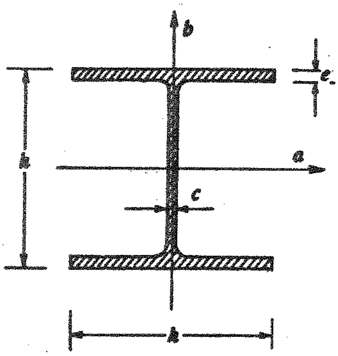

Fig. 4. h = 260 cm; e = 17.5 cm; c = 10 cm;
I_a = 14.919 cm^4; I_b = 5.135 cm^4

with regard to the effects of the horizontal seismic forces whose intensity, calculated in accordance with the Italian Standard, is shown in fig. 5. The moments M_g due to the non-seismic con-

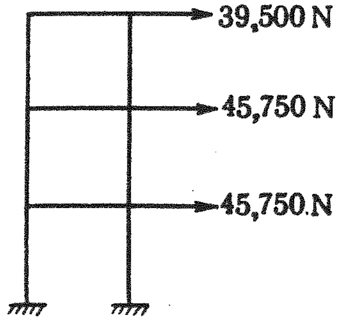

Fig. 5

stant load at the terminal cross-sections of the columns can be neglected, in a rough approximation, in comparison with the seismic moments M_s. Since corresponding columns have the same cross section at each floor, the most dangerous cross sections are those at the ends of the columns of the first floor. Fig. 6 shows the map of the columns and the position of the elastic center E and the seismic center S of that floor.

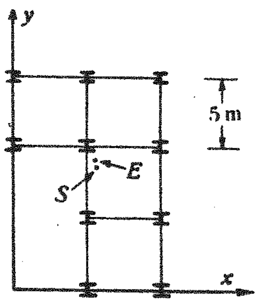

Fig. 6

The seismic elliptical domains have been computed by a procedure that makes use of the theory of the torsional ellipse of elasticity.

Then has been calculated the upper bound for alternating plasticity γ_A, by the technique previously exposed.

The value obtained has been $\gamma_A = 8.89$.

Upper and lower bounds for incremental collapse multiplier have been obtained through a systematic search for the optimum position of the center of rotation of the collapse mechanism. The best results have been: $\gamma_I = 9.03$, $\psi_I = 8.76$.

In this case it has been impossible to decide if the crisis is for incremental collapse or for alternating plasticity, but a very good estimation of the collapse multiplier /safety factor/: s has been obtained, namely $8.89 \geq s \geq 8.76$.

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EIGENVALUE PROBLEMS OF LINEAR ELASTICITY IN WHICH THE EIGENVALUES APPEAR IN THE BOUNDARY CONDITIONS

Pubblicazione N. 273

Eigenvalue Problems of Linear Elasticity in which the Eigenvalues appear in the Boundary Conditions

Manfredi ROMANO NAPLES ITALY

Let A be a properly regular domain of the space X^r, ∂A its boundary, S_i $(i=1,..s)$ and T_i $(i=1,..t)$ non overlapping parts of ∂A. We define the differential operator

$$L\,u = (-1)^m D^p a_{pq}(x) D^q u \qquad 0 \leq |p|, |q| \leq m$$

of order 2m, where the $\ell \times \ell$ matrices $a_{pq}(x)$ with complex entries belong to $C^\infty(X^r)$ and satisfy the identities $a_{pq}(x) = (-1)^{|p|+|q|}\overline{a_{qp}(x)}$ for any pair of multiindices p and q. Then the operator L is formally self-adjoint.

Consider the following eigenvalue problem

$$L\,u = \lambda\,u \tag{1}$$

$$M_{ik}\,u = 0 \qquad \text{on } S_i \quad (i=1,..s)\ (k=1,..\ \sigma_i) \tag{2}$$

$$M_{ik}\,u = \lambda\,N_{ik}\,u \qquad \text{on } T_i \quad (i=1,..t)\ (k=1,..\ \tau_i) \tag{3}$$

where M_{ik} and N_{ik} denote respectively differential operators of order less than 2m and less then m, with coefficients belonging to $C^\infty(\partial A)$, which are defined on the Sobolev Spaces $H_{2m}(A)$ and $H_m(A)$ in the sense of the traces.

By the Gauss - Green formula we have

$$B(u,w) + \langle \delta u,\ \gamma w \rangle = \lambda\,(u,w)_\circ \tag{4}$$

where

$$B(u,w) = (-1)^m \int_A \sum_{p,q} (-1)^{|p|} a_{pq}(x)\, D^q u\, D^p w\, dx \qquad \forall\, u,w \in H_m(A)$$

is the hermitian bilinear form associated to the operator L on A, δ and γ are differential operators of order 2m - 1 and m - 1 respectively, which are defined on $H_{2m}(A)$ and $H_m(A)$ in the sense of the traces, $\langle \cdot,\cdot \rangle$ denotes surface integration on δA and $(\cdot,\cdot)_\circ$ the scalar product in $H_o(A)$.

(°) This paper has been presented at the G.A.M.M. Conference held in Graz, April 5^{th} - 10^{th} 1976 and will be published on the Z.A.M.M.

Let us suppose that the boundary conditions (2) and (3) are such that the following identity holds

$$< \delta u, \gamma w > \;=\; b(u, w) - \lambda\, c(u, w) \tag{5}$$

where b and c are hermitian bilinear forms defined on $H_{m-1}(A) \times H_{m-1}(A)$ (i.e. in b and c do not appear derivatives of order greater then m - 1).

Let V be the subspace of $H_m(A)$ defined by the "stable" boundary conditions on S_i (i=1,..s), and such that $\overset{\circ}{H}_m(A) \subset V$ ($\overset{\circ}{H}_m(A)$ is the space of the functions $\omega \in H_m(A)$ with supp $\omega \subset A$).

The bilinear form c(u,w), to be hermitian, must have the form

$$c(u, w) = \sum_{i=1}^{t} < Q_i\, u, Q_i\, w >_i \tag{6}$$

where $<\cdot,\cdot>_i$ denotes surface integration on T_i and the differential operators Q_i are linear combinations of the N_{ik} (k=1,.. τ_i).

We define the two hermitian bilinear forms

$$\phi(u, w) = B(u, w) + b(u, w)$$

$$\psi(u, w) = (u, w)_o + c(u, w)$$

and suppose that $\phi(u,u)$ is coercive on V and $\psi(u,u)$ is positive on V. Then the variational eigenvalue problem

$$\phi(u, w) = \lambda\, \psi(u, w) \qquad \forall w \in V, \quad u \in V \tag{7}$$

is equivalent to problem (1)-(2)-(3). The problem (7) has a countable sequence of positive eigenvalues λ_k converging to zero (see [1] and [2]).

Upper bounds $\tilde{\lambda}_k$ to the first n eigenvalues λ_k can be computed by the Rayleigh - Ritz method, which can be fully proved for the problem (7) (see [1] and [2]), and are given by the roots of the following equation

$$\det \{ \phi(u_i, u_j) - \lambda\, \psi(u_i, u_j) \} = 0 \quad (i,j=1,..n) \tag{8}$$

where the u_i are n linearly independent vectors of V.

Let us now show how to compute lower bounds to the eigenvalues λ_k (k=1,..n). The Theory of Orthogonal Invariants [1] can be applied if we reduce problem (1)-(2)-(3) to an equivalent eigenvalue problem for an integral operator.

If we pose:

$$u(x) = v(x) + \sum_{j=1}^{t} h_j(x), \qquad \lambda\, u(x) = q(x), \qquad \lambda\, N_{jk}\, u(x) = b_{jk}(x) \tag{9}$$

we see that problem (1)-(2)-(3) can be split in the sum of the following t+1:

$$(P_\circ) \qquad L\, v = q \qquad \begin{cases} M_{ik}\, v = 0 & \text{on } S_i \\ M_{ik}\, v = 0 & \text{on } T_i \end{cases}$$

$$(P_j) \qquad L\, h_j = 0 \qquad \begin{cases} M_{ik}\, h_j = 0 & \text{on } S_i \\ M_{ik}\, h_j = 0 & \text{on } T_i \quad \text{for } i \neq j \\ M_{jk}\, h_j = b_{jk} & \text{on } T_j \end{cases}$$

$(j=1,\dots t)$

Problem $(P_\circ)$ has the following equivalent variational formulation

$$(P^x_\circ) \qquad \Phi(v\,,\,w) = (q\,,\,w)_\circ \qquad \forall w \in V, \quad v \in V, \quad q \in H_\circ(A)$$

The coerciveness hypothesis on $\Phi(u,u)$ allows to prove (see [1]) the existence of a positive compact operator $G : H_\circ(A) \to V$ such that

$$v = G\, q \tag{10}$$

is the solution of problem $(P_\circ)$.

The problems (P_j) have the following variational formulation

$$(P^x_j) \qquad \Phi(h_j\,,\,w) = \langle a_j\,,\,Q_j w\rangle_j = \langle Q^x_j\, a_j\,,\,w\rangle_j \qquad \forall w \in V, \quad h_j \in V, \quad a_j \in H_\circ(T_j)$$

where

$$a_j = \lambda\, Q_j\, u \tag{11}$$

and Q^x_j is the adjoint operator of Q_j. Q^x_j acts on a_j in the sense of distributions. The solution of these problems is given by

$$h_j = G\, Q^x_j\, a_j \tag{12}$$

From (9),(11),(10) and (12) we get the system ($\mu = \lambda^{-1}$, $\lambda \neq 0$)

$$\begin{aligned} \mu q &= G q + \sum_{i=1}^{t} G Q_i^x a_i \\ \mu a_j &= Q_j G q + \sum_{i=1}^{t} Q_j G Q_i^x a_i \end{aligned} \tag{13}$$

which is equivalent to problem (1)-(2)-(3). The operator G has the following integral representation : $G f(x) = \int_A g(x,\xi) f(\xi) d\xi$.

On the set $H = A \cup (\cup T_i)$ we define the measure $d\eta$ to be the Lebesgue measure $d\xi$ on A and the surface measure $d\beta_i$ on T_i. On the set $H \times H = (A\times A) \cup (\cup A\times T_i) \cup (\cup T_i\times A) \cup (\cup T_i\times T_j)$ we define the matrix kernel

$$\gamma(x , \xi) = \begin{cases} \gamma_{oo}(x,\xi) = g(x,\xi) & \text{on } A \times A \\ \gamma_{oi}(x,\xi) = Q_i(\xi)\, g(x,\xi) & \text{on } A \times T_i \\ \gamma_{io}(x,\xi) = Q_i(x)\, g(x,\xi) & \text{on } T_i \times A \\ \gamma_{ij}(x,\xi) = Q_j(x)\, Q_i(\xi)\, g(x,\xi) & \text{on } T_i \times T_j \end{cases} \tag{14}$$

and on H the function

$$z(x) = \begin{cases} q & \text{on } A \\ a_i & \text{on } T_i \end{cases}$$

Then we may write (13) in the form

$$\Gamma z(x) = \int_H \gamma(x , \xi)\, z(\xi)\, d\eta = \mu\, z(x) \tag{15}$$

The eigenvalue problem (15) is equivalent to problem (1)-(2)-(3) in the sense that his eigenvalues μ_k are the inverses of the λ_k. Now we may use the Theory of Orthogonal Invariants [1] to compute upper bounds to the μ_k, i.e. lower bounds to the λ_k. The trace of the operator Γ is given by

$$\operatorname{tr} \Gamma = \int_H \gamma(x , x)\, d\eta = \int_A g(\xi , \xi)\, d\xi + \sum_{i=1}^{t} \int_{T_i} \left. Q_i(x) Q_i(\xi) g(x,\xi) \right|_{x=\xi} d\beta_i .$$

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MANFREDI ROMANO

AN OPTIMAL ERROR ESTIMATE FOR EIGENVECTORS IN BUCKLING AND VIBRATION OF ELASTIC STRUCTURES

Pubblicazione N. 275

AN OPTIMAL ERROR ESTIMATE FOR EIGENVECTORS IN BUCKLING AND VIBRATION OF ELASTIC STRUCTURES.

Manfredi Romano

I. Introduction

The knowledge of the buckling and vibration modes plays a central role in post-buckling and vibration analysis of elastic structures. Therefore the estimate of the error connected with the approximate computation of the eigenvectors is of great importance.

The eigenvalue problems in the linear theory of elastic structures are usually formulated in terms of differential operators or in variational form. In any case it can be shown that they are equivalent to eigenvalue problems for compact hermitian operators, which in general have both positive and negative eigenvalues. In what follows we shall consider this type of problems in an abstract setting.

A critical review of some previous works on the error estimates for eigenvectors can be found in [1] . In the same paper the authors obtain an improved form of the upper bound due to G. Fichera [2] and, for the first time, two lower bounds. In the present paper a generalization of the original technique developed in [2] is provided. This new approach yields an expression of the upper bound which depends on a free parameter. Minimizing with respect to this parameter an optimal error bound is obtained.

An example has been worked out to compare the optimal bound with previous results, in a simple case in which the exact error is computable. Numerical evidence shows a drastic improvement. A relevant feature of the optimal bound comes out if we investigate the dependence on the eigenvalue estimates. All previous bounds are very critical in this respect : a small inaccuracy of the eigenvalue estimates results in a large increase of their value. The corresponding optimal bounds are subject only to small changes. This certainly is a basic advantage in numerical applications.

II. The optimal estimate

Let us consider an hermitian compact operator T on the complex Hilbert space H. We denote by $\{\mu_h^+\}$ ($\{\mu_j^-\}$) the non-increasing (non-decreasing) sequence of positive (negative) eigenvalues of T, each repeated according to its multiplicity, by $\{u_h^+\}$ ($\{u_j^-\}$) the sequence of the associated orhonormal eigenvectors. Moreover we denote by $(\cdot,\cdot)$ the inner product, by $\|\cdot\|$ the norm and by S_H the unit sphere in H. Let us define the projectors $P_h^+ = u_h^+ \otimes u_h^+$ and $P_j^- = u_j^- \otimes u_j^-$ ($\otimes$ = tensor product on $H \times H$) and the projector P_0 on ker T. The operator T has the following "spectral decomposition"

$$T = \sum_h \mu_h^+ P_h^+ + \sum_j \mu_j^- P_j^- \tag{1}$$

and every $f \in H$ has the representation

$$f = \sum_h P_h^+ f + \sum_j P_j^- f + P_0 f \tag{2}$$

Given $v \in S_H$ we define, following [2] , the error relative to the approximation of a subspace $U \subset H$ by means of v, as the distance of v from U. Hence, if we denote by P the projector on U and by $Q = I - P$ its orthogonal complement, the error is given by $\|Q v\|$.

Let $\tilde{\mu}_h^+$ ($\mu_j'^-$) and $\mu_h'^+$ ($\tilde{\mu}_j^-$) denote upper and lower bounds to μ_h^+ (μ_j^-).

We consider the case in which p positive eigenvalues(1) $\mu_k^+,\dots,\mu_{k+p-1}^+$ form a cluster separated from the others, i.e. we suppose that

$$\tilde{\mu}_{k+p}^+ \leqslant \mu_{k+p-1}'^+ \qquad\qquad \tilde{\mu}_k^+ \leqslant \mu_{k-1}'^+ \tag{3}$$

If $\mu_k^+ = \dots = \mu_{k+p-1}^+$ we obtain in particular the case of an eigenvalue of multiplicity p.

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(1) Would we consider here p negative eigenvalues $\mu_k^-,\dots,\mu_{k+p-1}^-$, then all the arguments that follow could be repeated identically, substituting only the sign - for +, interchanging the indexes j and h and the inqualities $<$ and $>$

Let us define $U = sp(u^+_k, \ldots, u^+_{k+p-1})$. For $v \in S_H$ we consider the equation

$$T\,v - \mu\,v = f \qquad \mu \in R \qquad f \in H \tag{4}$$

that is compatible iff $f \perp \ker(\,T - \mu\,I)$.

From (2) we have

$$v = \sum_h P^+_h\,v + \sum_j P^-_j\,v + P_\circ\,v \tag{5}$$

so that

$$Q\,v = v - P\,v = {\sum_h}' P^+_h\,v + \sum_j P^-_j\,v + P_\circ\,v \tag{6}$$

where ${\sum_h}'$ is the sum extended to the indexes $h \neq k, \ldots, k+p-1$.

Since, by (1),

$$P_\circ T\,v = T\,P_\circ v = 0, \quad P^+_h\,T\,v = T\,P^+_h\,v = \mu^+_h\,P^+_h\,v, \quad P^-_j\,T\,v = T\,P^-_j\,v = \mu^-_j\,P^-_j\,v$$

if $\mu \neq 0$, $\mu \neq \mu^-_j$ for every j, $\mu \neq \mu^+_h$ for $h \neq k, .., k+p-1$, from (4) follows

$$P_\circ\,v = -\frac{P_\circ\,f}{\mu} \qquad P^+_h\,v = \frac{P^+_h\,f}{\mu^+_h - \mu} \qquad P^-_j\,v = \frac{P^-_j\,f}{\mu^-_j - \mu} \tag{7}$$

for every j and for $h \neq k, \ldots, k+p-1$.

Substituting (7) in (6) we get

$$Q\,v = {\sum_h}' \frac{P^+_h\,f}{\mu^+_h - \mu} + \sum_j \frac{P^-_j\,f}{\mu^-_j - \mu} - \frac{P_\circ\,f}{\mu} \tag{8}$$

hence

$$\|Q\,v\|^2 = {\sum_h}' \frac{\|P^+_h\,f\|^2}{(\mu^+_h - \mu)^2} + \sum_j \frac{\|P^-_j\,f\|^2}{(\mu^-_j - \mu)^2} + \frac{\|P_\circ\,f\|^2}{\mu^2} \tag{9}$$

Since $\sum_h{}' \|P_h^+ f\|^2 + \sum_j \|P_j^- f\|^2 + \|P_\circ f\|^2 = \|Q f\|^2 \leqslant \|f\|^2$ and, from (4), $\|f\|^2 = \|T v\|^2 - 2\mu (Tv , v) + \mu^2$, if we define

$$\alpha = \| T v \| \qquad \beta = (Tv , v) \tag{10}$$

from (9) we get

$$\| Q v \|^2 \leqslant \frac{\beta^2 - 2\mu\alpha + \mu^2}{\min \{(\mu_h^+ - \mu)^2\}} \quad h \neq k,..,k+p-1 \tag{11}$$

We notice that by the Cauchy inequality $\beta \leqslant |\alpha|$.

Since the bound (11) holds for every $\mu \neq 0$ and $\mu \neq \mu_h^+$ (for $h \neq k,\dots,k+p-1$), we seek the value of μ that minimizes the right hand side, i.e. the product of the two functions

$$\beta^2 - 2\mu\alpha + \mu^2 \tag{12}$$

and

$$\left[\min \{(\mu_h^+ - \mu)^2\}\right]^{-1} \qquad h \neq k,..,k+p-1 \tag{13}$$

The function (12) has the absolute minimum $\beta^2 - \alpha^2$ for $\mu = \alpha$.

The function (13) has local minima in the middle points of the intervals between two consecutive distinct eigenvalues μ_h^+ for $h \neq k,\dots,k+p-1$.

Let us observe that $\beta \to \alpha \iff \alpha \to \mu_h^+ \iff v \to u_h^+$. The values of (12) are small when $\mu \sim \alpha$ and $\beta \sim \alpha \sim \mu_h^+$, what implies $\mu \sim \mu_h^+$. If $h \neq k,\dots,k+p-1$ when $\mu \to \mu_h^+$ the function (13) tends to $+\infty$. So we have two incompatible requirements. Hence to get non-trivial bounds we must assume that

$$\mu_{k+p}^+ < \alpha < \mu_{k-1}^+ \tag{14}$$

because $\mu \to \mu_h^+$ for $h = k,\dots,k+p-1$ does not imply divergence of (13).

Therefore we consider the case in which

$$\mu^{+}_{k+p} < \mu < \mu^{+}_{k-1} \tag{15}$$

By this assumption the bound (11) can be written in the form

$$||Q\, v||^{2} \leqslant \frac{\beta^{2} - 2\,\alpha\,\mu + \mu^{2}}{\min\{(\mu^{+}_{k-1} - \mu)^{2}, (\mu - \mu^{+}_{k+p})^{2}\}} \tag{16}$$

To obtain a computable optimal bound we assume that

$$\tilde{\mu}^{+}_{k+p} < \mu, \alpha < \mu'^{+}_{k-1} \tag{17}$$

and seek the value of μ that minimizes the bound

$$||Q\, v||^{2} \leqslant \frac{\beta^{2} - 2\,\alpha\,\mu + \mu^{2}}{\min\{(\mu'^{+}_{k-1} - \mu)^{2}, (\mu - \tilde{\mu}^{+}_{k+p})^{2}\}} \tag{18}$$

which is readily obtained from (16). If we define

$$\mu_{M} = \tfrac{1}{2}\,(\mu'^{+}_{k-1} + \tilde{\mu}^{+}_{k+p}) \tag{19}$$

($\mu_M = +\infty$ if $k = 1$), the bound (18) can be written in the form

$$||\, Q\, v\, ||^{2} \leqslant g\,(\,\mu\,) \tag{20}$$

where

$$g\,(\,\mu\,) = \begin{cases} g_{1}\,(\,\mu\,) & \text{for } \tilde{\mu}^{+}_{k+p} < \mu \leq \mu_{M} \\ g_{2}\,(\,\mu\,) & \text{for } \mu_{M} \leq \mu < \mu'^{+}_{k-1} \end{cases} \tag{21}$$

and

$$g_{1}\,(\,\mu\,) = \frac{\beta^{2} - 2\,\alpha\,\mu + \mu^{2}}{(\mu - \tilde{\mu}^{+}_{k+p})^{2}}\;, \qquad g_{2}\,(\,\mu\,) = \frac{\beta^{2} - 2\,\alpha\,\mu + \mu^{2}}{(\mu'^{+}_{k-1} - \mu)^{2}} \tag{22}$$

For both $g_1(\mu)$ and $g_2(\mu)$, if we pose respectively $\tilde{\mu}^{+}_{k+p} = \dot{\mu}$ and $\mu'^{+}_{k-1} = \dot{\mu}$, we have $(i = 1,2)$

$$D\, g_i(\mu) = \frac{2}{(\mu - \dot{\mu})^3} \left[\mu(\alpha - \dot{\mu}) - \beta^2 + \alpha\dot{\mu} \right] \tag{23}$$

$$D^2 g_i(\mu) = \frac{2}{(\mu - \dot{\mu})^4} \left[3\beta^2 - 4\alpha\dot{\mu} + \dot{\mu}^2 - 2\mu(\alpha - \dot{\mu}) \right] \tag{24}$$

From (23) we see that $Dg_i(\mu) = 0$ implies $\mu = \mu_\circ$ with

$$\mu_\circ = \frac{\beta^2 - \alpha\dot{\mu}}{\alpha - \dot{\mu}} \tag{25}$$

and from (24)

$$D^2 g_i(\mu_\circ) = \frac{2}{(\mu_\circ - \dot{\mu})^4} \left[\beta^2 - \alpha + (\alpha - \dot{\mu})^2 \right] > 0 \tag{26}$$

Substituting in (25) respectively $\tilde{\mu}^{+}_{k+p}$ and μ'^{+}_{k-1} for $\dot{\mu}$ we define

$$\tilde{\mu}_\circ = \frac{\beta^2 - \alpha\tilde{\mu}^{+}_{k+p}}{\alpha - \tilde{\mu}^{+}_{k+p}} \tag{27}$$

$$\mu'_\circ = \frac{\beta^2 - \alpha\mu'^{+}_{k-1}}{\alpha - \mu'^{+}_{k-1}} \tag{28}$$

We observe that $\tilde{\mu}_\circ$ ($\mu'_\circ$) is the point of minimum for g_1 (g_2) iff $\tilde{\mu}^{+}_{k+p} < \tilde{\mu}_\circ \leq \mu_M$ ($\mu_M \leq \mu'_\circ < \mu'^{+}_{k-1}$). It is easy to show that

$$\tilde{\mu}^{+}_{k+p} < \alpha < \mu_M \quad \Rightarrow \quad \tilde{\mu}^{+}_{k+p} < \tilde{\mu}_\circ \quad \text{and} \quad \mu'_\circ < \mu_M \tag{29}$$

$$\mu_M < \alpha < \mu'^{+}_{k-1} \quad \Rightarrow \quad \mu'_\circ < \mu'_{k-1} \quad \text{and} \quad \mu_M < \tilde{\mu}_\circ \tag{30}$$

Since $\alpha = \mu_M \quad \Rightarrow \quad \mu'_\circ \leq \mu_M \leq \tilde{\mu}_\circ$ from (29) and (30) we get

$$\mu^{+}_{k+p} \le \alpha \le \mu_M \quad \Rightarrow \quad \min g(\mu) = \begin{cases} g_1(\tilde{\mu}_o) & \text{if } \tilde{\mu}_o < \mu_M \\ g(\mu_M) & \text{if } \tilde{\mu}_o \quad \mu_M \end{cases} \tag{31}$$

$$\mu_M \le \alpha < \mu'^{+}_{k-1} \quad \Rightarrow \quad \min g(\mu) = \begin{cases} g_2(\mu'_o) & \text{if } \mu'_o > \mu_M \\ g(\mu_M) & \text{if } \mu'_o \quad \mu_M \end{cases} \tag{32}$$

Hence the optimal bound can be written in the form

$$\| Q v \|^2 \leqslant \min g(\mu) \tag{33}$$

where the right hand side can be computed by means of (31),(32),(22), (19),(27) and (28), when we have at our disposal the numerical values of β, α, $\tilde{\mu}^{+}_{k+p}$ and μ'^{+}_{k-1}.

A simple bound can be obtained from (18) if we assume $\mu = \alpha$ so to minimize the numerator of the right hand side. We get

$$\| Q v \|^2 \leqslant \frac{\beta^2 - \alpha^2}{\min\{(\alpha - \tilde{\mu}^{+}_{k+p})^2, (\mu'^{+}_{k-1} - \alpha)^2\}} \tag{34}$$

The bound (34) in many cases is quite near the optimal.

Analogous results can be easily proved by the same arguments for eigenvalue problems, formulated in terms of differential operators or in variational form, that can be shown to be equivalent to an eigenvalue problem for an hermitian compact operator.

III. Numerical results

We consider the eigenvalue problem

$$D^4 u - \lambda u = 0 \qquad\qquad u \in H_{4[0,1]} \quad \lambda \in R \tag{35}$$

with the boundary conditions $D^2 u(o) = D^2 u(1) = u(o) = u(1) = 0$, which comes out in the study of the free vibrations of a simply supported beam of constant cross section. As is well known problem (35) is solved in closed form. Its eigenvalues and the associated eigenfunctions are given by

$$\lambda_k = (k \pi)^4 \qquad u_k = \sin k\pi x \qquad k \in N \qquad x \in [0,1]$$

Problem (35) is equivalent to the following eigenvalue problem for a compact operator

$$T v - \mu v = 0 \qquad v \in H_{\circ[0,1]} \qquad \mu \in R \tag{36}$$

where

$$T v = \int_0^1 g (x , \xi) v (\xi) d \xi \tag{37}$$

with the hermitian kernel

$$g (x , \xi) = \begin{cases} \frac{1}{6} (2x\xi + x\xi^3 + x^3 \xi - \xi^3 - 3x^2 \xi) & \xi \leqslant x \\ \frac{1}{6} (2\xi x + \xi x^3 + \xi^3 x - x^3 - 3\xi^2 x) & x \leqslant \xi \end{cases}$$

The problems (36) and (37) have the same eigenvectors, while the eigenvalues are in the relation $\mu_k = \lambda_k^{-1}$.

The lower bounds μ_k' are obtained by the Rayleigh - Ritz method. It is convenient to consider the variational form of problem (35)

$$a (u , w) - \mu b (u , w) = 0 \qquad u \in W \qquad w \in W \tag{38}$$

where $\mu = \lambda^{-1}$, $W = \{w \in H_{2\ 0,1} \mid w(o) = w(1) = 0\}$ and

$$a (u , w) = \int_0^1 u w \, dx \qquad b (u , w) = \int_0^1 D^2u \; D^2w \, dx$$

As test functions $\{w_i\}$ ($i = 1,\ldots,n$) we choose piecewise cubic Hermite polynomials that satisfy the stable boundary conditions $w_i(o) = w_i(1) = 0$.

From (38) we get the finite dimensional eigenvalue problem

$$a\,(\,w_i\,,\,w_j\,)\,-\,\mu\cdot b\,(\,w_i\,,\,w_j\,)\,=\,0 \qquad (\,i,j\,=\,1,.,n\,) \tag{39}$$

whose eigenvalues are the lower bounds μ_k'.

Upper bounds $\tilde{\mu}_k$ are obtained by the method of orthogonal invariants [3]. Since

$$\sum_i \mu_i \;=\; \int_0^1 g\,(\,x\,,\,x\,)\,dx = \frac{1}{90}$$

we have

$$\mu_k \;=\; \frac{1}{90} + \sum_{i\,=\,1}^{n} \mu_i' - \mu_k' \qquad (\,k\,=\,1,\ldots,n\,)$$

We list the values of μ_k', μ_k and $\tilde{\mu}_k$ for $k = 1,2,3$ in table I and in table II respectively for $n = 10$ and $n = 16$.

TABLE I (n = 10)

Lower bounds	Exact eigenvalues	Upper bounds
$1.02637851\cdot 10^{-2}$	$1.02659822\cdot 10^{-2}$	$1.02828328\cdot 10^{-2}$
$6.39503165\cdot 10^{-4}$	$6.41623891\cdot 10^{-4}$	$6.58550782\cdot 10^{-4}$
$1.24751219\cdot 10^{-4}$	$1.26740522\cdot 10^{-4}$	$1.43798836\cdot 10^{-4}$

TABLE II (n = 16)

Lower bounds	Exact eigenvalues	Upper bounds
$1.02656446\cdot 10^{-2}$	$1.02659822\cdot 10^{-2}$	$1.02716801\cdot 10^{-2}$
$6.41290809\cdot 10^{-4}$	$6.41623891\cdot 10^{-4}$	$6.47326261\cdot 10^{-4}$
$1.26415027\cdot 10^{-4}$	$1.26740522\cdot 10^{-4}$	$1.32450478\cdot 10^{-4}$

Let v_1 and v_2 denote the first two eigenvectors of (39), for $n = 10$. The vectors v_1 and v_2 are Rayleigh - Ritz approximations for the eigenvectors u_1 and u_2 of problem (35).

The knowledge of the eigenvectors u_1 and u_2 allows to compute the exact value of the errors

$$\| Q_1 v_1 \| \quad = \quad 1.432748634 \cdot 10^{-4}$$

$$\| Q_2 v_2 \| \quad = \quad 2.312149680 \cdot 10^{-3}$$

where $Q_1 = I - P_1$ with $P_1 = u_1 \otimes u_1$ and $Q_2 = I - P_2$ with $P_2 = u_2 \otimes u_2$.

Next from (37) we compute

$$\alpha_1 = (Tv_1, v_1) = 1.02659820 \cdot 10^{-2} \qquad \alpha_2 = (Tv_2, v_2) = 6.41620830 \cdot 10^{-4}$$

$$\beta_1 = \| Tv_1 \| \quad = 1.02659821 \cdot 10^{-2} \qquad \beta_2 = \| Tv_2 \| \quad = 6.41622542 \cdot 10^{-4}$$

We consider the following upper bounds:

1) The upper bound given by E. Trefftz [4] ,(see also [1])

$$\| Q_1 v_1 \|^2 \leqslant \frac{\tilde{\mu}_1 - \alpha_1}{\tilde{\mu}_1 - \tilde{\mu}_2} \tag{T}$$

2) An upper bound due to H.F. Weinberger [5] ,(see also [1])

$$\| Q_2 v_2 \|^2 \leqslant \frac{\tilde{\mu}_2 - \alpha_2}{\tilde{\mu}_2 - \tilde{\mu}_3} + \frac{\tilde{\mu}_1 - \tilde{\mu}_3}{\tilde{\mu}_1 - \tilde{\mu}_2} \, \frac{\tilde{\mu}_1 - \alpha_1}{\tilde{\mu}_2 - \tilde{\mu}_3} \tag{W}$$

3) The upper bound given by G. Fichera [2]

$$\| Q_k v_k \|^2 \leqslant \frac{\beta^2 - 2 \mu_k' \alpha + \tilde{\mu}^2}{\min\{(\mu_{k-1}' - \tilde{\mu}_k)^2, (\mu_k' - \tilde{\mu}_{k+1})^2\}} \tag{F}$$

4) The improved version of (F) given in [1]

$$\| Q_k v_k \|^2 \leqslant \frac{\beta^2 - \alpha^2 + (\tilde{\mu}_k - \mu_k')^2}{\min\{(\mu_{k-1}' - \tilde{\mu}_k)^2, (\mu_k' - \tilde{\mu}_{k+1})^2\}} \tag{RE}$$

5) The optimal bound (33)

$$\| Q_k v_k \|^2 \leqslant \min g(\mu) \tag{R1}$$

6) The upper bound (35)

$$\| Q_k v_k \|^2 \leqslant \frac{\beta^2 - \alpha^2}{\min\{(\alpha - \tilde{\mu}_{k+1})^2, (\mu'_{k-1} - \alpha)^2\}} \qquad \text{(R2)}$$

Making use of the bounds of table I , we obtain the following numerical values:

$\| Q_1 v_1 \| \leq 4.184325061 \cdot 10^{-2}$ (T) $\quad \| Q_2 v_2 \| \leq 2.595689857 \cdot 10^{-1}$ (W)

$\| Q_1 v_1 \| \leq 6.513080511 \cdot 10^{-2}$ (F) $\quad \| Q_2 v_2 \| \leq 3.172507511 \cdot 10^{-1}$ (F)

$\| Q_1 v_1 \| \leq 1.989111486 \cdot 10^{-3}$ (RE) $\quad \| Q_2 v_2 \| \leq 3.854150824 \cdot 10^{-2}$ (RE)

$\| Q_1 v_1 \| \leq 1.551895362 \cdot 10^{-4}$ (R1) $\quad \| Q_2 v_2 \| \leq 2.977178509 \cdot 10^{-3}$ (R1)

$\| Q_1 v_1 \| \leq 1.551895399 \cdot 10^{-4}$ (R2) $\quad \| Q_2 v_2 \| \leq 2.977189312 \cdot 10^{-3}$ (R2)

With the better bounds of table II , we get:

$\| Q_1 v_1 \| \leq 2.433204001 \cdot 10^{-2}$ (T) $\quad \| Q_2 v_2 \| \leq 1.507983112 \cdot 10^{-1}$ (W)

$\| Q_1 v_1 \| \leq 3.660426971 \cdot 10^{-2}$ (F) $\quad \| Q_2 v_2 \| \leq 1.733403399 \cdot 10^{-1}$ (F)

$\| Q_1 v_1 \| \leq 6.463589175 \cdot 10^{-4}$ (RE) $\quad \| Q_2 v_2 \| \leq 1.221358839 \cdot 10^{-2}$ (RE)

$\| Q_1 v_1 \| \leq 1.549735687 \cdot 10^{-4}$ (R1) $\quad \| Q_2 v_2 \| \leq 2.910823808 \cdot 10^{-3}$ (R1)

$\| Q_1 v_1 \| \leq 1.550084410 \cdot 10^{-4}$ (R2) $\quad \| Q_2 v_2 \| \leq 2.910833895 \cdot 10^{-3}$ (R2)

The best bounds are obtained if we use the exact eigenvalues. We have:

$\| Q_1 v_1 \| \leq 1.496365699 \cdot 10^{-4}$ (T) $\qquad \| Q_2 v_2 \| \leq 2.526822124 \cdot 10^{-3}$ (W)

$\| Q_1 v_1 \| \leq 1.549165960 \cdot 10^{-4}$ (F) $\qquad \| Q_2 v_2 \| \leq 2.878544528 \cdot 10^{-3}$ (F)

$\| Q_1 v_1 \| \leq 1.549165958 \cdot 10^{-4}$ (R1) $\qquad \| Q_2 v_2 \| \leq 2.878543683 \cdot 10^{-3}$ (R1)

$\| Q_1 v_1 \| \leq 1.549165995 \cdot 10^{-4}$ (R2) $\qquad \| Q_2 v_2 \| \leq 2.878553120 \cdot 10^{-3}$ (R2)

III. Conclusions

In most applications only upper and lower bounds for the eigenvalues can be computed. The numerical results given above show that the bounds (T) and (W) are the best when the exact eigenvalues are used, but their dependence on the eigenvalue estimates is so critical that these bounds are very poor even when the approximation of the eigenvalues is good enough. The bound (RE) shows a better behaviour.

The optimal bound (33) and the simpler bound (34) give almost identical results and reveal a very smooth dependence on the eigenvalue estimates. Their relevant feature is that the worse are the eigenvalue estimates, the better is the improvement gained.

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Error estimates for eigenvector approximation in the theory of elastic vibrations

by ANNA ESPOSITO - MANFREDI ROMANO (Napoli)

SOMMARIO - *Si considerano alcune formule di maggiorazione dell'errore nell'approssimazione degli autovettori in problemi vibratori. Si presenta una versione migliorata di un già noto limite superiore nonché due limiti inferiori. Vengono forniti esempi numerici.*

1. Introduction.

In most problems of the linear theory of elastic vibrations the natural frequencies and the associated modes can be computed only for approximate solutions. While many effective numerical techniques have been developed and are currently applied to these problems, the estimate of the error is disregarded in most technical works. Such estimate is obtained for the natural frequencies when upper and lower bounds to the eigenvalues have been computed, since the modulus of the difference of these bounds is an upper bound to the error.

In the last years research work of many authors has made available effective methods for the computation of such bounds. The problem of error estimate in the approximate computation of eigenvectors is strictly connected to the previous one.

After the early works of KRILOV and TREFFTZ this problem has been studied with several different approaches, but only upper bounds to the error have been considered. From the point of view of the applications the knowledge of such bounds is often sufficient. Anyway when upper bounds are available it is impossible to know if the error estimate is good enough or very poor. The knowledge of a lower bound to the error allows to evaluate the accuracy of the estimate. In fact if a maximum value of the error is prescribed and the upper bound is greater than this maximum we cannot know if the error is really too big or the estimate is poor, until a lower bound is available.

In this case, if also the lower bound is greater than the maximum error, we have to improve the approximation, if not, we can try to improve the estimate.

This paper intends to be a contribute to the problem of the computation of upper and lower bounds to the error in the approximation of eigenvectors. The problem is posed in an abstract form which allows to get the maximum of generality of the results and to avoid the formal complexity of the specific notations of each particular case that fits in the general abstract setting.

We refer mainly to results of TREFFTZ, WEINBERGER and FICHERA.

These authors consider eigenvalue problems for a positive compact operator. We briefly review their approches and present the upper bounds in a form which allows comparison of the results. The computation of these bounds requires the knowledge of upper and lower bounds for the associated eigenvalues. Next we obtain an improved version of the upper bound given in [2] and, for the first time to our knowledge, lower bounds for the error.

The numerical example concerns with the case of a simply supported beam, in which the exact solution is known, and the bounds can be compared with the exact value of the error.

2. The abstract eigenvalue problem.

Eigenvalue problems in theory of linear elastic vibrations can be posed in terms of integral or differential operators, or in variational form. We consider briefly these three cases in an abstract setting to introduce the notations and the definitions that will be referred to in the next sections.

1) Let H be a complex Hilbert space. We shall denote by $(\cdot, \cdot)$ the inner product, by $\|\cdot\|$ the norm and by S_H the unit sphere in H. Let us consider a positive compact operator T in H and the following eigenvalue problem.

PROBLEM (T):

$$Tu-\mu u=0, \quad u\in H, \quad \mu\in C.$$

The characteristic set of T is a countable subset of R_0^+ and, if infinite, has zero as unique limit point. The eigenspaces are mutually orthogonal and have finite dimension which coincides with the algebraic multiplicity of the corresponding eigenvalues.

We shall denote by $\{\mu_h\}$ the (finite or infinite) non increasing sequence of the non-zero eigenvalues of T, each reapeated according to its multiplicity and by $\{u_h\}$ the sequence of the associated orthonormal eigenvectors. We shall denote by $\tilde{\mu}_k$ and μ'_k the best upper and lower bounds available for μ_k. Let P_h be the projector

$$P_h\, u = (u, u_h)\, u_h, \quad u \in H \tag{1}$$

and Q_h its orthogonal complement $I - P_h$.

For the operator T the following « spectral decomposition » holds:

$$T = \sum_h \mu_h P_h. \tag{2}$$

Let us suppose that, if the eigenvalues $\mu_k, \dots, \mu_{k+p-1}$ form a cluster separated from the others, i. e. $\tilde{\mu}_{k+p} < \mu'_{k+p-1}$ and $\tilde{\mu}_k < \mu'_{k-1}$, $U = sp\,(u_h, \dots, u_{k+p-1})$. We denote by P the projector on the subspace spanned by the corresponding eigenvalues, i. e.

$$Pu = \sum_{h=k}^{k+p-1} (u, u_h)\, u_h, \qquad u \in H \tag{3}$$

and by Q the projector $I - P$. If $\mu_k = \dots = \mu_{k+p-1}$ we obtain in particular the case of an eigenvalue of multiplicity p. The projector P_h, Q_h, P and Q commute with T. For every vector $v \in S_H$ the error in approximating U is defined (1) as the norm $\|Qv\|$ which is a measure of the distance of v from U.

2) Let L be a linear operator on H with domain D_L. Let V be a linear subvariety of D_L. We suppose that there exists a strictly positive compact operator T on H such that $T(H) \subset V$, $LTu = u$ $u \in H$ and $TLv = v$ $\forall v \in V$ (i. e. LT and TL are the identity operators on H and V respectively). Since $\ker T = \{0\}$ we have $v = \sum_h P_h v$ and the following

(1) In this definition we follow [2].

spectral decomposition for L

(4) $$Lv = \sum_h \lambda_k P_k v.$$

We consider the following eigenvalue problem

PROBLEM (L):

$$Lv - \lambda v = 0, \quad v \in V, \quad \lambda \in C - \{0\}.$$

Under the above hypotheses Problem (L) is equivalent to Problem (T) in the sense that the eigenvalues λ_i are the inverses of the μ_i.

We note that the eigenvalue problems $Lv - \lambda Mv = 0$ where M is a positive definite operator on H can be reduced to Problem (L).

We shall denote by $\widetilde{\lambda}_k$ and λ'_k the best lower and upper bounds available for λ_k.

3) Let K be a complex Hilbert space and $\|\cdot\|_K$ the norm in K. Let us denote respectively by $a(u, w)$ and $b(u, w)$ two positive definite bilinear forms on $H \times H$ and $K \times K$.

We suppose hat:

i) $K \subset H$

ii) The embedding J of K into H is compact.

iii) The quadratic form $b(\cdot)$ is coercive on a subspace W of K, i. e.

$$b(w) \geq c\,\|w\|_K, \quad \forall w \in W.$$

iv) W is dense in H (with respect to the norm of H).

We consider the eigenvalue problem

PROBLEM (E):

$$b(u, w) - \lambda a(u, w) = 0, \ \forall u \in W, \ \forall w \in W, \ \lambda \in C - \{0\}.$$

Under the above hypoteses Problem (E) is equivalent to Problem (T).

In fact it can be shown [2] that, if we take $b(u, w)$ as inner product in W, by the Riesz representation theorem, there exists a linear bounded

[2] See e. g. [3] where the proofs on pp. 85-86 are readily extended to our case.

operator G from H to W such that:

$$b(Gu, w) = a(u, w), \quad \forall u \in H, \quad \forall w \in H;$$

$$a(u, Gz) = a(Gu, z), \quad \forall u, z \in H.$$

Then

$$a(Gu, u) > 0, \quad \forall u \in H - \{0\}.$$

Problem (E) can be written in the form:

$$b(Gu, w) - \mu b(u, w) = 0, \quad \mu = \lambda^{-1} \in C, \quad \forall w \in W,$$

which is equivalent to

(5) $$Gu - \mu u = 0, \quad \mu \in C, \quad u \in H,$$

G, considered as an operator on H, is compact, because J is compact, but in general not hermitian.

We define the strictly positive operator A on H by:

$$(Au, w)_H = a(u, w)$$

and the strictly positive compact operator Γ on H by:

$$b(\Gamma u, w) = (u, w)_H, \quad \forall u \in H, \quad \forall w \in W.$$

Since $b(\Gamma Au, w)_H = a(u, w) = b(Gu, w)$ we see that

(6) $$G = \Gamma A.$$

Hence problem (5) can be written in the form

$$\Gamma Au - \mu u = 0, \quad \mu \in C, \quad u \in H,$$

which is equivalent to

(7) $$A^{1/2} \Gamma A^{1/2} z - \mu z = 0, \quad \mu \in C, \quad z \in H,$$

where $z = A^{1/2} u$. The operator $T = A^{1/2} \Gamma A^{1/2}$ is hermitian, strictly positive and compact on H.

For the eigenvalue Problem (E) we define the error with respect to the « energy norm » $|||v||| = b(v)$, $\forall v \in W$.

3. Review of some previous bounds.

In 1933 TREFFTZ published a paper [4] in which he considered the problem of eigenvectors approximation and the relevant error estimate. We deduce here his upper bound in a form which allows direct comparison with our results.

Suppose that $k=1$. From (3) we have:

$$\alpha=(Tv,v)=(TPv,Pv)+(TQv,Qv)\leq\mu_1\,||Pv||^2+(TQv,Qv). \tag{8}$$

By the maximum property of eigenvalues it is

$$\alpha\leq\mu_1,\quad (TQv,Qv)\leq\mu_{1+p}\,||Qv||^2. \tag{9}$$

Hence from (8) and (9)

$$||Pv||^2=\frac{\alpha}{\mu_1}-\frac{1}{\mu_1}(TQv,Qv)\geq\frac{\alpha}{\mu_1}-\frac{\mu_{1+p}}{\mu_1}\,||Qv||^2. \tag{10}$$

Next, since $||Pv||^2=1-||Qv||^2$, we get

$$||Qv||^2\leq\frac{\mu_1-\alpha}{\mu_1-\mu_{1+p}} \tag{11}$$

which gives a non trivial upper bound if $\mu_{1+p}<\alpha$. Since in general we have at our disposal only upper and lower bounds for μ_1 and μ_2, we may substitute in (11) $\tilde{\mu}_{1+p}$ for μ_{1+p} and, if $\tilde{\mu}_{1+p}<\alpha$, $\tilde{\mu}_1$ for μ_1, to get the following computable upper bound

$$||Qv||^2\leq\frac{\tilde{\mu}_1-\alpha}{\tilde{\mu}_1-\tilde{\mu}_{1+p}}\,. \tag{12}$$

The TREFFTZ bound can be obtained also with reference to Problem (L). In fact if for every $v\in V\cap S_H$ we define

$$\alpha=(Lv,v), \tag{13}$$

since

$$\lambda_1\leq\alpha,\quad (LQv,Qv)\geq\lambda_{1+p}\,||Qv||^2,$$

it follows (if $\alpha \leq \lambda_{1+p}$)

$$(14) \qquad ||Qv||^2 \leq \frac{\alpha - \lambda_1}{\lambda_{1+p} - \lambda_1} \leq \frac{\alpha - \tilde{\lambda}_1}{\tilde{\lambda}_{1+p} - \tilde{\lambda}_1}.$$

In the same way for Problem (E) we define for every $v \in S_W$

$$(15) \qquad \alpha = a(v, v),$$

and since $\alpha \leq \mu_1$ and $a(Qv, Qv) \leq \mu_{1+p} |||Qv|||^2$, it follows (if $\alpha > \mu_{1+p}$)

$$(16) \qquad |||Qv|||^2 \leq \frac{\mu_1 - \alpha}{\mu_1 - \mu_{1+p}} \leq \frac{\tilde{\mu}_1 - \alpha}{\tilde{\mu}_1 - \tilde{\mu}_{1+p}}.$$

These bounds have been extended by H. F. Weinberger [5] to include higher order eigenspaces. We give here a proof of his upper bound that is consistent with our presentation. We refer to Problem (T). For every $v \in S_H$ we have

$$(17) \qquad v = \sum_{h=1}^{k-1} P_h v + Pv + R_{k+p} v,$$

where $R_j = \sum_{h=j}^{\infty} P_h$. Hence

$$(18) \qquad \alpha = (Tv, v) \leq \sum_{h=1}^{k-1} \mu_h ||P_h v||^2 + \mu_k ||Pv||^2 + \mu_{k+p} ||R_{k+p} v||^2.$$

Since from (17) $||R_{k+p} v||^2 = 1 - ||Pv||^2 - \sum_{h=1}^{k-1} ||P_h v||^2$, we get

$$(19) \qquad \alpha - \mu_{k+p} \leq \sum_{h=1}^{k-1} (\mu_h - \mu_{k+p}) ||P_h v||^2 + (\mu_k - \mu_{k+p}) ||Pv||^2.$$

Substituting $1 - ||Qv||^2$ for $||Pv||^2$ in (19) we obtain:

$$(20) \qquad ||Qv||^2 \leq \frac{\mu_k - \alpha}{\mu_k - \mu_{k+p}} \left(1 + \frac{\sum_{h=1}^{k-1} (\mu_h - \mu_{k+p}) ||P_h v||^2}{\mu_k - \alpha} \right).$$

The inequality (20) holds for every $v \in S_H$. To obtain an upper bound we must replace the norms $||P_h v||^2$ at the right hand side.

To this end let us consider a set of $k-1$ orthonormal vectors $\{w_i\}$ $(i=1, \dots, k-1)$ such that $v \perp sp(w_1, \dots, w_{k-1})$.

We have

$$\sum_{i=1}^{k-1} \|P_h w_i\|^2 \leq 1 - \|P_h v\|^2, \qquad h \in N. \tag{21}$$

Let us define

$$\alpha_i = (Tw_i, w_i). \tag{22}$$

Since $w_i = \sum_{h=1}^{k-1} P_h w_i + R_k w_i$ we obtain:

$$\alpha_i \leq \sum_{h=1}^{k-1} \mu_h \|P_h w_i\|^2 + \mu_k \|R_k w_i\|^2, \tag{23}$$

and from (23)

$$\alpha_i - \mu_k \leq \sum_{h=1}^{k-1} (\mu_h - \mu_k) \|P_h w_i\|^2. \tag{24}$$

We sum this inequality for $i=1$ to $k-1$ and make use of (21) to get

$$\sum_{h=1}^{k-1} (\mu_h - \mu_k) \|P_h v\|^2 \leq \sum_{h=1}^{k-1} (\mu_h - \alpha_h). \tag{25}$$

From (25) in particular follows that

$$\sum_{h=1}^{k-1} \|P_h v\|^2 \leq \frac{\sum_{h=1}^{k-1} (\mu_h - \alpha_h)}{\mu_{k-1} - \mu_k} \tag{26}$$

From the inequalities (25) and (26) we deduce that

$$\sum_{h=1}^{k-1} (\mu_h - \mu_{k+p}) \|P_h v\|^2 \leq \tag{27}$$

$$\leq \sum_{h=1}^{k-1} (\mu_h - \alpha_h) + (\mu_k - \mu_{k+p}) \sum_{h=1}^{k-1} \|P_h v\|^2 \leq \sum_{h=1}^{k-1} (\mu_h - \alpha_h) \frac{\mu_{k-1} - \mu_{k+p}}{\mu_{k-1} - \mu_k}$$

Substituting (27) in (20) we get the bound

$$\|Qv\|^2 \leq \frac{\mu_k - \alpha}{\mu_k - \mu_{k+p}} + \frac{\mu_{k-1} - \mu_{k+p}}{\mu_{k-1} - \mu_k} \frac{\sum_{h-1}^{k-1} (\mu_h - \alpha_h)}{\mu_k - \mu_{k+p}}. \tag{28}$$

To make the right hand side of (28) small we have to choose $\alpha \sim \mu_k$ and $\alpha_h \sim \mu_h$ $(h=1, \dots, k-1)$. The orthonormal vectors $v, w_1, \dots, w_{k-1}$, that satisfy this requirements, can be obtained by the Rayleigh-Ritz method which gives lower bounds $\alpha_1, \dots, \alpha_{k-1}, \alpha$ for the eigenvalues $\mu_1, \dots, \mu_{k-1}, \mu_k$. From (28) we obtain a computable upper bound if $\alpha > \tilde{\mu}_{k+p}$.

In fact we could repeat the arguments that led to (28) substituting $\tilde{\mu}_h$ for μ_h $(h=1, \dots, k+p)$, to get

$$(29) \qquad \|Qv\|^2 \leq \frac{\tilde{\mu}_k - \alpha}{\tilde{\mu}_k - \tilde{\mu}_{k+p}} + \frac{\tilde{\mu}_{k-1} - \tilde{\mu}_{k+p}}{\tilde{\mu}_{k-1} - \tilde{\mu}_k} \frac{\sum_{h=1}^{k-1} (\tilde{\mu}_h - \alpha_h)}{\tilde{\mu}_k - \tilde{\mu}_{k+p}}$$

We must remark that the eigenvectors given by the Rayleigh-Ritz method may not satisfy the above specified orthogonality requirement.

This shortcoming is due to the numerical error that is often non negligible in eigenvectors computation. In this case the bound (29) cannot be assumed to hold. The same bound (29) can be proved for Problem (E) if we substitute $b(v, w)$ for (v, w) and $|||v|||$ for $\|v\|$.

For Problem (L) if we define $\alpha = (Lv, v)$ and $\alpha_i = (Lw_i, w_i)$ we get

$$(30) \qquad \|Qv\|^2 \leq \frac{\alpha_i - \tilde{\lambda}_k}{\tilde{\lambda}_{k+p} - \tilde{\lambda}_k} + \frac{\tilde{\lambda}_{k+p} - \tilde{\lambda}_{k-1}}{\tilde{\lambda}_k - \tilde{\lambda}_{k-1}} \frac{\sum_{h=1}^{k-1} (\alpha_h - \tilde{\lambda}_h)}{\tilde{\lambda}_{k+p} - \tilde{\lambda}_k}.$$

A different approach to the error estimate for eigenvectors approximation is due to FICHERA [2]. With reference to Problem (T) he considers the approximation of the eigenspaces associated to an eigenvalue μ_k of multiplicity p (i. e. $\mu_k = \mu_{k+1} = \dots = \mu_{k+p-1}$).

We shall reproduce here the proof of his upper bound which, with minor modifications, is adapted to the more general case of a cluster of p eigenvalues $\mu_k, \dots, \mu_{k+p-1}$.

For $v \in S_H$ and $\mu_j \in \{\mu_k, \dots, \mu_{k+p-1}\}$ we consider the following equation:

$$(31) \qquad Tv - \mu_j v = f_j,$$

that is compatible iff $f_j \perp \ker (T - \mu_j I)$.

We have

$$(32) \qquad v = \sum_h P_h v + P_0 v,$$

so that

$$Qv = v - Pv = \sum_h{}' P_h v + P_0 v, \tag{33}$$

where Σ' is the sum extended to the indexes $h \neq k, \dots, k+p-1$.

Since $P_0 Tv = TP_0 v = 0$ and $P_h Tv = TP_h v = \mu_h P_h v$ from (31) we deduce

$$P_0 v = -\frac{P_0 f_j}{\mu_j}, \quad P_h v = \frac{P_0 f_j}{\mu_h - \mu_j}. \tag{34}$$

Substituting (34) in (33) we get

$$Qv = \sum_h{}' \frac{P_h f_j}{\mu_h - \mu_j} - \frac{P_0 f_j}{\mu_j}. \tag{35}$$

Hence

$$\|Qv\|^2 = \sum_h{}' \frac{\|P_h f_j\|^2}{(\mu_h - \mu_j)^2} + \frac{\|P_0 f_j\|^2}{\mu_j^2}. \tag{36}$$

Since $\sum_h{}' \|P_h f_j\|^2 + \|P_0 f_j\|^2 = \|Qf_j\|^2 \leq \|f_j\|^2$ and from (31) $\|f_j\|^2 = \|Tv\|^2 - 2\mu_j (Tv, v) + \mu_j^2$, if we define

$$\beta = \|Tv\| \text{ and } \alpha = (Tv, v), \tag{37}$$

from (36) we get

$$\|Qv\|^2 \leq \frac{\beta^2 - 2\mu_j \alpha + \mu_j^2}{\min\{(\mu_{k-1} - \mu_j)^2, (\mu_j - \mu_{k+p})^2\}} \tag{38}$$

We observe that by the Cauchy inequality $\beta \geq \alpha$.

For any choice of $\mu_j \in \{\mu_k, \dots, \mu_{k+p-1}\}$, from (38) we get the computable upper bound in the form

$$\|Qv\|^2 \leq \frac{\beta^2 - 2\mu'_{k+p-1}\alpha + \tilde{\mu}_k^2}{\min\{(\mu'_{k-1} - \tilde{\mu}_k)^2, (\mu'_{k+p-1} - \tilde{\mu}_{k+p})^2\}}. \tag{39}$$

While (39) is valid for every $v \in S_H$, it can be non trivial only if $\tilde{\mu}_{k+p} < \alpha < \mu'_{k-1}$ [3]. The same bound can be proved for Problem (L).

[3] If $\mu_k = \mu_{k+1} =, \dots, = \mu_{k+p-1}$ so that $\mu'_{k+p-1} = \mu'_k$ from (39) we get the bound given by FICHERA.

If we define

$$\beta = \|Lv\|, \qquad \alpha = (Lv, v),$$

we get

$$\|Qv\|^2 \leq \frac{\beta^2 - 2\tilde{\lambda}_{k+p-1}\,\alpha + \lambda'^2_k}{\min\{(\tilde{\lambda}_k - \lambda'_{k-1})^2, (\tilde{\lambda}_{k+1} - \lambda'_{k+p-1})^2\}}. \tag{40}$$

For Problem (E) if we define

$$\beta = a\,(Gv, v), \qquad \alpha = a\,(v, v),$$

we get the same bound (39) with the norm $|||Qv|||$ at left hand side.

4. **Improvement of the upper bounds** (39), (40).

The upper bound (39) can be improved without any loss of generality. From (38) we get the following inequality

$$\|Qv\|^2 \leq \frac{\beta^2 - \alpha^2 + (\mu_j - \alpha)^2}{\min\{(\mu_{k-1} - \mu_j)^2, (\mu_j - \mu_{k+p})^2\}}. \tag{41}$$

Let us consider the following three cases:

i) $\alpha \leq \mu'_{k+p-1}$ then $(\mu_j - \alpha)^2 \leq (\tilde{\mu}_k - \alpha)^2$ for every μ_j and from (41) we get the upper bound

$$\|Qv\|^2 \leq \frac{\beta^2 - 2\tilde{\mu}_k\,\alpha + \tilde{\mu}_k^2}{\min\{(\mu'_{k-1} - \tilde{\mu}_k)^2, (\mu'_{k+p-1} - \tilde{\mu}_{k+p})^2\}}. \tag{42}$$

ii) $\tilde{\mu}_k \leq \alpha$ then $(\mu_j - \alpha)^2 \leq (\alpha - \mu'_{k+p-1})$ for every μ_j and from (41) we get

$$\|Qv\|^2 \leq \frac{\beta^2 - 2\mu'_{k+p-1}\,\alpha + \mu'^2_{k+p-1}}{\min\{(\mu'_{k-1} - \tilde{\mu}_k)^2, (\mu'_{k+p-1} - \tilde{\mu}_{k+p})^2\}} \tag{43}$$

iii) $\mu'_{k+p-1} \leq \alpha \leq \tilde{\mu}_k$ then $(\mu_j - \alpha)^2 \leq (\tilde{\mu}_k - \mu'_{k+p-1})^2$ and from (41) follows that

$$\|Qv\|^2 \leq \frac{\beta^2 - \alpha^2 + (\tilde{\mu}_k - \mu'_{k+p-1})^2}{\min\{(\mu'_{k-1} - \tilde{\mu}_k)^2, (\mu'_{k+p-1} - \tilde{\mu}_{k+p})^2\}}. \tag{44}$$

It is easy to see that each of the bounds (42), (43), and (44) is better than the bound (39). Numerical examples show that the improvement can be very pronounced.

Moreover the upper bounds (42) and (43) can be still improved in the following two cases:

i) for $\alpha \leq \mu'_{k+p-1}$, if $k=1$ or if $k>1$ and $\tilde{\mu}_{k+p-1}-\mu'_{k+p} \leq \mu'_{k-1}-\tilde{\mu}_k$ from (41) we deduce

$$(45) \qquad ||Qv||^2 \leq \frac{\beta^2-\alpha^2}{(\mu_{k+p-1}-\mu_{k+p})^2}+\frac{(\mu_{k+p-1}-\alpha)^2}{(\mu_{k+p-1}-\mu_{k+p})^2} .$$

Since $\tilde{\mu}_{k+p}<\alpha$ we get

$$(46) \qquad ||Q\ ||^2 \leq \frac{\beta^2-\alpha^2}{(\mu'_{k+p-1}-\tilde{\mu}_{k+p})^2}+\frac{(\tilde{\mu}_{k+p-1}-\alpha)}{(\tilde{\mu}_{k+p-1}-\tilde{\mu}_{k+p})^2} ,$$

that is a better upper bound than (42).

ii) For $\alpha \geq \tilde{\mu}_k$ if $\tilde{\mu}_{k-1}-\mu'_k<\mu'_{k+p-1}-\tilde{\mu}_{k+p}$ from (41) it follows that:

$$(47) \qquad ||Qv||^2 \leq \frac{\beta^2-\alpha^2}{(\mu_{k-1}-\mu_k)^2}+\frac{(\alpha-\mu_k)^2}{(\mu_{k-1}-\mu_k)^2} .$$

Since $\alpha \leq \mu'_{k-1}$ we have

$$(48) \qquad ||Qv||^2 \leq \frac{\beta^2-\alpha^2}{(\mu'_{k-1}-\tilde{\mu}_k)^2}+\frac{(\alpha-\mu'_k)^2}{(\mu'_{k-1}-\mu'_k)^2} ,$$

that is a better upper bound than (43).

5. Lower bounds.

We shall consider lower bounds to the error in approximating the eigenspace associated with an eigenvalue μ_k of multiplicity p, (i. e. $\mu_k=\ldots=\mu_{k+p-1}$).

For Problem (T), if $v \in S_H$, we have

$$(49) \qquad \alpha=(Tv,v)=\mu_k\,||Pv||^2+(TQv,Qv).$$

Since $||Pv||^2=1-||Qv||^2$ from (49) follows

$$\mu_k-\alpha=\mu_k\,||Qv||^2+(TQv, Qv),$$

so that

$$||Qv||^2\geq\frac{\mu_k-\alpha}{\mu_k}. \tag{50}$$

If $\alpha<\mu'_k$ from (50) we get the following computable non trivial lower bound

$$||Qv||^2\geq\frac{\mu'_k-\alpha}{\mu'_k}. \tag{51}$$

Similar results are readily obtained for Problem (L) and Problem (E). Lower bounds to the error can also be obtained on the basis of the approach due to FICHERA. In fact from (38) which, for $\mu_j=\mu_k$, is written in the form

$$||Qv||^2=\sum_h\frac{||P_h f||^2}{(\mu_h-\mu_k)^2}+\frac{||P_0 f||^2}{\mu_k^2},$$

since $Qf=f$, and $\max\{(\mu_h-\mu_k)^2, \mu_k^2\}=\max\{(\mu_1-\mu_k)^2, \mu_k^2\}$ we get

$$||Qv||^2\geq\frac{\beta^2-2\mu_k\,\alpha+\mu_k^2}{\max\{(\mu_1-\mu_k)^2, \mu_k^2\}}. \tag{52}$$

The following lower bounds are deduced by a procedure which is dual to the one used to get the upper bounds (42), (43), (44).
We consider three cases:

i) $\alpha\leq\mu'_k$.

$$||Qv||^2\geq\frac{\beta^2-2\mu'_k\,\alpha+\mu'^2_k}{\max\{(\tilde{\mu}_1-\mu'_k)^2, \tilde{\mu}_k^2\}}. \tag{53}$$

ii) $\tilde{\mu}_k\leq\alpha$.

$$||Qv||^2\geq\frac{\beta^2-2\tilde{\mu}_k\,\alpha+\tilde{\mu}_k^2}{\max\{(\tilde{\mu}_1-\mu'_k)^2, \tilde{\mu}_k^2\}}. \tag{54}$$

iii) $\mu'_k\leq\alpha\leq\tilde{\mu}_k$.

$$||Qv||^2\geq\frac{\beta^2-\alpha^2}{\max\{(\tilde{\mu}_1-\mu'_k)^2, \tilde{\mu}_k^2\}}. \tag{55}$$

The lower bounds (53) and (54) can be improved in the following two cases respectively

i) if $k=1$ or if $k>1$ and $\mu'_k>\tilde{\mu}_1-\mu'_k$ from (52) follows that

$$\|Qv\|^2 \geq \frac{\beta^2-\alpha^2}{\tilde{\mu}_k^2}+\frac{(\mu'_k-\alpha)^2}{\mu_k'^2}. \tag{56}$$

ii) If $k>1$ and $\mu'_1-\tilde{\mu}_k>\mu_k$ from (52) we get

$$\|Qv\|^2 \geq \frac{\beta^2-\alpha^2}{(\tilde{\mu}_1-\mu'_k)^2}+\frac{(\alpha-\tilde{\mu}_k)^2}{(\tilde{\mu}_1-\tilde{\mu}_k)^2}. \tag{57}$$

The lower bounds (53) - (57) can be proved in the same way for Problem (L) and Problem (E). We remark that the lower bound

$$\|Qv\|^2 \geq \frac{\beta^2-2\tilde{\mu}_k\,\alpha+\mu_k'^2}{\max\{(\tilde{\mu}_1-\mu'_k)^2, \tilde{\mu}_k^2\}}, \tag{58}$$

that could be obtained from (52) is worse than each of the bounds (53) - (55) and can be even negative.

6. Numerical results.

We consider the eigenvalue problem

$$D^4 u-\lambda u=0, \quad u\in H_4\,[0, 1], \quad \lambda\in R, \tag{59}$$

with the boundary conditions $u(0)=u(1)=Du(0)=Du(1)=0$, which comes out in the study of the free vibrations of a simply supported beam of constant cross section. Problem (59) can be solved in closed form. Its eigenvalues and the associated eigenvectors are given by

$$\lambda_k=(k\cdot\pi)^4, \quad u_k=\sin k\pi x, \quad k\in N, \quad x\in[0, 1]. \tag{60}$$

The variational form of problem (59) is

$$a(u, w)-\mu\, b(u, w)=0, \quad u\in W, \quad w\in W, \quad \mu\in R, \tag{61}$$

where $\mu=\lambda^{-1}$, $W=\{w\in H_2\,[0,1]\mid w(0)=w(1)=0\}$ and

(62) $$a(u,w)=\int_0^1 u\cdot w\,dx,\qquad b(u,w)=\int_0^1 D^2u\cdot D^2w\,dx.$$

The eigenvalue problem (59) and (60) are equivalent to the following one

(63) $$Tv-\mu v=0,\quad v\in H_0\,[0,1],\quad \mu\in R,$$

where T is the compact hermitian operator given by

(64) $$Tv=\int_0^1 g(x,\xi)\cdot v(\xi)\,d\xi$$

with

$$g(x,\xi)=\begin{cases}\frac{1}{6}(2x\xi+x\xi^3+x^3\xi-\xi^3-3x^2\xi) & \xi\leq x\\ \frac{1}{6}(2\xi x+\xi x^3+\xi^3x-x^3-3\xi^2x) & x\leq\xi.\end{cases}$$

The lower bounds μ'_k are obtained by the Rayleigh-Ritz method applied to the eigenvalue problem (61). As test functions we choose piecewise cubic Hermite polynomials $w_i\,(i=1,\dots,n)$ that satisfy the stable boundary conditions $w_i(0)=w_i(1)=0$.
From (60) we get the finite dimensional eigenvalue problem

(65) $$a(w_i,w_j)-\mu b(w_i,w_j)=0,\quad (i,j=1,\dots,n),$$

whose eigenvalues μ'_k are lower bounds for the corresponding μ_k.

The upper bounds $\tilde{\mu}_k$ are obtained by the method of orthogonal invariants. Since

$$\sum_i \mu_i=\int_0^1 g(x,x)\,dx=\frac{1}{90}$$

we have

$$\tilde{\mu}_k=\frac{1}{90}+\sum_{i=1}^{n}\mu'_i-\mu'_k\quad (k=1,\dots,n).$$

We list the values of μ'_k, μ_k and $\tilde{\mu}_k$ for $k=1,2,3$ in table I and in table II respectively for $n=6$ and for $n=10$.

TABLE I ($n = 6$).

Lower bounds	*Exact eigenvalues*	*Upper bounds*
$1.02493771 \cdot 10^{-2}$	$1.02659822 \cdot 10^{-2}$	$1.03375605 \cdot 10^{-2}$
$6.26720482 \cdot 10^{-4}$	$6.41623890 \cdot 10^{-4}$	$7.14903903 \cdot 10^{-4}$
$1.02880658 \cdot 10^{-4}$	$1.26740521 \cdot 10^{-4}$	$1.91064079 \cdot 10^{-4}$

TABLE II ($n = 10$).

Lower bounds	*Exact eigenvalues*	*Upper bounds*
$1.02637851 \cdot 10^{-2}$	$1.02659822 \cdot 10^{-2}$	$1.02828327 \cdot 10^{-2}$
$6.39503165 \cdot 10^{-4}$	$6.41623890 \cdot 10^{-4}$	$6.58550782 \cdot 10^{-4}$
$1.24751219 \cdot 10^{-4}$	$1.26740521 \cdot 10^{-4}$	$1.43798836 \cdot 10^{-4}$

Let v_1 and v_2 denote the first two eigenvectors of (65), for $n=6$. The knowledge of the eigenvectors u_1 and u_2 of problem (59) allows to compute the exact value of the errors:

$$\|Q_1 v_1\| = 1.10626717 \cdot 10^{-3} \qquad \|Q_2 v_2\| = 1.91324576 \cdot 10^{-2}$$

From (64) we compute

$$\alpha_1 = (Tv_1, v_1) = 1.02659697 \cdot 10^{-2} \qquad \alpha_2 = (Tv_2, v_2) = 6.41401958 \cdot 10^{-4}$$

$$\beta_1 = \|Tv_1\| = 1.02659760 \cdot 10^{-2} \qquad \beta_2 = \|Tv_2\| = 6.41507878 \cdot 10^{-4}$$

We shall denote by (T), (W), (F) and (RE) respectively the upper bounds (11) (TREFFTZ), (29) (WEINBERGER), (31) (FICHERA) and (44) (the improved bound of this paper), and by $(L1)$ and $(L2)$ the lower bounds (51) and (52). Making use of the values of μ'_k and $\widetilde{\mu}_k$ of Table I we obtain

(T) $\|Q_1 v_1\| \leq 8.62544183 \cdot 10^{-2}$ (W) $\|Q_2 v_2\| \leq 5.33309660 \cdot 10^{-1}$

(F) $\|Q_1 v_1\| \leq 1.42332145 \cdot 10^{-1}$ (F) $\|Q_2 v_2\| \leq 7.90694853 \cdot 10^{-1}$

(RE) $\|Q_1 v_1\| \leq 9.32506316 \cdot 10^{-3}$ (RE) $\|Q_2 v_2\| \leq 2.04175934 \cdot 10^{-1}$

(L1) $\|Q_1 v_1\| \geq 1.09696804 \cdot 10^{-3}$ (L1) $\|Q_2 v_2\| \geq 1.20040847 \cdot 10^{-3}$

(L2) $\|Q_1 v_1\| \geq 7.81053319 \cdot 10^{-4}$ (L2) $\|Q_2 v_2\| \geq 1.28495166 \cdot 10^{-2}$.

Let us now denote by v_1 and v_2 the first two eigenvectors of (65) for $n=10$. If we use the « energy norm » $|||v||| = b(v)$ we get

$$|||Q_1 v_1||| = 1.46299826 \cdot 10^{-2} \qquad |||Q_2 v_2||| = 5.75374511 \cdot 10^{-2}$$

and from (62)

$$\alpha_1 = a(v_1, v_1) = 1.02637852 \cdot 10^{-2} \qquad \alpha_2 = a(v_2, v_2) = 6.39503176 \cdot 10^{-4}$$

$$\beta_1 = a(Tv_1, v_1) = 1.02648835 \cdot 10^{-2} \qquad \beta_2 = a(Tv_2\, v_2) = 6.40561128 \cdot 10^{-4}$$

Making use of the values of Table II we obtain

(T) $|||Q_1 v_1||| \leq 4.44873087 \cdot 10^{-2}$ (W) $|||Q_2 v_2||| \leq 2.75655710 \cdot {}^{-1}$

(F) $|||Q_1 v_1||| \leq 6.69801625 \cdot 10^{-2}$ (F) $|||Q_2 v_2||| \leq 3.25779261 \cdot 10^{-1}$

(RE) $|||Q_1 v_1||| \leq 1.57584667 \cdot 10^{-2}$ (RE) $|||Q_2 v_2||| \leq 8.35929792 \cdot 10^{-2}$

(L1) $|||Q_1 v_1||| \geq 1.46030292 \cdot 10^{-2}$ (L1) $|||Q_2 v_2||| \geq 3.82364074 \cdot 10^{-3}$

(L2) $|||Q_1 v_1||| \geq 1.03442328 \cdot 10^{-2}$ (L2) $|||Q_2 v_2||| \geq 5.74911296 \cdot 10^{-2}$

The numerical results show that the improvement obtained by the bound (44) is really remarkable, especially when the eigenvalue estimates are not very accurate and the error is measured in the mean square norm $\|\cdot\|$.

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LA PRESSOFLESSIONE NEI MATERIALI NON RESISTENTI A TRAZIONE: ESISTENZA, UNICITA' ED APPROSSIMAZIONE DELLA SOLUZIONE

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LA FLESSIONE COMPOSTA NEI MATERIALI NON REAGENTI A TRAZIONE : Esistenza, Unicità ed Approssimazione della Soluzione.

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1. Introduzione

Il problema della sollecitazione di flessione composta nelle travi costituite da materiale non reagente a trazione e con comportamento elastico lineare a compressione, é certamente di eccezionale interesse per l'ingegnere strutturista. Esso rappresenta il classico modello di calcolo per le strutture in muratura ed in conglomerato cementizio, armato e non, delle flange bullonate rigide nei collegamenti delle strutture metalliche etc. Ciò nonostante non esiste nella letteratura una formulazione del problema che consenta di ottenere, nel caso generale, la dimostrazione dell'esistenza e dell'unicità della soluzione. Inoltre non si dispone di un procedimento generale di calcolo della soluzione. Il fatto che, nel caso generale, una soluzione approssimata venga usualmente ricercata per tentativi é motivato proprio dalla mancanza di un vero metodo. Nel presente lavoro si intende fornire una risposta rigorosa a queste due esigenze. Il problema é posto nell'ambito della teoria tecnica della trave, fondata sull'ipotesi che le sezioni rette rimangano piane a deformazione avvenuta. Questa ipotesi cinematica é da un lato suggerita dalla Teoria del Saint Venant, valida nel caso elastico lineare, dall'altro é confortata da numerose osservazioni sperimentali (si veda ad es. [1] cap.3).

La formulazione del problema unilaterale in termini di diseguaglianze variazionali, consente di ricondursi alla ricerca del minimo di un funzionale quadratico su un insieme convesso. Acquisita la dimostrazione dell'esistenza e dell'unicità della soluzione, sotto le opportune ipotesi di compatibilità sulla sollecitazione, si considera un algoritmo iterativo per l'approssimazione della soluzione mediante una tecnica di rilassamento. Si dimostra quindi la convergenza dell'algoritmo alla soluzione del problema.

2. Formulazione del problema

Si consideri una trave la cui sezione trasversale occupi un dominio piano Ω ed un sistema di riferimento ortonormale $\{O,x,y,z\}$ con l' asse z parallelo a quello baricentrico della trave. Nell'ambito della teoria tecnica della trave si suppone che l'unica componente di tensione presente nella sollecitazione di flessione composta sia la σ_z e che le sezioni trasversali rimangano piane a deformazione avvenuta, cioé che

$$\varepsilon_z = \phi_x y - \phi_y x + \delta \tag{1}$$

dove $\phi = (\phi_x, \phi_y)$ é il vettore curvatura dell'asse della trave nella sezione considerata e δ é la dilatazione in corrispondenza dell'origine O. Nel seguito σ_z ed ε_z saranno indicate con σ ed ε omettendo l'indice z. Con lettere greche si indicheranno le funzioni di quadrato sommabile secondo Lebesgue su Ω e con (σ,ε) il prodotto scalare in $L^2(\Omega)$, con lettere latine si indicheranno i vettori di R^3 e con il simbolo $u \cdot v$ il prodotto scalare usuale in R^3.

Posto

$$u = (\phi_x , \phi_y , \delta) \in R^3 \qquad (2)$$

$$t = (y , -x , 1) \in R^3 \qquad (3)$$

$$T u = t \cdot u \in L (\Omega) \qquad (4)$$

dove con $L(\Omega)$ si indica l'insieme delle funzioni lineari su Ω , la (1) può porsi nella forma

$$\varepsilon = T u \qquad (5)$$

Il principio dei lavori virtuali é espresso dalla seguente identità

$$(\sigma,\varepsilon) = (\sigma, Tu) = \int_\Omega \sigma t \cdot u \, d\Omega = u \cdot \int_\Omega \sigma t \, d\Omega = \phi_x \int_\Omega \sigma y \, d\Omega + \phi_y \int_\Omega -\sigma x \, d\Omega + \delta \int_\Omega \sigma \, d\Omega = u \cdot T^\times \sigma = u \cdot f \qquad (6)$$

dove

$$T^\times = \int_\Omega t(\cdot) \, d\Omega \qquad (7)$$

é l'operatore aggiunto di T, ed f é il vettore dei carichi

$$f = (M_x , M_y , N) \qquad (8)$$

La relazione

$$T^{\times}\sigma = f \tag{9}$$

esprime la condizione di equilibrio della sezione retta. Se si assume che il materiale é privo di resistenza a trazione ed ha comportamento elastico lineare a compressione con modulo E,[1] la legge costitutiva é espressa, in ogni punto di Ω, dalle relazioni

$$\begin{aligned} \sigma &= E\,\varepsilon \qquad &\text{se} \quad \varepsilon \leq 0 \\ \sigma &= 0 \qquad &\text{se} \quad \varepsilon > 0 \end{aligned} \tag{10}$$

che possono essere formulate equivalentemente in termini di complementarietà lineare

$$\sigma = E\,(\varepsilon - \eta) \qquad \eta \in L^2(\Omega) \tag{11}$$

$$\sigma \leq 0 \qquad \eta \geq 0 \qquad (\sigma, \eta) = 0 \tag{12}$$

Il problema in esame é espresso dalle (5),(9),(11) e (12) come un problema con vincoli unilaterali.

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(1) Nel seguito si considera, per semplicità di notazioni, il caso di materiale omogeneo, cioé con modulo E costante in Ω. L'estensione al caso di materiali non omogenei é ovviamente del tutto immediata.

Dalle (5),(9) e (11) si ha

$$\overset{\times}{T}\sigma = \overset{\times}{T} E (\varepsilon - \eta) = \overset{\times}{T} E (Tu - \eta) = f$$

e quindi in forma variazionale

$$E (Tu - \eta , Tv) = f \cdot v \qquad \forall \; v \in R^3 \tag{13}$$

Le (12) sono equivalenti alla diseguaglianza variazionale

$$(\sigma , \hat{\eta} - \eta) \leq 0 \qquad \forall \; \hat{\eta} \geq 0 \qquad \eta, \hat{\eta} \in L^2(\Omega)$$

che per mezzo delle (5) e (11) si può porre nella forma

$$E (Tu - \eta , \hat{\eta} - \eta) \leq 0 \qquad \forall \; \hat{\eta} \geq 0 \qquad \eta, \hat{\eta} \in L^2(\Omega) \tag{14}$$

La formulazione variazionale del problema, espressa dalle (13) e (14),rappresenta la condizione di minimo del funzionale quadratico

$$\Phi(u,\eta) = \frac{E}{2} \, \| Tu - \eta \|^2 - f \cdot u \qquad (u,\eta) \in R^3 \times L^2(\Omega) \tag{15}$$

sul cono chiuso e convesso

$$V = \{ (u,\eta) \mid \eta \geq 0 \} \tag{16}$$

Nel seguito si suppone $f \neq 0$. Infatti il caso $f = 0$ é banale: ogni coppia (u,η) tale che $Tu = \eta = \overset{+}{P} Tu$ é soluzione del problema. Con $\overset{+}{P}\alpha$ e $\overset{-}{P}\alpha$

si indicano le parti positiva e negativa di una funzione $\alpha \in L^2(\Omega)$, cioé

$$\alpha = \overset{+}{P}\alpha + \overset{-}{P}\alpha \qquad \overset{+}{P}\alpha \geq 0 \qquad \overset{-}{P}\alpha \leq 0$$

3. Esistenza ed Unicità

Si vuole dimostrare la seguente proposizione:

Condizione necessaria e sufficiente perché esista e sia unico il punto $(u_\circ,\eta_\circ)$ appartenente a V e tale che

$$\Phi(u_\circ,\eta_\circ) = \min_V \Phi(u,\eta) \tag{17}$$

é la seguente

$$f \cdot u < 0 \qquad \forall\, u \in C = \{ u \neq 0 : \| \overset{-}{P} Tu \| = 0 \} \tag{18}$$

Per ogni $u \in R^3$ il minimo del funzionale $\Phi(u,\eta)$ si ha per $\eta = \overset{+}{P}Tu$, cioé

$$\min_{\eta \geq 0} \Phi(u,\eta) \overset{*}{=} \Phi(u,\overset{+}{P}Tu) = \Psi(u) \tag{19}$$

dove

$$\Psi(u) = \frac{E}{2} \| \overset{-}{P} Tu \|^2 - f \cdot u \tag{20}$$

Basta quindi dimostrare che la (18) é condizione necessaria e sufficiente per l'esistenza e l'unicità del punto di minimo $u_\circ$ del funzionale $\Psi(u)$.

Dimostrazione dell'esistenza.

Sufficienza.

Dimostriamo il seguente risultato preparatorio:

Se la condizione (18) é verificata, per ogni successione $\{v_n\}$ di R^3 tale che

$$\| v_n \| \to + \infty \tag{21}$$

si ha

$$\lim\sup \Psi (v_n) = + \infty \tag{22}$$

Sia S la sfera unitaria di R^3 e consideriamo su di essa la successione

$$w_n = v_n / \| v_n \| \qquad \| w_n \| = 1 \tag{23}$$

Si ha

$$\Psi (v_n) = \frac{E}{2} \| v_n \|^2 \| P^{-} T w_n \|^2 - \| v_n \| \; f \cdot w_n \tag{24}$$

Dalla successione $\{w_n\}$, per la compattezza di S, se ne può estrarre una convergente $\{w_{n_k}\}$ tale che

$$\lim w_{n_k} = w_\circ \qquad \| w_\circ \| = 1 \tag{25}$$

Se $w_\circ \in C$ si ha $f \cdot w_\circ < 0$ e quindi dalla (24)

$$\lim \Psi (v_{n_k}) = \lim - \| v_{n_k} \| \; f \cdot w_\circ = + \infty \tag{26}$$

Se $w_o \in C$ si ha

$$\lim \Psi (v_{n_k}) = \lim \|v_{n_k}\|^2 \left(\frac{E}{2} \|P^- T w_{n_k}\|^2 - f \cdot w_o / \|v_{n_k}\| \right) = + \infty \tag{27}$$

Dalle (26) e (27) consegue la (22).

Sia ora $\{u_n\}$ una successione minimizzante $\Psi(u)$, cioé tale che

$$\lim \Psi (u) = \inf \Psi (u) < + \infty \tag{28}$$

La $\{u_n\}$ é limitata perché, se cosi non fosse, da essa se ne potrebbe estrarre una $\{u_{n_k}\}$ tale che

$$\|u_{n_k}\| \rightarrow + \infty$$

e quindi si avrebbe

$$\limsup \Psi (u_{n_k}) = + \infty$$

il che contraddice la (28).

Dalla $\{u_n\}$ si può allora estrarre una successione $\{u_{n_h}\}$ convergente per la quale si ha

$$\lim u_{n_k} = u_o$$

e per la continuitá di Ψ

$$\lim \Psi (u_{n_h}) = \Psi (u_o) = \min \Psi (u) \tag{29}$$

Necessarietà.

Mostriamo dapprima la necessarietà della seguente versione debole della condizione (18)

$$f \cdot u \leq 0 \qquad \forall\, u \in C = \{ u \neq 0 : \| P^{-} T u \| = 0 \} \tag{30}$$

Se la (30) non fosse verificata, esisterebbe un vettore $\bar{u} \in C$ tale che $f \cdot \bar{u} > 0$, e quindi

$$\Psi(\bar{u}) = - f \cdot \bar{u} < 0$$

Poiché se $\bar{u} \in C$ anche $\bar{u} \in C \ \forall\, \alpha > 0$, si avrebbe

$$\lim_{\alpha \to \infty} \Psi(\alpha\bar{u}) = \lim_{\alpha \to \infty} \alpha(-f \cdot \bar{u}) = -\infty$$

cioé

$$\inf \Psi(u) = -\infty$$

La necessarietà della (18) consegue dalla (30) e dalla proposizione seguente

$$f \cdot u = 0\, ,\ u \in C \quad \Longrightarrow \quad \inf \Psi(u) = -\infty \tag{31}$$

La dimostrazione della (31) presenta aspetti tecnici delicati. Essa può essere conseguita con una tecnica dovuta a G. Fichera. Per un'analisi approfondita dell'argomento si veda [2] pagg. 413 - 418.

Dimostrazione dell'unicità

Siano u_o ed u_o' due punti di minimo di $\Psi(u)$. Si avrebbe allora anche

$$\min \Phi (u,\eta) = \Phi (u_\circ,\eta_\circ) = \Phi (u'_\circ,\eta'_\circ) \tag{32}$$

con $$\eta_\circ = \overset{+}{P} T u_\circ \quad , \quad \eta'_\circ = \overset{+}{P} T u'_\circ \tag{33}$$

Se si pone

$$w = (u,\eta) \quad , \quad w_\circ = (u_\circ,\eta_\circ) \quad , \quad w'_\circ = (u'_\circ,\eta'_\circ) \tag{34}$$

$$b (w , w) = E \| T u - \eta \|^2 \tag{35}$$

$$\ell (w) = f \cdot u \tag{36}$$

si ha

$$\Phi (u , \eta) = \Phi (w) = \frac{1}{2} b (w , w) - \ell (w) \tag{37}$$

Dalle (32) si deduce che sono verificate le seguenti diseguaglianze variazionali equivalenti, per le (34), alle (13) e (14)

$$b (w_\circ , w - w_\circ) \geq \ell (w - w_\circ) \qquad w_\circ \in V, \quad \forall\, w \in V \tag{38}$$

$$b (w'_\circ , w - w'_\circ) \geq \ell (w - w'_\circ) \qquad w'_\circ \in V, \quad \forall\, w \in V \tag{39}$$

Sostituendo nelle (38) e (39) rispettivamente $w'_\circ$ e $w_\circ$ al posto di w, e sommando si ha

$$b (w_\circ - w'_\circ , w_\circ - w'_\circ) \leq 0$$

e quindi per le (33),(34) e (35)

$$\| \bar{P} T u_\circ - \bar{P} T u'_\circ \| = 0$$

cioé

$$P^{-} T u_o = P^{-} T u'_o \tag{40}$$

Mostriamo che, se la condizione (18) é soddisfatta, deve risultare

$$P^{-} T u_o = P^{-} T u'_o \neq 0 \tag{41}$$

Infatti se fosse $P^{-} T u_o = 0$, dalla (13) si avrebbe, ponendo $v = u_o$,

$$E (T u_o - \eta_o , T u_o) = E \| P^{-} T u_o \|^2 = f \cdot u_o = 0 \tag{42}$$

ma l'essere $f \cdot u = 0$, poiché $u_o \in C$, é in contrasto con la (18).
Dalla linearità delle funzioni $Tu = \varepsilon$ e dalle (40) e (41) consegue che deve risultare

$$u_o = u'_o$$

E' interessante osservare quale é il significato geometrico della condizione di compatibilità (18). Si riconosce che essa é equivalente alla classica condizione di equilibrio che impone che la risultante dei carichi applicata all'asse centrale sia di compressione ed intersechi il piano della sezione retta in un punto interno all'inviluppo convesso del dominio Ω [3].
Infatti la (19), per le (2),(3),(4) e (5), si può scrivere nella forma

$$M_x \phi_x + M_y \phi_y + N \delta < 0 \qquad \forall \ (\phi_x, \phi_y, \delta) \neq 0 \tag{43}$$

con

$$\phi_x y - \phi_y x + \delta \geq 0 \qquad \forall \ x,y \in \Omega \tag{44}$$

Dalla (43), ponendo $\phi_x = \phi_y = 0$ e $\delta > 0$ si ha

$$N < 0 \tag{45}$$

La (44) indica che i semipiani chiusi Σ^+ definiti dalle diseguaglianze

$$\phi_x y - \phi_y x + \delta \geq 0$$

al variare del vettore $u = (\phi_x, \phi_y, \delta)$, contengono tutti Ω. La loro intersezione é l'inviluppo convesso di Ω che si indica con il simbolo $\text{co}\ \Omega$. Dalla (43), dividendo per N, e tenendo presente la (45), si ha

$$\phi_x \frac{M_x}{N} + \phi_y \frac{M_y}{N} + \delta > 0 \tag{46}$$

e quindi $(- M_y/N\ ,\ M_x/N\ ) \subset (\ \text{co}\ \Omega\ )^\circ$, cioé il centro di sollecitazione é interno all'inviluppo convesso di Ω.

Nel caso in cui una parte della sezione é costituita da materiale elastico che reagisce a trazione, la legge costitutiva ha la forma seguente

$$\left.\begin{array}{lll} \sigma = E_1\,\varepsilon & \text{se} & \varepsilon \leq 0 \\ \sigma = 0 & \text{se} & \varepsilon > 0 \end{array}\right. \quad \text{su } \Omega_1$$

$$\sigma = E_2\,\varepsilon \qquad \text{su } \Omega_2$$

con $\Omega_1 \cup \Omega_2 = \Omega$ e $(\ \Omega_1 \cap \Omega_2\ )^\circ = \emptyset$.

In questo caso il funzionale Φ assume la forma :

$$\Phi (u,\eta) = \frac{E_1}{2} \|T u - \eta\|^2_{\Omega_2} + \frac{E_2}{2} \|T u\|^2_{\Omega_1} - f \cdot u$$

Le dimostrazioni di esistenza e di unicità della soluzione valgono ancora, omettendo la condizione di compatibilità (18).

4. Approssimazione della soluzione.

La tecnica di soluzione del problema di minimo (15),(16) é fondata su un metodo di rilassamento per la minimizzazione del funzionale $\Phi(u,\eta)$.
Si consideri il seguente schema iterativo:

$$\min_u \Phi (u, \eta_n) = \Phi (u_{n+1},\eta_n) \tag{47}$$

$$\min_{\eta \geq 0} \Phi (u_{n+1},\eta) = \Phi (u_{n+1},\eta_{n+1}) = \Psi (u_{n+1}) \tag{48}$$

L' esistenza e l'unicità del minimo (47) sono facilmente dimostrabili con argomenti classici. Quelle del minimo (48) sono conseguenza della (19).
Il procedimento può essere inizializzato ponendo $\eta = 0$.
Si intende ora dimostrare la convergenza dell'algoritmo definito dalle (47) e (48) verso la soluzione del problema. Poiché $\eta_n = P^+ T u_n$, il procedimento iterativo si formula in termini di u al modo seguente:

$$\min_v \Phi (v,P^+Tu_n) = \Phi (u_{n+1},P^+Tu_n) \tag{49}$$

La (49) definisce l'operatore Q tale che

$$Q u_n = u_{n+1} \tag{50}$$

Per la continuità di $\Phi, \overset{+}{P}$ e T, anche l'operatore Q é continuo.

Per ogni vettore $\bar{u}$ tale che

$$Q\,\bar{u} = \bar{u} \tag{51}$$

dalla (49) si ha

$$\min_{v} \Phi (v, \overset{+}{P} T \bar{u}) = \Phi (\bar{u}, \overset{+}{P} T \bar{u}) = \Psi (\bar{u}) \tag{52}$$

e, per la (19),

$$\Psi (\bar{u}) = \Psi (u_{o})$$

Per l' unicità della soluzione si ha infine

$$\bar{u} = u_{o} \tag{53}$$

L'operatore Q ha quindi un unico punto unito che coincide con il vettore u_{o} minimizzante $\Psi(u)$.

Mostriamo ora che la successione

$$\{u_n\} = \{Q^n u_1\} \qquad (Q^n = \text{iterato } n_{mo} \text{ di } Q) \tag{54}$$

generata dall'algoritmo iterativo, é limitata qualunque sia $u_1 \in R^3$.

Infatti, se cosi non fosse, esisterebbe almeno una successione estratta da essa $\{u_{n_k}\}$ tale che

$$\| u_{n_k} \| \rightarrow +\infty$$

ma allora (vedi (21) e (22)) sarebbe

$$\limsup \Psi (u_{n_k}) = + \infty$$

il che é assurdo perché la successione $\Psi(u_n)$ é non-crescente. Infatti dalla (49) si deduce che $\Psi(u_n) \geq \Psi(u_{n+1})$.

Essendo la $\{u_n\}$ limitata, da essa si può estrarre almeno una successione $\{u_{n_h}\}$ convergente ad un limite $\bar{u}$. Dalla diseguaglianza

$$\| \bar{u} - Q\bar{u} \| \leq \| \bar{u} - u_{n_h} \| + \| Q\bar{u} - Qu_{n_{h-1}} \|$$

passando al limite per $h \to + \infty$ si ha, per la continuità di Q e per la (53),

$$\bar{u} = Q \bar{u} = u_\circ$$

Ogni successione estratta dalla (54) e convergente ha pertanto come limite $u_\circ$. Ne consegue che, per ogni $u_1 \in R^3$,

$$\{u_n\} = \{Q^n u_1\} \to u_\circ$$

e la dimostrazione della convergenza é acquisita.

5. Conclusioni

La dimostrazione della convergenza del metodo iterativo di calcolo é strettamente connessa a quella dell'esistenza e dell'unicità della soluzione, che ne é quindi indispensabile premessa. L'algoritmo considerato può essere facilmente utilizzato dal punto di vista pratico. E' tuttavia ben più conveniente applicare un nuovo metodo iterativo di rapida convergenza, detto di rilassamento "geometrico", che é presentato nel lavoro [4] dove se ne dimostra la convergenza. L'esplicita formulazione del procedimento di calcolo per questi due metodi ed un confronto della loro efficacia numerica, sono presentati in [5] .

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AN ALGORITHM OF FAST CONVERGENCE FOR A CLASS OF UNILATERAL PROBLEMS

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AN ALGORITHM OF FAST CONVERGENCE FOR A CLASS OF UNILATERAL PROBLEMS

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1. Introduction

A large number of relevant mechanical models lead to the study of unilateral problems, depending on the constraint's conditions and/or the constitutive relations being expressed by inequalities. In recent times, starting with the solution of the Signorini problem by G. Fichera (1) in 1963, much work has been done in this field and many theoretical and computational results have been contributed. A comprehensive collection of available approximation methods for unilateral problems can be found in a recent book by R. Glowinski, J.L. Lions and R. Trémolièrs (2). However the effectiveness of general methods, when applied to special problems, may reveal to be very poor.

A new iterative method for a class of unilateral problems, which includes a number of cases of high interest in structural engineering, is presented here. This method, that will be referred to as the "geometric" relaxation, is developed with reference to a sample problem, namely the bending and compression of a beam of elastic material without tensile strenght. Existence, uniqueness and approximation of the solution by a standard relaxation method have been discussed, for this structural problem, in a previous paper (3).

On this basis the proposed algorithm is analized and its convergence proved. Numerical evidence shows that a comparison between the standard relaxation and the proposed "geometric" relaxation method, reveals a quite drastic improvement of the convergence rate in the latter case. The detailed computational procedure and comparative numerical results for the two methods, with reference to the sample problem, can be found in (4).

2. The unilateral problem

Let Ω be a domain of R^2 and $L^2(\Omega)$ the space of square integrable functions on Ω. We denote by T the linear bounded operator

$$T : u \in R^3 \rightarrow Tu = t \cdot u \in L(\Omega) \subset L^2(\Omega) \tag{1}$$

where $t = (x,y,1)$, $(x,y) \in \Omega$ and $L(\Omega)$ is the space of the real valued linear functions defined on Ω.

Consider the problem of the minimum of the quadratic functional

$$\Phi(u,\eta) = \frac{E}{2} \|T u - \eta\|^2 - f \cdot u \qquad (u,\eta) \in R^3 x L^2(\Omega) \tag{2}$$

on the closed convex cone

$$V = \{ (u,\eta) : \eta \geq 0 \} \tag{3}$$

which is equivalent to the unilateral problem arising in the technical beam theory in the case of bending and compression of elastic materials with no tensile strenght (3) (E = Joung modulus in compression).

Let us denote by $P^+\alpha$ and $P^-\alpha$ the positive and negative parts of a function $\alpha \in L^2(\Omega)$. Since for every $u \in R^3$, the minimum of the functional

$\Phi(u,\eta)$ is attained for $\eta = P^{+}Tu$, i.e.

$$\min_{\eta \geqslant 0} \Phi (u,\eta) = \Phi (u,P^{+}Tu) \tag{4}$$

the problem (2),(3) is equivalent to the problem of the minimum of the functional

$$\Psi (u) = \Phi (u,P^{+}Tu) = \frac{E}{2} \| P^{-}Tu \|^{2} - f \cdot u \qquad u \in R^{3} \tag{5}$$

The necessary and sufficient condition for the existence and uniqueness of a minimizing vector $u_{\circ}$ for $\Psi(u)$, is given by

$$f \cdot u < 0 \qquad u \in C = \{ u \neq 0 : \| P^{-}Tu \| = 0 \} \tag{6}$$

The proof of this proposition is given in (3).

We notice that if

$$\min \Psi (u) = \Psi (u_{\circ}) \tag{7}$$

then by (4)

$$\min \Phi (u,\eta) = \Phi (u_{\circ},\eta_{\circ}) = \Phi (u_{\circ},P^{+}Tu_{\circ}) \tag{8}$$

Let us define the following subsets of Ω

$$\Omega_{j}^{+} = \{ (x,y) \in \Omega : T u_{j} \geq 0 \} \tag{9}$$

$$\Omega_{j}^{-} = \{ (x,y) \in \Omega : T u_{j} \leq 0 \} \tag{10}$$

and the subspaces of $L^2(\Omega)$

$$\Sigma_j^+ = \{ \varepsilon \in L^2(\Omega) \quad : \text{supp } \varepsilon \subset \Omega_j^+ \} \tag{11}$$

$$\Sigma_j^- = \{ \varepsilon \in L^2(\Omega) \quad : \text{supp } \varepsilon \subset \Omega_j^- \} \tag{12}$$

We denote by P_j^+ and P_j^- the orthogonal projectors on Σ_j^+ and Σ_j^- respectively.

3. The "geometric" relaxation method

Let us consider the following iterative scheme

$$\min_u \; \Phi (u, P_j^+ Tu) = \Phi (u_{j+1}, P_j^+ Tu_{j+1}) \tag{13}$$

The existence and uniqueness of the point of minimum u_{j+1} is readily proved by standard arguments, if the vector u_j is such that meas $\Omega_j^- \neq 0$. We notice that if meas $\Omega_j^- = 0$ the problem (13) has no solution. If meas $\Omega_j^- \neq 0$ it follows that also meas $\Omega_{j+1}^- \neq 0$. In fact, since u_{j+1} is the solution of the minimum problem (13), we have

$$f \cdot u_{j+1} = \| P_j^- Tu_{j+1} \|^2 > 0$$

while, by the compatibility condition (6), if meas $\Omega_{j+1}^- = 0$ it should be

$$f \cdot u_{j+1} < 0$$

Hence we may define the operator Q such that

$$Q\, u_j \;=\; u_{j+1} \tag{14}$$

which is continuous by the continuity of Φ, P_j^+ and T.
We shall prove that the algorithm (13) is convergent to the solution u_o if the iterative procedure is initialized with a vector u_1 such that meas $\Omega_1^- \neq 0$, i.e.

$$\{u_j\} \;=\; \{Q^j\, u_1\} \;\rightarrow\; u_o \qquad (Q^j = j_{th} \text{ iterate of } Q) \tag{15}$$

Let us now show that the operator Q has a unique fixed point, which coincides with the solution vector u_o. If

$$u_h \;=\; Q\, u_h \tag{16}$$

from (13) follows that

$$\min_u \;\Phi\,(\, u, P_h^+ T u \,) \;=\; \Phi\,(\, u_h, P_h^+ T u_h \,) \;=\; \Psi\,(\, u_h \,) \tag{17}$$

and from (4)

$$\Psi\,(\, u_h \,) \;=\; \Psi\,(\, u_o \,) \tag{18}$$

The uniqueness of the solution then implies

$$u_h \;=\; u_o \tag{19}$$

Let us prove that the sequence $\{u_j\}$ is bounded.
Indeed we have that

$$c_j \, \| u_{j+1} \|^2 \leq \| P_j^- \, T u_{j+1} \|^2 = f \cdot u_{j+1} \leq \| f \| \, \| u_{j+1} \| \tag{20}$$

and hence

$$\| u_{j+1} \| \leq \frac{1}{c_j} \| f \| \qquad\qquad c_j > 0$$

The boundedness of $\{u_j\}$ is then established if we show that there exists a positive constant c such that, for every j,

$$c_j \geq c > 0$$

To this end it is enough to prove that the sequence $\{\text{meas } \Omega_j\}$ is bounded below by a positive constant. This fact is easily deduced from the geometric interpretation of the compatibility condition (6) (see paper (3)), which ensures the existence in Ω of an internal point that belongs to every set of the family $\{\Omega_j^-\}$.
By the boundedness of $\{u_j\}$ there exists at least a subsequence $\{u_{j_k}\}$ which converges to a limit $\bar{u}$. From the inequality

$$\| \bar{u} - Q \, \bar{u} \| \leq \| \bar{u} - u_{j_k} \| + \| Q \, \bar{u} - Q \, u_{j_{k-1}} \| \tag{21}$$

taking the limit for $k \to +\infty$, we get from (16) and (19) and by the continuity of Q

$$\bar{u} = Q\,\bar{u} = u_\circ \tag{22}$$

Therefore every converging subsequence of $\{u_j\}$ has the limit $u_\circ$. It follows that

$$\{u_j\} \to u_\circ$$

and the convergence of the algorithm is proved.

4. Conclusions

Let us observe that, for the sample problem (2),(3), we have

$$\Phi\,(\,u, P_j^+ Tu\,) = \frac{E}{2}\,\|\,Tu - P_j^+ Tu\,\|_\Omega^2 - f \cdot u =$$

$$= \frac{E}{2}\,\|\,P_j^- Tu\,\|_\Omega^2 - f \cdot u = \frac{E}{2}\,\|\,Tu\,\|_{\bar{\Omega}_j}^2 - f \cdot u$$

Hence the iterative scheme (13) can be written in the form

$$\min_u \frac{E}{2}\,\|Tu\|_{\bar{\Omega}_j}^2 - f \cdot u = \frac{E}{2}\,\|\,Tu_{j+1}\|_{\bar{\Omega}_j}^2 - f \cdot u_{j+1} \tag{24}$$

which shows that the solution of the unilateral problem is approximated by solving a sequence of uncostrained minimum problems for the quadratic function $\|\,Tu\|^2 - f \cdot u$ on the subdomains $\bar{\Omega}_j$ of Ω. Therefore at each step of the iterative algorithm the "geometry" of the domain changes.

The exceptionally good rate of convergence exibited by this method (see (4)) makes it a powerful computational tool, that can be easily programmed on a computer. We notice that its convenience will be very much appreciated also if manual computations are performed or grafical methods applied.

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THE NUMERICAL PERFORMANCE OF A NEW ITERATIVE METHOD FOR UNILATERAL PROBLEMS OF STRUCTURAL MECHANICS

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THE NUMERICAL PERFORMANCE OF A NEW ITERATIVE METHOD FOR UNILATERAL PROBLEMS OF STRUCTURAL MECHANICS

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1. Introduction

A new iterative method for the approximation of the solution of unilateral problems in structural mechanics, has been presented in a previous paper (1), with reference to the engineering problem of bending and compression of a beam of elastic material without tensile strenght. Existence, uniqueness and convergence of a classical relaxation method have been proved, for this problem, in the paper (2). The convergence proof for the new "geometric" relaxation method has been given in (1).
The explicit formulations of these two methods and the comparison of the convergence rates, are presented in this paper. It is shown that the "geometric" relaxation algorithm is much more efficient than the classical one. Its convergence rate is exceptionally good. Numerical results are given for a simple case in which the exact solution is known. Extensive numerical experiences confirm, without exceptions, the same behaviour observed in this particular case. When the "geometric" method is applied, three or four steps are sufficient to obtain a really satisfactory approximation. The classical relaxation has a much poorer efficiency.

2. Formulation of the problem

Let us consider a beam whose cross-section occupies a plane domain Ω and an orthogonal reference frame $\{O,x,y,z\}$, with the z axis parallel to the centroidal axis of the beam. In the context of the technical beam theory it is assumed that only the stress component $\sigma_z = \sigma$ is different from zero and that the cross-sections remain plane after deformation. Hence the deformation component $\varepsilon_z = \varepsilon$ is given by

$$\varepsilon = -\phi_y x + \phi_x y + \delta \tag{1}$$

where $\phi = (\phi_x, \phi_y)$ is the curvature vector of the beam and δ is the unitary extension at the origin O. If we define [1]

$$u = (-\phi_y, \phi_x, \delta) \tag{2}$$

$$t = (x, y, 1) \tag{3}$$

$$T u = t \cdot u \in L (\Omega) \tag{4}$$

where $L(\Omega)$ denotes the space of the linear functions on Ω, the relation (1) can be written in the form

$$\varepsilon = T u \tag{5}$$

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(1) Latin letters will denote vectors of R^3, and $u \cdot v$ the usual scalar product in R^3. Greek letters denote square integrable functions on Ω and (σ,ε) the scalar product in $L^2(\Omega)$.

The virtual work principle is expressed by the following identity

$$(\sigma,\varepsilon) = (\sigma,Tu) = \int_\Omega \sigma t\cdot u \, d\Omega = u \cdot \int_\Omega \sigma t \, d\Omega =$$
$$= -\phi_y \int_\Omega \sigma y \, d\Omega + \phi_x \int_\Omega \sigma x \, d\Omega + \delta \int_\Omega \sigma \, d\Omega = u \cdot T^\times \sigma = u \cdot f \tag{6}$$

where

$$T^\times = \int_\Omega t(\cdot) \, d\Omega \tag{7}$$

is the adjoint operator of T and f is the load vector

$$f = (-M_y, M_x, N) \tag{8}$$

The relation

$$T^\times \sigma = f \tag{9}$$

is the condition of equilibrium for the cross-section of the beam. In the paper (1) has been shown that the problem of bending and compression for a beam of elastic material without tensile strenght is equivalent to the minimum problem for the "energy" functional

$$\Phi (u,\eta) = \frac{E}{2} \|Tu - \eta\|^2 - f \cdot u \qquad (u,\eta) \in R^3 \times L^2(\Omega) \tag{10}$$

on the closed convex cone

$$V = \{ (u,\eta) : \eta \geq 0 \} \tag{11}$$

for which existence and uniqueness have been proved.
Let $(u_\circ,\eta_\circ)$ be the vector of $R^3 \times L^2(\Omega)$ which minimizes $\Phi(u,\eta)$. For a given vector u_h of R^3 we define the subsets of Ω

$$\Omega_h^+ = \{ (x,y) \in \Omega : T\, u_h \geq 0 \} \tag{12}$$

$$\Omega_h^- = \{ (x,y) \in \Omega : T\, u_h \leq 0 \} \tag{13}$$

and the subspaces of $L^2(\Omega)$

$$\Sigma_h^+ = \{ \varepsilon \in L^2(\Omega) \ : \ \text{supp}\ \varepsilon \subset \Omega_h^+ \} \tag{14}$$

$$\Sigma_h^- = \{ \varepsilon \in L^2(\Omega) \ : \ \text{supp}\ \varepsilon \subset \Omega_h^- \} \tag{15}$$

By P_h^+ and P_h^- we shall denote the orthogonal projectors on Σ_h^+ and Σ_h^- respectively. From (10) we see that

$$\eta_\circ = P_\circ^+ \, T\, u_\circ \tag{16}$$

Hence the solution of problem (10),(11) is completely characterized by the vector $u_\circ$.

3. The classical relaxation method

The solution of the problem (10),(11) can be approximated (see (2)) constructing a minimizing sequence $\{u_h\} \to u_\circ$ by the following iterative algorithm

$$\min_u \Phi(u, P_h^+ T u_h) = \Phi(u_{h+1}, P_h^+ T u_h) =$$
$$= \frac{E}{2} \| T u_{h+1} - P_h^+ T u_h \|^2 - f \cdot u_{h+1} \tag{17}$$

The minimum problem (17) is equivalent to the variational equation

$$E (T u_{h+1} - P_h^+ T u_h, T v) = f \cdot v \qquad \forall v \in R^3 \tag{18}$$

Since, by definition, for every $\alpha \in L^2(\Omega)$

$$(\alpha, T v) = T^\times \alpha \cdot v \tag{19}$$

from (18) we get

$$T^\times E T u_{h+1} - T^\times E P_h^+ T u_h = f \tag{20}$$

and, solving for u_{h+1},

$$u_{h+1} = (T^\times E T)^{-1} (f + T^\times E P_h^+ T u_h) \tag{21}$$

From (4) and (7) it follows that

$$J = T^{\times} E T = \int_{\Omega} E\, t \otimes t \, d\Omega \tag{22}$$

where the symbol $\otimes$ denotes the tensor product in R^3.

From (3) we have

$$t \otimes t = \begin{vmatrix} x^2 & xy & x \\ xy & y^2 & y \\ x & y & 1 \end{vmatrix} \tag{23}$$

and, substituting in (22), we get

$$J = T^{\times} E T = \begin{vmatrix} I_x & I_{xy} & S_x \\ I_{xy} & I_y & S_y \\ S_x & S_y & A \end{vmatrix} \tag{24}$$

that is the inertia matrix of the cross-section of the beam.

Similar arguments show that

$$J_h^+ = T^{\times} E P_h^+ T = \int_{\Omega_h^+} E\, t \otimes t \, d\Omega \tag{25}$$

that is the inertia matrix of the part Ω_h^+ of Ω.

Therefore the relation (21) can be written in the form

$$u_{h+1} = J^{-1} (f + J_h^+ u_h) \tag{26}$$

that gives an explicit formulation of the algorithm (17).

This method requires the inversion of the matrix J and, at each step, the construction of the matrix J_h^+. It can be initialized e.g. choosing the first vector u_1 in such a way that $J_h^+ = 0$. Then we have

$$u_2 = J^{-1} f \tag{27}$$

4. The "geometric" relaxation method

In the paper (1) the following iterative method to construct a minimizing sequence $\{u_h\} \to u_\circ$ for the problem (10),(11), has been presented.

$$\begin{aligned} \min \ \Phi (u, P_h^+ T u) &= \Phi (u_{h+1}, P_h^+ T u_{h+1}) = \\ &= \frac{E}{2} \left\| T u_{h+1} - P_h^+ T u_{h+1} \right\|^2 - f \cdot u_{h+1} \end{aligned} \tag{28}$$

This minimum problem is equivalent to the variational equation

$$E (T u_{h+1} - P_h^+ T u_{h+1}, T v) = f \cdot v \qquad \forall\, v \in R^3 \tag{29}$$

From (19) and (30) we get

$$T^\times E P_h^- T u_{h+1} = f \tag{31}$$

and, solving for u_{h+1},

$$u_{h+1} = (T^\times E P_h^- T)^{-1} f \tag{32}$$

From (7), (15) and by the definition of P_h^-, we have

$$J_h^- = T^{\times} E P_h^- T = \int_{\Omega_h^-} E\, t \otimes t \, d\Omega \tag{33}$$

that is the inertia matrix of the part Ω_h^- of Ω. Hence the relation (32) can be put in the form

$$u_{h+1} = (J_h^-)^{-1} f \tag{34}$$

which is the explicit formulation of the algorithm (28). It requires, at each step, the construction of the matrix J_h^- and its inversion. The process can be initialized e.g. choosing $\Omega_h^- = \Omega$, so that

$$u_2 = J^{-1} f \tag{35}$$

5. Comparison of the two methods

An extensive comparative analysis of the numerical efficiency of the two methods, gives in every case the same typical result. Therefore to give a precise idea of the general behaviour, it is sufficient to consider a very simple problem in which the exact solution is known.

Let us consider a beam of rectangular cross-section with basis b, height h and the reference frame $\{0,x,y\}$ with the origin in the center. It is well known that elementary arguments show that if

$$f = (0, N\frac{h}{3}, N) \qquad N < 0$$

then

$$u_o = \left(0, \frac{8N}{Ebh^2}, 0 \right)$$

so that $Tu_o = 0$ for $y = 0$, i.e. the neutral axis coincides with the x axis. Let us now show the numerical results relative to the classical relaxation (CR) and the "geometric" relaxation (GR) methods. In the following table we give the value of the distance of the neutral axis from the exact position as a function of the number of iteration steps (h=1).

Step nr.	CR	GR
1	0.25	0.25
2	0.189	0.10
3	0.153	0.025
4	0.129	0.002174
5	0.110	0.00001866
6	0.096	0.0000000015
10	0.060	
30	0.010	
50	0.00244	
100	0.000075	
150	0.00000234	

We see that in 3 ÷ 4 steps the GR method gives a very accurate approximation. The same result is obtained by the CR method with 20 ÷ 60 steps. The "geometric" relaxation is really a powerful computational method which can be easily programmed on a computer.

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COMPATIBILITY UNDER UNILATERAL CONSTRAINTS

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COMPATIBILITY UNDER UNILATERAL CONSTRAINTS

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Summary

We consider the compatibility problem for given strain distributions and constraints' displacements in a structure with unilateral constraints. A general necessary and sufficient condition of compatibility is proved by a constructive method which has a sound structural interpretation. The analysis is carried out in an abstract context which allows to get a general result applicable to a wide class of similar problems.

1. Introduction

The compatibility problem in linear bilateral structural mechanics has been investigated in a previous paper by one of the authors, (1) and a general variational condition for the compatibility of given strain distributions and constraints' displacements has been proved, under minimal assumptions on the data.
In the present paper the corresponding result for the compatibility problem under unilateral constraints is established.
The method of analysis is the natural extension of the one adopted for the linear bilateral problem. It is suggested by the following observation: " a given strain distribution and constraints' displacement is compatible if and only if, when imposed as assigned distorsions on an elastic structure, no elastic strain energy is induced in the structure."

The simple trick of using an " auxiliary " elastic problem, to get a constructive proof of the fundamental existence result, reveals to be successfull also in the present unilateral context.
The abstract formulation of boundary value problems of linear structural mechanics, founded upon the concept of virtual work, which was developed in (1), will be followed to give an unifying character to the treatment. It can be applied to different problems falling into the range of the theory.
The mathematical tools of functional analysis are used in the development, but with a language which emphasizes the application to structural mechanics.

2. Background definitions and results

Let us recall here the essential elements of the abstract formulation of boundary value problems of linear structural mechanics presented in (1).
The involved fields are assumed to be elements of suitably defined Hilbert spaces which are placed in duality by the bilinear form " virtual work ".
Quoting explicitly the mechanical interpretation of the vector fields placed in duality, we consider the following pairs of dual spaces:

V , V'	displacement , force
W , W'	stress , strain

A prime denotes the dual Hilbert space.
To give an abstract definition of " interior " and " boundary " values of the displacement and force fields, we consider the closed subspace V_o of V consisting of the displacement fields which " vanish in a boundary layer ". The boundary values of the displa-

cement fields will be the elements of the orthogonal complement Γ of $V_\circ$ in V. Interior and boundary force fields will then belong to the dual spaces so that we have:

$V_\circ$, $V_\circ'$	" interior " displacement , force
Γ , Γ'	" boundary " displacement , force

We shall denote by " int " and " bnd " the orthogonal projectors of V onto $V_\circ$ and Γ.
A class of boundary value problems is then defined by a triplet of bounded linear operators:

$T \; : \; V \to W'$	deformation operator
$T^\times \; : \; W \to V_\circ'$	" interior " equilibrium operator
$B \; : \; W \to \Gamma'$	" boundary " equilibrium operator

such that the range of T is closed in W' and the following Green's formula holds:

$$\{T^\times\sigma, \mathrm{int}\, u\} + \langle B\sigma, \mathrm{bnd}\, u\rangle = (Tu, \sigma) \qquad \forall\, u \in V,\ \forall\, \sigma \in W$$

where the symbols $\{\cdot,\cdot\}$, $\langle\cdot,\cdot\rangle$, $(\cdot,\cdot)$ denote the duality pairings in $V_\circ' \times V_\circ$, $\Gamma' \times \Gamma$, $W' \times W$.
They have the mechanical meaning of virtual works of the corresponding dual fields, so that the identity above is the abstract formulation of the virtual work principle of linear structural mechanics.

3. Unilateral compatibility problem

We consider the class of unilateral problems in which unilateral constraints conditions are imposed on the boundary displacements. In abstract terms we assume that boundary displacements belong to a closed convex cone $\Gamma^+ \subset \Gamma$. The set of admissible displacement fields will accordingly be defined to be the closed convex cone :

$$V^+ = \{u \in V : \text{bnd } u \in \Gamma^+\}$$

Since the virtual work of constraints' reactions must be non negative, the corresponding set of admissible stress fields will be the closed convex cone :

$$W^+ = \{\sigma \in W : \langle B\sigma, \text{bnd } u\rangle \geq o \quad \forall\, u \in V^+\}$$

We may now formulate the following :

<u>Unilateral compatibility problem</u>

Given a strain distribution $\varepsilon \in W'$, find the solution set :

$$V^+(\varepsilon) = \{u \in V^+ : Tu = \varepsilon\}$$

which amounts in looking for the (possibly empty) set of admissible displacement fields to which corresponds the given strain field. Let us denote by :

$$\Sigma_o = \{\sigma \in W : T^\times \sigma = o\}$$

the closed subspace of self-equilibrated stress fields and by :

$$\Sigma_o^+ = \Sigma_o \cap W^+$$

the closed convex cone of admissible self-equilibrated stress fields. A complete solution of the unilateral compatibility problem is provided by the following :

Unilateral compatibility theorem

The solution set $V^+(\varepsilon)$ is non empty if and only if the virtual work of the given strain field $\varepsilon \in W'$ for any admissible self-equilibrated stress field $\sigma \in \Sigma_o^+$ is non negative, that is :

$$(\varepsilon,\sigma) \geq o \qquad \forall\, \sigma \in \Sigma_o^+$$

We shall give a constructive proof of this result which is based on the following observation:
" let an anelastic deformation field be imposed on an elastic structure with unilateral constraints, in absence of applied loads.
If no elastic strain energy is induced in the structure, the stress field and hence the elastic strain field vanishes. As a consequence the displacement solution is indipendent on the assumed elasticity and the associated strain field is exactly the imposed anelastic deformation."
We remark that the necessity of the unilateral compatibility condition stated above is a trivial consequence of the virtual work principle.
The non trivial part of the proof concerns the sufficiency and, as suggested by the previous observation, will be based on the following auxiliary elastic boundary value problem :

$$T^\times\sigma = o \qquad \sigma \in W$$

$$Tu = \varepsilon + j\sigma \qquad u \in V^+$$

$$\{B\sigma,\mathrm{bnd}(v - u)\} \geq o \qquad \forall\, v \in V^+$$

The elasticity j can be an arbitrary positive definite symmetric bounded linear operator from W onto W'. In the case when W = W' it can be conveniently choosen to be the identity. In the general case the most obvious candidate is the Riesz isometry from W onto W'.
The boundary conditions can be equivalently rewritten as:

$$\langle B\sigma, \text{bnd } v\rangle \geq o \qquad \forall\, v \in V^+$$
$$\langle B\sigma, \text{bnd } u\rangle = o$$

which are characteristic of unilateral contact problems.
The firt one imposes that $\sigma \in W^+$ and the other that the virtual work of constraints' reactions for the boundary displacements must vanish.
To prove the existence of a solution of the elastic problem above, we remark that it can be equivalently formulated as the minimum problem of the associated elastic strain energy functional over the set of admissible displacements :

$$\min \frac{1}{2}\{\| Tu - \varepsilon \|^2 \ /\ u \in V^+\}$$

where $\| \cdot \|$ denotes the norm in W'.
Indeed if $u_\circ \in V^+$ is a displacement solution of this minimum problem, the stress solution will be : $\sigma_\circ = j^{-1}(Tu_\circ - \varepsilon)$ so that :

$$(T(v - u), \sigma_\circ) \geq o \qquad \forall\, v \in V^+$$

and by the virtual work principle :

$$\{T^\times \sigma_\circ, \text{int}(v - u_\circ)\} + \langle B\sigma_\circ, \text{bnd}(v - u_\circ)\rangle \geq o \qquad \forall\, v \in V^+$$

or equivalently :

$T^{\times}\sigma_\circ = o$

$<B\sigma_\circ, \text{bnd } u_\circ> = o$

$<B\sigma_\circ, \text{bnd } v> \geq o \qquad \forall\, v \in V^+$

The converse is proved following the same steps backwards.
Let us now observe that this minimum problem can be interpreted as the minimum distance problem of the strain field $\varepsilon \in W'$ from the closed convex cone :

$TV^+ = \{Tu \in W' : u \in V^+\}$

The Riesz projection theorem in Hilbert spaces ensures the existence of a solution $u_\circ \in V^+$.
The last step of the proof consists in showing that the minimum distance is in fact zero if the unilateral compatibility condition holds.
Indeed, since $\sigma_\circ$ is self-equilibrated, we have :

$(Tu_\circ, \sigma_\circ) = (\varepsilon, \sigma_\circ) + \|\sigma_\circ\|^2 = o \qquad (\ \|\cdot\|\ \text{is the norm in } W\ )$

Now, noting that $\sigma_\circ \in \Sigma_\circ^+$, the unilateral compatibility condition implies that :

$(\varepsilon, \sigma_\circ) \geq o$

and hence :

$\|\sigma_\circ\|^2 \leq o$

so that the stress solution vanishes and : $\varepsilon = Tu_\circ$.
The solution set of the unilateral compatibility problem will then be given by :

$$V^+(\varepsilon) = \{u_o + R\} \cap V^+$$

where $R = \{u \in V : Tu = o\}$ denotes the closed subspace of rigid displacement fields.

4. General boundary conditions

The previous analysis can be readily extended to the case of general boundary conditions, including mixed (traction and displacement) bilateral and unilateral contact problems.
To this end let us consider a pair of complementary orthogonal projectors P,Q of Γ, a closed convex cone $\Gamma^+ \subset Q\Gamma$ and a given displacement field $w \in V$.
The set of admissible displacements is then defined by :

$$V^+ = \{u \in V : Q\mathrm{bnd}\, u \in Q\mathrm{bnd}\, w + \Gamma^+\}$$

which is a closed convex cone with vertex at $w \in V$.
The projector Q identifies the subspace $Q\Gamma$ of boundary displacements on which boundary conditions are imposed.
The boundary values $Q\mathrm{bnd}\, w$ of the given displacement field $w \in V$ yield the assigned constraints' displacements.
The set of admissible stress fields is then defined by :

$$W^+ = \{\sigma \in W : \langle Q'B\sigma, Q\mathrm{bnd}(v - w)\rangle \geq o\} \qquad \forall\, v \in V^+$$

where Q' is the dual projector of Q on Γ' and $Q'B\sigma$ are the constraints' reactions associated with the stress field $\sigma \in W$.
We can now formulate the general :

Unilateral compatibility problem

Given a strain distribution $\varepsilon \in W'$, a displacement field $w \in V$

and an orthogonal projector Q of Γ, find the solution set :

$$V^{+}(\varepsilon,w,Q) = \{u \in V^{+} : Tu = \varepsilon\}$$

Denoting the subspace of self-equilibrated stress fields by :

$$\Sigma_o = \{\sigma \in W : T^{\times}\sigma = o,\ P'B\sigma = o\}$$

where $P'B\sigma$ are the boundary loads, we can state the corresponding:

Unilateral compatibility theorem

The solution set $V^{+}(\varepsilon,w,Q)$ is non empty if and only if the virtual work of the given strain field $\varepsilon \in W'$ for any admissible self equilibrated stress field $\sigma \in \Sigma_o^{+}$ is no less than the virtual work of the corresponding constraints' reactions $Q'B\sigma \in Q'\Gamma'$ for the assigned constraints' displacements $Q\text{bnd}\ w \in Q\Gamma$, that is

$$(\varepsilon,\sigma) \geq \langle Q'B\sigma, Q\text{bnd}\ w\rangle \qquad \forall\ \sigma \in \Sigma_o^{+} = \Sigma_o \cap W^{+}$$

The proof of this more general result can be carried out following exactly the same lines of reasoning as in the previous special case and will not be reproduced here explicitly.
We only quote that the auxiliary elastic boundary value problem is now :

$$T^{\times}\sigma = o \qquad \sigma \in W^{+}$$

$$P'B\sigma = o$$

$$Tu = \varepsilon + j\sigma \qquad u \in V^{+}$$

$$\langle Q'B\sigma, Q\text{bnd}(v - u)\rangle \geq o \qquad \forall\ v \in V^{+}$$

and that the boundary condition can be equivalently rewritten as:

$$\langle Q'B\sigma, Qbnd(v - w)\rangle \geq o \qquad \forall\, v \in V^+$$

$$\langle Q'B\sigma, Qbnd(u - w)\rangle = o$$

which are characteristic of unilateral contact problems in presence of an assigned constraints' displacement.

5. Concluding remarks

Let us remark here the main features of the analysis presented in this paper. The leading idea which lies behind the rigorous proof of the unilateral compatibility condition has a sound mechanical interpretation.

In fact an heuristic version of the proof could be easily explained to any structural engineer without any knowledge of functional analysis.

A relevant consequence of the basic idea is that the abstract analysis which has been developed can be applied to any structural model, including three-dimensional continuum, two-dimensional plates and shells and one-dimensional beam theory.

Moreover, by giving a different physical interpretation to the involved fields and to the governing operators, similar problems from other fields of mathematical physics can be immediately included.

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BOUNDS TO THE NATURAL FREQUENCIES FOR A VIBRATION PROBLEM OF STRUCTURAL MECHANICS

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BOUNDS TO THE NATURAL FREQUENCIES FOR A VIBRATION PROBLEM OF STRUCTURAL MECHANICS

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1. Introduction

Many vibration problems of structural mechanics are formulated as eigenvalue problems for differential operators, in which the eigenvalues appear also in the boundary conditions. A general method to compute bounds to the natural frequencies in such cases has been presented in (1)
We shall consider in this paper a particular problem which is of basic interest in the seismic analysis of tall structures supporting an heavy rigid body. This is the case, e.g., of airport towers, of columns bearing a vessel etc.. It is often possible to suppose that the elastic structure behaves as an equivalent cantilever beam with a concentrated mass on a rigid arm at its end. The analysis of this problem, while shows all the relevant features of the general method, allows to get a clear idea of the explicit procedure to be followed in each particular case.
Upper bounds to the natural frequencies are computed by the Rayleigh - Ritz method with reference to the variational formulation of the problem. The original eigenvalue problem is then reduced to an equivalent one for an integral operator and the theory of orthogonal invariants (2), is ap-

plied to compute the lower bounds. We give the explicit analytic solution and present a numerical example.

2. Formulation of the problem

Let us consider a cantilever beam of lenght ℓ_1 with flexural rigidity k and mass per unit lenght $\bar{m}$. Its free end is connected to an heavy rigid body which can be considered as a concentrated mass $\bar{M}$ at distance ℓ_2 on a rigid arm. We shall denote by z the centroidal axis of the beam, by v the deflection function, by ω the natural frequency and by n the axial force.

The problem of small stationary vibrations, taking in to account only the flexural deformations, is formulated in the following way

$$k\, D_z^4\, v - n\, D_z^2\, v = \omega^2\, \bar{m}\, v$$

$$v(o) = D_z\, v(o) = 0$$

$$k\, D_z^2\, v(\ell_1) = -\, n\, \ell_2\, D_z\, v(\ell_1) + \omega^2\, \bar{M}\, \ell_2\, (v(\ell_1) + \ell_2 D_z v(\ell_1))$$

$$k\, D_z^3\, v(\ell_1) = n\, D_z\, v(\ell_1) - \omega^2\, \bar{M}\, (v(\ell_1) + \ell_2 D_z v(\ell_1))$$

Introducing the reference quantities $\ell_\circ$, $m_\circ$, $k_\circ$ and defining

$$\alpha^2 = \frac{n\,\ell_1^2}{k} \qquad \lambda = \frac{\omega^2\,\ell_1\,\ell_\circ^2\,m_\circ}{k} \qquad m = \frac{\bar{m}\,\ell_1^3}{m_\circ\,\ell_\circ^3} \qquad M = \frac{\bar{M}\,\ell_1^2}{m_\circ\,\ell_\circ^3}$$

$$\ell = \frac{\ell_1}{\ell_\circ} \qquad \varepsilon = \frac{\ell_2}{\ell_1} \qquad x = \frac{z}{\ell} \in I = (0,1) \qquad u = \frac{v}{\ell}$$

we get the adimensional form of the problem,

Problem P_1:

$$D_x^4 u - \alpha^2 D_x^2 u = \lambda m u \qquad \alpha,\lambda \in C$$

$$u(o) = D_x u(o) = 0$$

$$D_x^2 u(1) + \alpha^2 \varepsilon D_x u(1) = \lambda M \varepsilon (u(1) + \varepsilon D_x u(1))$$

$$D_x^3 u(1) - \alpha^2 D_x u(1) = - \lambda M (u(1) + \varepsilon D_x u(1))$$

We notice that if $n \geq 0$, α is real. When $n < 0$, α is imaginary and we shall write $\alpha = i\beta$.

The eigenvalue problem P_1 has the following variational formulation,

Problem P_2

$$b (u,w) = \lambda a (u,w) \qquad \forall w \in V, u \in V$$

where

$$V = \{ w \in H_2 : w(o) = D w(o) = 0 \}$$

$$b(u,w) = \int_I D^2u D^2w dx + \alpha^2 \int_I Du Dw dx + \alpha^2 \varepsilon Du(1) Dw(1)$$

$$a(u,w) = \int_I m u w dx + M (u(1) + \varepsilon Du(1)) (w(1) + \varepsilon Dw(1))$$

We shall suppose that β^2 is not an eigenvalue of the problem

$$D^4 u + \beta^2 D^2 u = 0$$

$$u(o) = D u(o) = 0$$

$$D^2 u(1) - \beta^2 \varepsilon D u(1) = 0$$

$$D^3 u(1) + \beta^2 D u(1) = 0$$

i.e. that it does not correspond to a critical value of the axial force. Then $b(u,u)$ is coercive on V and since $a(u,u)$ is positive definite on V, the eigenvalue problem P_2 has a countable sequence of positive eigenvalues λ_h.

To the problem P_2 we can apply the Rayleigh - Ritz method to compute upper bounds to the λ_h. If we choose ν linearly independent functions w_h which belong to V, upper bounds to the first ν eigenvalues are given by the roots of the determinant equation

$$\det \{b (w_h, w_k) - \lambda\, a (w_h, w_k)\} = 0 \qquad (h,k=1,\ldots\nu)$$

The w_h can be choosen to be piecewise cubic Hermite polynomials that vanish with their first derivatives at $x = 0$.

To obtain lower bounds to the λ_h we can then apply the method of orthogonal invariants (2). To this end it is necessary to formulate the eigenvalue problem P_1 as an equivalent problem for an integral operator.

3. <u>The equivalent problem for an integral operator</u>

If we define

$$q = \lambda\, m\, u \in L^2(I) \qquad a = \lambda\, M\, (\, u(1) + Du(1)\,) \in R \tag{1}$$

from problem P_1 we get the associated "static" problem,

Problem P_3:

$$D^4 u - \alpha^2 D^2 u = q$$

$$u(o) = D\, u(o) = 0$$

$$D^2 u(1) + \alpha^2 D\, u(1) = \varepsilon\, a$$

$$D^3 u(1) - \alpha^2 D\, u(1) = -\, a$$

which is a non - homogeneous boundary value problem for a non - homo-geneous equation.

Let us now assume

$$u = v + h \tag{2}$$

to split the problem P_3 in the following two, Problem P_3':

$$D^4 v - \alpha^2 D^2 v = q$$

$$v(o) = Dv(o) = 0$$

$$D^2 v(1) + \alpha^2 \varepsilon D v(1) = 0$$

$$D^3 v(1) - \alpha^2 D v(1) = 0$$

which is a homogeneous boundary value problem for a non - homogeneous equation, and

Problem P_3'':

$$D^4 h - \alpha^2 D^2 h = 0$$

$$h(o) = D h(o) = 0$$

$$D^2 h(1) + \alpha^2 \varepsilon D h(1) = \varepsilon a$$

$$D^3 h(1) - \alpha^2 D h(1) = - a$$

which is a non - homogeneous boundary value problem for a homogeneous equation.

The problem P'_3 has the following variational formulation

$$b(v,w) = (q,w)_\circ \qquad \forall w \in V, \quad v \in V, \quad q \in L^2(I) \tag{3}$$

where $(\cdot,\cdot)_\circ$ denotes the scalar product in $L^2(I)$.

Since $b(v,v)$ is coercive on V, there exists a positive compact operator G such that

$$v = G\,q \tag{4}$$

which has the integral representation

$$G\,q = \int_I g(x,t)\, q(t)\, dt \tag{5}$$

The symmetric kernel $g(x,t)$ is given by

$$g(x,t) = \begin{cases} c\, g_1(x,t) & x \leq t \\ c\, g_2(x,t) & x \geq t \end{cases} \tag{6}$$

with $g_2(x,t) = g_1(x,t)$. For $n \geq 0$ we have

$$\begin{aligned} g_1(x,t) = {} & (1+\alpha\varepsilon)(2e^{\alpha(1-x)} + 2e^{\alpha(1-t)} - e^{\alpha(1-x-t)} - e^{\alpha(1+x-t)} + 2e^{\alpha}(\alpha x-1)) \\ & + (1-\alpha\varepsilon)(-2e^{-\alpha(1-x)} - 2e^{-\alpha(1-t)} + e^{-\alpha(1-x-t)} + e^{-\alpha(1+x-t)} + 2e^{-\alpha}(\alpha x+1)) \end{aligned} \tag{7}$$

$$c^{-1} = 2\,\alpha^3\,(e^{\alpha}(1+\alpha\varepsilon) + e^{-\alpha}(1-\alpha\varepsilon))$$

and for $n < 0$,

$$g_1(x,t) = \varepsilon\beta\ (2\cos\beta(x-1)+2\cos\beta(t-1)-2\cos\beta-\cos\beta(x+t-1))-$$

$$-\ (2\sin\beta(x-1)+2\sin\beta(t-1)+2\sin\beta-\sin\beta(x+t-1))- \tag{9}$$

$$-\ \varepsilon\beta\ (2\beta x\sin\beta+\cos\beta(x-t+1)) + (2\beta x\cos\beta-\sin\beta(x-t+1))$$

$$c^{-1} = \beta^3\ (\ 2\varepsilon\beta\ \sin\beta - 2\ \cos\beta\ ) \tag{10}$$

The problem P_3'' has the variational formulation

$$b\ (\ h,w\ ) = a\ (\ w(1) + \varepsilon\ D\ w(1)\ ) \qquad \forall\ w \in V,\ \ a \in R \tag{11}$$

If we adopt the functional notation for the Dirac δ distribution, it follows by definition

$$a\ (\ w(1) + \varepsilon\ D\ w(1)\ ) = (\ a\ \delta(1), w + \varepsilon\ D\ w\ )_\circ = (\ a\ (\ \delta(1) - \varepsilon\ D\ \delta(1)\ ), w\ )_\circ$$

so that (11) can be written in the form

$$b\ (\ h,w\ ) = (\ (\ \delta(1) - \varepsilon\ D\ \delta(1)\ )\ a, w\ )_\circ \tag{12}$$

Hence, from (4), we get

$$h\ (x) = G\ (\ \delta(1) - \varepsilon\ D\ \delta(1)\ )\ a \tag{13}$$

If we define $\mu = \lambda^{-1}$ and substitute (4) and (13) in (2) and (1), taking account of (5), we obtain

$$\begin{aligned} \mu\, q &= m \int_I g(x,t)\, q(t)\, dt + m\,(\, g(x,1) + \varepsilon\, D_t\, g(x,1)\,)\, a \\ \mu\, a &= M\,(\int_I g(1,t)\, q(t)\, dt + \varepsilon \int_I D_x\, g(1,t)\, q(t)\, dt\,) + \\ &\quad + M\,(\, g(1,1) + \varepsilon\, D_t\, g(1,1) + \varepsilon\, D_x\, g(1,1) + \varepsilon^2\, D_x D_t\, g(1,1)\,)\, a \end{aligned} \tag{14}$$

We define on the set I x I the measure $d\eta$ to be the product $dt \times d\delta(1)$ of the Lebesgue and Dirac measures, and the function

$$z = (\,\bar{q},\bar{a}\,) \qquad \bar{q} = q\, m^{-\frac{1}{2}}, \quad \bar{a} = a\, M^{-\frac{1}{2}}$$

Hence the eigenvalue problem (14) can be written in the form

$$\Gamma\, z = \int_{I\times I} \gamma\,(\,x,t\,)\, z\,(\,t\,)\, d\eta = \mu\, z \tag{15}$$

where $\gamma(x,t)$ is the symmetric matrix kernel

$$\gamma(x,t) = \begin{cases} \gamma_{11}(x,t) = m\, g(x,t) \\ \gamma_{12}(x,t) = (mM)^{\frac{1}{2}}\,(g(x,t) + \varepsilon D_t g(x,t)) \\ \gamma_{21}(x,t) = (mM)^{\frac{1}{2}}\,(g(x,t) + \varepsilon D_x g(x,t)) \\ \gamma_{22}(x,t) = M\,(g(x,t) + \varepsilon(D_x g(x,t) + D_t g(x,t)) + \varepsilon^2 D_x D_t g(x,t)) \end{cases} \tag{16}$$

The integral operator Γ has a countable sequence of positive eigenvalues $\mu_h = \lambda_h^{-1}$ converging to zero.

4. Computation of an orthogonal invariant

If we denote by λ_h' the upper bounds to the λ_h obtained by the Rayleigh - Ritz method, the lower bounds λ_h'' can be computed when an orthogonal invariant of Γ is known (2). We shall consider here the simplest one that is

$$J = \operatorname{tr} \Gamma = \sum_{i=1}^{\infty} \mu_i$$

Then we have

$$\lambda_h'' = \left(J - \sum_{i=1}^{\nu} (\lambda_i')^{-1} + (\lambda_h')^{-1} \right)^{-1} \qquad (h=1,\ldots\nu)$$

The trace J of Γ is given by

$$J = \operatorname{tr} \Gamma = \int_{I\times I} \gamma(x,x)\, d\eta = m \int_I g(x,x)\, dx + M\,(g(1,1) + \varepsilon(D_x g(1,1) +$$

$$+ D_t g(1,1)) + \varepsilon^2 D_x D_t g(1,1)) = m\, J_1 + M\, J_2$$

For $n > 0$ we have

$$J_1 = c\,((1+\alpha\varepsilon)(e^{\alpha}\,(2\alpha^2-6\alpha+7)+e^{-\alpha}-8)+(1-\alpha\varepsilon)(e^{-\alpha}\,(2\alpha^2+6\alpha+7)+e^{\alpha}-8))$$

$$c^{-1} = 4\,\alpha^4\,(e^{\alpha}\,(1+\alpha\varepsilon) + e^{-\alpha}\,(1-\alpha\varepsilon))$$

$$J_2 = 4\alpha c\,((1+\alpha\varepsilon)(e^{\alpha}\,(\alpha\varepsilon+\alpha-1)) + (1-\alpha\varepsilon)(e^{-\alpha}\,(\alpha\varepsilon+\alpha+1)))$$

For $n < 0$ we have

$$J_1 = d\,(\sin\beta\,(\beta^3 - 3\beta\varepsilon + 3\beta) + \cos\beta\,(4 + \beta^2(3\varepsilon - 1)) - 4) \tag{23}$$

$$d^{-1} = 2\,\beta^4\,(\cos\beta - \beta\varepsilon\sin\beta) \tag{24}$$

$$J_2 = 2\beta d\,(\sin\beta\,(1 - \beta^2(\varepsilon + \varepsilon^2)) - \beta\cos\beta) \tag{25}$$

We notice that for $\alpha = \beta = 0$ it results

$$J_1 = \frac{1}{12} \qquad\qquad J_2 = \frac{1}{3} + \varepsilon + \varepsilon^2$$

A numerical example developed for $\nu = 30$ and

$$\ell_\circ = \ell_1 = 1 \qquad m_\circ = \bar{m} \qquad M = 0.21758 \qquad \varepsilon = 0.141 \qquad \alpha = \beta = 0$$

has given the following results

$$1.52409051 \leq \lambda_1 \leq 1.52409296$$

$$3.80634097 \leq \lambda_2 \leq 3.80648190$$

$$6.51181750 \leq \lambda_3 \leq 6.51388420$$

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ACCADEMIA NAZIONALE DEI LINCEI
RENDICONTI DELLA CLASSE
DI SCIENZE FISICHE, MATEMATICHE E NATURALI

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Sulla soluzione di problemi strutturali in presenza di legami costitutivi unilaterali

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Meccanica. — *Sulla soluzione di problemi strutturali in presenza di legami costitutivi unilaterali* (*). Nota (**) di GIOVANNI ROMANO (***) e MANFREDI ROMANO (***), presentata dal Corrisp. E. GIANGRECO.

SUMMARY. — A general model for the analysis of a class of structural problems with unilateral constitutive relations is formulated and its consistency is proved. Two iterative methods for the numerical solution of this class of problems are considered and their convergence properties are analyzed. Applications to problems of relevant interest in structural engineering are briefly discussed.

1. PREMESSA

Nel campo dell'ingegneria strutturale è di grande rilievo il problema del calcolo di strutture costituite da materiali il cui comportamento è descritto con soddisfacente approssimazione da un modello di tipo unilaterale.

In tale classe di problemi rientra ad esempio il calcolo di strutture in conglomerato (armato o meno) quando la fessurazione a trazione del materiale gioca un ruolo rilevante nella determinazione dello stato tensionale.

Di grande interesse tecnico è inoltre l'analisi delle sollecitazioni e dei conseguenti quadri fessurativi nelle strutture in muratura (archi, volte).

Un'altra vasta problematica che conduce a formulare modelli strutturali con comportamento costitutivo di tipo unilaterale è quella relativa alla risposta incrementale di strutture in campo elastico – perfettamente plastico.

L'analisi di tale classe di problemi viene affrontata in questa Nota formulando un modello generale che consente di includere in una trattazione unitaria una varietà di problemi applicativi. Con riferimento a tale modello si fornisce una dimostrazione dei risultati concernenti l'esistenza e l'unicità della soluzione.

Si mostra che l'esistenza di una soluzione è legata al rispetto di una condizione di compatibilità sui carichi, espressa in termini di lavoro virtuale.

L'analisi di tale questione con riferimento al problema generale del minimo di un funzionale quadratico su di un convesso è stata affrontata per la prima volta da G. Fichera che ha fornito i risultati fondamentali di esistenza e di unicità [1].

Nel presente contesto si è seguita una via dimostrativa diretta, considerando il problema di minimo del funzionale non lineare dell'energia potenziale del sistema. È da rilevare che si fa riferimento a modelli strutturali

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discretizzati per i quali lo spazio delle configurazioni ammissibili è di dimensione finita. L'estensione dei risultati di esistenza ed unicità al caso del continuo è un problema tutt'ora aperto.

Per il calcolo numerico della soluzione di problemi strutturali con legame costitutivo di tipo unilaterale si può applicare un metodo di rilassamento di tipo classico, consistente nell'imporre alternativamente il rispetto della condizione di equilibrio e del legame costitutivo.

Applicazioni di tale metodo sono state proposte da M. Capurso [2] per i problemi elasto-plastici incrementali e dagli autori [3] nel caso della pressoflessione nei materiali non resistenti a trazione.

In questa Nota si presenta la formulazione del metodo con riferimento al modello generale considerato e se ne fornisce la dimostrazione della convergenza.

È da osservare che la rapidità di convergenza risulta però in molti casi insoddisfacente. Tale inconveniente è in generale tanto più accentuato quanto più rilevante è il comportamento unilaterale del sistema.

Si propone quindi un nuovo metodo iterativo, detto di rilassamento geometrico, e si dimostra che ogni successione convergente generata dall'algoritmo tende alla soluzione del problema unilaterale.

Tale metodo, pur non avendo un campo di applicabilità che comprende tutti i problemi unilaterali descritti dal modello generale, costituisce un valido strumento di calcolo numerico. Esso può infatti essere applicato con successo ad una vasta classe di problemi strutturali, mostrando in ogni caso una grande rapidità di convergenza.

A conclusione della Nota si discutono brevemente due esempi di applicazione a problemi di particolare interesse nell'ingegneria strutturale.

2. Formulazione del modello

Sia V lo spazio lineare degli spostamenti ammissibili della struttura, che assumeremo di dimensione finita, e V′ lo spazio duale dei carichi agenti su di essa. Con W e W′ si denotano lo spazio di Hilbert delle deformazioni e quello duale degli stati tensionali. Indicando con

$$\mathrm{def} : \mathrm{V} \to \mathrm{W}$$

l'operatore lineare che associa ad ogni spostamento $u \in \mathrm{V}$ la deformazione corrispondente $\mathrm{def}\, u \in \mathrm{W}$, il legame costitutivo è definito dalle relazioni:

$$(1) \qquad \sigma = \mathrm{S}\,(\mathrm{def}\, u - \varepsilon) \qquad u \in \mathrm{V} \qquad \sigma \in \mathrm{W}' \qquad \varepsilon \in \mathrm{K}$$

$$(\sigma\,, \eta - \varepsilon) \leq 0 \qquad \forall \eta \in \mathrm{K}$$

dove K è un cono chiuso e convesso di $\mathrm{W}, \mathrm{S} : \mathrm{W} \to \mathrm{W}'$ è un operatore costitutivo del tipo elastico (cioè simmetrico e coercivo) e $(\, , \,)$ denota il prodotto scalare in W.

La legge costitutiva (1) è equivalente al problema di minima distanza:

$$\min_{\varepsilon \in K} \| \operatorname{def} u - \varepsilon \|_S = \| \operatorname{def} u - \Pi \operatorname{def} u \|_S$$

dove $\| \cdot \|_S$ denota la norma indotta da S in W, che si dirà la norma in energia, e Π è il proiettore ortogonale in energia di W su K.

La condizione di equilibrio è espressa in termini di lavoro virtuale da

$$(2) \qquad (\sigma, \operatorname{def} v) = l(v) \qquad \forall v \in V \qquad l \in V'$$

dove $l(v)$ è il lavoro virtuale dei carichi.

La relazione costitutiva (1) e la condizione di equilibrio (2) costituiscono la formulazione variazionale del problema di minimo

$$(P1) \qquad \min \{ \Phi(u, \varepsilon) / u \in V, \varepsilon \in K \}$$

dove

$$\Phi(u, \varepsilon) = \tfrac{1}{2} \| \operatorname{def} u - \varepsilon \|_S^2 - l(u) \, .$$

Considerando il funzionale non lineare

$$\Psi(u) = \tfrac{1}{2} \| \operatorname{def} u - \Pi \operatorname{def} u \|_S^2 - l(u)$$

che rappresenta l'energia potenziale del sistema, si può mostrare che il problema P1 è equivalente al seguente problema di minimo non condizionato

$$(P2) \qquad \min \{ \Psi(u) / u \in V \} \, .$$

A tal fine si noti che se (u_0, ε_0) è soluzione del problema P1 deve aversi $\varepsilon_0 = \Pi \operatorname{def} u_0$ e quindi $\Psi(u_0) = \Phi(u_0, \varepsilon_0)$. Se esistesse un $\bar{u} \in V$ tale che $\Psi(\bar{u}) < \Psi(u_0)$, ponendo $\bar{\varepsilon} = \Pi \operatorname{def} \bar{u}$, si avrebbe $\Phi(\bar{u}, \bar{\varepsilon}) = \Psi(\bar{u}) < \Psi(u_0) = \Phi(u_0, \varepsilon_0)$ contro l'ipotesi, e quindi u_0 è soluzione di P2.

Viceversa sia $\bar{u}$ soluzione del problema P2. Se esistesse (u_0, ε_0) tale che $\Phi(u_0, \varepsilon_0) < \Phi(\bar{u}, \bar{\varepsilon})$ risulterebbe $\Psi(u_0) = \Phi(u_0, \varepsilon_0) < \Phi(\bar{u}, \bar{\varepsilon}) = \Psi(\bar{u})$ contro l'ipotesi, e quindi $(\bar{u}, \bar{\varepsilon})$ è soluzione di P1.

Dimostriamo ora il seguente:

TEOREMA DI ESISTENZA. Detto $V_k = \{ u \in V : \operatorname{def} u \in K \}$ *il cono chiuso e convesso degli spostamenti ammissibili cui corrisponde uno stato tensionale nullo, il problema* P2 *ammette soluzione se il carico verifica la condizione di compatibilità*:

$$(C1) \qquad l(u) \leq 0 \qquad \forall u \in V_k$$

$$l(u) = 0 \iff u \in R$$

ove $R = \{ u \in V_k : \operatorname{def} u = 0 \}$ *è il sottospazio degli spostamenti rigidi ammissibili.*

Si premette un risultato tecnico:

LEMMA. *Detto* $\tilde{P}$ *il proiettore ortogonale su* R *e* $\tilde{Q} = I - \tilde{P}$ *il proiettore complementare, si ha*

$$\| \tilde{Q} u_n \| \to +\infty \;\Rightarrow\; \max \lim \Psi(u_n) = +\infty$$

Dimostrazione. Si consideri una successione di versori $w_n = \| \tilde{Q}u_n \|^{-1} \tilde{Q}u_n$ convergente al limite w_0. Si distinguono due casi:

i) $w_0 \in V_k$.

Essendo $\tilde{P}w_0 = \lim \tilde{P}w_n = 0$ si ha $w_0 \in R$ e quindi $l(w_0) < 0$. Ne segue che:

$$\lim \Psi(\tilde{Q}u_n) \geq - \lim \| \tilde{Q}u_n \| \, l(w_0) = + \infty$$

ii) $w_0 \bar{\in} V_k$.

In tal caso si ha ancora

$$\lim \Psi(\tilde{Q}u_n) = \lim \| \tilde{Q}u_n \|^2 \{ \tfrac{1}{2} \| \operatorname{def} w_0 - \Pi \operatorname{def} w_0 \|_S^2 - \| \tilde{Q}u_n \|^{-1} l(w_n) \} = + \infty \; .$$

in quanto $\lim (\operatorname{def} w_n - \Pi \operatorname{def} w_n) = \operatorname{def} w_0 - \Pi \operatorname{def} w_0 \neq 0$.

Osservando infine che $\Psi(u) = \Psi(\tilde{Q}u) \; \forall u \in V$, si ottiene il risultato voluto.

Per conseguire la dimostrazione del teorema di esistenza si consideri una successione $\{u_n\}$ minimizzante per $\Psi(u)$, cioè tale che

$$\lim \Psi(u_n) = \inf \{ \Psi(u) / u \in V \} .$$

In virtù del Lemma precedente la successione $\{\tilde{Q}u_n\}$ è limitata. Se ne può dunque estrarre una convergente $\{\tilde{Q}u_{n_k}\} \to u_0$. Per la continuità di Ψ si ha:

$$\lim \Psi(u_{n_k}) = \lim \Psi(Qu_{n_k}) = \Psi(u_0) = \inf \{ \Psi(u) / u \in V \}$$

e quindi u_0 è soluzione del problema P2.

È peraltro facile verificare che per l'esistenza di una soluzione del problema P2 è necessario che:

$$l(u) \leq 0 \qquad \forall u \in V_k .$$

Infatti se esistesse un $u^* \in V_k$ tale che $l(u^*) > 0$, si avrebbe:

$$\lim_{\alpha \to +\infty} \Psi(\alpha u^*) = - \infty$$

Per quanto concerne l'unicità della soluzione sussiste il seguente

TEOREMA DI UNICITÀ. *Il problema* P2 *ammette un'unica soluzione in termini di stato tensionale.*

Si osservi infatti che, denotando con z la coppia $(u, \varepsilon) \in V \times W$, il funzionale $\Phi(u, \varepsilon)$ può scriversi nella forma $\frac{1}{2} b(z, z) - l(u)$, dove $b(z, z) = \| \operatorname{def} u - \varepsilon \|_S^2$.

Se $z_0 = (u_0, \varepsilon_0)$ e $z_0' = (u_0', \varepsilon_0')$ sono soluzioni del problema P2, si ha:

$$b(z_0, z - z_0) \geq l(u - u_0)$$

$$b(z_0', z - z_0') \geq l(u - u_0')$$

da cui $b(z_0 - z_0', z_0 - z_0') = 0$ e quindi essendo $\varepsilon_0 = \Pi \operatorname{def} u_0$ ed $\varepsilon_0' = \Pi \operatorname{def} u_0'$ dalla definizione della forma quadratica b segue che:

$$\operatorname{def} u_0 - \Pi \operatorname{def} u_0 = \operatorname{def} u_0' - \Pi \operatorname{def} u_0' .$$

Denotando con $\sigma(u) = S(\operatorname{def} u - \Psi \operatorname{def} u)$ lo stato tensionale associato al campo di spostamenti u, si ha infine

$$\sigma(u_0) = \sigma(u_0')$$

3. METODI DI CALCOLO

Nel formulare metodi di calcolo si assume, senza ledere la generalità, che non vi siano spostamenti rigidi ammissibili.

Il problema unilaterale si scrive allora nella forma:

$$\text{(P)} \qquad \min\{\Phi(u, \varepsilon)/u \in V, \varepsilon \in K\} = \min\{\Psi(u)/u \in V\}$$

con la condizione di compatibilità

$$\text{(C)} \qquad l(u) < 0 \qquad \forall u \in V_k, \qquad u \neq 0 .$$

La soluzione del problema P può essere ottenuta con un metodo di rilassamento di tipo classico, consistente nell'imporre alternativamente la condizione di equilibrio ed il rispetto del legame costitutivo, secondo lo schema iterativo:

$$\min_{u \in V} \Phi(u, \varepsilon_n) = \Phi(u_{n+1}, \varepsilon_n)$$

$$\min_{\varepsilon \in K} \Phi(u_{n+1}, \varepsilon) = \Phi(u_{n+1}, \varepsilon_{n+1}) = \Psi(u_{n+1}) .$$

Poiché $\varepsilon_n = \Pi \operatorname{def} u_n$, l'algoritmo A tale che $u_{n+1} = A u_n$ è definito dal problema di minimo:

$$\text{(3)} \qquad \min_{u \in V} \Phi(u, \Pi \operatorname{def} u_n) = \Phi(u_{n+1}, \Pi \operatorname{def} u_n) .$$

Si vuole ora mostrare che ogni punto di compattezza della successione $\{u_n\}$ generata dall'algoritmo A è soluzione del problema P. A tal fine si osservi che sussistono le diseguaglianze:

$$\text{(4)} \qquad \Psi(u) = \Phi(u, \Pi \operatorname{def} u) \geq \Phi(Au, \Pi \operatorname{def} u) \geq$$
$$\geq \Phi(Au, \Pi \operatorname{def} Au) = \Psi(Au) .$$

La successione $\{\Psi(u_n)\}$ è dunque non crescente e poiché, in virtù del lemma del par. 2, $\|u_n\| \to +\infty \Rightarrow \lim \Psi(u_n) = +\infty$, ne segue che la successione $\{u_n\}$ è limitata.

L'algoritmo A è inoltre di discesa stretta per il funzionale Ψ in quanto

$$\Psi(\mathrm{A}u) = \Psi(u) \quad \Rightarrow \quad u = \mathrm{A}u = u_0$$

dove u_0 è soluzione del problema P.

Infatti, se $\Psi(\mathrm{A}u) = \Psi(u)$, dalla (4) segue che

$$\Phi(\mathrm{A}u, \Pi \operatorname{def} u) = \Phi(u, \Pi \operatorname{def} u). \tag{5}$$

Osservando che, per la definizione dell'algoritmo A,

$$\min_{v \in \mathrm{V}} \Phi(v, \Pi \operatorname{def} u) = \Phi(\mathrm{A}u, \Pi \operatorname{def} u). \tag{6}$$

e che tale problema di minimo ammette un'unica soluzione, dalle (5) e (6) si deduce che $\mathrm{A}u = u$ e

$$\min_{v \in \mathrm{V}} \Phi(\mathrm{V}, \Pi \operatorname{def} u) = \Phi(u, \Pi \operatorname{def} u). \tag{7}$$

La (7) è equivalente alla condizione variazionale di equilibrio

$$(\mathrm{S}(\operatorname{def} u - \Pi \operatorname{def} u), \operatorname{def} v) = l(v) \qquad \forall v \in \mathrm{V}.$$

Essendo inoltre soddisfatto il legame costitutivo con $\varepsilon = \Pi \operatorname{def} u$ ne segue che $u = \mathrm{A}u = u_0$.

Dalla monotonia della $\{\Psi(u_n)\}$, per ogni estratta $\{u_{n_k}\}$, si ha inoltre

$$\inf \Psi(u_{n_k}) = \lim \Psi(u_{n_k}) \leq \lim \Psi(u_k) = \inf \Psi(u_k) \leq \inf \Psi(u_{n_k})$$

e dunque

$$\lim \Psi(u_{n_k}) = \lim \Psi(u_n). \tag{8}$$

Se $\{u_{n_k}\} \to \bar{u}$, per la continuità di A, $\{\mathrm{A}u_{n_k}\} \to \mathrm{A}\bar{u}$ e quindi dalla (8):

$$\lim \Psi(u_{n_k}) = \Psi(\bar{u}) = \lim \Psi(\mathrm{A}u_{n_k}) = \Psi(\mathrm{A}\bar{u}).$$

Pertanto $\bar{u}$ è soluzione del problema P. Si può concludere che ogni successione convergente estratta dalla $\{u_n\}$ tende ad una soluzione del problema P. In particolare, se tale problema ammette un'unica soluzione u_0, la successione $\{u_n\}$ converge ad u_0.

Un pregio del metodo di calcolo descritto è quello di essere applicabile a qualsiasi problema che rientri nel modello generale. Purtroppo però la sua rapidità di convergenza risulta tanto più insoddisfacente quanto più accentuato è il comportamento unilaterale. A causa di ciò, proprio in molti casi di grande interesse in ingegneria strutturale, esso richiede un notevole tempo di calcolo e talvolta risulta praticamente inapplicabile.

Si espone ora un nuovo metodo iterativo, detto di rilassamento geometrico, che, pur non essendo generale come il precedente, può essere applicato con successo ad una vasta classe di problemi strutturali. In tali casi esso è sempre caratterizzato da una grande rapidità di convergenza.

Con riferimento al problema P, per ogni $u \in V$ sia definito un proiettore ortogonale in energia $P(u)$ di W su un sottospazio $W(u)$, e tale che

$$P(u) \operatorname{def} u = \Pi \operatorname{def} u \qquad \forall u \in V .$$

Si consideri quindi il problema di minimo

(9) $$\min_{u \in V} \tfrac{1}{2} \| \operatorname{def} u - P(u_n) \operatorname{def} u \|_S^2 - l(u)$$

e si definisca il sottospazio

$$V_n = \{u \in V : \operatorname{def} u = P(u_n) \operatorname{def} u\} .$$

Il problema (9) ammette un'unica soluzione u_{n+1} se e solo se $V_n = \{0\}$. Si osservi che $V_n = \{0\} \Rightarrow P(u_n) \neq I$, dove I è l'identità su W.

Mostriamo ora che

(10) $$P(u_n) \neq I \Rightarrow P(u_{n+1}) \neq I .$$

Infatti, essendo u_{n+1} soluzione del problema (9), se $P(u_n) \neq I$, si ha

(11) $$l(u_{n+1}) = \| \operatorname{def} u_{n+1} - P(u_n) \operatorname{def} u_{n+1} \|_S^2 > 0 .$$

Se fosse $P(u_{n+1}) = I$, risulterebbe

$$\operatorname{def} u_{n+1} = P(u_{n+1}) \operatorname{def} u_{n+1} = \Pi \operatorname{def} u_{n+1} ,$$

cioè $u_{n+1} \in V_k - \{0\}$.

Per la condizione di compatibilità C si avrebbe allora $l(u_{n+1}) < 0$, il che contraddice la (11).

Si supponga ora che

(12) $$P(u_n) \neq I \Rightarrow V_n = \{0\} \qquad \forall n .$$

Dalla (10) e dalla (12) segue che, scelto u_1 tale che $P(u_1) \neq I$, risulta

$$V_n = \{0\} \qquad \forall n .$$

Se è verificata l'ipotesi (12) il problema (9) definisce quindi l'algoritmo

$$u_{n+1} = A u_n .$$

Si vuole ora dimostrare che, se la successione $\{u_n\}$ generata dall'algoritmo A converge, il suo limite $\bar{u}$ è soluzione del problema P. Si ha infatti

$$\min_{u \in V} \tfrac{1}{2} \| \operatorname{def} u - P(u_n) \operatorname{def} u \|_S^2 - l(u) =$$
$$= \tfrac{1}{2} \| \operatorname{def} u_{n+1} - P(u_n) \operatorname{def} u_{n+1} \|_S^2 - l(u_{n+1})$$

e, passando al limite,

(13) $$\min_{u \in V} \tfrac{1}{2} \| \operatorname{def} u - P(\bar{u}) \operatorname{def} u \|_S^2 - l(u) =$$
$$= \tfrac{1}{2} \| \operatorname{def} \bar{u} - P(\bar{u}) \operatorname{def} \bar{u} \|_S^2 - l(\bar{u}) .$$

Poiché $P(\bar{u})\,\mathrm{def}\,\bar{u} = \Pi\,\mathrm{def}\,\bar{u}$, la (13) è equivalente alla condizione variazionale di equilibrio

$$S(\mathrm{def}\,\bar{u} - \Pi\,\mathrm{def}\,\bar{u})\,,\,\mathrm{def}\,v) = l(v) \qquad \forall v \in V$$

ed essendo verificato il legame costitutivo con $\varepsilon = \Pi\,\mathrm{def}\,\bar{u}$, ne segue che $\bar{u}$ è soluzione del problema P.

4. Analisi di strutture costituite da materiali non resistenti a trazione

Si considera, a titolo d'esempio il problema del calcolo di strutture monodimensionali piane soggette a flessione composta e costituite da materiale elastico lineare a compressione e non resistente a trazione.

Il legame costitutivo è pertanto definito, in ogni punto x della generica sezione trasversale, delle relazioni

$$\sigma(x) = E\,\mathrm{def}\,u(x) \qquad \text{per} \quad \mathrm{def}\,u(x) \leq 0$$

$$\sigma(x) = 0 \qquad \text{per} \quad \mathrm{def}\,u(x) \geq 0$$

dove σ è la tensione normale, E il modulo di elasticità in compressione, u un campo di spostamenti ammissibili e $\mathrm{def}\,u$ la corrispondente dilatazione nella direzione dell'asse della struttura.

Nell'ambito della teoria tecnica della trave, si considera valido il principio di conservazione delle sezioni piane.

Identificando gli spazi W e W' delle deformazioni e delle tensioni con lo spazio $L^2(\Omega)$ delle funzioni di quadrato sommabile nel dominio Ω della struttura, il legame costitutivo è espresso in termini globali dalle relazioni:

$$\sigma = E(\mathrm{def}\,u - \delta) \qquad u \in V \qquad \delta \in K$$

$$(\sigma\,,\,\varepsilon - \delta) < 0 \qquad \forall \varepsilon \in K$$

dove V è lo spazio di dimensione finita degli spostamenti ammissibili e K è il cono chiuso e convesso delle dilatazioni positive.

Simulando la fessurazione a trazione del materiale col campo di dilatazioni anelastiche $\delta \in K$, si riconosce quindi che il modello strutturale in esame costituisce un caso particolare di quello generale formulato in questa Nota.

Ad esso possono dunque applicarsi i metodi di calcolo considerati. Per quanto concerne il metodo di rilassamento geometrico si osservi che $P(u)$ è in questo caso definito come il proiettore ortogonale sul sottospazio $W(u)$ costituito dai campi di dilatazione il cui supporto coincide con quello della parte positiva di $\mathrm{def}\,u$.

L'algoritmo iterativo corrispondente a tale definizione del proiettore consiste nel considerare, ad ogni passo, il problema dell'equilibrio elastico di una struttura a «geometria variata» coincidente con quella parte della struttura data che risulta compressa al passo precedente.

Esempi di applicazione numerica al problema della pressoflessione [4] ed al calcolo delle sollecitazioni in strutture ad arco [5], nel caso di materiale non resistente a trazione, mostrano una rapida convergenza dell'algoritmo di rilassamento geometrico. In effetti una soluzione sufficientemente approssimata si ottiene in $4 \sim 6$ passi di iterazione. È da rilevare che il calcolo della soluzione degli stessi problemi strutturali, affrontato con il metodo di rilassamento classico, richiede in genere un numero di passi di iterazione da 10 a 20 volte superiore.

5. ANALISI INCREMENTALE DI STRUTTURE ELASTOPLASTICHE

La risposta incrementale di una struttura in campo elastico–perfettamente plastico è descritta dalle relazioni costitutive

$$\dot{\sigma} = \mathrm{S}\,(\mathrm{def}\,\dot{u} - \dot{\varepsilon}_p) \qquad \dot{u} \in \mathrm{V},\ \dot{\varepsilon}_p \in \mathrm{K}$$

$$(\sigma\,,\,\dot{\varepsilon} - \dot{\varepsilon}_p) \leq \sigma \qquad \forall \dot{\varepsilon} \in \mathrm{K}$$

e dalla condizione di equilibrio

$$(\dot{\sigma}\,,\,\mathrm{def}\,\dot{v}) = l\,(\dot{v}) \qquad \forall \dot{v} \in \mathrm{V}$$

dove S è la rigidezza elastica e K è il cono chiuso e convesso definito localmente dalle seguenti proprietà. Sia $\mathrm{D}\,(x)$ il dominio elastico (chiuso e convesso nello spazio degli stati tensionali) nel punto x della struttura e $\partial\mathrm{D}\,(x)$ la sua frontiera. Allora:

$\mathrm{K}\,(x) = \{0\}$ se $\sigma\,(x)$ è interno a $\mathrm{D}\,(x)$,

$\mathrm{K}\,(x)$ è il cono delle normali esterne a $\mathrm{D}\,(x)$ nel punto $\sigma\,(x)$ se $\sigma\,(x) \in \partial\mathrm{D}\,(x)$.

Si riconosce pertanto che il problema dell'analisi incrementale di una struttura con comportamento elastico–perfettamente plastico rientra nel modello generale considerato in questa Nota e può formularsi come problema di minimo

$$\min \{\tfrac{1}{2} \|\,\mathrm{def}\,\dot{u} - \Pi\,\mathrm{def}\,\dot{u}\,\|_{\mathrm{S}}^2 - l\,(\dot{u}) / \dot{u} \in \mathrm{V}\}\,.$$

Per applicare a questo problema l'algoritmo solutivo fornito dal metodo di rilassamento geometrico si considera il proiettore $\mathrm{P}\,(\dot{u})$ di W sul sottospazio $\mathrm{W}\,(\dot{u})$ generato da $\Pi\,\mathrm{def}\,\dot{u}$, definito dal problema di minima distanza

$$\min_{\alpha} \|\,\mathrm{def}\,\dot{v} - \alpha\,\Pi\,\mathrm{def}\,\dot{u}\,\|_{\mathrm{S}} = \|\,\mathrm{def}\,\dot{v} - \mathrm{P}\,(\dot{u})\,\mathrm{def}\,\dot{u}\,\|_{\mathrm{S}}$$

con α reale.

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SUL CALCOLO DELLE STRUTTURE AD ARCO NON RESISTENTI A TRAZIONE

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SUL CALCOLO DELLE STRUTTURE AD ARCO NON RESISTENTI A TRAZIONE

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SOMMARIO

Si considera il problema della determinazione dello stato tensionale in strutture ad arco costituite da materiali non resistenti a trazione. Si osserva che la presenza di zone fessurate può modificare in modo rilevante la distribuzione delle caratteristiche della sollecitazione rispetto a quella determinata con una soluzione elastica. Per la soluzione del problema strutturale, in presenza di un legame costitutivo unilaterale, si applicano due metodi di rilassamento proposti dagli autori in un precedente lavoro. Alcuni esempi numerici per archi circolari e parabolici consentono il confronto dei metodi di calcolo e mostrano un caratteristico effetto di adattamento della struttura in conseguenza della fessurazione.

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I. Introduzione

In ingegneria strutturale è di grande interesse l'analisi dello stato tensionale in strutture realizzate con materiali, quali il conglomerato e le murature, che hanno una trascurabile resistenza a trazione.
Un classico esempio è costituito dal problema della determinazione delle tensioni normali nella sollecitazione di pressoflessione. Un' analisi generale di tale problema e la discussione di metodi iterativi per il calcolo numerico della soluzione sono state presentate in [1] , [2] e [3] .
E' da rilevare peraltro che, nella letteratura tecnica, ai fini della valutazione delle caratteristiche della sollecitazione in strutture iperstatiche, si assume sempre un comportamento elastico lineare del materiale. Tale ipotesi è a priori inaccettabile, in quanto la presenza di zone fessurate modifica la geometria della struttura effettivamente reagente e può quindi alterare in modo rilevante la distribuzione delle caratteristiche della sollecitazione.
Un' analisi corretta del problema conduce a formulare un modello strutturale con legame costitutivo unilaterale che rientra in una tematica generale affrontata dagli autori in [4] .
Per il calcolo numerico della soluzione sono stati proposti ed analizzati in [4] due metodi di rilassamento. Il primo, di tipo classico, consiste nell'imporre alternativamente la condizione di equilibrio elastico ed il rispetto del legame costitutivo. Il secondo, detto di rilassamento geometrico, consiste nel considerare ad ogni passo del procedimento iterativo il problema dell'equilibrio elastico per una struttura a geometria variata coincidente con la parte della struttura data che risulta compressa al passo precedente.
In questa nota si considera in particolare il problema del calcolo delle sollecitazioni in strutture ad arco e si presentano alcuni esempi di calcolo numerico della soluzione.
Il confronto dei due metodi di calcolo mostra che il metodo di rilassamento geometrico è caratterizzato da una grande rapidità di convergenza e richiede un numero di passi di iterazione da 10 a 20 volte inferiore di quello richiesto, a parità di approssimazione, dal metodo di rilassamento di tipo classico.
Gli esempi numerici riportati pongono in evidenza inoltre che, nelle strutture pressoinflesse, la distribuzione reale delle caratteristiche della sollecitazione può essere notevolmente diversa da quella valutata assumendo il comportamento elastico dell'intera struttura.

Di conseguenza anche i quadri fessurativi ed i valori delle tensioni possono risultare molto diversi. E' da rilevare che nei casi in cui l'ipotesi di comportamento elastico conduce a determinare una linea delle pressioni non tutta interna all'arco, il che comporterebbe la non esistenza della soluzione, l'analisi corretta può mostrare che in realtà la linea delle pressioni è tutta interna all'arco. Infatti la fessurazione del materiale produce una ridistribuzione delle sollecitazioni e quindi l'adattamento della struttura.

2. Il problema strutturale

Si consideri una struttura ad arco la cui linea d'asse Γ sia contenuta in un piano e si denotino con s l'ascissa curvilinea su Γ e con $\Omega(s)$ la sezione all'ascissa s. Nel piano di $\Omega(s)$ sia definito un riferimento cartesiano $\{O,x,y\}$ con l'origine nel baricentro di $\Omega(s)$ e l'asse y nel piano cui appartiene Γ.
Si assume, nell'ambito della teoria tecnica della trave, che le sezioni trasversali rimangano piane a deformazione avvenuta.
Con σ ed ε si denoteranno rispettivamente le tensioni normali su Ω e le dilatazioni nella direzione della tangente a Γ.
Per materiali non resistenti a trazione, assumendo un comportamento elastico lineare a compressione, il legame costitutivo si scrive

$$\begin{aligned} \sigma &= 0 && \text{per} \quad \varepsilon \geq 0 \\ \sigma &= E\,\varepsilon && \text{per} \quad \varepsilon \leq 0 \end{aligned}$$

Il problema dell'equilibrio elastico della struttura si pone come problema di minimo del funzionale energia complementare :

$$\min \frac{1}{2E} \int_{\Gamma} \int_{\Omega} \sigma^2 \, d\Omega \, ds - R(\delta) \tag{1}$$

sotto le condizioni

$$\sigma \geq 0 \qquad \text{e} \qquad \sigma \in \Sigma$$

dove Σ denota la varietà delle tensioni normali in equilibrio con i carichi

ed $R(\delta)$ è il lavoro virtuale delle reazioni vincolari associate alle tensioni σ per i corrispondenti cedimenti δ.
L'esistenza di una soluzione del problema (1) è legata al rispetto di condizioni di compatibilità sui carichi. E' però da rilevare che una verifica a priori di queste condizioni non è possibile se non in alcuni semplici casi particolari. Nel formulare metodi di calcolo è pertanto necessario supporre che la soluzione esista e verificare quindi a posteriori tale ipotesi attraverso l'effettiva determinazione della soluzione.

3. Metodi di calcolo

Il calcolo della soluzione del problema strutturale considerato può essere conseguito mediante l'applicazione di due metodi di rilassamento che sono stati proposti e discussi in [4] con riferimento ad una classe generale di problemi strutturali in presenza di legami costitutivi di tipo unilaterale. In quella nota sono state discusse le proprietà di convergenza di tali metodi. Nel seguito se ne fornisce una formulazione esplicita con riferimento al calcolo delle strutture ad arco.

a) Il metodo di rilassamento classico
Il metodo consiste nel risolvere, ad ogni passo del procedimento iterativo, il problema dell'equilibrio elastico della struttura considerandola soggetta ai carichi assegnati e ad un campo di distorsioni coincidente con quello delle dilatazioni positive determinate al passo precedente.
Il procedimento può essere in ogni caso inizializzato assumendo nullo il campo delle distorsioni impresse.
Nel caso di strutture ad arco è conveniente formulare il problema dell'equilibrio elastico come problema di minimo del funzionale energia complementare, in quanto ciò conduce a risolvere un sistema di equazioni lineari di ordine pari al grado di iperstaticità della struttura. La formulazione in termini di minimo del funzionale energia potenziale richiederebbe infatti per la descrizione del campo di spostamenti ammissibili della struttura discretizzata, ed a parità di approssimazione, un numero di parametri indipendenti proporzionale a quello degli elementi nei quali è suddiviso l'arco. Ad ogni passo bisognerebbe risolvere quindi un sistema di equazioni lineari di ordine molto più elevato.

Si assumerà che la funzione $\varepsilon(x,y)$ è lineare in ogni sezione in quanto ciò rappresenta, nel caso delle usuali strutture ad arco, un'approssimazione senz'altro accettabile. Si può allora scrivere

$$\varepsilon\,(x,y) = ax + by + c = e \cdot \rho$$

dove

$$e = (\,a,\,b,\,c\,) \qquad\qquad \rho = (\,x,\,y,\,1\,)$$

L'algoritmo iterativo è definito dal problema di minimo

$$\min \frac{1}{2} \cdot \int_\Gamma \int_\Omega \frac{1}{E}\,(\,\sigma + E\,\varepsilon_n^+\,)^2\, d\Omega\;\; ds \;-\; R(\,\delta\,) \qquad \text{con}\;\; \sigma \in \Sigma \tag{2}$$

dove ε_n^+ è la parte positiva del campo di dilatazioni ε_n determinate al passo precedente.
Se N è il grado di iperstaticità della struttura, l'insieme dei campi di tensione autoequilibrati a cui è associata una dilatazione lineare su ogni sezione trasversale, costitutisce un sottospazio di dimensione N. Scelta allora una base $\{\sigma_i\}$ $i = 1,\ldots N$, ogni elemento è rappresentabile come combinazione lineare $X_i\sigma_i$ dove X_i sono le incognite iperstatiche.
La soluzione del problema (2) appartiene dunque alla varietà lineare dei campi di tensione del tipo

$$\sigma_{on} + X_i\,\sigma_i$$

dove σ_{on} è un campo di tensioni in equilibrio con i carichi e tale che la somma $\sigma_{on}/E + \varepsilon_n^+$ della dilatazione elastica e della distorsione impressa sia una funzione lineare su ogni sezione trasversale.
La formulazione variazionale del problema (2) è allora espressa dal seguente sistema lineare di ordine N

$$\int_\Gamma \int_\Omega (\,\sigma_{on}/E + \varepsilon_n^+ + X_i\,\sigma_i/E\,)\,\sigma_k\, d\Omega\;\; ds = R(\,\delta\,) \qquad k = 1,\ldots N \tag{3}$$

dove $R_k(\delta)$ è il lavoro virtuale delle reazioni vincolari associate al campo di tensioni autoequilibrate σ_k.

Introducendo le rappresentazioni

$$\begin{aligned} \sigma_{on}/E \; + \; \varepsilon_n^{+} \; &= \; c_{on} \cdot \rho \\ \sigma_i/E \; &= \; c_i \cdot \rho \end{aligned} \qquad (4)$$

si ha

$$\varepsilon_{n+1} \; = \; (\, c_{on} + X_i \, c_i \,) \cdot \rho$$

Per calcolare le incognite iperstatiche X_i, utilizzando le (4), il sistema (3) si pone nella forma :

$$\int_\Gamma \int_\Omega E \, (\, c_{on} + X_i \, c_i \,) \cdot \rho \, (\, c_k \cdot \rho \,) \, d\Omega \, ds \; = \; R_k(\delta) \qquad (5)$$

Poichê risulta

$$(\, c_{on} + X_i \, c_i \,) \cdot \rho \, (\, c_k \cdot \rho \,) \; = \; (\, \rho \otimes \rho \,) \, (\, c_{on} + X_i \, c_i \,) \cdot c_k$$

dove il simbolo $\otimes$ denota il prodotto tensoriale, definendo la matrice di inerzia de la sezione trasversale

$$J \; = \; \int_\Omega E \, \rho \otimes \rho \, d\Omega \; = \; \int_\Omega E \begin{vmatrix} x^2 & xy & x \\ xy & y^2 & y \\ x & y & 1 \end{vmatrix} d\Omega \; = \; \begin{vmatrix} I_x & I_{xy} & S_x \\ I_{xy} & I_y & S_y \\ S_x & S_y & A \end{vmatrix}$$

il sistema (5) si scrive

$$X_i \int_\Gamma J \, c_i \cdot c_k \, ds \; = \; - \int_\Gamma J \, c_{on} \cdot c_k \, ds \; + \; R_k(\delta) \qquad (6)$$

Si osservi che :

$$\begin{aligned} J \, c_i \; &= \; \int_\Omega E \, \rho \otimes \rho \, d\Omega \; c_i \; = \; \int_\Omega E \, (\, \rho \cdot c_i \,) \, \rho \, d\Omega \; = \\ &= \; \int_\Omega \sigma_i \, \rho \, d\Omega \; = \; (\, -M_{iy}, \, M_{ix}, \, N_i \,) \; = \; f_i \end{aligned} \qquad (7)$$

dove M_{ix}, M_{iy}, N_i sono rispettivamente i momenti flettenti e lo sforzo normale relativi al campo di tensioni σ_i.
Analogamente

$$J\, c_{on} = \int_\Omega \sigma_{on}\, \rho\, d\Omega + \int_\Omega E\, \varepsilon_n^+\, \rho\, d\Omega =$$

$$= (-M_{oy}, M_{ox}, N_o) + (-M_{ny}^+, M_{nx}^+, N_n^+) = f_o + f_n^+ \tag{8}$$

dove M_{ox}, M_{oy}, N_o sono le caratteristiche in equilibrio con i carichi e M_{nx}^+, M_{ny}^+, N_n^+ quelle corrispondenti alle tensioni $E\, \varepsilon_n^+$.
Dalle (7) e (8), sostituendo in (6), si ha:

$$X_i \int_\Gamma J^{-1} f_i \cdot f_k\, ds = - \int_\Gamma J^{-1} (f_o + f_n^+) \cdot f_k\, ds + R_k(\delta) \tag{9}$$

e ponendo :

$$Q = |Q_{ik}| \qquad Q_{ik} = \int_\Omega J^{-1} f_i \cdot f_k\, ds$$

$$d_n = |d_{nk}| \qquad d_{nk} = - \int_\Omega J^{-1} (f_o + f_n^+) \cdot f_k\, ds + R_k(\delta)$$

$$X = |X_i|$$

il sistema (9) si scrive nella forma matriciale

$$Q\, X = d_n \tag{10}$$

Detta $X_{(n+1)}$ la soluzione del sistema (10), il campo di distorsioni ε_{n+1}^+ da considerare al passo successivo del procedimento iterativo è costituito dalla parte positiva del campo di dilatazioni

$$\varepsilon = J^{-1} ((f_o + f_n^+) + X_i\, f_i) \cdot \rho \tag{11}$$

b) Il metodo di rilassamento geometrico

Il procedimento iterativo relativo a questo metodo consiste nel risolvere ad ogni passo il problema dell'equilibrio elastico per una struttura a geometria variata coincidente con quella parte della struttura data che risulta compressa al passo precedente.

Denotando con Ω_n^- la parte della sezione trasversale Ω che risulta compressa al passo n-esimo, l'algoritmo iterativo è definito dal problema di minimo:

$$\min \frac{1}{2} \int_\Gamma \int_{\Omega_n^-} \frac{1}{E} \sigma^2 \, d\Omega \, ds - R(\delta) \qquad \sigma \in \Sigma \tag{12}$$

Il procedimento iterativo può essere inizializzato ponendo $\Omega_1^- = \Omega$.

La formulazione variazionale del problema (12) conduce al sistema di N equazioni lineari:

$$\int_\Gamma \int_{\Omega_n^-} \frac{1}{E} (\sigma_{on} + X_i \sigma_{in}) \sigma_{kn} \, d\Omega \, ds = R_k(\delta) \tag{13}$$

dove σ_{on} è un campo di tensioni, lineare su Ω_n^-, ed in equilibrio con i carichi e σ_{in} sono i campi di tensioni associati alle X_i unitarie.

Introducendo le rappresentazioni

$$\sigma_{on}/E = c_{on} \cdot \rho \qquad \sigma_{in}/E = c_{in} \cdot \rho \qquad \text{su } \Omega_n^- \tag{14}$$

si ha

$$\varepsilon = (c_{on} + X_i c_{in}) \rho \tag{15}$$

Per calcolare le incognite X_i, tramite le (14), il sistema (13) si pone nella forma

$$\int_{\Omega_n} E (c_{on} + X_i c_{in}) \cdot \rho (c_k \cdot \rho) \, d\Omega \, ds = R_k(\delta) \tag{16}$$

Se si denota con J_n la matrice d'inerzia di Ω_n^-, definita da

$$J_n = \int_{\Omega_n^-} E \rho \otimes \rho \, d\Omega$$

il sistema (16) può scriversi

$$X_i \int_\Gamma J_n c_{in} \cdot c_{kn} \, ds = - \int_\Gamma J_n c_{on} \cdot c_{kn} \, ds + R_k(\delta) \tag{17}$$

Si ha :

$$J_n c_{in} = \int_{\Omega_n^-} E \rho \otimes \rho \, d\Omega \; c_{in} = \int_{\Omega_n^-} E (c_{in} \cdot \rho) \rho \, d\Omega =$$

$$= \int_{\Omega_n^-} \sigma_{in} \rho \, d\Omega = (-M_{iy}, M_{ix}, N_i) = f_i \tag{18}$$

$$J_n c_{on} = \int_{\Omega_n^-} E \rho \otimes \rho \, d\Omega \; c_{on} = \int_{\Omega_n^-} E (c_{on} \cdot \rho) \rho \, d\Omega =$$

$$= \int_{\Omega_n^-} \sigma_{on} \rho \, d\Omega = (-M_{oy}, M_{ox}, N_o) = f_o \tag{19}$$

e, sostituendo in (17),

$$X_i \int_\Gamma J_n^{-1} f_i \cdot f_k \, ds = - \int_\Gamma J_n^{-1} f_o \cdot f_k \, ds + R_k(\delta) \tag{20}$$

Ponendo

$$Q_n = |Q_{nik}| \qquad Q_{nik} = \int_\Gamma J_n^{-1} f_i \cdot f_k \, ds$$

$$d_n = |d_{nk}| \qquad d_{nk} = - \int_\Gamma J_n^{-1} f_o \cdot f_k \, ds + R_k(\delta)$$

si ottiene la forma matriciale del sistema (20)

$$Q \; X = d_n \tag{21}$$

Detta $X_{(n+1)}$ la soluzione di tale sistema, dalla (15) si ha

$$\varepsilon_{n+1} = J_n^{-1} (f_o + X_{(n+1)} f_i) \rho \tag{22}$$

La struttura da considerare al passo successivo è quindi individuata dalla parte Ω^-_{n+1} delle sezioni Ω in cui la ε_{n+1} risulta negativa.

4. L'arco parabolico

Si considera un arco parabolico incastrato alle imposte, simmetrico e simmetricamente caricato, la cui linea d'asse Γ ha rappresentazione parametrica

$$\underline{r}(x) = (x, d(1 - x^2/a^2)) \qquad -a \leq x \leq a$$

Ponendo

$$b = a^2/(2d) \qquad e \qquad k(x) = (b^2 + x^2)^{\frac{1}{2}}$$

si ha

$$\left| \frac{ds}{dx} \right| = \| \underline{r}'(x) \| = \frac{k(x)}{b}$$

$$\underline{t} = \frac{1}{k(x)} (b, -x)$$

$$\underline{n} = -\frac{1}{k(x)} (x, b)$$

dove s è l'ascissa curvilinea su Γ, l'apice denota la derivazione rispetto ad x, $\underline{t}$ ed $\underline{n}$ sono i versori della tangente e della normale a Γ.
Si assumerà che le sezioni trasversali ed il carico siano simmetrici rispetto al piano contenente la linea d'asse dell'arco. Tale ipotesi è peraltro verificata nella grande maggioranza dei casi di interesse applicativo.
Se si scelgono come incognite iperstatiche X_1 ed X_2 lo sforzo normale ed il momento flettente nella sezione di chiave, le corrispondenti caratteristiche della sollecitazione in funzione del parametro x sono date da:

$$N_1 = \frac{b}{k(x)} \qquad M_1 = -\frac{x^2}{2b}$$

$$N_2 = 0 \qquad M_2 = 1$$

Le caratteristiche di sforzo normale N_0 e di momento flettente M_0 dovute ai carichi (positivi se rivolti verso il basso), hanno le seguenti espressioni per $0 \leq x \leq a$

a) Per il peso proprio espresso come carico uniforme p distribuito sulla linea d'asse (per archi a sezione costante)

$$N_o = - p/(2b)\ (x/k(x)\ (x\ k(x) + b^2(\ln|x + k(x)| - \ln b)))$$

$$M_o = - p/b\ (x/2\ (x\ k(x) + b^2(\ln|x + k(x)| - \ln b)) + 1/3\ (b^3 - k^3(x)))$$

b) Per il carico dovuto al peso proprio del riempimento dalla quota d'imposta a quella di chiave con un materiale di peso specifico γ

$$N_o = - \frac{\gamma}{6b} \frac{x}{k\ (x)} \qquad M_o = - \frac{\gamma}{24\ b}\ x^4$$

c) Per un carico uniforme verticale q

$$N_o = - \frac{x^2}{k\ (x)} \qquad M_o = - q\ \frac{x^2}{2}$$

d) Per una forza verticale F agente sulla sezione di chiave dell'arco

$$N_o = - F\ \frac{x}{k\ (x)} \qquad M_o = - F\ x$$

Se si denotano con Δ e ϕ rispettivamente i cedimenti orizzontali (positivi verso l'esterno) e di rotazione (positivi se antiorari) delle sezioni d'imposta, si ha

$$R_1\ (\delta) = \Delta - d\ \phi \qquad R_2\ (\delta) = \phi$$

5. L'arco circolare

Si consideri ora un arco circolare, incastrato alle imposte, simmetrico e simmetricamente caricato, la cui linea d'asse Γ ha rappresentazione parametrica:

$$\underline{r}\ (\alpha) = r\ (\sin\alpha,\ \cos\alpha) \qquad -\beta \leq \alpha \leq \beta \qquad 0 < \beta \leq \pi/2$$

Si ha

$$\left| \frac{ds}{d\alpha} \right| = \| \underline{r}'(\alpha) \| = r$$

$$\underline{t} = (\cos\alpha, -\sin\alpha)$$

$$\underline{n} = -(\sin\alpha, \cos\alpha)$$

dove s è l'ascissa curvilinea su Γ, l'apice denota la derivazione rispetto ad α, $\underline{t}$ ed $\underline{n}$ sono i versori della tangente e della normale a Γ.
Nelle stesse ipotesi e con la stessa scelta delle incognite iperstatiche del paragrafo precedente, si ha

$$N_1 = \cos\alpha \qquad M_1 = -r(1-\cos\alpha)$$

$$N_2 = 0 \qquad M_2 = 1$$

Le caratteristiche N_o ed M_o dovute ai carichi hanno le seguenti espressioni per $0 \leq \alpha \leq \pi/2$

a) Per il peso proprio espresso come carico uniforme p distribuito sulla linea d'asse (per archi a sezione costante)

$$N_o = -p\, r\, \alpha \sin\alpha \qquad M_o = -p\, r^2 (\alpha \sin\alpha + \cos\alpha - 1)$$

b) Per il carico dovuto al peso proprio del riempimento dalla quota d'imposta a quella di chiave con un materiale di peso specifico γ

$$N_o = -\gamma\, r^2 \sin\alpha (\alpha - \sin\alpha)$$

$$M_o = -\gamma\, r^3 \left(\alpha \sin\alpha - \frac{1}{2}\sin^2\alpha + \cos\alpha - 1\right)$$

c) Per un carico uniforme verticale q

$$N_o = - q\, r \sin^2\alpha \qquad M_o = - \frac{1}{2} q\, r^2 \sin^2\alpha$$

d) Per una forza verticale F agente sulla sezione di chiave dell'arco.

$$N_o = - F \sin\alpha \qquad M_o = - F\, r \sin\alpha$$

Se si denotanò con Δ e ϕ rispettivamente i cedimenti orizzontali (positivi verso l'esterno) e di rotazione (positivi se antiorari) delle sezioni d'imposta, si ha

$$R_1(\delta) = \Delta - r\,\phi \qquad R_2(\delta) = \phi$$

6. Esempi numerici

Si riportano alcuni esempi di calcolo per archi simmetrici con linea d'asse parabolica o circolare ed a sezione rettangolare.
Per il calcolo degli integrali estesi alla linea d'asse dell'arco, si è considerata una discretizzazione in 60 elementi con interpolazione quadratica delle caratteristiche della sollecitazione in ogni elemento.
In tutti gli esempi l'arco è di sezione costante di altezza h = 100 cm e tutti i carichi sono riferiti ad 1 cm di larghezza dell'arco.
Gli archi parabolici hanno luce 2a = 10 m e freccia d = 4 m, quelli circolari hanno raggio r = 5 m.
I metodi di rilassamento sono stati in ogni caso inizializzati in modo da fornire al primo passo la soluzione elastica. Nelle figure relative ad ogni esempio di calcolo sono rappresentate, su metà arco,le linee delle pressioni ottenute ad ogni passo di iterazione e la distribuzione finale delle zone fessurate.

Esempio 1.
Si ha un arco parabolico soggetto ai seguenti carichi :

peso proprio p = 0.25 Kg/cm
forza concentrata in chiave F = 400 Kg

La soluzione elastica fornisce i seguenti valori delle incognite iperstatiche

$$X_1 = -514.73 \text{ Kg} \qquad X_2 = 44246.13 \text{ Kgcm}$$

La linea delle pressioni corrispondente a tale soluzione è esterna all'arco in prossimità della sezione di chiave (fig. 1).
Applicando il metodo di rilassamento geometrico in 6 passi di iterazione si ottiene

$$X_1 = -588.57 \text{ Kg} \qquad X_2 = 27215.24 \text{ Kgcm}$$

con 5 cifre significative esatte. In soluzione la linea delle pressioni è tutta interna all'arco. Il valore massima della tensione normale si ha nella sezione di chiave, ed è $\sigma_{max} = 52.19$ Kg/cm^2.
La fessurazione si verifica in due ristrette zone inferiori in chiave ed alle imposte, ed in un'ampia zona superiore alle reni.
Il metodo di rilassamento classico è praticamente inapplicabile perchè la convergenza è così lenta che dopo 200 iterazioni non si raggiunge ancora una soddisfacente approssimazione.

Esempio 2.
Si ha un arco parabolico soggetto ai seguenti carichi :

peso proprio $p = 0.25$ Kg/cm
peso del riempimento con materiale di peso specifico $\gamma = 1.8 \cdot 10^{-3}$ Kg/cm^3
forza concentrata in chiave $F = 300$ Kg

La soluzione elastica fornisce i seguenti valori delle incognite iperstatiche

$$X_1 = -439.26 \text{ Kg} \qquad X_2 = 32001.31 \text{ Kgcm}$$

La linea delle pressioni corrispondente è esterna all'arco in prossimità della sezione di chiave (fig. 2).
Applicando il metodo di rilassamento geometrico in 7 passi di iterazione si ottiene

$$X_1 = -482.78 \text{ Kg} \qquad X_2 = 21775.55 \text{ Kgcm}$$

con 5 cifre significative esatte. In soluzione la linea delle pressioni è tutta interna all'arco. Il valore massimo della tensione normale si ha nella sezione di chiave, ed è $\sigma_{max} = 32.87$ Kg/cm^2.
La fessurazione si verifica in una ristretta zona inferiore in chiave ed in una zona superiore alle reni. Non si ha fessurazione alle imposte.
Il metodo di rilassamento classico è insodc'sfacente, come si evince dalla fig. 3, dove sono riportate le linee delle pressioni relative ai primi 90 passi di iterazione e, per confronto, quella corrispondente alla soluzione ottenuta con il metodo di rilassamento geometrico.

Esempio 3.
Si ha un arco parabolico soggetto ai seguenti carichi :

peso proprio $p = 0.25$ Kg/cm
peso del riempimento con materiale di peso specifico $\gamma = 1.8 \cdot 10^{-3}$ Kg/cm^3
sovraccarico uniforme $q = 0.20$ Kg/cm
cedimento orizzontale alle imposte $\Delta = 0.05$ cm

La soluzione eıastica fornisce i seguenti valori delle incognite iperstatiche

$$X_1 = -117.52 \text{ Kg} \qquad X_2 = 8863.13 \text{ Kgcm}$$

La linea delle pressioni corrispondente è esterna all'arco in un'ampia zona intorno alla sezione di chiave (fig. 4).
Applicando il metodo di rilassamento geometrico in 4 passi di iterazione si ottiene

$$X_1 = -143.73 \text{ Kg} \qquad X_2 = 4751.58 \text{ Kgcm}$$

con 5 cifre significative esatte. In soluzione la linea delle pressioni è tutta interna all'arco. Il valore massimo delle tensioni normali è $\sigma_{max} = 2.83$ Kg/cm^3.
La fessurazione si verifica in due ampie zone, una inferiore in chiave ed una superiore all'imposta.
Il metodo di rilassamento classico fornisce praticamente la stessa soluzione dopo 40 iterazioni(fig 5).

Esempio 4.
Si ha un arco circolare soggetto ai seguenti carichi :

peso proprio p = 0.25 Kg/cm
peso del riempimento con materiale di peso specifico γ = 1.8 10^{-3} Kg/cm^3
sovraccarico uniforme q = 0.20 Kg/ m
forza concentrata in chiave F = 100 Kg

La soluzione elastica fornisce i seguenti valori dellè incognite iperstatiche

$$X_1 = - 255.43 \text{ Kg} \qquad X_2 = 20227.88 \text{ Kgcm}$$

La linea delle pressioni corrispondente è esterna all'arco in prossimità della sezione di chiave(fig. 6).
Applicando il metodo di rilassamento geometrico in 7 passi di iterazione si ottiene

$$X_1 = - 278.73 \text{ Kg} \qquad X_2 = 12843.14 \text{ Kgcm}$$

con 5 cifre significative esatte. Il valore massimo della tensione normale si ha nella sezione di chiave ed è σ_{max} = 23.78 Kg/cm^2.
La fessurazione si verifica inferiormente in chiave ed alla imposte, ed in un'ampia zona superiore alle reni.

Esempio 5.
Si ha un arco circolare soggetto ai seguenti carichi :

peso proprio p = 0.25 Kg/cm
peso del riempimento con materiale di peso specifico γ = 1.8 10^{-3} Kg/cm^3
sovraccarico uniforme q = 0.30 Kg/cm
forza concentrata in chiave F = 20 Kg

La soluzione elastica fornisce i seguenti valori delle incognite iperstatiche

$$X_1 = - 211.28 \text{ Kg} \qquad X_2 = 9035.70 \text{ Kgcm}$$

La linea delle pressioni corrispondente è tutta interna all'arco. La massima tensione normale si ha in chiave ed è $\sigma_{max} = 9.74 \ Kg/cm^2$.
Applicando il metodo di rilassamento geometrico (fig. 7) dopo 5 passi di iterazione si ha

$$X_1 = - 215.44 \ Kg \qquad X_2 = 7691.63 \ Kgcm$$

con 5 cifre significative esatte. Il valore massimo della tensione normale si verifica alla imposte ed e $\sigma_{max} = 8.72 \ Kg/cm^2$. In chiave si ha $\sigma_{max} = 5.02 \ Kg/cm^2$.
La fessurazione si ha in un'ampia zona inferiore in chiave ed in ristrette zone inferiori alle imposte e superiori alle reni.
Il metodo di rilassamento classico fornisce praticamente la stessa soluzione dopo 50 passi di iterazione (fig. 8).
Gli esempi riportati mostrano una notevole influenza della fessurazione del materiale sulla distribuzione delle caratteristiche della sollecitazione, in particolare sul valore del momento flettente in chiave. L'effetto è quello di una ridistribuzione delle sollecitazioni con diminuzione dell'eccentricità nella sezione di chiave.
Nei primi quattro esempi si mostra che, mentre la soluzione elastica dà luogo ad una linea delle pressioni che risulta esterna all'arco in una zona intorno alla sezione di chiave, il che renderebbe impossibile l'equilibrio, alla soluzione corretta corrispondeuna linea delle pressioni tutta interna, con valori relativamente modesti delle massime tensioni normali.
Nell'ultimo esempio, dove la soluzione elastica fornisce una linea della pressioni tutta interna all'arco, l'effetto della fessurazione è quello di ridurre sensibilmente le tensioni massime nella sezione di chiave.

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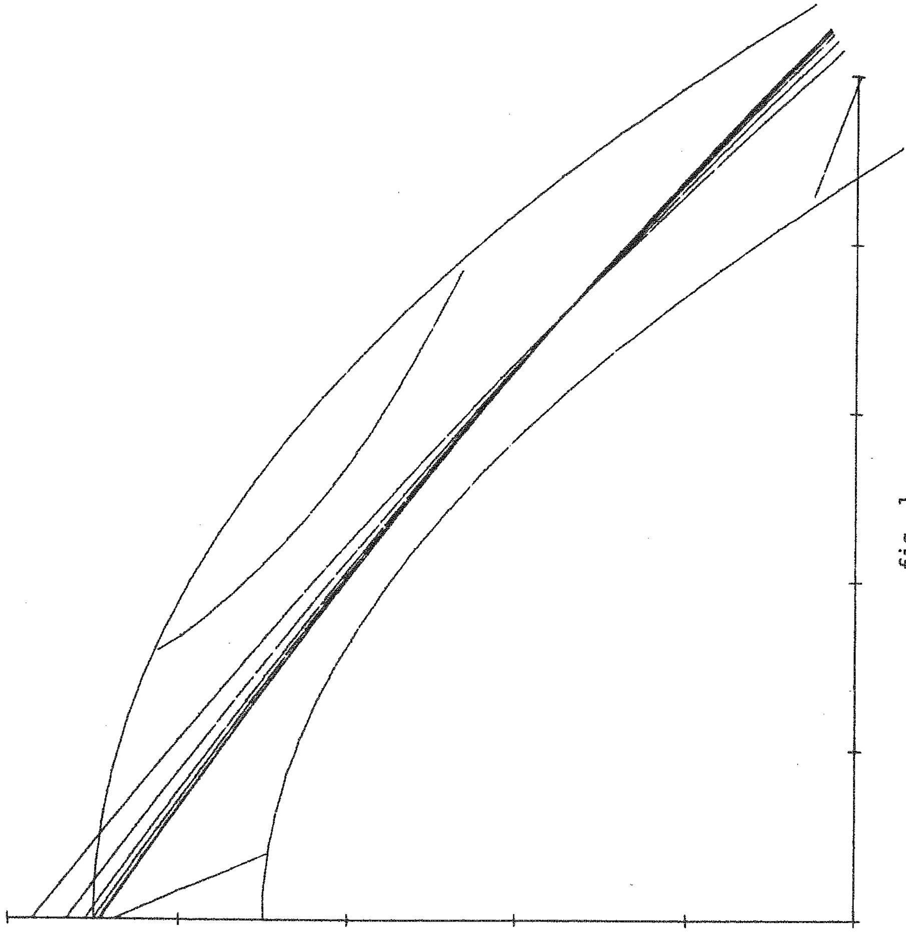

fig. 1

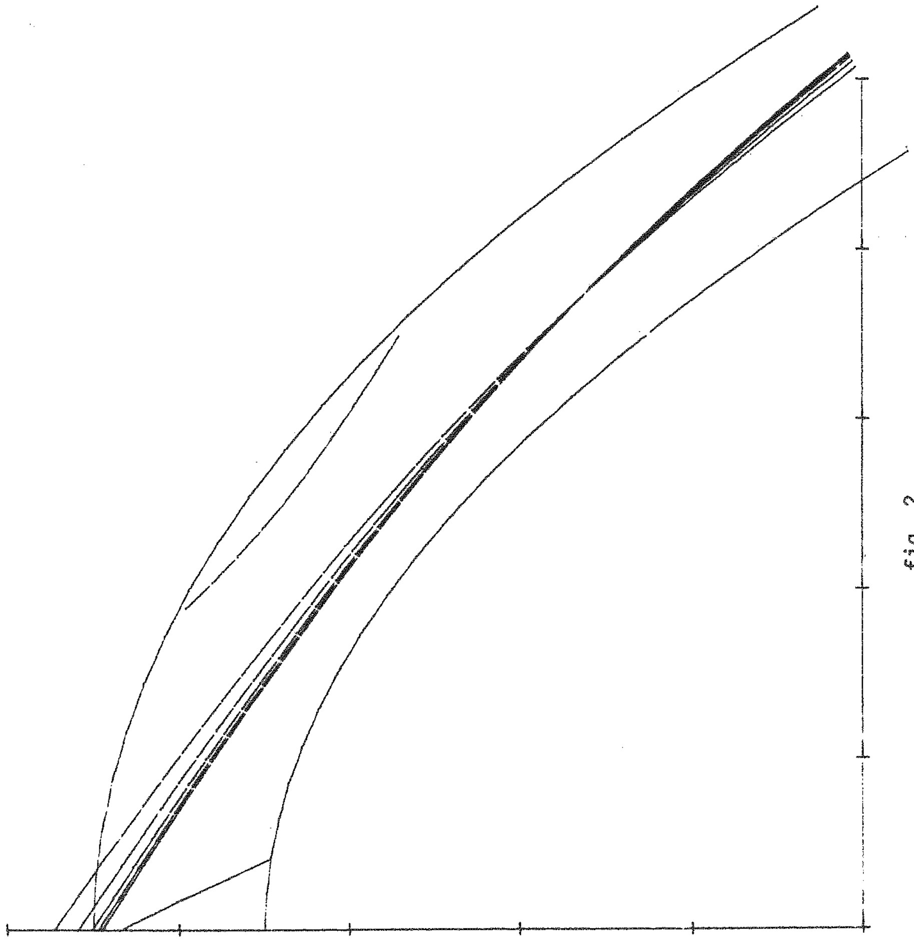

fig. 2

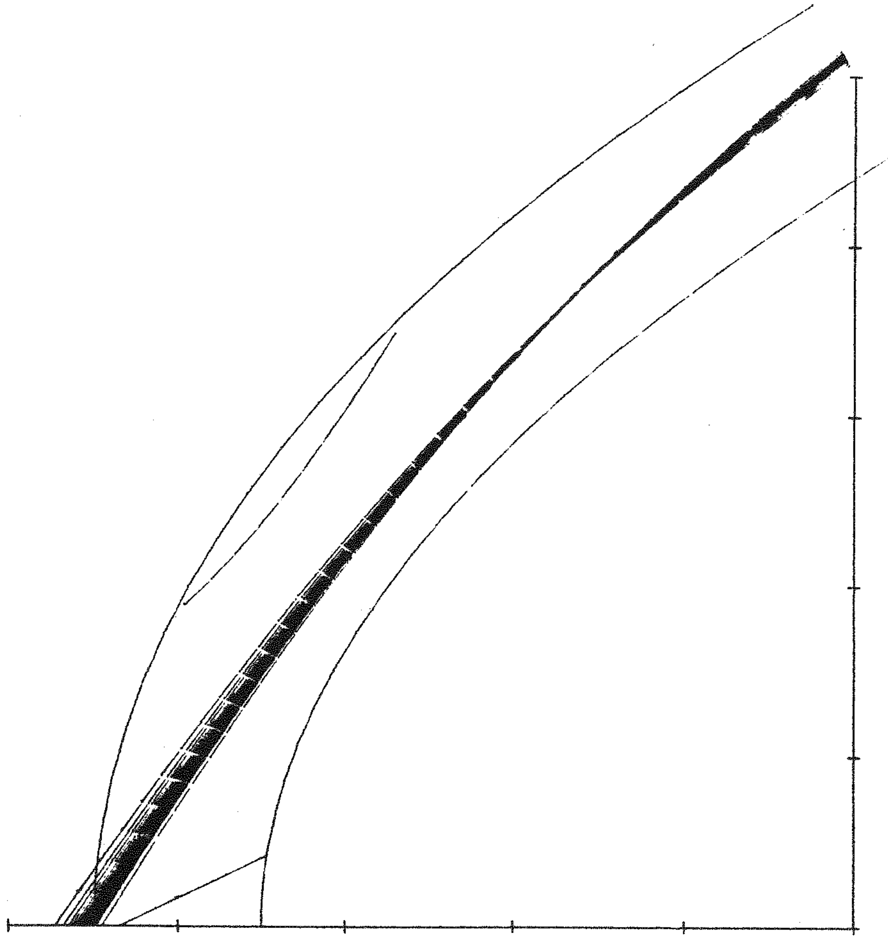

fig. 3

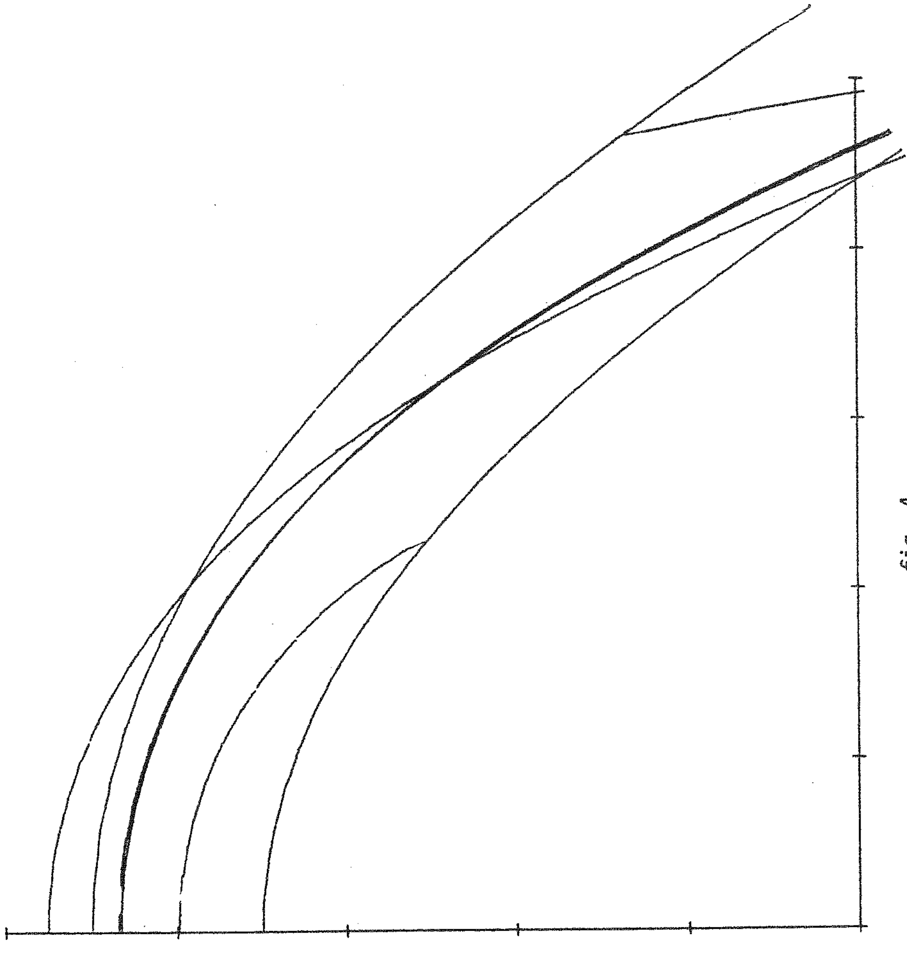

fig. 4

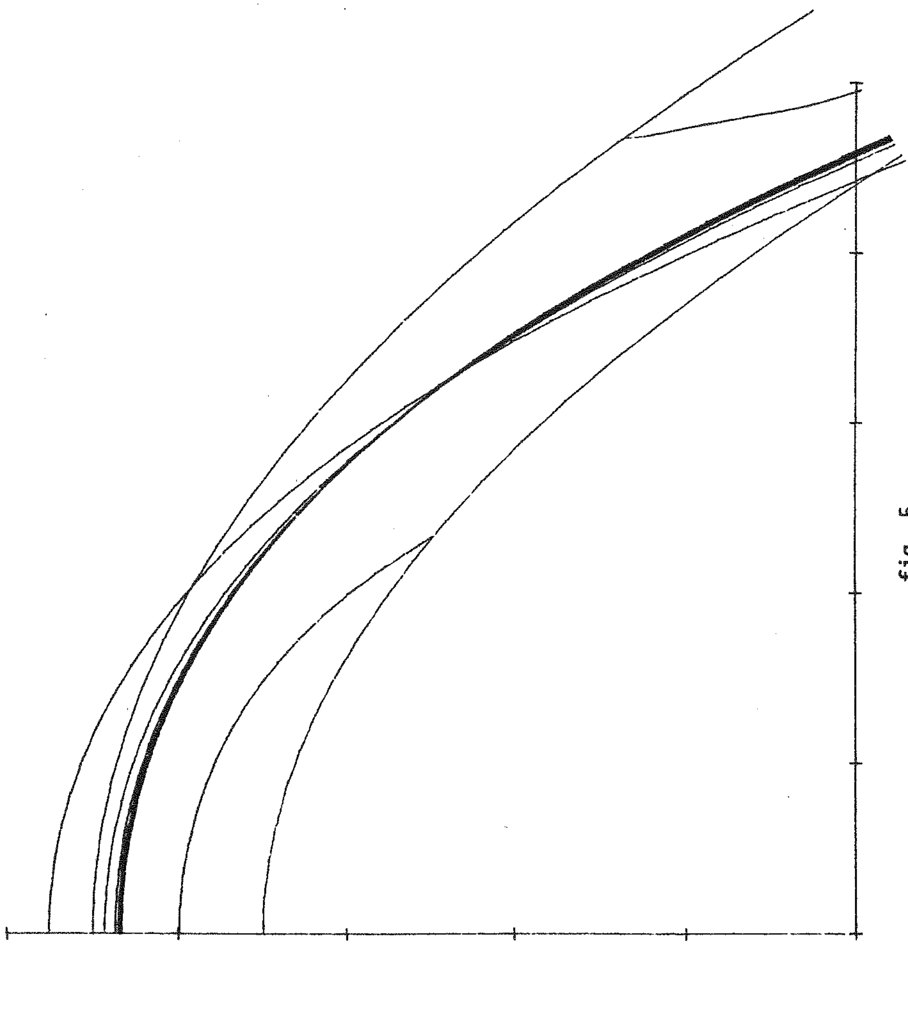

fig. 5

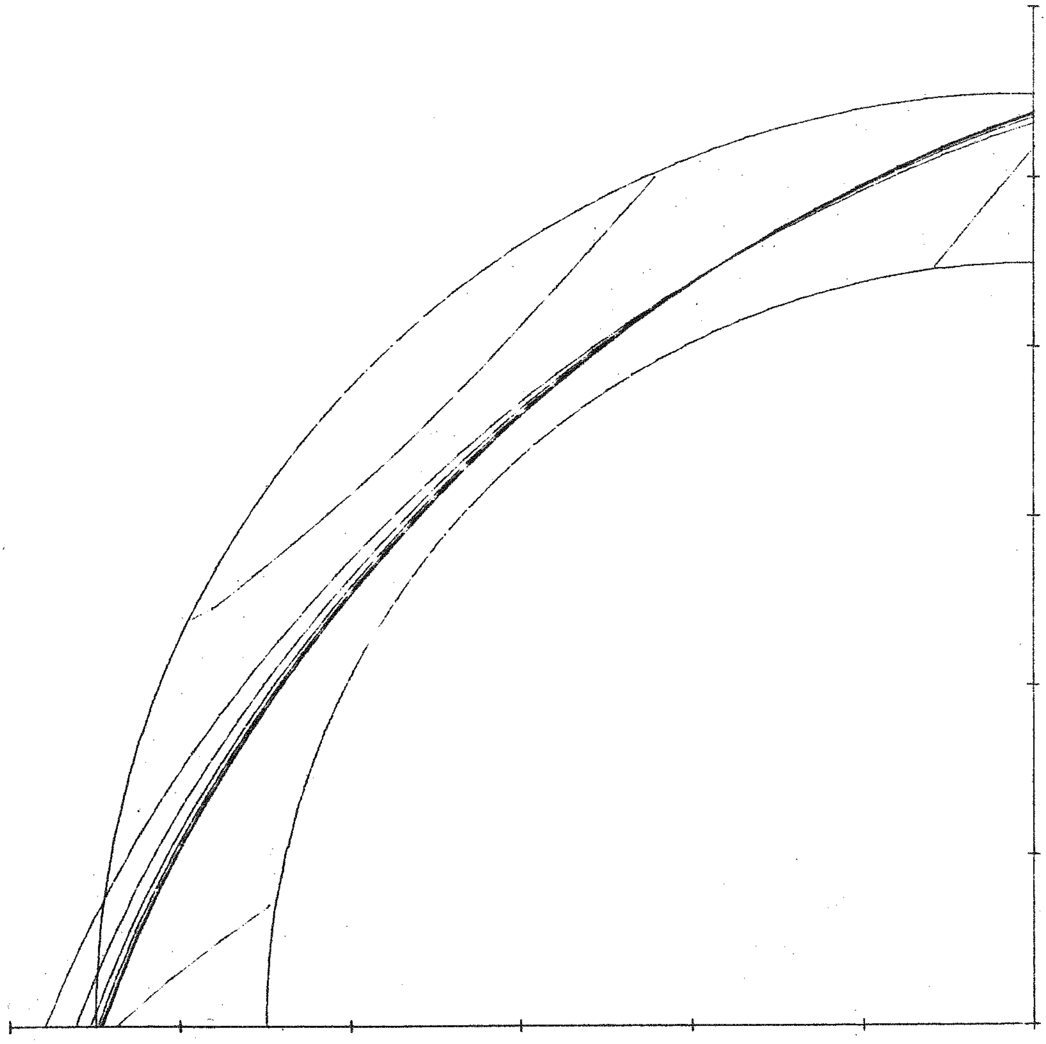

fig. 6

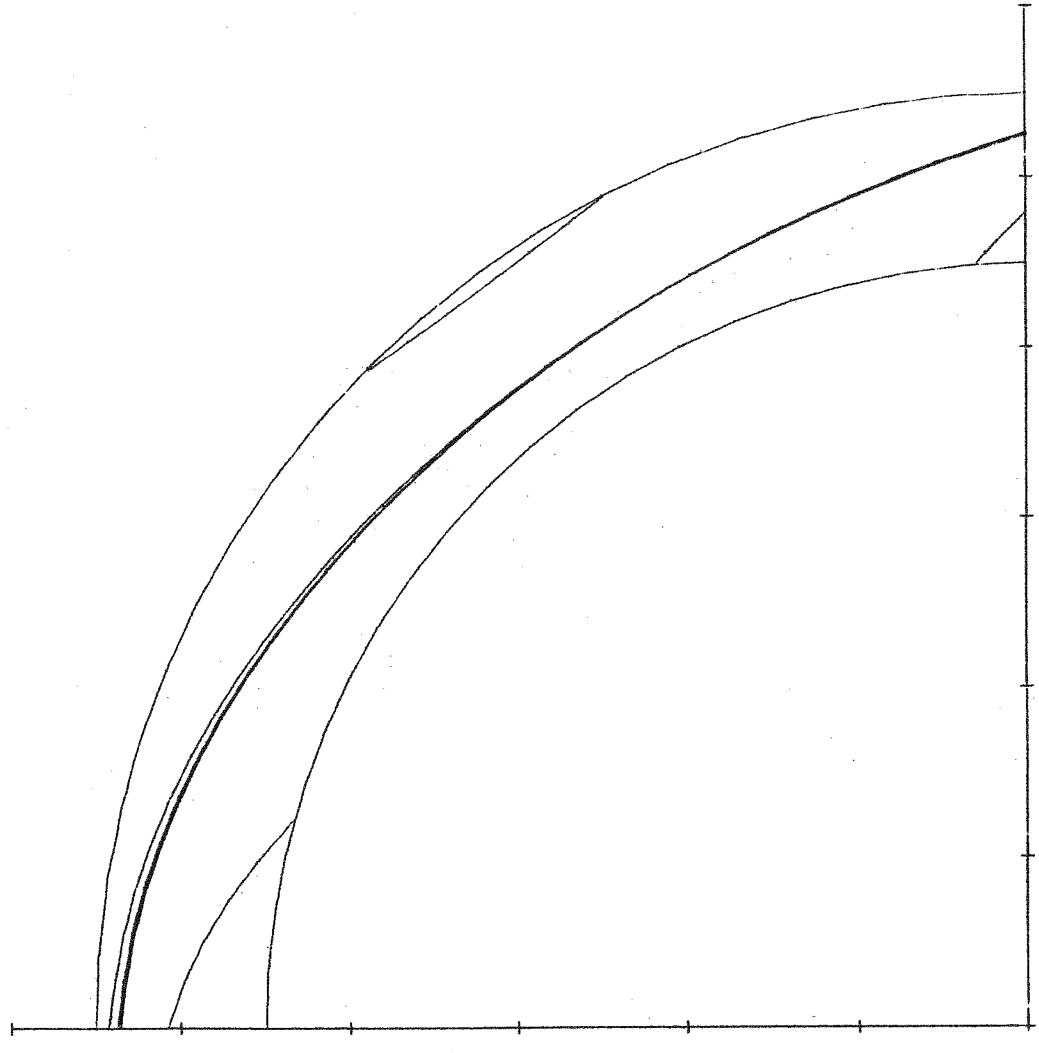

fig. 7

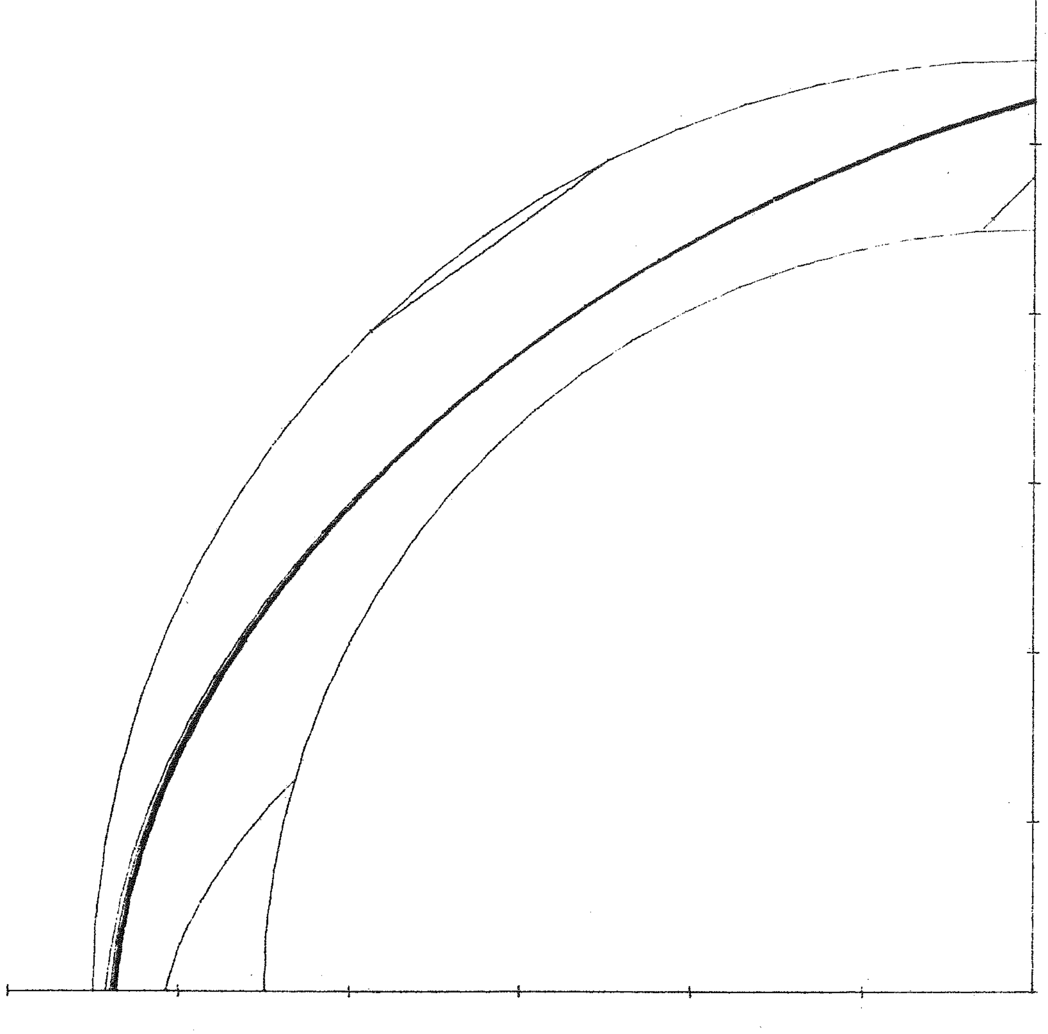

fig. 8

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ON THE FOUNDATION OF VARIATIONAL PRINCIPLES IN LINEAR STRUCTURAL MECHANICS

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ON THE FOUNDATION OF VARIATIONAL PRINCIPLES IN LINEAR STRUCTURAL MECHANICS

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Abstract

A general variational theory of linear structural mechanics founded upon the concept of virtual work duality is formulated. The complementary problems of equilibrium and compatibility are discussed and their equivalent variational forms are derived. The general analysis is then applied to the formulation of the fundamental variational principles of linear elastostatics which are proved to be necessary and sufficient conditions for the solution of the elastic boundary value problem.

1 – Introduction

Variational principles and methods play a fundamental role in structural mechanics. Basic theoretical results, such as existence and uniqueness of the solution of boundary value problems, and effective computational methods are in fact derived from variational formulations of structural problems. However the importance of variational statements goes far beynd their use as convenient tools for the analysis and the solution of structural problems. In fact the variational formulation can be the only correct way to state the problem itself.

From a mathematical point of view, variational formulations are founded upon the concept of duality pairing between the fields involved in the analysis. In the context of linear structural mechanics the bilinear form "virtual work" yields the duality pairing between the dual fields of displacements and forces and of strains and stresses.

It is then natural to introduce the basic concept of virtual work duality from the very beginning of the development of a variational theory of structural mechanics.

This approach is followed in the present paper. A general formulation of the virtual work principle is the main tool for the analysis of the complementary boundary value problems of equilibrium and compatibility.

The variational formulations of these problems are discussed in detail and are proved to be necessary and sufficient conditions for the solution.

The basic existence results are obtained by means of auxiliary elastic boundary value problems, following an idea first presented in [1].

The results of the analysis are then applied to the formulation of the fundamental variational principles of linear elastostatics which are thus proved to be necessary and sufficient conditions for the solution of the elastic problem.

2 – The virtual work duality

In linear structural mechanics, the set of configurations of a structure is conveniently described by the linear space V of its displacements from the reference configuration. A force acting on the stucture is then defined to be a linear functional on the displacement space V.

The values of the linear functional "force" have the mechanical meaning of virtual work of the force for the corresponding displacement field.

The force space is accordingly defined as the dual of the displacement space V. When the space V is endowed with a suitable topology, the force space will be defined as its topological dual V', the space of bounded linear functionals on V.

This is equivalent to require that the virtual work must depend continuously on the displacement field.

We assume that all the functional spaces involved in the analysis are (possibly finite dimensional) Hilbert spaces.

Since it is fundamental in mechanics to distinguish between rigid and distorsion displacements, we have to define a strain space and a deformation operator:

$$T : V \rightarrow W$$

which associates to any displacement field $u \in V$ the corresponding strain field $Tu \in W$.

In order to establish the main existence results, we require that T is a bounded linear operator with a closed range in W.

The following inequality must then hold:

$$\| T \| \; \| u \| \geq \| Tu \| \geq k \; \inf \{ \| u - v \| / v \in R \} \qquad u \in V \qquad (2.1)$$

where $k > o$ and $\| \cdot \|$ denotes the norm in the corresponding Hilbert space. The kernel of T:

$$R = \{ u \in V : Tu = o \}$$

is the subspace of the rigid displacements of the structure.

The mechanical interpretation of (2.1) states that, if rigid displacements are ruled out, there is a continuous correspondence between the displacement fields and the associated strain fields.

The stress space W' is defined as the dual space of W, and the equilibrium operator:

$$T' : W' \rightarrow V'$$

is the dual of T, according to the identity:

$$(T'\sigma, u) = (\sigma, Tu) \qquad u \in V, \sigma \in W' \qquad (2.2)$$

where the symbol (,) denotes the duality pairing.

To any stress field $\sigma \in W'$ the operator T' associates the corresponding equilibrated force $T' \sigma \in V'$.

The identity (2.2) is the general abstract statement of the virtual work principle of linear structural mechanics.

3 – Equilibrium and Compatibility problems

To formulate boundary value problems we have to distinguish between boundary and interior terms. To this end let us consider the closed subspace V_o of V, which consists of the displacement fields vanishing in a boundary layer of the domain of the structure, and its orthogonal complement Γ.

We shall refer to V_o and Γ respectively as the interior and the boundary displacement space.

Correspondently the force space W' can be splitted in the complementary orthogonal subspaces V_o' and Γ', respectively the interior and the boundary force space.

Denoting the orthogonal projectors on V_o and Γ by:

$$\begin{array}{ll} \text{int} \;: V \to V_o & \text{interior displacement operator} \\ \text{bnd} : V \to \Gamma & \text{boundary displacement operator} \end{array}$$

the duality relations:

$$\begin{aligned} T^* &= (\text{int})' \;\; T' = (T\ \text{int})' \\ B &= (\text{bnd})'\, T' = (T\ \text{bnd})' \end{aligned}$$

define the bounded linear operators:

$$\begin{array}{ll} T^* : W' \to V_o' & \text{interior equilibrium operator} \\ B \;\;: W' \to \Gamma' & \text{boundary equilibrium operator} \end{array}$$

which associate to any stress field $\sigma \in W'$ the corresponding equilibrated interior (body) force $T^* \sigma \in V_o'$ and boundary force $B\sigma \in \Gamma'$.

The operator T^* is usually said to be the "formal adjoint" of T.

The virtual work identity (2.2) may then be rewritten by splitting the interior and the boundary terms:

$$(T^* \sigma, \text{int}\ u) + (B\sigma, \text{bnd}\ u) = (\sigma, Tu) \tag{3.1}$$

In the formulation of linear problems constraints can be imposed on the displacement fields both at the interior and on the boundary of the structure. Accordingly we shall consider two pairs of complementary projectors:

$$\begin{array}{ll} P + P_c = I & (\text{identity in } V_o) \\ Q + Q_c = I & (\text{identity in } \Gamma) \end{array} \tag{3.2}$$

so that the virtual work identity may be rewritten:

$$\begin{aligned} &(P_c'\, T^*\sigma, \text{int}\ u) + (Q_c'\, B\sigma, \text{bnd}\ u) = \\ &= (\sigma, Tu) - (T^*\sigma, P\ \text{int}\ u) - (B\sigma, Q\ \text{bnd}\ u) \end{aligned} \tag{3.3}$$

where an apex denotes the dual projectors.

Complementary equilibrium and compatibility problems are then formulated as follows:

P.1 – Equilibrium problem

Given a body force field $b \in V_o'$ and a traction field on the boundary $t \in \Gamma'$, find the set Σ of the stress fields $\sigma \in W'$ such that:

$$\begin{aligned} P_c'\, T^*\sigma &= P_c'\, b \\ Q_c'\, B\sigma &= Q_c'\, t \end{aligned} \tag{3.4}$$

P.2 – Compatibility problem

Given a strain field $\varepsilon \in W$ and a displacement field $w \in V$, find the set U of the displacement fields $u \in V$ such that:

$$\begin{aligned} Tu &= \varepsilon \\ P \operatorname{int} u &= P \operatorname{int} w \\ Q \operatorname{bnd} u &= Q \operatorname{bnd} w \end{aligned} \qquad (3.5)$$

where $P \operatorname{int} w$ and $Q \operatorname{bnd} w$ are the displacements imposed to the constraints at the interior and on the boundary of the domain of the structure. It is apparent that the solution sets Σ and U are (possibly empty) linear varieties.

4 – Variational formulations

The complementary problems P.1 and P.2 may be equivalently formulated in variational form.

Indeed, if we define the applied load functional $\ell \in V'$ by:

$$\ell(u) = (P'_c\, b, \operatorname{int} u) + (Q'_c\, t, \operatorname{bnd} u) \qquad u \in V$$

the equilibrium relations (3.4) are equivalent to the variational condition:

$$(\sigma, Tu) - (T^*\sigma, P \operatorname{int} u) - (B\sigma, Q \operatorname{bnd} u) = \ell(u) \qquad (4.1)$$

where $u \in V$, a result that follows easily from the virtual work identity in the form (3.3). We remark that the second and third term in (4.1) are the virtual works of the reactions $P' T^*\sigma$ and $Q' B \sigma$ of the interior and boundary constraints, in equilibrium with the stress field $\sigma \in W'$.

Further if we consider the linear variety of the admissible displacements:

$$L = \{u \in V : P \operatorname{int} u = P \operatorname{int} w,\ Q \operatorname{bnd} u = Q \operatorname{bnd} w\}$$

and the parallel subspace of admissible additional displacements:

$$L_o = \{u \in V : P \operatorname{int} u = o,\ Q \operatorname{int} u = o\}$$

an equivalent formulation of (3.4) and (4.1) is given by:

$$(\sigma, T(u - w)) = \ell(u - w) \qquad u \in L$$

or

$$(\sigma, Tv) = \ell(v) \qquad v \in L_o \qquad (4.2)$$

In fact, by virtue of (3.3), (4.2) may be rewritten as

$$(P'_c\,(T^*\sigma - b), \text{int } u) + (Q'_c\,(B\sigma - t), \text{bnd } u) = o \qquad u \in L_o$$

which implies (3.4). The implication (3.4) $\Rightarrow$ (4.1) is easily established by means of (3.3). Now trivially (4.1) $\Rightarrow$ (4.2) and the equivalence above is proved.

A variational formulation of the compatibility problem P.2 may be obtained by proving that the relations (3.5) are equivalent to the variational condition:

$$\begin{aligned}&(\sigma\,, \varepsilon) - (T^*\sigma, P \text{ int } w) - (B\sigma, Q \text{ bnd } w) = \\ &= (P'_c\, T^*\sigma, \text{int } u) + (Q'_c\, B\sigma, \text{bnd } u) \qquad \sigma \in W'\end{aligned} \tag{4.3}$$

The implication (3.5) $\Rightarrow$ (4.3) is an immediate consequence of (3.3). The converse result is deeper: in fact it is intimately related to the problem of the existence of a solution of the compatibility problem P.2. We shall establish the implication (4.3) $\Rightarrow$ (3.5) at the end of the following section.

5 – Existence conditions

In the discussion of the existence of a solution of the equilibrium and of the compatibility problem we shall follow a procedure, first presented in [1], which rests on the formulation of suitable "auxiliary" elastic boundary value problems.

First we shall deal with the equilibrium problem to show that:

E.1 The solution set Σ of the equilibrium problem P.1 is a non empty linear variety if and only if the following condition is met:

$$\ell\,(u) = o \qquad\qquad u \in R_o = R \cap L_o \tag{5.1}$$

where R_o is the subspace of the admissible additional rigid displacements. The proof of the necessity of the condition (5.1) follows easily from (4.2). The sufficiency can be established by considering the elastic problem:

$$\begin{aligned}&P'_c\, T^*\,\sigma = P'_c\, b\\ &Q'_c\, B\,\sigma = Q'_c\, t\\ &Tu = C\,\sigma\\ &P \text{ int } u = o\\ &Q \text{ bnd } u = o\end{aligned} \tag{5.2}$$

where $C : W' \rightarrow W$ is the elastic compliance operator, which is symmetric and coercive:

$$C = C' \quad \text{and} \quad (C\sigma, \sigma) \geqq k \, \| \sigma \|^2 \qquad \sigma \in W'$$

with $k \geqq o$.

The corresponding potential energy functional is:

$$\phi\ (u) = \tfrac{1}{2} \, \| Tu \|_S^2 - \ell\ (u) \qquad u \in L_o$$

where $\| \cdot \|_S$ denotes the energy norm induced in V by the elastic stiffness operator $S = C^{-1} : W \rightarrow W'$.

The elastic problem above is then equivalent to the minimum problem:

$$\min \ \{\phi\ (u) \,/\, u \in L_o\} \tag{5.3}$$

It is well known that, under the validity of (2.1), if the condition (5.1) is satisfied, this problem admits a solution $u_o \in L_o$ which is unique to within an arbitrary admissible additional rigid displacement field.

The stress solution $\sigma_o = S\,Tu_o$ is then unique and we have:

$$(\sigma_o, Tv) = \ell\ (v) \quad v \in L_o \ \Leftrightarrow \ \sigma_o \in \Sigma$$

Hence $\Sigma = \sigma_o + \Sigma_o$ where:

$$\begin{aligned} \Sigma_o &= \{\sigma \in W' : P'_c\, T^*\sigma = o,\ Q'_c\, B\sigma = o\} = \\ &= \{\sigma \in W' : (\sigma, Tv) = o \quad v \in L_o\} \end{aligned}$$

is the subspace of selfequilibrated stress fields.

The existence condition for the compatibility problem is given by:

E.2 The solution set U of the compatibility problem P.2 is a non empty linear variety if and only if the following condition is met:

$$\begin{aligned} (\sigma, \varepsilon) &= (T^*\sigma, P \text{ int } w) + (B\sigma, Q \text{ bnd } w) = \\ &= (\sigma, Tw) \qquad \sigma \in \Sigma_o \end{aligned} \tag{5.4}$$

The necessity of (5.4) follows immediately from (4.3).

The proof of the sufficiency is obtained by considering the elastic problem:

$$\begin{aligned} &P'_c\, T^*\sigma = o \\ &Q'_c\, B\,\sigma = o \\ &Tu = \varepsilon + C\sigma \\ &P \text{ int } u = P \text{ int } w \\ &Q \text{ bnd } u = Q \text{ bnd } w \end{aligned}$$

which is equivalent to the minimum problem:

$$\min \ \{ \tfrac{1}{2} \| Tu - \epsilon \|^2_S \ / \ u \in L \} \qquad (5.5)$$

Under the validity of (2.1) the linear variety T $\{L\}$ is closed and the problem (5.5) admits an unique solution Tu_o which is the point of T $\{L\}$ of minimum distance from ε.

Denoting by $\sigma_o = S(Tu_o - \varepsilon)$ the stress solution of (5.5), we have that:

$$(\sigma_o, T(u - u_o)) = o \qquad u \in L \qquad (5.6)$$

Now choosing $u = w$ and setting $Tu_o = \varepsilon + C\sigma_o$ in (5.6), we get:

$$(\sigma_o, T(w - u_o)) = - \| \sigma_o \|^2_C + (\sigma_o, Tw) - (\sigma_o, \varepsilon) = o \qquad (5.7)$$

Since $\sigma_o \in \Sigma_o$, from (5.4) and (5.7) we infer that $\sigma_o = o$ and hence

$$\varepsilon = Tu_o \qquad u_o \in L \qquad (5.8)$$

The solution set of the compatibility problem is then the linear variety:

$$U = u_o + R_o$$

Finally we remark that with the result (5.8) we have proved the implication: (5.4) → (3.5), and since (4.3) → (5.4) trivially, we have also proved that (4.3) → (3.5).

6 – Variational Principles of Linear Elastostatics

In the previous section we have proved the equivalence of the equilibrium problem P.1 and of the compatibility problem P.2 respectively to the variational statements (4.1), (4.2) and (4.3), (5.4).

Depending on the choice of the variational formulation of the equilibrium and of the compatibility problem, different variational principles can be esthablished in linear structural mechanics.

On this basis we give here a short presentation of the fundamental variational principles of linear elastostatics.

A general variational theory of incremental elastic-plastic structural problems is developed in [2].

Let us then consider the following

Linear Problem of Elastostatics

Given a body force $b \in V'_o$, a boundary traction $t \in \Gamma'$, an imposed (thermal) distorsion $\delta \in W$ and the interior and boundary constraints displacements P int w and Q bnd w, find the triplet $(u_o, \varepsilon_o, \sigma_o) \in V \times W \times W'$ which satisfy the conditions:

equilibrium	$P_c' T^*\sigma = P_c' b$
	$Q_c' B\sigma = Q_c' t$
compatibility	$Tu = \varepsilon + \delta$
	$P \text{ int } u = P \text{ int } w$
	$Q \text{ bnd } u = Q \text{ bnd } w$
constitutive relation	$\sigma = S\varepsilon$

Assuming the variational statements (4.1) and (4.3) for the equilibrium and the compatibility conditions, the elastic problem may be written:

equilibrium (6.1)

$$(\sigma, Tv) - (T^*\sigma, P \text{ int } v) - (B\sigma, Q \text{ bnd } v) - \ell(v) = o \qquad v \in V$$

compatibility (6.2)

$$-(\tau, \varepsilon + \delta) - (T^*\tau, P \text{ bnd}(u - w)) - (B\tau, Q \text{ bnd}(u - w)) +$$
$$+ (\tau, Tu) = o \qquad \tau \in W'$$

constitutive relation

$$(S\varepsilon - \sigma, \eta) = o \qquad \eta \in W \tag{6.3}$$

which can be easily seen to be the Euler conditions for the potential functional:

$$F(u, \varepsilon, \sigma) = \tfrac{1}{2}\| \varepsilon \|_S^2 + (\sigma, Tu - \varepsilon - \delta) - \ell(u) +$$
$$- (T^*\sigma, P \text{ int}(u - w)) - (B\sigma, Q \text{ bnd}(u - w)) \tag{6.4}$$

The variational conditions (6.1) – (6.3) are then equivalent to the free variational principle:

$$\partial F(u, \varepsilon, \sigma)(v, \eta, \tau) = o \qquad (v, \eta, \tau) \in V \times W \times W'$$

that has been first proposed by Hu [4] and Washizu [5].

Assuming the fulfillment of the constitutive relation, setting $\varepsilon = C\sigma$ in (6.2), it is apparent that (6.1) and (6.2) become the Euler conditions of the Reissner [6] functional:

$$F(u, \sigma) = -\tfrac{1}{2}\| \sigma + S\delta \|_C^2 + (\sigma, Tu) - \ell(u) + \tag{6.5}$$
$$- (T^*\sigma, P \text{ int}(u - w)) - (B\sigma, Q \text{ bnd}(u - w))$$

Further on, if the constitutive relation $\sigma = S\varepsilon$ and the compatibility condition $\varepsilon = Tu - \delta$ with $u \in L$ are satisfied, the equilibrium condition in the form (4.2) gives

$$(S(Tu - \delta), Tv) = \ell(v) \qquad v \in L_o \tag{6.6}$$

which is the Euler condition for the potential energy functional:

$$F(u) = \tfrac{1}{2}\| Tu - \delta \|_S^2 - \ell(u) \qquad u \in L \tag{6.7}$$

Since F (u) is convex, the condition (6.6) is equivalent to the minimum problem:

$$\min \{F(u) \;/\; u \in L\} \tag{6.8}$$

Assuming the fulfillment of the equilibrium condition $\sigma \in \Sigma$ and of the constitutive relation : $\varepsilon = C\sigma$, the compatibility condition in the form (5.4):

$$-(\tau, C\sigma + \delta) + (\tau, Tw) = o \qquad \tau \in \Sigma_o \tag{6.9}$$

gives the Euler condition for the complementary energy functional:

$$\begin{aligned} F(\sigma) &= -\tfrac{1}{2}\| \sigma - S\delta \|_C^2 + (\sigma, Tw) - (\ell, w) = \\ &= -\tfrac{1}{2}\| \sigma - S\delta \|_C^2 + (T^*\sigma, P \text{ int } w) + (B\sigma, Q \text{ bnd } w) \end{aligned} \tag{6.10}$$

Since F (σ) is concave, the condition (6.9) is equivalent to the maximum problem:

$$\max \{F(\sigma) \;/\; \sigma \in \Sigma\}$$

We remark that at the solution $(u_o, \varepsilon_o, \sigma_o)$ of the elastic problem all the functional above assume the same value:

$$F(u_o, \varepsilon_o, \sigma_o) = F(u_o, \sigma_o) = F(u_o) = F(\sigma_o)$$

Finally we point out that in the relevant literature variational principles are, as a rule, formulated as necessary conditions for the solution of boundary value problems.

For a general and comprehensive account of the most recent contributions in this field we refer to [3].

It is however the sufficiency aspect of the variational principles that play a basic role in the proof of existence results and in the formulation of computational methods.

The proof of the sufficiency of the variational principles of linear structural mechanics can be obtained on the basis of the results presented in this paper, since the variational formulations of the equilibrium and of the compatibility problems have been shown to be equivalent statements of the corresponding boundary value problems.

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EQUILIBRIUM AND COMPATIBILITY UNDER INTERNAL AND EXTERNAL CONVEX CONSTRAINS

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EQUILIBRIUM AND COMPATIBILITY UNDER EXTERNAL AND INTERNAL CONVEX CONSTRAINTS

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Abstract

The paper deals with the analysis of structural models subject to external and internal contraints which define a closed convex set of admissible displacement fields and a closed convex set of admissible stress distributions.
A variational condition on the load distribution is proved to be necessary and sufficient for the equilibrium under the prescribed convex constraints.
The geometric compatibility problem is then investigated and the condition to be satisfied by the strain field is provided.

1. Introduction

In the small displacement theory of structural mechanics non linear equilibrium and compatibility problems may arise as a consequence of non affine external contraint conditions on the displacement fi_ elds and/or of non linear internal constraint conditions on the st_ ress distributions.

Both situations do arise in usual models of structural engineering so that an investigation on the equilibrium and on the geometric compatibility problems, intended to extend the known results for the linear case, is of a major interest.

The most general context in which it appears to be possible to carry out a comprehensive analysis in these directions, is that of convex structural analysis where the sets of the admissible displacement fi_ elds and stress distributions are assumed to be convex and closed.

2. The structural model

We consider a general structural model in the framework of the small displacement theory, in which non linear geometric effects are negle_ cted.

The kinematic description of the model is then defined by the linear space V of the displacement fields from a reference configuration, the strain space W and the deformation operator :

$T : V \to W$

which maps the displacement fields into the corresponding strain fields.

The dual static description is obtained from the kinematic one via the concept of virtual work.

Force and stress distributions are defined to be linear functionals resp. on the displacement and the strain spaces, and then are elements of the dual spaces V' and W'.

The values of these linear functionals resp. yield the external and the internal virtual works, both denoted by the same symbol $\langle\cdot,\cdot\rangle$.

The dual equilibrium operator $T' : W' \rightarrow V'$

is defined by the identity :

$$\langle\sigma, Tv\rangle = \langle T'\sigma, v\rangle \qquad \text{for any } \sigma \in W' \ , \ v \in V \tag{2.1}$$

and the following orthogonality relations are assumed to hold:

$$R(T) = N(T')^{\perp} \tag{2.2}$$

$$R(T') = N(T)^{\perp} \tag{2.3}$$

where $R(\cdot)$ and $N(\cdot)$ resp. denote the range and the null space.

The elements of the subspace $N(T)$ are the rigid displacements and the elements of $N(T')$ are the selfequilibrated stress distributions, in absence of constraints.

2.1 External constraints

We shall consider the structure to be subjected to a set of external contraints defined by:

- kinematic description

  The set of the admissible displacements is a closed convex set $C \subset V$.

- static description

  At a configuration $u \in C$ the set of the admissible constraint reactions is the inward normal cone to C at u.

Denoting by $N_C(u)$ the outward normal cone to C at u and by $P_C(u)$ the inward tangent cone to C at u, we have:

$P_C(u) = N_C(u)^-$ the negative polar cone

and hence:

$$\langle r,v\rangle \geq o \quad \text{for any} \quad -r \in N_C(u) \ , \ v \in P_C(u) \tag{2.4}$$

that is : the admissible constraint reactions at a configuration $u \in C$ must perform a non negative virtual work for any displacement field in the tangent cone to C at u.

The properties above qualify the external constraints as perfect (i.e. firm and frictionless) convex constraints.

2.2 Internal constraints

We shall assume that the stress distributions in the structure must belong to a closed convex set Q of admissible stresses. No constitutive properties are defined for the material, that is no relation is assumed between the strain fields and the stress distributions.

3. Equilibrium

In this formal context we look for the condition to be imposed on an applied load distribution $\ell \in V'$ acting on the structure at a configuration $u \in C$ in order that equilibrium does hold, that is : there exists an admissible constraint reaction

$$-\ r \in N_C(u) \text{ such that } \langle \ell + r,v\rangle = o \quad \text{for any } v \in N(T) \tag{3.1}$$

namely, the virtual work performed by the external force distribution, which is the sum of the active load and of the constraint reactions, is zero for any rigid displacement.

In absence of internal constraints on the stress distributions, the condition to be imposed on the load distribution is given by : $\langle \ell, v\rangle \leq o$ for any $v \in N(T) \cap P_C(u)$ (3.2)

In fact (2.4) and (3.1) imply (3.2).

To prove the converse let us rewrite (3.2) as:

$$\ell \in \{N(T) \cap P_C(u)\}^- = N(T)^\perp + P_C(u)^- = R(T') + N_C(u)$$

which ensures that it's possible to find a constraint reaction $-r \in N_C(u)$ and a stress distribution $\sigma \in W'$ such that:

$$\ell + r = T'\sigma \qquad \text{or equiv.} \qquad (3.3)$$

$$\langle \ell, v\rangle + \langle r, v\rangle = \langle \sigma, Tv\rangle \qquad \text{for any } v \in V \text{ , or equiv.} \qquad (3.4)$$

$$\langle \ell, v\rangle \leq \langle \sigma, Tv\rangle \qquad \text{for any } v \in P_C(u) \qquad (3.5)$$

The closed convex set of all the stress distributions which satisfy the equilibrium condition (3.5) is denoted by $\Sigma(\ell,u)$.

Let us now investigate about the possibility of equilibrium under the internal constraint condition $\sigma \in Q$.

In other terms we look for a necessary and sufficient condition ensuring that the convex set $\Sigma(\ell,u) \cap Q$ is not empty.

The answer to such a question is provided by the following:

Equilibrium theorem

The set $\Sigma(\ell,u) \cap Q$ is not empty if and only if one of the following equivalent conditions is satisfied:

$$\langle \ell, v\rangle \leq \sup\{\langle \tau, Tv\rangle \ / \ \tau \in Q\} \qquad \text{for any } v \in P_C(u) \qquad (3.6)$$

$$\langle \ell, v\rangle \leq \langle \tau, Tv\rangle \quad \text{for any } \tau \in Q \text{ , } v \in P_C(u) \text{ such that } Tv \in N_Q(\tau) \qquad (3.7)$$

namely, the external virtual work of the load distribution must be not greater than the maximum admissible internal virtual work for any displacement field in the tangent cone to C at u.

First we prove that (3.6) is equivalent to (3.7), by observing that:

$<\tau_o, Tv> = \sup \{ <\tau, Tv> \ / \ \tau \in Q \}$

when $\tau_o \in Q$ is such that $Tv \in N_Q(\tau_o)$

The proof of the theorem is then carried out as follows:

- the only if part is trivial since (3.5) clearly implies (3.6) when $\sigma \in Q$.
- the if part will be proved on the basis of a preliminary result:

<u>Lemma</u>

Let us denote by $\{ \tau + P_Q(\tau) \}$ the tangent conical manifold to Q at $\tau \in Q$. Then the following conditions are equivalent:

a) $\Sigma(\ell, u) \cap Q$ is not empty

b) $\Sigma(\ell, u) \cap \{ \tau + P_Q(\tau) \}$ is not empty for any $\tau \in Q$

Clearly a) implies b) since:

$\{ \tau + P_Q(\tau) \} \supset Q$ for any $\tau \in Q$

The converse implication, that b) implies a), will be proved by contradiction, showing that if a) is false then b) is false too. In fact if $\Sigma(\ell, u) \cap Q$ is empty it should exist a $\tau_o \in Q$ such that $\Sigma(\ell, u) \cap \{ \tau_o + P_Q(\tau_o) \}$ is empty.

Having established this preliminary result we go back to the proof of the if part of the equilibrium theorem.

We shall prove that (3.7) implies a).

To this end it is sufficient to rewrite (3.7) in the form:

$<\ell - T'\tau, v> \leq o$ for any $\tau \in Q$, $v \in P_C(u)$ such that $Tv \in N_Q(\tau)$

or equivalently:

$$\begin{aligned} \ell - T'\tau \in \{ P_C(u) \cap T^{-1}(N_Q(\tau)) \}^- &= \\ = P_C(u) + T'(N_Q(\tau)^-) &= \\ = N_C(u) + T'(P_Q(\tau)) & \quad \text{for any } \tau \in Q \end{aligned}$$

hence :

$$\ell \in N_C(u) + T'\{ \tau + P_Q(\tau)\} \qquad \text{for any } \tau \in Q$$

which is equivalent to b) and, by the Lemma, to a).

3.1 Conical internal constraints

It is interesting to specialize the equilibrium condition to the case when the set of admissible stress distributions is a convex cone.

In this case we have that:

$$N_Q(\tau) = Q^- \qquad \text{for any } \tau \in Q$$

and hence:

$$\langle \tau, Tv \rangle \leq o \qquad \text{for any } \tau \in Q \text{ , } Tv \in N_Q(\tau)$$

so that the equilibrium condition becomes:

$$\langle \ell, v \rangle \leq o \qquad \text{for any } v \in P_C(u) \text{ such that } Tv \in Q^-$$

4. Compatibility

Let us now investigate about a condition ensuring that a given strain field $\varepsilon \in W'$ is geometrically compatible, namely that it exists an admissible displacement field $u \in C$ such that:

$$\varepsilon = Tu$$

To this end let us denote by:

$$\Sigma_o(u) = \{ \sigma \in W' \text{ such that } \langle \sigma, Tv \rangle \geq o \text{ for any } v \in P_C(u)\}$$

the closed convex cone of the selfequilibrated stress distributions at the configuration $u \in C$.

The answer to our question is given by the following:

Compatibility theorem

A strain field $\varepsilon \in W'$ is geometrically compatible if and only if it satisfies the condition:

$$\langle\tau,\varepsilon\rangle \geq \langle\tau,Tv\rangle = \langle r(\tau),v\rangle \quad \text{for any } v\in C,\ \tau\in\Sigma_o(v) \tag{4.1}$$

proof:

Let us first remark that, by definition we have:

$$\Sigma_o(v) = \{TP_C(v)\}^+ \quad \text{the positive polar cone} \tag{4.2}$$

Now if $\varepsilon = Tu$ with $u\in C$, the condition (4.1) can be written:

$$\langle\tau,T(u - v)\rangle \geq o \qquad \text{for any } v\in C,\ \tau\in\Sigma_o(v)$$

which is an immediate consequence of (4.2), since clearly:

$u - v \quad P_C(v)$ for any $v\in C$.

This proves the only if part.

To prove the if part we rewrite (4.1) as:

$$\varepsilon - Tv\in\Sigma_o(v)^+ = TP_C(v) \quad \text{for any } v\in C$$

or equivalently:

$$\varepsilon\in T\{v + P_C(v)\} \text{ for any } v\in C \tag{4.3}$$

We shall prove that (4.3) implies in fact that:

$$\varepsilon\in TC \tag{4.4}$$

To this end we first remark that the set $T^{-1}(\varepsilon)$ is the affine manifold:

$u_o + N(T)$ where $u_o\in V$ is such that $Tu_o = \varepsilon$

From (4.3) it follows that such an u_o does exist and the set:

$\{u_o + N(T)\}\cap\{v + P_C(v)\}$ is not empty for any $v\in C$.

Hence, by the same argument used in the proof of the Lemma in par.3, we infer that the set:

$\{u_o + N(T)\}\cap C$ is not empty

whence (4.4) and the if part is proved.

Remark

In the application of the variational condition (4.1) as a sufficient condition it is obviously possible to choose, among the elements of C, those which are significant, in the sense that they belong to a subset C_e of C such that the family:

$\{\Sigma_o(v) \;/\; v \in C_e\}$

is exhaustive, namely it includes all the selfequilibrated stress distributions.

4.1 Conical external constraints

Let us consider the special case when C is a conical manifold:

$$C = w_o + C_o \qquad w_o \in V$$

where C_o is a closed convex cone.

The elements belonging to the largest affine manifold L included in C are the maximally constrained configurations of the structure. Trivially $w_o \in L$.

The closed convex cone of the selfequilibrated stress distributions at a configuration $w \in L$ is indipendent from the special choice of w in L. In fact:

$$\Sigma_o(w) = \Sigma_o = \{ <\tau, Tv> \geq o \quad \text{for any } v \in C_o\} = (TC_o)^+$$

Moreover it is easy to show that:

$$\Sigma_o(u) \subset \Sigma_o \qquad \text{for any } u \in C$$

Hence we get that Σ_o is exhaustive and the compatibility condition (4.1) reduces to (1),(2):

$$<\tau, \varepsilon> \geq <\tau, Tw> = \ll r(\tau), w> \qquad \text{for any } \tau \in \Sigma_o$$

where $w \in L$ is any fixed maximally constrained configuration.

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ELASTOSTATICS OF STRUCTURES WITH UNILATERAL CONDITIONS ON STRESS AND DISPLACEMENT FIELDS

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Summary

A general analysis of the elastostatic problem for structures with unilateral conditions on the stress distributions and on the displacement fields is developed.

The unilateral external constraints are assumed to define a convex conical manifold of admissible displacement fields.

Linear elastic materials with a convex constitutive condition on the stress are considered.

Anelastic strain are assumed to develop according to a convex coniugacy rule which generalizes the standard normality rule of perfect plasticity.

A complete theoretical scheme of the constitutive properties of the material is developed on this basis.

The existence of a convex and differentiable elastic strain energy is proved and the expression of the complementary elastic energy is given.

Two general results yielding the equilibrium and the geometric compatibility conditions under external and internal convex constraints are invoked to formulate the basic variational principles governing the elastostatic problem.

The minimum principles for the potential and the complementary energy functionals and the related error bounding techniques, extending the classical results in linear elasticity, are established.

It is shown that, under suitable regularity assumptions, namely the additivity of the involved subdifferentials, the stress formulation yields the existence of the solution for the elastostatic problem.

1. INTRODUCTION

In structural mechanics the analysis of models in which unilateral conditions are imposed by external and internal constraints is of the greatest interest.

As a matter of fact, in the reality, geometric external constraints on the displacement fields are often of a unilateral type.

Moreover constitutive models with convex conditions on the internal stress distributions may simulate in an effective way the behaviour of a number of interesting structural materials.

Among these we way mention the "no tension" materials and the "no compression" membrane type models.

Rock mechanics problems and the analysis of concrete and masonry structures are important fields of application in structural engineering.

We present here a general theory of structural models in which the unilateral external constraints are assumed to define a convex conical manifold of admissible displacement fields.

The material properties are characterized by a convex yield condition on the stress distributions.

An elastic constitutive model is considered by splitting the total strain response into the sum of a linear elastic and an anelastic part.

For the latter a convex conjugacy rule, which generalizes the standard normality rule of perfect plasticity, is assumed.

In this respèct it is worth noting that the violation of such a rule can be shown to lead to generally non consistent physical models (G.Romano, M.Romano.[1]).

The constitutive scheme is analyzed in the general context of convex analysis and the existence of a convex and differentiable elastic strain energy is proved.

The properties of the proximity operators, introduced by J.J. Moreau.[2], are the basic tools in this investigation.

The displacement and the stress formulations of the elastostatic problem are developed on the basis of general results concerning the equilibrium and the geometric compatibility under external and internal convex constraints (G.Romano, M.Romano.[3]).

The minimum principles for the potential and the complementary energy are shown to be necessary and sufficient conditions for the solution of the elastostatic problem and the related error bounding techniques are established.

The existence of the solution of the elastostatic problem is proved under the assumption that the subdifferentials of the indicator functions of the convex sets of the admissible and of the equilibrated stress distributions do have the additivity property.

The proof is founded upon a previous result, due to one of the authors (G.Romano.[8]), establishing the sufficiency of the principle of the minimum complementary energy, for linear elastic materials under unilateral external constraints.

2. GENERALITIES

We shall develop the analysis of the structural model in the geometrical context of the small displacements theory in which non linear geometric effects are neglected and hence velocity fields and displacements from a reference configuration can be identified.

The linear spaces of displacement and strain fields will be respectively denoted by V and W and the dual spaces of external forces and of internal stress distributions will be denoted by V' and W'.

The duality pairings representing the external and the internal virtual works will both be denoted by the symbol $\langle \cdot , \cdot \rangle$.

The deformation operator : $T : V \rightarrow W$

mapping the displacement fields into the corresponding strain fields, and the dual equilibrium operator : $T' : W' \rightarrow V'$

mapping the internal stress distributions into the corresponding external force distributions, are related by the virtual work identity:

$$\langle \sigma , Tv \rangle = \langle T'\sigma , v \rangle \qquad \text{for any} \qquad v \in V \; , \quad \sigma \in W'$$

The following orthogonality relations will be assumed to hold :

$$R(T) = N(T')^{\perp} ,$$

$$R(T') = N(T)^{\perp} ,$$

where R and N denote the range and the null space, respectively.

When endowing the spaces V and W with a Hilbert space topology, the above orthogonality relations can be proved under the assumption that the range R(T) of T is closed.

3. GEOMETRIC CONSTRAINTS

We consider the structure to be subject to external unilateral constraints such that the set of the admissible displacements is a convex conical manifold : $C = w_o + C_o \quad , \quad w_o \in V$,

where C_o is a closed convex cone with vertex at the origin. Denoting by L the largest affine manifold included in C, the displacements $w \in L$ will correspond to maximally constrained configurations (m.c.c.).

The mechanical definition of perfect constraints states that the admissible reactions at a configuration $u \in C$ must perform a non-negative virtual work for any variation of displacement compatible with the unilateral external constraints.

In geometric terms, this means that the admissible reactions r must belong to the inward cone to C at u, that is :

$$- r \in \theta\chi_C(u) \ , \tag{3.1}$$

where the symbol θ denotes the subdifferential operator and χ is the indicator function of C. The condition (3.1) is then equivalent to :

$$<r, v - u> \geq o \qquad \text{for any} \qquad v \in C \ .$$

We remark that at an m.c.c. :

$$- \theta \ \chi_C(w) = C_\circ^+ ,$$

with $C_\circ^+$ the positive polar cone of $C_\circ$.

4. CONSTITUTIVE PROPERTIES

We shall analyze a constitutive model for materials in which the admissible stress distributions are required to belong to a closed convex set.

The material response is assumed to be the sum of a linear elastic strain field and of an anelastic one which satisfies the standard normality rule.

The theory will be developed in the general context of convex analysis and the basic properties of the constitutive model will be derived by a suitable extension of some ideas and results due to J.J.Moreau.[2].

Let us consider a pair of conjugate convex functions f and g, defined on the strain and on the stress space, respectively.

By assuming that f and g are the pointwise suprema of their affine minorants, the conjugacy relation can be written as :

$$f\ (\delta) = \sup\ \{\ \langle\tau,\delta\rangle - g(\tau)\ /\ \tau \in W'\} \qquad \delta \in W, \tag{4.1}$$

$$g\ (\sigma) = \sup\ \{\ \langle\sigma,\eta\rangle - f(\eta)\ /\ \eta \in W\ \} \qquad \sigma \in W'\ . \tag{4.2}$$

We recall that σ and δ are conjugate with respect to f and g if one of the following equivalent relations holds :

i) $f(\delta) + g(\sigma) = \langle\sigma,\delta\rangle$,

ii) $\sigma \in \theta\, f(\delta)$ i.e. $f(\eta) - f(\delta) \geq \langle\sigma,\eta - \delta\rangle$ for any $\eta \in W$,

iii) $\delta \in \theta\, g(\sigma)$ i.e. $g(\tau) - g(\sigma) \geq \langle\tau - \sigma,\delta\rangle$ for any $\tau \in W'$.

Denoting by $S : W \to W'$ the elastic stiffness operator assumed to be linear, positive definite and symmetric, and by $A = S^{-1}$ the elastic compliance operator, the material response is assumed to be :

$$\varepsilon = A\sigma + \delta \quad \text{or} \qquad (4.3)$$

$$\sigma = S(\varepsilon - \delta)$$

where the anelastic strain δ satisfies a generalized normality rule :

$$\delta \in \theta\, g(\sigma) \qquad (4.4)$$

A suitable extension of Moreau's definition of the proximity operator (J.J.Moreau.[2]) allows to derive the general properties of the constitutive model.

To this end let us define the following proximity operators [(1)]:

a) $\sigma = \text{prox}_{gA}(S\varepsilon)$ is the solution of the minimum problem :

$$\min \{\tfrac{1}{2}\ \| S\varepsilon - \tau \|^2_A + g(\tau)\ /\ \tau \in W'\} ,$$

(1) It can be proved that the minimum problems below admit an unique solution (J.J.Moreau.[2]).

i.e., σ satisfies

$$\frac{1}{2} \| S\varepsilon - \sigma \|_A^2 + g(\sigma) = \xi(S\varepsilon) \ , \qquad (4.5)$$

b) $\delta = \mathrm{prox}_{fS}(\varepsilon)$ is the solution of the minimum problem :

$$\min \{ \frac{1}{2} \| \varepsilon - \eta \|_S^2 + f(\eta) \ / \ \eta \in W \}$$

i.e. δ satisfies

$$\frac{1}{2} \| \varepsilon - \delta \|_S^2 + f(\delta) = \phi(\varepsilon) \qquad (4.6)$$

where $\| \cdot \|_S$ and $\| \cdot \|_A$ denote the norm in the energy of S and A respectively and ξ and ϕ are the functionals that associate to ε the value of the minima (4.5) and (4.6)

We remark that the proximity operator is in fact a generalization of the orthogonal projector and reduces to it when the involved convex function is the indicator of a closed convex set.

By virtue of a classical result of J.L.Lions.[4], (C.Baiocchi, A.Capelo.[5]), the minimum problems (4.5) and (4.6) are equivalent to the variational inequalities :

$$g(\tau) - g(\sigma) \geq \langle \tau - \sigma, \varepsilon - A\sigma \rangle \quad \text{for any} \quad \tau \in W', \qquad (4.7)$$

$$f(\eta) - f(\delta) \geq S(\varepsilon - \delta), \eta - \delta \rangle \quad \text{for any} \quad \eta \in W \ , \qquad (4.8)$$

which, by the definition of the subdifferential operator, can be written as :

$$\delta = \varepsilon - A\sigma \in \theta \ g(\sigma) \ ,$$

$$\sigma = S(\varepsilon - \delta) \in \theta\, f(\delta).$$

The additive decomposition of the total strain field in the elastic part $A\sigma$ and the anelastic part δ, which satisfies the generalized normality rule (4.4), is thus uniquely defined by the proximity operators a) and b).

The functionals $\phi(\varepsilon)$ and $\xi(S\varepsilon)$, defined by (4.5) and (4.6), result to be convex and differentiable and their gradients are the proximity operators :

$$\sigma = \text{grad}\ \phi(\varepsilon) = \text{prox}_{gA}(S\varepsilon), \tag{4.9}$$

$$\delta = \text{grad}\ \xi(S\varepsilon) = \text{prox}_{fS}(\varepsilon). \tag{4.10}$$

Since from (4.9) σ is the gradient of ϕ with respect to ε, the potential $\phi(\varepsilon)$ has the meaning of the elastic strain energy of the material.

We shall give hereafter an explicit proof of (4.9) which provides some inequalities useful in the sequel.

To this end we first observe that, for any pair of strain fields ε and $\varepsilon_\circ$, denoting by δ , $\delta_\circ$ the associated anelastic strains, the following identity holds :

$$\begin{aligned}\phi(\varepsilon) - \phi(\varepsilon_\circ) &= \phi(\varepsilon) + \phi(\varepsilon_\circ) - 2\phi(\varepsilon_\circ) = \\ &= \phi(\varepsilon) + \phi(\varepsilon_\circ) - \|\varepsilon_\circ - \delta_\circ\|_S^2 - 2f(\delta_\circ) = \\ &= \phi(\varepsilon) + \phi(\varepsilon_\circ) - 2f(\delta_\circ) +\end{aligned}$$

$$+ \langle S(\varepsilon_o - \delta_o), (\varepsilon - \varepsilon_o) - (\delta - \delta_o) - (\varepsilon - \delta) \rangle .$$

Now, from (4.6), we have :

$$\phi(\varepsilon) + \phi(\varepsilon_o) - 2f(\delta_o) = \tfrac{1}{2} \| (\varepsilon - \delta) - (\varepsilon_o - \delta_o) \|_S^2 +$$

$$+ \langle S(\varepsilon_o - \delta_o), \varepsilon - \delta \rangle + f(\delta) - f(\delta_o),$$

and then, setting :

$$\sigma = S(\varepsilon - \delta) \quad \text{and} \quad \sigma_o = S(\varepsilon_o - \delta_o),$$

we get :

$$\phi(\varepsilon) - \phi(\varepsilon_o) = \tfrac{1}{2} \| \sigma - \sigma_o \|_A^2 - \langle \sigma_o, \delta - \delta_o \rangle +$$

$$+ f(\delta) - f(\delta_o) + \langle \sigma_o, \varepsilon - \varepsilon_o \rangle \qquad (4.11)$$

From (4.11) and the inequality (4.8) finally it follows that :

$$\phi(\varepsilon) - \phi(\varepsilon_o) \geq \tfrac{1}{2} \| \sigma - \sigma_o \|_A^2 + \langle \sigma_o, \varepsilon - \varepsilon_o \rangle , \qquad (4.12)$$

and , a fortiori :

$$\phi(\varepsilon) - \phi(\varepsilon_o) \geq \langle \sigma_o, \varepsilon - \varepsilon_o \rangle , \qquad (4.13)$$

and, interchanging the roles of ε and ε_o :

$$\phi(\varepsilon) - \phi(\varepsilon_o) \leq \langle \sigma, \varepsilon - \varepsilon_o \rangle , \qquad (4.14)$$

From (4.13) and (4.14) we get :

$$0 \leq \phi(\varepsilon) - \phi(\varepsilon_o) - \langle \sigma_o, \varepsilon - \varepsilon_o \rangle \leq \langle \sigma - \sigma_o, \varepsilon - \varepsilon_o \rangle \leq$$

$$\leq \| \varepsilon - \varepsilon_o \|_S^2 \qquad (4.15)$$

where the last inequality follows from the non expansion property of the proximity operators (J.J.Moreau.[2]).

From the inequalities (4.15) we see that : $\sigma_\circ = \text{grad}\ \phi(\varepsilon_\circ)$.

Then by (4.13) we get the convexity of ϕ.

The convex conjugate functional of ϕ is defined by :

$$\psi(\sigma) = \sup \{ \langle\sigma,\eta\rangle - \phi(\eta) \ / \ \eta \in W \}.$$

The concave and differentiable functional : $\langle\sigma,\eta\rangle - \phi(\eta)$ attains its maximum at $\eta = \varepsilon = A\sigma + \delta$ since its gradient at this point is : $\sigma - \text{grad}\ \phi(\varepsilon) = 0$.

The explicit expression of the functional ψ is thus given by :

$$\psi(\sigma) = \langle\sigma,\varepsilon\rangle - \phi(\varepsilon) = \| \sigma \|^2_A + \langle\sigma,\delta\rangle - \tfrac{1}{2}\| \sigma \|^2_A - f(\delta) =$$
$$= \tfrac{1}{2}\| \sigma \|^2_A + g(\sigma) \tag{4.16}$$

By the conjugacy relation and (4.16) we get :

$$\varepsilon \in \theta\psi(\sigma) = A\sigma + \theta\ g(\sigma), \tag{4.17}$$

whence we infer that $\psi(\sigma)$ is the complementary elastic energy of the material. We remark that, while ϕ is differentiable, ψ is differentiable if and only if the function g is.

Let us now consider the special constitutive model in which the stress distributions are assumed to belong to a closed convex set Q. We may infer the basic properties from the general scheme developed above by setting :

$$g(\sigma) = \chi_Q(\sigma) = \begin{cases} 0 & \text{if } \sigma \in Q , \\ +\infty & \text{if } \sigma \notin Q , \end{cases} \tag{4.18}$$

where χ_Q is the indicator function of the convex Q.

The conjugate function of g turns out to be the support function f of the convex Q :

$$f(\delta) = \sup \{ <\tau,\delta> / \ \tau \in Q \},$$

and the conjugacy relations give :

$$\sigma \in Q ,$$

$$\delta \in \theta\chi_Q(\sigma) ,$$

where $\theta\chi_Q(\sigma)$ is the outward normal cone to Q at σ.

Bv substituting (4.18) into (4.5) we get :

$$\inf \{ \tfrac{1}{2} \| \tau - S\varepsilon \|_A^2 \ / \ \tau \in Q \} = \tfrac{1}{2} \| \sigma - S\varepsilon \|_A^2 = \xi(S\varepsilon) ,$$

and then :

$$\sigma = \mathrm{proj}_Q(A;S\varepsilon) ;$$

namely, the stress distribution is the orthogonal projection of $S\varepsilon$, in the energy of A, on Q.

An interesting special case is met when the set Q of admissible stress distributions is a closed convex cone.

Such a model can effectively be adopted to simulate the response of materials without tensile strength and with a very high compressive strength. Rock mechanics problems and the analysis of concrete and masonry structures are fields of application in structural engineering.

In this case the conjugate function f turns out to be the indicator function of the negative polar Q^- of Q, and we have :

$$\delta = \mathrm{proj}_{Q^-}(S;\varepsilon);$$

namely, the anelastic strain is the orthogonal projection of the total strain ε, in the energy of S, on Q^-.

5. THE ELASTOSTATIC PROBLEM

The elastostatic problem for the structural model defined above is formulated as follows :

Given a load distribution $\ell \in V'$,

and a prescribed strain field $\varepsilon \in W$

Find an admissible displacement field $u \in C$, (5.1)

and an admissible stress distribution $\sigma \in Q$, (5.2)

satisfying the constitutive property : $\sigma = \text{grad}\, \phi(Tu - \varepsilon)$ (5.3)

and the equilibrium condition : $\langle \ell, v \rangle \geq \langle \sigma, Tv \rangle$

for any $v \in \{\theta\, \chi_C(u)\}^-$, (5.4)

where $\{\theta\, \chi_C(u)\}^-$ is the negative polar cone of the outward normal cone to C at u, that is, the closed cone generated by the admissible variations of displacements from u.

The convex set of all stress distributions satisfying (5.4) will be denoted by $\Sigma_\ell(u)$.

Then (5.2) and (5.4) are equivalent to : $\sigma \in \Sigma_\ell(u) \cap Q$.

The condition to be imposed to the load distribution so that the convex set $\Sigma_\ell(u) \cap Q$ be not empty, is given by (G.Romano,M.Romano[3]) :

$$<\ell,v> \leq \text{supp}_Q(Tv) \quad \text{for any} \quad v \in \{\theta\ \chi_C(u)\}^- \qquad (5.5)$$

where supp $(\cdot)$ denotes the support function :

$$\text{supp}_Q(Tv) = \sup \{<\sigma,Tv> \ / \ \sigma \in Q\}.$$

The equilibrium condition (5.5) can be stated as follows :

for any admissible variation of displacement field, the external virtual work must be not greater than the maximum internal virtual work.

5.1 Displacement formulation

By substituting (5.3) into (5.4) we get the variational condition to be satisfied by the displacement field $u \in C$:

$$<\text{grad}\ \phi(Tu - \varepsilon),Tv> \geq <\ell,v> \quad \text{for any} \quad v \in \{\theta\ \chi_C(u)\}^- . \qquad (5.6)$$

Defining the potential energy functional :

$$\Phi(u) = \phi(Tu - \varepsilon) - <\ell,u> \qquad (5.7)$$

the condition (5.6) can be written as :

$$- \text{grad}\ \Phi(u) \in \theta\ \chi_C(u) \ ; \qquad (5.8)$$

namely : the gradient of the potential energy functional at u must belong to the inward cone to C at u.

It is easily seen that (5.8) is equivalent to the minimum problem :

$$\Phi(u) = \min \{ \Phi(v) \ / \ v \in C \}. \qquad (5.9)$$

A direct proof of the existence of the solution for the minimum problem above was not available up to now.

A proof for the finite dimensional case, under linear geometric constraints has been given in (G.Romano,M.Romano.[9]).

We shall give an existence proof for the solution of the elastostatic problem, for the infinite dimensional case, in the next paragraph.

5.2 Stress formulation

The elastostatic problem may alternatively be set in terms of the stress distributions which are in equilibrium, under the applied load $\ell \in V'$, in a maximally constrained configuration $w \in L$.

In fact it can be proved that (G.Romano.[8]) :

The condition : $\quad -\varepsilon + Tw \in \theta\, \chi_{\Sigma_\ell}(\sigma) \ , \quad \sigma \in \Sigma_\ell = \Sigma_\ell(w),$ (5.10)

or, equivalent : $\quad \langle \tau - \sigma, \varepsilon - Tw \rangle \geq 0$ for any $\tau \in \Sigma_\ell$,

implies that :

i) the strain field ε is compatible, that is, there exists an admissible displacement field $u \in C$ such that $\varepsilon = Tu$.

i) the stress distribution $\sigma \in \Sigma_\ell$ is in equilibrium at $u \in C$ under the action of the load ℓ, i.e., $\sigma \in \Sigma_\ell(u)$.

Now, by setting :

$$\varepsilon = \varepsilon^* + A\sigma + \delta \quad \text{with} \quad \sigma \in Q \ , \quad \delta \in \theta\, \chi_Q(\sigma),$$

we get :

$$\varepsilon \in \varepsilon^* + \theta\psi(\sigma) \tag{5.11}$$

Moreover, setting :

$$r(\sigma) = T'\sigma - \ell$$

and, defining the complementary energy functional :

$$\Psi(\sigma) = \psi(\sigma) - \langle\sigma,\varepsilon^*\rangle - \langle r(\sigma),w\rangle \tag{5.12}$$

the conditions (5.10) and (5.11) can be written as :

$$0 \in \theta(\Psi + \chi_{\Sigma_\ell})(\sigma). \tag{5.13}$$

This is equivalent to the minimum problem :

$$\Psi(\sigma) = \min \{ \Psi(\tau) / \ \tau \in \Sigma_\ell \} \tag{5.14}$$

which can also be written as :

$$\Psi(\sigma) = \min \{\tfrac{1}{2} \|\tau\|_A^2 - \langle\tau,\varepsilon^*\rangle - \langle r(\tau),w\rangle / \tau \in \Sigma_\ell \cap Q \} \tag{5.15}$$

This minimum problem admits an unique solution if and only if the convex set $\Sigma_\ell \cap Q$ is not empty, i.e. iff the condition (5.5) is satisfied.

To prove that the solution of (5.15) is also a solution of the elastostatic problem, we can follow the above steps backwards.

It has to be remarked that, to deduce (5.12) from (5.13), we need the validity of the additive property of the subdifferentials.

Sufficient conditions can be found in (J.J.Moreau.[6]) and an attitude to overcome this difficulty has been proposed in (B.Nayroles.[7]).

5.3 Error bound

From the inequality (4.12) we get :

$$\Phi(u) - \Phi(u_\circ) = \phi(u) - \phi(u_\circ) + \langle \ell, u - u_\circ \rangle \geq$$

$$\geq \tfrac{1}{2} \|\sigma(u) - \sigma(u_\circ)\|_A^2 + \langle \ell, u - u_\circ \rangle + \langle \sigma_\circ, T(u-u_\circ) \rangle .$$

Analogous developments for the functional Ψ yield :

$$\Psi(\sigma) - \Psi(\sigma_\circ) = \psi(\sigma) - \psi(\sigma_\circ) - \langle r(\sigma) - r(\sigma_\circ), w \rangle \geq$$

$$\geq \tfrac{1}{2} \| \sigma - \sigma_\circ \|_A^2 + \langle \sigma - \sigma_\circ, Tu_\circ \rangle - \langle r(\sigma) - r(\sigma_\circ), w \rangle ,$$

where $\sigma(u) = \text{grad}\ \phi(Tu)$, $u \in C$ and $\sigma \in \Sigma_\ell$.

If $u_\circ$, $\sigma_\circ$ is the solution of the elastostatic problem, then $\sigma(u_\circ) = \sigma_\circ$, and the last terms in the above inequalities, by virtue of (5.4) and (5.10), are non-negative. Moreover :

$$\Phi(u_\circ) + \Psi(\sigma_\circ) = \langle \sigma_\circ, Tu_\circ \rangle - \langle \ell, u_\circ \rangle - \langle r(\sigma_\circ), w \rangle = \langle r(\sigma_\circ), u_\circ - w \rangle = 0 .$$

Hence, adding the above inequalities, we get the extimate :

$$\Phi(u) + \Psi(\sigma) \geq \tfrac{1}{2} \| \sigma(u) - \sigma_\circ \|_A^2 + \tfrac{1}{2} \| \sigma - \sigma_\circ \|_A^2 ,$$

for any $u \in C$, $\sigma \in \Sigma_\ell$.

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SULLA DEFORMABILITA' A TAGLIO DELLE TRAVI DI PARETE SOTTILE

Pubblicazione N. 312

SULLA DEFORMABILITA' A TAGLIO DELLE TRAVI DI PARETE SOTTILE

Giovanni Romano
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Sommario

Si mostra che la teoria classica che tiene conto della deformabilita' a taglio delle travi di parete sottile non fornisce risultati corretti nel caso di travi con sezione trasversale priva di assi di simmetria.
Viene formulato un modello coerente con le ipotesi di base della teoria tecnica e si dimostra che le direzioni principali della matrice di deformabilità a taglio non coincidono in generale con quelle della matrice di deformabilità a flessione.
Vengono presentati degli esempi numerici per sezioni chiuse ed aperte che mettono in luce le differenze rispetto alla teoria classica.

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1. Introduzione

In molti problemi di ingegneria strutturale ed in particolare nella analisi sismica di edifici con nuclei irrigidenti, è necessaria la valutazione della deformabilità a taglio di travi di parete sottile con sezioni chiuse ed aperte.

In questo lavoro si intende portare un contributo allo studio di questo classico problema, mostrando che la formulazione presente nella letteratura non è corretta per travi la cui sezione trasversale sia priva di assi di simmetria, in quanto fondata sull'ipotesi che le deformazioni da taglio e da flessione possano essere valutate separatamente nei due piani principali d'inerzia. Ciò è evidentemente suggerito dalla teoria del taglio alla Saint Venant, nella quale però gli spostamenti trasversali sono associati alle sole deformazioni da flessione, mentre le deformazioni da taglio producono soltanto ingobbamento delle sezioni rette.

Nella teoria tecnica i vincoli sulle rotazioni sono imposti in termini di spostamenti associati alla deformazioni flessionali, e quindi anche le deformazioni da taglio producono spostamenti trasversali.

La formulazione della teoria in un sistema di riferimento generale mostra che non è possibile disaccoppiare gli effetti della flessione e del taglio nei due piani principali d'inerzia. Infatti le rispettive matrici di deformabilità possono non avere le stesse direzioni principali.

Esempi numerici indicano che, per sezioni prive di assi di simmetria, l'orientamento delle direzioni principali per le due matrici di deformabilità può essere notevolmente diverso.

2. Modello cinematico e relazioni costitutive

Si consideri una trave di parete sottile ad asse rettilineo ed un sistema di riferimento ortogonale O,x,y,z con l'asse z parallelo all'asse della trave e gli assi x ed y arbitrari.

Si introducono le seguenti notazioni :

A = sezione della trave

k = versore dell'asse della trave

r = vettore posizione nel piano della sezione

u = spostamento trasversale del centro di taglio

ϕ = rotazione flessionale della sezione

δ = scorrimento trasversale della sezione

ε = dilatazione longitudinale

La derivazione rispetto all'ascissa z sarà indicata con un apice.

I simboli $\cdot$, x, $\otimes$, denoteranno rispettivamente il prodotto scalare, vettoriale e tensoriale.

Il modello cinematico a base della teoria tecnica è definito dalle relazioni

$$\varepsilon = k \times \phi' \cdot r \qquad (2.1)$$

$$\delta = k \times \phi + u' \qquad (2.2)$$

Si denoti con E il modulo di Joung e con G quello di elasticità tangenziale del materiale. E' conveniente inoltre introdurre i vettori

$$\tau = (\tau_{zx}, \tau_{zy}) \qquad \gamma = (\gamma_{zx}, \gamma_{zy})$$

I legami costitutivi sono allora espressi dalle relazioni

$$\sigma = E\ \varepsilon \tag{2.3}$$

$$\tau = G\ \gamma \tag{2.4}$$

dove σ è la tensione normale.

3. Dualità

Il lavoro virtuale interno per unità di lunghezza nella teoria delle travi è dato da

$$\int_A \sigma\ \varepsilon^* dA \ + \ \int_A \tau \cdot \gamma^* dA \tag{3.1}$$

Nel caso di sollecitazione di flessione e taglio, assumendo come parametri cinematici indipendenti Φ' e δ, come enti duali si hanno rispettivamente il momento flettente M ed il taglio T che sono legati dalla relazione di equilibrio

$$T = k \cdot x\ M' \tag{3.2}$$

Il lavoro virtuale interno per unità di lunghezza può quindi essere espresso nella forma

$$M \cdot \phi'^* + T \cdot \delta^* \tag{3.3}$$

dove

$$M \cdot \phi'^{*} = \int_A \sigma \, \varepsilon^{*} \, dA \tag{3.4}$$

e

$$T \cdot \delta^{*} = \int_A \tau : \gamma^{*} dA \tag{3.5}$$

Le relazioni (3.4) e (3.5) consentono di esprimere i parametri cinematici ϕ' e δ come funzioni esplicite di M e T e di ottenere quindi le rappresentazioni delle matrici di deformabilità a flessione ed a taglio.

Infatti facendo uso della (2.3) e della (2.1), dalla (3.4) si ha

$$M \cdot \phi'^{*} = \int_A E \, \varepsilon \, \varepsilon^{*} \, dA =$$

$$= \int_A E \, (r \otimes r) \, dA \, (k \times \phi') \cdot (k \times \phi')^{*} \tag{3.6}$$

Introducendo la matrice d'inerzia elastica della sezione retta

$$I_E = \int_A E \, r \otimes r \, dA \tag{3.7}$$

la (3.6) si può scrivere nella forma

$$k \times M = I_E \, (k \times \phi') \tag{3.8}$$

che fornisce la classica relazione tra momento flettente e curvatura flessionale.

Definendo la matrice di deformabilità flessionale C_f come l'inversa della I_E, dalla (3.8) si ha la relazione

$$k \times \phi' = C_f \, (k \times M) \tag{3.9}$$

che esprime ϕ' in funzione di M.

Per ottenere una espressione esplicita del parametro di scorrimento δ si osserva che dalla (3.5) si ha

$$T^{*} \cdot \delta = \int_A \tau^{*} \cdot \tau / G \, dA = \int_c q^{*} q \, / \, (Gb) \, ds \tag{3.10}$$

dove b è la lunghezza della corda, s è l'ascissa curvilinea sulla linea media c della sezione e q è il flusso del vettore τ attraverso la corda all'ascissa s.

Espressioni esplicite del flusso q in funzione del taglio T si ricavano nel seguito per il caso delle sezioni sottili aperte e quello delle sezioni sottili chiuse.

4. Sezioni aperte

Dalla relazione di equilibrio interno

$$\operatorname{div} \tau = - \sigma' \tag{4.1}$$

facendo uso delle (2.3), (2.1), (3.2) e della (3.7), si ha

$$\operatorname{div} \tau = - C_f T \cdot E r \qquad (4.2)$$

Il flusso q uscente dalla parte A* della sezione retta, per la (4.2), è dato da

$$q = \int_{A*} \operatorname{div} \tau \, dA = - C_f S_E \cdot T \qquad (4.3)$$

dove $S_E = \int_{A*} E r \, dA$ è il vettore momento statico di A*.
Sostituendo la (4.3) nella (3.8) si ottiene

$$T^* \cdot \delta = (C_f \int_c (S_e \otimes S_e)/(Gb) \, ds \, C_f) \, T^* \cdot T \qquad (4.4)$$

e quindi

$$\delta = C_f \Gamma C_f T \qquad (4.5)$$

dove

$$\Gamma = \int_c (S_E \otimes S_E)/(Gb) \, ds \qquad (4.6)$$

Se si definisce la matrice di deformabilità a taglio

$$C_t = C_f \Gamma C_f \qquad (4.7)$$

la relazione (4.5) si può scrivere nella forma

$$\delta = C_t T \qquad (4.8)$$

La matrice C_t è simmetrica ed ha direzioni principali che in generale non coincidono con quelle della C_f.

5. Sezioni chiuse

Per motivi di semplicità formale si considerano sezioni chiuse monoconnesse (sezioni tùbolari). I risultati si estendono in modo ovvio al caso di sezioni pluriconnesse.
Scelta una corda di riferimento b_o e detto q_o il flusso in corrispondenza di essa, posto

$$q = q_a + q_o \tag{5.1}$$

dalle (4.1) e (4.2) si ha

$$q_a = - C_f \, S_E \cdot T \tag{5.2}$$

e quindi dalla (3.8)

$$T^* \cdot \delta = \int_c (q_a + q_o)^* (q_a + q_o)/(Gb) \, ds \tag{5.3}$$

Il flusso q_o è determinato dalla relazione di congruenza

$$\int_c \gamma \, ds = \int_c (q_a + q_o)/(Gb) \, ds = 0 \tag{5.4}$$

ed ha quindi la rappresentazione

$$q_o = - \int_c q_a/(Gb)\, ds \,/\, \int_c 1/(Gb)\, ds \tag{5.5}$$

Per la condizione (5.4) la (5.3) diventa

$$T^* \cdot \delta = \int_c (q_a - q_o)\, q_a^*/(Gb)\, ds \tag{5.6}$$

Sostituendo la (5.5) nella (5.6) si ha

$$T^* \cdot \delta = \int_c (q_a\, q_a^*)/(Gb)\, ds -$$

$$- \int_c q_a/(Gb)\, ds \int_c q_a^*/(Gb)\, ds \,/\, \int_c 1/(Gb)\, ds \tag{5.7}$$

Facendo uso della (5.2) si ottiene la seguente espressione per la matrice di deformabilità a taglio

$$C_t = C_f \left(\int_c (S_E \otimes S_E)/(Gb)\, ds -\right.$$

$$\left. - \int_c S_E/(Gb)\, ds \otimes \int_c S_E/(Gb)\, ds \,/\, \int_c 1/(Gb)\, ds\right) C_f \tag{5.8}$$

Anche la matrice C_t definita dalla (5.8) è simmetrica ed ha direzioni principali che in generale non coincidono con quelle della matrice C_f.

6. Esempi numerici

Sono stati elaborati esempi numerici per tre sezioni sottili aperte e tre sezioni sottili chiuse prive di assi di simmetria.

I risultati sono consegnati nelle figure da 1 a 6. Le linee medie delle sezioni sono tracciate a tratto e punto. Le direzioni principali delle matrici di deformabilità a flessione C_f sono indicate con linee continue, quelle delle matrici di deformabilità a taglio C_t con linee tratteggiate.

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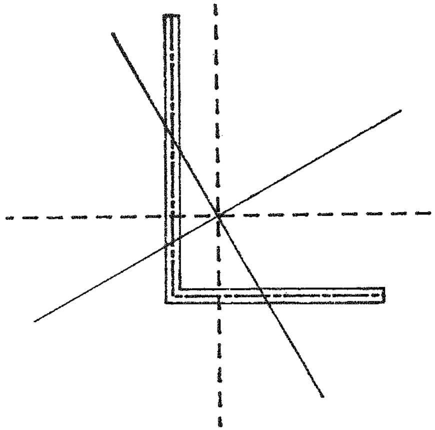

Fig. 1

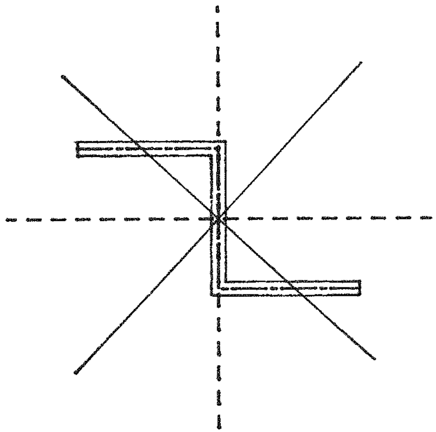

Fig. 2

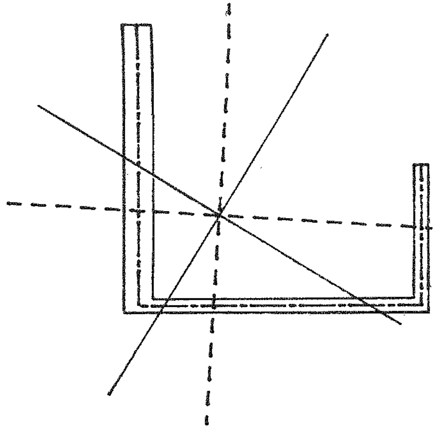

Fig. 3

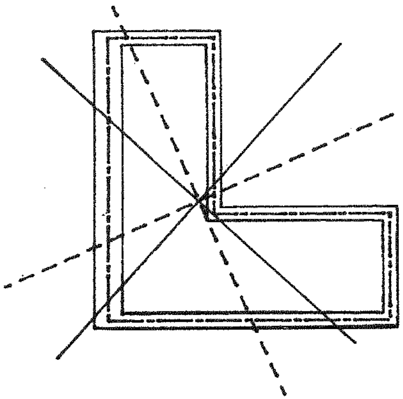

Fig. 4

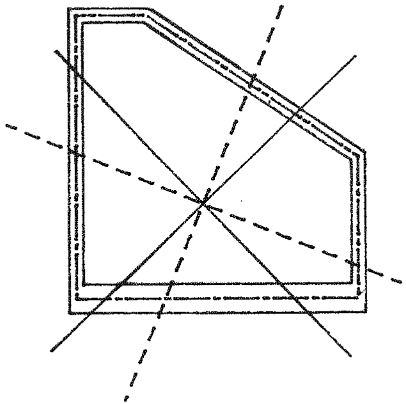

Fig. 5

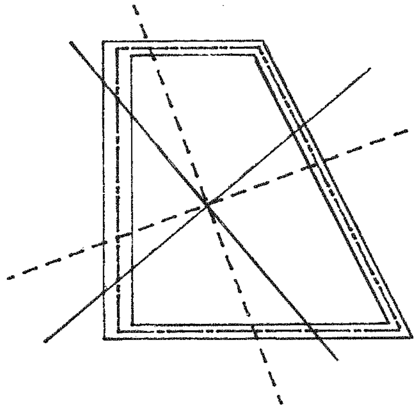

Fig. 6

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AN ALGEBRAIC APPROACH TO LINEAR ELASTOSTATICS

Pubblicazione N. 313

AN ALGEBRAIC APPROACH TO LINEAR ELASTOSTATICS

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ABSTRACT

An algebraic theory of linear elastic structures is presented. The concept of duality is emphasized to be the basic tool for the formulation of a well posed model in structural mechanics. Two general direct decomposition properties of the relevant linear spaces are proved and are shown to yield a simple direct proof of the existence and uniqueness of the solution of the linear elastostatic problem, in the displacement and in the stress formulation.

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1. INTRODUCTION

The elastostatic problem in linear structural mechanics has been a classical object of investigation and is rigthly considered a well established chapter of structural analysis.

Although the basic principles and methods are well known since long ago, only rather recently the existence theory has been satisfactorily developed in the context of functional analysis (see e.g. (1),(2)).

In this respect it should be noticed that a rigorous treatment of the existence problem in a topological framework requires a quite deep mathematical background and the recourse to sophisticated technical arguments which aren't usually available to mainly mechanical minded people.

Moreover in most situations the topological arguments needed in the analysis cannot be given a sound mechanical interpretation.

Further on it should be emphasized that the analytical complexity of mechanical problems, when set in a functional topological context, is readily sweeped out if reference is made to finite dimensional models.

As a matter of fact, very often the essential aspects of a mechanical problem are hidden in an overriding multitude of analytical details.

Two basic ideas stem from these observations :

1) The algebraic structure of a mechanical problem yields the basic informations on the model to be investigated.

In fact the primitive concepts in mechanics are of a purely algebraic nature.

2) The role of duality is fundamental to introduce non observable objects in mechanics: force and stress distributions have to be considered as derived notions, the geometric entities being the basic ones.

In the light of these observations we shall present here an algebraic approach to linear elastostatics which yields the fundamental existence and uniqueness results and the main properties of the model without any explicit recourse to topological arguments.

The basic tools in the development of the classical displacement and stress formulations are two direct decomposition theorems which are established in the appendix.

2. BASIC NOTIONS

In the definition of the formal structure of a mechanical theory one has readily to be convinced that the essential aspects of the basic notions are of an algebraic nature.

Starting from the definition of the observable objects - configurations, displacements, strains and related rate quantities - the non observable ones - forces and stresses - are introduced by means of duality pairings, via the concept of virtual power.

Since we shall deal with the linearized theory, where displacements from a reference configuration and velocity fields are assumed to be identified, the following set of elements can be regarded as basic for the definition of the structural model :

a) An affine space of configurations which can be identified with the linear space V of displacements from a given reference configuration.

b) A linear subspace V_r of "rigid" displacements.

c) A "tangent deformation" linear operator :

$T : V \rightarrow W$

mapping the displacement space into a "strain space" W and such that :

$N(T) = V_r$

that is the null space of T is exactly V_r

d) The linear space of "external" force distributions, defined to be the dual V' of V , i.e. the space of linear functionals on V.

e) The linear space of "internal" stress distributions defined to be the dual W' of W .

We shall denote with the same symbol $\langle \cdot , \cdot \rangle$ the duality pairings between V' and V and between W' and W . The value of the linear functionals force and stress distributions, will respectively be called the "external" and the "internal" virtual work .

f) The "equilibrium" operator :

$T' : W' \rightarrow V'$

which is the dual of T :

$\langle T'\sigma , v \rangle = \langle \sigma , Tv \rangle \qquad$ for any $v \in V$, $\sigma \in W'$

Affine perfect constraints may be considered by introducing the following definitions :

g) An affine variety $L \subseteq V$ of admissible displacements :

$L = w + L_o$

where w is any element in V and L_o is the linear subspace of the variations of admissible displacements .

h) The linear subspace of constraint reactions :

$L_o \;=\; \{ r \in V' : \langle r,v \rangle = o \quad \text{for any } v \in L_o \}$

Constraints reactions are thus characterized as the force distributions which perform no virtual work for any variation of admissible displacement field.

3. ORTHOGONALITY RELATIONS

For the development of the theory the following orthogonality properties are essential :

i) $N(T) = R^{\perp}(T')$

ii) $N(T') = R^{\perp}(T)$

iii) $R(T) = N^{\perp}(T')$

iv) $R(T') = N^{\perp}(T)$

where R( ) and N( ) denote respectively the range and the null space and the suffix $\perp$ the orthogonal complement.

While the first two properties are readily proved to be a direct consequence of the definition of dual operators, the proof of the last two is anything but trivial in the general case.

If the relevant spaces V and W are endowed with an Hilbert space topology, and the duality pairings are compatible with this topological structure, then iii) and iv) can be proved by taking the orthogonal complements of i) and ii) , under the assumption that R(T) is a closed subspace.

This is the standard way to get these results in the functional analysis approach.

A well known closure result for R(T) , in the context of linear continuum mechanics where T = symgrad, is given by the Korn s inequalities (1) .

In a purely algebraic context the four relations above always do hold true (3).

Unfortunately if V' and W' are the algebraic duals of V and W , that is no continuity requirement is made, the force and the stress distribution spaces result to be too large for a convenient development of the theory.

For instance, there is no hope to establish a one to one correspondence between the strain and the stress space as required in elasticity, unless a finite dimensional model is assumed.

We feel that these shortcomings are sure symptoms that the analytical framework of the mathematical theory is not adequate to a convenient analysis of the mechanical problem.

An emphasis in this direction has also been given in a very interesting paper by B.Nayroles (4).

Since we are mostly interested in the mechanical aspects of the theory, we shall simply assume in the sequel that the above orthogonality relations hold true.

Anyway a full mathematical consistency, in the context of functional analysis can easily be achieved by assuming that :

i) The functional spaces are Hilbert.

ii) The range of T is closed.

iii) The affine variety L is closed.

4. EQUILIBRIUM

A load distribution $l \in V'$ is said to be in equilibrium if it performs no virtual work for any rigid variation of admissible displacement, namely:

$$\langle l,v\rangle = o \quad \text{for any} \quad v \in L_o \cap N(T) \tag{4.1}$$

or equivalently :

$$l \in (L_o \cap N(T))^{\perp}$$

Representation theorem

If $l \in V'$ is in equilibrium, then there exists a stress distribution $\sigma \in W'$ and a constraint reaction $r \in L_o^{\perp}$ such that :

$$l + r = T'\sigma \tag{4.2}$$

proof :

$$l \in (L_o \cap N(T))^{\perp} = L_o^{\perp} + N^{\perp}(T) = L_o^{\perp} + R(T')$$

and hence the result.

The converse statement can be easily proved noting that (4.2) is equivalent to the variational condition :

$<l,v> + <r,v> = <T'\sigma,v> = <\sigma,Tv>$ for any $v \in V$

and, since $r \in L_o$, also equivalent to :

$<l,v> = <\sigma,Tv>$ for any $v \in L_o$ (4.3)

The stress distributions satisfying (4.3) are said to be in equilibrium with l , and belong to the affine variety :

$\Sigma_l = \sigma_l + \Sigma_o$

where σ_l is an element of Σ_l and :

$\Sigma_o = \{ \sigma \in W' : <\sigma,Tv> = o$ for any $v \in L_o \} = (TL_o)$

is the linear subspace of selfequilibrated stress distributions.

It is easily shown that the constraint reactions satisfying (4.2) belong to the affine variety :

$R_l = r_l + T'\Sigma_o$

The representation theorem above ensures that if l is in equilibrium, the affine varieties Σ_l and R_l are not empty .

5. COMPATIBILITY

A strain field $\varepsilon \in W$ is said to be compatible if there exists an admissible displacement field $u \in L$ such that :

$\varepsilon = Tu$

or equivalently if : $\varepsilon \in TL$

A necessary and sufficient condition for the compatibility of a strain field is given by the following :

Compatibility theorem

A strain field $\varepsilon \in W$ is compatible if and only if :

$<\tau,\varepsilon> = <r(\tau),w>$ for any $\tau \in \Sigma_o$

where w is any element of L ,

that is : the internal virtual work performed by any selfequilibrated stress distribution for the given strain field must be equal to the external virtual work performed by the corresponding selfequilibrated constraint reaction for an admissible displacement field.

proof :

since : $\tau \in \Sigma_o$ we have that : $r(\tau) = T'\tau$ and hence :

$<\tau,\varepsilon> = <T'\tau,w> = <\tau,Tw>$ for any $\tau \in \Sigma_o$

or equivalently :

$<\tau,\varepsilon - Tw> = o$ for any $\tau \in \Sigma_o$

which , being : $\Sigma_o = (TL_o)^{\perp}$

implies : $\varepsilon - Tw \in \Sigma_o^{\perp} = TL_o$

whence : $\varepsilon \in TL$.

6. LINEAR ELASTICITY

Linear elastic structures are characterized by the existence of a symmetric , positive definite constitutive operator which establishes a one to one linear mapping between the strain and the stress space:

$S : W \to W'$ elastic stiffness

$<S\varepsilon_1,\varepsilon_2> = <S\varepsilon_2,\varepsilon_1>$ for any $\varepsilon_1,\varepsilon_2 \in W$

$<S\varepsilon,\varepsilon> \geq o$ for any $\varepsilon \in W$

$<S\varepsilon,\varepsilon> = o \to \varepsilon = o$

The inverse operator $C = S^{-1}$ is the elastic compliance .

7. LINEAR ELASTOSTATICS

The linear elastostatic problem in structural mechanics is stated as follows :

Given an equilibrated load distribution $l \in V'$

and a prescribed strain field $\varepsilon \in W$

Find an admissible displacement field $u \in L$.

and a stress distribution $\sigma \in \Sigma_1$

such that : $F\sigma + \varepsilon = Tu$.

The equilibrium condition on l is clearly necessary for the existence of a solution, since otherwise Σ_1 would be empty.

It will be shown, on the basis of simple algebraic arguments, that this condition is also sufficient .

The analysis will be carried out in the context of the two classical approaches to the elastostatic problem : the displacement and the stress formulation .

The basic arguments are founded upon two direct decomposition results proved in the appendix .

8. DISPLACEMENT FORMULATION

Let us consider the displacement field as the basic unknown , so that the problem consists to :

Find $u \in L$ such that : $S(Tu - \varepsilon) \in \Sigma_1$

Let us first analyze the associated homogeneous problem, when : $L = L_o$ and $\varepsilon = o$.

The variational formulation of the equilibrium condition gives :

$\langle l,v\rangle = \langle STu,Tv\rangle$ for any $v \in L_o$

Setting :

$K = T'ST$ the structural stiffness operator

we get the following equivalent statement of the homogeneous elastostatic problem :

Find $u_o \in L_o$ such that : $Ku_o - l \in L_o^{\perp}$

Existence of a solution is then readily inferred from the equilibrium condition and the direct decomposition result provided by theorem 1 of the appendix :

$l \in (L_o \cap N(T))^{\perp} = L_o^{\perp} \oplus KL_o$

that is :

any equilibrated load distribution can be uniquely decomposed in the sum of a constraint reaction and of an admissible elastic response (i.e. an elastic response to an admissible displacement field)

It is also apparent that the displacement solution is unique to within an additional admissible displacement field which doesn't give rise to an elastic response of the structure, i.e. an additional admissible rigid displacement field .

The previous discussion can be readily extended to the general case by setting :

$u = w + v_o$ with $w \in L$ and $v_o \in L_o$

$f = l - T'S\varepsilon - T'STw$ equivalent external force

In fact the variational condition of equilibrium :

$\langle l,v \rangle = \langle S(Tu - \varepsilon),Tv \rangle$ for any $v \in L_o$

is equivalent to :

$Kv_o - f \in L_o^{\perp}$

and since : $f \in (L_o \cap N(T))^{\perp}$

an appeal to theorem 1 of the appendix again yields the desired result .

9. STRESS FORMULATION

If we consider the stress distribution as the basic unknown , the problem will consist to :

Find $\sigma \in \Sigma_1$ such that : $C\sigma + \varepsilon \in TL$

Analyzing first the homogeneous case, let us set :

$\sigma = \sigma_1 + \sigma_o$ with $\sigma_1 \in \Sigma_1$ and $\sigma_o \in \Sigma_o$.

The compatibility problem will then be written as :

Find $\tau_o \in \Sigma_o$ such that : $C\sigma_1 + C\tau_o \in TL_o = \Sigma_o^{\perp}$

Now, by theorem 2 of the appendix we have that :

$C\sigma_1 \in W = C\Sigma_o \oplus \Sigma_o^{\perp} = C\Sigma_o \oplus TL_o$

that is :

any deformation field can be uniquely decomposed in the sum of a compatible one and of an elastic one corresponding to a selfequilibrated stress distribution.

This yields the existence and uniqueness result for the stresss solution of the homogeneous problem.

To extend this result to the general case it is sufficient to set:

$\eta = C\sigma_1 + \varepsilon - Tw$

so that the compatibility problem may be rewritten :

Find $\tau_o \in \Sigma_o$ such that : $C\tau_o + \eta \in TL_o = \Sigma_o^{\perp}$

Since $\eta \in W$, theorem 2 of the appendix again yields the desired result .

It is worth noting that the previous result can be also derived on the basis of theorem 3 of the appendix by setting $\Omega = \Sigma_o^{\perp} = TL_o$ to get : $W' = \Sigma_o \oplus STL_o$

that is :

any stress distribution can be uniquely decomposed in the sum of a selfequilibrated one and of one corresponding to a compatible elastic strain field .

APPENDIX

We shall present here some direct decomposition results concerning the relevant spaces in the linear elastostatics and the related operators.

To this end the following simple properties of the stiffness operator $K = T'ST$ will be needed :

i) K is symmetric and positive definite

ii) $R(K) = N(K)$

iii) $N(K) = N(T)$

iv) $R(K) = R(T')$

DIRECT DECOMPOSITION THEOREMS

Let L_o be a linear subspace of V.

LEMMA 1.1

$(KL_o)^{\perp} \cap L_o = N(K) \cap L_o$

proof :

$v \in L_o \cap (KL_o)^{\perp} \rightarrow \langle Kv,v\rangle = o \; , \; v \in L_o \rightarrow \quad Kv = o \; , \; v \in L_o$

$v \in L_o \cap N(K)$

LEMMA 1.2

$KL_o \cap L_o^{\perp} = o$

proof : $f \in KL_o \cap L_o^{\perp} \rightarrow$ exists $v \in L_o$ such that $f = Kv \; , \; \langle Kv,v\rangle = o \rightarrow Kv = o \rightarrow f = o$.

THEOREM 1

$$(L_o \cap N(K))^{\perp} = L_o^{\perp} \oplus KL_o$$

The symbol $\oplus$ denotes, as usual, the direct sum of subspaces .

proof :

taking the orthogonal complements in the statement of LEMMA 1.1, by the trivial intersection property of LEMMA 1.2 we get the result .

Let Σ_o be a linear subspace of W' .

LEMMA 2.1

$$(C\Sigma_o)^{\perp} \cap \Sigma_o = \{o\}$$

proof :

$$\tau \in (C\Sigma_o)^{\perp} \cap \Sigma_o \rightarrow \langle C\tau, \tau\rangle = o \rightarrow \tau = o$$

LEMMA 2.2

$$C\Sigma_o \cap \Sigma_o^{\perp} = \{o\}$$

proof :

$\varepsilon \in C\Sigma_o \cap \Sigma_o^{\perp} \rightarrow$ exists $\tau \in \Sigma_o$ such that :

$\varepsilon = C\tau$, $\langle\tau, C\tau\rangle = o \rightarrow \tau = o$.

THEOREM 2

$$W = C\Sigma_o \oplus \Sigma_o^{\perp}$$

proof :

taking the orthogonal complements in the statement of LEMMA 2.1 , by the trivial intersection property of LEMMA 2.2 , we get the result .

THEOREM 3

$W' = (C\Sigma_o)^{\perp} \oplus \Sigma_o$

proof :

interchange the roles of LEMMA 2.1 and LEMMA 2.2 in the proof of THEOREM 2 .

Two further direct decomposition formulas involving the elastic stiffness operator can be easily derived by considering a linear subspace Ω of W :

$W' = S\Omega \oplus \Omega^{\perp}$

$W = (S\Omega)^{\perp} \oplus \Omega$

It has to be remarked however that these last formulas can also be directly derived from THEOREM 2 and THEOREM 3 by noting that:

$(C\Sigma_o)^{\perp} = S\,\Sigma_o^{\perp}$

$(S\,\Sigma_o^{\perp})^{\perp} = C\,\Sigma_o$

and setting : $\Sigma_o^{\perp} = \Omega$

A further result can be derived by interchanging the roles of LEMMA 1.1 and LEMMA 1.2 in THEOREM 1, to get :

$V = L_o \oplus (KL_o)^{\perp} = L_o \oplus K^{-1}(L_o^{\perp})$

with : $L_o \cap (KL_o)^{\perp} = L_o \cap N(K)$

that is, in terms of the homogeneous elastostatic problem :

any displacement field can be decomposed in the sum of an admissible one and of one which gives rise to a selfequilibrated elastic response .

The decomposition is unique to within an admissible additional rigid displacement .

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EIGENVECTOR ESTIMATES AND APPLICATION TO SOME PROBLEMS OF STRUCTURAL ENGINEERING

Manfredi Romano

Computation of approximate eigenmodes is required in many problems of of structural engineering, so that the relevant error estimates are of great practical importance. This paper presents the complete proof of new error estimates which are based on an original technique due to G. Fichera. In the seismic analysis of structures a number of natural frequencies and associated vibration modes are computed by means of a Rayleigh - Ritz, finite element approximation. It is shown that the error estimates of this paper can be applied to the modal analysis of framed structures. To this end is presented a method to obtain an explicit representation of the Green operator that is suitable for computation by numerical procedures also in the case of very complex structures, as real buildings are.

1. INTRODUCTION

This paper deals with the problem of computing error estimates for eigenvectors in vibration problems of elastic structures. We confine ourselves to structural problems in which the Green operator can be explicitly constructed. In the first part of the paper a complete proof of new estimates, which give upper and lower bounds to the error, is given. The upper estimate was first proposed by the author in [5] as an improved version of the estimate due to G. Fichera [3]. The lower estimate was first presented in [1]. In the second part it is presented a method to obtain an explicit representation of the Green operator in a form which is suitable for computation by numerical procedures, also in the case of very complex structures.

2. BASIC ERROR REPRESENTATION IN AN ABSTRACT SETTING

Since we have in mind application to vibration problems, we shall suppose that the Green operator T of the structure is a positive compact operator on the complex Hilbert space H. Moreover to simplify the notations we shall suppose that all the eigenvalues are simple. All the arguments that follow can be easily extended to the more general case of hermitian compact operators with multiple positive and negative eigenvalues.

We shall denote by $\{\mu_h\}$ the decreasing sequence of eigenvalues of T and by $\{u_h\}$ an orthonormal sequence of associated eigenvectors. We denote by $(\cdot,\cdot)$ the inner product, by $\| \;\|$ the norm and by S_H the unit sphere in H. Let us consider the projectors $P_h = u_h \otimes u_h$ ($\otimes$ = tensor product on HxH) and the orthogonal projector P_o on ker T. The operator T has the following spectral decomposition

$$T = \Sigma \, \mu_h \, P_h \tag{2.1}$$

and every $v \in H$ has the representation

$$v = \Sigma \, P_h \, v + P_o \, v \tag{2.2}$$

Given $v \in S_H$ we define the error relative to the approximation of a subspace U of H by means of v, as the distance of v from U. Hence if we denote by P the projector on U and by Q = I - P its orthogonal complement, the error is given by $\|Qv\|$ .We shall consider the approximation of the eigenspace associated to the eigenvalue μ_k, so that $U = sp(u_k)$, $P = P_k$ and

$$Qv = v - Pv = \sum_{h \neq k} P_h v + P_o v \tag{2.3}$$

Let us denote by Σ the spectrum of T and by Γ the resolvent set. Let Λ be the set $\{\mu_k\}$. Given $v \in S_H$, for every $\mu \in \Gamma \cup \Lambda$, we set

$$Tv - \mu v = f \tag{2.4}$$

From (2.1) we have

$$P_o Tv = TP_o v = 0, \qquad P_h Tv = TP_h v = \mu_h P_h v$$

so that, substituting (2.2) in (2.4), it follows

$$P_o v = - \frac{P_o f}{\mu}, \qquad P_h v = \frac{P_h f}{\mu_h - \mu} \quad h \neq k \tag{2.5}$$

By means of (2.5) and (2.3), we get

$$Qv = \sum_{h \neq k} \frac{P_h f}{\mu_h - \mu} - \frac{P_o f}{\mu} \tag{2.6}$$

hence

$$\| Qv \|^2 = \sum_{h \neq k} \frac{\| P_h f \|^2}{(\mu_h - \mu)^2} + \frac{\| P_o f \|^2}{\mu^2} \tag{2.7}$$

that is the representation we wanted to obtain.

3. THE UPPER ESTIMATE

We have

$$\sum_{h \neq k} \| P_h f \|^2 + \| P_o f \|^2 = \| Qf \|^2 \leq \| f \|^2 \tag{3.1}$$

$$\| f \|^2 = (Tv, Tv) - 2\mu\, (Tv, v) + \mu^2 \tag{3.2}$$

If we define

$$\beta = \|Tv\| \qquad \alpha = (Tv,v) \tag{3.3}$$

the relation (3.2) can be written in the form

$$\|f\|^2 = \beta^2 - 2\mu\alpha + \mu^2 = \beta^2 - \alpha^2 + (\mu - \alpha)^2 \tag{3.4}$$

Making use of (3.1) and (3.4), from the representation (2.7), we get the following upper estimate for $\|Qv\|^2$

$$\|Qv\|^2 \leq \frac{\beta^2 - \alpha^2 + (\mu - \alpha)^2}{\min\limits_{\gamma \notin \Sigma - \Lambda} (\gamma - \mu)^2} = g(\mu) \tag{3.5}$$

Let us notice that by the Cauchy inequality we have $\beta^2 - \alpha^2 \geq 0$.

To obtain an optimal upper estimate we seek the minimum of $g(\mu)$. We shall prove that a necessary condition for the estimate (3.5) to be non trivial, is that

$$\mu \;,\; \alpha \notin \;]\, \mu_{k+1} \;,\; \mu_k \,[\tag{3.6}$$

Let us first suppose that μ doesn't satisfy the condition (3.6). From (2.1) and (2.2) we have

$$\beta^2 = \sum_h \mu_h^2 \;\|P_h v\|^2, \qquad \alpha = \sum_n \mu_h \|P_h v\|^2 \tag{3.7}$$

Substituting (3.7) in (3.4) we see that the numerator of $g(\mu)$ can be written in the form

$$\sum_h (\mu_h^2 - 2\mu\mu_h) \|P_h v\|^2 + \mu^2 \tag{3.8}$$

which is a linear functional in the variables $||P_h v||^2$ on the closed convex polyhedron defined by the inequalities

$$|| P_h v||^2 \geq 0, \qquad \sum_h ||P_h v||^2 \leq 1 \tag{3.9}$$

The extreme points of this convex set are given by the intersections of the linear variety defined by the relation $\sum_h ||P_h v||^2 = 1$ with the coordinate axes, and by the origin. On the i-th coordinate axis it is $||P_i v||^2 = 1$ and $||P_j v||^2 = 0$ for every $j \neq i$, hence the functional (3.8) has the value $(\mu_i - \mu)^2$. In the origin it is $||P_i v||^2 = 0$ for every i, and the functional (3.8) has the value μ^2. Since μ doesn't satisfy (3.6) the values $(\mu_i - \mu)^2$ for $i \neq k$ and μ^2 are never less than the denominator of $g(\mu)$, what implies that

$$\inf_\mu \; g(\mu) \geq 1 \tag{3.10}$$

If μ satisfies the condition (3.6) and α doesn't it is easy to see from (3.5) that again $(\mu - \alpha)^2$ is not less than the denominator of $g(\mu)$, what gives $g(\mu) \geq 1$, and the proof is complete.

We have now to seek the minimum of $g(\mu)$ under the condition (3.6). Let us denote by γ_1 and γ_2 respectively a lower bound to μ_k and an upper bound to μ_{k+1} and by $\delta = (\gamma_1 + \gamma_2)$ the middle point of the interval (γ_1, γ_2).

To obtain computable upper estimates we must substitute the condition (3.6) by the stronger one

$$\mu, \alpha \notin \;]\, \gamma_2 \, , \, \gamma_1 \, [\tag{3.11}$$

Let us define

$$g_k(\mu) = \frac{\beta^2 - \alpha^2 + (\mu - \alpha)^2}{(\mu - \gamma_k)} \qquad k = 1,2 \tag{3.12}$$

We have $g(\mu) = g_2(\mu)$ if $\gamma_2 < \alpha \leq \delta$ and $g(\mu) = g_1(\mu)$ if $\delta \leq \alpha < \gamma_1$. Hence we shall consider the following two cases.

Case 1 :- $\gamma_2 < \alpha \leq \delta$

In this case the function $g(\mu) = g_2(\mu)$ has the minimum in the point

$$\mu = \mu_{02} = \alpha + \frac{\beta^2 - \alpha^2}{\alpha - \gamma_2}$$

If $\mu_{02} \leq \delta$, i.e. if $\beta^2 - \alpha^2 \leq (\delta - \alpha)\ (\alpha - \gamma_2)$, the minimum of $g(\mu)$ is

$$g(\mu_{02}) = \frac{\beta^2 - \alpha^2}{(\alpha - \gamma_2) + \beta^2 - \alpha^2} < 1 \qquad (3.13)$$

If $\mu_{02} \geq \delta$, the minimum of $g(\mu)$ is given by

$$g(\delta) = \frac{\beta^2 - \alpha^2 + (\delta - \alpha)^2}{(\gamma_1 - \gamma_2)/2} \qquad (3.14)$$

Case 2 : $\delta \leq \alpha < \gamma_1$

In this case the function $g(\mu) = g_1(\mu)$ has the minimum in the point

$$\mu = \mu_{01} = \alpha - \frac{\beta^2 - \alpha^2}{\gamma_1 - \alpha}$$

If $\mu_{01} \geq \delta$, i.e. if $\beta^2 - \alpha^2 \geq (\alpha - \phi)\ (\gamma_1 - \alpha)$, the minimum of $g(\mu)$ is

$$g(\mu_{01}) = \frac{\beta^2 - \alpha^2}{(\gamma_1 - \alpha)^2 + \beta^2 - \alpha^2} < 1 \qquad (3.15)$$

If $\mu_{01} \leq \delta$, the minimum of $g(\mu)$ is given by (3.14).

The optimal upper estimate is given by (3.13), (3.14) and (3.15) and requires the knowledge of α and β , defined by (3.3), of the lower bound γ_1 and of the upper bound γ_2. These bounds must be good enough to separate the eigenvalue μ_k from the others.

4. THE LOWER ESTIMATE

Let us set in (2.4) $\mu = \mu_k$. Then we have

$$Pf = P(Tv - \mu_k v) = 0, \qquad Qf = f$$

It follows that

$$\sum_{h\neq k} \|P_k f\|^2 + \| P_o f \|^2 = \| Qf \|^2 = \|f\|^2 \tag{4.1}$$

From (2.7) and (4.1) we get the non trivial lower estimate

$$\| Qv \|^2 \qquad \frac{\beta^2 - \alpha^2 + (\mu_k - \alpha)^2}{\max\{(\mu_1 - \mu_k)^2, \mu_k^2\}} \tag{4.2}$$

Let us now suppose that we have at our disposal a lower bound μ_k' and an upper bound μ_k'' that separate the eigenvalue μ_k from the others. Then we obtain non trivial computable lower estimates from (4.2) in the following way. We substitute the denominator of (4.2) with the greater number given by

$$\max \{(\mu_k'' - \mu_k'), \mu_k''\}$$

and the quantity $(\mu_k - \alpha)^2$ at the numerator of (4.2) with smaller numbers according to the scheme

$$(\mu_k' - \alpha)^2 \qquad \text{if} \quad \alpha \leq \mu_k'$$

$$(\mu_k'' - \alpha)^2 \qquad \text{if} \quad \mu_k'' \leq \alpha$$

$$0 \qquad \text{if} \quad \mu_k' \leq \alpha \leq \mu_k''$$

5. A CLASS OF STRUCTURAL PROBLEMS

Approximate vibration modes of elastic structures are usually computed by a Rayleigh - Ritz, finite element procedure. We consider here structures whose elements can described by a beam model. This class is of great interest in engineering practice.

We want to show how to construct an explicit representation of the Green function for the relevant vibration problem. Let ω be the angular frequency. We set $\lambda = \omega^2$, $\mu = \lambda^{-1}$ and introduce the following notations

n	number of beam elements
$v(x)$	n - vector valued displacement function of the beams
s	r - vector of the Lagrangean parameters in the finite element approximation
V	space of admissible displacement functions
k_i	elastic compliance of the cross sections of the i-th beam
m_i	distributed mass on the axis of the i-th beam
ℓ_i	lenght of the axis of the i-th beam
M	concentrated mass matrix, such that λMs is the vector of the associated inertia forces

The eigenvalue problem has the following variational formulation

$$B\,(v\;,\;v^*) = \lambda\;(\;Ms\cdot s^* + m\,(v,v^*)\;) \qquad (v,v^*) \in V \times R^r \tag{5.1}$$

where $B\,(v\;,\;v^*)$ is the bilinear form of the internal virtual work.

If we define

$$q_i = \lambda\; m_i\; v_i, \qquad f_i = \lambda\;(Ms)_i \tag{5.2}$$

and substitute in (5.1), we get the variational formulation of the associated static problem in the form

$$B\ (v\ ,\ v^*) = f \cdot s + (q, v^*) \tag{5.3}$$

The problem (5.3) can be split in the following two

$$B\ (v'\ ,\ v^*) = (q, v^*) \tag{5.4}$$

$$B\ (v''\ ,\ v^*) = f \cdot s \tag{5.5}$$

Hence

$$v = v' + v'', \qquad\qquad s = s' + s'' \tag{5.6}$$

Let us now introduce on the beam axes the adimensional abscissa $x = z_i/\ell_i$, so that integrations will be performed on the interval (0,1).

From the differential formulation of the problem (5.4) we know that v_i' must be of the form

$$v_i'(x) = C_i s' \cdot t(x) \ell_i + \ell_i^3 k_i \int_0^1 \bar{g}(x,y)\ q(y)\ dy \tag{5.7}$$

where C_i is the Boolean matrix that maps s' in the vector of nodal parameters for the i-th beam and t(x) is the vector of displacement functions associated to the nodal parameters. By C_i' we shall denote the traspose of C_i. The function $\bar{g}(x,y)$ is the Green kernel for the clamped beam. Solving (5.4) for s' we find that

$$s' = E\ C_j'\ \ell_j^2 \int_0^1 t(y)\ q_i(y)\ dy \tag{5.8}$$

where E is the compliance matrix of the finite element model of the structure.

From the differential formulation of the problem (5.5) we know that v_i''

must be of the form

$$v''_i\ (x) = C_i s''\ t(x)\ell_i \tag{5.9}$$

and solving (5.5) for s" we get

$$s'' = E\ f \tag{5.10}$$

We may now substitute (5.8) in (5.7) and (5.10) in (5.9) to get from (5.6) and (5.2) an equivalent eigenvalue problem for an integral operator. To this aim let us consider the two sets $A_1 = \underset{i}{X}\ (0,1)$ and $A_2 = \{a\}$ where a is a real number. On A_1 we have the Lebesgue measure dx=dy, on {a} the Dirac measure da. If we define $A = A_1 \cup A_2$, the eigenvalue problem can be put in form

$$H\ z = \int_A h(x,y)\ z(y)\ dy = \lambda z \tag{5.11}$$

where z = (v;s) and the matrix kernel h(x,y) is defined as follows

$$\begin{aligned}
&h_{11}(x,y) = \{\ell_i\ g_{ij}(x,y)\ m_j \ell_j^2\} && \text{on } A_1 \times A_1 \\
&h_{12}(x,a) = \{M\ E\ \ell_i C'_i\ t(x)\} && \text{on } A_1 \times A_2 \\
&h_{21}(a,y) = \{E\ C'_i\ t(y)\ m_i \ell_i^2\} && \text{on } A_2 \times A_1 \\
&h_{22}(a,a) = \{E\ M\} && \text{on } A_2 \times A_2
\end{aligned} \tag{5.12}$$

and

$$g_{ij}(x,y) = C_i\ E\ C'_j\ t(y)\cdot t(x) + \delta_{ij}\ \ell_j\ k_j\ \bar{g}(x,y) \tag{5.13}$$

In (5.12) and (5.13) sum on repeated j is implicit and δ_{ij} is the unit matrix.

Let a normalized r - vector c be an approximate eigenvector computed by the finite element method and define

$$w_i(x) = C_i c \ t(x)\ell_i, \qquad z = (w;c) \tag{5.14}$$

We want to compute

$$\beta^2 = (Hz \ , \ Hz)/(z \ , \ z)$$

$$\alpha = (Hz \ , \ z)/(z \ , \ z)$$

By means of (5.14), (5.11), (5.12) and (5.13) we get

$$(z \ , \ z) = (P_2 + I)\ c \cdot c$$

$$(Hz \ , \ z) = (P_2EP_1 + H_1 + MEP_2 + EP_1 + EM)\ c \cdot c$$

$$(Hz \ , \ Hz) = (P_1EP_2EP_1 + H_3 + 2H_2EP_1 + MEP_2EM + 2ME(P_2EP_1 + H_1))\ c \cdot c$$

where I is the unit matrix and the matrices $E, P_1, P_2, H_1, H_2, H_3$ are defined by the following relations

$$E = C_i' \ K \ C_i$$

$$P_1 = C_i' \ m_i \ \ell_i^3 \ L \ C_i$$

$$P_2 = C_i' \ \ell_i^3 \ L \ C_i$$

$$H_1 = C_i' \ m_i \ \ell_i^7 \ k_i \ G_1 \ C_i$$

$$H_2 = C'_i \; m_i \; \ell_i \; k_i \; G_2 \; C_i$$

$$H_3 = C'_i \; m_i^2 \; \ell_i^{11} \; k_i^2 \; G_3 \; C_i$$

In the above definitions the sum on the repeated indexes i is implicit. The matrices K, L, G_1, G_2, G_3 can be easily computed from the following explicit relations

$$K = \int_0^1 D^2 t(x) \otimes D^2 t(x) \; dx$$

$$L = \int_0^1 t(x) \otimes t(x) \; dx$$

$$G_1 = \int_0^1 \int_0^1 \bar{g}(x,y) \; t(x) \otimes t(y) \; dy \; dx$$

$$G_2 = \int_0^1 \left(\int_0^1 \bar{g}(x,y) \; t(y) \; dy \otimes t(x) \right) dx$$

$$G_3 = \int_0^1 \left(\int_0^1 \bar{g}(x,y) \; t(y) \; dy \otimes {}_0^1 \bar{g}(x,y) \; t(y) \; dy \right) dx$$

The matrices E and P_1 are respectively the compliance and the mass matrix of the finite element model of the structure and hence must be known in order to compute the approximating vector c. All the matrices $E, P_1, P_2, H_1, H_2, H_3$ can be easily computed by techniques that are typical of computer programming, without performing the matrix products that appear in their definitions. Indeed the matrices C_i are really never created, with great saving of time and memory requirements.

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EIGENFREQUENCIES ESTIMATES FOR STRUCTURES WITH NON-PRISMATIC ELEMENTS

M. Cuomo, A. Greco, M. Romano

In this paper it is presented a method for estimating lower bounds to the eigenfrequencies of elastic frames with non-uniform members, based on Fichera's method of Orthogonal Invariants of positive compact operators. To this end the explicit representation of the Green Operator for the structure is obtained. It is also shown how the computational effort needed to get lower bounds can be minimized on the basis of some decomposition properties that are proved in the paper.

1. INTRODUCTION

A great deal of attention has been devoted in the last decades to the estimation of lower bounds to the eigenfrequencies of elastic structures (see, e.g. [1,2,3,4]).

In the present paper elastic frames with non-uniform members are considered, and the lower bounds are obtained applying Fichera's theory of Orthogonal Invariants of positive compact operators. This method has been applied to a simply supported beam in [5], and to plane frames with constant cross-sections in [6]. More recently some authors have considered the case of non-uniform members applying different methods [7,8].

We shall confine ourselves to the case of beams whose height is an affine function of the centroidal abscissa. In fact this case covers a wide range of structural problems, and the explicit computation of the Green function and of the related invariants is not too cumbersome.

The computation of upper bounds to the periods of vibration, is based on a result that seems to be of interest in itself. Indeed it is proved that the sum of the periods of vibration of elastic frames is equal to the sum of two terms with a clear mechanical meaning. The first is the sum of the finite number of periods of vibration computed by the Rayleigh-Ritz method, taking a set of test functions that is a basis for the kernel of the differential operator of equilibrium. The second is the sum of the series of the periods of vibration of each beam considered as clamped.

If the upper bounds to the periods of vibration are obtained employing the orthogonal invariant of degree one and order one, that is the trace of the Green operator, the difference between each Rayleigh-Ritz lower bound and the corresponding upper bound proves to be given by the second term defined above.

This result allows some qualitative considerations that may help to improve the choice of the finite element decomposition in the Rayleigh-Ritz method.

Two numerical examples show that, by the procedure proposed in the paper, satisfactory estimates for the leading eigenfrequencies of engineering structures may be obtained with a reasonable computational effort.

2. FORMULATION OF THE VIBRATION PROBLEM

While most of the results will be proved in an abstract setting, the method will be developed with reference to a frame structure composed of m straight beams.

For each beam z_i will denote the centroidal abscissa and l_{oi} its length. It is convenient to introduce the adimensional abscissa

$$x_i = z_i / l_i \quad x_i \in [0,1]. \qquad (i = 1,\dots, m)$$

It is supposed that for each beam the cross-sectional inertia and area are given by the following functions of x (fig. 1)

$$I_i = I_{oi}\,(1-\vartheta_i x)^3$$

$$(i = 1, \dots, m)$$

$$A_i = A_{oi}\,(1-\vartheta_i x)$$

therefore the flexural and axial stiffnesses have the form

$$\chi_i(x) = \frac{E\, I_{oi}}{l_i^4}\,(1-\vartheta_i x)^3$$

$$v_i(x) = \frac{E\, A_{oi}}{l_i^2}\,(1-\vartheta_i x)$$

where E is the Young Modulus.

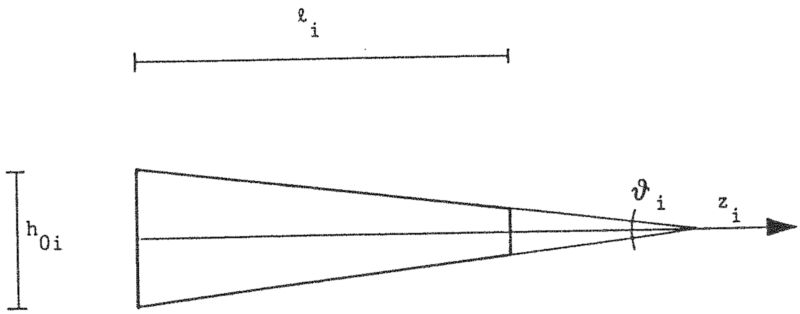

Fig. 1

The mass per unit length for each beam is assumed to be given by

$$m_i(x) = m_{oi}(1-\vartheta_i x) + m_{ci}$$

where m_{ci} accounts for the inertia of non-structural elements that may be connected to the beam. Let's define

$$\mathbf{P}_i(x) = \begin{bmatrix} m_i(x) & 0 \\ 0 & m_i(x) \end{bmatrix}$$

and

$$\mathbf{P} = \mathrm{diag}\,[\,\mathbf{P}_i(x)]$$

Let $u_{ti}(x)$ and $u_{ai}(x)$ be the transversal and axial components of the displacement functions $\mathbf{u}_i(x)$, so that

$$\mathbf{u}_i(x) = [u_{ti}(x), u_{ai}(x)]$$

and define the vector

$$\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m]$$

Let $B(\mathbf{u},\mathbf{w})$ be the bilinear form of the elastic energy that for frame structures is defined on the Sobolev space $H_{2[0,1]}$ and takes the form

$$B(\mathbf{u},\mathbf{w}) = \sum_{i=1}^{m}\left(\int_0^1 \chi_i(x)\, u''_{ti}(x)\, w''_{ti}(x)\, dx + \int_0^1 v_i(x)\, u'_{ai}(x)\, w'_{ai}(x)\, dx\right) \qquad (1)$$

The vibration problem in variational form is set in the subspace U of the admissible beam displacement fields that satisfy all the kinematical constraints. These constraints are expressed by r linear relations imposed on the essential boundary values of the displacement functions for each beam.

The variational formulation of the vibration problem is then given by the relation

$$B(\mathbf{u},\mathbf{w}) = \lambda\,(\mathbf{P}\mathbf{u},\mathbf{w}) \qquad \mathbf{u}\in U \qquad \forall \mathbf{w}\in U \qquad (2)$$

where λ denotes the square of the (circular) frequency ω.

The eigenvalue problem (2) is equivalent to an eingenvalue problem for a positive compact operator. In fact, defining

$$\mathbf{q} = \lambda\, \mathbf{P}\mathbf{u} \qquad (3)$$

relation (2) can be written in the form

$$B(\mathbf{u},\mathbf{w}) = (\mathbf{q},\mathbf{w}) \qquad \mathbf{u}\in U \quad \forall \mathbf{w}\in U \tag{4}$$

The solution to this problem is given by

$$\mathbf{u} = G\mathbf{q} \tag{5}$$

where the Green operator G turns out to be positive and compact [9].

Substituting λ **Pu** for **q** in (5) the following eigenvalue problem which is equivalent to problem (2) is obtained

$$\Gamma\mathbf{u} = \mu\mathbf{u} \tag{6}$$

where $\mu = \lambda^{-1}$ and $\Gamma = GP$

Since the matrix operator **P** is positive and diagonal the operator Γ is a positive compact operator so that the eigenvalue problem (2) has a countable set of positive eigenvalues $\lambda_i = \mu_i^{-1}$.

3. EIGENVALUE ESTIMATES

Lower bounds μ''_h to the first n periods of vibration can be obtained applying the Rayleigh-Ritz method to the eigenvalue problem (2).

Upper bounds μ^*_h are given by the following relation

$$\mu^*_h = \sum_{i=1}^{\infty} \mu_i - \sum_{i=1}^{n} \mu''_i + \mu''_h \qquad (h = 1,\dots, n) \tag{7}$$

In fact

$$\mu^*_h = \sum_{i=1}^{\infty} \mu_i - \sum_{i=1}^{n} \mu''_i + \mu''_h = \sum_{\substack{i=1\\ i\neq h}}^{\infty} \mu_i - \sum_{\substack{i=1\\ i\neq h}}^{n} \mu''_i + \mu_h >$$

$$> \sum_{\substack{i=1\\ i\neq h}}^{\infty} \mu_i - \sum_{\substack{i=1\\ i\neq h}}^{n} \mu_i + \mu_h > \mu_h$$

The infinite sum in (7) is given by [9]

$$\sum_{i=1}^{\infty} \mu_i = \operatorname{tr} \Gamma \tag{8}$$

so that the integral operator Γ needs to be evaluated.

4. EXPLICIT REPRESENTATION OF THE GREEN OPERATOR

Let us consider the decomposition of the space of admissible displacements U in the direct sum of two subspaces U' and U" that are orthogonal in the energy scalar product, i.e.

$$\begin{aligned} &U = U' \oplus U'' \\ &B(\mathbf{u}',\mathbf{u}'') = 0 \qquad \forall \mathbf{u}' \in U',\ \forall \mathbf{u}'' \in U'' \end{aligned} \tag{9}$$

Problem (4) can then be split in the following two

$$B(\mathbf{u}',\mathbf{w}') = (\mathbf{q},\mathbf{w}') \qquad \mathbf{u}' \in U',\ \forall \mathbf{w}' \in U' \tag{10}$$

and

$$B(\mathbf{u}'',\mathbf{w}'') = (\mathbf{q},\mathbf{w}'') \qquad \mathbf{u}'' \in U'',\ \forall \mathbf{w}'' \in U'' \tag{11}$$

The solutions for problems (10) and (11) are given by

$$\mathbf{u}' = G'\,\mathbf{q} \tag{12}$$

and

$$\mathbf{u}'' = G''\,\mathbf{q} \tag{13}$$

where G' and G" are positive compact Green operators. Hence from (9), (12) and (13) it follows that

$$\mathbf{u} = \mathbf{u}' + \mathbf{u}'' = G'\,\mathbf{q} + G''\,\mathbf{q} = (G' + G'')\,\mathbf{q} \tag{14}$$

and

$$G = G' + G'' \tag{15}$$

In order to find an explicit representation of the operators G' and G" the subspaces U' and U" must be defined.

Let U' be the subspace of the displacement functions that satisfy homogeneous boundary conditions at each beam, i.e., the displacement functions for the clamped beams.

The elements of U' have the following representation

$$\mathbf{u}'(x) = G'\,\mathbf{q} = \int_0^1 \mathbf{g}'\,(x,\xi)\,\mathbf{q}(\,\xi)\,d\,\xi \tag{16}$$

where the matrix kernel $\mathbf{g}'\,(x,\xi)$ is given by

$$\mathbf{g}'\,(x,\xi) = \text{diag}\,[\,\mathbf{g}'_i\,(x,\xi)] \qquad (i = 1, \ldots, m)$$

and

$$\mathbf{g}'_i\,(x,\xi) = \begin{bmatrix} g'_{ti}\,(x,\xi) & 0 \\ 0 & g'_{ai}\,(x,\xi) \end{bmatrix}$$

Explicit expressions of the Green kernels $g'_t\,(x,\xi)$ and $g'_a\,(x,\xi)$ for the transversal and axial displacements of the clamped beam with variable cross-section can be found in [11].

The orthogonal subspace U" is defined by the condition

$$B(\mathbf{u}',\mathbf{u}") = 0 \qquad \forall\,\mathbf{u}' \in U'$$

Making use of (1), and integrating by parts, the previous relation becomes

$$B(\mathbf{u}',\mathbf{u}") = (\mathbf{u}',L\mathbf{u}") \qquad \forall\,\mathbf{u}' \in U' \tag{17}$$

since all the boundary terms vanish, due to the definition of U'. The differential operator L is defined as

$$L = \text{diag}\,[\mathbf{L}_i] \tag{18}$$

where

$$\mathbf{L}_i = \begin{bmatrix} D^2\,\chi_i(x)\,D^2 & 0 \\ 0 & -\,D\,\nu_i(x)\,D \end{bmatrix}$$

From relation (17) it follows

$$L\mathbf{u}" = \mathbf{0} \tag{19}$$

so that the subspace U" must be contained in the kernel of the operator L, the dimension of which is 6m. Besides the subspace U" is also contained in the space U of admissible displacements defined by r kinematical constraints. Hence it results

$$U'' = \ker L \cap U$$

and dim U" = n = 6m - r where n is the total number of the free kinematical parameters which will be collected in the vector **d**.

A convenient choice of the set of base functions $\sigma(x)$, $\varepsilon(x)$ allows to get the following representation of the elements v of ker L

$$\mathbf{v}(x) = \mathbf{A}(x)\,\mathbf{s} \tag{20}$$

where

$$\mathbf{A} = \operatorname{diag}\,[\mathbf{A}_i]$$

$$\mathbf{A}_i(x) = \begin{bmatrix} \sigma_i(x) & 0 \\ 0 & \varepsilon_i(x) \end{bmatrix}$$

$$\mathbf{s} = [s_1, s_2, \ldots, s_m]$$

being s_i the vector of the 6 geometric kinematical parameters at the ends of the i-th beam. Explicit representations for the base functions can be found in [11].

Expressing the 6m components of s as functions of the n components of **d** by means of the r linear kinematical constraints that define the space U, the following relation is obtained

$$\mathbf{s} = \mathbf{Q}\,\mathbf{d} \tag{21}$$

so that the functions **u"** of U" take the form

$$\mathbf{u}''(x) = \mathbf{A}(x)\,\mathbf{Q}\,\mathbf{d} \tag{22}$$

The vector **d** is determined as solution of the equilibrium problem

$$B(\mathbf{u}'',\mathbf{w}'') = (\mathbf{q}, \mathbf{w}'') \qquad \mathbf{u}'' \in U'', \quad \forall \mathbf{w}'' \in U'' \tag{23}$$

on the n- dimensional space U".

Using the representation (22) and the definition (1), problem (23) yields

$$\mathbf{Q}^* \int_0^1 \mathbf{N}^*(x)\ \mathbf{T}(x)\ \mathbf{N}(x)\,dx\,\mathbf{Q}\,\mathbf{d} - \mathbf{Q}^* \int_0^1 \mathbf{A}^*(x)\ \mathbf{q}(x)\,dx = 0 \tag{24}$$

where

$$\mathbf{N} = \text{diag}\,[\mathbf{N}_i] \qquad\qquad \mathbf{T} = \text{diag}\,[\mathbf{T}_i]$$

$$\mathbf{N}_i = \begin{bmatrix} D^2\,\sigma^*_i & 0 \\ 0 & D\,\varepsilon_i^* \end{bmatrix} \qquad \mathbf{T}_i = \begin{bmatrix} \chi_i(x) & 0 \\ 0 & \nu_i(x) \end{bmatrix}$$

so that

$$\mathbf{d} = \mathbf{K}^{-1}\,\mathbf{Q}^* \int_0^1 \mathbf{A}^*(x)\,\mathbf{q}(x)\,dx \tag{25}$$

The stiffness matrix $\mathbf{K}$ of the structure is given by

$$\mathbf{K} = \mathbf{Q}^*\,\text{diag}\,[\mathbf{K}_i]\,\mathbf{Q} \tag{26}$$

where

$$\mathbf{K}_i = \int_0^1 \mathbf{N}_i(x)\,\mathbf{T}_i(x)\,\mathbf{N}_i(x)\,dx =$$

$$= \int_0^1 \begin{bmatrix} \chi_i\,(x)\,D^2\sigma_i \otimes D^2\sigma_i & 0 \\ 0 & \nu_i\,(x)\,D\varepsilon_i \otimes D\varepsilon_i \end{bmatrix} dx \tag{27}$$

The matrix $\mathbf{K}_i$ is explicitly given in [11].

Making use of (25) and (22), the functions u''(x) take the form

$$\mathbf{u}''(x) = G''\,\mathbf{q}(x) = \int_0^1 g''\,(\,x,\xi\,)\,\mathbf{q}(\xi)\,d\xi \tag{28}$$

where the kernel g'' (x,ξ) is given by the 2m x 2m matrix

$$g''\,(x,\xi) = \mathbf{A}(x)\,\mathbf{Q}\,\mathbf{K}^{-1}\,\mathbf{Q}^*\,\mathbf{A}^*\,(\xi) \tag{29}$$

The representations (16) and (28) of G' and G'' allow to obtain an explicit form for the operator Γ, since

$$\Gamma = GP = G'P + G''P = \Gamma' + \Gamma'' \tag{30}$$

Therefore the sum of the periods of vibration of the structure is given by

$$\sum_{i=1}^{\infty} \mu_i = \operatorname{tr} \Gamma = \operatorname{tr} \Gamma' + \operatorname{tr} \Gamma'' \tag{31}$$

The result (31) holds due to the fact that the space U has been decomposed in two subspaces that are orthogonal in energy.

5. COMPUTATION OF LOWER BOUNDS - THE RAYLEIGH-RITZ METHOD

In order to get lower bounds to the periods of vibration it is convenient to apply the Rayleigh-Ritz method taking as trial functions a basis of the subspace U".

In fact the substitution of $\lambda''\mathbf{Pu''}$ for $\mathbf{q}$ in (11) leads to the n-dimensional eigenvalue problem

$$B(\mathbf{u''},\mathbf{w''}) = \lambda'' (\mathbf{Pu''},\mathbf{w''}) \qquad \mathbf{u''} \in U'' \qquad \forall \mathbf{w''} \in U'' \tag{32}$$

which, by (13), is equivalent to

$$\Gamma'' \mathbf{u''} = \mu'' \mathbf{u''} \qquad \mathbf{u''} \in U''$$

where $\mu'' = (\lambda'')^{-1}$ and $\Gamma'' = G''\mathbf{P}$. Therefore

$$\sum_{i=1}^{n} \mu''_i = \operatorname{tr} \Gamma'' \tag{33}$$

Substituting in (32) the representation (28) of $\mathbf{u}''$, and recalling definition (27), the following matrix eigenvalue problem is obtained

$$\mathbf{Kd} = \lambda'' \mathbf{Md} \tag{34}$$

where

$$\mathbf{M} = \mathbf{Q}^* \int_0^1 \mathrm{A}^*(x)\, \mathrm{P}(x)\, \mathrm{A}(x)\, dx\, \mathbf{Q} \tag{35}$$

The matrix $\mathbf{M}$ is the so-called consistent mass matrix of the structure, which is explicitly computed in [11].

6. COMPUTATION OF THE UPPER BOUNDS

Making use of the relations (7), (31) and (33), the upper bounds to the periods of vibration, are given by

$$\mu^*_h = \mathrm{tr}\,\Gamma - \mathrm{tr}\,\Gamma'' + \mu''_h = \mathrm{tr}\,\Gamma' + \mu''_h \qquad (36)$$

so that

$$\mu^*_h - \mu''_h = \mathrm{tr}\,\Gamma' \qquad (37)$$

The term tr Γ' has a clear mechanical meaning. Indeed, substituting in (10) λ' **Pu'** for **q**, the following eigenvalue problem on the space U' is obtained

$$B(\mathbf{u}',\mathbf{w}') = \lambda'\,(\mathbf{Pu}',\mathbf{w}') \qquad \mathbf{u}' \in U' \quad \forall\,\mathbf{w}' \in U' \qquad (38)$$

which, owing to (12), is equivalent to

$$\Gamma'\,\mathbf{u}' = \mu'\,\mathbf{u}' \qquad (39)$$

The sum of the eigenvalues of problem (39) is equal to tr Γ' [1] and from the definition of U' it follows that

$$\mathrm{tr}\,\Gamma' = \sum_{i=1}^{m} \sum_{h=1}^{\infty} \mu'^{\,i}_{h} \qquad (40)$$

where $\mu'^{\,i}_{h}$ denotes the h-th period of vibration of the i-th beam considered as clamped.

7. NUMERICAL APPLICATIONS

The method proposed has been applied to the elastic frames with non-uniform elements shown in figs. 2,3. In both cases the first 15 eigenvalues have been estimated.The upper bounds have been computed employing a Sturm-sequence technique developed by Gupta [10]. The lower bounds have been obtained by formula (36).

For the structure of fig. 2, a discretization in 165 degrees of freedom has been adopted. The results of the computation are listed in table I. The first 10 eigenvalues have been separated and the bracketing is especially good for the leading eigenfrequencies.

The structure of fig. 3 has been discretized in 162 degrees of freedom. A lower number of eigenvalues has been separated with respect to the previous case, as can be seen from table II. This is due to the fact that the overall stiffness of the structure is large in comparison with the stiffnesses of the simple beams considered as clamped.

As a consequence the term $\mathrm{tr}\,\Gamma'$, which gives the amplitude of the bounding intervals, is large in comparison with the values of the leading eigenvalues.

8. CONCLUSIONS

An effective method for deriving lower bounds to the eigenfrequencies of elastic frames with non-uniform members has been described. It is based on a general property, namely that the sum of the periods of vibration of the structure is given by the sum of the periods computed by the Rayleigh-Ritz method, with test functions belonging to the kernel of the differential operator of equilibrium, plus the sum of the periods of vibration of the beams considered as clamped.

The validity of this result is based upon the decomposition of the displacement function in two terms, which are orthogonal in energy.

Each upper bound is obtained by adding to the corresponding lower bound the constant quantity defined by (40), that is computationally easy to evaluate. In the case of frames with constant cross-sectional members, this term is fairly trivial to evaluate, since it results ([5])

$$\mathrm{tr}\,\Gamma' = \sum_{i=1}^{m} (m_i / 420\chi_i)$$

Moreover it is observed that there is no need to compute the sum of the Rayleigh-Ritz periods but it is necessary to evaluate only the periods for which the corresponding upper bounds are required.

It is finally observed that, in order to obtain "good" lower bounds for the relevant eigenfrequencies, it is necessary to make the sum of the periods of vibration of the clamped beams as small as possible. This can be achieved, without increasing the size of the Rayleigh-Ritz problem, through an appropriate choice of the finite element decomposition of the frame.

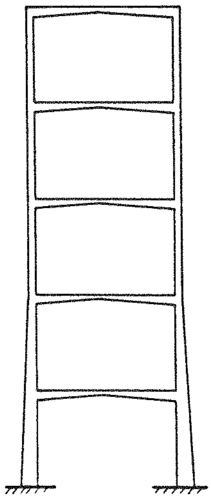

Fig.2

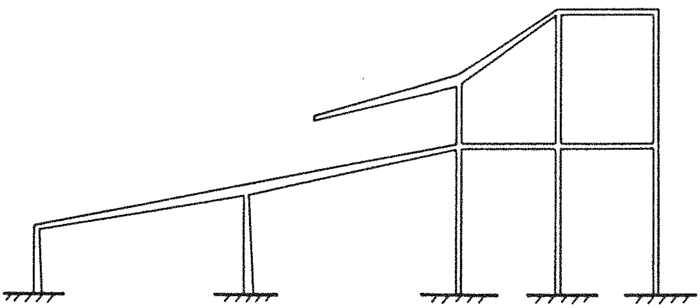

Fig.3

TABLE I

PERIODS $\mu = 1/\omega^2$ $\quad$ $\mathrm{tr}\,\Gamma' = 948.5571881514$

LOWER BOUNDS	n.	UPPER BOUNDS
1800160.9339203200	1	1801109.4911084700
184881.9212992560	2	185830.4784874080
57857.4448859298	3	58806.0020740812
25360.8073655916	4	26309.3645537430
22366.6780710326	5	23315.2352591840
15663.4614306678	6	16612.0186188192
12811.2085365552	7	13759.7657247066
11537.5110392629	8	12486.0682274143
9019.7479876143	9	9968.3051757657
7609.7962575854	10	8558.3534457368
3579.6500493356	11	4528.2072374870
3413.0228051159	12	4361.5799932673
2522.4455949230	13	3471.0027830744
2335.2817342675	14	3283.8389224189
2304.1607536699	15	3252.7178935213

FREQUENCIES $f = 1/(2\pi\sqrt{\mu})$

UPPER BOUNDS	n.	LOWER BOUNDS
.0001186218	1	.0001185905
.0003701459	2	.0003692000
.0006616686	3	.0006563104
.0009993983	4	.0009812167
.0010641905	5	.0010423180
.0012716754	6	.0012348350
.0014061287	7	.0013567962
.0014817129	8	.0014243190
.0016758029	9	.0015940776
.0018244569	10	.0017203825
.0026601115	11	.0023651407
.0027242724	12	.0024098955
.0031689051	13	.0027014234
.0032934456	14	.0027773410
.0033156124	15	.0027905950

TABLE II

PERIODS $\mu = 1/\omega^2$ $\quad$ $\mathrm{tr}\,\Gamma' = 1515.6647437373$

LOWER BOUNDS	n.	UPPER BOUNDS
235562.6040625770	1	237078.2688063150
91187.3358511770	2	92703.0005949143
51188.9939343000	3	52704.6586780373
27595.7592418897	4	29111.4239856271
13980.1674058841	5	15495.8321496214
12858.6522075394	6	14374.3169512768
10710.3792749003	7	12226.0440186376
9352.8438481747	8	10868.5085919120
7247.7711107745	9	8763.4358545118
6219.6605858294	10	7735.3253295667
4121.4144648781	11	5637.0792086154
2956.7107204586	12	4472.3754641959
2694.1125991983	13	4209.7773429356
2184.4393458344	14	3700.1040895717
1960.9242412650	15	3476.5889850023

FREQUENCIES $f = 1/(2\pi\sqrt{\mu})$

UPPER BOUNDS	n.	LOWER BOUNDS
.0003279193	1	.0003268694
.0005270513	2	.0005227250
.0007034477	3	.0006932592
.0009580738	4	.0009327997
.0013460585	5	.0012785352
.0014035323	6	.0013274755
.0015378632	7	.0014393856
.0016456910	8	.0015266348
.0018694674	9	.0017001331
.0020180726	10	.0018095927
.0024791168	11	.0021197911
.0029269528	12	.0023798578
.0030662832	13	.0024529606
.0034052589	14	.0026164543
.0035940964	15	.0026992522

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APPROXIMATION ERROR IN SEISMIC MODAL ANALYSIS OF FRAME STRUCTURES

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Sommario - *L'errore connesso alla usuale discretizzazione strutturale nell'analisi sismica modale delle strutture intelaiate viene calcolato mediante il confronto con i risultati della analisi esatta del modello continuo. Si mostra che, per alcuni tipi di strutture, anche quando l'errore in termini di parametri spettrali non è molto rilevante quello relativo alle sollecitazioni può essere inaccettabile. Viene suggerito un semplice criterio per migliorare l'approssimazione del modello discreto.*

Summary - *In the present paper the error related to the usual FE discretization in seismic analysis of frames is computed by comparison with the results of the "exact" analysis of the continuous structure. It is shown that, for certain classes of structures, even when the error on the spectral parameters is not too large the one on the internal stresses can be quite unacceptable. A simple criterion for improving the FE approximation is suggested.*

1. INTRODUCTION

The procedures of seismic dynamic analysis commonly adopted in the design of structures make use of FE discretizations of the continuous model. The displacements are approximated by means of shape functions which satisfy compatibility but not equilibrium conditions along the elements. In the case of frame structures the so called Hermite polynomials are employed.

Upper bounds for the eigenfrequencies are then evaluated on the basis of the Rayleigh - Ritz Method, and the relevant error estimates can be obtained by means of well established techniques [1,2,3,4,5].

In previous works [6,7,8,9] error bounds for eigenvalues and eigenvectors have been evaluated for frame structures.

In this paper the error in terms of displacements and stress fields connected with the Rayleigh - Ritz analysis is directly computed for general frames. The case of beams with uniform cross-section will be considered, since, for this class of structures, the exact solution of the dynamic problem is available.

The spectral parameters for the continuous frame model are computed using a method due to Wittrick and Williams which allows the separation of the eigenvalues [10,11].

A rigorous proof of this method has been recently given allowing a better understanding of the results and the evaluation of the eigenvectors in the case the algebraic problem related with Wittrick and Williams algorithm becomes singular [12].

The modal dynamic analysis for the continuous frame model is developed in order to obtain the "exact" structural response.

In the following only elastic structures will be considered and no kind of dissipation will be taken into account according to the assumptions adopted in the standard seismic analysis of space frames.

2. THE VIBRATION PROBLEM FOR AN ELASTIC SYSTEM

The dynamic equilibrium equation can be formulated in variational form as

$$E(u,\tilde{u}) + M(\ddot{u},\tilde{u}) = (q(x,t),\tilde{u}) \quad u \in V \ \ \forall \tilde{u} \in V \qquad (1)$$

where a dot denotes differentiation with respect to time and

u	is the displacements field
V	is the space of the admissible displacements
$E(u,\tilde{u})$	is the bilinear form of the elastic energy
$M(u,\tilde{u})$	is the bilinear form of inertia
$q(x,t)$	is a time-dependent external load.

In the case of seismic analysis the external load is given by

$$q(x,t) = P(x)\, R(x)\, \ddot{u}_g(t)$$

where

$\ddot{u}_g$	is the ground acceleration vector
$R(x)$	is a rigid body motion operator
$P(x)$	is the mass distribution operator.

Problem (1) is associated to the following eigenvalue problem

$$M(u,\tilde{u}) = \mu E(u,\tilde{u}) \quad u \in V \ \ \forall \tilde{u} \in V \qquad (2)$$

where $\mu = 1/\omega^2$ and ω is the natural circular frequency of vibration of the structure.

Integrating by parts, by a standard procedure, problem (2) can be cast in the equivalent form

$$L_\mu u = 0 \quad u \in V$$

where L_μ is the differential operator of dynamic equilibrium for the system and the solution u has to satisfy also the natural boundary conditions. If the inertia term vanishes, the operator L_μ reduces to the static equilibrium operator L.

For vibration problems of elastic structures the bilinear forms $E(u,\tilde{u})$ and $M(u,\tilde{u})$ turn out to be symmetric and positive definite, so that problem (2) has real positive eigenvalues. Eigenvectors associated to distinct eigenvalues are orthogonal with respect to the energy scalar product defined by $E(u,\tilde{u})$.

Moreover, since the quadratic form $M(u,u)$ is completely continuous with respect to $E(u,u)$ [13], problem (2) has a countable set of eigenvalues which have finite multiplicity and can be ordered in a sequence converging to zero.

We shall denote by $\{\mu_i\}$ the non-increasing sequence of eigenvalues, each counted according to its multiplicity, and by $\{w_i\}$ the associated sequence of eigenvectors normalized with respect to the energy norm, i.e.

$$E(w_i, w_i) = 1$$

By the orthogonality property, for $\mu_i \neq \mu_k$, it follows that

$$E(w_i, w_k) = 0$$
$$M(w_i, w_k) = 0.$$

3. MODAL SUPERPOSITION ANALYSIS FOR THE CONTINUOUS SYSTEM

Under the hypotheses listed in the previous paragraph any vector $u \in V$ has the following Fourier expansion [4]

$$u(x,t) = \sum_n E\Big(u(x,t), w_n(x)\Big) w_n(x) = \sum_n y_n(t)\, w_n(x) \quad u \in V \tag{3}$$

which implies the Parseval's equations

$$\begin{aligned} E(u,\bar{u}) &= \textstyle\sum_n E(u,w_n)E(\bar{u},w_n) \\ M(u,\bar{u}) &= \textstyle\sum_n \mu_n E(u,w_n)E(\bar{u},w_n) \end{aligned} \tag{4}$$

Making use of these relations the dynamic equilibrium equation (1) can be written as

$$\begin{aligned} \textstyle\sum_n E(u,w_n)E(\bar{u},w_n) + \sum_n \mu_n E(\ddot{u},w_n)E(\bar{u},w_n) \\ = \Big(q(x,t),\bar{u}\Big) \quad \forall \bar{u} \in V \end{aligned} \tag{5}$$

In order to compute the coefficients $y_i(t)$ of the Fourier expansion we set $u = w_i$ in equation (5), so that it takes the form

$$E(u,w_i) + \mu_i E(\ddot{u},w_n) = \Big(q(x,t), w_i(x)\Big) \tag{6}$$

Substituting the expansion (3) in (6) the following modal decomposition is derived

$$y_i(t) + \mu_i \ddot{y}_i(t) = \Big(q(x,t), w_i(x)\Big) \tag{7}$$

Expression (7), which is valid both in the infinite and in the finite dimensional case, can be used either for evolutive analysis, in the case the loading function $q(t)$ is known, or for seismic analysis with the method of response spectra.

In thé latter case, the maximum contribution of the i-th mode to the displacement field is given by

$$u_{i_{max}}(x) = y_{i_{max}} w_i(x) \tag{8}$$

where

$$y_{i_{max}} = g_i \omega_i \max_t V_i(t,\omega_i) \tag{9}$$

In this formula $V_i(t,\omega_i)$ is the modal earthquake response integral and g_i the excitation factor defined as

$$g_i = \Big(q(x,t), w_i(x)\Big) = \Big(P(x)R(x)\ddot{u}_g, w_i(x)\Big) \tag{10}$$

The maximum value of the response integral is by definition the spectral pseudo-velocity $S_v(\omega_i)$ so that, substituting $S_a(\omega_i)$ for $\omega_i S_v(\omega_i)$, in terms of the pseudo-acceleration response spectrum expression (8) takes the form

$$u_{i_{max}}(x) = g_i\, S_a(\omega_i)\, w_i(x) \tag{11}$$

In order to superpose contributions of different modes, according to international seismic codes, the following conventional rule has been adopted

$$u_{max} = \Big(\sum_{i=1}^{r} |u_{i_{max}}(x)|^2\Big)^{1/2}$$

where r is the number of modes considered in the analysis.

4. SEISMIC MODAL ANALYSIS FOR FRAME STRUCTURES

The standard approximation to the seismic response of frame structures requires the knowledge of a set of leading spectral parameters and of the relevant excitation factors.

In what follows the computational scheme for seismic analysis of both the continuous model and the usual Rayleigh - Ritz approximation will be briefly sketched.

4.1 THE CONTINUOUS MODEL

The values of the eigenfrequencies are obtained by means of the Wittrick and Williams algorithm. This method allows to count the number p of eigenvalues μ_i which are smaller than a given number δ [10,11].

Let us denote by

- p_1 the number of eigenvalues smaller than δ of the restriction of problem (2) to the subspace $V_1 \subset V$ of the admissible displacements for the clamped beams;
- p_2 the number of eigenvalues smaller than δ of the restriction of problem (2) to the finite dimensional subspace $V_2 \subset V$ that, with respect to the indefinite scalar product $C(u,\bar{u}) = M(u,\bar{u}) - \delta E(u,\bar{u})$, is the C-orthogonal complement of V_1 [12].

If δ is not an eigenvalue of the restriction of problem (2) to V_1, it can be proved that

$$p = p_1 + p_2 \tag{12}$$

The algebraic eigenvalue problem, that is the restriction of problem (2) to the finite dimensional subspace V_2, is given by

$$(M_\delta - \mu E_\delta)d = 0 \tag{13}$$

where the mass and stiffness matrices M_δ and E_δ are defined as follows

$$\begin{aligned} M_{\delta_{ij}} &= M(\psi_{\delta_i}, \psi_{\delta_j}) \\ E_{\delta_{ij}} &= E(\psi_{\delta_i}, \psi_{\delta_j}) \end{aligned} \tag{14}$$

and ψ_{δ_i} are shape functions that belong to the kernel of the operator L_δ.

On the basis of the additive count property expressed by formula (12) a procedure that allows to separate and to compute the eigenvalues of the continuous system is readily implemented.

Let us suppose that no one of the eigenvectors w_i belongs to V_1. This requirement can always be simply satisfied by a suitable choice of the subspace V_1.

Under the above hypotheses the eigenfunctions of problem (2) normalized with respect to the "energy" norm have the following representation

$$w_i(x) = \alpha A(\mu_i)\, d_i \tag{15}$$

where

- d_i are the eigenvectors of problem (13) for $\delta = \mu_i$
- α is a normalizing factor given by $(E_{\mu_i} d_i \cdot d_i)^{-1}$
- $A(\mu_i)$ is a linear operator containing the shape functions

The modal excitation factors are evaluated by substituting the representation (15) into (10).

4.2 THE DISCRETE MODEL

The Rayleigh - Ritz approximation to the vibration problem is based on the restriction of the eigenvalue problem (2) to a finite dimensional subspace W of V.

For frame structures the subspace W has dimension equal to the number n of the nodal parameters and is generated by a system of n piecewise Hermite polynomial shape functions $v_i \in Ker L$.

The Rayleigh - Ritz matrix eigenvalue problem takes the form

$$M s = \mu E s \tag{16}$$

where the mass and stiffness matrices M and E are defined by

$$\begin{aligned} M_{ij} &= M(v_i, v_j) \\ E_{ij} &= E(v_i, v_j) \end{aligned} \tag{17}$$

Let $\{s_i\}$ be the set of eigenvectors of problem (16) normalized with respect to the "energy" norm so that $E\, s_i \cdot s_i = 1$. Then the eigenfunctions of the restriction of problem (2) to the subspace W are given by

$$w_i(x) = B(x)\, s_i \tag{18}$$

where $B(x)$ is the matrix of the piecewise Hermite polynomial shape functions. Therefore the modal excitation factor can be obtained by substituting expression (18) in formula (10).

5. ERRORS EVALUATION IN SEISMIC ANALYSIS

As it is readily recognized from formula (11), the errors on the expected maximum displacements and internal forces evaluated by a Rayleigh - Ritz discrete model are related to the accuracy of the eigenfunction approximation.

In particular for the usual discretization which makes use of Hermite polynomials, the error can be expected to be large when, for one of the leading eigenmodes of the structure, some members of the frame vibrate with a shape close to that of a clamped beam.

In fact, for these members, the piecewise Hermite polynomials cannot correctly model the eigenfunctions. Such situations arise when one of the leading eigenfrequencies of the frame is close enough to the eigenfrequency ω_c of some of its elements considered as clamped.

The phenomenon described will be illustrated through numeric examples.

In the evaluation of seimic response the design spectrum of italian code has been used. In all the applications the direction considered for the earthquake is the horizontal one.

5.1 NUMERICAL RESULTS

The first example refers to a plane frame with one foundation at a lower level. In table I the first 5 exact eigenfrequencies of the structure are listed . Comparison of the results of the Rayleigh - Ritz approximation with the exact model shows that stresses are overestimated in the short columns and underestimated in the long one. For this element the maximum error for the bending moment is of 56%.

As it can be observed from table I the first two frequencies are the most important in the seismic response of the structure and they lie relatively close to the eigenfrequency ω_c of the long column with clamped ends.

The eigenmode estimated using Rayleigh - Ritz analysis differ substantially from the exact one. As a consequence the factors g_i are quite different for the first two modes (table I). Figure 1 presents the displacements $u_{i_{max}}$ given by formula (11) for these modes.

From the analysis of the data it appears that, in the case of structures with one of the leading eigenfrequency close to the frequency ω_c of some element, noticeable errors in the seismic analysis may occur.

It is therefore necessary to refine the FE mesh in order to allow a more accurate description of the displacements along the critical members. In the above example it is sufficient to insert one nodal point in the long column to reduce drastically the errors (table II). This improvement is appearent from the plots of $u_{i_{max}}$ in fig. 2.

TABLE I

Mode Number	1	2	3	4	5
Exact Frequency ω	3.2037	4.1057	8.1144	8.5811	8.988
Excitation Factors (Exact Analysis)	16.524	14.702	1.396	0.384	0.238
Excitation Factors (Approx. Analysis)	21.074	3.634	0.175	0.125	0.257

Frequency of the long column $\omega_c = 4.4787$

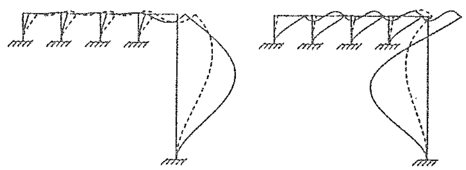

Fig.1 - I and II mode of vibration. Thick line : exact solution Dashed line : Rayleigh - Ritz approximation

TABLE II

Error on the Frequencies

Mode Number	1	2	3	4	5
One element per beam (15 degrees of freedom)	0.157	0.225	0.986	0.943	1.133
One extra nodal point (18 degrees of freedom)	0.010	0.003	0.341	0.891	0.846

Maximum Errors on Displacements and Internal Stresses $|\alpha_{appr} - \alpha_{exac}|/\alpha_{exac}$

	Horiz. Displ.	Vert. Displ.	Joint Rotat.	Bend. Mom.	Shear Force	Axial Force
15 d.o.f.'s	0.266	0.675	0.333	0.561	0.371	0.684
18 d.o.f.'s	0.005	0.001	0.020	0.046	0.169	0.696

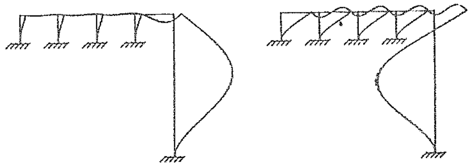

Fig.2 - I and II Mode of Vibration of the Frame with one extra node in the long column
Thick line : exact solution - Dashed line : approx. solution

A second example refers to a plane frame with bracings in the first span. Table III shows the first 6 exact eigenfrequencies and the frequency of vibration ω_c of the beams 1 and 2 considered as clamped. The latter value lies well above the first two eigefrequencies, which are the leading ones for seimic analysis.

However, also in this case, large errors for the displacements and stresses are found in the Rayleigh - Ritz approximation which in the critical element underestimates the bending moment by 48% (Table IV).

This is due to the fact that the mode shapes of the beams 1 and 2 have a substantial component of the mode of vibration of the clamped beam which is lost in the Hermite approximation (fig. 3).

The maximum error on the bending moments occurs in pillar 1 while in the stiffer part of the structure Rayleigh - Ritz estimates are somewhat larger than the exact values (fig. 4). In this case it is needed to double the number of degrees of freedom in order to obtain satisfactory estimates.

TABLE III

Mode Number	1	2	3	4	5	6
Exact Frequency ω	3.1275	3.9064	5.0943	5.5721	6.4534	6.982
Excitation Factors (Exact Analysis)	27.339	19.555	2.328	1.965	1.991	5.388
Excitation Factors (Approx. Analysis)	30.268	10.254	2.432	0.818	3.677	0.027

Frequency of the columns 1,2 ω_c = 5.6845

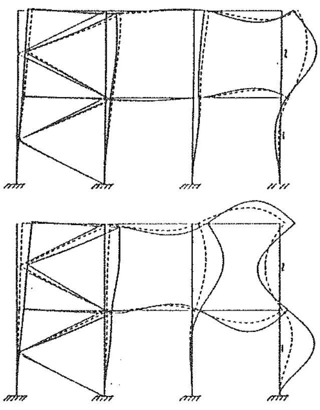

Fig.3 - I and II Mode of Vibration
Thick line : exact solution - Dashed line : approx. solution

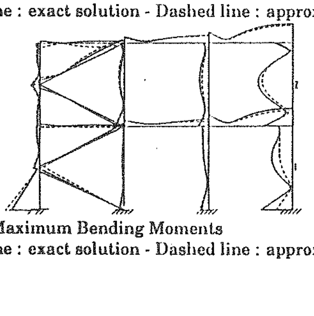

Fig.4 - Maximum Bending Moments
Thick line : exact solution - Dashed line : approx. solution

TABLE IV

Error on the Frequencies

Mode Number	1	2	3	4	5	6
30 d.o.f.'s	0.088	0.147	0.338	0.355	0.399	0.416
60 d.o.f.'s	0.083	0.052	0.094	0.121	0.077	0.148

Maximum Errors on Displacements and Internal Stresses $|\alpha_{appr} - \alpha_{exac}|/\alpha_{exac}$

	Horiz. Displ.	Vert. Displ.	Joint Rotat.	Bend. Mom.	Shear Force	Axial Force
30 d.o.f.'s	0.244	0.924	1.114	1.382	2.082	1.377
60 d.o.f.'s	0.221	0.222	0.260	0.581	0.591	0.309

6. CONCLUSIONS

The evaluation of the errors which arise from approximate FE seismic analysis of frames by direct comparison with an exact solution of the continuous model has been presented.

It has been observed that, even when the errors on the eigenfrequencies are contained within acceptable values, those on diplacements and stresses can be quite large. Moreover their values can be underestimated by the standard discrete analysis so that particular care has to be taken in the design.

An effective and very cheap method to overcome this difficulty is to insert extra nodal points in the critical elements.

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OMAGGIO
A
GIULIO CERADINI

NOTE SCIENTIFICHE
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PROOF of the WITTRICK and WILLIAMS ALGORITHM

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1. Introduction

It is a great pleasure to dedicate this work to Prof. Giulio Ceradini on the occasion of his 70^{th} birthday.

The aim of this paper is to provide a general proof of a very useful invariance property for the sign count of eigenvalues. The proof will be carried out in an abstract setting and then specialized to eigenvalue problems of linear elasticity.

On the basis of this invariance result it is possible to achieve the eigenvalue separation in an important class of vibration and buckling problems of continuum structural mechanics.

In the special context of structural analysis of elastic frames, this property has been first formulated by W.H. Wittrick and F.W. Williams for vibration problems in 1970 [1] and for buckling problems in 1973 [2] (see also [3]).

In spite of a very skillful intuition their proof cannot be considered satisfactory since, expecially for continuous structural systems, it is based on some propositions which are assumed to be true on *physical grounds*.

The rigorous proof provided in this paper allows to get a better understanding of the basic invariance property and provides useful hints to device effective procedures for the eigenvector computation in the singular cases [4].

2. The abstract eigenvalue problem

Let us consider the eigenvalue problem :

$$A(v,w) \;=\; \mu\, B(v,w) \qquad v \in V - \{0\} \quad \forall\, w \in V \qquad (P)$$

where $A(v,w)$ and $B(v,w)$ are sesquilinear functionals defined on a Hilbert space H and V is a subspace of H.

Let us make the following assumption :

Hypothesis 1. *The sesquilinear functionals $A(v,w)$ and $B(v,w)$ are hermitian on the subspace V.*

By hypothesis 1 the eigenvalue problem (P) has real eigenvalues (if any).

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Let us put the question :

Problem 1. *How many eigenvalues of problem (P) are greater than a given positive number δ ?*

It is worth noting that the solution of this problem provides an effective tool for the separation and the explicit numerical computation of the eigenvalues of problem (P).

In order to give an answer to the question posed by problem 1 we shall make the following assumptions :

Hypothesis 2. *The quadratic form $B(v,v)$ is positive definite on the subspace V of H. Further the subspace V has a Hilbert space structure with the energy scalar product $\langle\langle u, w\rangle\rangle = B(v,w)$ and the associated norm defined by the relation $|||v|||^2 = B(v,v)$.*

By hypothesis 2 the eigenvectors of problem (P) associated to distinct eigenvalues are orthogonal in energy.

If hypothesis 2 holds true we may define the Rayleigh quotient :

$$R(v) = \frac{A(v,v)}{B(v,v)} \qquad v \in V - \{0\} \tag{1}$$

In terms of the functional $R(v)$ the eigenvalue problem (P) can be written in the form :

$$\nabla_w R(v) = 0 \qquad \forall\, w \in V \quad v \in V - \{0\} \quad \mu = R(v) \tag{2}$$

Hypothesis 3. *The Rayleigh quotient $R(v)$ is bounded on V.*

By hypothesis 3 we get the following two properties :

i) Recursive definition of the eigenvalues (see [5] pag. 38-39)

ii) Poincarè theorem (1890) on the independent characterization of the eigenvalues (see [5] pag. 55-56).

Let us recall that the index of a quadratic functional is defined as the largest dimension of a subspace of V on which it is positive definite.

A well known consequence of Poincarè theorem is the following :

Index property. *The number of positive eigenvalues of problem (P) is equal to the index of $A(v,v)$.*

This property suggests that the solution of Problem 1 can be achieved by considering an equivalent eigenvalue problem with a shifted spectrum. In fact if we set :

$$\mu = \beta + \delta \tag{3}$$

and define

$$C(v,w) = A(v,w) - \delta\, B(v,w) \tag{4}$$

the eigenvalue Problem (P) turns into the following one :

$$C(v,w) = \beta\, B(v,w) \qquad v \in V - \{0\} \quad \forall\, w \in V. \tag{E}$$

It is easy to show that the eigenvalue problem (E) has the same eigenvectors of the problem (P) and that his spectrum is obtained from that of the problem (P) by a shift of amount $-\delta$ along the real axis.

Hence we may equivalently restate Problem 1 in the following form :

Problem 2. *How many positive eigenvalues has problem (E)?*

The solution of this problem is non trivial if the space V is infinite dimensional.

3. The main theorem

The solution of problem 2 can be achieved by solving two problems of the same kind posed in suitable subspaces of V.

In order to prove this property it is necessary to provide some preliminary assumptions and results.

The bilinear functional $C(v,w)$ defines on the space V an indefinite scalar product. The nullspace K of $C(v,w)$ on V is defined by the relation :

$$u \in K \iff C(u,w) = 0 \quad w \in V \tag{5}$$

The bilinear functional $C(v,w)$ is said to be degenerate on V iff $\dim K > 0$.
From the definitions (4) and (5) it follows that $C(v,w)$ is degenerate on V iff δ is an eigenvalue of problem (P). The nullspace K is then the eigenspace associated to δ.

For any subspace $U \in H$ we shall denote by $U^{\perp}$ its $C-$ orthogonal complement in H, i.e.

$$u \in U^{\perp} \iff C(u,w) = 0 \qquad u \in H \quad \forall\, w \in U \tag{6}$$

Let us consider a subspace V_1 of V and its $C-$orthogonal complement V_2 in V defined as :

$$V_2 = V_1^{\perp} \cap V \tag{7}$$

so that

$$v \in V_2 \iff C(v,w) = 0 \quad v \in V \quad \forall\, w \in V_1 \tag{8}$$

The restrictions of problem (P) to the subspaces V_1 and V_2 of V give rise to the eigenvalue problems :

$$A(v_1,w_1) = \mu\, B(v_1,w_1) \qquad v_1 \in V_1 - \{0\} \quad \forall\, w_1 \in V_1 \tag{P_1}$$

$$A(v_2,w_2) = \mu\, B(v_2,w_2) \qquad v_2 \in V_2 - \{0\} \quad \forall\, w_2 \in V_2 \tag{P_2}$$

and by (3) and (4) to the associated problems :

$$C(v_1,w_1) = \beta\, B(v_1,w_1) \qquad v_1 \in V_1 - \{0\} \quad \forall\, w_1 \in V_1 \tag{E_1}$$

$$C(v_2, w_2) = \beta\, B(v_2, w_2) \qquad v_2 \in V_2 - \{0\} \quad \forall\, w_2 \in V_2 \tag{E_2}$$

The eigenvalue problems (E_1) and (E_2) have the same eigenvectors of the problems (P_1) and (P_2) respectively. Their spectra are obtained from those of the problems (P_1) and (P_2) by a shift $-\delta$ along the real axis.

We shall denote by K_1 and K_2 the nullspaces of the restrictions of $C(v, w)$ to the subspaces V_1 and V_2, i.e.

$$u_1 \in K_1 \iff C(u_1, w_1) = 0 \quad u_1 \in V_1 \quad \forall\, w_1 \in V_1 \tag{9}$$

$$u_2 \in K_2 \iff C(u_2, w_2) = 0 \quad u_2 \in V_2 \quad \forall\, w_2 \in V_2 \tag{10}$$

By definition (6) we have :

$$K_1 = V_1 \cap V_1^{\perp} \tag{11}$$

$$K_2 = V_2 \cap V_2^{\perp} \tag{12}$$

The nullspaces K_1 and K_2 are non trivial iff the parameter δ is an eigenvalue of problems (P_1) and (P_2) respectively.

Lemma 1. *The subspace $V_1 \cap V_2$ is given by the nullspace K_1 of $C(v, w)$ in V_1, i.e.*

$$K_1 = V_1 \cap V_2$$

Proof.

$$u \in K_1 \iff C(u, w) = 0 \qquad u \in V_1, \forall\, w \in V_1 \iff u \in V_1 \cap V_2$$

Hypothesis 4. *Any vector of the space V can be represented as the sum of a vector of the subspace V_1 and a vector of the subspace V_2, i.e.*

$$V = V_1 + V_2$$

From hypothesis 4 and lemma 1 it follows that, iff the bilinear functional $C(v, w)$ is non degenerate on V_1, the space V is the direct sum of the subspaces V_1 and V_2, i.e.

$$V = V_1 \oplus V_2 \tag{13}$$

Lemma 2. *The nullspace K is contained in K_2. In addition, if hypothesis 4 is satisfied, it results :*

$$K_2 = K. \tag{14}$$

Proof. If in the definition of the nullspace K we put $w = w_1,\ w_1 \in V_1$ and $w = w_2,\ w_2 \in V_2$, respectively we get

$$u \in K \Rightarrow C(u, w_1) = 0 \quad u \in V \quad \forall\, w_1 \in V_1 \Rightarrow u \in V_2$$
$$u \in K \Rightarrow C(u, w_2) = 0 \quad u \in V \quad \forall\, w_2 \in V_2 \Rightarrow u \in V_2^{\perp}$$

so that from (12)

$$K \subset K_2. \tag{15}$$

Moreover, from the hypothesis 4 and the C – orthogonality property $C(u_2, w_1) = 0$, we get

$$u_2 \in K_2 \;\Rightarrow\; C(u_2, w) = C(u_2, w_2) = 0 \quad u_2 \in V_2 \;\forall\, w \in V \;\Rightarrow\; u_2 \in K$$

so that

$$K \supset K_2 \tag{16}$$

and the proof is complete.

We are now ready to prove the main invariance property for the sign count of the eigenvalues.

$$n = n_1 + n_2 \tag{20}$$

Proof. On the basis of representations (17) - (19) it is easy to show [6] that the spaces $V,\ V_1,\ V_2$ can be decomposed in the following direct sums of C–orthogonal subspaces :

$$V = V^+ \oplus V^- \oplus K \tag{21}$$
$$V_1 = V_1^+ \oplus V_1^- \oplus K_1 \tag{22}$$
$$V_2 = V_2^+ \oplus V_2^- \oplus K_2 \tag{23}$$

where $V^-,\ V_1^-,\ V_2^-$ are subspaces on which the quadratic functional $C(v,v)$ is negative definite. Since $K_1 = \{0\}$, from (13), (14), (22) and (23) we have :

$$V = (V_1^+ \oplus V_2^+) \oplus (V_1^- \oplus V_2^-) \oplus K \tag{24}$$

By the C – orthogonality of V_1 and V_2, if we define :

$$U^+ = V_1^+ \oplus V_2^+ \tag{25}$$
$$U^- = V_1^- \oplus V_2^- \tag{26}$$

we get the following decomposition of V

$$V = U^+ \oplus U^- \oplus K \tag{27}$$

Theorem 1. *Let us suppose that problems* (E), (E$_1$), *and* (E$_2$) *have a finite number of positive eigenvalues that we shall denote by* n, n$_1$ *and* n$_2$ *respectively. By the index property we may then define the subspaces* $V^+,\ V_1^+,\ V_2^+$ *to be subspaces of maximal dimension on which the quadratic functional* $C(v,v)$ *is positive definite in* $V,\ V_1,\ V_2$ *respectively.*
Moreover we shall assume that the spaces $V,\ V_1,\ V_2$ *have the following decompositions*

$$V = V^+ + (V^+)^\perp \tag{17}$$
$$V_1 = V_1^+ + (V_1^+)^\perp \tag{18}$$
$$V_2 = V_2^+ + (V_2^+)^\perp \tag{19}$$

Then, if the bilinear functional $C(u,w)$ *is non-degenerate on* V_1, *i.e. if* $K_1 = \{0\}$, *the following sign count relation holds :*

where $C(v, v)$ is positive definite on U^+ and negative definite on U^-. From (25) it follows that :

$$\dim U^+ = n_1 + n_2. \tag{28}$$

Let us now prove that :

$$\dim U^+ = \dim V^+ = n. \tag{29}$$

Suppose first that :

$$\dim U^+ > \dim V^+;$$

then there would exist a non-trivial vector $v \in U^+$ such that

$$C(v, w) = 0 \quad \forall\, w \in V^+.$$

From (21) it follows that :

$$v = U^+ \cap (V^- \oplus K)$$

whence

$$C(v, v) = 0 \text{ and } v = 0$$

what contradicts the hypothesis. Then it must be $\dim U^+ \leq \dim V^+$. In the same way it is readily proved that $\dim U^+ \geq \dim V^+$, so that equality (29) follows. From (28) and (29) we get the sign count relation (20).

It is important to point out that the sign count invariance property (20) allows to solve Problem (E) if the space V_1 can be chosen so that the solutions of Problems (E_1) and (E_2) are computable.

4. Eigenvalue Problems of Linear Elasticity

The vibration or buckling problems of linear elasticity can be put in the form of problem (P). To simplify the exposition we shall assume that the quadratic functional $A(v, v)$ is positive definite, so that all the eigenvalues are positive.

In the vibration and buckling problems the bilinear form $A(v, w)$ takes account of the inertia effects and of the geometry changes respectively. In both cases the positive definite quadratic functional $B(v, v)$ provides the elastic energy associated with the deformation of the structure.

For the eigenvalue problems of linear elasticity the requirements of hypotheses 1 and 2 are obviously satisfied. Moreover the Rayleigh quotient fulfills the following fundamental :

Compactness property

For any $\varepsilon > 0$ there exists a finite dimensional subspace W of V such that :

$$|\, R(v)\, | < \varepsilon \tag{C}$$

for every $v \neq 0$ that belongs to the orthogonal complement of W in H.

By property (C) the hypothesis 3 is satisfied. Moreover the basic spectral properties of elasticity are implied [5].

Namely we have that :

i) The eigenvalue problem (P) has a countable set of eigenvalues;

ii) The non zero eigenvalues have finite multiplicity and, if each of them is counted according to its multiplicity, they can be ordered in a non increasing sequence $\{\mu_i\}$ converging to zero.

iii) If we denote by $\{u_i\}$ the orthonormal (in energy) sequence of eigenvectors associated to the eigenvalues $\{\mu_i\}$, any vector v of V has the spectral decomposition

$$v = \sum_{i=1}^{\infty} B(v, u_i)\, u_i \tag{30}$$

iv) The bilinear functionals $A(v, w)$ and $B(v, w)$ have the Parseval decompositions :

$$A(v, w) = \sum_{i=1}^{\infty} \mu_i B(v, u_i) B(w, u_i) \tag{31}$$

$$B(v, w) = \sum_{i=1}^{\infty} B(v, u_i) B(w, u_i) \tag{32}$$

From (4), (31) and (32) we get the following representation of the bilinear functional $C(v, w)$:

$$C(v, w) = \sum_{i=1}^{\infty} (\mu_i - \delta) B(v, u_i)\, B(w, u_i) \tag{33}$$

and of the quadratic functional $C(v, v)$:

$$C(v, v) = \sum_{i=1}^{\infty} (\mu_i - \delta)\, [B(v, u_i)]^2 \tag{34}$$

Let us recall that if the eigenvectors u_h and u_k of problem (P) are associated to distinct eigenvalues, i.e. $\mu_h \neq \mu_k$, it results :

$$A(u_h, u_k) = B(u_h, u_k) = 0 \tag{35}$$

so that :

$$C(u_h, u_k) = A(u_h, u_k) - \delta\, B(u_h, u_k) = 0 \tag{36}$$

By the relation (36) the eigenspaces associated to distinct eigenvalues are C–orthogonal in V.

The spectral decomposition (34) shows that the subspace V^+ is given by the direct sum of the eigenspaces U_i associated to the eigenvalues of problem (P) that are greater than δ. It follows that V^+ is finite dimensional and non-zero iff $0 < \delta < \mu_1$.

Let us consider the non-trivial case in which we have :

$$\mu_{n+h+1} < \delta < \mu_1 \qquad n, h \geq 0 \qquad\qquad \delta = \mu_{n+1} = \ldots = \mu_{n+h}$$

From (30) and (34) it can be readily seen that :

$$V^+ = \bigoplus_{i=1}^{n} U_i \tag{37}$$

$$V^- = \bigoplus_{i=n+h+1}^{\infty} U_i \tag{38}$$

$$K = \bigoplus_{i=n+1}^{n+h} U_i \tag{39}$$

and

$$(V^+)^\perp = \bigoplus_{i=n+1}^{\infty} U_i \tag{40}$$

so that :

$$V = V^+ \oplus (V^+)^\perp \tag{41}$$

The relation (41) shows that the hypothesis (17) of theorem 1 is satisfied. Identical results are obviously valid for the eigenvalue problems (E_1) and (E_2) so that the hypotheses (18) and (19) are fulfilled too.

From the spectral decomposition (33) we see that the bilinear functional $C(v, w)$ is non degenerate on V_1 iff the shifting parameter δ does not coincide with an eigenvalue of problem (P_1). If this singular case does occur a special attention must be paid in the eigenvectors computation [4].

Finally it remains to show that hypothesis 4 is satisfied too. The proof of this property will depend on the specific mathematical features of the structural model under consideration.

In particular for frame structure it can be shown [4] that hypothesis 4 is satisfied if V_1 is defined to be the subspace of admissible displacements corresponding to zero nodal parameters and if the shifting δ does not coincide with an eigenvalue of problem (P_1). The $C-$orthogonal complementary space $(V_1)^\perp$ will then be the finite dimensional kernel of the matrix differential operator of dinamical equilibrium.

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