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Geometric continuum mechanics

Giovanni Romano · Raffaele Barretta · Marina Diaco

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Abstract Geometric Continuum Mechanics (GCM) is a new formulation of Continuum Mechanics (CM) based on the requirement of Geometric Naturality (GN). According to GN, in introducing basic notions, governing principles and constitutive relations, the sole geometric entities of space-time to be involved are the metric field and the motion along the trajectory. The additional requirement that the theory should be applicable to bodies of any dimensionality, leads to the formulation of the Geometric Paradigm (GP) stating that push-pull transformations are the natural comparison tools for material fields. This basic rule implies that rates of material tensors are Lie-derivatives and not derivatives by parallel transport. The impact of the GP on the present state of affairs in CM is decisive in resolving questions still debated in literature and in clarifying theoretical and computational issues. As a consequence, the notion of Material Frame Indifference (MFI) is corrected to the new Constitutive Frame Invariance (CFI) and reasons are adduced for the rejection of chain decompositions of finite elasto-plastic strains. Geometrically consistent notions of Rate Elasticity (RE) and Rate Elasto-Visco-Plasticity (REVP) are formulated and consistent relevant computational methods are designed.

G. Romano (✉) · R. Barretta · M. Diaco
Department of Structures for Engineering and Architecture, University of Naples Federico II, via Claudio 21, 80125 Naples, Italy
e-mail: romano@unina.it

R. Barretta
e-mail: rabarret@unina.it

M. Diaco
e-mail: diaco@unina.it

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1 Introduction

The initial acceptance of the theory of relativity was due to HILBERT, KLEIN, POINCARÉ and MINKOWSKI, knowledgeable and authoritative scientists, who were intrigued by the new approach by EINSTEIN and shared his enthusiasm. The definitive triumph of the theory was sealed by the prediction and interpretation of physical phenomena which the classical theory was unable to explicate.

There are similarities between that situation and the present one concerning Continuum Mechanics (CM). In both cases one is faced with a well-consolidated theory rich of implications and interpretations of experimental facts, but with some unexplicable difficulties and paradoxes. In both cases the right idea comes by collecting hints and partial answers by earlier proposals made by valuable researchers.

Relativistic Mechanics (RM) is based on the assessment that the light signal *in vacuum* is the speediest communication tool to synchronize clocks of different observers. Geometric Continuum Mechanics

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(GCM) is based on the assessment that push-pull transformations are the tool to compare material tensor fields at displaced placements along the trajectory or at the same placement as seen by different observers.

As RM collapses into classical mechanics when the magnitudes of all velocities at play are much smaller than the speed of light in *vacuum*, so the new GCM reproduces the engineering linearized approximation when geometric non-linearities are sufficiently small.

The need for a guideline is indeed especially felt when the fully nonlinear theory is dealt with. The geometric approach provides, however, proper definitions of basic concepts also in the small displacement range, as exemplified in Sect. 15 dealing with *material homogeneity*, and convenient coordinate-free treatments as in [3–7, 13, 32–35, 37, 44].

Basic building blocks for the development of Non-Linear Continuum Mechanics (NLCM) have been put in place long time ago [54, 55]. This approach was highly influential in drawing the track for subsequent contributions [16, 24, 25, 47, 56]. Significant progress was also made to fulfill the demanding requests of Computational Mechanics [8, 11, 12, 20, 23].

In formulating the methodological framework illustrated in the present contribution, the book [21] should be cited as especially significant. Although still adopting notions and methods taken from the essentially algebraic treatment developed in [53, 54], that book should rightly be considered a landmark turning point towards a more genuinely geometric point of view.

A special mention is also deserved by computationally oriented papers [26] and [49] which, at about the same time, provided a clear hint towards the need for a consistent geometric approach to constitutive theory.

Applications of differential geometry to CM, still in the framework proposed in [53–55] and since then followed in literature up to now, have recently been contributed in [45, 46, 58] and in the book [14].

The ideas and the methods outlined in the sequel are *not* intended to dress a fashionable suit over well-known treatments and results. Rather they have been developed under the pressing need of clarifying basic issues, such as the notions of stress rate, of rate elastic and inelastic constitutive laws, of conservativeness of elastic models and the request of frame indifference for material fields and constitutive relations.

Under this aspect, we have primarily followed the fascination of what Piero Villaggio in [57] calls *the ineffable pleasure of free intellectual research*.

Indeed any attempt to fulfill the demands posed by the exigence of overcoming unclear statements and unmotivated assumptions, readily leads to the conclusion that a *brand new* theoretical and computational framework is needed.

Previous contributions to NLCM can there be embedded, adapted or corrected to fit into a consistent geometrical scheme capable of answering in a satisfactory way still pending questions and to formulate effective computational strategies, as delineated in Sect. 3.

2 Motivation

The reason why a scholar in Continuum Mechanics, who is wishing to face problems in the nonlinear geometric range, should learn basic differential geometry, is that linear algebra and linear calculus can only work in the context of a (small displacements) linearized theory.

If this simple observation is not deemed to be sufficient to convince that the conceptual effort needed to manage these fundamentals is worth to be sustained, answering the following questions might help.

Consider an inflatable *rubber balloon*. The lower dimensionality of this body makes the governing rules for a non-linear analysis to follow in a natural way.

1. A first task is the comparison of stress fields before and after the inflation, that is between displaced and distorted configurations of the balloon. How to perform the comparison? How to evaluate the rate of variation of the stress field?
2. A further question concerns the formulation of the constitutive model of the rubber balloon, assumed to be elastic. What are the state variables and what the response? How to verify that the rubber is the same in two points on the balloon?
3. A third issue of investigation is the comparison between constitutive relations detected by different observers performing tests on the balloon's rubber.
4. A last question concerns the proposal of an algorithm for automatic computation of the dynamical trajectory of the inflated rubber balloon, under the thrust of the outgoing air.

Who is willing to give himself an answer, at least one of a purely methodological character, to these questions might find interesting to compare his own conclusions with the ones provided below.

3 New ideas and notions

A careful analysis of the motivating questions leads to conclude that a new geometric approach to Non-Linear Continuum Mechanics (NLCM) is required.

A main effort is to be directed in eliminating standard but otherwise arbitrary choices (such as parallel transport by translation in EUCLID space). Unmotivated special assumptions in adopted procedures is a major obstacle in extending the treatment from the usual 3D body model to include the 1D and 2D engineering models of wires and membranes (the balloon is a thin shell which is conveniently modeled by as a 2D membrane). The simulation of changes of observer needs also a complete revisitation of previous incorrect treatments leading to negative conclusions about feasibility of rate formulations in elasticity and to unacceptable enunciations that rate-elastic materials should necessarily be isotropic [54].

Constitutive relations require new rate formulations in which the stress (and possibly other internal variables) is the state variable and the output is an additive list of various kind of stretching (elastic, plastic, viscous, thermal). These innovative issues will be discussed in detail in Sects. 20, 21.

Leading new notions illustrated in this paper are the following.

1. Definition of *spatial* and *material* tensor bundles over the events manifold and corresponding fields.
2. *Geometric Naturality* (GN) according to which, in introducing basic notions, governing principles and constitutive relations, the metric properties of space-time and the motion along the trajectory should be the sole geometric entities involved.
3. *Dimensionality Independence* (DI) requiring that all notions and results of the field theory should be directly applicable to bodies of any dimensionality.
4. *Geometric Paradigm* (GP) stating that only material tensors can be involved in constitutive relations and that the rule for comparison between material tensors is the push-pull according to the relevant diffeomorphic transformation.
5. *Constitutive Frame Invariance* (CFI) which corrects the principle of Material Frame Indifference (MFI) enunciated in [22, 54]. The CFI states that material tensors are EUCLID frame-invariant and that constitutive relations must be EUCLID frame-invariant.

The first item requires a kinematic framework for GCM based on a space-time formulation and provides a brand new definition of material fields on the trajectory, with no recourse to reference placements, see Remark 1.

The second item, although basic, has never been explicitly stated in a proper geometric form.

The third item is quite reasonable but its requirement has been not fulfilled in most treatments.

The fourth item is a logical consequence of the previous requirements. Its motivation is more subtle when a dimensional coincidence between the body and the ambient space occurs, but is self-proposing for lower dimensional bodies. To grasp the motivation, it should be observed that comparisons between material tensors must be made in either one of the following circumstances.

A. Between material tensors based on two particles at the same time instant, as seen by a single observer.
B. Between material tensors based on same particle at two time instants, as seen by a single observer.
C. Between material tensors based on same particle at the same time instant, as seen by observers in relative motion.

In case A, which occurs for instance in the definition of homogeneous material properties in a body, the comparison tool is an isometric invertible linear transformation between the tangent spaces at the base points. To see this, try to argue about how to compare the stretching and the stress tensors at two points of a curved membrane, at a given time instant, and then consult Sect. 15 below.

In case B, the natural way to perform the comparison consists in considering the evolution diffeomorphism between two placements of a body along a trajectory (whether real or virtual), and in performing the push-pull transformations according to the induced isomorphism between corresponding tangent spaces.

In case C, the comparison is again performed in a natural way by a push-pull transformation according to the map relating the points of view of distinct observers.

The first adoption of the rule dictated by the GP dates back to the mid of eighteenth century being implicit in EULER's notion of stretching and in his celebrated formula providing the expression in terms of the velocity field [30]. The rule has been however often violated in more recent times, with the consequence that the development of CM has been brought out of the right geometric track.

The fifth item is the consistent reformulation of the geometrically and physically improper notion of Material Frame Indifference (MFI), as thoroughly discussed in Sect. 19.

4 New results

The conceptual clarity of the geometric approach and the effectiveness of its adoption become evident as soon as it is applied to formulate constitutive relations, to discuss basic issues such as time independence, time invariance, frame invariance, integrability conditions and conservation of elastic energy, and to design algorithms for the implementation of computational methods.

Fictitious difficulties faced with in the last decades are eliminated by adopting the Geometric Paradigm (GP) which leads to formulate rate constitutive relations for elasto-visco-plasticity (and similar models of material behavior) in a direct and definite way and resolves the long lasting debate about rates of material tensors by giving a unique, simple and well-defined answer. A first exposure of this new approach was contributed in [29] with explicit reference to a new model of covariant hypo-elasticity, an issue playing a fundamental role in constitutive theory.

In fact, the statement about non-integrability of the simplest non-covariant hypo-elastic law [50], based on the analysis performed in [9] and credited in [54], led to discard rate constitutive relations in computational formulations and suggested to introduce a finite formulation based on the multiplicative decomposition of the deformation gradient into subsequent plastic and elastic transformations [18, 19]. This decomposition has gained an increasing favor notwithstanding many debates and criticisms and the questionable physical meaning and the geometric inconsistency of a finite measure of plastic strain, discussed below in Sect. 9. All these matters stem out of a purely algebraic treatment in which geometric features of the non-linear problem are not properly taken into account, since involved tangent spaces at displaced material points are treated as they were coincident or could be superposed by translation. Although the temptation to perform parallel transports by translation might be hard to be resisted when dealing with 3D bodies (and in fact such a translation is performed in [53, 54] and in subsequent contributions), a quick look at the situation for lower dimensional bodies, depicted in Fig. 6, reveals that this track cannot be followed in the construction of a theory embracing bodies of any dimensionality.

The geometric approach, with the ensuing Geometric Paradigm, provides the natural framework for Non-Linear Continuum Mechanics (NLCM), restores to the models of rate-elasticity, of rate elasto-visco-plasticity, and to rate models describing phase transformations or growth phenomena in biomechanics, a basic and effective role in the analysis of material behavior, in a full non-linear range and for bodies of any dimensionality, and draws clear methodological guidelines.

5 Preliminary notions

The investigation about transformations from a given manifold $\mathbb{M}$ to another one $\mathbb{N}$ is a basic task in continuum mechanics. For instance, one is faced with this task when dealing with motions, changes of observer and computational schemes.

A manifold is a geometric object which generalizes the notion of a curve, surface or ball in the EUCLID space. It is characterized by a family of local charts which are differentiable and invertible maps onto open sets in model linear space, say $\mathcal{R}^n$. Then n is the manifold dimension. The inverse maps provide local coordinate systems. Velocities of parametrized curves through a point $\mathbf{x} \in \mathbb{M}$ on a manifold, are tangent vectors at that point and describe the tangent linear space $T_{\mathbf{x}}\mathbb{M}$. The dual space of real-valued linear maps on $T_{\mathbf{x}}\mathbb{M}$ is denoted by $T_{\mathbf{x}}^*\mathbb{M} = (T_{\mathbf{x}}\mathbb{M})^*$ and its elements are called covectors at $\mathbf{x} \in \mathbb{M}$.

To a smooth transformation $f : \mathbb{M} \mapsto \mathbb{N}$ it corresponds, at each point $\mathbf{x} \in \mathbb{M}$, a linear infinitesimal transformation $T_{\mathbf{x}} f : T_{\mathbf{x}}\mathbb{M} \mapsto T_{f(\mathbf{x})}\mathbb{N}$ between the tangent spaces, called the *differential*, whose action on the tangent vector $\mathbf{u}_{\mathbf{x}} := \partial_{s=0}\mathbf{c}(s) \in T_{\mathbf{x}}\mathbb{M}$ to a curve $\mathbf{c} : \mathcal{R} \mapsto \mathbb{M}$, at the point $\mathbf{x} = \mathbf{c}(0)$, is defined by

$$T_{\mathbf{x}} f \cdot \mathbf{u}_{\mathbf{x}} = \partial_{s=0}(f \circ \mathbf{c})(s).$$

A dot $\cdot$ denotes linear dependence on subsequent arguments belonging to linear spaces. A circle $\circ$ denotes composition of maps. A chochét $\langle , \rangle$ denotes the bilinear, non-degenerate duality between pairs of dual linear spaces $(T_{\mathbf{x}}\mathbb{M}, T_{\mathbf{x}}^*\mathbb{M})$ or $(T_{f(\mathbf{x})}\mathbb{N}, T_{f(\mathbf{x})}^*\mathbb{N})$. The dual linear map

$$(T_{\mathbf{x}} f)^* : T^*_{f(\mathbf{x})}\mathbb{N} \mapsto T^*_{\mathbf{x}}\mathbb{M},$$

is defined by the identity

$$\langle T_{\mathbf{x}} f \cdot \mathbf{u}_{\mathbf{x}}, \mathbf{w}_{f(\mathbf{x})} \rangle = \big\langle \mathbf{u}_{\mathbf{x}}, (T_{\mathbf{x}} f)^* \cdot \mathbf{w}_{f(\mathbf{x})} \big\rangle,$$

for any $\mathbf{u}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbb{M}$ and $\mathbf{w}_{f(\mathbf{x})} \in T_{f(\mathbf{x})}\mathbb{N}$.

The tangent bundle $T\mathbb{M}$ and the cotangent bundle $T^*\mathbb{M}$ are disjoint unions respectively of the linear tangent spaces and of the dual spaces based at points of the manifold.

The global transformation between tangent bundles $Tf : T\mathbb{M} \mapsto T\mathbb{N}$ is called the *tangent transformation*. The operator T, acting on manifolds and on maps between them, is named the *tangent functor*.

Zeroth order tensors are just real-valued functions. Second order tensors at $\mathbf{x} \in \mathbb{M}$ are bilinear maps on pairs of vectors or covectors based at that point. They are named covariant, contravariant or mixed depending on whether the arguments are both vectors, both covectors or a vector and a covector.

The corresponding linear tensor spaces at $\mathbf{x} \in \mathbb{M}$ are denoted by $\mathrm{FUN}(T_{\mathbf{x}}\mathbb{M})$, $\mathrm{COV}(T_{\mathbf{x}}\mathbb{M})$, $\mathrm{CON}(T_{\mathbf{x}}\mathbb{M})$, $\mathrm{MIX}(T_{\mathbf{x}}\mathbb{M})$. First order covariant tensors are covectors and first order contravariant tensors are tangent vectors. Second order tensors at $\mathbf{x} \in \mathbb{M}$ are equivalently defined as linear operators from a tangent or cotangent space to another such space at that point:

$$(\mathbf{s}_{\mathrm{COV}})_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{M} \mapsto T^*_{\mathbf{x}}\mathbb{M} \in \mathrm{COV}(T_{\mathbf{x}}\mathbb{M}),$$
$$(\mathbf{s}_{\mathrm{CON}})_{\mathbf{x}} : T^*_{\mathbf{x}}\mathbb{M} \mapsto T_{\mathbf{x}}\mathbb{M} \in \mathrm{CON}(T_{\mathbf{x}}\mathbb{M}),$$
$$(\mathbf{s}_{\mathrm{MIX}})_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{M} \mapsto T_{\mathbf{x}}\mathbb{M} \in \mathrm{MIX}(T_{\mathbf{x}}\mathbb{M}).$$

A covariant tensor $\mathbf{g}_{\mathbf{x}} \in \mathrm{COV}(T_{\mathbf{x}}\mathbb{M})$ is non-degenerate:

$$\mathbf{g}_{\mathbf{x}}(\mathbf{u}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}}) = 0 \quad \forall \mathbf{w}_{\mathbf{x}} \in T_{\mathbf{x}}\mathbb{M} \quad \Longrightarrow \quad \mathbf{u}_{\mathbf{x}} = \mathbf{0}_{\mathbf{x}}.$$

The corresponding linear operator $\mathbf{g}_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{M} \mapsto T^*_{\mathbf{x}}\mathbb{M}$ is then invertible and provides a tool to change tensorial type (alterations). The most important alterations are those which transform covariant or contravariant tensors into mixed ones and vice versa.

$$(\mathbf{s}_{\mathrm{COV}})_{\mathbf{x}} \in \mathrm{COV}(T_{\mathbf{x}}\mathbb{M}) \quad \Longrightarrow$$
$$\mathbf{g}_{\mathbf{x}}^{-1} \cdot (\mathbf{s}_{\mathrm{COV}})_{\mathbf{x}} \in \mathrm{MIX}(T_{\mathbf{x}}\mathbb{M}),$$
$$(\mathbf{s}_{\mathrm{CON}})_{\mathbf{x}} \in \mathrm{CON}(T_{\mathbf{x}}\mathbb{M}) \quad \Longrightarrow$$
$$(\mathbf{s}_{\mathrm{CON}})_{\mathbf{x}} \cdot \mathbf{g}_{\mathbf{x}} \in \mathrm{MIX}(T_{\mathbf{x}}\mathbb{M}).$$

Symmetry of covariant or contravariant tensors means invariance of their values under an exchange of the two arguments. A pseudo-metric tensor is a non-degenerate covariant tensor which is symmetric, i.e.

$$\mathbf{g}_{\mathbf{x}}(\mathbf{u}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}}) = \mathbf{g}(\mathbf{w}_{\mathbf{x}}, \mathbf{u}_{\mathbf{x}}).$$

A metric tensor $\mathbf{g}_{\mathbf{x}} \in \mathrm{COV}(T_{\mathbf{x}}\mathbb{M})$ is symmetric and positive definite, i.e. such that

$$\mathbf{u}_{\mathbf{x}} \neq \mathbf{0} \quad \Longrightarrow \quad \mathbf{g}_{\mathbf{x}}(\mathbf{u}_{\mathbf{x}}, \mathbf{u}_{\mathbf{x}}) > 0.$$

A tensor bundle $\mathrm{TENS}(T\mathbb{M})$ is the disjoint union of tensor fibers which are linear tensor spaces based at points of the manifold.

A bundle is characterized by the projection operator $\boldsymbol{\pi} : \mathrm{TENS}(T\mathbb{M}) \mapsto \mathbb{M}$ which assigns to each element $\mathbf{s}_{\mathbf{x}} \in \mathrm{TENS}(T_{\mathbf{x}}\mathbb{M})$ of the bundle the corresponding base point $\mathbf{x} \in \mathbb{M}$. The fibers $\boldsymbol{\pi}^{-1}(\mathbf{x})$ are the inverse images of the projection and are assumed to be related each-other by diffeomorphic transformations, so that they are all of the same dimension.

A *tensor field* is a map $\mathbf{s} : \mathbb{M} \mapsto \mathrm{TENS}(T\mathbb{M})$ from a manifold $\mathbb{M}$ to a tensor bundle $\mathrm{TENS}(T\mathbb{M})$ such that a point $\mathbf{x} \in \mathbb{M}$ is mapped to a tensor based at the same point, i.e. such that $\boldsymbol{\pi} \circ \mathbf{s}$ is the identity map on $\mathbb{M}$. In geometrical terms it is said that a tensor field is a *section* of a tensor bundle.

A transformation $f : \mathbb{M} \mapsto \mathbb{N}$ maps a curve on $\mathbb{M}$ into a curve in $\mathbb{N}$ and, under suitable assumptions, scalar, vector and covector fields from $\mathbb{M}$ onto $f(\mathbb{M}) \subset \mathbb{N}$ (push forward $\uparrow$) and vice versa (pull back $\downarrow$).[1]

A synopsis is provided below. Assumptions of differentiability and of invertibility of the differential, are claimed whenever needed by the formulae [27].

Push forward from $\mathbb{M}$ on $f(\mathbb{M})$, $f : \mathbb{M} \mapsto \mathbb{N}$ injective.

$$\psi : \mathbb{M} \mapsto \mathcal{R}, \quad (f\uparrow\psi)_{f(\mathbf{x})} = \psi_{\mathbf{x}},$$
$$\mathbf{v} : \mathbb{M} \mapsto T\mathbb{M}, \quad (f\uparrow\mathbf{v})_{f(\mathbf{x})} = T_{\mathbf{x}} f \cdot \mathbf{v}_{\mathbf{x}},$$
$$\mathbf{v}^* : \mathbb{M} \mapsto T^*\mathbb{M},$$
$$\big\langle f\uparrow\mathbf{v}^*, \mathbf{w}\big\rangle_{f(\mathbf{x})} = \big\langle \mathbf{v}^*_{\mathbf{x}}, (T_{\mathbf{x}} f)^{-1} \cdot \mathbf{w}_{f(\mathbf{x})} \big\rangle$$

[1] In differential geometry these are respectively denoted by low and high asterisks $_*;^*$ [51]. This standard notation leads however to consider too many similar stars in the geometric sky, i.e. push, pull, duality, HODGE operator.

Pull back from $f(\mathbb{M})$ to $\mathbb{M}$.

$$\phi : \mathbb{N} \mapsto \mathcal{R}, \quad (f\downarrow\phi)_{\mathbf{x}} = \phi_{f(\mathbf{x})},$$
$$\mathbf{w} : \mathbb{N} \mapsto T\mathbb{N}, \quad (f\downarrow\mathbf{w})_{\mathbf{x}} = (T_{\mathbf{x}}f)^{-1} \cdot \mathbf{w}_{f(\mathbf{x})},$$
$$\mathbf{w}^* : \mathbb{N} \mapsto T^*\mathbb{N}, \quad \langle f\downarrow\mathbf{w}^*, \mathbf{v}\rangle_{\mathbf{x}} = \langle \mathbf{w}^*_{f(\mathbf{x})}, T_{\mathbf{x}}f \cdot \mathbf{v}_{\mathbf{x}}\rangle.$$

Push-pull relations for second order covariant, contravariant and mixed tensors, are defined so that their scalar values be invariant and are given by the formulae

$$(f\downarrow\mathbf{s}_{\mathrm{COV}})_{\mathbf{x}} = (T_{\mathbf{x}}f)^* \cdot (\mathbf{s}_{\mathrm{COV}})_{f(\mathbf{x})} \cdot T_{\mathbf{x}}f \in \mathrm{COV}(T_{\mathbf{x}}\mathbb{M}),$$
$$(f\uparrow\mathbf{s}_{\mathrm{CON}})_{f(\mathbf{x})} = T_{\mathbf{x}}f \cdot (\mathbf{s}_{\mathrm{CON}})_{\mathbf{x}} \cdot (T_{\mathbf{x}}f)^* \in \mathrm{CON}(T_{f(\mathbf{x})}\mathbb{N}),$$
$$(f\uparrow\mathbf{s}_{\mathrm{MIX}})_{f(\mathbf{x})} = T_{\mathbf{x}}f \cdot (\mathbf{s}_{\mathrm{MIX}})_{\mathbf{x}} \cdot (T_{\mathbf{x}}f)^{-1} \in \mathrm{MIX}(T_{f(\mathbf{x})}\mathbb{N}).$$

These transformation rules play an important role in CM since, as recalled below in Sects. 11, 12, 13, the metric tensor is covariant and the dual stress tensor is contravariant. The transformation of mixed tensors does not preserve symmetry with respect to a metric tensor, unless the transformation is isometric, Sect. 17.

A morphism F over f is a pair of maps (F, f) between tensor bundles and their base manifolds, that preserve the tensorial fibers, as expressed by the commutative diagram

$$\begin{tikzcd} \mathrm{TENS}(T\mathbb{M}) \arrow[r, "F"] \arrow[d, "\boldsymbol{\pi}_{\mathbb{M}}"'] & \mathrm{TENS}(T\mathbb{N}) \arrow[d, "\boldsymbol{\pi}_{\mathbb{N}}"] \\ \mathbb{M} \arrow[r, "f"] & \mathbb{N} \end{tikzcd} \quad \Longleftrightarrow$$
$$\boldsymbol{\pi}_{\mathbb{N}} \circ F = f \circ \boldsymbol{\pi}_{\mathbb{M}}.$$

Morphisms that are invertible and differentiable with the inverse, are named *diffeomorphisms*. Important instances of diffeomorphisms are the displacements from a placement of a body to another one, changes of observer, and straightening out maps, Sects. 6, 16, 18. On the other hand, differentiable maps which are *not* diffeomorphisms are, for instance, immersions and projections, Sect. 6.

6 Kinematics and observers

In introducing basic issues of Continuum Kinematics (CK) an emphasis is put on the essential geometrical ingredients of the theory and on the roles they play. The container is the four dimensional events manifold E which is connected and without boundary.

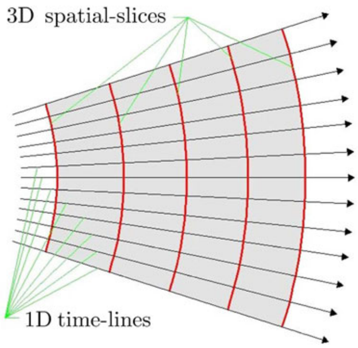


Fig. 1 Space-time splitting

Physical experience tells us that tests performed by an observer are concerned with measurements which, as time goes on, are performed on a trajectory, detected as a set of events sharing some definite properties and fulfilling a characteristic conservation law, such as mass or electric charge conservation.

An observer performs a double foliation of the 4D events manifold into complementary 3D spatial-slices (*isochronous* events) and 1D time-lines (*isotopic* events), Fig. 1.

Accordingly, the tangent space $T_{\mathbf{e}}\mathrm{E}$ at any event $\mathbf{e} \in \mathrm{E}$ is split into a complementary pair of a 3D time-vertical subspace $V_{\mathbf{e}}\mathrm{E}$ (tangent to a spatial-slice) and a 1D time-horizontal subspace $H_{\mathbf{e}}\mathrm{E}$ (tangent to a time-line) generated by a time arrow $\mathbf{Z}_{\mathbf{e}} \in T_{\mathbf{e}}\mathrm{E}$.

The time-vertical subbundle $V\mathrm{E}$ (horizontal $H\mathrm{E}$) of the tangent bundle $T\mathrm{E}$ is the disjoint union of all time-vertical subspaces $V_{\mathbf{e}}\mathrm{E}$ (horizontal $H_{\mathbf{e}}\mathrm{E}$). These are respectively called *spatial* bundle and *time* bundle.

The integral lines $\mathbf{z} \in \mathrm{C}^1(\mathcal{Z}; \mathrm{E})$ of the field $\mathbf{Z} : \mathrm{E} \mapsto T\mathrm{E}$ of time-arrows, are solutions of the differential equation $\partial_{\lambda=0}\mathbf{z}(\lambda) = \mathbf{Z}$. These lines define a foliation whose disjoint 1D leaves are made of *isotopic* events.

An *observer* is expressed in geometrical terms by assigning a field of time-arrows $\mathbf{Z} : \mathrm{E} \mapsto T\mathrm{E}$ and a real-valued *time-function* $t_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}; \mathcal{Z})$ which assigns to each event $\mathbf{e} \in \mathrm{E}$ the corresponding time instant $t = t_{\mathrm{E}}(\mathbf{e}) \in \mathcal{Z}$ which is a real scalar having the physical dimension of time (being *Zeit* the German for *Time*).[2]

The projection according to the time-function makes the events manifold E into the time-bundle whose fibers are sets of *isochronous* (*simultaneous*) events.

The action of an observer may be represented as a rank-one linear projector defined by the tensor product

$$\mathbf{P}_{\mathcal{Z}} := dt_{\mathrm{E}} \otimes \mathbf{Z} : T\mathrm{E} \mapsto H\mathrm{E}.$$

Without loss of generality we may assume *tuning*, i.e. that

$$\partial_{\lambda=0}(t_{\mathrm{E}} \circ \mathbf{z})(\lambda) = \langle dt_{\mathrm{E}}, \mathbf{Z}\rangle = 1.$$

Any tangent vector field $\mathbf{X} : \mathrm{E} \mapsto T\mathrm{E}$ is split into a *horizontal* time-component

$$\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{X} = (dt_{\mathrm{E}} \otimes \mathbf{Z}) \cdot \mathbf{X} = \langle dt_{\mathrm{E}}, \mathbf{X}\rangle \mathbf{Z} \in H\mathrm{E},$$

and a (*time-vertical*) space-component

$$\mathbf{P}_{\mathcal{S}} \cdot \mathbf{X} = \mathbf{X} - \mathbf{P}_{\mathcal{Z}} \cdot \mathbf{X} = \mathbf{X} - \langle dt_{\mathrm{E}}, \mathbf{X}\rangle \mathbf{Z} \in V\mathrm{E}.$$

Then $\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{Z} = \mathbf{Z}$. The characteristic properties $\mathbf{P}_{\mathcal{Z}} \cdot \mathbf{P}_{\mathcal{Z}} = \mathbf{P}_{\mathcal{Z}}$ and $\mathbf{P}_{\mathcal{S}} \cdot \mathbf{P}_{\mathcal{S}} = \mathbf{P}_{\mathcal{S}}$ are easily verified. Spatial vectors are in the kernel of the time-differential dt_{E} since

$$\mathbf{P}_{\mathcal{S}} \cdot \mathbf{X} = \mathbf{X} \quad \Longleftrightarrow \quad \langle dt_{\mathrm{E}}, \mathbf{X}\rangle = 0.$$

In classical mechanics, the events manifold E is assumed to be an affine EUCLID 4D manifold.

The time-vertical subspaces detected by observers are parallel one another and identified with a model 3D affine ambient space $\mathcal{S}$. The time-arrows field is also assumed to be generated by translation of a given one so that time-horizontal 1D subspaces detected by observers are parallel one another and the time parameter can be assumed to be the same for all observers, Fig. 2.

An EUCLID observer defines a one-to-one correspondence, in geometric terms a *trivialization* $\boldsymbol{\gamma} \in \mathrm{C}^1(\mathrm{E}; \mathcal{S} \times \mathcal{Z})$, between the events time-bundle $t_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}; \mathcal{Z})$ and the Cartesian product $\mathcal{S} \times \mathcal{Z}$, with Cartesian projector $\boldsymbol{\pi}_{\mathcal{Z}} \in \mathrm{C}^1(\mathcal{S} \times \mathcal{Z}; \mathcal{Z})$, which is fiber respecting, i.e. such that $\boldsymbol{\pi}_{\mathcal{Z}} \circ \boldsymbol{\gamma} = t_{\mathrm{E}}$ [14].

[2]The time-function $t_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}; \mathcal{Z})$ is assumed to be a projection, i.e. surjective with a surjective tangent map.

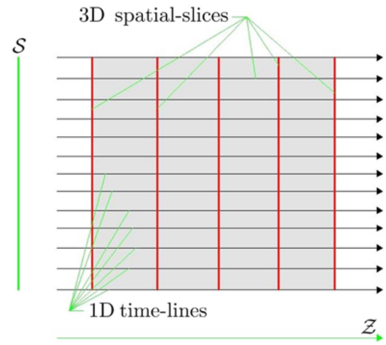


Fig. 2 Euclid space-time splitting

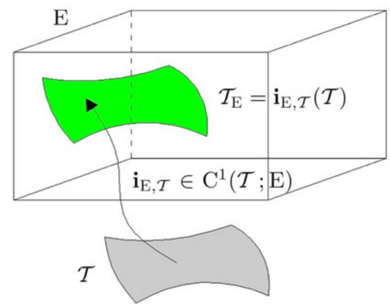


Fig. 3 Trajectory immersion map

In this way all spatial slices are identified with the affine space $\mathcal{S}$ and all time-lines are identified with the time axis $\mathcal{Z}$. These identifications play a basic role in the theory, allowing for definitions of parallel transport in space-time and of spatial motion.

7 Trajectory, body and motion

An observer makes measurements on events belonging to an immersed trajectory $\mathcal{T}_{\mathrm{E}}$, which is a submanifold (possibly lower dimensional) of the events manifold E. Lower dimensional trajectories are considered in the dynamics of 1D or 2D continuum engineering models (wires or membranes).

It is convenient to think of the *trajectory* as a manifold $\mathcal{T}$ which is the domain of an immersion map $\mathbf{i}_{\mathrm{E},\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{E})$ whose image is the immersed trajectory $\mathcal{T}_{\mathrm{E}} = \mathbf{i}_{\mathrm{E},\mathcal{T}}(\mathcal{T}) \subset \mathrm{E}$,[3] as sketched in Fig. 3.

This setting renders it clear that events in the trajectory $\mathcal{T}$ may be detected by a number of free coor-

[3]An immersion is a injective map whose tangent map is injective too.

dinates equal to the dimension $\dim \mathcal{T} \leq \dim \mathcal{T}_\mathrm{E}$, while events in the immersed trajectory $\mathcal{T}_\mathrm{E}$ will be detected by a number of coordinates equal to the dimension $\dim \mathcal{T}_\mathrm{E}$ but fulfilling suitable nonlinear constraints to reduce dimensionality.

Our choice is to avoid the *a priori* introduction of a physically undetectable *body manifold* $\mathcal{B}$ and rather deduce the notions of body manifold and *material particles* from the testable ones of trajectory and motion, as illustrated below.

By this approach, the treatment is more directly related to physical experience and to laboratory measurements and has the mathematical advantage of developing the theory in the 4D space-time manifold.

The choice of space-time as container manifold leads to the simple and general rule expressed by the *Geometric Paradigm* (GP) governing transformations of material tensor fields under the action of the motion or under a change of observer. Significant examples will be given in Sects. 10, 16, 17.

In the trajectory manifold $\mathcal{T}$, the time-fibration $t_\mathcal{T} \in \mathrm{C}^1(\mathcal{T}; \mathcal{Z})$ defined by the composition

$$t_\mathcal{T} := t_\mathrm{E} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}},$$

associates, with each trajectory-event, the time instant pertaining to the immersed event.

The corresponding fibers $t_\mathcal{T}^{-1}(t)$ with $t \in \mathcal{Z}$, when immersed in the events manifold, represent the body-placements $\mathbf{i}_{\mathrm{E},\mathcal{T}}(t_\mathcal{T}^{-1}(t))$.

The motion $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}; \mathcal{T})$ is a one-parameter family of transformations of the trajectory manifold in itself, which preserves simultaneity of events. This property is illustrated by the following commutative diagram

$$\begin{tikzcd}
\mathcal{T}_\mathrm{E} \arrow[r, "\boldsymbol{\varphi}_\alpha^\mathrm{E}"] & \mathcal{T}_\mathrm{E} \\
\mathcal{T} \arrow[u, "\mathbf{i}_{\mathrm{E},\mathcal{T}}"] \arrow[r, "\boldsymbol{\varphi}_\alpha"] \arrow[d, "t_\mathcal{T}"'] & \mathcal{T} \arrow[u, "\mathbf{i}_{\mathrm{E},\mathcal{T}}"'] \arrow[d, "t_\mathcal{T}"] \\
\mathcal{Z} \arrow[r, "\mathrm{SH}_\alpha"] & \mathcal{Z}
\end{tikzcd} \iff$$

$$t_\mathcal{T} \circ \boldsymbol{\varphi}_\alpha = \mathrm{SH}_\alpha \circ t_\mathcal{T}.$$

The time-shift $\mathrm{SH}_\alpha \in \mathrm{C}^1(\mathcal{Z}; \mathcal{Z})$ is defined by

$$\mathrm{SH}_\alpha(t) := t + \alpha \quad \forall \alpha, t \in \mathcal{Z}.$$

The description of a trajectory is conveniently made by means of a parametric representation involving a local system of coordinates in space-time. Hence the submanifold $\mathcal{T}_\mathrm{E}$ and the immersed motion $\boldsymbol{\varphi}_\alpha^\mathrm{E} \in \mathrm{C}^1(\mathcal{T}_\mathrm{E}; \mathcal{T}_\mathrm{E})$ are the direct objects of investigations in mechanics, rather than the trajectory $\mathcal{T}$ itself and the motion $\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}; \mathcal{T})$.

The *trajectory velocity* is the tangent vector field given by $\mathbf{v} := \partial_{\alpha=0} \boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}; T\mathcal{T})$. Its immersion in space-time $\mathbf{v}_\mathrm{E} = \mathbf{i}_{\mathrm{E},\mathcal{T}} \uparrow \mathbf{v} = \partial_{\alpha=0} \boldsymbol{\varphi}_\alpha^\mathrm{E} \in \mathrm{C}^1(\mathcal{T}_\mathrm{E}; T\mathrm{E})$ is defined by the commutative diagram

$$\begin{tikzcd}
\mathcal{T}_\mathrm{E} \arrow[r, "\mathbf{v}_\mathrm{E}"] & T\mathrm{E} \\
\mathcal{T} \arrow[u, "\mathbf{i}_{\mathrm{E},\mathcal{T}}"] \arrow[r, "\mathbf{v}"] & T\mathcal{T} \arrow[u, "T\mathbf{i}_{\mathrm{E},\mathcal{T}}"']
\end{tikzcd} \iff$$

$$\mathbf{v}_\mathrm{E} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}} = T\mathbf{i}_{\mathrm{E},\mathcal{T}} \circ \mathbf{v}.$$

Accordingly the immersed trajectory velocity is split into a spatial and a time component

$$\mathbf{v}_\mathrm{E} = \mathbf{v}_\mathcal{S} + \mathbf{v}_\mathcal{Z}, \quad \text{with } \mathbf{v}_\mathcal{S} = \mathbf{P}_\mathcal{S} \cdot \mathbf{v}_\mathrm{E},\ \mathbf{v}_\mathcal{Z} = \mathbf{P}_\mathcal{Z} \cdot \mathbf{v}_\mathrm{E}.$$

Taking the time derivative of the simultaneity preserving property gives

$$\langle dt_\mathrm{E}, \mathbf{v}_\mathrm{E} \rangle = \left\langle dt_\mathrm{E}, \partial_{\alpha=0} \boldsymbol{\varphi}_\alpha^\mathrm{E} \right\rangle = \partial_{\alpha=0} \mathrm{SH}_\alpha \circ t_\mathrm{E} = 1,$$

so that

$$\mathbf{v}_\mathcal{Z} = \mathbf{P}_\mathcal{Z} \cdot \mathbf{v}_\mathrm{E} = \mathbf{Z},$$

and thus the spatial component $\mathbf{v}_\mathcal{S}$ provides a complete knowledge of the trajectory velocity.

The immersed trajectory velocity $\mathbf{v}_\mathrm{E} = \mathbf{i}_{\mathrm{E},\mathcal{T}} \uparrow \mathbf{v}$ is the direct object of investigations in mechanics since the immersed trajectory is evaluated as the integral manifold of the vector field $\mathbf{v}_\mathrm{E} \in \mathrm{C}^1(\mathcal{T}_\mathrm{E}; T\mathrm{E})$.

Events related by the motion are the elements of a class of equivalence defined by the equivalence relation

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{E} \mid \exists \alpha \in \mathcal{Z} : \mathbf{e}_2 = \boldsymbol{\varphi}_\alpha(\mathbf{e}_1),$$

which foliates the trajectory, as depicted in Fig. 4.

A **material particle** is a line (a one-dimensional manifold) whose elements are evolution-related trajectory events.

The **body manifold** is the quotient manifold (of dimension $n = \dim \mathcal{T} - 1$) induced by the foliation of the trajectory manifold.

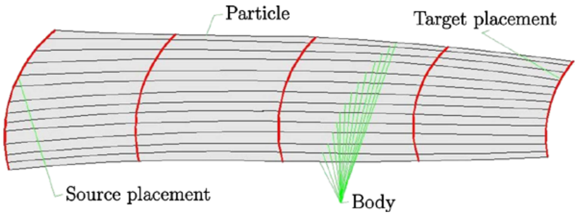


Fig. 4 Trajectory, body and particles

A **body placement** is a fiber of simultaneous events in the immersed trajectory.

8 Spatial and material fields

By definition of the time-projection in the trajectory manifold $t_{\mathcal{T}} := t_{\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}}$ and by injectivity of the map $T\mathbf{i}_{\mathrm{E},\mathcal{T}} \in \mathrm{C}^1(T\mathcal{T}; T\mathrm{E})$, being $dt_{\mathcal{T}} := dt_{\mathrm{E}} \circ T\mathbf{i}_{\mathrm{E},\mathcal{T}}$ we infer the equivalence

$$dt_{\mathcal{T}} = 0 \quad \Longleftrightarrow \quad dt_{\mathrm{E}} = 0,$$

i.e. the immersion $\mathbf{i}_{\mathrm{E},\mathcal{T}} : \mathcal{T} \mapsto \mathrm{E}$ preserves isochronism.

The spatial bundle $\boldsymbol{\pi}_{\mathrm{E}} \in \mathrm{C}^1(V\mathrm{E}_{\mathcal{T}_{\mathrm{E}}}; \mathcal{T}_{\mathrm{E}})$ is the time-vertical subbundle of the restriction of the tangent bundle $\boldsymbol{\pi}_{\mathrm{E}} \in \mathrm{C}^1(T\mathrm{E}; \mathrm{E})$, to the immersed trajectory manifold $\mathcal{T}_{\mathrm{E}}$. Time-vertical vectors $\mathbf{u}_{\mathrm{E}} \in V\mathrm{E}_{\mathcal{T}_{\mathrm{E}}}$ are characterized by the property

$$\langle dt_{\mathrm{E}}, \mathbf{u}_{\mathrm{E}} \rangle = 0.$$

The trajectory bundle is the tangent bundle $\boldsymbol{\pi}_{\mathcal{T}} \in \mathrm{C}^1(T\mathcal{T}; \mathcal{T})$ to the trajectory manifold.

The material bundle is the time-vertical subbundle $\boldsymbol{\pi}_{\mathcal{T}} \in \mathrm{C}^1(V\mathcal{T}; \mathcal{T})$ of the trajectory bundle. Time-vertical vectors $\mathbf{u}_{\mathcal{T}} \in V\mathcal{T}$ are characterized by the property

$$\langle dt_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}} \rangle = 0.$$

Spatial vector fields $\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathcal{T}_{\mathrm{E}}; V\mathrm{E}_{\mathcal{T}_{\mathrm{E}}})$ are sections of the spatial bundle $\boldsymbol{\pi}_{\mathrm{E}} \in \mathrm{C}^1(V\mathrm{E}_{\mathcal{T}_{\mathrm{E}}}; \mathcal{T}_{\mathrm{E}})$.

Spatial tensor fields are constructed over the spatial bundle. At each event $\mathbf{e} \in \mathcal{T}_{\mathrm{E}}$ of the immersed trajectory, with $t = t_{\mathrm{E}}(\mathbf{e})$, they act in a multi-linear way on vectors tangent or cotangent to the spatial fiber $t_{\mathrm{E}}^{-1}(t)$.

Material vector fields $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; V\mathcal{T})$ are sections of the material bundle $\boldsymbol{\pi}_{\mathcal{T}} \in \mathrm{C}^1(V\mathcal{T}; \mathcal{T})$.

Material tensor fields are constructed over the material bundle. At each event $\mathbf{e} \in \mathcal{T}$ of the trajectory, with $t = t_{\mathcal{T}}(\mathbf{e})$, they act in a multi-linear way on vectors tangent or cotangent to the body placement $t_{\mathcal{T}}^{-1}(t)$.

Sections $\mathbf{s}_{\mathcal{T}_{\mathrm{E}}} \in \mathrm{C}^1(\mathcal{T}_{\mathrm{E}}; \mathrm{TENS}(V\mathcal{T}_{\mathrm{E}}))$ of the immersed material bundle $\boldsymbol{\pi}_{\mathcal{T}_{\mathrm{E}}} \in \mathrm{C}^1(V\mathcal{T}_{\mathrm{E}}; \mathcal{T}_{\mathrm{E}})$ will still be called *material* tensor fields.

A volume form is a material tensor field of alternating n-order tensors, defined to within a scalar multiple, with n dimension of the body manifold [27].

Material tensor fields are the main geometric issues in Continuum Mechanics.

Remark 1 The geometric definition of *spatial* and *material* tensor fields given above should be taken as carefully distinct from the homonymic fields in literature. These latter are usually respectively defined to be fields in the body manifold $\mathcal{B}$ (material fields) and in the current placement (spatial fields), see e.g. [48]. According to the new approach, both *material* and *spatial* fields are based on the immersed trajectory manifold. These are the fields of direct interest in Continuum Mechanics, susceptible of describing properties related to the body motion. The difference between them is that material vector fields are tangent to the immersed trajectory at fixed time, i.e. to placements of the body, while spatial fields are tangent to the events manifold at fixed time, i.e. to spatial slices. Examples of material fields are stress, stressing and stretching tensor fields, the heat flux vector field and the scalar fields of temperature and thermodynamical potentials. Spatial fields are forces, virtual velocities and accelerations. The space-time velocity field $\mathbf{v} \in \mathrm{C}^1(\mathcal{T}; T\mathcal{T})$ and its immersion $\mathbf{v}_{\mathrm{E}} \in \mathrm{C}^1(\mathcal{T}_{\mathrm{E}}; T\mathrm{E})$ are neither material nor spatial, but the component $\mathbf{v}_{\mathcal{S}} \in \mathrm{C}^1(\mathcal{T}_{\mathrm{E}}; V\mathrm{E})$, which brings an equivalent information, is a spatial field.

This new definition of spatial and material tensor fields, is physically meaningful and geometrically clear. Its evidence is shadowed by the practice of considering trajectory manifolds having the same dimensionality of the events manifold. Anyway, when lower dimensional trajectory manifolds, such as those pertaining to wires or membranes, are investigated, the need for a clear distinction between spatial and material tensor fields based on the trajectory and between

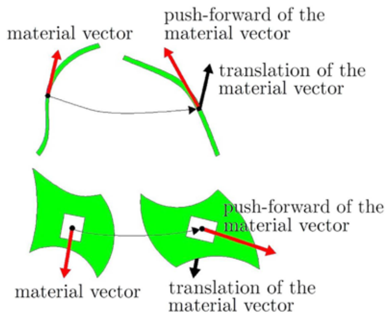


Fig. 5 Push and parallel transport of material vectors

the relevant comparison rules, respectively by parallel transport[4] and by push as dictated by the Geometric Paradigm (GP), becomes evident, as sketched in Fig. 5. There, spatial vectors got by parallel transport of immersed material vectors are black arrows, while pushed material vectors are red arrows.

The new definition of material and spatial fields has no relation with the so called EULER and LAGRANGE descriptions in Mechanics. The discussion provided in [28] points out that the former description amounts in performing time-derivatives along the motion (LIE derivative for material fields and parallel derivatives for spatial fields) by splitting the space-time velocity field into spatial and time components. This procedure is feasible only in special fluid-dynamics problems with reference to which it was originally introduced [15].

9 Material metric field

The change of geometric properties of a continuous body, moving from a source placement to a target placement in the trajectory, can be measured by means of a *space-time metric* tensor field $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}; \mathrm{Cov}(T\mathrm{E}))$. It is the pointwise sum of pull-back of metric tensors on the spatial bundle $\mathbf{g}_{\mathcal{S}} \in \mathrm{C}^1(\mathrm{E}; \mathrm{Cov}(V\mathrm{E}))$ and on the time bundle $\mathbf{g}_{\mathcal{Z}} \in \mathrm{C}^1(\mathrm{E}; \mathrm{Cov}(H\mathrm{E}))$

$$\mathbf{g}_{\mathrm{E}} := \mathbf{P}_{\mathcal{S}}\!\downarrow\!\mathbf{g}_{\mathcal{S}} + \mathbf{P}_{\mathcal{Z}}\!\downarrow\!\mathbf{g}_{\mathcal{Z}}.$$

The explicit expression for $\mathbf{a}, \mathbf{b} \in T\mathrm{E}$ is

$$\mathbf{g}_{\mathrm{E}}(\mathbf{a}, \mathbf{b}) = \mathbf{g}_{\mathcal{S}}(\mathbf{P}_{\mathcal{S}}\mathbf{a}, \mathbf{P}_{\mathcal{S}}\mathbf{b}) + \mathbf{g}_{\mathcal{S}}(\mathbf{P}_{\mathcal{Z}}\mathbf{a}, \mathbf{P}_{\mathcal{Z}}\mathbf{b}).$$

The spatial metric field $\mathbf{g} \in \mathrm{C}^1(\mathcal{T}_{\mathrm{E}}; \mathrm{Cov}(V\mathrm{E}_{\mathcal{T}_{\mathrm{E}}}))$ is the restriction of the space-time metric field $\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E}; \mathrm{Cov}(T\mathrm{E}))$ to the immersed trajectory and to the time-vertical bundle $V\mathrm{E}$.

The trajectory metric field is the pulled-back space-time metric field $\mathbf{i}_{\mathrm{E},\mathcal{T}}\!\downarrow\!\mathbf{g}_{\mathrm{E}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{Cov}(T\mathcal{T}))$ to the trajectory manifold, given for $\mathbf{a}, \mathbf{b} \in T\mathcal{T}$ by

$$(\mathbf{i}_{\mathrm{E},\mathcal{T}}\!\downarrow\!\mathbf{g}_{\mathrm{E}})(\mathbf{a}, \mathbf{b}) = \mathbf{g}_{\mathrm{E}}(T\mathbf{i}_{\mathrm{E},\mathcal{T}} \cdot \mathbf{a}, T\mathbf{i}_{\mathrm{E},\mathcal{T}} \cdot \mathbf{b}).$$

The material metric field is the restriction $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{Cov}(V\mathcal{T}))$ of the trajectory metric to the time-vertical bundle $V\mathcal{T}$, defined by

$$\mathbf{g}_{\mathcal{T}} := (\mathbf{i}_{\mathrm{E},\mathcal{T}}\!\downarrow\!\mathbf{g}_{\mathrm{E}}) = \mathbf{i}_{\mathrm{E},\mathcal{T}}\!\downarrow \cdot\, \mathbf{g}_{\mathrm{E}} \cdot \mathbf{i}_{\mathrm{E},\mathcal{T}}.$$

The material metric plays a basic role in Continuum Mechanics since measurements of the length of curves in body's placements are based on it. The material metric is deduced form the spatial metric by restricting the argument tangent vectors to spatial immersions of material tangent vectors.

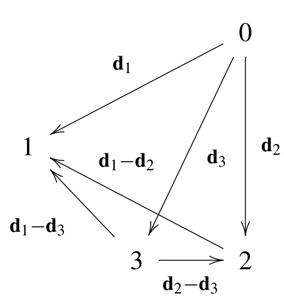
To get a full metric information from experimental data, the following geometric construction is adopted. The EUCLID norm fulfills the parallelogram identity

$$2\big(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\big) = \|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2,$$

with $\mathbf{a}, \mathbf{b} \in V\mathcal{T}$ material vectors. From the knowledge of the lengths $\|\mathbf{a}\|, \|\mathbf{b}\|, \|\mathbf{a}-\mathbf{b}\|$ of the sides of a triangle the length of the sum $\|\mathbf{a}+\mathbf{b}\|$ is inferred. Then the polarization formula

$$\mathbf{g}_{\mathcal{T}}(\mathbf{a}, \mathbf{b}) := \frac{1}{4}\big(\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\big)$$

defines a symmetric, positive definite, twice covariant material tensor.[5] A description of the material metric is provided by considering a non-degenerate simplex



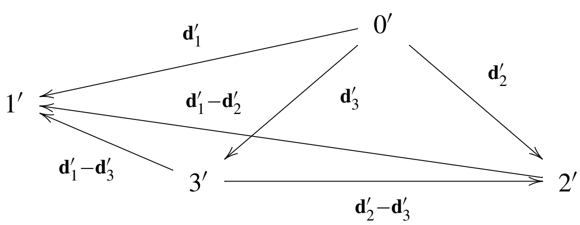
[4] The introduction of the geometric notions of connection and parallel transport can be avoided by restricting oneself to transport by translation in EUCLID space.

[5] The proof is due to FRÉCHET, VON NEUMANN, JORDAN, see [59]. Validity of parallelogram identity is an assumption stronger than the one of validity of PYTHAGORAS' theorem.

For a body dimension $n = 1, 2, 3$, the simplex is a segment, a triangle or a tetrahedron, respectively with $C_{n+1,2} = (n+1)n/2$ sides ($C_{2,2} = 1$ ($n = 1$), $C_{3,2} = 3$ ($n = 2$), $C_{4,2} = 6$ ($n = 3$)). The binomial coefficient $C_{n+1,2}$ gives also the number of components of the symmetric GRAM matrix $G_{ij} := \mathbf{g}_{\mathcal{T}}(\mathbf{d}_i, \mathbf{d}_j)$, $i, j = 1, \ldots, n$ of the metric tensor with respect to the basis of material vectors $\mathbf{d}_1, \ldots, \mathbf{d}_n \in V\mathcal{T}$. The material metric tensor is expressed in terms of the side-lengths by the formula

$$\mathbf{g}_{\mathcal{T}}(\mathbf{d}_i, \mathbf{d}_j) := \frac{1}{2}\big(\|\mathbf{d}_i\|^2 + \|\mathbf{d}_j\|^2 - \|\mathbf{d}_i - \mathbf{d}_j\|^2\big),$$

On each particle, in going from a source to a target placement along the evolution starting time $t \in I$ and ending at time $\tau = t + \alpha \in I$, the finite stretch is experimentally evaluated by comparing the lengths of the edges of a simplex in the tangent space to the source placement with the lengths of the edges of the transformed simplex in the tangent space to the target placement, as depicted hereafter.



By definition, the vectors $\mathbf{d}'_1, \ldots, \mathbf{d}'_n$ of the transformed basis in the target placement are related to the vectors $\mathbf{d}_1, \ldots, \mathbf{d}_n$ of the basis in the source placement by the tangent map $T\boldsymbol{\varphi}_\alpha$ to the transformation. The length measurements permit to evaluate the GRAM matrix

$$\begin{aligned}&\mathbf{g}_{\mathcal{T}}\big(\mathbf{d}'_i, \mathbf{d}'_j\big) - \mathbf{g}_{\mathcal{T}}(\mathbf{d}_i, \mathbf{d}_j)\\ &\quad = \mathbf{g}_{\mathcal{T}}(T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_i, T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_j) - \mathbf{g}_{\mathcal{T}}(\mathbf{d}_i, \mathbf{d}_j)\end{aligned}$$

By linearity of the tangent map $T\boldsymbol{\varphi}_\alpha$ and bilinearity of the metric tensor, the notion of finite stretch is independent of the choice of a particular simplex. Indeed, introducing the pull-back of the material metric tensor

$$(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g}_{\mathcal{T}})(\mathbf{d}_i, \mathbf{d}_j) := \mathbf{g}_{\mathcal{T}}(T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_i, T\boldsymbol{\varphi}_\alpha \cdot \mathbf{d}_j),$$

the basis $\mathbf{d}_1, \ldots, \mathbf{d}_n$ may be eliminated from the formula to get the tensorial measure of the finite stretch in passing from the placement at time $t \in I$ to the one at time $\tau = t + \alpha \in I$

$$\boldsymbol{\eta}_\alpha := \frac{1}{2}(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{g}_{\mathcal{T}} - \mathbf{g}_{\mathcal{T}}).$$

with the factor $\frac{1}{2}$ introduced for convenience, see Sect. 13. This is the GREEN material *strain* (or *stretch*) field.

10 Time-rates of material and spatial fields

The Geometric Paradigm (GP) leads to the following new notions of time-rates for material and spatial fields.

A material field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{TENS}(V\mathcal{T}))$ has a time-rate expressed by its LIE derivative along the motion

$$\dot{\mathbf{s}}_{\mathcal{T}} = \mathcal{L}_{\mathbf{v}}\mathbf{s}_{\mathcal{T}} := \partial_{\alpha=0}(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{s}_{\mathcal{T}}),$$

where

$$(\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{s}_{\mathcal{T}}) := \boldsymbol{\varphi}_\alpha{\downarrow} \cdot \mathbf{s}_{\mathcal{T}} \cdot \boldsymbol{\varphi}_\alpha,$$

with the operator $\boldsymbol{\varphi}_\alpha{\downarrow}$ pointwise defined in Sect. 5 for the various kinds of tensors.

Accordingly, *time invariance* along the motion is expressed by the condition

$$\mathcal{L}_{\mathbf{v}}\mathbf{s}_{\mathcal{T}} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{s}_{\mathcal{T}} = (\boldsymbol{\varphi}_\alpha{\downarrow}\mathbf{s}_{\mathcal{T}}).$$

A spatial field $\mathbf{s}_{\mathrm{E}} \in \mathrm{C}^1(\mathcal{T}_{\mathrm{E}}; \mathrm{TENS}(V\mathrm{E}_{\mathcal{T}_{\mathrm{E}}}))$ has a time-rate expressed by a parallel derivative along the motion

$$\dot{\mathbf{s}}_{\mathrm{E}} = \nabla_{\mathbf{v}_{\mathrm{E}}}\mathbf{s}_{\mathrm{E}} := \partial_{\alpha=0}\big(\boldsymbol{\varphi}^{\mathrm{E}}_\alpha{\Downarrow}\mathbf{s}_{\mathrm{E}}\big),$$

performed according to a given parallel transport $\Uparrow$ in the events manifold, where

$$\big(\boldsymbol{\varphi}^{\mathrm{E}}_\alpha{\Downarrow}\mathbf{s}_{\mathrm{E}}\big) := \boldsymbol{\varphi}^{\mathrm{E}}_\alpha{\Downarrow} \cdot \mathbf{s}_{\mathrm{E}} \cdot \boldsymbol{\varphi}^{\mathrm{E}}_\alpha.$$

Accordingly, *time invariance* along the motion is expressed by the condition

$$\nabla_{\mathbf{v}}\mathbf{s}_{\mathrm{E}} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{s}_{\mathrm{E}} = \big(\boldsymbol{\varphi}^{\mathrm{E}}_\alpha{\Downarrow}\mathbf{s}_{\mathrm{E}}\big).$$

Remark 2 In evaluating time-rates of material and spatial fields defined on lower dimensional bodies it is not allowed to split the velocity as sum of time and spatial components to get the decompositions

$$\mathcal{L}_{\mathbf{v}_\mathrm{E}} \mathbf{s}_{\mathcal{T}_\mathrm{E}} = \mathcal{L}_{\mathbf{Z}} \mathbf{s}_{\mathcal{T}_\mathrm{E}} + \mathcal{L}_{\mathbf{v}_\mathcal{S}} \mathbf{s}_{\mathcal{T}_\mathrm{E}},$$
$$\nabla_{\mathbf{v}_\mathrm{E}} \mathbf{s}_\mathrm{E} = \nabla_{\mathbf{Z}} \mathbf{s}_\mathrm{E} + \nabla_{\mathbf{v}_\mathcal{S}} \mathbf{s}_\mathrm{E}.$$

Indeed the partial time derivative $\mathcal{L}_{\mathbf{Z}} \mathbf{s}_{\mathcal{T}_\mathrm{E}} = \nabla_{\mathbf{Z}} \mathbf{s}_\mathrm{E}$ or the partial space-derivative $\mathcal{L}_{\mathbf{v}_\mathcal{S}} \mathbf{s}_{\mathcal{T}_\mathrm{E}}$ and $\nabla_{\mathbf{v}_\mathcal{S}} \mathbf{s}_\mathrm{E}$ are performable when the vectors $\mathbf{v}_\mathcal{S}$ or $\mathbf{v}_\mathcal{Z} = \mathbf{Z}$ are transversal to the immersed trajectory. Previous treatments of stress-rates [21] and definitions of acceleration [16, 54] are instead based on this decomposition and are therefore confined to 3D bodies.

Remark 3 The previous definition of time-rate of material tensors is in accord with the proposal, made by ARGYRIS in [2], of natural (or symplectic) stress components. Indeed, the contravariant stress tensor is expressed in terms of the symplectic components by the polarization formula

$$\boldsymbol{\sigma}_\mathcal{T}(\mathbf{d}_i^*, \mathbf{d}_j^*) := \frac{1}{2}\big(\boldsymbol{\sigma}_\mathcal{T}(\mathbf{d}_i^*) + \boldsymbol{\sigma}_\mathcal{T}(\mathbf{d}_j^*) - \boldsymbol{\sigma}_\mathcal{T}(\mathbf{d}_i^* - \mathbf{d}_j^*)\big),$$

where $\boldsymbol{\sigma}_\mathcal{T}(\mathbf{d}_i^*) := \boldsymbol{\sigma}_\mathcal{T}(\mathbf{d}_i^*, \mathbf{d}_i^*)$ and $\mathbf{d}_i^* := \mathbf{g}_\mathcal{T} \mathbf{d}_i$ with $i, j = 1, \ldots, n$, being n the body dimension and $\mathbf{d}_i$ the sides of a non-degenerate symplex. Taking the time derivative of the components, evaluated on a flying symplex along the flow $\mathbf{Fl}^\mathbf{v}$ we get

$$\begin{aligned}\partial_{\lambda=0} \boldsymbol{\sigma}_\mathcal{T}\big(T\mathbf{Fl}^\mathbf{v}_\lambda \cdot \mathbf{d}_i^*\big) &= \partial_{\lambda=0}\big(\mathbf{Fl}^\mathbf{v}_\lambda \downarrow \boldsymbol{\sigma}_\mathcal{T}\big)\big(\mathbf{d}_i^*\big) \\ &= \mathcal{L}_\mathbf{v} \boldsymbol{\sigma}_\mathcal{T}\big(\mathbf{d}_i^*\big) = \dot{\boldsymbol{\sigma}}_\mathcal{T}\big(\mathbf{d}_i^*\big).\end{aligned}$$

Remark 4 Time-invariance of a material tensor field is a *natural* property, depending only on the motion which is an essential ingredient of the theory. On the contrary, time-invariance of a spatial tensor field is *not* a natural property, being dependent on the choice of a connection on the events manifold. For instance, time-invariance of the trajectory velocity is not natural. Accordingly, the notion of acceleration is also not natural, being connection dependent.

11 Stretching field

The *material projector* $\boldsymbol{\Pi} \in \mathrm{C}^1(V\mathrm{E}_{\mathcal{T}_\mathrm{E}}; V\mathcal{T})$ from the spatial bundle onto the material bundle is defined by the identity

$$\mathbf{g}_\mathcal{T}(\boldsymbol{\Pi} \cdot \mathbf{a}_\mathrm{E}, \mathbf{b}) = \mathbf{g}(\mathbf{a}, \mathbf{i}_{\mathrm{E},\mathcal{T}} \uparrow \mathbf{b}),$$
$$\forall \mathbf{a}_\mathrm{E} \in V\mathrm{E}_{\mathcal{T}_\mathrm{E}}, \mathbf{b} \in V\mathcal{T}.$$

Then $\boldsymbol{\Pi}^A = \mathbf{i}_{\mathrm{E},\mathcal{T}} \uparrow \in \mathrm{C}^1(V\mathcal{T}; V\mathrm{E}_{\mathcal{T}_\mathrm{E}})$ is the *space-time immersion*, $(\mathbf{g}_\mathcal{T}, \mathbf{g})$-adjoint, with $\boldsymbol{\Pi} \cdot \boldsymbol{\Pi}^A = \mathrm{ID}_{V\mathcal{T}}$.

Given a mixed tensor field $\mathbf{L} \in \mathrm{C}^1(V\mathrm{E}_{\mathcal{T}_\mathrm{E}}; V\mathrm{E}_{\mathcal{T}_\mathrm{E}})$ and a pair of material vectors $\mathbf{a}, \mathbf{b} \in V\mathcal{T}$, we have that

$$\mathbf{g}\big(\mathbf{L} \cdot \boldsymbol{\Pi}^A \cdot \mathbf{a}, \boldsymbol{\Pi}^A \cdot \mathbf{b}\big) = \mathbf{g}_\mathcal{T}\big(\boldsymbol{\Pi} \cdot \mathbf{L} \cdot \boldsymbol{\Pi}^A \cdot \mathbf{a}, \mathbf{b}\big),$$

with $\boldsymbol{\Pi} \cdot \mathbf{L} \cdot \boldsymbol{\Pi}^A \in \mathrm{C}^1(\mathcal{T}; \mathrm{MIX}(V\mathcal{T}))$ mixed material tensor field. According to EULER's formula, the stretching field in a placement of a body is determined by the spatial velocity field at the evaluation time, the material stretching tensor field being defined by [30]

$$\begin{aligned}\boldsymbol{\varepsilon}(\mathbf{v}_\mathcal{S}) &= \frac{1}{2}\mathcal{L}_\mathbf{v} \mathbf{g}_\mathcal{T} = \mathbf{i}_{\mathrm{E},\mathcal{T}} \downarrow \frac{1}{2}(\mathcal{L}_\mathbf{v} \mathbf{g}) \\ &= \boldsymbol{\Pi} \cdot \frac{1}{2}(\mathcal{L}_\mathbf{v} \mathbf{g}) \cdot \boldsymbol{\Pi}^A = \mathbf{g}_\mathcal{T} \cdot \boldsymbol{\Pi} \cdot \mathbf{D}(\mathbf{v}_\mathcal{S}) \cdot \boldsymbol{\Pi}^A,\end{aligned}$$

where $\mathbf{D}(\mathbf{v}_\mathcal{S}) \in \mathrm{C}^1(\mathcal{T}_\mathrm{E}; \mathrm{MIX}(V\mathrm{E}))$ is expressed by the extended EULER's formula

$$\mathbf{D}(\mathbf{v}_\mathcal{S}) := \mathrm{sym}(\nabla \mathbf{v}_\mathcal{S}) + \mathbf{G}(\mathbf{v}_\mathcal{S}) + \mathbf{A}(\mathbf{v}_\mathcal{S})$$

with $\mathbf{G}(\mathbf{v}_\mathcal{S}), \mathbf{A}(\mathbf{v}_\mathcal{S}) \in \mathrm{C}^1(\mathcal{T}_\mathrm{E}; \mathrm{MIX}(V\mathrm{E}))$ defined by

$$\mathbf{g} \circ \mathbf{G}(\mathbf{v}_\mathcal{S}) := \frac{1}{2} \nabla_{\mathbf{v}_\mathcal{S}} \mathbf{g},$$
$$\mathbf{A}(\mathbf{v}_\mathcal{S}) := \mathrm{sym}\big(\mathrm{TORS}(\mathbf{v}_\mathcal{S})\big).$$

These terms are tensorial in $\mathbf{v}_\mathcal{S}$ and vanish when the metric-compatible and torsion-free LEVI-CIVITA connection associated with $\mathbf{g}$ is adopted [30].

A velocity field is isometric if the corresponding stretching field vanishes. The same reasoning may be applied to a virtual motion along a virtual trajectory $\delta\mathcal{T}$ in a virtual events manifold $\delta\mathrm{E} := \mathrm{E}(t) \times \Lambda$ at a fixed time instant, say $t \in I$, with Λ line of virtual-time instants. The virtual stretching due to a virtual velocity field $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}; V\mathrm{E})$ is then given by

$$\boldsymbol{\varepsilon}(\delta\mathbf{v}) = \mathbf{i}_{\mathrm{E},\mathcal{T}} \downarrow \frac{1}{2}(\mathcal{L}_{\delta\mathbf{v}} \mathbf{g}) = \mathbf{g}_\mathcal{T} \cdot \boldsymbol{\Pi} \cdot \mathbf{D}(\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A.$$

12 Duality pairing

The duality between a contravariant material tensor $\boldsymbol{\sigma} \in \mathrm{CON}(V\mathcal{T})$ and a covariant $\boldsymbol{\varepsilon} \in \mathrm{COV}(V\mathcal{T})$ material tensor is defined as the linear invariant of the mixed tensor field $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^A \in \mathrm{MIX}(V\mathcal{T})$, that is

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle := J^1\big(\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^A\big),$$

with the *adjoint* $\boldsymbol{\varepsilon}^A \in \mathrm{CON}(V\mathcal{T})$ defined by the identity

$$\boldsymbol{\varepsilon}(\mathbf{a}, \mathbf{b}) = \boldsymbol{\varepsilon}^A(\mathbf{b}, \mathbf{a}),$$

for all $\mathbf{a}, \mathbf{b} \in V\mathcal{T}$. The corresponding mixed tensors are given by

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma} \cdot \mathbf{g}_{\mathcal{T}} \in \mathrm{MIX}(V\mathcal{T}),$$
$$\mathbf{E} = \mathbf{g}_{\mathcal{T}}^{-1} \cdot \boldsymbol{\varepsilon} \in \mathrm{MIX}(V\mathcal{T})$$

with the $\mathbf{g}_{\mathcal{T}}$-*adjoint* $\mathbf{E}^A \in \mathrm{MIX}(V\mathcal{T})$ defined by the identity

$$\mathbf{g}_{\mathcal{T}}(\mathbf{E}\mathbf{a}, \mathbf{b}) = \mathbf{g}_{\mathcal{T}}(\mathbf{E}^A\mathbf{b}, \mathbf{a}).$$

Hence $\boldsymbol{\varepsilon}^A = \mathbf{g}_{\mathcal{T}} \cdot \mathbf{E}^A$ and $\boldsymbol{\sigma} = \boldsymbol{\Sigma} \cdot \mathbf{g}_{\mathcal{T}}^{-1}$ so that

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^A = \boldsymbol{\Sigma} \cdot \mathbf{g}_{\mathcal{T}}^{-1} \cdot \mathbf{g}_{\mathcal{T}} \cdot \mathbf{E}^A = \boldsymbol{\Sigma} \cdot \mathbf{E}^A.$$

By symmetry and positive definiteness of the bilinear form

$$\begin{aligned}\mathbf{g}_{\mathrm{MIX}}(\boldsymbol{\Sigma}, \mathbf{E}) &= \mathbf{g}_{\mathrm{MIX}}(\mathbf{E}, \boldsymbol{\Sigma}) \\ &:= J^1(\boldsymbol{\Sigma} \cdot \mathbf{E}^A) = J^1\big(\mathbf{E}^A \cdot \boldsymbol{\Sigma}\big),\end{aligned}$$

the duality pairing induces a metric tensor in $\mathrm{MIX}(V\mathcal{T})$.

13 Equilibrium

We denote by $\mathcal{H}_{\boldsymbol{\Omega}}$ a linear space of virtual velocity fields in the placement $\boldsymbol{\Omega}$, endowed with a suitable HILBERT topology and by $\mathcal{H}^*_{\boldsymbol{\Omega}}$ the dual space [38, 43]. According to the original definition, enunciated by JOHANN BERNOULLI in a letter on 26 February 1715 to PIERRE VARIGNON, the equilibrium of a force system acting on a body, at a given time instant, is characterized by the property that there is no duality interaction (virtual power) between the force system $\mathbf{f} \in \mathcal{H}^*_{\boldsymbol{\Omega}}$ and any virtual isometric velocity field

$$\langle \mathbf{f}, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}} = 0, \quad \forall \delta\mathbf{v} \in \mathcal{H}_{\boldsymbol{\Omega}} : \mathbf{D}(\delta\mathbf{v}) = 0.$$

Stress fields in the body are introduced by duality, as LAGRANGE multipliers of the constraint defined by the linear subspace of virtual isometric velocities.

The Virtual Power Principle (VPP) states that there exists a material tensor field $\boldsymbol{\sigma} \in \mathrm{C}^1(\mathcal{T}; \mathrm{CON}(V\mathcal{T}))$, the KIRCHHOFF stress, such that

$$\langle \mathbf{f}, \delta\mathbf{v} \rangle_{\boldsymbol{\Omega}} = \int_{\boldsymbol{\Omega}} \big\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}) \big\rangle \mathbf{m}.$$

The virtual power performed, at time $t \in I$, by the equilibrated force system $\mathbf{f} \in \mathcal{H}^*_{\boldsymbol{\Omega}}$ interacting with any virtual velocity field $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}; V\mathrm{E})$, is then equal to the integral of the virtual power per unit mass performed by the stress field interacting with the induced virtual stretching field times the mass form $\mathbf{m} \in \mathrm{C}^1(\mathcal{T}; \mathrm{VOL}(V\mathcal{T}))$, the integral being extended over the body placement $\boldsymbol{\Omega}$ at that time. The KIRCHHOFF mixed stress field is given by

$$\mathbf{K} := \boldsymbol{\sigma} \cdot \mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1\big(\mathcal{T}; \mathrm{MIX}(V\mathcal{T})\big).$$

Then

$$\begin{aligned}\big\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\delta\mathbf{v}) \big\rangle &:= J^1\big(\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\delta\mathbf{v})^A\big) \\ &= \mathbf{g}_{\mathrm{MIX}}\big(\mathbf{K} \cdot \mathbf{g}_{\mathcal{T}}^{-1}, \mathbf{g}_{\mathcal{T}} \cdot \boldsymbol{\Pi} \cdot \mathbf{D}(\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A\big) \\ &= \mathbf{g}_{\mathrm{MIX}}\big(\mathbf{K}, \boldsymbol{\Pi} \cdot \mathbf{D}(\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A\big).\end{aligned}$$

Denoting by $\boldsymbol{\mu} \in \mathrm{C}^1(\mathcal{T}; \mathrm{VOL}(V\mathcal{T}))$ the material volume form associated with the material metric tensor $\mathbf{g}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{COV}(V\mathcal{T}))$ and by $\rho \in \mathrm{C}^1(\mathcal{T}; \mathrm{FUN}(V\mathcal{T}))$ the scalar mass density on the trajectory, so that $\mathbf{m} = \rho\boldsymbol{\mu}$, the CAUCHY stress $\mathbf{T}$ is given by $\mathbf{T} := \rho\mathbf{K}$.

Then, the inner products $\mathbf{g}_{\mathrm{MIX}}(\mathbf{K}, \boldsymbol{\Pi} \cdot \mathbf{D}(\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A)$ and $\mathbf{g}_{\mathrm{MIX}}(\mathbf{T}, \boldsymbol{\Pi} \cdot \mathbf{D}(\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A)$ provide the internal virtual power per unit mass and unit volume, respectively.

Symmetry of the covariant stretching tensor $\boldsymbol{\varepsilon}(\delta\mathbf{v}) \in \mathrm{COV}(V\mathcal{T})$ and $\mathbf{g}_{\mathcal{T}}$-symmetry of the mixed stretching tensor $\boldsymbol{\Pi} \cdot \mathbf{D}(\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A \in \mathrm{MIX}(V\mathcal{T})$ entail corresponding symmetries of the contravariant stress $\boldsymbol{\sigma} \in \mathrm{CON}(V\mathcal{T})$ and of the mixed tensors expressing KIRCHHOFF $\mathbf{K} \in \mathrm{MIX}(V\mathcal{T})$ and CAUCHY $\mathbf{T} \in \mathrm{MIX}(V\mathcal{T})$ stresses, the anti-symmetric part being inessential in performing virtual power.

The equality in the statement of the VPP is the extension to functional spaces of the well-known orthogonality property concerning the kernel of a linear operator and the image of its dual, in linear algebra. The extension requires however topological concepts and results from Functional Analysis.[6]

14 Green's formula

In getting GREEN formula the first item is STOKES formula applied to a body placement $\boldsymbol{\Omega}$

$$\int_{\boldsymbol{\Omega}} d(\boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot \mathbf{h}) = \oint_{\partial\boldsymbol{\Omega}} \boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot \mathbf{h},$$

where $\boldsymbol{\mu}_{\boldsymbol{\Omega}} \in \mathrm{C}^1(\boldsymbol{\Omega}; \mathrm{COV}(V\mathcal{T}))$ is the material volume-form on the trajectory, $\mathbf{h} \in \mathrm{C}^1(\boldsymbol{\Omega}; V\mathcal{T})$ is a material vector field and d is the exterior derivative on the placement manifold.

The boundary integral may be rewritten by resorting to the equality

$$\boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot \mathbf{h} = \mathbf{g}_{\mathcal{T}}(\mathbf{h}, \mathbf{n}_{\partial\boldsymbol{\Omega}})\boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot \mathbf{n}_{\partial\boldsymbol{\Omega}} = \mathbf{g}_{\mathcal{T}}(\mathbf{h}, \mathbf{n}_{\partial\boldsymbol{\Omega}})\boldsymbol{\mu}_{\partial\boldsymbol{\Omega}},$$

where $\boldsymbol{\mu}_{\partial\boldsymbol{\Omega}} := \boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot \mathbf{n}_{\partial\boldsymbol{\Omega}}$ is the induced volume-form on the boundary $\partial\boldsymbol{\Omega}$ of $\boldsymbol{\Omega}$ with $\mathbf{n}_{\partial\boldsymbol{\Omega}} \in \mathrm{C}^1(\mathcal{T}; V\mathcal{T})$ time-vertical outward normal.

The second item is the differential homotopy formula which, when applied to the material volume-form $\boldsymbol{\mu}_{\boldsymbol{\Omega}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{COV}(V\mathcal{T}))$, writes

$$\mathcal{L}_{\mathbf{h}}\boldsymbol{\mu}_{\boldsymbol{\Omega}} = d(\boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot \mathbf{h}) + (d\boldsymbol{\mu}_{\boldsymbol{\Omega}}) \cdot \mathbf{h}.$$

Being $d\boldsymbol{\mu}_{\boldsymbol{\Omega}} = 0$ by maximality of the material volume-form, the divergence of a material vector field in a placement is defined, in terms of LIE derivative or of exterior derivative, by

$$(\operatorname{div}\mathbf{h})\boldsymbol{\mu}_{\boldsymbol{\Omega}} := \mathcal{L}_{\mathbf{h}}\boldsymbol{\mu}_{\boldsymbol{\Omega}} = d(\boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot \mathbf{h}).$$

The output is the general expression of GAUSS *divergence theorem*

$$\int_{\boldsymbol{\Omega}} (\operatorname{div}\mathbf{h})\boldsymbol{\mu}_{\boldsymbol{\Omega}} = \oint_{\partial\boldsymbol{\Omega}} \mathbf{g}_{\mathcal{T}}(\mathbf{h}, \mathbf{n}_{\partial\boldsymbol{\Omega}})\boldsymbol{\mu}_{\partial\boldsymbol{\Omega}}.$$

The third item is the definition of *spatial divergence* of a material tensor field. With reference to a mixed tensor field $\mathbf{T} \in \mathrm{C}^1(\mathcal{T}; \mathrm{MIX}(V\mathcal{T}))$, the $\mathbf{g}_{\mathcal{T}}$-adjoint $\mathbf{T}^A \in \mathrm{C}^1(\mathcal{T}; \mathrm{MIX}(V\mathcal{T}))$ is defined by the identity for all $\mathbf{u}, \mathbf{w} \in \mathrm{C}^1(\mathcal{T}; V\mathcal{T})$

$$\mathbf{g}_{\mathcal{T}}(\mathbf{T}^A \cdot \mathbf{w}, \mathbf{u}) = \mathbf{g}_{\mathcal{T}}(\mathbf{T} \cdot \mathbf{u}, \mathbf{w}).$$

The *spatial divergence* $\operatorname{Div}\mathbf{T} \in \mathrm{C}^1(\mathcal{T}; V\mathrm{E}_{\mathcal{T}_\mathrm{E}})$ is then defined by a formal LEIBNIZ rule

$$\begin{aligned}\mathbf{g}(-\operatorname{Div}\mathbf{T}, \delta\mathbf{v}) &:= \mathbf{g}_{\mathrm{MIX}}(\mathbf{T}, \boldsymbol{\Pi} \cdot \nabla\delta\mathbf{v} \cdot \boldsymbol{\Pi}^A) \\ &\quad - \operatorname{div}((\mathbf{T}^A \cdot \boldsymbol{\Pi}) \cdot \delta\mathbf{v}),\end{aligned}$$

where $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}; V\mathrm{E}_{\mathcal{T}_\mathrm{E}})$ is a virtual velocity field.

The definition is well-posed because the sum of the two terms at the l.h.s. is tensorial in the field $\delta\mathbf{v} \in \mathrm{C}^1(\mathcal{T}; V\mathrm{E}_{\mathcal{T}_\mathrm{E}})$ as can be proven by a direct application of the tensoriality criterion in [27, Lemma 1.2.1 p. 28].

Setting $\mathbf{h} = \mathbf{T}^A \cdot \boldsymbol{\Pi} \cdot \delta\mathbf{v}$ in GAUSS divergence theorem, it follows that

$$\begin{aligned}&\int_{\boldsymbol{\Omega}} \operatorname{div}((\mathbf{T}^A \cdot \boldsymbol{\Pi}) \cdot \delta\mathbf{v})\boldsymbol{\mu}_{\boldsymbol{\Omega}} \\ &\quad = \oint_{\partial\boldsymbol{\Omega}} \boldsymbol{\mu}_{\boldsymbol{\Omega}} \cdot (\mathbf{T}^A \cdot \boldsymbol{\Pi} \cdot \delta\mathbf{v}) \\ &\quad = \oint_{\partial\boldsymbol{\Omega}} \mathbf{g}_{\mathcal{T}}(\mathbf{T}^A \cdot \boldsymbol{\Pi} \cdot \delta\mathbf{v}, \mathbf{n}_{\partial\boldsymbol{\Omega}})\boldsymbol{\mu}_{\partial\boldsymbol{\Omega}} \\ &\quad = \oint_{\partial\boldsymbol{\Omega}} \mathbf{g}_{\mathcal{T}}(\mathbf{T} \cdot \mathbf{n}_{\partial\boldsymbol{\Omega}}, \boldsymbol{\Pi} \cdot \delta\mathbf{v})\boldsymbol{\mu}_{\partial\boldsymbol{\Omega}}.\end{aligned}$$

By definition of spatial divergence $\operatorname{Div}\mathbf{T} \in \mathrm{C}^1(\mathcal{T}; V\mathrm{E}_{\mathcal{T}_\mathrm{E}})$ and assuming symmetry of $\mathbf{T}$, we get GREEN's formula

$$\begin{aligned}&\int_{\boldsymbol{\Omega}} \mathbf{g}_{\mathrm{MIX}}(\mathbf{T}, \boldsymbol{\Pi} \cdot \operatorname{sym}(\nabla\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A)\boldsymbol{\mu}_{\boldsymbol{\Omega}} \\ &\quad = -\int_{\boldsymbol{\Omega}} \mathbf{g}(\operatorname{Div}\mathbf{T}, \delta\mathbf{v})\boldsymbol{\mu}_{\boldsymbol{\Omega}} \\ &\qquad + \oint_{\partial\boldsymbol{\Omega}} \mathbf{g}(\boldsymbol{\Pi}^A \cdot \mathbf{T} \cdot \mathbf{n}_{\partial\boldsymbol{\Omega}}, \delta\mathbf{v})\boldsymbol{\mu}_{\partial\boldsymbol{\Omega}}.\end{aligned}$$

The virtual power principle then yields

$$\begin{aligned}\langle\mathbf{f}, \delta\mathbf{v}\rangle_{\boldsymbol{\Omega}} &= \int_{\boldsymbol{\Omega}} \mathbf{g}_{\mathrm{MIX}}(\mathbf{T}, \boldsymbol{\Pi} \cdot \mathbf{D}(\delta\mathbf{v}) \cdot \boldsymbol{\Pi}^A)\boldsymbol{\mu}_{\boldsymbol{\Omega}} \\ &= -\int_{\boldsymbol{\Omega}} \mathbf{g}(\operatorname{Div}\mathbf{T}, \delta\mathbf{v})\boldsymbol{\mu}_{\boldsymbol{\Omega}} \\ &\quad + \oint_{\partial\boldsymbol{\Omega}} \mathbf{g}(\boldsymbol{\Pi}^A \cdot \mathbf{T} \cdot \mathbf{n}_{\partial\boldsymbol{\Omega}}, \delta\mathbf{v})\boldsymbol{\mu}_{\partial\boldsymbol{\Omega}} \\ &\quad + \int_{\boldsymbol{\Omega}} \mathbf{g}((\mathbf{G}^A + \mathbf{A}^A) \cdot \boldsymbol{\Pi}^A \cdot \mathbf{T} \cdot \boldsymbol{\Pi}, \delta\mathbf{v})\boldsymbol{\mu}_{\boldsymbol{\Omega}}.\end{aligned}$$

[6] For bodies of maximal dimension, the VPP is a proved theorem [27].

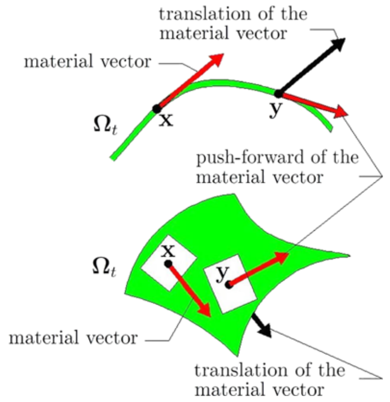


Fig. 6 Push and translation of a material vector

15 Homogeneity

Although the notion of homogeneity of a material tensor field is of clear theoretical and technical interest in CM, a proper definition is not readily available in literature. According to the geometric point of view, the comparison of material tensors at simultaneous events on the trajectory, must be made by push. Motivated by the fact that homogeneity and invariance under change of EUCLID observers, in relative isometric motion, should be related notions, we put the following definition.

A material field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{TENS}(V\mathcal{T}))$ is *two-point homogeneous* if there exist at least an isometric isomorphism between the tangent spaces at two points in a body placement, such that the values of the material field at the two points are related by push-pull according to this isometry.

If this property holds for each pair of material points in a body placement, the material field is called *homogeneous*. The temptation of comparing the values of a material field, at two material points in a body placement, by parallel transport in space, should readily be abandoned, just by giving a look at Fig. 6.

16 Frame changes

A change of observer is a diffeomorphic transformation $\boldsymbol{\zeta}_{\mathrm{E}} \in \mathrm{C}^1(\mathrm{E};\mathrm{E})$ of the events manifold onto itself, i.e. a transformation which is differentiable and invertible together with its tangent map $T\boldsymbol{\zeta}_{\mathrm{E}} \in \mathrm{C}^1(T\mathrm{E};T\mathrm{E})$.

The induced transformation $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\boldsymbol{\zeta}})$ is a time-bundle diffeomorphism between the trajectories seen by different observers, as depicted in the commutative diagram

$$\begin{tikzcd} \mathrm{E} \arrow[r, shift left, "\boldsymbol{\zeta}_{\mathrm{E}}"] & \mathrm{E} \arrow[l, shift left, "\boldsymbol{\zeta}_{\mathrm{E}}^{-1}"] \\ \mathcal{T} \arrow[u, "\mathbf{i}_{\mathrm{E},\mathcal{T}}"] \arrow[r, shift left, "\boldsymbol{\zeta}"] & \mathcal{T}_{\boldsymbol{\zeta}} \arrow[u, "\mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\zeta}}}"'] \arrow[l, shift left, "\boldsymbol{\zeta}^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T} \arrow[r, shift left, "\boldsymbol{\zeta}"] \arrow[d, "t_{\mathcal{T}}"] & \mathcal{T}_{\boldsymbol{\zeta}} \arrow[l, shift left, "\boldsymbol{\zeta}^{-1}"] \arrow[d, "t_{\mathcal{T}_{\boldsymbol{\zeta}}}"] \\ \mathcal{Z} \arrow[r, leftrightarrow, "\mathrm{ID}_{\mathcal{Z}}"] & \mathcal{Z} \end{tikzcd}$$

$$\iff \begin{cases} \boldsymbol{\zeta}_{\mathrm{E}} \circ \mathbf{i}_{\mathrm{E},\mathcal{T}} = \mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\zeta}}} \circ \boldsymbol{\zeta}, \\ t_{\mathcal{T}_{\boldsymbol{\zeta}}} \circ \boldsymbol{\zeta} = t_{\mathcal{T}}. \end{cases}$$

A material tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{TENS}(V\mathcal{T}))$ is *frame invariant* if, under the action of a transformation $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\boldsymbol{\zeta}})$, it transforms according to the push

$$\mathbf{s}_{\mathcal{T}_{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{s}_{\mathcal{T}}.$$

The pushed motion is defined by the commutative diagram

$$\begin{tikzcd} \mathcal{T}_{\boldsymbol{\zeta}} \arrow[r, "(\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi})_\alpha"] & \mathcal{T}_{\boldsymbol{\zeta}} \\ \mathcal{T} \arrow[u, "\boldsymbol{\zeta}"] \arrow[r, "\boldsymbol{\varphi}_\alpha"] & \mathcal{T} \arrow[u, "\boldsymbol{\zeta}"'] \end{tikzcd} \iff (\boldsymbol{\zeta}\!\uparrow\!\boldsymbol{\varphi})_\alpha = \boldsymbol{\zeta} \circ \boldsymbol{\varphi}_\alpha \circ \boldsymbol{\zeta}^{-1}.$$

In a change of EUCLID observer the frame transformation is an isometry $\boldsymbol{\zeta}_{\mathrm{E}}^{\mathrm{ISO}} \in \mathrm{C}^1(\mathrm{E};\mathrm{E})$ i.e. $\mathbf{g}_{\mathrm{E}} = \boldsymbol{\zeta}_{\mathrm{E}}^{\mathrm{ISO}}\!\uparrow\!\mathbf{g}_{\mathrm{E}}$. The trajectory transformation $\boldsymbol{\zeta}_{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\boldsymbol{\zeta}})$ is then an isometry too. Indeed from the equality

$$\mathbf{i}_{\mathrm{E},\mathcal{T}}\!\uparrow\!\boldsymbol{\zeta}_{\mathrm{ISO}}\!\downarrow = \boldsymbol{\zeta}_{\mathrm{E}}^{\mathrm{ISO}}\!\downarrow\!\mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\zeta}}}\!\uparrow,$$

it follows that

$$\begin{aligned} \boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\mathbf{g}_{\mathcal{T}} &= \boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!(\mathbf{i}_{\mathrm{E},\mathcal{T}}\!\downarrow\!\mathbf{g}_{\mathrm{E}}) = \mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\zeta}}}\!\downarrow\!(\boldsymbol{\zeta}_{\mathrm{E}}^{\mathrm{ISO}}\!\uparrow\!\mathbf{g}_{\mathrm{E}}) \\ &= \mathbf{i}_{\mathrm{E},\mathcal{T}_{\boldsymbol{\zeta}}}\!\downarrow\!\mathbf{g}_{\mathrm{E}} = \mathbf{g}_{\mathcal{T}_{\boldsymbol{\zeta}}}. \end{aligned}$$

17 Frame-invariance

A trajectory tensor field $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T};\mathrm{TENS}(V\mathcal{T}))$ is *frame-invariant* under the action of a trajectory transformation $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T};\mathcal{T}_{\boldsymbol{\zeta}})$ if

$$\mathbf{s}_{\mathcal{T}_{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}\!\uparrow\!\mathbf{s}_{\mathcal{T}}.$$

The trajectory velocity $\mathbf{v} := \partial_{\alpha=0}\boldsymbol{\varphi}_\alpha \in \mathrm{C}^1(\mathcal{T}; T\mathcal{T})$ is frame-invariant since the transformed velocity $\mathbf{v}_\zeta := \partial_{\alpha=0}(\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi})_\alpha \in \mathrm{C}^1(\mathcal{T}_\zeta; T\mathcal{T}_\zeta)$ fulfills the relation

$$\mathbf{v}_\zeta := \partial_{\alpha=0}(\boldsymbol{\zeta}\uparrow\boldsymbol{\varphi})_\alpha = \partial_{\alpha=0}(\boldsymbol{\zeta} \circ \boldsymbol{\varphi}_\alpha \circ \boldsymbol{\zeta}^{-1}) \\ = T\boldsymbol{\zeta} \circ \partial_{\alpha=0}\boldsymbol{\varphi}_\alpha \circ \boldsymbol{\zeta}^{-1} = \boldsymbol{\zeta}\uparrow\mathbf{v}.$$

Naturality of LIE derivative with respect to push

$$\boldsymbol{\zeta}\uparrow(\mathcal{L}_\mathbf{v}\mathbf{s}_\mathcal{T}) = \mathcal{L}_{\boldsymbol{\zeta}\uparrow\mathbf{v}}(\boldsymbol{\zeta}\uparrow\mathbf{s}_\mathcal{T}),$$

and frame-invariance $\mathbf{v}_\zeta = \boldsymbol{\zeta}\uparrow\mathbf{v}$ of the velocity, ensure that invariance of a material tensor field $\mathbf{s}_\mathcal{T}$ with respect to a trajectory transformation $\boldsymbol{\zeta} \in \mathrm{C}^1(\mathcal{T}; \mathcal{T}_\zeta)$, implies invariance of its LIE derivative

$$\mathbf{s}_{\mathcal{T}_\zeta} = \boldsymbol{\zeta}\uparrow\mathbf{s}_\mathcal{T} \quad \Longrightarrow \quad \mathcal{L}_{\mathbf{v}_\zeta}\mathbf{s}_{\mathcal{T}_\zeta} = \boldsymbol{\zeta}\uparrow(\mathcal{L}_\mathbf{v}\mathbf{s}_\mathcal{T}).$$

EUCLID frame-invariance of trajectory tensor fields is invariance under transformations $\boldsymbol{\zeta}_{\mathrm{ISO}} \in \mathrm{C}^1(\mathcal{T}; \mathcal{T}_{\zeta_{\mathrm{ISO}}})$ that are isometric, i.e. such that

$$\mathbf{g}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \boldsymbol{\zeta}_{\mathrm{ISO}}\uparrow\mathbf{g}_\mathcal{T}.$$

The material metric tensor is EUCLID frame-invariant by definition. EUCLID frame-invariance of material tensors is a basic axiom of GCM.

18 Straightening out

A straightening-out map is a diffeomorphism which transforms the trajectory into a Cartesian product $\boldsymbol{\Omega} \times \mathcal{Z}$, with $\boldsymbol{\Omega} \subset \mathcal{R}^m$ a domain. Moreover the motion is transformed into a *time-translation* i.e. a degenerate motion $\mathrm{SHIFT}_\alpha \in \mathrm{C}^1(\boldsymbol{\Omega} \times \mathcal{Z}; \boldsymbol{\Omega} \times \mathcal{Z})$ defined by

$$\mathrm{SHIFT}_\alpha(\mathbf{x}, t) := (\mathbf{x}, t + \alpha), \quad \mathbf{x} \in \boldsymbol{\Omega}, \quad t, \alpha \in \mathcal{Z}.$$

This procedure, which in the space-time framework is the notion conceptually equivalent to the choice of a reference domain $\boldsymbol{\Omega}$, plays a basic role in computational mechanics because it allows to perform linear operations, such as integration over a time interval, by virtue of push to a straightened-out trajectory segment and to get the result at the end time by pull back of the computational outcome to the actual trajectory.

19 Constitutive laws

Constitutive relations are here designed to model the mechanical material response detected in laboratory tests in such a way that the time evolution of the stress tensor field, fulfilling equilibrium and constitutive properties, is uniquely defined by the knowledge of the evolution of the data (i.e. applied forces, imposed displacements, thermal variations, etc.). The constitutive behavior of material models is verified by carefully designed laboratory experiments, and by their theoretical interpretation. The mathematical expression, at each event along the trajectory, involves the time-rate of the material metric tensor (the *stretching*), the stress tensor and the time-rate of the stress (the *stressing*) and additional material tensors (*internal variables*), simulating micro-structural changes, and their time-rates.

In accordance with the treatment exposed for the material metric field, the time-rates of the stress tensor and of internal tensorial variables must be evaluated as LIE derivatives along the motion. At difference, however, no expression in terms of parallel derivatives is available, in general, because the parallel transport along the trajectory does not preserve time-verticality, for lower dimensional bodies. In fact EULER's stretching formula finds the reasons of its validity in the fact that the material metric tensor is the pull-back, to the material bundle $V\mathcal{T}$, of the spatial metric tensor which is defined in the whole events manifold [30].

To consider a theoretical framework suitable for investigating a sufficiently large class of material behaviors for engineering applications, a constitutive law is assumed to be a relation involving a constitutive operator $\mathbf{C}$, according to the following definition [28].

Definition 1 (Constitutive operator) A constitutive operator $\mathbf{C}$ is a fiber preserving (and possibly multivalued) correspondence between material tensor bundles, whose domain and codomain are WHITNEY[7] products of material tensor bundles.

The property of fiber preservation means that the constitutive relation is *local*, in the sense that material tensor fields based at an event on the trajectory

[7] The WHITNEY product of tensor bundles with projection $\boldsymbol{\pi}_{\mathbb{M},\mathbb{N}} \in \mathrm{C}^1(\mathbb{N}; \mathbb{M})$ and $\boldsymbol{\pi}_{\mathbb{M},\mathbb{H}} \in \mathrm{C}^1(\mathbb{H}; \mathbb{M})$ over the same base manifold $\mathbb{M}$, is the product bundle fulfilling the condition $\mathbb{N} \times_\mathbb{M} \mathbb{H} := \{(\mathbf{n}, \mathbf{h}) \in \mathbb{N} \times \mathbb{H} \mid \boldsymbol{\pi}_{\mathbb{M},\mathbb{N}}(\mathbf{n}) = \boldsymbol{\pi}_{\mathbb{M},\mathbb{H}}(\mathbf{h})\}$ [27].

are related to material tensor fields also based at that same event on the trajectory. In this respect, we observe that *non-local* constitutive relations require to perform linear operations (such as space-integration) involving material tensors based at distinct simultaneous events on the trajectory. In non-local theories a transport tool is then required to bring all material tensors to be based at the same point. The result is however *not* natural since the choice of a transport tool is non-uniquely defined, as evident in lower dimensional continua.

To simplify, but without loss of generality, we will consider a single material tensor field

$$\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1\big(\mathcal{T}; \mathrm{TENS}(V\mathcal{T})\big),$$

in the domain of the constitutive operator. Since all tensor fields and operators considered in the sequel are material, the subscript $\mathcal{T}$ will be dropped, whenever unnecessary.

Constitutive frame-invariance (CFI) is expressed by the following property of the constitutive operator

$$\mathbf{C}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}} = \boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\mathbf{C}_{\mathcal{T}}.$$

Explicitly the condition writes

$$\begin{aligned}\mathbf{C}_{\mathcal{T}_{\zeta_{\mathrm{ISO}}}}(\boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\mathbf{s}_{\mathcal{T}}) &= (\boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\mathbf{C}_{\mathcal{T}})(\boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\mathbf{s}_{\mathcal{T}})\\ &= \boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\big(\mathbf{C}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}})\big).\end{aligned}$$

This means that material tensor fields, fulfilling the constitutive relation, must be still related by the law after an EUCLID frame-transformation.[8]

Constitutive time-invariance in a time interval $I \subset \mathcal{Z}$ is expressed by the following property of the constitutive operator

$$\mathbf{C}_{\mathcal{T}} = \boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{C}_{\mathcal{T}}, \quad \forall \alpha \in I.$$

Explicitly the condition writes

$$\mathbf{C}_{\mathcal{T}}(\boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{s}_{\mathcal{T}}) = (\boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{C}_{\mathcal{T}})(\boldsymbol{\varphi}_\alpha\!\uparrow\!\mathbf{s}_{\mathcal{T}}) = \boldsymbol{\varphi}_\alpha\!\uparrow\!\big(\mathbf{C}_{\mathcal{T}}(\mathbf{s}_{\mathcal{T}})\big).$$

This means that time-invariant material tensor fields, fulfilling the constitutive relation at a time $t \in \mathcal{Z}$, are still related by the law at later times $t + \alpha \in \mathcal{Z}$.

[8] CFI substitutes the notion of Material Frame Indifference stated in [54] by the equality $\mathbf{C}_{\mathcal{T}} = \boldsymbol{\zeta}_{\mathrm{ISO}}\!\uparrow\!\mathbf{C}_{\mathcal{T}}$ in which the change of constitutive operator due to the change of observer is not taken into account [31].

Constitutive homogeneity is expressed by the property that, at distinct points $\mathbf{x}, \mathbf{y} \in \boldsymbol{\Omega}$ in a body placement, the constitutive operators are related by push-pull according to an isometric isomorphism, i.e.

$$\mathbf{C}_{\mathbf{y}} = \mathbf{L}^{\mathrm{ISO}}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\mathbf{C}_{\mathbf{x}}.$$

Explicitly the condition writes

$$\begin{aligned}\mathbf{C}_{\mathbf{y}}\big(\mathbf{L}^{\mathrm{ISO}}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\mathbf{s}_{\mathbf{x}}\big) &= \big(\mathbf{L}^{\mathrm{ISO}}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\mathbf{C}_{\mathbf{x}}\big)\big(\mathbf{L}^{\mathrm{ISO}}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\mathbf{s}_{\mathbf{x}}\big)\\ &= \mathbf{L}^{\mathrm{ISO}}_{\mathbf{y},\mathbf{x}}\!\uparrow\!\big(\mathbf{C}_{\mathbf{x}}(\mathbf{s}_{\mathbf{x}})\big).\end{aligned}$$

where $\mathbf{s}_{\mathcal{T}} \in \mathrm{C}^1(\mathcal{T}; \mathrm{TENS}(V\mathcal{T}))$ denotes a list of material tensor fields.

This means that material tensor fields, that are $(\mathbf{x}, \mathbf{y})$-homogeneous, will fulfill the constitutive relation at $\mathbf{y} \in \boldsymbol{\Omega}$ if they fulfill the law $\mathbf{x} \in \boldsymbol{\Omega}$ and vice versa.

20 Rate-elasticity

A basic model of constitutive behavior is provided by the rate-elastic law[9] which expresses, at each event in the trajectory, the *elastic stretching* tensor as a linear response to a *stressing* tensor by means of a tangent operator of elastic compliance $\mathbf{H}(\boldsymbol{\sigma})$ which depends non-linearly on the *stress* tensor

$$\mathbf{el} = \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}.$$

The *elastic stretching* tensor **el** has the physical dimension of the reciprocal of a time. It is *not* a LIE derivative along the motion, unless a *purely rate-elastic* behavior is assumed, so that $\mathbf{el} = \frac{1}{2}\mathcal{L}_{\mathbf{v}}\mathbf{g}_{\mathcal{T}}$. The *stressing* tensor field is the LIE derivative along the motion of the stress field

$$\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{v}}\boldsymbol{\sigma} := \partial_{\alpha=0}(\boldsymbol{\varphi}_\alpha\!\downarrow\!\boldsymbol{\sigma}).$$

By the results in Sect. 17, frame invariance of the stress field implies that the stressing tensor field is frame-invariant too.

If the rate-elastic operator is the fiber-derivative (i.e. the derivative with respect to the stress at a fixed event in the trajectory) of a stress-dependent

[9] An hypo-elastic model was introduced by TRUESDELL in [52] with a different definition. The new formulation of rate elasticity was first contributed in [29].

and tensor-valued potential, $d_F \boldsymbol{\Phi} = \mathbf{H}$, the constitutive relation is called CAUCHY integrable. If in addition the latter potential is the fiber-derivative of a stress dependent and scalar-valued potential, $d_F E^* = \boldsymbol{\Phi}$, the model is called GREEN integrable and hence $d_F^2 E^* = d_F \boldsymbol{\Phi} = \mathbf{H}$. The CAUCHY integrability is equivalent to the former of the following symmetry conditions

$$\langle d_F \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma} \cdot \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma} \rangle = \langle d_F \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta\boldsymbol{\sigma} \cdot \delta_2\boldsymbol{\sigma}, \delta_1\boldsymbol{\sigma} \rangle,$$
$$\langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta_1\boldsymbol{\sigma}, \delta_2\boldsymbol{\sigma} \rangle = \langle \mathbf{H}(\boldsymbol{\sigma}) \cdot \delta_2\boldsymbol{\sigma}, \delta_1\boldsymbol{\sigma} \rangle,$$

and both are equivalent to GREEN integrability [29].

Let us now apply the definition given in Sect. 19 to the new rate-elastic law.

Constitutive frame-invariance of the tangent operator of elastic compliance means that

$$\mathbf{H}_{\mathcal{T}_{\boldsymbol{\zeta}_{\mathrm{ISO}}}} = \boldsymbol{\zeta}_{\mathrm{ISO}} \uparrow \mathbf{H}_{\mathcal{T}}.$$

The condition assures that, if a stretching tensor is related to a stress and a stressing tensor, then an isometrically pushed stretching tensor will be related to the pushed stress and stressing tensors.

The simplest rate-elastic law, widely adopted in early engineering computations in 3D isotropic elasticity, is expressed, in terms of the KIRCHHOFF mixed stress tensor $\mathbf{K} = \boldsymbol{\sigma} \circ \mathbf{g}_{\mathcal{T}}$, by

$$\mathbf{El} := \mathbf{H}^{\mathrm{MIX}}(\mathbf{K}) \cdot \dot{\mathbf{K}},$$

with the elastic compliance tangent operator given by

$$\mathbf{H}^{\mathrm{MIX}}(\mathbf{K}) := \frac{1}{2\mu}\mathbb{I} - \frac{\nu}{E}\mathbf{I} \otimes \mathbf{I}.$$

Here $\mathbf{El} = \mathbf{g}_{\mathcal{T}}^{-1} \circ \mathbf{el}$ is the mixed elastic stretching, and $\dot{\mathbf{K}} := \dot{\boldsymbol{\sigma}} \cdot \mathbf{g}_{\mathcal{T}}$ is the mixed alteration of KIRCHHOFF stressing. The operators $\mathbb{I}$ and $\mathbf{I}$ denote the fiberwise linear transformations of the material bundles $\mathrm{MIX}(V\mathcal{T})$ and $V\mathcal{T}$ onto themselves, respectively defined by the property of being the identity in each fiber.

We underline that, unlike $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{v}}\boldsymbol{\sigma}$, the rate $\dot{\mathbf{K}}$ is *not* the LIE-derivative of the KIRCHHOFF stress along the motion, since this would involve also the LIE derivative of the material metric. Indeed

$$\mathcal{L}_{\mathbf{v}}\mathbf{K} = \mathcal{L}_{\mathbf{v}}(\boldsymbol{\sigma} \cdot \mathbf{g}_{\mathcal{T}}) = (\mathcal{L}_{\mathbf{v}}\boldsymbol{\sigma}) \cdot \mathbf{g}_{\mathcal{T}} + \boldsymbol{\sigma} \cdot (\mathcal{L}_{\mathbf{v}}\mathbf{g}_{\mathcal{T}}).$$

The fulfillment of CAUCHY integrability condition is assured since the fiber derivative vanishes identically, $d_F \mathbf{H}(\mathbf{K}) = \mathbf{O}$. The property of GREEN integrability then follows by $\mathbf{g}_{\mathcal{T}}$-symmetry of $\mathbf{H}(\mathbf{K})$. The frame-invariance property is also readily verified [31].

The GREEN integrability of the rate-elastic law expressed in terms of KIRCHHOFF stress tensor, and the property of mass conservation, assure fulfillment of conservation of elastic energy. This basic notion is enunciated in terms of paths in the functional space of stress fields along the motion. This is an important and innovative procedure emerging from the new theory in accord with the *Geometric Paradigm* (GP).

Conservation of elastic energy is expressed by the condition

$$\int_{\mathcal{T}_I} \langle \boldsymbol{\sigma}, \mathbf{H}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}} \rangle \mathbf{m} = 0,$$

with $\mathcal{T}_I$ trajectory segment corresponding to a time interval $I = [t_1, t_2]$, for any *stress path along the motion* $\boldsymbol{\sigma} \circ \boldsymbol{\varphi} : I \mapsto \mathrm{C}^1(\mathcal{T}; \mathrm{CON}(V\mathcal{T}))$ that is *covariantly closed* i.e. any stress path fulfilling the condition

$$\boldsymbol{\sigma} = \boldsymbol{\varphi}_{t_2 - t_1} \downarrow \boldsymbol{\sigma} = \boldsymbol{\varphi}_{t_2 - t_1} \downarrow \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\varphi}_{t_2 - t_1}.$$

This condition is equivalent to require that the stress path along any particle, when pushed to a straightened-out trajectory, becomes a cycle.

The result concerning conservativeness is of great interest in computational mechanics, when dealing with geometrically non-linear problems. It eventually settles the many debates about the troublesome lack of conservativeness of the simplest rate-elastic law.

Symmetry of contravariant and covariant material tensors, such as the stress $\boldsymbol{\sigma} \in \mathrm{C}^1(\mathcal{T}; \mathrm{CON}(V\mathcal{T}))$ and the elastic stretching $\mathbf{el} \in \mathrm{C}^1(\mathcal{T}; \mathrm{COV}(V\mathcal{T}))$, is clearly preserved under push performed according to a straightening out map or to a frame transformation. On the contrary $\mathbf{g}_{\mathcal{T}}$-symmetry of mixed material tensors, such as the stress $\mathbf{K} \in \mathrm{C}^1(\mathcal{T}; \mathrm{MIX}(V\mathcal{T}))$ is not preserved under non-isometric transformations. It is therefore advisable to transform symmetric material tensors to their contravariant or covariant form prior to push constitutive relations under frame transformations or finite displacements.

21 Elasticity, hyper-elasticity and elasto-visco-plasticity

The beauty and the power of the geometric approach find a special evidence in the formulation of constitutive equations and in the investigation about their properties. These features are exemplified hereafter by the formulation of the most usual laws of material behavior in the geometrically non-linear range to show that their expressions are identical to the ones adopted in the geometrically linearized theory [42]. Indeed, the linearization assumptions amount to consider a straightened-out trajectory and hence LIE derivatives collapse into usual partial time-derivative.

We recall that a superimposed dot on material tensor fields denotes a LIE derivative along the motion, i.e. $\dot{\boldsymbol{\sigma}} := \mathcal{L}_{\mathbf{v}}\boldsymbol{\sigma} := \partial_{\alpha=0}(\boldsymbol{\varphi}_\alpha \downarrow \boldsymbol{\sigma})$.

An elastic (hyper-elastic) constitutive model is a rate-elastic model which is time-invariant and CAUCHY (GREEN) integrable, with an invertible CAUCHY potential $\boldsymbol{\Phi}$, so that

$$\mathbf{el} = d_F \boldsymbol{\Phi}(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, \qquad (\mathbf{el} = d_F^2 E^*(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}),$$

with time-invariance expressed by

$$\boldsymbol{\Phi} = \boldsymbol{\varphi}_\alpha \uparrow \boldsymbol{\Phi}, \qquad (E^* = \boldsymbol{\varphi}_\alpha \uparrow E^*).$$

An elasto-visco-plastic model of constitutive behavior, which is of primary applicative interest in NLCM, is described by the relations

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{v}_{\mathcal{S}}) = \mathbf{el} + \mathbf{vp}, & \text{stretching additivity,} \\ \mathbf{el} = d_F^2 E^*(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, & \text{elastic law,} \\ \boldsymbol{vp} \in \partial_F \mathcal{F}(\boldsymbol{\sigma}), & \text{visco-plastic flow rule,} \end{cases}$$

where E^* is the scalar stress elastic potential corresponding to GREEN integrability, ∂_F is the fiber-subdifferential of the extended real-valued convex visco-plastic potential $\mathcal{F}$ (i.e. the subdifferential at a fixed event in the trajectory) and $\mathbf{vp}$ is the *visco-plastic stretching* tensor.
The visco-plastic flow rule may equivalently be expressed by the fiberwise variational inequality

$$\mathcal{F}(\overline{\boldsymbol{\sigma}}) - \mathcal{F}(\boldsymbol{\sigma}) \geq \langle \mathbf{vp}, \overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \rangle,$$

for any $\overline{\boldsymbol{\sigma}} \in \mathcal{K}$, with $\mathcal{K} = \operatorname{dom} \mathcal{F}$ the *elastic domain*. Neither the elastic stretching $\mathbf{el}$ nor the visco-plastic stretching $\mathbf{vp}$ are defined as LIE derivatives of a material tensor field along the motion. The superimposed dot usually adopted in literature is therefore a misleading notation.

An elasto-plastic model of constitutive behavior is described by the law

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{v}_{\mathcal{S}}) = \mathbf{el} + \mathbf{pl}, & \text{stretching additivity,} \\ \mathbf{el} = d_F^2 E^*(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, & \text{elastic law,} \\ \mathbf{pl} \in \partial_F \sqcup_{\mathcal{K}} (\boldsymbol{\sigma}) = \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}), & \text{plastic flow rule,} \end{cases}$$

where E^* is the scalar stress elastic potential corresponding to GREEN integrability, ∂_F is the fiber-subdifferential of the convex indicator $\sqcup_{\mathcal{K}}$ of the elastic domain $\mathcal{K}$ at $\boldsymbol{\sigma} \in \mathcal{K}$ and $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})$ is the outward normal cone, with $\mathbf{pl}$ *plastic stretching* tensor. The plastic flow rule may equivalently be expressed by the fiberwise variational inequality

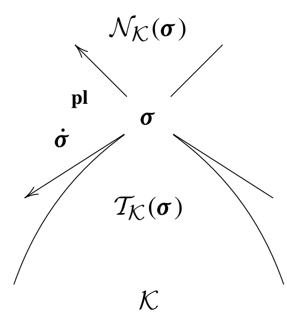
$$\langle \mathbf{pl}, \overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \rangle \leq 0, \quad \boldsymbol{\sigma} \in \mathcal{K}, \quad \forall \overline{\boldsymbol{\sigma}} \in \mathcal{K}.$$

An incremental elasto-plastic model is described by

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{v}_{\mathcal{S}}) = \mathbf{el} + \mathbf{pl}, & \text{stretching additivity,} \\ \mathbf{el} = d_F^2 E^*(\boldsymbol{\sigma}) \cdot \dot{\boldsymbol{\sigma}}, & \text{elastic law,} \\ \mathbf{pl} \in \mathcal{N}_{\mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})}(\dot{\boldsymbol{\sigma}}), & \text{rate plastic flow rule.} \end{cases}$$

The rate plastic flow rule, which is more stringent than the plastic flow rule of the previous model, implies the following fiberwise orthogonality condition, known as PRAGER's complementarity rule:

$$\langle \mathbf{pl}, \dot{\boldsymbol{\sigma}} \rangle = 0, \qquad \mathbf{pl} \in \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}), \quad \dot{\boldsymbol{\sigma}} \in \mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma}),$$



Here $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma}), \mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})$ are the normal and the tangent cones to the elastic domain $\mathcal{K}$ at the stress point $\boldsymbol{\sigma} \in \mathcal{K}$ and $\mathcal{N}_{\mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})}(\dot{\boldsymbol{\sigma}})$ is the normal cone to $\mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})$ at the stress-rate point $\dot{\boldsymbol{\sigma}} \in \mathcal{T}_{\mathcal{K}}(\boldsymbol{\sigma})$.

Frame indifference of these constitutive relations may be assessed under conditions analogous of the one expressed for the tangent operator of elastic compliance.

Although computational issues will not be discussed in this treatment, we remark that the design of computational procedures in the geometric nonlinear range is conveniently performed by a suitable discretization of the trajectory manifold and by envisaging iterative algorithms for the solution of the equilibrium problem and of the constitutive relation. To this end, at each event of the discretized trajectory, the constitutive relation is pushed to a straightened-out trajectory. Then, pushed LIE derivatives become partial time derivatives and linear differential or integral operations may be carried out. The results of these linear operations remain confined to this *computation chamber* and only fields pertaining to the final time-instant are physically meaningful, when pulled-back to the trajectory manifold. This observation entails that finite elastic and plastic strains cannot have any physical role in constitutive laws in the nonlinear geometric range, contrary to widely adopted proposals in literature, such as the elasto-plasticity model based on the multiplicative decomposition of the deformation gradient.

It is remarkable that in the new theory no body manifold $\mathcal{B}$ needs to be considered in formulating constitutive relations. Reference configurations, more properly named straightened-out trajectories in the space-time context, are computational tools providing a location for linear calculus. The ensuing theoretical procedure is in fact analogous to the one adopted in computational codes based on FEM (Finite Element Method).

22 Dynamics

To underline the role of *Geometric Naturality*, as a general rule pertaining to field theories, we briefly describe here a formulation of Continuum Dynamics (CD) which develops according to that rule [40, 41].

The foundations of CD are laid down in the most general way by means of a variational principle concerning the trajectory and the relevant evolution operator. The principle may be put in the following standard geometric form.

The Action Principle is a variational principle to be fulfilled by the action integral over the trajectory with the variations made by displacing the trajectory in the container manifold. To this end a lifted trajectory is considered to be a submanifold of the state-space manifold defined as the velocity-time (or the covelocity-time) manifold and an action one-form on the state-space is devised by lifting the LAGRANGE scalar functional from the trajectory in the events to the lifted trajectory into the state-space manifold.

Under suitable assumptions, the Action Principle may be localized to provide the EULER-LAGRANGE differential equation and the ERDMANN-WEIESTRASS corner conditions at singular points. No geometric connection in the state-space manifold enters into the theory until this stage and hence it can be affirmed that CD may be founded in a natural way in terms of the motion and of the LAGRANGE functional, without any additional assumptions. An equivalent principle can be formulated by imposing that variations of trajectory leave the energy functional invariant, to get a generalized form of MAUPERTUIS *Least Action Principle*, in which conservation of energy along the trajectory is not assumed but recovered as a natural condition, as illustrated in [39].

The introduction of a linear connection provides a valuable tool of investigation about the properties of the trajectory evolution fulfilling the Action Principle, the choice of a special connection being a question of convenience. For instance, a curvilinear coordinate system induces an associated path-independent parallel transport and a corresponding linear connection which has vanishing torsion and curvature forms. The adoption of a LEVI-CIVITA connection induce a torsion-free and metric connection with a non-vanishing curvature. In this respect, we underline that only the torsion of the linear connection enters in the equations of Dynamics. An important example is provided by POINCARÉ's law of Dynamics which is the outcome of taking the path-independent parallel transport induced by a mobile reference system associated with curvilinear coordinates. In this case, the torsion form is equal to the opposite of the LIE bracket and hence the *structure coefficients* (components of the LIE brackets of pairs of basis vector fields) appear into the equation of Dynamics [41]. The LEVI-CIVITA connection on the trajectory and the LAGRANGE functional given by the kinetic energy per unit mass, lead to generalized EULER and D'ALEMBERT laws of Dynamics. The standard formulations are recovered in the Euclid space endowed with the parallel transport by translation.

23 Discussion

In NLCM, the evaluation of the strain tensor field involves the determination of the tangent map $T\boldsymbol{\varphi}_\alpha$ only to within composition with transformations which are linear and isometric in each tangent fiber.

Nonetheless, this map, denoted by $\mathbf{F} = T\boldsymbol{\varphi}_\alpha$ in [54] and named deformation or transplacement gradient, has been considered as the basic kinematic variable entering in the elastic law in most treatments of CM. A correction to eliminate inessential isometric transformations is there performed by a *reduction* procedure based on the principle of Material Frame Indifference (MFI).

A critical analysis on the issue has recently been carried out in [31]. It is there put into evidence that MFI involves an improper equality between tensor fields observed in different frames. The *Geometric Paradigm* (GP) dictates instead that the correct geometric tool for the comparison should be a push according to the trajectory transformation. The geometrically proper notion of *Constitutive Frame Invariance* (CFI) was just introduced to substitute the untenable notion of MFI, as illustrated in Sect. 19.

A major shortcoming of the approach followed in [54] is that it leads to formulate rate constitutive laws in terms of the time rate $\dot{\mathbf{F}} = \partial_{\alpha=0} T\boldsymbol{\varphi}_\alpha$ of the tangent map along the motion. This time-rate is evaluated by performing a translation of a tangent vector along a particle, a procedure which cannot be extended to lower dimensional models of continua, see Fig. 6. Moreover, this evaluation does not comply with the *Geometric Naturality* (GN) principle because it involves the choice of the parallel transport by translation as preferred one.

On the other hand, the natural procedure to be followed in evaluating the time rate of the tangent map should consist in performing a pull-back along the motion, of the material vector $T\boldsymbol{\varphi}_\alpha \cdot \mathbf{h}$ tangent to the target placement, to the corresponding time-independent vector $\mathbf{h}$ tangent to the source placement. This procedure will clearly yield a vanishing result.

These considerations lead to the conclusion that the time derivative $\dot{\mathbf{F}}$, requiring an unnatural choice of a parallel transport and being inapplicable to lower dimensional bodies, does not fulfill the basic principles of *Geometric Naturality* (GN) and *Dimensionality Independence* (DI) and hence, contrary to common usage, cannot appear in constitutive relations.

It follows that the multiplicative decomposition of the deformation gradient into the chain of an elastic and an inelastic homomorphisms, which has gained a vast popularity after his proposal in [18], is misformulated from the physico-geometrical point of view. This observation, together with known troubles concerning the definition and the interpretation of natural and intermediate configurations [36], should convince that this constitutive modeling must be abandoned.

24 Conclusions

The turning point in the development of NLCM should be the ascertainment of the central role played by notions and concepts from basic differential geometry, which should be learned and put at the center of any subsequent deepening of or special progress in the matter.

The adoption of a natural conceptual procedure, according to which principles and notions are introduced on the sole basis of essential geometric ingredients of the field theory, translates into simple general rules to be followed. Sure guidelines are thus provided for the statement of general principles, for the formulation of constitutive relations, for the design of experimental verifications and for the detection of suitable computational algorithms.

A main issue is that material tensor fields must be compared by transformation according to push-pull along diffeomorphic displacement maps. In the ensuing theory of Geometric Continuum Mechanics (GCM), a central role is played by the material metric tensor field and by the theoretical notion of stretching field.

This field is pointwise evaluated by measuring the rate of elongation of the edges of a non-degenerate simplex in a tangent material space, a definition susceptible of direct experimental measurements, for instance by strain gauges. The geometric theory leads to:

new notions and proper definitions of

spatial and material fields
time-rates of material fields
time-invariance of material fields
frame-invariance of material fields

new consistent statements of

constitutive frame-invariance
conservation of elastic energy
rate elasticity
rate elasto-visco-plasticity

critical analysis and rejection of the notions of

material frame-indifference
multiplicative elasto-plastic decomposition
finite elastic stretch
finite visco-plastic stretch

new general formulations of

EULER's *stretching formula*
Virtual Power Principle
Action Principle of Continuum Dynamics

making their expressions in terms of any linear connection in the events manifold, available for applications.

From the computational point of view the geometric theory provides a firm ground to the formulation of effective iterative algorithms based on push-pull transformations between the actual trajectory and a straightened out trajectory segment playing the role of *computation chamber*. Therein linear operations can be performed and time-integrals of elastic and plastic stretching can be evaluated in finite step solution algorithms.

The geometric theory provides a consistent framework to deal with problems in Bio-Mechanics involving large deformations, growth and remodeling, as in investigations on soft tissues [10, 17]. In these contexts the new theory is self-proposing as valid replacement of treatments affected by troublesome physical interpretations related to the assumption of a multiplicative decomposition of the deformation gradient [1].

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