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On the theory of material inhomogeneities

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Abstract

The theory of material inhomogeneities, according to Noll's approach, is revisited in detail and its interpretations in the recent literature on material anelastic behaviors, growth and phase transition phenomena are critically discussed.

Keywords: Elasticity; Inhomogeneity; Differential geometry

1. Introduction

The simulation of material inhomogeneities, growth phenomena and phase-transition or defect propagation in continuous bodies are presently an issue of increasing interest in the literature on mechanics and thermodynamics. Most contributions are based on mere rephrasing, in simplified form, of the theory of material inhomogeneities developed by Walter Noll (see Truesdell and Noll, 1965; Noll, 1968) and, with a more deep geometrical treatment, by Wang (1968). This theory was almost completely neglected until about 20 years later when it was referred to by Epstein and Maugin (1990). The point of view exposed in Epstein and Maugin (1990, 1996) and Maugin and Epstein (1998), is strictly related to the Kondo–Kröner–Lee decomposition in finite plasticity, according to which the differential of the change of configuration at a point may be decomposed in a plastic and a subsequent elastic part. These papers were intended to contribute an extension of Eshelby's results concerning the continuum theory of lattice defects and the evaluation of the force acting on defects in a nonlinear elastic medium (see Eshelby, 1951, 1956, 1975). The theory of inhomogeneities developed by Noll and Wang consists in fact in a comprehensive study of the differential geometric aspects of the anelastic–elastic chain decomposition formerly envisaged by Kondo (1955) and Kröner (1960) and later adopted in the context of finite plasticity by Lee (1969).

A major difficulty in detecting these analogies is the comparison between different notations and formal expositions. In this respect we must notice that the term *material inhomogeneities* has been inadequately adopted to denote anelastic transformations, while its usual meaning refers to a nonuniform constitutive law. In recent treatments anelastic phenomena have been simulated by linear transformations (between tan-

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gent spaces) dubbed with picturesque names depending on the authors' taste and the special physical motivation. In fact the *uniform reference* map (Noll, 1968; Epstein and Maugin, 1996; Maugin and Epstein, 1998; Maugin, 2002), the *plastic transformation* (Lee, 1969), the *transplant* map (Epstein, 2002), and the *relaxed stance* baptized in (Di Carlo and Quiligotti, 2002), are just nicknames for what engineers would simply refer to as anelastic transformations, or their inverse. We shall not try to give up with the tradition and will adopt the denomination *transplant* map which has a flavour of surgical operation towards a well improved beauty.

2. The transplant map

We summarize hereafter the essentials of Noll's theory of inhomogeneities, a nice exercise in differential geometry, by adopting a coordinate-free exposition which, in our opinion, simplifies the original formalism. In a differential geometric description, the ambient space $\{\mathbb{S}, \mathbf{g}\}$ is a differentiable manifold endowed with a Riemannian metric and a path-independent parallel transport. Classically the ambient space is the Euclidean space and the parallel transport is the translation. The body is identified with an embedded submanifold $\mathbb{M} \subset \mathbb{S}$, dubbed the *reference placement*, having the same dimension as the ambient space. Then its tangent bundle $\mathbb{TM}$ is the collection of the linear tangent spaces $\mathbb{T}_{\mathbf{m}}\mathbb{S}$ whose elements are the vectors based at $\mathbf{m} \in \mathbb{M}$ and its cotangent bundle $\mathbb{T}^*\mathbb{M}$, is the collection of the dual spaces $\mathbb{T}^*_{\mathbf{m}}\mathbb{S}$ whose elements are the co-vectors (linear forms) based at $\mathbf{m} \in \mathbb{M}$. A map $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M};\mathbb{S})$ is called a C^1-*morphism* of $\mathbb{M}$ into $\mathbb{S}$. A C^1-*diffeomorphism* is an invertible C^1-morphism such that $\boldsymbol{\varphi}^{-1} \in \mathrm{C}^1(\boldsymbol{\varphi}(\mathbb{M});\mathbb{M})$. A configuration of the body in the ambient space $\{\mathbb{S}, \mathbf{g}\}$ is a diffeomorphism $\boldsymbol{\varphi} \in \mathrm{C}^1(\mathbb{M};\mathbb{S})$ of the reference body manifold onto a space submanifold $\boldsymbol{\varphi}(\mathbb{M}) \subset \mathbb{S}$, dubbed the placement of the body in the ambient space. The differential of the configuration map is a smooth field of invertible tensors $d\boldsymbol{\varphi} \in \mathrm{C}(\mathbb{M}; BL(\mathbb{TM}; \mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}\mathbb{S}))$ where $\mathbb{T}_{\boldsymbol{\varphi}(\mathbb{M})}\mathbb{S}$ is the restriction to $\boldsymbol{\varphi}(\mathbb{M})$ of the tangent bundle $\mathbb{TS}$. A transplant is a smooth field $\mathbf{P} \in \mathrm{C}^1(\mathbb{M}; BL(\mathbb{TM};\mathbb{TM}))$ of invertible tensors $\mathbf{P}(\mathbf{m}) \in BL(\mathbb{T}_{\mathbf{m}}\mathbb{M};\mathbb{T}_{\mathbf{m}}\mathbb{M})$. We recall that a tensor field *lives at points* which means that the scalar values taken by a tensor field, over the argument vector fields, at a point depend only on the values of the vector fields at that point (Spivak, 1979). The inverse transplant $\mathbf{P}^{-1} \in \mathrm{C}^1(\mathbb{M}; BL(\mathbb{TM};\mathbb{TM}))$ is the field of inverse tensors $\mathbf{P}^{-1}(\mathbf{m}) \in BL(\mathbb{T}_{\mathbf{m}}\mathbb{M};\mathbb{T}_{\mathbf{m}}\mathbb{M})$.

2.1. Transplant induced connection

A transplant induces in the material manifold $\mathbb{M}$ a path-independent parallel transport defined by $S^{\mathbf{P}}(\mathbf{x},\mathbf{y}) := \mathbf{P}^{-1}(\mathbf{x})S(\mathbf{x},\mathbf{y})\mathbf{P}(\mathbf{y}),\ \mathbf{x},\mathbf{y} \in \mathbb{M}$ where S is the path-independent parallel transport in the space manifold (the translation in the Euclidean space). The transplant $\mathbf{P}(\mathbf{y})$ pushes forward a referential material line-element (a tangent vector) $\mathbf{v}(\mathbf{y}) \in \mathbb{T}_{\mathbf{y}}\mathbb{M}$ to the corresponding transplanted material line-element, which is then translated to the base point $\mathbf{x} \in \mathbb{M}$ by $S(\mathbf{x},\mathbf{y})$. The inverse transplant $\mathbf{P}^{-1}(\mathbf{x})$ pulls back this transplanted material line-element at $\mathbf{x} \in \mathbb{M}$ to the referential material line element $S^{\mathbf{P}}(\mathbf{x},\mathbf{y})\mathbf{v}(\mathbf{y}) \in \mathbb{T}_{\mathbf{x}}\mathbb{M}$ which is the final result of the parallel transport. This parallel transport induces in $\mathbb{M}$ a connection ∇ which can be computed as follows (Marsden and Hughes, 1988). Let $\mathbf{c} \in \mathrm{C}^1(\mathscr{R};\mathbb{M})$ be a material curve thru the material point $\mathbf{m} = \mathbf{c}(t)$ with tangent $\mathbf{t} = \dot{\mathbf{c}}(t) = d_{\tau|\tau=t}\mathbf{c}(\tau) \in \mathbb{T}_{\mathbf{m}}\mathbb{M}$. The parallel transport $S^{\mathbf{P}}_{t,\tau}(\mathbf{c}) \in BL(\mathbb{T}_{\mathbf{c}(\tau)}\mathbb{M};\mathbb{T}_{\mathbf{c}(t)}\mathbb{M})$, along the material curve $\mathbf{c} \in \mathrm{C}^1(\mathscr{R};\mathbb{M})$, from $\mathbf{c}(\tau)$ to $\mathbf{c}(t)$, is defined by

$$S^{\mathbf{P}}_{t,\tau}(\mathbf{c}) := \mathbf{P}^{-1}(\mathbf{c}(t))S(\mathbf{c}(t),\mathbf{c}(\tau))\mathbf{P}(\mathbf{c}(\tau)).$$

The covariant derivative measures the rate of variation of the parallel transported material line element as the material point starts moving along a material direction and is then evaluated according to the formula:

$$\begin{aligned}\nabla_{\mathbf{t}}\mathbf{v}(\mathbf{m}) &= d_{\tau|\tau=t}S^{\mathbf{P}}_{t,\tau}(\mathbf{c})\mathbf{v}(\mathbf{c}(\tau)) = \mathbf{P}^{-1}(\mathbf{c}(t))d_{\tau|\tau=t}S(\mathbf{c}(t),\mathbf{c}(\tau))(\mathbf{P}(\mathbf{c}(\tau))\mathbf{v}(\mathbf{c}(\tau))) = \mathbf{P}^{-1}(\mathbf{m})d_{\mathbf{t}}(\mathbf{P}(\mathbf{m})\mathbf{v}(\mathbf{m}))\\ &= \mathbf{P}^{-1}(\mathbf{m})d_{\mathbf{t}}(\mathbf{P}(\mathbf{m}))\mathbf{v}(\mathbf{m}) + d_{\mathbf{t}}\mathbf{v}(\mathbf{m}).\end{aligned}$$

Here d denotes the covariant derivative according to the standard Riemannian connection in the space manifold (i.e., the directional derivative if the material manifold is the Euclidean space). The covariant derivative according to the material affine connection induced by the transplant map is then written as

$$\nabla\mathbf{v}(\mathbf{m}) = \mathbf{P}^{-1}(\mathbf{m})d(\mathbf{P}(\mathbf{m})\mathbf{v}(\mathbf{m})) \iff \mathbf{P}\nabla_{\mathbf{t}}\mathbf{v} = d_{\mathbf{t}}(\mathbf{P}\mathbf{v}).$$

A comparison of notations can be made with the treatment in (Noll, 1968), where $\mathbf{P}^{-1}$ is denoted by $\mathbf{K}$ and dubbed the *uniformity map*, and with the component expression provided in (Epstein, 2002) where $\mathbf{P}$ is our $\mathbf{P}^{-1} = \mathbf{K}$. In our opinion, the choice made in this paper is to be preferred. Indeed the connection defined by our $\mathbf{P}$ is defined in the reference manifold and is related to the metric induced by the transplant in the reference manifold (see Section 2.2). The covariant derivative at a point depends only on the tangent vector to the curve $\mathbf{c}$ at that point and is then well-defined even if parallel transports of a vector along different curves, joining the base point to another point, provide different final vectors.

Remarkably, if the parallel transport is path-independent, the connection enjoys special properties, described below. As is well-known, two tensor fields can be associated with an affine connection ∇, the Cartan torsion and the Riemann-Christoffel curvature, respectively defined by:

$$\begin{aligned} \text{TORS}(\mathbf{a},\mathbf{b}) &:= \nabla_{\mathbf{a}}\mathbf{b} - \nabla_{\mathbf{b}}\mathbf{a} - [\mathbf{a},\mathbf{b}], \\ \text{CURV}(\mathbf{a},\mathbf{b},\mathbf{c}) &:= \nabla_{\mathbf{a}}\nabla_{\mathbf{b}}\mathbf{c} - \nabla_{\mathbf{b}}\nabla_{\mathbf{a}}\mathbf{c} - \nabla_{[\mathbf{a},\mathbf{b}]}\mathbf{c}, \end{aligned}$$

where $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{TM}$ and $[\mathbf{a},\mathbf{b}] = \mathscr{L}_{\mathbf{a}}\mathbf{b} = -[\mathbf{b},\mathbf{a}]$ is the Lie bracket of the vector fields $\mathbf{a},\mathbf{b} \in \mathbb{TM}$ defined by, $[\mathbf{a},\mathbf{b}]f := (d_{\mathbf{a}}d_{\mathbf{b}} - d_{\mathbf{b}}d_{\mathbf{a}})f$, with $f \in C^2(\mathbb{M};\mathscr{R})$ scalar field (see Choquet-Bruhat, 1970; Abraham et al., 1988; Romano, 2001). These tensor fields provide a measure of the lack of symmetry of the second covariant derivative of differentiable scalar and vector fields, being

$$\begin{aligned} (\nabla^2_{\mathbf{ba}} - \nabla^2_{\mathbf{ab}})f &= \text{TORS}(\mathbf{a},\mathbf{b})f, \\ (\nabla^2_{\mathbf{ab}} - \nabla^2_{\mathbf{ba}})\mathbf{c} &= \text{CURV}(\mathbf{a},\mathbf{b},\mathbf{c}) - \nabla_{\text{TORS}(\mathbf{a},\mathbf{b})}\mathbf{c}. \end{aligned}$$

The tensoriality of the curvature field implies that it vanishes identically if the parallel transport associated with the connection is path-independent. Indeed, to compute the curvature at any point, by tensoriality we may extend the vectors at that point to arbitrary vector fields on which the covariant derivatives may be computed and the result is independent of the chosen extension. If the extension is performed by means of a path-independent parallel transport, all the covariant derivatives vanish and the result follows. The same argument shows that the torsion may be computed as the opposite of the Lie bracket of the parallel-transported vector fields. The transplant-induced parallelism is path-independent, so that the associated curvature vanishes, i.e., $\text{CURV} = 0$.

Remark 1. According to Noll (1968), a body is *locally homogeneous* if the transplant map $\mathbf{P} \in C^1(\mathbb{M}; BL(\mathbb{TM};\mathbb{TM}))$ can be integrated to a transplant morphism $\boldsymbol{\eta}_{\mathbf{P}} \in C^1(\mathbb{M};\mathbb{M})$ with differential $d\boldsymbol{\eta}_{\mathbf{P}} \in C(\mathbb{M}; BL(\mathbb{TM};\mathbb{TM}))$ equal to the transplant map, i.e., $\mathbf{P} = d\boldsymbol{\eta}_{\mathbf{P}}$. The transplanted body is dubbed *homogeneous* if the transplant map is a C^1-*diffeomorphism*. A necessary and sufficient condition for local homogeneity is the vanishing of the torsion, i.e., $\text{TORS} = 0$. Indeed any pair of vectors $\mathbf{a}$, $\mathbf{b}$ may be extended, by translation, to a pair of vector fields in $\mathbb{TM}$, still denoted by $\mathbf{a}$, $\mathbf{b}$ so that $[\mathbf{a},\mathbf{b}] = 0$. Hence, by the tensoriality of the torsion, we have that

$$\mathbf{P}\,\text{TORS}(\mathbf{a},\mathbf{b}) = d_{\mathbf{a}}(\mathbf{Pb}) - d_{\mathbf{b}}(\mathbf{Pa}) = (d_{\mathbf{a}}\mathbf{P})\mathbf{b} - (d_{\mathbf{b}}\mathbf{P})\mathbf{a} = 0,$$

which is the integrability condition in a linear space. The above definitions, although rising interesting questions from a mathematical point of view, appear inadequate in a mechanical context since the customary meaning of the term *homogeneity* has nothing to do with topological properties of a material body.

2.2. Transplant induced metric

A Riemannian metric may be defined in the transplanted body by setting:

$$\mathbf{g}_{\mathbb{M}}(\mathbf{a},\mathbf{b}) := \mathbf{g}(\mathbf{Pa},\mathbf{Pb}), \quad \forall \mathbf{a},\mathbf{b} \in \mathbb{TM}.$$

In the Riemannian manifold $\{\mathbb{M},\mathbf{g}_{\mathbb{M}}\}$ we may consider the associated Levi-Civita connection, denoted by $\nabla^{\mathbb{M}}$, uniquely defined by requiring the fulfilment of the following two conditions which mimic standard properties of the Euclidean space (Petersen, 1998):

(i) $\text{TORS}^{\mathbb{M}}(\mathbf{a},\mathbf{b}) = \nabla^{\mathbb{M}}_{\mathbf{a}}\mathbf{b} - \nabla^{\mathbb{M}}_{\mathbf{b}}\mathbf{a} - [\mathbf{a},\mathbf{b}] = 0, \quad \forall \mathbf{a},\mathbf{b} \in \mathbb{TM},$

(ii) $\nabla^{\mathbb{M}}\mathbf{g}_{\mathbb{M}} = 0.$

Condition (i) requires that the torsion of the connection vanishes, i.e., that the second covariant derivative of any differentiable scalar field be symmetric. Condition (ii) requires that the covariant derivative of the metric vanishes. The Riemannian manifold $\{\mathbb{M},\mathbf{g}_{\mathbb{M}}\}$ is *flat* if the corresponding Riemann-Christoffel curvature tensor field vanishes, that is if:

$$\text{CURV}^{\mathbb{M}}(\mathbf{a},\mathbf{b},\mathbf{c}) := \nabla^{\mathbb{M}2}{}_{\mathbf{ab}}\mathbf{c} - \nabla^{\mathbb{M}2}{}_{\mathbf{ba}}\mathbf{c} = \nabla^{\mathbb{M}}_{\mathbf{a}}\nabla^{\mathbb{M}}_{\mathbf{b}}\mathbf{c} - \nabla^{\mathbb{M}}_{\mathbf{b}}\nabla^{\mathbb{M}}_{\mathbf{a}}\mathbf{c} - \nabla^{\mathbb{M}}_{[\mathbf{a},\mathbf{b}]}\mathbf{c} = 0, \quad \forall \mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{TM}.$$

The vanishing of the curvature $\text{CURV}^{\mathbb{M}}$ ensures that there exists a (local) transplant configuration map $\boldsymbol{\varphi}_{\mathbb{M}} \in C^1(\{\mathbb{M},\mathbf{g}_{\mathbb{M}}\};\{\mathbb{S},\mathbf{g}\})$ which is a (local) Riemannian isometry between the transplanted body $\{\mathbb{M},\mathbf{g}_{\mathbb{M}}\}$ and a placement $\{\boldsymbol{\varphi}_{\mathbb{M}}(\mathbb{M}) \subset \mathbb{S},\mathbf{g}\}$ of the body in the Euclidean ambient space. This amounts to require that the transplant induced metric be equal to the pull-back of the space metric tensor according to the (local) configuration map, defined by

$$(\boldsymbol{\varphi}^*_{\mathbb{M}}\mathbf{g})(\mathbf{a},\mathbf{b}) := \mathbf{g}(\mathrm{d}\boldsymbol{\varphi}_{\mathbb{M}}\mathbf{a},\mathrm{d}\boldsymbol{\varphi}_{\mathbb{M}}\mathbf{b}) = \mathbf{g}_{\mathbb{M}}(\mathbf{a},\mathbf{b}), \qquad \forall \mathbf{a},\mathbf{b} \in \mathbb{TM},$$

with $\mathrm{d}\boldsymbol{\varphi}_{\mathbb{M}} \in C^1(\mathbb{M};BL(\mathbb{TM};\mathbb{TS}))$ differential of $\boldsymbol{\varphi}_{\mathbb{M}}$. If a transplant configuration map exists, it is unique to within the left-composition with an isometric transformation in the ambient space $\{\mathbb{S},\mathbf{g}\}$ and the transplant metric field $\mathbf{g}_{\mathbb{M}}$ is said to be (locally) *kinematically compatible* with the Euclidean ambient space metric $\mathbf{g}$. Indeed, if $\boldsymbol{\varphi}_{\mathbb{M}} \in C^1(\mathbb{M};\mathbb{S})$ and $\boldsymbol{\chi}_{\mathbb{M}} \in C^1(\mathbb{M};\mathbb{S})$ are two transplant configuration maps, we have that

$$\boldsymbol{\varphi}^*_{\mathbb{M}}\mathbf{g} = \boldsymbol{\chi}^*_{\mathbb{M}}\mathbf{g} = \mathbf{g}_{\mathbb{M}},$$

and hence $\boldsymbol{\varphi}_{\mathbb{M}} \circ \boldsymbol{\chi}^{-1}_{\mathbb{M}} \in C^1(\mathbb{S};\mathbb{S})$ and $\boldsymbol{\chi}_{\mathbb{M}} \circ \boldsymbol{\varphi}^{-1}_{\mathbb{M}} \in C^1(\mathbb{S};\mathbb{S})$ are isometric transformations in the ambient space $\{\mathbb{S},\mathbf{g}\}$. As proven in (Romano, 2001; Romano et al., 2006c), their differentials are constant fields of linear isometries.

Theorem 2. *Let $\mathbb{M} \subset \mathbb{S}$ be a connected open set and $\boldsymbol{\varphi} \in C^2(\mathbb{M};\mathbb{S})$ be a diffeomorphic transformation such that the associated* GREEN *strain tensor field $\frac{1}{2}(\boldsymbol{\varphi}^*\mathbf{g} - \mathbf{g})$ is constant on $\mathbb{M}$. Then the differential $d\boldsymbol{\varphi} \in C^1(\mathbb{M};BL(\mathbb{TM};\mathbb{TS}))$ is constant on $\mathbb{M}$. In particular, if the transformation $\boldsymbol{\varphi} \in C^2(\mathbb{M};\mathbb{S})$ is isometric (i.e., $\boldsymbol{\varphi}^*\mathbf{g} = \mathbf{g}$) then the differential $d\boldsymbol{\varphi}$ is a constant linear isometry $\mathbf{Q} \in BL(\mathbb{TM};\mathbb{TS})$ so that*

$$\boldsymbol{\varphi}(\mathbf{x}) - \boldsymbol{\varphi}(\mathbf{y}) = \mathbf{Q}(\mathbf{x} - \mathbf{y}), \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{M}.$$

Proof. Let $\mathbf{h}_1,\mathbf{h}_2,\mathbf{h} \in \mathbb{TM}$ be arbitrary constant fields. By assumption:

$$\partial_{\mathbf{h}}\mathbf{g}(\partial_{\mathbf{h}_1}\boldsymbol{\varphi}(\mathbf{x}),d_{\mathbf{h}_2}\boldsymbol{\varphi}(\mathbf{x})) = \mathbf{g}(d^2_{\mathbf{hh}_1}\boldsymbol{\varphi}(\mathbf{x}),d_{\mathbf{h}_2}\boldsymbol{\varphi}(\mathbf{x})) + \mathbf{g}(d^2_{\mathbf{hh}_2}\boldsymbol{\varphi}(\mathbf{x}),d_{\mathbf{h}_1}\boldsymbol{\varphi}(\mathbf{x})) = 0.$$

By exchanging $\mathbf{h}_1$ with $\mathbf{h}$ and $\mathbf{h}_2$ with $\mathbf{h}$ we get two more relations, so that

(i) $\mathbf{g}(d^2_{\mathbf{hh}_1}\boldsymbol{\varphi}(\mathbf{x}),d_{\mathbf{h}_2}\boldsymbol{\varphi}(\mathbf{x})) + \mathbf{g}(d_{\mathbf{h}_1}\boldsymbol{\varphi}(\mathbf{x}),d^2_{\mathbf{hh}_2}\boldsymbol{\varphi}(\mathbf{x})) = 0,$

(ii) $\mathbf{g}(d^2_{\mathbf{h}_1\mathbf{h}}\boldsymbol{\varphi}(\mathbf{x}),d_{\mathbf{h}_2}\boldsymbol{\varphi}(\mathbf{x})) + \mathbf{g}(d_{\mathbf{h}}\boldsymbol{\varphi}(\mathbf{x}),d^2_{\mathbf{h}_1\mathbf{h}_2}\boldsymbol{\varphi}(\mathbf{x})) = 0,$

(iii) $\mathbf{g}(d^2_{\mathbf{h}_2\mathbf{h}_1}\boldsymbol{\varphi}(\mathbf{x}),d_{\mathbf{h}}\boldsymbol{\varphi}(\mathbf{x})) + \mathbf{g}(d_{\mathbf{h}_1}\boldsymbol{\varphi}(\mathbf{x}),d^2_{\mathbf{h}_2\mathbf{h}}\boldsymbol{\varphi}(\mathbf{x})) = 0.$

Since the second directional derivative is symmetric, it follows that

$$\mathbf{g}(d^2_{\mathbf{h}_1\mathbf{h}_2}\boldsymbol{\varphi}(\mathbf{x}),d_{\mathbf{h}}\boldsymbol{\varphi}(\mathbf{x})) = 0.$$

Hence by the nonsingularity of $d\boldsymbol{\varphi}(\mathbf{x})$ we have that

$$d^2_{\mathbf{h}_1\mathbf{h}_2}\boldsymbol{\varphi}(\mathbf{x}) = 0 \iff d^2\boldsymbol{\varphi}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbb{M},$$

and by the connectedness of $\mathbb{M}$ we infer that $d\boldsymbol{\varphi}$ is a constant field. □

To investigate about the relation between the two connections ∇ and $\nabla^{\mathbb{M}}$, let us compute the Riemannian covariant derivative by the Koszul formula:

$$2\mathbf{g}_{\mathbb{M}}(\nabla^{\mathbb{M}}_{\mathbf{a}}\mathbf{b},\mathbf{c}) = d_{\mathbf{a}}(\mathbf{g}_{\mathbb{M}}(\mathbf{b},\mathbf{c})) + d_{\mathbf{b}}(\mathbf{g}_{\mathbb{M}}(\mathbf{c},\mathbf{a})) - d_{\mathbf{c}}(\mathbf{g}_{\mathbb{M}}(\mathbf{a},\mathbf{b})) + \mathbf{g}_{\mathbb{M}}([\mathbf{a},\mathbf{b}],\mathbf{c}) - \mathbf{g}_{\mathbb{M}}([\mathbf{b},\mathbf{c}],\mathbf{a}) + \mathbf{g}_{\mathbb{M}}([\mathbf{c},\mathbf{a}],\mathbf{b}).$$

If ∇ is any affine connection which preserves the metric $\mathbf{g}_{\mathbb{M}}$, i.e., such that $\nabla\mathbf{g}_{\mathbb{M}} = 0$, we have that

$$d_{\mathbf{a}}(\mathbf{g}_{\mathbb{M}}(\mathbf{b},\mathbf{c})) = \mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\mathbf{b},\mathbf{c}) + \mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\mathbf{c},\mathbf{b}).$$

Then, recalling that, by definition:

$$\text{TORS}(\mathbf{a},\mathbf{b}) := \nabla_{\mathbf{a}}\mathbf{b} - \nabla_{\mathbf{b}}\mathbf{a} - [\mathbf{a},\mathbf{b}],$$

a direct substitution into Koszul formula reveals that:

$$2\mathbf{g}_{\mathbb{M}}(\nabla^{\mathbb{M}}_{\mathbf{a}}\mathbf{b},\mathbf{c}) = 2\mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\mathbf{b},\mathbf{c}) + \mathbf{g}_{\mathbb{M}}(\text{TORS}(\mathbf{c},\mathbf{a}),\mathbf{b}) + \mathbf{g}_{\mathbb{M}}(\text{TORS}(\mathbf{c},\mathbf{b}),\mathbf{a}) + \mathbf{g}_{\mathbb{M}}(\text{TORS}(\mathbf{b},\mathbf{a}),\mathbf{c}).$$

This result, which is equivalent to the Formula 34.11 in Truesdell and Noll (1965), yields the following implication:

$$\text{TORS} = 0 \Rightarrow \nabla = \nabla^{\mathbb{M}},$$

and provides the proof of a well known result in Riemannian geometry: the uniqueness of the Levi-Civita connection (which is torsionless and preserves the metric $\mathbf{g}_{\mathbb{M}}$). It is easy to see that the connection defined by the transplant map preserves the metric $\mathbf{g}_{\mathbb{M}}$ since (Truesdell and Noll, 1965, Formula 34.10)

$$\begin{aligned}\mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\mathbf{b},\mathbf{c}) + \mathbf{g}_{\mathbb{M}}(\nabla_{\mathbf{a}}\mathbf{c},\mathbf{b}) &= \mathbf{g}(\mathbf{P}\nabla_{\mathbf{a}}\mathbf{b},\mathbf{P}\mathbf{c}) + \mathbf{g}(\mathbf{P}\nabla_{\mathbf{a}}\mathbf{c},\mathbf{P}\mathbf{b}) = \mathbf{g}(d_{\mathbf{a}}(\mathbf{P}\mathbf{b}),\mathbf{P}\mathbf{c}) + \mathbf{g}(d_{\mathbf{a}}(\mathbf{P}\mathbf{c}),\mathbf{P}\mathbf{b}) = d_{\mathbf{a}}\mathbf{g}(\mathbf{P}\mathbf{b},\mathbf{P}\mathbf{c})\\ &= d_{\mathbf{a}}\mathbf{g}_{\mathbb{M}}(\mathbf{b},\mathbf{c}).\end{aligned}$$

This property can also be inferred by checking that the parallel transport induced by the transplant preserves the metric $\mathbf{g}_{\mathbb{M}}$:

$$\begin{aligned}\mathbf{g}_{\mathbb{M}}(S^{\mathbf{p}}(\mathbf{x},\mathbf{y})\mathbf{a},S^{\mathbf{p}}(\mathbf{x},\mathbf{y})\mathbf{b}) &= \mathbf{g}_{\mathbb{M}}(\mathbf{P}^{-1}(\mathbf{x})S(\mathbf{x},\mathbf{y})\mathbf{P}(\mathbf{y})\mathbf{a},\mathbf{P}^{-1}(\mathbf{x})S(\mathbf{x},\mathbf{y})\mathbf{P}(\mathbf{y})\mathbf{b}) = \mathbf{g}(S(\mathbf{x},\mathbf{y})\mathbf{P}(\mathbf{y})\mathbf{a},S(\mathbf{x},\mathbf{y})\mathbf{P}(\mathbf{y})\mathbf{b})\\ &= \mathbf{g}(\mathbf{P}(\mathbf{y})\mathbf{a},\mathbf{P}(\mathbf{y})\mathbf{b}) = \mathbf{g}_{\mathbb{M}}(\mathbf{a},\mathbf{b}).\end{aligned}$$

From the properties $\text{TORS}^{\mathbb{M}} = 0$ and $\text{CURV} = 0$ and the previous results, we get the following chain of implications:

$$\text{TORS} = 0 \Rightarrow \nabla = \nabla^{\mathbb{M}} \Rightarrow \begin{cases}\text{TORS} = \text{TORS}^{\mathbb{M}} = 0,\\ \text{CURV}^{\mathbb{M}} = \text{CURV} = 0.\end{cases}$$

Since the torsion TORS is considered to be a measure of the dislocation density induced by the transplant (Nye, 1953), we may infer, with Davini (2001), that the vanishing of the dislocation density implies the local compatibility of the transplant metric. On the other hand the compatibility of the transplant metric does not imply the vanishing of the dislocation density since the isometric (rotational) part of the transplant map provides a nonvanishing torsion, unless the transplant is a constant field of rotations. To see this, we observe that, if the torsion vanishes, the transplant map $\mathbf{P} \in C^1(\mathbb{M}; BL(\mathbb{TM};\mathbb{TM}))$ will admit a potential $\boldsymbol{\eta}_{\mathbf{P}} \in C^1(\mathbb{M};\mathbb{M})$ and the compatibility of the transplant-induced metric implies by Theorem 2 that the transplant map is a constant isometric field.

3. Conclusions

The comprehensive presentation of Noll's theory of inhomogeneities developed in the paper suggests some remarks about the way it has been referred to in the recent literature concerning anelastic material behavior, phase transition and growth phenomena in continuum mechanics. As a first remark, the interpretation of the torsion of the connection, defined by the transplant-induced parallel transport, as a measure of the density of dislocations, is questionable since the transplant map is intended to describe anelastic transformations of any kind, such as thermal effects, material growth or phase transition phenomena. Moreover the knowledge of the torsion is not sufficient to recover the field of transplant maps itself which remains an undectable kinematic descriptor of the material behavior. Indeed, in the framework of continuum mechanics, no experimental test can be designed to accomplish this task. A second remark should be addressed to the very foundation of the anelastic behavior of materials based on the assignment of linear transformations of tangent spaces of the material manifold. Indeed all modern contributions, strongly influenced by Truesdell and Noll's impressive treatise on nonlinear field theories (Truesdell and Noll, 1965), follow the wake of the so-called

Kondo–Kröner–Lee or Noll–Wang decomposition of the deformation gradient which envisages an unnatural ordering of events in modeling plastic–elastic transformations. This geometrical picture of material behavior appears, at first sight, elegant and fascinating on the mathematical side. However, the underlying model is based on a chain decomposition which assumes an ordering of material responses that is physically unsound. Moreover this model requires the introduction of an intermediate local state, commonly dubbed, with misleading terminology, a configuration (intermediate, fictitious, conceptual, and so on). This is the source of serious troubles, both conceptual and operational, as witnessed by the many, vain, attempts to get rid of its undesirable collateral effects. But worse things are to come. Strange and unidentified mechanical objects are flying into the theory: the *plastic spin* trouble has provoked may headaches to scholars and several ineffective remedies have been suggested. We underline that the denomination of the uniformity (or transplant) map as *elastic inhomogeneity* has hidden its real significance of *anelastic transformation* (see Epstein and Maugin, 1996; Maugin and Epstein, 1998). Anelastic phenomena in materials are best modeled by considering the reference placement of the body as a Riemannian manifold endowed with two metric tensor fields. The former simulates the changes in length of the material fibers of each tangent space due to the configuration map describing the change of placement of the body in space. The latter takes account of the change of length due to anelastic phenomena. According to this point of view, the metric approach to material behavior, recently developed in (Romano et al., 2006a,b), provides a physically sound foundation for the theoretical description of anelastic phenomena in continuum mechanics.

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