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Algorithmic tangent stiffness in elastoplasticity and elastoviscoplasticity: A geometric insight

Giovanni Romano *, Raffaele Barretta, Marina Diaco

University of Naples Federico II, Via Claudio 21, 80125 Naples, Italy

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ABSTRACT

The algorithmic, or consistent, tangent stiffness was introduced to improve the asymptotic convergence rate of the iterative correction algorithm for the evolutive analysis of elastoplastic structures. The original approach is based on a formulation of the elastoplastic law in terms of a plastic multiplier with an analysis which, in general, requires an operator inversion. A geometric description of the method, based on hypersurface theory, is proposed here to provide a clear picture of the algorithmic properties. An estimate of the tangent stiffness associated with finite step elastoplastic and elastoviscoplastic constitutive models is given. It is based on the properties of the projection operator on the elastic domain and avoids operator inversions retaining the beneficial properties of the original one.

1. Introduction

The evolution of solids and structures characterized by an elastic behavior and by an inelastic flow provides a highly nonlinear structural problem due to the constitutive law. The solution algorithm is based on an initial linear elastic guess followed by an iterative correction devoted to the annihilation of the residual of the structural response by evaluating, in an suitable fashion, the tangent behavior of the nonlinear constitutive response (see e.g. Ortiz and Popov, 1985; Simo and Taylor, 1985, 1986; Simo et al., 1985; Ortiz and Simo, 1986; Auricchio et al., 1992; Hofstetter et al., 1993; Ristinmaa and Tryding, 1993; Matzenmiller and Taylor, 1994; Simo, 1998; Simo and Hughes, 1998; Han and Reddy, 1999; Armero and Pérez-Foguet, 2002a,b). The process of reduction, below a given tolerance, of an appropriate norm of the residual could be performed either by assuming a purely elastic behavior or by evaluating the tangent stiffness according to the rate inelastic problem. The former assumption leads to an unconditionally stable iterative algorithm but it could be affected by an unacceptably low convergence rate. The latter choice has a better performance in terms of convergence but still does not enjoy the asymptotic convergence rate of Newton-like iterative schemes (Luenberger, 1984). To overcome these shortcomings a different approach was proposed by Simo and Taylor (1985). The algorithmic tangent stiffness there introduced takes into account the fact that the stress field violates the yield condition in correspondence of the algorithmic trials. The problem is here analyzed from a geometric point of view by endowing the stress space with the inner product provided by the complementary elastic strain energy. Exact and approximate expressions of the algorithmic tangent stiffness in associative elastoplasticity are discussed in terms of the geometric properties of the convex elastic domains described by piecewise smooth yield functions. The analysis is extended to elastoviscoplastic models in which the threshold stress is evaluated according to the projection on the convex elastic domain.

2. Constitutive models and algorithms

Let us consider a continuous body whose material behavior is described by an elastoplastic or elastoviscoplastic model. The ambient euclidean space is E and V is the associated affine space of translations. In a geometrically linearized theory, strains and strain rates at a point of the body belong to the linear space $\mathbb{D}$ of twice covariant symmetric tensors on V. Stresses are in the dual linear space $\mathbb{S}$ of twice contravariant symmetric tensors. The relevant virtual work is provided by the duality pairing $\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \rangle$ between $\boldsymbol{\sigma} \in \mathbb{S}$ and $\boldsymbol{\varepsilon} \in \mathbb{D}$. The elastic response of the material is described by linear symmetric and positive definite operators $\mathbf{C} \in BL(\mathbb{S};\mathbb{D})$ and $\mathbf{E} = \mathbf{C}^{-1} \in BL(\mathbb{D};\mathbb{S})$, respectively the elastic compliance and the elastic stiffness. The symbol $BL()$ stands for bounded linear map. The linear elastic compliance induces in $\mathbb{S}$ a metric tensor $\mathbf{g_C}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) := \langle \mathbf{C}\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \rangle$ and the associated norm $\|\boldsymbol{\sigma}\|_{\mathbf{C}} := \sqrt{\mathbf{g_C}(\boldsymbol{\sigma}, \boldsymbol{\sigma})}$. A piecewise regular convex yield function $\varphi : \mathbb{S} \to \mathscr{R}$ defines with its zero-level set $\mathscr{K} := \{\boldsymbol{\sigma} \in \mathbb{S} | \varphi(\boldsymbol{\sigma}) \leqslant 0\}$ the convex elastic domain in the stress space $\mathbb{S}$. The outward normal cone at $\boldsymbol{\sigma} \in \mathscr{K}$ is the

* Corresponding author.
E-mail addresses: romano@unina.it (G. Romano), rabarret@unina.it (R. Barretta), diaco@unina.it (M. Diaco).

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convex set $\mathcal{N}_{\mathscr{K}}(\boldsymbol{\sigma}) := \{\boldsymbol{\varepsilon} \in \mathbb{D} : \langle \boldsymbol{\varepsilon}, \boldsymbol{\tau} - \boldsymbol{\sigma} \rangle \leqslant 0, \quad \forall \boldsymbol{\tau} \in \mathscr{K}\}$. At points internal to $\mathscr{K}$, the normal cone $\mathcal{N}_{\mathscr{K}}(\boldsymbol{\sigma})$ degenerates to the null set. The reader is referenced to Rockafellar (1975) for an exhaustive exposition about Convex Analysis.

2.1. Constitutive laws

In the realm of geometrically linearized theories of elastoviscoplastic material behavior, the total strain is additively decomposed as the sum of an elastic and an anelastic part. The former is related to the stress by the linear elastic compliance tensor, while the latter may be represented as an element of the subdifferential of a convex potential at the stress point (Romano et al., 1993):

$$\dot{\boldsymbol{\varepsilon}} \in \mathbf{C}\boldsymbol{\sigma} + \partial\varphi(\boldsymbol{\sigma}).$$

The perfectly plastic behavior is modeled by setting $\varphi = \sqcup_{\mathscr{K}}$ the convex indicator of the convex elastic domain $\mathscr{K}$:

$$\sqcup_{\mathscr{K}}(\boldsymbol{\sigma}) = \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \in \mathscr{K}, \\ +\infty & \text{if } \boldsymbol{\sigma} \notin \mathscr{K}, \end{cases}$$

so that $\partial\varphi(\boldsymbol{\sigma}) = \mathcal{N}_{\mathscr{K}}(\boldsymbol{\sigma})$. To model the viscoplastic behavior, the convex potential is defined as the composition of a Young function $m : \mathscr{R} \to \mathscr{R}$, which takes account of the relaxation time τ, with a convex interdiction function $g : \mathbb{S} \to \mathscr{R} \cup +\infty$. Perzyna's model is got setting $m(x) = 0$ for $x < 0$ and $m(x) = \frac{1}{2\tau}x^2$ for $x \geqslant 0$ and the interdiction function given by the difference between a yield function and the plastic threshold (Perzyna, 1963). By assuming that the function g at a point $\boldsymbol{\sigma} \in \mathbb{S}$ is the distance in complementary elastic energy of $\boldsymbol{\sigma}$ from the elastic domain:

$$g(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma})\|_{\mathbf{C}},$$

the constitutive model proposed by Duvaut and Lions (1972) is obtained.

Remark 2.1. Hardening behaviors may be taken into account by considering a model of generalized standard elastoplastic or elastoviscoplastic material. According to the original proposal in Halphen and Nguyen (1975), the elastic domain $\mathscr{K}$ is a convex set in the product space of stresses and thermodynamical affinities. The flow rule is expressed by the normality, of the plastic flow and of the rate of change of internal variables, to the elastic domain (see e.g. Halphen and Nguyen, 1975; Nguyen, 1977; Martin, 1975, 1981; Eve et al., 1990; Martin and Nappi, 1990; Romano et al., 1992). A treatment of generalized standard materials in the framework of Convex Analysis may be found in Romano et al. (1993).

2.2. Elastoplastic algorithmic constitutive law

We consider a discrete time integration scheme, denoting by $\boldsymbol{\varepsilon}_0, \ \mathbf{p}_0 \in \mathbb{D}$ the total and the plastic strain and by $\boldsymbol{\sigma}_0 \in \mathbb{S}$ the stress tensor provided by the approximate solution of the structural problem at the end of a time-step. In the subsequent time-step, the iterative algorithm for the solution of the nonlinear elastoplastic or elastoviscoplastic problem provides a sequence of total strains, starting with a purely elastic initial guess of the structural response. We denote by $\boldsymbol{\varepsilon} \in \mathbb{D}$ the total strain predicted at the current iteration and by the pair $\mathbf{p} \in \mathbb{D}, \ \boldsymbol{\sigma} \in \mathbb{S}$ the corresponding approximate solution of the constitutive law in terms of plastic strain and stress tensor. Let us set: $\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0, \ \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0$, $\Delta\boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_{\mathrm{TR}} := \boldsymbol{\sigma}_{\mathbf{o}} + \mathbf{E}\Delta\boldsymbol{\varepsilon}$ the trial stress. By integrating the flow rule according to a fully implicit scheme, the elastoplastic algorithmic constitutive law writes as:

$$\begin{cases} \mathbf{E}\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\sigma} + \mathbf{E}\Delta\mathbf{p}, \\ \Delta\mathbf{p} \in \mathcal{N}_{\mathscr{K}}(\boldsymbol{\sigma}), \end{cases} \iff \quad \boldsymbol{\sigma}_{\mathrm{TR}} \in \boldsymbol{\sigma} + \mathcal{N}^{\mathbf{C}}_{\mathscr{K}}(\boldsymbol{\sigma}),$$

where $\mathcal{N}^{\mathbf{C}}_{\mathscr{K}}(\boldsymbol{\sigma}) := \{\overline{\sigma} \in \mathbb{S} : \mathbf{g}_{\mathbf{C}}(\overline{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma}) \leqslant 0, \quad \forall \boldsymbol{\tau} \in \mathscr{K}\}$ is the normal cone according to the metric $\mathbf{g}_{\mathbf{C}}$. We set $\mathbb{S}_{\mathbf{C}} := \{\mathbb{S}, \mathbf{g}_{\mathbf{C}}\}$. To obtain the expression of the elastoplastic algorithmic tangent stiffness it is expedient to rewrite the constitutive law in terms of the orthogonal projector $\mathbf{P}_{\mathscr{K}}$ in $\mathbb{S}_{\mathbf{C}}$ onto the elastic domain $\mathscr{K}$ (Fig. 1): $\boldsymbol{\sigma} = \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}), \ \Delta\mathbf{p} = \mathbf{C}(\boldsymbol{\sigma}_{\mathrm{TR}} - \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))$.

By taking the derivative with respect to the evolution parameter, the formula $\boldsymbol{\sigma} = \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})$ yields the incremental law $\dot{\boldsymbol{\sigma}} = (d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))\mathbf{E}\dot{\boldsymbol{\varepsilon}}$. The linear operator

$$\mathbf{K}^{\mathrm{EP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) := (d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))\mathbf{E}$$

is the elastoplastic algorithmic tangent stiffness. The evaluation of the elastoplastic algorithmic tangent stiffness contributed in Simo and Taylor (1985) is based on the formulation of the discrete elastoplastic constitutive law: $\boldsymbol{\sigma}_0 + \mathbf{E}\Delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\Delta\mathbf{p}$, with $\Delta\mathbf{p} = \lambda\nabla\varphi(\boldsymbol{\sigma})$, in terms of a plastic multiplier λ. The algebraic reasoning is summarized hereafter. Rewriting the discrete elastoplastic constitutive law as: $\mathbf{C}\boldsymbol{\sigma} - \mathbf{C}\boldsymbol{\sigma}_0 + \lambda\nabla\varphi(\boldsymbol{\sigma}) = \Delta\boldsymbol{\varepsilon}$ and taking the derivative with respect to the evolution parameter, the incremental law: $\mathbf{C}\dot{\boldsymbol{\sigma}} + \dot{\lambda}\nabla\varphi(\boldsymbol{\sigma}) + \lambda\nabla^2\varphi(\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\varepsilon}}$, is got, where $\nabla^2\varphi(\boldsymbol{\sigma})$ is the Hessian of the yield function φ. Under plastic loading, the stress point $\boldsymbol{\sigma}$ is bound to move along the boundary of the elastic domain, so that: $\mathbf{g}(\dot{\boldsymbol{\sigma}}, \nabla\varphi(\boldsymbol{\sigma})) = 0$ and this gives the expression of the plastic multiplier rate: $\dot{\lambda} = \frac{\mathbf{g}(\mathbf{H}(\boldsymbol{\sigma})\dot{\boldsymbol{\varepsilon}}, \nabla\varphi(\boldsymbol{\sigma}))}{\mathbf{g}(\mathbf{H}(\boldsymbol{\sigma})\nabla\varphi(\boldsymbol{\sigma}), \nabla\varphi(\boldsymbol{\sigma}))}$, with $\mathbf{H}(\boldsymbol{\sigma}) := [\mathbf{C} + \lambda\nabla^2\varphi(\boldsymbol{\sigma})]^{-1}$. Let us set $\beta(\boldsymbol{\sigma}_{\mathrm{TR}}) := \mathbf{g}(\mathbf{H}(\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))\nabla\varphi(\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})), \nabla\varphi(\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})))$ and $\mathbf{N}_{\mathbf{H}}(\boldsymbol{\sigma}_{\mathrm{TR}}) := \mathbf{H}(\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))\nabla\varphi(\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))$. Substituting the expression of $\dot{\lambda}$ in the incremental law above, and observing that $\boldsymbol{\sigma} = \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})$, with $\boldsymbol{\sigma}_{\mathrm{TR}}$ the trial stress point, the elastoplastic algorithmic tangent stiffness, defined by $\dot{\boldsymbol{\sigma}} = \mathbf{K}^{\mathrm{EP}}(\boldsymbol{\sigma}_{\mathrm{TR}})\dot{\boldsymbol{\varepsilon}}$, is expressed as

$$\mathbf{K}^{\mathrm{EP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) := \mathbf{H}(\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})) - \frac{\mathbf{N}_{\mathbf{H}}(\boldsymbol{\sigma}_{\mathrm{TR}}) \otimes \mathbf{N}_{\mathbf{H}}(\boldsymbol{\sigma}_{\mathrm{TR}})}{\beta(\boldsymbol{\sigma}_{\mathrm{TR}})}.$$

2.3. Elastoviscoplastic algorithmic constitutive law

By a time integration according to a fully implicit scheme, the flow rule proposed by Duvaut and Lions (1972) writes:

$$\begin{cases} \boldsymbol{\sigma}_0 + \mathbf{E}\Delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\Delta\mathbf{p}, \\ \mathbf{E}\Delta\mathbf{p} = \frac{\Delta t}{\tau}(\boldsymbol{\sigma} - \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma})), \end{cases} \iff \quad \boldsymbol{\sigma}_{\mathrm{TR}} = \boldsymbol{\sigma} + \frac{\Delta t}{\tau}(\boldsymbol{\sigma} - \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma})),$$

where $\boldsymbol{\sigma}_{\mathrm{TR}} := \boldsymbol{\sigma}_{\mathbf{o}} + \mathbf{E}\Delta\boldsymbol{\varepsilon}$ is the trial stress and τ is the relaxation time of the material. To get the elastoviscoplastic algorithmic tangent stiffness we note that $\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}) = \mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})$, see Fig. 2. Then, by taking the derivative with respect to the evolution parameter, the formula $\boldsymbol{\sigma} = \left(1 + \frac{\Delta t}{\tau}\right)^{-1}\left(\boldsymbol{\sigma}_{\mathrm{TR}} + \frac{\Delta t}{\tau}\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})\right)$ yields the incremental law $\dot{\boldsymbol{\sigma}} = \mathbf{K}^{\mathrm{EVP}}(\boldsymbol{\sigma}_{\mathrm{TR}})\dot{\epsilon}$. The linear operator $\mathbf{K}^{\mathrm{EVP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) := (1 + \frac{\Delta t}{\tau})^{-1}(\mathbf{I} + \frac{\Delta t}{\tau}\, d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))\mathbf{E}$ is the elastoviscoplastic algorithmic tangent stiffness.

Remark 2.2. By integrating according to a fully implicit scheme the flow rule introduced in Perzyna (1963), the elastoviscoplastic algorithmic constitutive law writes:

$$\begin{cases} \boldsymbol{\sigma}_0 + \mathbf{E}\Delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\Delta\mathbf{p}, \\ \Delta\mathbf{p} = \frac{\Delta t}{\tau}\langle g(\boldsymbol{\sigma})\rangle\nabla g(\boldsymbol{\sigma}), \end{cases} \iff \quad \boldsymbol{\sigma}_{\mathrm{TR}} = \boldsymbol{\sigma} + \frac{\Delta t}{\tau}\langle g(\boldsymbol{\sigma})\rangle\mathbf{E}\nabla g(\boldsymbol{\sigma}),$$

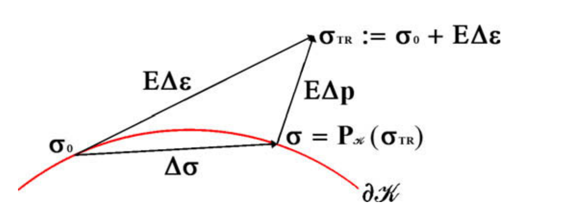

Fig. 1. Geometric scheme of the discrete elastoplastic constitutive law.

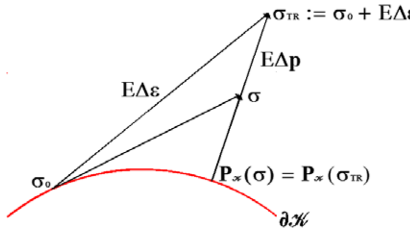

Fig. 2. Geometric scheme of the Duvaut and Lions discrete elastoviscoplastic constitutive law.

where $\langle g(\boldsymbol{\sigma})\rangle := \frac{1}{2}(g(\boldsymbol{\sigma}) + |g(\boldsymbol{\sigma})|)$ is Macaulay's bracket. Outside the elastic range the function g is positive so that Perzyna's algorithmic constitutive law becomes $\boldsymbol{\sigma}_{\mathrm{TR}} = \boldsymbol{\sigma} + \frac{\Delta t}{\tau} g(\boldsymbol{\sigma})\mathbf{E}\nabla g(\boldsymbol{\sigma})$. Taking the derivative with respect to the evolution parameter, the incremental law is then given by:

$$\mathbf{E}\dot{\varepsilon} = \dot{\boldsymbol{\sigma}} + \frac{\Delta t}{\tau}(\nabla g(\boldsymbol{\sigma}) \otimes \nabla g(\boldsymbol{\sigma}))\dot{\boldsymbol{\sigma}} + \frac{\Delta t}{\tau} g(\boldsymbol{\sigma})\mathbf{E}\nabla^2 g(\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}.$$

For sufficiently small values of the ratio $\frac{\Delta t}{\tau}$, the linear operator

$$\left(\mathbf{I} + \frac{\Delta t}{\tau}(\nabla g(\boldsymbol{\sigma}) \otimes \nabla g(\boldsymbol{\sigma})) + \frac{\Delta t}{\tau} g(\boldsymbol{\sigma})\mathbf{E}\nabla^2 g(\boldsymbol{\sigma})\right)^{-1}\mathbf{E},$$

is Perzyna's algorithmic tangent stiffness at the stress point solution of the nonlinear equation $\psi(\boldsymbol{\sigma}) := \boldsymbol{\sigma} + \frac{\Delta t}{\tau} g(\boldsymbol{\sigma})\mathbf{E}\nabla g(\boldsymbol{\sigma}) = \boldsymbol{\sigma}_{\mathrm{TR}}$. The geometric approach proposed in this paper may be applied if the solution of the nonlinear equation $\psi(\boldsymbol{\sigma}) = \boldsymbol{\sigma}_{\mathrm{TR}}$ is expressible in terms of the projection in $\mathbb{S}_{\mathbf{C}}$ of the trial stress $\boldsymbol{\sigma}_{\mathrm{TR}}$ onto the elastic domain.

3. A geometric formula

As shown above, the evaluation of the algorithmic tangent stiffness, in elastoplasticity: $\mathbf{K}^{\mathrm{EP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) := (d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))\mathbf{E}$ and in Duvaut and Lions elastoviscoplasticity: $\mathbf{K}^{\mathrm{EVP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) := \left(1 + \frac{\Delta t}{\tau}\right)^{-1}(\mathbf{I} + \frac{\Delta t}{\tau} d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}))\mathbf{E}$, requires the knowledge of the derivative of the nonlinear projector $d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})$ in complementary elastic energy norm, on the elastic domain $\mathscr{K}$. To evaluate the expression of $d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}})$ we consider a special foliation of the space $\mathbb{S}_{\mathbf{C}}$ induced by the boundary $\partial\mathscr{K}$ of the convex elastic domain $\mathscr{K}$. Each folium of the foliation is a hypersurface parallel to $\partial\mathscr{K}$, obtained by shifting its points outward in the normal direction of a fixed amount. In Fig. 3, $\partial\mathscr{K}^r$ is the folium passing through $\boldsymbol{\sigma}_{\mathrm{TR}}$. Such a foliation is said to be generated by the level sets of the distance function $r \in \mathrm{C}^2(\mathbb{S}_{\mathbf{C}}; \mathscr{R})$ from $\partial\mathscr{K}$, with the property that (see e.g. Petersen, 1998): $\|\nabla_{\mathbf{C}} r(\mathbf{s})\|_{\mathbf{C}} = 1$ for any $\mathbf{s} \in \mathbb{S}_{\mathbf{C}}$. The unit normal in $\mathbb{S}_{\mathbf{C}}$ to a folium is the gradient, in complementary elastic energy, of the distance function: $\mathbf{n}(\mathbf{s}) = \nabla_{\mathbf{C}} r(\mathbf{s})$. The derivative of the projector $\mathbf{P}_{\mathscr{K}}$ is conveniently computed by considering only tangent directions to the relevant folium since the derivative vanishes along the normal direction. To this end we denote by $r(\mathbf{s}) := \|\mathbf{s} - \mathbf{P}_{\mathscr{K}}(\mathbf{s})\|_{\mathbf{C}}$, the distance in complementary elastic energy between a stress point $\mathbf{s} \in \mathbb{S}_{\mathbf{C}}$ and its projection on $\mathscr{K}$, and by $\partial\mathscr{K}^r$ the folium passing through $\mathbf{s}$. Hence we have that $\mathbf{s} = \mathbf{P}_{\mathscr{K}}(\mathbf{s}) + r(\mathbf{s})\mathbf{n}(\mathbf{s}), \quad \mathbf{s} \in \partial\mathscr{K}^r$, where $\mathbf{n}(\mathbf{s})$ is the outward unit normal in $\mathbb{S}_{\mathbf{C}}$ to the folium $\partial\mathscr{K}^r$. Denoting by $\mathbb{T}_{\partial\mathscr{K}^r}(\mathbf{s})$ the tangent hyperplane to the folium $\partial\mathscr{K}^r$ at the point $\mathbf{s} \in \partial\mathscr{K}^r$, taking the derivative along tangents vectors to the folium and observing that $r(\mathbf{s})$ is a constant function on $\partial\mathscr{K}^r$, we get $\mathbf{h} = d\mathbf{P}_{\mathscr{K}}(\mathbf{s})\mathbf{h} + r(\mathbf{s})\mathbf{S}(\mathbf{s})\mathbf{h}$, for any $\mathbf{h} \in \mathbb{T}_{\partial\mathscr{K}^r}(\mathbf{s})$, where $\mathbf{S}(\mathbf{s})$ is the shape operator of the folium passing through $\mathbf{s} \in \mathbb{S}_{\mathbf{C}}$, defined as the $\mathbf{g}_{\mathbf{C}}$-symmetric Hessian $\mathbf{S} := \nabla_{\mathbf{C}}\mathbf{n} = \nabla^2_{\mathbf{C}} r$ of the distance function in $\mathbb{S}_{\mathbf{C}}$. From the equality $\mathbf{g}_{\mathbf{C}}(\mathbf{Sn}, \mathbf{h}) = \mathbf{g}_{\mathbf{C}}(\mathbf{Sh}, \mathbf{n}) = \mathbf{g}_{\mathbf{C}}(\nabla_{\mathbf{C}}\mathbf{n} \cdot \mathbf{h}, \mathbf{n}) = \frac{1}{2} d_{\mathbf{h}}\mathbf{g}_{\mathbf{C}}(\mathbf{n}, \mathbf{n}) = 0$, for any $\mathbf{h} \in \mathbb{S}_{\mathbf{C}}$, it follows that $\mathbf{S}(\mathbf{s})\mathbf{n}(\mathbf{s}) = 0$. Let us denote by $\mathbf{\Pi}(\mathbf{s})$ the linear orthogonal projector in $\mathbb{S}_{\mathbf{C}}$ on $\mathbb{T}_{\partial\mathscr{K}^r}(\mathbf{s})$, so that $\mathbf{\Pi}(\mathbf{s})\mathbf{h} = (\mathbf{I} - \mathbf{n}(\mathbf{s}) \otimes_{\mathbf{g_C}} \mathbf{n}(\mathbf{s}))\mathbf{h} = \mathbf{h} - \mathbf{g}_{\mathbf{C}}(\mathbf{h}, \mathbf{n}(\mathbf{s}))\mathbf{n}(\mathbf{s}) = \mathbf{h}$, for any $\mathbf{h} \in \mathbb{S}_{\mathbf{C}}$, with $\otimes_{\mathbf{g_C}}$ the tensor product in $\mathbb{S}_{\mathbf{C}}$. Then, given that $d\mathbf{P}_{\mathscr{K}}(\mathbf{s})\mathbf{n}(\mathbf{s}) = 0$ and $\mathbf{S}(\mathbf{s})\mathbf{n}(\mathbf{s}) = 0$, the relation $\mathbf{h} = d\mathbf{P}_{\mathscr{K}}(\mathbf{s})\mathbf{h} + r(\mathbf{s})\mathbf{S}(\mathbf{s})\mathbf{h}$, for any $\mathbf{h} \in \mathbb{T}_{\partial\mathscr{K}^r}(\mathbf{s})$, may be rewritten as:

$$d\mathbf{P}_{\mathscr{K}}(\mathbf{s}) = \mathbf{\Pi}(\mathbf{s}) - r(\mathbf{s})\mathbf{S}(\mathbf{s}).$$

The application of this formula requires the knowledge of the implicit analytical expression of the folium $\partial\mathscr{K}^r$ in terms of the yield function $\varphi : \mathbb{S} \to \mathscr{R}$ which describes the elastic domain. A simple and effective approximation may be got by replacing the hypersurface $\partial\mathscr{K}^r$ with the $\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})$-level set of the yield function as depicted in Fig. 4. This procedure leads to the exact evaluation of the derivative of the nonlinear projector when the level sets of the yield function are homothetic hypersurfaces, as in Von Mises criterion.

3.1. Approximate algorithmic tangent stiffness

Let us recall that, if $d_{\mathbf{h}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})$ is the directional derivative at $\boldsymbol{\sigma}_{\mathrm{TR}} \in \mathbb{S}_{\mathbf{C}}$ along the vector $\mathbf{h} \in \mathbb{S}_{\mathbf{C}}$, the gradient $\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \in \mathbb{S}_{\mathbf{C}}$ at $\boldsymbol{\sigma}_{\mathrm{TR}} \in \mathbb{S}_{\mathbf{C}}$ is defined by $\mathbf{g}_{\mathbf{C}}(\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}), \mathbf{h}) = d_{\mathbf{h}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})$, for any $\mathbf{h} \in \mathbb{S}_{\mathbf{C}}$. The shape operator $\mathbf{S}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})$ of the $\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})$-level set of the yield function is given by: $\mathbf{S}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) = \nabla_{\mathbf{C}}\mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})$, where $\mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) := \frac{\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})}{\|\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}}$ is the unit normal in $\mathbb{S}_{\mathbf{C}}$. Denoting by $\mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) := \mathbf{I} - \mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) \otimes_{\mathbf{g_C}} \mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})$ the linear orthogonal projector in $\mathbb{S}_{\mathbf{C}}$ on the tangent hyperplane to the level set of $\boldsymbol{\varphi}$ at $\boldsymbol{\sigma}_{\mathrm{TR}} \in \mathbb{S}_{\mathbf{C}}$, the directional derivative of $\mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})$ along a tangent direction $\mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h}$ is given by:

$$\begin{aligned}
\mathbf{S}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h} &= \nabla_{\mathbf{C}}\mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) \cdot \mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h} \\
&= \left(\frac{\nabla^2_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \cdot \mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h}}{\|\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}} - \frac{\mathbf{g}(\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}), \nabla^2_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \cdot \mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h})}{\|\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|^3_{\mathbf{C}}}\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\right) \\
&= \frac{1}{\|\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}}(\mathbf{I} - \mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) \otimes_{\mathbf{g_C}} \mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}))\nabla^2_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \cdot \mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h} \\
&= \frac{1}{\|\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}}\mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\nabla^2_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h},
\end{aligned}$$

where $\nabla^2_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})$ is the Hessian in $\mathbb{S}_{\mathbf{C}}$ of φ: $\mathbf{g}_{\mathbf{C}}(\nabla^2_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{h}_1, \mathbf{h}_2) := d_{\mathbf{h}_2}\mathbf{g}_{\mathbf{C}}(\nabla_{\mathbf{C}}\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}), \mathbf{h}_1)$, for any $\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{S}_{\mathbf{C}}$. The approximate algorithmic tangent stiffness in elastoplasticity

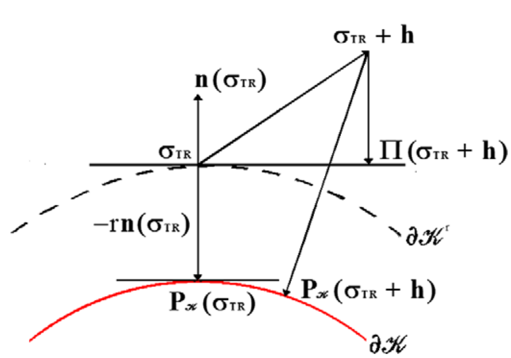

Fig. 3. Parallel hypersurface and projectors.

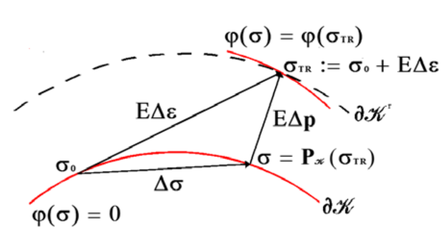

Fig. 4. Approximation of the parallel hypersurface.

and in Duvaut and Lions elastoviscoplasticity are respectively given by:

$$\mathbf{K}_{\varphi}^{\mathrm{EP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) = (\boldsymbol{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) - r(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{S}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}))\mathbf{E},$$

$$\mathbf{K}_{\varphi}^{\mathrm{EVP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) = \left(1 + \frac{\Delta t}{\tau}\right)^{-1} \left(\mathbf{I} + \frac{\Delta t}{\tau}(\boldsymbol{\Pi}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) - r(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{S}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}))\right)\mathbf{E}.$$

Their symmetry is a consequence of the symmetry of $\boldsymbol{\Pi}_{\varphi}\mathbf{E}$ and $\mathbf{S}_{\varphi}\mathbf{E}$. Indeed, due to the symmetry of $\mathbf{E}$ and to the $\mathbf{g}_{\mathbf{C}}$-symmetry of $\boldsymbol{\Pi}_{\varphi}$ and $\mathbf{S}_{\varphi}$, we may write that:

$$\begin{aligned}\langle \boldsymbol{\Pi}_{\varphi}\mathbf{E}\varepsilon_1, \boldsymbol{\varepsilon}_2\rangle &= \langle \mathbf{E}\mathbf{C}\boldsymbol{\Pi}_{\varphi}\mathbf{E}\varepsilon_1, \boldsymbol{\varepsilon}_2\rangle = \langle \mathbf{C}\boldsymbol{\Pi}_{\varphi}\mathbf{E}\varepsilon_1, \mathbf{E}\varepsilon_2\rangle = \langle \mathbf{C}\boldsymbol{\Pi}_{\varphi}\mathbf{E}\varepsilon_2, \mathbf{E}\varepsilon_1\rangle \\ &= \langle \boldsymbol{\Pi}_{\varphi}\mathbf{E}\varepsilon_2, \boldsymbol{\varepsilon}_1\rangle,\end{aligned}$$

so that the linear operator $\boldsymbol{\Pi}_{\varphi}\mathbf{E}$ is symmetric. Analogously, the symmetry of $\mathbf{S}_{\varphi}\mathbf{E}$ is proven.

4. Conclusions

A simple geometrical approach has been proposed to evaluate the elastoplastic algorithmic tangent stiffness in the evolutive analysis of structural problems based on the elastic prediction/plastic correction method. The analysis extends to the elastoviscoplastic model in which the threshold stress is evaluated according to the projection on the convex elastic domain. Indeed a crucial role is played by the evaluation of the derivative of the nonlinear projector on the elastic domain: $d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}) = \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}}) - r(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{S}(\boldsymbol{\sigma}_{\mathrm{TR}})$. This formula naturally leads to the following observations. At flat points of the yield surface the shape operator vanishes so that $d\mathbf{P}_{\mathscr{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}) = \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})$. The elastoplastic algorithmic tangent stiffness is then equal to the continuous tangent stiffness: $\mathbf{K}^{\mathrm{EP}}(\boldsymbol{\sigma}_{\mathrm{TR}}) = (\boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}}) - r(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{S}(\boldsymbol{\sigma}_{\mathrm{TR}}))\mathbf{E} = \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})\mathbf{E} = \mathbf{K}_{\mathrm{RATE}}^{\mathrm{EP}}(\boldsymbol{\sigma}_{\mathrm{TR}})$. At non-flat points the convexity of the elastic domain implies that the elastoplastic algorithmic tangent stiffness is smaller than the continuous one (the shape operator is positive definite) leading to improved convergence rates. This considerations provide a direct and simple motivation of why and when a better convergence is got by adopting the elastoplastic algorithmic tangent stiffness instead of the rate tangent stiffness. An effective estimate is got by substituting the $\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})$-level set of the yield function in place of the folium $\partial\mathscr{K}^r$, thus leading to an expression which can be computationally convenient. For yield functions with homothetically expanding level sets (for instance Von Mises criterion) the new formula gives the exact algorithmic tangent stiffness. Numerical evidence by implementations in a computer code will be provided in a forthcoming paper.

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